Functions

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Functions

- A function f from a set A to a set B is an assignment of exactly one element of B to each element of A.
- We write
- $f(a) = b$
- if b is the unique element of B assigned by the function f to the element a of A.

- If f is a function from A to B, we write
- $f: A \to B$
- (note: Here, "$\to$" has nothing to do with if… then)

Functions

- If $f:A \to B$, we say that A is the domain of f and B is the codomain of f.

- If $f(a) = b$, we say that b is the image of a and a is the pre-image of b.

- The range of $f:A \to B$ is the set of all images of all elements of A.

- We say that $f:A \to B$ maps A to B.

Functions

- Let us take a look at the function $f:P \to C$ with
- P = {Vijay, Arvind, Praveen, Bhushan}
- C = {GJ, Delhi, MP, AP}

- f(Vijay) = GJ
- f(Arvind) = Delhi
- f(Praveen) = MP
- f(Bhushan) = AP

- Here, the range of f is C.

Functions

- Let us re-specify f as follows:

- f(Vijay) = GJ
- f(Arvind) = AP
- f(Praveen) = MP
- f(Bhushan) = AP

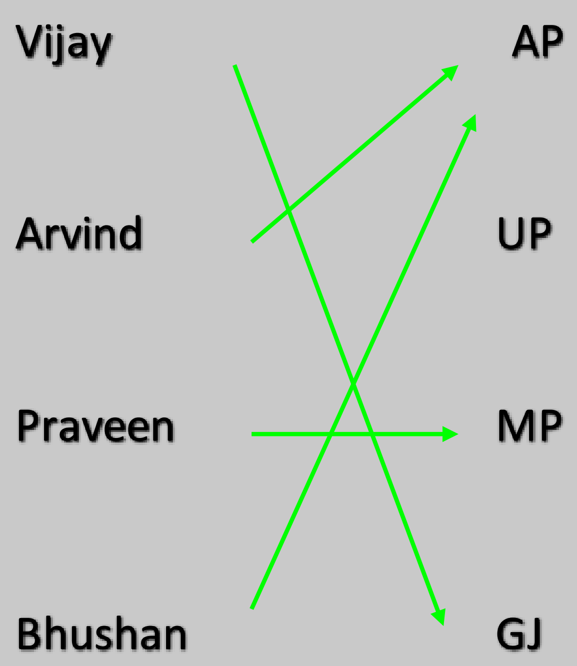
- Is f still a function? yes

What is its range? {GJ, AP, MP}

Functions

- Other ways to represent f:

x	f(x)
Vijay	GJ
Arvind	AP
Praveen	MP
Bhushan	AP



Functions

- If the domain of our function f is large, it is convenient to specify f with a formula, e.g.:

- f:**R**→**R**
- f(x) = 2x

- This leads to:
- f(1) = 2
- f(3) = 6
- f(-3) = -6
- …

Functions

- We already know that the range of a function $f:A \rightarrow B$ is the set of all images of elements $a \in A$.

- If we only regard a subset $S \subseteq A$, the set of all images of elements $s \in S$ is called the image of S.

- We denote the image of S by f(S):

- $f(S) = \{f(s) \mid s \in S\}$

Functions

- Let us look at the following well-known function:
- f(Vijay) = GJ
- f(Arvind) = AP
- f(Praveen) = MP
- f(Bhushan) = AP

- What is the image of S = {Vijay, Arvind} ?
- f(S) = {GJ, AP}

- What is the image of S = {Arvind, Bhushan} ?
- f(S) = {AP}

Properties of Functions

- A function $f:A \to B$ is said to be one-to-one (or injective), if and only if

- $\forall x, y \in A\ (f(x) = f(y) \to x = y)$

- In other words: f is one-to-one if and only if it does not map two distinct elements of A onto the same element of B.

Properties of Functions

- And again…
- f(Vijay) = GJ
- f(Arvind) = AP
- f(Praveen) = MP
- f(Bhushan) = AP
- Is f one-to-one?

- No, Arvind and Bhushan are mapped onto the same element of the image.

g(Vijay) = GJ
g(Arvind) = UP
g(Praveen) = MP
g(Bhushan) = AP

Is g one-to-one?

Yes, each element is assigned a unique element of the image.

Properties of Functions

- How can we prove that a function f is one-to-one?
- Whenever you want to prove something, first take a look at the relevant definition(s):
- $\forall x, y \in A\ (f(x) = f(y) \rightarrow x = y)$

- Example:
- $f: \mathbf{R} \rightarrow \mathbf{R}$
- $f(x) = x^2$

- Disproof by counterexample:
- $f(3) = f(-3)$, but $3 \neq -3$, so f is not one-to-one.

