# [B] POISSON DISTRIBUTION

This distribution can be derived as a limiting case of the binomial distribution by making $n$ very large ($n \to \infty$) and $p$ very small ($p \to 0$)

If $n \to \infty$ and $p \to 0$ then $np$ always remains finite ; say $m$

$\therefore \; np = m$

$\therefore \; p = m/n$ $\qquad \therefore \; q = 1 - p = 1 - m/n$

Now ; for a Binomial distribution

$$P(x = r) = {}^nC_r \, p^r q^{n-r}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \left(\frac{m}{n}\right)^r \left[1 - \frac{m}{n}\right]^{n-r}$$

$$= \frac{m^r}{r!} \times \frac{n(n-1)(n-2)\ldots(n-r+1)}{n^r} \times \frac{(1 - m/n)^n}{(1 - m/n)^r}$$

$$= \frac{m^r}{r!} \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\ldots\left(\frac{n-r+1}{n}\right) \frac{(1 - m/n)^n}{(1 - m/n)^r}$$

$$= \frac{m^r}{r!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{r-1}{n}\right) \frac{\left\{(1 - m/n)^{-n/m}\right\}^{-m}}{(1 - m/n)^r}$$

$$\text{-------------} \; (*)$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e \quad ; \quad \lim_{n\to\infty}\left(1+\frac{q}{n}\right)^{A}=1$$

Now ; as $n\to\infty$ ; each of the $(r-1)$ factors

$$\left.\begin{array}{l}\left(1-\frac{1}{n}\right),\left(1-\frac{2}{n}\right),\ \cdots\ \left(1-\frac{r-1}{n}\right)\to 1 \\ \text{Also } \left(1-\frac{m}{n}\right)^{r}\to 1 \text{ and } \left\{\left(1-\frac{m}{n}\right)^{-\frac{n}{m}}\right\}^{-m}\to e^{-m}\end{array}\right\}$$

Hence ; when $n\to\infty$ ; from eq$^n$ (*) ; we have ;

$$P(r)=\frac{m^r e^{-m}}{r!}$$

Thus ; the probability f$^{ns}$ of the poisson distribution is

$$\boxed{P(r)=\frac{e^{-m}m^r}{r!}}\ \boxed{=\frac{e^{-\lambda}\lambda^r}{r!}} \quad ; r=0,1,2,\cdots$$

where $\boxed{m=\lambda=np}$

Note :→ The sum of the Probability is 1 for $r=0,1,2,3,\cdots n$

$$P(0)+P(1)+P(2)+\cdots+P(n)+\cdots$$

$$=e^{-m}+\frac{m e^{-m}}{1!}+\frac{m^2 e^{-m}}{2!}+\frac{m^3e^{-m}}{3!}+\cdots$$

$$=e^{-m}\left\{1+\frac{m}{1!}+\frac{m^2}{2!}+\frac{m^3}{3!}+\cdots\right\}$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$$

$$= e^{-m} e^{m}$$

$$= 1$$

## Recurrence Formula for the Poisson Distribution :→

For Poisson Distribution

$$\left.\begin{aligned} P(r) &= \frac{m^r e^{-m}}{r!} \\ P(r+1) &= \frac{m^{r+1} e^{-m}}{(r+1)!} \end{aligned}\right\}$$

Hence,

$$\frac{P(r+1)}{P(r)} = \frac{m \cdot r!}{(r+1)!} = \frac{m}{r+1}$$

$$\Rightarrow \boxed{P(r+1) = \frac{m}{r+1} \cdot P(r)}$$

where $r = 0, 1, 2, 3, \ldots$

## Mean and Variance of the Poisson Distribution

### Constants of Poisson Distribution

| $r$ | $P(r)$ | $rP(r)$ | $r^2P(r)$ |
|---|---|---|---|
| 0 | $e^{-m}$ | 0 | 0 |
| 1 | $me^{-m}$ | $1 \cdot m \cdot e^{-m}$ | $me^{-m}$ |
| 2. | $\frac{m^2 e^{-m}}{2!}$ | $\frac{2 \cdot m^2 e^{-m}}{2!}$ | $\frac{2^2 m^2 e^{-m}}{2!}$ |
| 3. | $\frac{m^3 e^{-m}}{3!}$ | $\frac{3 \cdot m^3 \cdot e^{-m}}{3!}$ | $\frac{3^2 m^2 e^{-m}}{3!}$ |
| 4. | $\frac{m^4 e^{-m}}{4!}$ | $\frac{4 \cdot m^4 e^{-m}}{4!}$ | $\frac{4^2 m^2 e^{-m}}{4!}$ |
| ⋮ | ⋮ | ⋮ | ⋮ |

