2.1. CONTINUOUS RANDOM VARIBLE

A random variable X which can take every value in the domain or when its range R is an interval then X is continuous random variable.

Example :

1. Age
2. Height
3. Weight
4. Temperature

2.2. PROBABILITY DENSITY FUNCTION

The probability density function of random variable X is defined as

$$f_x(x) = P(x \leq X \leq x + \delta x)/\delta x$$

for small interval $(x, x + \delta x)$ of length dx around the point x

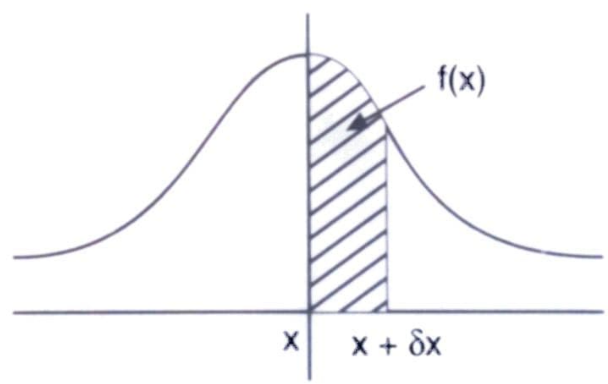

Fig. 2.1

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

which represent the area between the curve $y = f(x)$, x axis and the ordinate at $x = a$ and $x = b$ since total probability is unity.

i.e., $$\int_{-\infty}^{\infty} f(x)dx = 1$$

The probability density function (p.d.f) of a random variable X usually denoted by $f_x(x)$ or simply $f(x)$ has following properties.

1. $f(x) \geq 0, -\infty < x < \infty$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

2.3. CUMULATIVE DISTRIBUTION (DISTRIBUTION FUNCTION)

If X is a random variable, then $P(X \leq x)$ is called the cumulative distribution (c.d.f) or simply distribution function and it is denoted by $F(x)$.

$$\therefore \quad F(x) = P(X \leq x)$$

$$F(x) = P(X \leq \mathbf{x}) = \int_{-\infty}^{x} f(x)\,dx$$

2.4. EXPECTATION OF RANDOM VARIABLE

If X is a continuous random variable, then the expectation of the random variable X as defined by

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\,dx$$

The expected value of X^2 is defined as

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$E(X)$ is also called mean of X

Properties

1. If X is random variable and a is constant then
   (i) $E(a) = a$
   (ii) $E(aX) = aE(X)$
   (iii) $E(X - \bar{X}) = 0$
2. If X and Y are two random variables then
   $$E(X + Y) = E(X) + E(Y)$$
3. $E(XY) = E(X)\, E(Y)$ if X and Y are two independent random variable.
4. If $y = a + bx$ where a and b are constants then
   $$E(Y) = E(aX + b) = aE(X) + b$$

2.5. VARIANCE AND STANDARD DEVIATION OF CONTINUOUS RANDOM VARIABLE

Variance of x is defined as

$$\text{Var}(X) = V(X)$$

$$= E(X - \bar{X})^2 = E(X^2) - [E(X)]^2$$

Standard deviation of random varaible x is denoted by S.D(x) and is defined as

$$\text{S.D.}(x) = \sigma = \sqrt{V(X)} = +\sqrt{E(X)^2 - [E(X)]^2}$$

Example 2.1. A continuous random variables X has a probability density function defined by

$$f(x) = \begin{cases} \frac{1}{16}(3+x)^2 \text{ if } -3 \le x - 1 \\ \frac{1}{16}(6-2x^2) \text{ if } -1 \le x < 1 \\ \frac{1}{16}(3-x)^2 \text{ if } 1 < x \le 3 \\ 0 \qquad \text{elsewhere} \end{cases}$$

Verify that $f(x)$ is a density function and also find the mean of the random variable X.