Properties of Functions

- ... and yet another example:

- $f: \mathbf{R} \to \mathbf{R}$
- $f(x) = 3x$

- One-to-one: $\forall x, y \in A\ (f(x) = f(y) \to x = y)$
- To show: $f(x) \neq f(y)$ whenever $x \neq y$ (indirect proof)

- $x \neq y$

$\Leftrightarrow 3x \neq 3y$

$\Leftrightarrow f(x) \neq f(y)$,

so if $x \neq y$, then $f(x) \neq f(y)$, that is, f is one-to-one.

Properties of Functions

- A function $f:A \rightarrow B$ with $A,B \subseteq R$ is called strictly increasing, if
- $\forall x,y \in A \ (x < y \rightarrow f(x) < f(y))$,
- and strictly decreasing, if
- $\forall x,y \in A \ (x < y \rightarrow f(x) > f(y))$.

- Obviously, a function that is either strictly increasing or strictly decreasing is one-to-one.
-

Properties of Functions

- A function $f:A\to B$ is called onto, or surjective, if and only if for every element $b\in B$ there is an element $a\in A$ with $f(a) = b$.

- In other words, f is onto if and only if its range is its entire codomain.

- A function $f: A\to B$ is a one-to-one correspondence, or a bijection, if and only if it is both one-to-one and onto.

- Obviously, if f is a bijection and A and B are finite sets, then $|A| = |B|$.

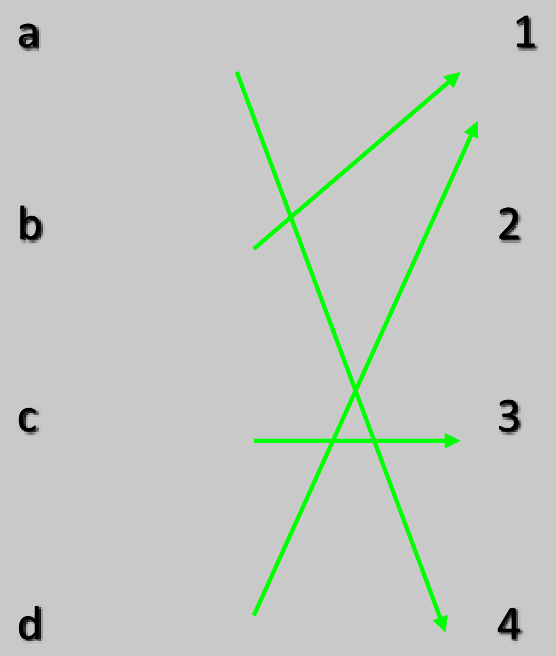
Properties of Functions

- Examples:

- In the following examples, we use the arrow representation to illustrate functions $f:A \rightarrow B$.

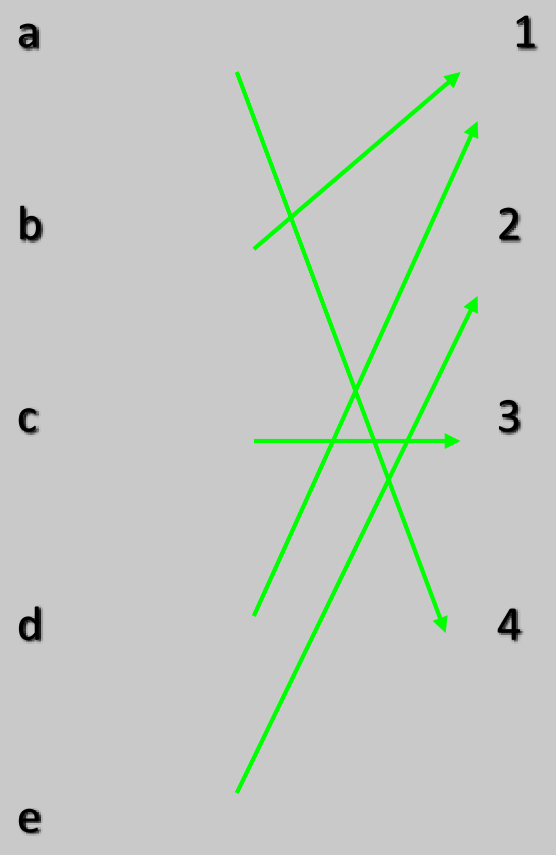
- In each example, the complete sets A and B are shown.

