The Jet Metric

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Abstract. In order to define a metric on jet space, linear scale space is considered from a statistical standpoint. Given a scale σ, the scale space solution can be interpreted as maximizing a certain Gaussian posterior probability, related to a particular Tikhonov regularization. The Gaussian prior, which governs this solution, in fact induces a Mahalanobis distance on the space of functions. This metric on the function space gives, in a rather precise way, rise to a metric on n-jets. The latter, in turn, can be employed to define a norm on jet space, as the metric is translation invariant and homogeneous. Recently, [1] derived a metric on jet space and our results reinforce his findings, while providing a totally different approach to defining a scale space jet metric.

1 Introduction

Feature vectors made up of n-jets or other Gaussian derivative features are used in many image analysis and processing applications. Examples can be found in image retrieval and matching, representation, and image labelling and segmentation tasks (see for instance [2–9]). In many of the previous applications, the need arises to define a distance between such measurement vectors for which one should define a metric in this multidimensional space of measurements. The decision of which metric to actually use can be based on a study of the problem at hand, its optimality with respect to the given task, the right prior knowledge, educated guessing, etc. It is however also of interest to consider the possibility of defining such a metric in 'empty' space in a principled way, *i.e.*, provide a metric given the only thing we know is that the derivative measurements are obtained by means of, say, Gaussian apertures.

1.1 Outline

Section 2 provides a sketch of a recently proposed approach to deriving a metric on jet space. Subsequently, Section 3 provides a new approach to it based on statistical considerations, starting from a known scale space regularization framework, and relating this to a statistical interpretation. In four steps, this section identifies the Gaussian prior governing scale space regularization, discusses the Mahalanobis distance induced by it, suggests a marginalization of the prior to define distances in subspaces, and finally defines the general *n*-jet metric using the latter suggestion. Section 4 concludes the paper.

Finally, as a note, this work should not be confused with the work presented in [10–12], which deal with an a priori metric in scale space itself, or rather its absence. In addition, we should note that this work is not concerned with distances between distribution, which may dealt with using information geometry.

2 [1]'s Jet Norm

Recently, as part of a considerably larger effort, an attempt to define the proper distance on scale space *n*-jets—with a focus on 2-jets—has been undertaken in [1] (*cf.* [13]). The author defines a norm on jet space, which induces the desired metric. In order to find the appropriate jet space norm, several requirements that should characterize it are specified, e.g., it is supposed to be invariant to translation, rotation, and reflection of the image domain from which the jet is derived and also an invariance to constant image offsets is assumed. The latter implies, for instance, that the zeroth order jet should be of no influence on the norm and therefore we are in fact dealing with a semi-norm.

However, these requirements are, indeed, requirements on images and not on jets. In order to relate jets to images, the authors use the language of metamerism, which relates a collection of image measurements to the class of images that give rise to precisely these observations [14–16]. Another point to relating jets to images is that in image space there is a 'natural' choice for the so-called scale space norm based on which the actual scale space jet norm can then be constructed. Given an image $I : \mathbb{R}^n \to \mathbb{R}$, the scale space $\|\cdot\|_\sigma$ norm takes on something like the form of a weighted L^2 norm

$$\|I\|_\sigma = \left(\int_{\mathbb{R}^d} g_\sigma(x) I^2(x) - \left(\int_{\mathbb{R}^d} g_\sigma(x) I(x) dx \right)^2 dx \right)^{\frac{1}{2}}, \tag{1}$$

where g_σ is the Gaussian kernel at scale σ. Note that the latter part in the definition of the norm makes sure that it is invariant to constant image offsets and therefore, in fact, we are dealing with a semi-norm again.

The idea is now to take a unique representative from the metameric class of images related to the jet and define its norm to be equal to the norm of this unique representative. The choice made in [1] is the function from the class that minimizes the scale space norm. This turns out to be a polynomial of the same degree as the jet is. Without going into any additional details concerning the exact derivation, we merely provide the end result, in which only the norm for the 2-jet of two-dimensional images is fully worked out.

Let c_{uv} be the two-dimensional image derivative at scale σ at the origin, which is given by

$$c_{uv} = \int_{\mathbb{R}^2} \frac{\partial^{u+v}}{\partial x^u \partial y^v} g_\sigma(x) I(x) dx \tag{2}$$

and let the corresponding 2-jet be denoted by J_2:

$$J_2 = (c_{00}, c_{01}, c_{10}, c_{11}, c_{02}, c_{20})^T \, . \tag{3}$$

The 2-jet norm is now given by

$$\|J_2\| = \sigma(c_{10}^2 + c_{01}^2 + \tfrac{1}{2}\sigma^2(c_{20}^2 + c_{11}^2 + c_{02}^2))^{\frac{1}{2}} \, . \tag{4}$$

It is straightforward to check that this norm fulfills the requirements mentioned earlier. As always, the norm also induces a metric on the space. Given two jets J_2 and Y_2, $\|J_2 - Y_2\|$ gives the distance between the two.