Solution. Since $f(x)$ is density function, $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{-3} f(x)dx + \int_{-3}^{-1} f(x)dx + \int_{-1}^{1} f(x)dx + \int_{1}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx$$

$$= \int_{-\infty}^{-3} 0.dx + \int_{-3}^{-1} \frac{1}{16}(3+x)^2 dx + \int_{-1}^{1}(6-2x^2)dx + \int_{1}^{3}\frac{1}{16}(3-x)^2 dx + \int_{3}^{\infty} 0.dx$$

$$= \frac{1}{16}\int_{-3}^{-1}(3+x)^2 dx + \int_{-1}^{1}(6-2x^2)dx + \frac{1}{16}\int_{1}^{3}(3-x)^2 dx$$

$$= \frac{1}{16}\left\{\left[\frac{(3+x)^3}{3}\right]_{-3}^{-1} + \left[6x - \frac{2x^3}{3}\right]_{-1}^{1} - \left[\frac{(3-x)^3}{3}\right]_{1}^{3}\right\}$$

$$= \frac{1}{16}\left\{\left[\frac{8}{3} - 0\right] + \left[\left(6 - \frac{2}{3}\right) - \left(-6 + \frac{2}{3}\right) - \left(0 - \frac{8}{3}\right)\right]\right\} = 1$$

$$\therefore \qquad \int_{-\infty}^{\infty} f(x)dx = 1,$$

Hence $f(x)$ is a density function.

Mean of the random variable X is

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \frac{1}{16}\int_{-3}^{-1} x(3+x)^2 dx + \frac{1}{16}\int_{1}^{-1} x(6-2x^2)dx + \frac{1}{16}\int_{1}^{3} x(3-x)^2 dx$$

$$= \frac{1}{16}\int_{-3}^{-1} x\,(9 + x^2 + 6x)\,dx + 0 + \frac{1}{16}\int_{1}^{3} x\,(9 + x^2 - 6x)\,dx$$

since the integrand of the second integral is odd function.

$$= \frac{1}{16}\int_{-3}^{-1} (9x + x^3 + 6x^2)dx + \frac{1}{16}\int_{1}^{3} (9x + x^3 - 6x^2)dx$$

$$= \frac{1}{16}\left\{\left[\frac{9x^2}{2} + \frac{x^4}{4} + \frac{6x^3}{3}\right]_{-3}^{-1} + \left[\frac{9x^2}{2} + \frac{x^4}{4} - \frac{6x^3}{3}\right]_{1}^{3}\right\}$$

$$= \frac{1}{16}\left\{\left[\left(\frac{9}{2} - \frac{81}{2}\right) + \left(\frac{1}{4} - \frac{81}{4}\right) + \left(\frac{-6}{3} - \frac{-162}{3}\right)\right] + \left[\left(\frac{81}{2} - \frac{9}{2}\right) + \left(\frac{81}{4} - \frac{1}{4}\right) - \left(\frac{162}{3} - \frac{1}{3}\right)\right]\right\}$$

$$= 0$$

Therefore, the mean of the random variableb X is zero.

Example 2.2. A continuous random variable X has

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for } -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

represents the density, find the mean and standard deviation of X.

Solution. If $f(x)$ is density function, then it satisfies

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{-1} f(x)dx + \int_{-1}^{1} f(x)dx + \int_{1}^{\infty} f(x)dx$$

$$= \int_{-\infty}^{-1} 0.dx + \int_{-1}^{1} \frac{1}{2}(x+1)dx + \int_{1}^{\infty} 0.dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + x\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1^2}{2} + 1\right) - \left(\frac{(-1)^2}{2} - 1\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{3}{2}\right) - \left(-\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\cdot\frac{4}{2} = 1$$

Hence, $$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for } -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$ is a density function.

Mean of the random variable X is

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

$$= \int_{-\infty}^{-1} 0.dx + \int_{-1}^{1} x \cdot \frac{1}{2}(x+1).dx + \int_{1}^{\infty} 0.dx$$

$$= \frac{1}{2}\int_{-1}^{1} (x^2 + x) \cdot dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{3} - \frac{-1}{3}\right) + \left(\frac{1}{2} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{3}$$

Therefore the mean of the random variable X is $\frac{1}{3}$

$\therefore$ The variance of the random variable X is

$$\text{Var}(X) = E(X^2) - \{E(X)\}^2$$

Now

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)dx$$

$$= \int_{-1}^{1} x^2 f(x)dx$$

$$= \int_{-1}^{1} x^2 \frac{1}{2}(x+1)dx$$

$$= \frac{1}{2}\int_{-1}^{1} (x^3 + x^2)dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{4} + \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}+\frac{1}{3}-\frac{1}{4}+\frac{1}{3}\right]$$

$$= \frac{1}{3}$$

Now, $$Var(X) = E(X^2) - \{E(X)\}^2$$

$$= \frac{1}{3}-\left(\frac{1}{3}\right)^2 = \frac{1}{3}-\frac{1}{9} = \frac{2}{9}$$