Properties of Functions



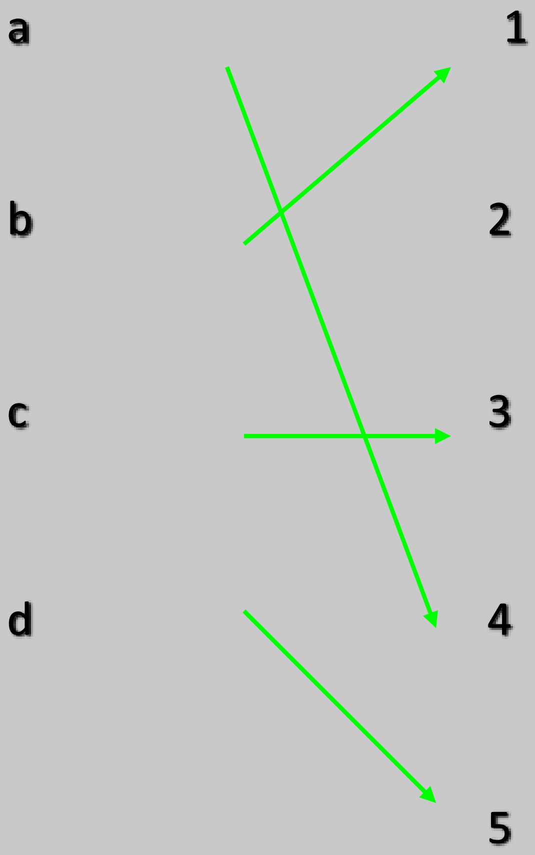
- Is f injective?
- No.
- Is f surjective?
- No.
- Is f bijective?
- No.

Properties of Functions



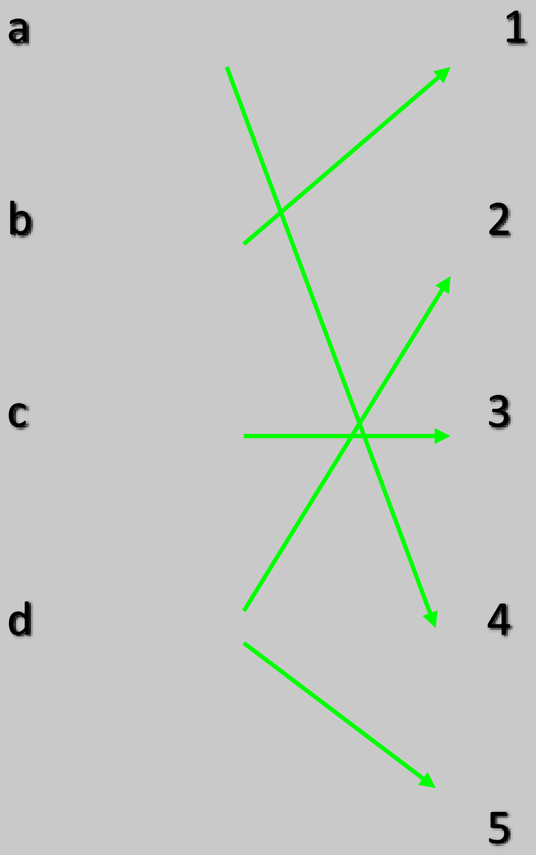
- Is f injective?
- No.
- Is f surjective?
- Yes.
- Is f bijective?
- No.

Properties of Functions



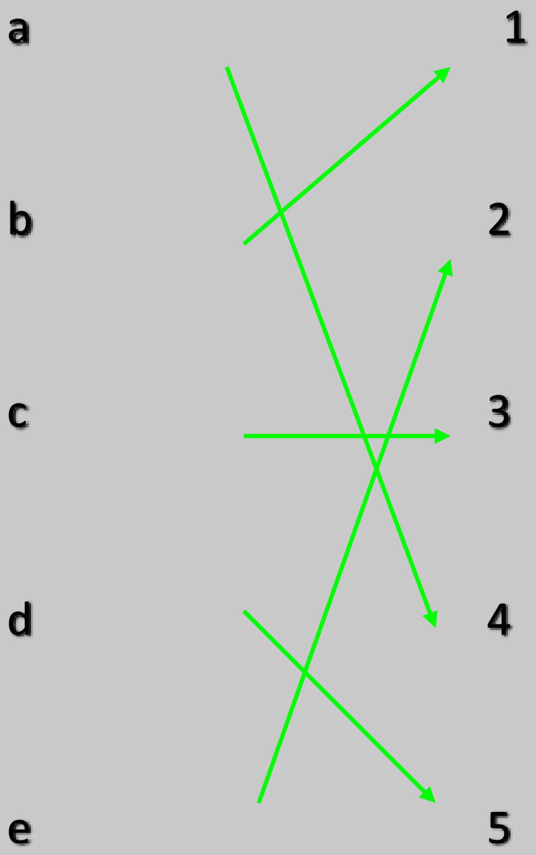
- Is f injective?
- Yes.
- Is f surjective?
- No.
- Is f bijective?
- No.

