1 Problem 3

$$\begin{aligned} \min_{y} \quad & 2y_1 \\ \text{s.t.} \quad & \begin{pmatrix} 0 & 5y_1 & 0 \\ 5y_1 & 3y_2 & 0 \\ 0 & 0 & 4y_1+2 \end{pmatrix} \end{aligned}$$

The top-left entry is 0 $\Rightarrow$ the first row and column are zeros, which means $y_1 = 0$ in any feaasible solution. The feasible solutions are: $y_1 = 0, y_2 \geq 0$. So the primal optimum is 0.

$$y_1 \begin{pmatrix} 0 & 5 & 0 \\ 5 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} + y_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \succeq 0$$

to get the dual:

$$\begin{aligned} & \max -2X_{33} \\ \text{s.t.} \quad & 5X_{12} + 5X_{21} + 4X_{33} = 2 \\ & 3X_{22} = 0 \\ & X \succeq 0 \end{aligned}$$

$X_{22} = 0$ and $X \succeq 0 \Rightarrow X_{12} = X_{21} = 0 \Rightarrow X_{33} = \frac{1}{2}$, and the optimal value for this dual SDP is -1.
The duality gap is not zero in the optimum points.

2 Problem 4

3 Problem 5

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2 \\ \text{s.t } Cx = d \end{aligned}$$

It can be written as:

$$\begin{aligned} \min_{x} x^T A^T Ax - 2b^T Ax \\ \text{s.t } Cx = d \end{aligned}$$

x^* is a solution if $Cx^* = d$ and $\|Ax_p - b\|_2^2 \leq \|Ax - b\|_2^2$ holds for any vector x that satisfies $Cx = d$
Primal solution:

$$x_p = (A^T A)^{-1} A^T b$$

Lagrangian:

$$L(x, \lambda) = \inf_x \left\{ x^T A^T Ax - 2b^T Ax + \lambda^T (Cx - d) \right\}$$

$$\frac{\partial L(x,\lambda)}{\partial x} = 2A^T Ax - 2A^T b + C^T \lambda$$

$$\frac{\partial L(x,\lambda)}{\partial x} = 0 \Rightarrow x^* = (A^T A)^{-1}(A^T b - \tfrac{1}{2}C^T \lambda)$$

KKT conditions:

1. $\lambda \geq 0$
2. $2A^T Ax^* - 2A^T b + C^T \lambda = 0$
3. $Cx^* - d = 0$

Dual solution:

$$\begin{bmatrix} x_d \\ \lambda \end{bmatrix} = \begin{bmatrix} 2A^T A & C^T \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2A^T b \\ d \end{bmatrix}$$

4 Problem 6

1)

$$\begin{gathered} \min_{x \in \mathbb{R}^n} C^T x \\ \text{s.t } Ax = b \\ x \geq 0 \end{gathered}$$

Lagrangian:

$$\begin{gathered} g(y, z) = \inf L(x, y, z) := C^T x + y^T (AX - b) - z^T x \\ 0 = grad_x L(x, y, z) = C + A^T y - z \\ g(y, z) = -y^T b \end{gathered}$$

Then dual problem:

$$\begin{gathered} \max_{y,z} g(y, z) \\ \text{s.t } C + A^T y - z = 0 \\ y \geq 0, z \geq 0 \end{gathered}$$

Which is equivalent to:

$$\begin{gathered} \max_{y} -y^T b \\ \text{s.t } C + A^T y = 0 \\ y \geq 0 \end{gathered}$$

KKT conditions:

1. $z, y \geq 0$
2. $-x^* \leq 0$
3. $Ax^* - b = 0$
4. $C + A^T y - z = 0$
5. $z^* x^* = 0$

2)

$$\min_{x \in \mathbb{R}^n} C^T x - \tau \sum \log(x_i)$$
$$\text{s.t } Ax = b$$

$$L(x, \lambda) = C^T x - \tau \sum \log(x_i) + \lambda^T (Ax - b) \ grad_x L = 0 = C - \begin{bmatrix} \frac{\tau}{x_1} \\ \cdots \\ \frac{\tau}{x_n} \end{bmatrix} + A^T \lambda$$