### Mean

$$\mu = \sum_{r=0}^{\infty} r \cdot P(r)$$

$$= \sum_{r=0}^{\infty} r \cdot \frac{e^{-m} m^r}{r!}$$

$$= me^{-m} + \frac{2m^2e^{-m}}{2!} + \frac{3m^3e^{-m}}{3!} + \frac{4m^4e^{-m}}{4!} + \cdots$$

$$= m e^{-m} \left\{ 1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots \right]$$

$$= m e^{-m} \{ e^{m} \}$$

$$\boxed{\text{Mean} = m}$$

(*) Variance $\sigma^2 = \sum_{r=0}^{\infty} r^2 P(r) - \mu^2$

$$= \sum_{r=0}^{\infty} r^2 \cdot \frac{m^r \cdot e^{-m}}{r!} - m^2$$

$$= m e^{-m} + \frac{2^2 m^2 e^{-m}}{2!} + \frac{3^2 m^3 e^{-m}}{3!} + \cdots - m^2$$

$$= m e^{-m} \left\{ 1 + 2m + \frac{3}{2!} m^2 + \frac{4}{3!} m^3 + \cdots \right] - m^2$$

$$= m e^{-m} \left\{ 1 + \frac{(1+1) m}{1!} + \frac{(1+2) m^2}{2!} + \frac{(1+3) m^3}{3!} + \cdots \right] - m^2$$

$$= m e^{-m} \left\{ \left[ 1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots \right] + \left[ m + \frac{2m^2}{2!} + \frac{3m^3}{3!} + \cdots \right] \right] - m^2$$

$$= m e^{-m} \left\{ \left[ 1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots \right] + m \left[ 1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots \right] \right\} - m^2$$

$$= m e^{-m} \{ e^{m} + m e^{m} \} - m^2$$

$$= m e^{-m} e^{m} \{ 1 + m \} - m^2$$

$$= m (1+m) - m^2$$

$$= m + m^2 - m^2$$

$$= m$$

Hence ; For the poisson Distribution ;

$$\boxed{\text{Mean} = \text{Variance} = m}$$

**Note** :→ If $p$ is small and $n$ is large then we use poisson distribution.

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Ex-1 If the variance of the poisson distribution is 2; find the Probabilities for $r = 1,2,3,4$ from the recurrence relation of the poisson distribution

Sol :- $\lambda$ : the Parameter of Poisson dist
= variance
= 2

Recurrence relation for the poisson dist is

$$P(r+1) = \frac{\lambda}{r+1} P(r) = \frac{2}{r+1} P(r)$$

Now $P(r) = \frac{m^r e^{-m}}{r!} \Rightarrow P(0) = \frac{e^{-2}}{0!} = e^{-2} = 0.1353$

Putting $r = 0, 1, 2, 3, \ldots$ ; we get

$P(1) = 2 \cdot P(0) = 2 \times 0.1353 = 0.2706$

$P(2) = \frac{2}{2} \cdot P(1) = 0.2706$

$P(3) = \frac{2}{3} P(2) = \frac{2}{3} \times 0.2706 = 0.1804$

$P(4) = \frac{2}{4} P(3) = \frac{1}{2} \times 0.1804 = 0.902$

Ex-2 Using poissons distribution; find the probability that the aces of spades will be drawn from a pack of well-shuffled cards at least once in 104 consecutive trials
(given $e^{-2} = 0.1353$)

$sol^n$ :→ We have

$$P(r) = \frac{e^{-\lambda} \lambda^r}{r!} \quad ; \quad r = 0, 1, 2, 3, \ldots$$

Here $\lambda = n \cdot p = 104 \times \frac{1}{52} = 2$

∴ Probability of drawing an ace of spades at least once

$$= P(1) + P(2) + \cdots + P(52)$$

$$= 1 - P(0)$$

$$= 1 - \frac{e^{-2}(1)}{0!}$$

$$= 1 - e^{-2}$$

$$= 1 - 0.1353$$

$$= 0.8647$$

Ex-3 :→ If the probability that an individual suffers a bad reaction from a certain injection is 0.001 determine the probability that out of 2000 individuals

