

④ THEORETICAL DISTRIBUTIONS

→ In this chapter we shall study the following univariate probability distributions.

{1} Binomial Distribution

{2} Poisson Distribution

{3} Normal Distribution

The first two distributions are discrete probability distributions and the third is a continuous probability distribution.

[A] BINOMIAL DISTRIBUTION

Binomial distribution is also known as the Bernoulli distribution. This distribution can be used under the following conditions:

{1} An experiment consists of a finite number of repeated trials.

{2} Each trial has only two possible mutually exclusive outcomes which are known as a "success" and a "failure"

{3} All the (different) trials are independent i.e. the result of any trial ; is not affected in any way by the preceding trials

ctual does not affect the result of succeeding trials.

(i) The probability of success in any trial is p and is constant for each trial. The Probability of a failure denoted by q ; and is equal to $1-p$.

The sequence of trials under the above assumption is also known as a Bernoulli trials.

for ex: we know that the probability of getting a head or a tail on tossing a coin is $\frac{1}{2}$.

If the coin is tossed thrice; the probability of getting one head and two tails can be combined as

H-T-T, T-H-T, T-T-H.

The probability of each one of these being:

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$\text{i.e } \left(\frac{1}{2}\right)^3$$

their total probability shall be:

$$3 \left(\frac{1}{2}\right)^3$$

similarly ; if a trial is repeated n -times and if p is the probability of a success and q , that of a failure ; then the probability of r successes and $(n-r)$ failures is given by :

$$P^r \cdot q^{n-r}$$

But these r successes and $(n-r)$ failures can occur in any of the ${}^n C_r$ ways in each of which the probability is same.

Thus ; the probability of r successes is

$${}^n C_r P^r q^{n-r}$$

→ Let n be the total numbers of repeated trials ; p be the probability of a success in a trial and q , be the probability of its failure.

Let r be a random variable which denotes the number of successes in n trials.

The possible values of r are $0, 1, 2, 3, 4, \dots, n$.

We are interested in finding the probability of r successes out of n trials.

$$\text{i.e } P(r)$$

To find this probability ; we assume that the first r trials are successes and remaining $(n-r)$ trials are failures. Since different trials are assumed to be independent ; the probability of this sequence is ;

$$\underbrace{p \cdot p \cdot p \cdots p}_{r \text{ times}} \cdot \underbrace{q \cdot q \cdot q \cdots q}_{(n-r) \text{ times}}$$

$$\text{i.e } p^r q^{n-r}$$

Since out of n trials any r trials can be success ; the number of sequences showing any r trials as success and remaining $(n-r)$ trials as failures is : ${}^n C_r$; where the probability of r successes in each trial is : $p^r q^{n-r}$.

Hence the required probability is

$$P(r) = n C_r p^r q^{n-r}$$



where $p+q=1$ and $r=0, 1, 2, \dots, n$.

The distribution \star is called the binomial probability distribution.

Note :-

The successive probabilities $P(r)$ in \star for $r=0, 1, 2, \dots, n$ are:

$$n C_0 p^0 q^n; n C_1 p^1 q^{n-1}; n C_2 p^2 q^{n-2}; n C_3 p^3 q^{n-3}; \dots \\ n C_n p^n q^0.$$

which are the terms in the binomial expansion of $(q+p)^n$.

Hence this distribution is known as a Binomial distribution.

\star Recurrence formula for the Binomial Distribution :-

In a Binomial distribution

$$\left\{ \begin{array}{l} P(r) = n C_r q^{n-r} p^r = \frac{n!}{(n-r)! r!} q^{n-r} p^r \\ P(r+1) = n C_{r+1} q^{n-r-1} p^{r+1} = \frac{n!}{(n-r-1)! (r+1)!} q^{n-r-1} p^{r+1} \end{array} \right.$$

Now;

$$\frac{P(r+1)}{P(r)} = \frac{(n-r)!}{(n-r-1)!} \times \frac{r!}{(r+1)!} \times \frac{p}{q},$$

$$= \frac{(n-r)(n-r-1)!}{(n-r-1)!} \times \frac{r!}{(r+1)r!} \times \frac{p}{q},$$

$$= \frac{(n-r)}{r+1} \times \frac{p}{q}$$

$$\Rightarrow P(r+1) = \frac{(n-r)}{r+1} \times \frac{p}{q} \cdot P(r)$$

which is the required recurrence formula.