Properties of Functions



- Is f injective?
- No! f is not even a function!

Properties of Functions



- Is f injective?
- Yes.
- Is f surjective?
- Yes.
- Is f bijective?
- Yes.

Inversion

- An interesting property of bijections is that they have an **inverse function**.

- The **inverse function** of the bijection $f:A \to B$ is the function $f^{-1}:B \to A$ with
- $f^{-1}(b) = a$ whenever $f(a) = b$.

Inversion

Example:

$f(a) = 1$
$f(b) = 2$
$f(c) = 3$
$f(d) = 4$
$f(e) = 5$

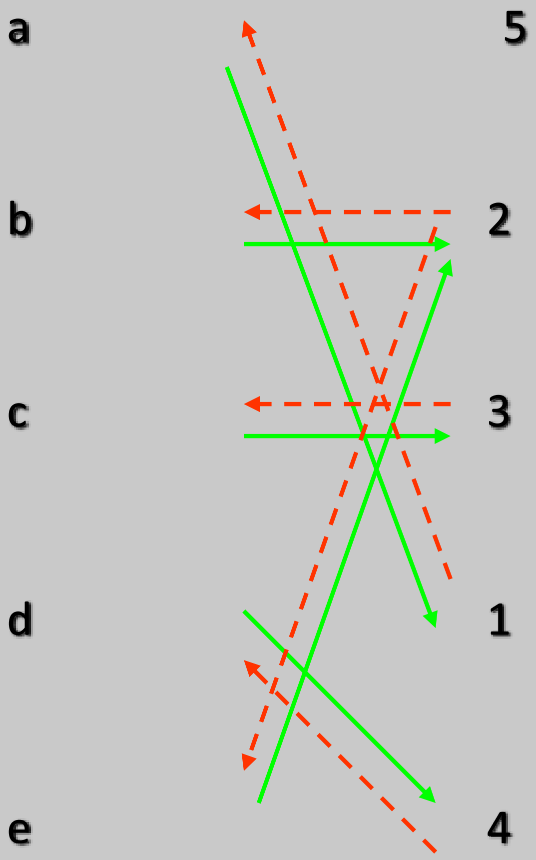
Clearly, f is bijective.

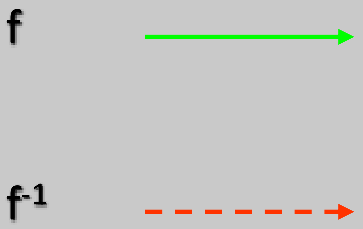
The inverse function f^{-1} is given by:

$f^{-1}(1) = a$
$f^{-1}(2) = b$
$f^{-1}(3) = c$
$f^{-1}(4) = d$
$f^{-1}(5) = e$

Inversion is only possible for bijections (= invertible functions)

Inversion





• $f^{-1}: C \to P$ is no function, because it is not defined for all elements of C and assigns two images to the pre-image 2.

Sum and Product of a Function

- Let f1 and f2 be function from A to R, then f1+f2 and f1*f2 are also functions from A to R defined for all $x \in A$ by

- $(f1+f2)(x) = f1(x)+f2(x)$
- and
- $(f1*f2)(x) = f1(x)*f2(x)$

Example

- Let f1 = x^4+2x^2+1 and f2 = $2-x^2$ then
- (f1+f2)(x) = f1(x)+f2(x)
- (f1+f2)(x) = $\{x^4+2x^2+1\} + \{2-x^2\}$
- (f1+f2)(x) = x^4+x^2+3

- (f1*f2)(x) = f1(x)*f2(x)
- (f1*f2)(x) = $\{x^4+2x^2+1\} * \{2-x^2\}$
- (f1*f2)(x) = $2x^4-x^6+4x^2-2x^4+2-x^2$
- (f1*f2)(x) = $-x^6+3x^2+2$

Composition

- The **composition** of two functions $g:A \rightarrow B$ and $f:B \rightarrow C$, denoted by $f \circ g$, is defined by

- $(f \circ g)(a) = f(g(a))$

- This means that
  - **first**, function g is applied to element $a \in A$, mapping it onto an element of B,
  - **then**, function f is applied to this element of B, mapping it onto an element of C.
  - **Therefore**, the composite function maps from A to C.