<a> exactly 3 and

<b> more than 2 individuals will suffer a bad reaction ( given $e^{-2} = 0.136$ )

Sol:- Here $P = 0.001$ ; $n = 2000$

$\lambda = nP = 2000 \times 0.001 = 2$

According to poissons distribution

$$P(r) = \frac{e^{-\lambda} \lambda^{r}}{r!}$$

(a) $P(3) = \frac{e^{-2} (2)^3}{3!} = \frac{8}{6} \times e^{-2} = \frac{4}{3} \times 0.136 = 0.1804$

or $1 - P(r \leq 2)$

(b) $P(r > 2) = P(3) + P(4) + P(5)$

$= 1 - [P(0) + P(1) + P(2)]$

$= 1 - \left\{ \frac{e^{-2}}{0!} + \frac{e^{-2} \cdot 2}{1!} + \frac{e^{-2}\, 2^2}{2!} \right\}$

$= 1 - e^{-2} \{1 + 2 + 2\}$

$= 1 - (0.136)(5)$

$= 1 - 0.680$

$= 0.32$

Ex-4 Between the hours 2 P.M. and 4 P.M. The average number of phone calls per minute coming into the switch board of a company is 2.35. Find the probability that during one particular minute; there will be at most 2 phone calls. [given $e^{-2.35} = 0.095374$]
(by using poisson dist)

Soln:-

If the variable x denotes the number of telephone calls per minute; then x will follow Poisson distribution with parametre $m = 2.35$ and Probability fn ;

$$P(x=r) = \frac{e^{-m} m^r}{r!} = \frac{e^{-2.35} \times (2.35)^r}{r!}$$

The Prob. that during one particular minute there will be at most 2 phone calls is given by

$$P(x \leq 2) = P(0) + P(1) + P(2)$$

$$= e^{-2.35} \left\{ 1 + 2.35 + \frac{(2.35)^2}{2!} \right\}$$

$$= 0.095374 \{ 1 + 2.35 + 2.76125 \}$$

$$= 0.095374 \times 6.11125$$

$$= 0.5828543$$

**Ex-5** It is known from past experience that in a certain plant there are on the average 4 industrial accidents per month. Find the probability that in a given year there will be less than 4 accidents by assuming poisson distribution ($e^{-4} = 0.0183$)

**Sol:→** In the usual notations; given $m = 4$

If the variable x denotes the number of accidents in the plant per month; then by poisson distribution;

$$P(X=r) = \frac{e^{-m}\, m^r}{r!} = \frac{e^{-4}\,(4)^r}{r!}$$

The required Prob. that there will be less than 4 accidents is given by;

$$P(X<4) = P(0) + P(1) + P(2) + P(3)$$

$$= e^{-4}\left\{1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!}\right\}$$

$$= e^{-4}\{1 + 4 + 8 + 10.67\}$$

$$= 0.0183 \times 23.67$$

$$= 0.4332$$

Ex-6 If 5% of the electric bulbs manufactured by a company are defective; use poisson distribution to find the probability that in a sample of 100 bulbs.
<i> none is defective
<ii> 5 bulbs will be defective.

(given : $e^{-5} = 0.007$)

sol :→ Here given ; $n = 100$
$P$ = Prob. of a defective bulb = 5% = 0.05

{ since $P$ is small and $n$ is large ; we may approximate the given dist. by poisson dist. }

The parameter $m$ of poisson dist. is :

$$m = nP = 100 \times 0.05 = 5$$

Let $x$ denote the number of defective bulbs in a sample of 100 then

$$P(X = r) = \frac{e^{-m} m^r}{r!} = \frac{e^{-5} 5^r}{r!}$$

<i> The prob. that none of the bulbs is defective is given by :

$$P(X = 0) = e^{-5} = 0.0067$$

<ii> The prob. of 5 defective bulbs is given by

$$P(x=5) = \frac{e^{-5}\,5^5}{5!} = \frac{0.0067 \times 625}{24}$$

$$= \frac{4.375}{24}$$

$$= 0.1754.$$