Applying this formula successively we can find: $P(1), P(2), P(3), \dots$; if $P(0)$ is known.

④ →

r	$P(r)$	$r \cdot P(r)$	$r^2 P(r)$
0	q^n	0	0
1	$nC_1 q^{n-1} p$	$nC_1 q^{n-1} p$	$1^2 nC_1 q^{n-1} p$
2	$nC_2 q^{n-2} p^2$	$2 nC_2 q^{n-2} p^2$	$2^2 nC_2 q^{n-2} p^2$
3	$nC_3 q^{n-3} p^3$	$3 nC_3 q^{n-3} p^3$	$3^2 nC_3 q^{n-3} p^3$
4	1	1	1
5	1	1	1
6	1	1	1
n	p^n	$n p^n$	$n^2 p^n$

(*) Mean and Variance of the Binomial Distribution :-

(*) Mean $\mu = \sum r \cdot p(r)$

$$= \sum_{r=0}^n r \cdot nC_r q^{n-r} p^r$$

$$= 0 + 1 \cdot nC_1 q^{n-1} p + 2 nC_2 q^{n-2} p^2 + 3 nC_3 q^{n-3} p^3 \\ + \dots + n nC_n p^n$$

$$= n q^{n-1} p + \frac{2 n(n-1)}{2!} q^{n-2} p^2 + \frac{3 n(n-1)(n-2)}{3!} q^{n-3} p^3 \\ + \dots + n p^n$$

$$= n q^{n-1} p + n(n-1) q^{n-2} p^2 + \frac{n(n-1)(n-2)}{2} q^{n-3} p^3 \\ + \dots + n p^n$$

$$= np \left\{ q^{n-1} + (n-1) q^{n-2} p + \frac{(n-1)(n-2)}{2 \cdot 1} q^{n-3} p^2 \right. \\ \left. + \dots + p^{n-1} \right\}$$

$$= np \left\{ nC_0 q^{n-1} + n-1 C_1 q^{n-2} p + n-1 C_2 q^{n-3} p^2 + \dots + n-1 C_{n-1} p^{n-1} \right\}$$

$$= np (q + p)^{n-1}$$

$$\text{Mean} = np$$

$$(\because p+q=1)$$

$$\textcircled{*} \text{ Variance } \sigma^2 = \sum_{r=0}^n r^2 p(r) - [\text{Mean}]^2$$

$$= \sum_{r=0}^n [r+r(r-1)] p(r) - \mu^2$$

$$= \sum_{r=0}^n r \cdot p(r) + \sum_{r=0}^n r(r-1) p(r) - \mu^2$$

$$= \mu + \sum_{r=0}^n r(r-1) p(r) - \mu^2$$

$$= \mu + \sum_{r=2}^n r(r-1) p(r) - \mu^2$$

(\therefore contribution due to $r=0$
and $r=1$ is zero)

$$= \mu + \sum_{r=2}^n r(r-1) n C_r q^{n-r} p^r - \mu^2$$

$$= \mu - \mu^2 + \sum_{r=2}^n r(r-1) n C_r q^{n-r} p^r$$

$$= \mu - \mu^2 + \left\{ 2 \cdot 1 n C_2 q^{n-2} p^2 + 3 \cdot 2 n C_3 q^{n-3} p^3 + \dots + n(n-1) n C_n p^n \right\}$$

$$= \mathcal{U} - \mathcal{U}^2 + \left\{ \frac{2 \cdot 1 \cdot n(n-1)}{2!} q^{n-2} p^2 \right. \\ \left. + \frac{3 \cdot 2 \cdot n(n-1)(n-2)}{3!} q^{n-3} p^3 \right. \\ \left. + \dots + n(n-1)p^n \right\}$$

$$= \mathcal{U} - \mathcal{U}^2 + \left\{ n(n-1) q^{n-2} p^2 + n(n-1)(n-2) q^{n-3} p^3 \right. \\ \left. + \dots + n(n-1)p^n \right\}$$

$$= \mathcal{U} - \mathcal{U}^2 + n(n-1)p^2 \left\{ q^{n-2} + (n-2)q^{n-3} p \right. \\ \left. + \dots + p^{n-2} \right\}$$

$$= \mathcal{U} - \mathcal{U}^2 + n(n-1)p^2 \left\{ (n-2)C_0 q^{n-2} + (n-2)C_1 q^{n-3} p \right. \\ \left. + \dots + (n-2)C_{n-2} p^{n-2} \right\}$$