Composition

- Example:

- $f(x) = 7x – 4$, $g(x) = 3x$,
- $f: \mathbf{R} \rightarrow \mathbf{R}$, $g: \mathbf{R} \rightarrow \mathbf{R}$

- $(f \circ g)(5) = f(g(5)) = f(15) = 105 – 4 = 101$

- $(f \circ g)(x) = f(g(x)) = f(3x) = 21x - 4$

Composition

- Composition of a function and its inverse:
- $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$
- The composition of a function and its inverse is the **identity function** $i(x) = x$.

Graphs

- The **graph** of a function $f:A \rightarrow B$ is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.

- The graph is a subset of $A \times B$ that can be used to visualize f in a two-dimensional coordinate system.

Floor and Ceiling Functions

- The floor and ceiling functions map the real numbers onto the integers ($\mathbf{R} \to \mathbf{Z}$).

- The floor function assigns to $r \in \mathbf{R}$ the largest $z \in \mathbf{Z}$ with $z \leq r$, denoted by $\lfloor r \rfloor$.

- **Examples:** $\lfloor 2.3 \rfloor = 2$, $\lfloor 2 \rfloor = 2$, $\lfloor 0.5 \rfloor = 0$, $\lfloor -3.5 \rfloor = -4$

- The ceiling function assigns to $r \in \mathbf{R}$ the smallest $z \in \mathbf{Z}$ with $z \geq r$, denoted by $\lceil r \rceil$.

- **Examples:** $\lceil 2.3 \rceil = 3$, $\lceil 2 \rceil = 2$, $\lceil 0.5 \rceil = 1$, $\lceil -3.5 \rceil = -3$

Steps to find the inverse of a function

1. Replace the function f(x) by y in the equation describing the function.

2. Swap the variables x and y, i.e., replace x by y and vice versa.

3. Solve for y.

4. Replace y by $f^{-1}(x)$.

let f(x) = x+1

step 1 : y = x+1

step 2 : x = y+1

step 3 : x-1 = y

Final Answer : f^(-1) (x) = x-1

Exercise-1

- Find the inverse of the function **$f(x) = x^3+2$**.

- Solution:
- Step1: $y= x^3+2$
- Step2: $x= y^3+2$
- Step3: $x-2=y^3$

  $y=(x-2)^{1/3}$
- Step4: **$f^{-1}(x) = (x-2)^{1/3}$**

Exercise-2

- Find the inverse of the function **$f(x) = (x+1)/x$**.
- Solution:
- Step1: $y= (x+1)/x$
- Step2: $x= (y+1)/y$
- Step3: $xy= y+1$

  $xy-y=1$

  $y(x-1)=1$

  $y=1/(x-1)$
- Step4: **$f^{-1}(x) = 1/(x-1)$**

Exercise-3

- Find the inverse of the function **$f(x) = (x-2)^{1/3}$**.

f^(-1) (x)= x^(3) + 2

Exercise-4

- Find the inverse of the function **f(x) = (7+4x)/(6-5x)**.

inverse = (6x-7)/(5y+4)

Exercise-5

- Find the inverse of the function **$f(x) = 4e^{(6x+2)}$**.

Inverse = [(4th root of y)-2]/4

Properties of Inverse function

- The inverse function of a function is unique.
- The inverse function of a bijective function is also bijective.
- If $f:A \rightarrow B$ and $g:B \rightarrow C$ are two bijective functions, then $g \circ f:A \rightarrow C$ is also a bijective function and $(g \circ f)^{-1}=f^{-1} \circ g^{-1}$.
- If $f(x)=y$, then $f^{-1}(y)=x$.
- $f \circ g = g \circ f$, only when $f^{-1} = g$ or $g^{-1} = f$ and $f \circ g(x) = g \circ f(x) = x$.

Exercise-6

- Let A=B=Z and let f:A→B be defined by f(a)=a+1, for a∈A. Is f invertible?

→ **YES !**

Exercise-7

- Let R be the set of real numbers and let f:R→R be defined by $f(x)=x^2$. Is f invertible?

→ **No :(**

Exercise-8

- Let the functions f: A→B, g: B→C and h: C→D be defined by the following set of rules:
- f(a)=2, f(b)=1, f(c)=2.
- g(1)=y, g(2)=x, g(3)=w.
- h(x)=4, h(y)=6, h(z)=4, h(w)=5.

Then (A) Find the composition function $h \circ g \circ f$? A ⟶ D

(B) Determine if each function is one-to-one.

⟶ **No, as in D, 4 is assigned to two numbers of domain.**

(c) Determine if each function is onto.

⟶ **No, as in D, 4 and 5 are not included in the range.**

Pigeon Hole Principle

This has been left blank for your self study. 🙂

Thank You