3 Prior Induced Jet Metric

The derivation of the norm in [1] and its induced metric, may involve several nontrivial steps to some. An important one, for instance, being the choice of representative function from the metameric class defined by the jet. As [1] remarks, the function taking on the maximum scale space norm is not uniquely determined and is therefore excluded. However, this does not necessarily imply that the minimum should be considered. Moreover, it may be questioned whether it should actually be the scale space norm to base the right choice of representative on, given that the definition of a jet norm should anyway be based on such metameric representative.

In this section, a different approach to defining a jet norm is proposed. Linear scale space is considered from a statistical standpoint from which it is clear that a certain Gaussian prior in the space of functions actually governs its behavior. This prior induces a Mahalanobis distance on this space, which in turn can be used to define a metric on n-jets. Finally, as the metric is translation invariant and homogeneous, it can also be used to define a norm on scale space jets. For two-dimensional images, as it turns out, the jet metric defined is the same as the one given in [1].

3.1 Gaussian Prior

In [17], scale space is related to a specific instance of Tikhonov regularization [18]. The regularized scale space image I_σ at scale σ associated to the initial image I on $\mathbb{R}^d$ minimizes the functional E defined as

$$E[\Upsilon] := \frac{1}{2}\int_{\mathbb{R}^d} (\Upsilon(x) - I(x))^2 + \sum_{i=1}^{\infty}\sum_{|N|=i} \frac{\sigma^{2i}}{2N!}\left(\frac{\partial^{|N|}\Upsilon(x)}{\partial x^N}\right)^2 \, dx \,, \tag{5}$$

where

$$N = N_1, \ldots, N_d \tag{6}$$

is a multi-index used to denote derivatives of order

$$|N| = \sum_{i=1}^{d} N_i \tag{7}$$

and $N!$ equals the product over all factorials of all d indices, *i.e.*,

$$N! = \prod_{i=1}^{d} N_i! \,, \tag{8}$$

where d is the dimensionality of the images in the space. The first term on the right hand side penalizes deviations of the function Υ from the given image I, while the second part is the regularization term for Υ, not involving I. It is this latter part that can be readily interpreted as a prior P on the space of image through a multiplication by -1 and subsequently exponentiating it (see for example [19, 20]):

$$P(I) = \frac{1}{Z}\exp\int_{\mathbb{R}^d} -\sum_{i=1}^{\infty}\sum_{|N|=i} \frac{\sigma^{2i}}{2N!}\left(\frac{\partial^{|N|}I(x)}{\partial x^N}\right)^2 \, dx \,. \tag{9}$$

This is a well-known *Gaussian* prior used, for instance, in areas that are concerned with kernel methods and machine learning [21] (*cf.* [22]).

3.2 Scale Space Mahalanobis Distance

Given that P defines a Gaussian prior on image space, a straightforward choice of metric on this space is the Mahalanobis distance induced by the covariance structure of the Gaussian distribution, which is in analogy with the finite dimensional case [23]. That is, given a covariance matrix C on the space, the Mahalanobis distance d_M between two vectors x and y is given by

$$d_M(x, y) = \sqrt{(x-y)^{\mathrm{t}} C^{-1} (x-y)} \,. \tag{10}$$

Initially, the fact that one needs the inverse of the covariance matrix, may seem troublesome as the Gaussian density specified in Equation (9) is given on an infinite dimensional space for which it may not be readily clear how to define such inverses. Luckily Gaussian densities are actually defined by means of their inverted covariance matrices and Equation (9) directly provides us with the appropriate scale space Mahalanobis distance d_S, which is defined for two images I and Υ as

$$d_S(I,\Upsilon) = \sqrt{\int_{\mathbb{R}^d} \sum_{i=1}^{\infty} \sum_{|N|=i} \frac{\sigma^{2i}}{2N!} \left(\frac{\partial^{|N|} I(x)}{\partial x^N} - \frac{\partial^{|N|} \Upsilon(x)}{\partial x^N} \right)^2 dx} \,. \tag{11}$$

Note that in the previous equation, the summation and integration go over all the image derivatives and all spatial locations, however there is no covariance between any of these. In other words, cross terms between image I and image Υ involving different locations or different multi-indices for the respective images do not occur, only squares of differences between corresponding components of I and Υ do, *i.e.*, $\frac{\partial^{|N|} I(x)}{\partial x^N}$ and $\frac{\partial^{|N|} \Upsilon(x)}{\partial x^N}$ for the same choice of x and N. In a sense, the covariance matrix and its inverse are diagonal.