$\therefore$ Standard deviation of $X = \dfrac{\sqrt{2}}{3}$

Example 2.3. If the probability density function

$$f(x) = \begin{cases} kx^3 & \text{if } 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

Find the value 'k' and find the probability between $x = \frac{1}{2}$ and $x = \frac{3}{2}$.

Solution. From the given data, $f(x) = \begin{cases} kx^3 & \text{if } 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$

If $f(x)$ is a density function, then it satisfies $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_{-\infty}^{0} f(x)dx + \int_{0}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx = 1$$

$$\Rightarrow \quad k \cdot \int_0^3 x^3 dx = 1 \Rightarrow \left[\frac{x^4}{4}\right]_0^3 = 1 \Rightarrow k\left[\left(\frac{3^4}{4}-0\right)\right] = 1$$

$$\Rightarrow \quad \frac{81}{4}k = 1$$

$$\therefore \quad k = \frac{4}{81}$$

Now, $$f(x) = \begin{cases} \frac{4}{81}x^3 & \text{if } 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

(*i*) $P\left(\frac{1}{2} \le x \le \frac{3}{2}\right)$

$$P\left(\frac{1}{2} \le X \le \frac{3}{2}\right) = \int_{\frac{1}{2}}^{\frac{3}{2}} f(x)dx$$

$$= \frac{4}{81}\int_{\frac{1}{2}}^{\frac{3}{2}} x^3 dx$$

$$= \frac{4}{81}\left[\frac{x^4}{4}\right]_{\frac{1}{2}}^{\frac{3}{2}} = \frac{1}{81}[x^4]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{1}{81}\left[\left(\frac{3}{2}\right)^4 - \left(\frac{1}{2}\right)^4\right] = \frac{1}{81}\left[\frac{80}{16}\right]$$

$$= \frac{5}{81} = 0.0617$$

Example 2.4. Is the function defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 2 \\ \dfrac{3+2x}{18} & \text{if } 2 \le x \le 4 \\ 0 & \text{if} \quad x > 4 \end{cases}$$

a probability density function? Find the probability that a variate having $f(x)$ as density function will fall in the interval $(2 \le X \le 3)$.

Solution. Given $f(x)$ is

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 2 \\ \dfrac{3+2x}{18} & \text{if } 2 \le x \le 4 \\ 0 & \text{if} \quad x > 4 \end{cases}$$

If it is a density function, then it satisfies $\int_{-\infty}^{+\infty} f(x)dx = 1$

Now, $$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{2} f(x)dx + \int_{2}^{4} f(x)dx + \int_{4}^{\infty} f(x)dx$$

$$= \int_{-\infty}^{0} 0dx + \int_{2}^{4}\frac{3+2x}{18}dx + \int_{4}^{\infty} f(x)dx$$

$$= 0 + \frac{3}{18}[x]_2^4 + \frac{2}{18}\left[\frac{x^2}{2}\right]_2^4 + 0 = \frac{1}{6}[4-2] + \frac{1}{18}[4^2 - 2^2]$$

$$= \frac{1}{3} + \frac{2}{3} = 1$$

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

Hence, $f(x) = \begin{cases} 0 & \text{if } x < 2 \\ \dfrac{3+2x}{18} & \text{if } 2 \le x \le 4 \\ 0 & \text{if } x > 4 \end{cases}$ is a density function

Now $P(2 \le X \le 3) = \int_{2}^{3} f(x)dx$

$$= \int_{2}^{3} \frac{3+2x}{18} dx$$

$$= \frac{3}{18}[x]_2^3 + \frac{2}{18}\left[\frac{x^2}{2}\right]_2^3$$

$$= \frac{1}{6}[3-2] + \frac{1}{18}[3^2 - 2^2]$$

$$= \frac{1}{6} + \frac{5}{18} = \frac{8}{18} = \frac{4}{9}$$

$$= 0.44$$