

2.1. CONTINUOUS RANDOM VARIABLE

A random variable X which can take every value in the domain or when its range R is an interval then X is continuous random variable.

Example :

1. Age
2. Height
3. Weight
4. Temperature

2.2. PROBABILITY DENSITY FUNCTION

The probability density function of random variable X is defined as

$$f_x(x) = P(x \leq X \leq x + \delta x) / \delta x$$

for small interval $(x, x + \delta x)$ of length dx around the point x

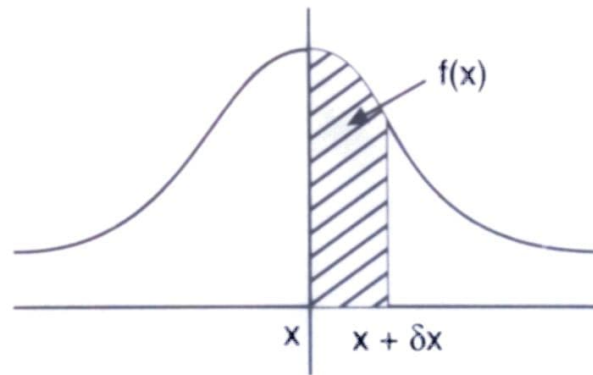


Fig. 2.1

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

which represent the area between the curve $y = f(x)$, x axis and the ordinate at $x = a$ and $x = b$ since total probability is unity.

i.e.,
$$\int_{-\infty}^{\infty} f(x) dx = 1$$

The probability density function (p.d.f) of a random variable X usually denoted by $f_x(x)$ or simply $f(x)$ has following properties.

1. $f(x) \geq 0, -\infty < x < \infty$

2. $\int_{-\infty}^{\infty} f(x) dx = 1$

2.3. CUMULATIVE DISTRIBUTION (DISTRIBUTION FUNCTION)

If X is a random variable, then $P(X \leq x)$ is called the cumulative distribution (c.d.f) or simply distribution function and it is denoted by $F(x)$.

$$\therefore F(x) = P(X \leq x)$$

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(x) dx$$

2.4. EXPECTATION OF RANDOM VARIABLE

If X is a continuous random variable, then the expectation of the random variable X as defined by

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

The expected value of X^2 is defined as

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$E(X)$ is also called mean of X

Properties

1. If X is random variable and a is constant then

- (i) $E(a) = a$

- (ii) $E(aX) = aE(X)$

- (iii) $E(X - \bar{X}) = 0$

2. If X and Y are two random variables then

$$E(X + Y) = E(X) + E(Y)$$

3. $E(XY) = E(X) E(Y)$ if X and Y are two independent random variable.

4. If $y = a + bx$ where a and b are constants then

$$E(Y) = E(aX + b) = aE(X) + b$$

2.5. VARIANCE AND STANDARD DEVIATION OF CONTINUOUS RANDOM VARIABLE

Variance of x is defined as

$$\text{Var}(X) = V(X)$$

$$= E(X - \bar{X})^2 = E(X^2) - [E(X)]^2$$

Standard deviation of random variable x is denoted by S.D(x) and is defined as

$$\text{S.D.}(x) = \sigma = \sqrt{V(X)} = \sqrt{E(X)^2 - [E(X)]^2}$$

Example 2.1. A continuous random variables X has a probability density function defined by

$$f(x) = \begin{cases} \frac{1}{16}(3+x)^2 & \text{if } -3 \leq x < -1 \\ \frac{1}{16}(6-2x^2) & \text{if } -1 \leq x < 1 \\ \frac{1}{16}(3-x)^2 & \text{if } 1 < x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

Verify that $f(x)$ is a density function and also find the mean of the random variable X .

Solution. Since $f(x)$ is density function, $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^{-3} f(x)dx + \int_{-3}^{-1} f(x)dx + \int_{-1}^1 f(x)dx + \int_1^3 f(x)dx + \int_3^{\infty} f(x)dx \\ &= \int_{-\infty}^{-3} 0 \cdot dx + \int_{-3}^{-1} \frac{1}{16}(3+x)^2 dx + \int_{-1}^1 (6-2x^2)dx + \int_1^3 \frac{1}{16}(3-x)^2 dx + \int_3^{\infty} 0 \cdot dx \\ &= \frac{1}{16} \int_{-3}^{-1} (3+x)^2 dx + \int_{-1}^1 (6-2x^2)dx + \frac{1}{16} \int_1^3 (3-x)^2 dx \\ &= \frac{1}{16} \left\{ \left[\frac{(3+x)^3}{3} \right]_{-3}^{-1} + \left[6x - \frac{2x^3}{3} \right]_{-1}^1 - \left[\frac{(3-x)^3}{3} \right]_1^3 \right\} \\ &= \frac{1}{16} \left\{ \left[\frac{8}{3} - 0 \right] + \left[\left(6 - \frac{2}{3} \right) - \left(-6 + \frac{2}{3} \right) - \left(0 - \frac{8}{3} \right) \right] \right\} = 1 \end{aligned}$$

$$\therefore \int_{-\infty}^{\infty} f(x)dx = 1,$$

Hence $f(x)$ is a density function.