3.3 Marginalization of the Scale Space Metric

Given that none of the components in the prior model governing scale space are correlated, if one is interested in a metric only involving a subset of the components, it may be reasonable to simply restricting the distance calculation to this subset in order to provide a metric in this lower-dimensional space. A similar situation arises in three-dimensional Euclidean space, when one is merely interested in the first two dimensions. In that case, the distance between points would be given by considering their distance in the two-dimensional Euclidean space corresponding to the first two coordinates, the third dimension is simply discarded in the calculation of the metric.

Clearly, Euclidean space corresponds to a covariance that equals the identity matrix and a deviation from the identity as covariance may render the approach suggested invalid. However, realizing that restricting the distance calculations to a subspace of the original space actually means that a *marginalization* of the distribution has taken place, we can further substantiate the approach sketched above. A result on the marginalization of Gaussian processes [23] states that if X is normally distributed, any set of components of X is distributed in a multivariate normal way for which the means, variances, and covariances are obtained by taking the corresponding components of the original mean vector and covariance matrix. Therefore, one can simply pick out the locations x and multi-indices N from Equation (11) in which one is interested and only integrate or sum over these.

3.4 The *n*-Jet Metric

In order to define any n-jet metric coming from any d-dimensional images, the spatial location has to be restricted to the origin and the derivative order has to bounded by n, which would directly lead to the following metric

$$d'_J(J_n, Y_n) = \sqrt{\sum_{i=1}^{n} \sum_{|N|=i} \frac{\sigma^{2i}}{2N!} \left(\frac{\partial^{|N|} I(0)}{\partial x^N} - \frac{\partial^{|N|} \Upsilon(0)}{\partial x^N} \right)^2} , \tag{12}$$

in which the jets J_n and Y_n are obtained from the images I and Υ, respectively.

However, we are not necessarily interested in a metric on image derivatives at scale zero. The scale specific prior was used because, the images were assumed to be scale space regularized at that scale. Therefore, the interest is not with Equation (12), but rather with its adaptation into which σ blurred images are substituted, *i.e.*

$$\begin{aligned} d_J(J_n, Y_n) &= \sqrt{\sum_{i=1}^{n} \sum_{|N|=i} \frac{\sigma^{2i}}{2N!} \left(\frac{\partial^{|N|} I_\sigma(0)}{\partial x^N} - \frac{\partial^{|N|} \Upsilon_\sigma(0)}{\partial x^N} \right)^2} \\ &= \sqrt{\sum_{i=1}^{n} \sum_{|N|=i} \frac{\sigma^{2i}}{2N!} \left(c_N - \gamma_N \right)^2} , \end{aligned} \tag{13}$$

in which the coefficients c_N and γ_N are the jet coefficients corresponding to derivative multi-index N, which is the general form of the coefficients given in Equation (2). Equation (13) defines our general jet metric for which it is easy to demonstrate that it equals [1]'s metric on 2-jet space for 2-dimensional images—or generally any higher-order jets on two-dimensional images.

Note that d_J is both translation invariant and homogeneous, *i.e.*,

$$d_J(J_n, Y_n) = d_J(J_n + Q_n, Y_n + Q_n) \tag{14}$$

and

$$d_J(cJ_n, cY_n) = |c| d_J(J_n, Y_n) \tag{15}$$

for all n-jets J_n, Y_n, and Q_n, and constant $c \in \mathbb{R}$, which implies that $d_J(J_n, 0)$ is a norm on jet space. We also should remark that the requirements imposed in

[1] on the norm and the metric are 'automatically' fulfilled by our approach and they do not have to be enforced explicitly.

Besides, note also that when dealing with *scale-normalized* derivatives—*i.e.*, dimensionless measurements, the expression for metric becomes scale independent:

$$d_J(J_n, Y_n) = \sqrt{\sum_{i=1}^{n} \sum_{|N|=i} \frac{1}{2N!} \left(c'_N - \gamma'_N\right)^2 \, dx}\,, \tag{16}$$

where c'_N and γ'_N are the scale-normalized jet coefficients. In this way, jets from different scales may actually be compared to each other using the jet metric.

4 Conclusion and End Remarks

Through an approach that substantially differs from the initial one suggested in [1], a definition of a scale space jet metric in 'empty' space was proposed. Our findings support the original definition of this metric, strengthening the validity of it. The statistical approach employed to come to our derivation of the metric may also be of interest in itself, as it provides a general tool to formulate more exotic metrics, norm, distances, or even similarity measures.

One remark that should be made is that in the Gaussian case, the choice of the Mahalanobis distance may seem obvious. However, one should realize that other choices may be possible as well. One of these is to consider image space endowed with a Gaussian density as a realization of Gauss space [24]. Such an approach would, however result in highly nonlinear behavior, but we have no argument yet to discard this option beforehand. Another option might be to consider the so-called canonical distance measure proposed in [25]. Even though the latter is normally used in supervised learning algorithms, it might provide additional insight with respect to our current approach.

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