$$= \mathcal{U} - \mathcal{U}^2 + n(n-1)p^2 [q + p]^{n-2}$$

$$= \mathcal{U} - \mathcal{U}^2 + n(n-1)p^2 \quad (\because p+q=1)$$

$$= np - n^2 p^2 + n(n-1)p^2$$

$$= np \left\{ 1 + (n-1)p - np \right\}$$

$$= np \{ 1 - p^2 \}$$

$$\text{Variance} = npq.$$

$$\text{where; Variance} = \sigma^2$$

→ standard deviation of the binomial distribution is

$$\sqrt{npq} = \sigma.$$

$$nC_r = \frac{n!}{(n-r)!r!}$$

Ex-1 Ten unbiased coins are tossed simultaneously.
Find the Probability of obtaining.

- (1) Exactly 6 heads
- (2) At least 8 heads
- (3) No head
- (4) At least one head
- (5) Not more than three heads
- (6) At least 4 heads.

Sol: \rightarrow If P denotes the probability of a head ;
then

$$P = q = \frac{1}{2}$$

Here $n = 10$

If the random variable x denotes the number of heads ; then by Binomial - Probability law ; the Probability of r head is given by :

$$\begin{aligned} P(r) &= P(X = r) = nC_r P^r \cdot q^{n-r} \\ &= 10C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{10-r} \\ &= 10C_r \left(\frac{1}{2}\right)^{10} \end{aligned}$$

$$\therefore P(r) = \frac{1}{1024} 10C_r \quad \dots \quad *$$

(1) Required Prob. = $P(6)$

$$= \frac{1}{1024} \cdot 10C_6$$

$$= \frac{252}{1024}$$

$$= \frac{63}{256}$$

(2) Required Prob. = $P(X \geq 8) = P(8) + P(9) + P(10)$

$$= \frac{1}{1024} \left\{ 10C_8 + 10C_9 + 10C_{10} \right\}$$

$$= \frac{1}{1024} \left\{ 45 + 10 + 14 \right\}$$

$$= \frac{56}{1024}$$

$$= \frac{7}{128}$$

(3) Req. Prob. $P(X = 0) = P(0)$

$$= \frac{1}{1024} \cdot 10C_0$$

$$= \frac{1}{1024}$$

(4) Req. Prob. = $P(1) + P(2) + P(3) + \dots + P(10)$

$$= P[\text{At least one head}]$$

$$= 1 - P[\text{no head}]$$

$$= 1 - PC(0)$$

$$= 1 - \frac{1}{1024}$$

$$= \frac{1023}{1024}$$

55 Req. Prob. $P(X \leq 3) = PC(0) + PC(1) + PC(2) + PC(3)$

$$= \frac{1}{1024} \{ 10C_0 + 10C_1 + 10C_2 + 10C_3 \}$$

$$= \frac{1}{1024} \{ 1 + 10 + 45 + 120 \}$$

$$= \frac{176}{1024} = \frac{11}{64}$$

56 Req. Prob $P(X \geq 4) = PC(4) + PC(5) + \dots + PC(10)$

$$= \frac{1}{1024} \{ 10C_4 + 10C_5 + \dots + 10C_{10} \}$$

$$= \frac{53}{64}$$

OR

$$= 1 - P(X \leq 3)$$

$$= 1 - \{ PC(0) + PC(1) + PC(2) + PC(3) \}$$

$$= 1 - \frac{11}{64} \quad (\because \text{by } 55)$$

$$= \frac{53}{64}$$

Ex-2 A pair of dice is thrown 10 times IF getting a doublet (same number on both) is considered a success ; Find the probability of
 (1) 4 successes.
 (2) No. success.

Sol: Here $n = 10$

A doublet can be obtained when a pair of dice is thrown in

(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)

i.e in 6 ways.

$$\therefore p = \frac{6}{36} = \frac{1}{6}$$

$$\therefore q = 1-p = 1 - \frac{1}{6} = \frac{5}{6}$$

since ; $P(X=r) = {}^n C_r p^r \cdot q^{n-r}$

$$(1) P(4 \text{ success}) = {}^{10} C_4 \left(\frac{5}{6}\right)^6 \left(\frac{1}{6}\right)^4$$

$$= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} \cdot \frac{(5)^6}{(6)^{10}}$$

$$= 210 \times \frac{(5)^6}{(6)^{10}}$$

$$= 7 \times \frac{(5)^7}{(6)^9}$$

$$= \frac{7}{36} \left(\frac{5}{6}\right)^7 = 0.054$$

$$\begin{aligned} \text{Ex-2} \quad P(\text{no success}) &= PC_0 \\ &= 10C_0 \left(\frac{5}{6}\right)^{10} \\ &= \left(\frac{5}{6}\right)^{10} = 0.1615 \end{aligned}$$

Ex-3 The probability of a man hitting a target is $\frac{1}{4}$. If he fires 7 times; what is the probability of his hitting the target at least twice?