Mean of the random variable X is

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \frac{1}{16} \int_{-3}^{-1} x(3+x)^2 dx + \frac{1}{16} \int_{-1}^1 x(6-2x^2)dx + \frac{1}{16} \int_1^3 x(3-x)^2 dx \end{aligned}$$

After integration, the power of x will be odd so that putting limits in it will be canceled out as both limit's values are same and of opposite sign.

$$= \frac{1}{16} \int_{-3}^{-1} x(9 + x^2 + 6x) dx + 0 + \frac{1}{16} \int_1^3 x(9 + x^2 - 6x) dx$$

since the integrand of the second integral is odd function.

$$= \frac{1}{16} \int_{-3}^{-1} (9x + x^3 + 6x^2) dx + \frac{1}{16} \int_1^3 (9x + x^3 - 6x^2) dx$$

$$= \frac{1}{16} \left\{ \left[\frac{9x^2}{2} + \frac{x^4}{4} + \frac{6x^3}{3} \right]_{-3}^{-1} + \left[\frac{9x^2}{2} + \frac{x^4}{4} - \frac{6x^3}{3} \right]_1^3 \right\}$$

$$= \frac{1}{16} \left\{ \left[\left(\frac{9}{2} - \frac{81}{2} \right) + \left(\frac{1}{4} - \frac{81}{4} \right) + \left(\frac{-6}{3} - \frac{-162}{3} \right) \right] + \left[\left(\frac{81}{2} - \frac{9}{2} \right) + \left(\frac{81}{4} - \frac{1}{4} \right) - \left(\frac{162}{3} - \frac{1}{3} \right) \right] \right\}$$

$$= 0$$

Therefore, the mean of the random variable X is zero.

Example 2.2. A continuous random variable X has

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for } -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

represents the density, find the mean and standard deviation of X .

Solution. If $f(x)$ is density function, then it satisfies

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\begin{aligned} \Rightarrow \int_{-\infty}^{\infty} f(x) dx &= \int_{-\infty}^{-1} f(x) dx + \int_{-1}^1 f(x) dx + \int_1^{\infty} f(x) dx \\ &= \int_{-\infty}^{-1} 0 dx + \int_{-1}^1 \frac{1}{2}(x+1) dx + \int_1^{\infty} 0 dx \\ &= \frac{1}{2} \left[\frac{x^2}{2} + x \right]_{-1}^1 \\ &= \frac{1}{2} \left[\left(\frac{1^2}{2} + 1 \right) - \left(\frac{(-1)^2}{2} - 1 \right) \right] \\ &= \frac{1}{2} \left[\left(\frac{3}{2} \right) - \left(-\frac{1}{2} \right) \right] \\ &= \frac{1}{2} \cdot \frac{4}{2} = 1 \end{aligned}$$

Hence,

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for } -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases} \text{ is a density function.}$$

Mean of the random variable X is

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_{-\infty}^{-1} 0 \cdot dx + \int_{-1}^1 x \cdot \frac{1}{2}(x+1) \cdot dx + \int_1^{\infty} 0 \cdot dx \\ &= \frac{1}{2} \int_{-1}^1 (x^2 + x) \cdot dx \\ &= \frac{1}{2} \left[\frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^1 \\ &= \frac{1}{2} \left[\left(\frac{1}{3} - \frac{-1}{3} \right) + \left(\frac{1}{2} - \frac{1}{2} \right) \right] \\ &= \frac{1}{3} \end{aligned}$$

Therefore the mean of the random variable X is $\frac{1}{3}$

\therefore The variance of the random variable X is

$$\text{Var}(X) = E(X^2) - \{E(X)\}^2$$

Now

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{+\infty} x^2 f(x) dx \\ &= \int_{-1}^1 x^2 f(x) dx \\ &= \int_{-1}^1 x^2 \frac{1}{2}(x+1) dx \\ &= \frac{1}{2} \int_{-1}^1 (x^3 + x^2) dx \\ &= \frac{1}{2} \left[\frac{x^4}{4} + \frac{x^3}{3} \right]_{-1}^1 \\ &= \frac{1}{2} \left[\left(\frac{1}{4} + \frac{1}{3} \right) - \left(\frac{1}{4} - \frac{1}{3} \right) \right] \end{aligned}$$

$$= \frac{1}{2} \left[\frac{1}{4} + \frac{1}{3} - \frac{1}{4} + \frac{1}{3} \right]$$

$$= \frac{1}{3}$$

Now,

$$\text{Var}(X) = E(X^2) - \{E(X)\}^2$$

$$= \frac{1}{3} - \left(\frac{1}{3} \right)^2 = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$$

$$\therefore \text{Standard deviation of } X = \frac{\sqrt{2}}{3}$$

Example 2.3. If the probability density function

$$f(x) = \begin{cases} kx^3 & \text{if } 0 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

Find the value 'k' and find the probability between $x = \frac{1}{2}$ and $x = \frac{3}{2}$.

