Proof by Contrapositive

DISCRETE MATHEMATICS

19BCE245 AAYUSH SHAH

Contents

Overview

What is ContraPositive

and its truth table

Logic is the relationship that leads, on the basis of a series of other propositions, to the acceptance of one proposal. In logic, negating both terms and reversing the direction path produces the contrapositive of a conditional statement.

the contrapositive of the statement “if X, then Y” is “if not Y, then not X”. A statement and its contrapositive are logically equivalent, in that context, that if the statement is true, then its contrapositive is also true and vice versa.

Rule of Inference

Brief Introduction

A logical type consisting of a function that takes propositions, analyses their syntax, and returns a conclusion is called a rule of inference or transformation rule. Proof by Contrapositive is also a rule of inference which is used in proofs in mathematics. Sometimes, more often than not, If the contrapositive is easier to prove than the original conditional statement itself, this approach of proof by Contrapositive is preferred.

TruthTable of contrapositive

A truth table is a mathematical table which is used in the logic. By the use of this truth table, the relevance of proof by contrapositive can be demonstrated. In which it is shown that p → q and ˜q → ˜p have the same truth values in all cases :

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Applications

Real-life Applications of Proof by contrapositive

Proving irrationality of $\sqrt{2}$

When If it is easier to determine the truth of the contrapositive than the truth of the statement itself, The contrapositive of a statement can be a powerful tool to prove mathematical theorems as it always has the same truth value i.e. truth or falsity. A proof by contraposition or contrapositive is a direct proof of a statement's contrapositive. However, it is also possible to use indirect approaches, such as proof by contradiction, with contraposition, for example in proof of the irrationality of the square root of 2. The statement that "If $\sqrt{2}$ is rational, then it can be expressed as an irreducible fraction" can be made through the concept of a rational number. This statement is valid because it is the restatement of a definition. "If $\sqrt{2}$ cannot be represented as an irreducible fraction, then it is not rational" is the contrapositive of that statement. Like the initial argument, this contrapositive is also valid. Therefore, if it can be shown that $\sqrt{2}$ cannot be represented as an irreducible fraction, then $\sqrt{2}$ must not be a rational number. By contradiction, the latter can be proven.

Square root of non-square Integer

The previous example used the contrapositive of a definition to illustrate a theorem. By proving the contrapositive of the theorem's argument, one may also prove a theorem. In order to show that if a positive integer N is a non-square number, its square root is irrational, we can prove that if a positive integer $\sqrt{N}$ has a rational square root, then N is a square number. This can be illustrated by setting N equal to the rational expression p/q with p and q being positive integers without a common prime factor, and squaring to get N = p^2/q^2 and nothing that since N is a positive integer q = 1 so that N = p^2 , which is a square number.

Modus tollens

This concept of Proof by Contrapositive is also used in Modus tollens. Modus tollens (i.e. denying the consequent) is a deductive argument form and a rule of inference. The form of "If P, then Q. Not Q. Therefore, not P." Taken by the Modus tollens. It is an application of the fact that if a statement is true, the contrapositive is also true. The type shows that P's inference implies Q to Q's negation, implying that P's negation is a true statement.

Applications

in various Mathematical Frameworks

Probability Calculus

Contrapositive is an instance of the Bay’s theorem that can be represented in a particular form as an equation which uses contrapositive's application.

Subjective Logic

In subjective logic, contrapositive represents an example of the subjective Bayes’ theorem expressed as an equation.

Probability Calculus

Contrapositive is an instance of the Bay's theorem that can be represented in a particular form as :

$$\Pr(\neg P \mid \neg Q) = \frac{\Pr(\neg Q \mid \neg P)\, a(\neg P)}{\Pr(\neg Q \mid \neg P)\, a(\neg P) + \Pr(\neg Q \mid P)\, a(P)}.$$

The conditional probability Pr(˜Q | P) generalises the logical statement P→˜Q in the equation above, i.e. we can also attach some probability to the statement in addition to assigning TRUE or FALSE. The term a(P) denotes the prior probability of P. Suppose that Pr(˜Q | P) = 1 is equivalent to P→˜Q which is TRUE, and that Pr(˜Q | P) = 0 is equivalent to P→˜Q which is FALSE. It is then easy to see that Pr(˜P | ˜Q) = 1 it is Real, i.e. that. This is because Pr(˜Q | P) = 1 – Pr(Q | P) that's why the fraction on the right-hand side of the above equation is equal to 1, and is therefore Pr(˜P| ˜Q) equivalent to ˜Q→˜P being TRUE.

Therefore, the theorem of Bayes' reflects a generalization of contraposition.

Subjective Logic

In subjective logic, contrapositive represents an example of the subjective Bayes' theorem expressed as :

$$(\omega^{A}_{P\tilde{|}Q}, \omega^{A}_{P\tilde{|}\neg Q}) = (\omega^{A}_{Q|P}, \omega^{A}_{Q|\neg P})\, \widetilde{\phi}\; a_P\,,$$

where $(\omega^{A}_{P\tilde{|}Q}, \omega^{A}_{P\tilde{|}\neg Q})$ denotes a pair of binomial conditional opinions given by source A. And the parameter aP denotes a pair of binomial conditional opinions given by source A.

This subjective of Bayes' is a generalisation of both Theorem of Bayes' and contraposition.

Conclusion

Since contrapositive proof includes negating such logical proof, statements, one must be vigilant. If the statements are complex at all, negation can be very difficult. However, often there are some negative statements already in the given proposal, the natural choice is proof by contrapositive. Although it is possible to use direct proof exclusively , there are times when contrapositive proof is far simpler.

Thank You !

References :
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https://en.wikipedia.org/wiki/Contraposition
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