Sol: Here $p = \frac{1}{4}$ $n = 7$

$$\therefore q = 1 - \frac{1}{4} = \frac{3}{4}$$

The probability of hitting the target twice

$$= P(X \geq 2)$$

$$= PC(2) + PC(3) + PC(4) + \dots + PC(7)$$

$$= 1 - PC(0) - PC(1)$$

$$= 1 - nC_0 p^0 q^7 - nC_1 p^1 q^6$$

$$= 1 - 7C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^7 - 7C_1 \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^6$$

$$= 1 - \left(\frac{3}{4}\right)^7 - 7 \cdot \frac{1}{4} \left(\frac{3}{4}\right)^6$$

$$= 1 - \frac{3^7}{4^7} (3+7)$$

$$= 1 - \frac{7290}{16384} = \frac{9094}{16384} = 0.5551$$

Ex-4 Eight coins are thrown simultaneously
Find the chance of obtaining at least six heads.

Sol: When a coin is thrown;

Let P = the prob. of getting head
 q = " " " " " tail

$$\therefore P = q = \frac{1}{2} \quad \text{and } n = 8$$

$$\therefore P(\text{at least 6 heads}) = P(X \geq 6)$$

$$= P(6 \text{ heads}) + P(7 \text{ heads}) + P(8 \text{ heads})$$

$$= {}^8C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^2 + {}^8C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right) + {}^8C_8 \left(\frac{1}{2}\right)^8$$

$$= \left(\frac{1}{2}\right)^8 \left\{ {}^8C_6 + {}^8C_7 + {}^8C_8 \right\}$$

$$= \frac{1}{256} \left\{ \frac{8 \times 7}{2 \times 1} + \frac{8}{1} + 1 \right\}$$

$$= \frac{1}{256} \left\{ 28 + 8 + 1 \right\}$$

$$= \frac{37}{256}$$

$$= 0.144$$

Ex-5 If the Probability that a man aged 60 will live to be 70 is 0.65 what is the Probability that out of 10 men now 60 at least 7 will live to be 70?

Sol: Let P = Prob. of living up to 70

$$= 0.65$$

$$= \frac{65}{100} = \frac{13}{20}$$

$$\therefore q = \text{Prob. of dying} = 1 - \frac{13}{20} = \frac{7}{20}$$

Here $n = \text{total no. of men} = 10$
and at least 7 will live to be 70

\therefore There are following four possibilities.

$$\text{Req. Prob. } P(X \geq 7) = P(7) + P(8) + P(9) + P(10)$$

$$= P_1 + P_2 + P_3 + P_4$$

(1) $P_1 = \text{The prob. of 7 living and 3 dying}$

$$= 10C_7 p^7 q^3$$

$$= \frac{10!}{7! 3!} \left(\frac{13}{20}\right)^7 \left(\frac{7}{20}\right)^3$$

$$= \frac{120}{(20)^{10}} (13)^7 (7)^3 = 0.25222$$

12 P_2 = The Prob. of 8 living and 2 dying

$$= 10 C_8 p^8 q^2$$

$$= \frac{10!}{8! 3!} \left(\frac{13}{20}\right)^8 \left(\frac{7}{20}\right)^2$$

$$= \frac{45}{(20)^{10}} (13)^8 (7)^2 = 0.17565$$

13 P_3 = The Prob. of 9 living and 1 dying

$$= 10 C_9 p^9 q^1$$

$$= \frac{10!}{9! 1!} \left(\frac{13}{20}\right)^9 \left(\frac{7}{20}\right)^1$$

$$= \frac{10}{(20)^{10}} (13)^9 (7)^1 = 0.072492$$

14 P_4 = The Prob. of 10 living.

$$= 10 C_{10} p^{10}$$

$$= 1 \cdot \left(\frac{13}{20}\right)^{10} = 0.013463$$

Hence Required Probability = $P_1 + P_2 + P_3 + P_4$

$$= 0.5139$$