Solution. From the given data, $f(x) = \begin{cases} kx^3 & \text{if } 0 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$

If $f(x)$ is a density function, then it satisfies $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\Rightarrow \int_{-\infty}^0 f(x) dx + \int_0^3 f(x) dx + \int_3^{\infty} f(x) dx = 1$$

$$\Rightarrow k \cdot \int_0^3 x^3 dx = 1 \Rightarrow \left[\frac{x^4}{4} \right]_0^3 = 1 \Rightarrow k \left[\left(\frac{3^4}{4} - 0 \right) \right] = 1$$

$$\Rightarrow \frac{81}{4} k = 1$$

$$\therefore k = \frac{4}{81}$$

Now,

$$f(x) = \begin{cases} \frac{4}{81} x^3 & \text{if } 0 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$(i) P\left(\frac{1}{2} \leq x \leq \frac{3}{2}\right)$$

$$\begin{aligned}
 P\left(\frac{1}{2} \leq X \leq \frac{3}{2}\right) &= \int_{\frac{1}{2}}^{\frac{3}{2}} f(x) dx \\
 &= \frac{4}{81} \int_{\frac{1}{2}}^{\frac{3}{2}} x^3 dx \\
 &= \frac{4}{81} \left[\frac{x^4}{4} \right]_{\frac{1}{2}}^{\frac{3}{2}} = \frac{1}{81} [x^4]_{\frac{1}{2}}^{\frac{3}{2}} \\
 &= \frac{1}{81} \left[\left(\frac{3}{2}\right)^4 - \left(\frac{1}{2}\right)^4 \right] = \frac{1}{81} \left[\frac{80}{16} \right] \\
 &= \frac{5}{81} = 0.0617
 \end{aligned}$$

Example 2.4. Is the function defined by

$$f(x) = \begin{cases} 0 & \text{if } x < 2 \\ \frac{3+2x}{18} & \text{if } 2 \leq x \leq 4 \\ 0 & \text{if } x > 4 \end{cases}$$

a probability density function? Find the probability that a variate having $f(x)$ as density function will fall in the interval $(2 \leq X \leq 3)$.

Solution. Given $f(x)$ is

$$f(x) = \begin{cases} 0 & \text{if } x < 2 \\ \frac{3+2x}{18} & \text{if } 2 \leq x \leq 4 \\ 0 & \text{if } x > 4 \end{cases}$$

If it is a density function, then it satisfies $\int_{-\infty}^{+\infty} f(x) dx = 1$

Now,

$$\begin{aligned}
 \int_{-\infty}^{+\infty} f(x) dx &= \int_{-\infty}^2 f(x) dx + \int_2^4 f(x) dx + \int_4^{\infty} f(x) dx \\
 &= \int_{-\infty}^2 0 dx + \int_2^4 \frac{3+2x}{18} dx + \int_4^{\infty} 0 dx \\
 &= 0 + \frac{3}{18} [x]_2^4 + \frac{2}{18} \left[\frac{x^2}{2} \right]_2^4 + 0 = \frac{1}{6} [4-2] + \frac{1}{18} [4^2 - 2^2] \\
 &= \frac{1}{3} + \frac{2}{3} = 1
 \end{aligned}$$

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

Hence,

$$f(x) = \begin{cases} 0 & \text{if } x < 2 \\ \frac{3+2x}{18} & \text{if } 2 \leq x \leq 4 \\ 0 & \text{if } x > 4 \end{cases} \text{ is a density function}$$

Now

$$\begin{aligned} P(2 \leq X \leq 3) &= \int_2^3 f(x)dx \\ &= \int_2^3 \frac{3+2x}{18} dx \\ &= \frac{3}{18} [x]_2^3 + \frac{2}{18} \left[\frac{x^2}{2} \right]_2^3 \\ &= \frac{1}{6} [3-2] + \frac{1}{18} [3^2 - 2^2] \\ &= \frac{1}{6} + \frac{5}{18} = \frac{8}{18} = \frac{4}{9} \\ &= 0.44 \end{aligned}$$