⊛. THEORETICAL DISTRIBUTIONS ⊛

→ In this chapter we shall study the following univariate probability distributions.

<1> Binomial Distribution
<2> Poisson Distribution
<3> Normal Distribution.

The first two distributions are discrete probability distributions and the third is a continuous probability distribution.

[A] BINOMIAL DISTRIBUTION

Binomial distribution is also known as the Bernoulli distribution. This distribution can be used under the following conditions:

<1> An experiment consists of a finite number of repeated trials.

<2> Each trial has only two possible mutually exclusive out comes which are known as a "success" and a "failure"

<3> All the (Different) trials are independent i.e the result of any trial; is not affected in any way by the preceding trials

and does not affect the result of succeeding trials.

4) The probability of success in any trial is p and is constant for each trial. The Probability of a failure denoted by q ; and is equal to 1 - p.

The sequence of trials under the above assumption is also known as a Bernoulli trials.

for ex : We know that the probability of getting a head or a tail on tossing a coin is $\frac{1}{2}$.

If the coin is tossed thrice ; the probability of getting one head and two tails can be combined as

H-T-T , T-H-T , T-T-H.

The probability of each one of these being :

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$i.e \quad \left(\frac{1}{2}\right)^3$$

their total probability shall be :

$$3\left(\frac{1}{2}\right)^3$$

similarly; if a trial is repeated n-times and if p is the probability of a success and q that of a failure; then the probability of r successes and $(n-r)$ failures is given by:

$$p^r \cdot q^{n-r}$$

But these r successes and $(n-r)$ failures can occur in any of the ^nC_r ways in each of which the probability is same.

Thus; the probability of r successes is

$$^nC_r \, p^r \, q^{n-r}$$

→ Let n be the total numbers of repeated trials; p be the probability of a success in a trial and q be the probability of its failure.

Let r be a random variable which denotes the number of successes in n trials.
The possible values of r are $0, 1, 2, 3, 4, ----, n$.

We are intereseted in finding the Probability of r successes out of n trials

i.e $P(r)$

To find this probability; we assume that the first r trials are successes and remaining $(n-r)$ trials are failurs. since different trials are assumed to be independent; the probability of this sequence is;

$$\underbrace{p \cdot p \cdot p \cdot \ldots p}_{r \text{ times}} \cdot \underbrace{q \cdot q \cdot q \cdot \ldots q}_{(n-r) \text{ times}}$$

i.e $p^r q^{n-r}$

since out of n trials any r trials can be success; the number of sequences showing any r trials as success and remaining $(n-r)$ trials as failures is: ${}^nC_r$; where the probability of r successes in each trial is : $p^r q^{n-r}$

Hence the required probability is

$$\boxed{P(r) = {}^nC_r\, p^r q^{n-r}} \quad \text{-----} \circledast$$

where $p+q=1$ and $r=0,1,2,\dots n$.

The distribution $\circledast$ is called the binomial probability distribution.

Note :→

The successive probabilities $P(r)$ in $\circledast$ for $r=0,1,2,\dots n$ are:

${}^nC_0\, p^0 q^n$; ${}^nC_1\, p^1 \cdot q^{n-1}$; ${}^nC_2\, p^2 q^{n-2}$; ${}^nC_3\, p^3 q^{n-3}$; ----

${}^nC_n\, p^n q^0$.

which are the terms in the binomial expansion of $(q+p)^n$.

Hence this distribution is known as a Binomical distribution.

$\circledast$ Recurrence formula for the Binomial Distribution :→

In a Binomical distribution

$$\begin{cases} P(r) = {}^nC_r\, q^{n-r} p^r = \dfrac{n!}{(n-r)!\, r!}\, q^{n-r} p^r \\ P(r+1) = {}^nC_{r+1}\, q^{n-r-1} p^{r+1} = \dfrac{n!}{(n-r-1)!\,(r+1)!}\, q^{n-r-1} p^{r+1} \end{cases}$$

Now;

$$\frac{P(r+1)}{P(r)} = \frac{(n-r)!}{(n-r-1)!} \times \frac{r!}{(r+1)!} \times \frac{p}{q}$$

$$= \frac{(n-r)(n-r-1)!}{(n-r-1)!} \times \frac{r!}{(r+1)\,r!} \times \frac{p}{q}$$

$$= \frac{(n-r)}{r+1} \times \frac{p}{q}$$

$$\Rightarrow \boxed{P(r+1) = \frac{(n-r)}{r+1} \times \frac{p}{q} \cdot P(r)}$$

which is the required recurrence formula. Applying this formula successively we can find: $P(1), P(2), P(3)$ --- ; if $P(0)$ is known.

(c) →

r	$P(r)$	$r \cdot P(r)$	$r^2 P(r)$
0	q^n	0	0
1	${}^nC_1\, q^{n-1}\, p$	${}^nC_1\, q^{n-1}\, p$	$1^2\, {}^nC_1\, q^{n-1}\, p$
2	${}^nC_2\, q^{n-2}\, p^2$	$2\, {}^nC_2\, q^{n-2}\, p^2$	$2^2\, {}^nC_2\, q^{n-2}\, p^2$
3	${}^nC_3\, q^{n-3}\, p^3$	$3\, {}^nC_3\, q^{n-3}\, p^3$	$3^2\, {}^nC_3\, q^{n-3}\, p^3$
⋮	⋮	⋮	⋮
n	p^n	$n p^n$	$n^2 p^n$

* Mean and Variance of the Binomial Distribution :→

* Mean $\mu = \Sigma r \cdot P(r)$

$$= \sum_{r=0}^{n} r \cdot {}^nC_r \, q^{n-r} p^r$$

$$= 0 + 1 \cdot {}^nC_1 q^{n-1} p + 2\,{}^nC_2 q^{n-2} p^2 + 3\,{}^nC_3 q^{n-3} p^3 + \cdots + n\,{}^nC_n p^n$$

$$= n q^{n-1} p + 2 \frac{n(n-1)}{2!} q^{n-2} p^2 + 3 \frac{n(n-1)(n-2)}{3!} q^{n-3} p^3 + \cdots + n p^n$$

$$= n q^{n-1} p + n(n-1) q^{n-2} p^2 + \frac{n(n-1)(n-2)}{2} q^{n-3} p^3 + \cdots + n p^n$$

$$= np \left\{ q^{n-1} + (n-1) q^{n-2} p + \frac{(n-1)(n-2)}{2 \cdot 1} q^{n-3} p^2 + \cdots + p^{n-1} \right\}$$

$$= np \left\{ {}^{n-1}C_0 q^{n-1} + {}^{n-1}C_1 q^{n-2} p + {}^{n-1}C_2 q^{n-3} p^2 + \cdots + {}^{n-1}C_{n-1} p^{n-1} \right\}$$

$$= np (q + p)^{n-1}$$

$$\boxed{\text{Mean} = np}$$ $(\because p+q=1)$

* Variance $\sigma^2 = \sum_{r=0}^{n} r^2 P(r) - [\text{Mean}]^2$

$= \sum_{r=0}^{n} [r + r(r-1)] P(r) - \mu^2$

$= \sum_{r=0}^{n} r.P(r) + \sum_{r=0}^{n} r(r-1)P(r) - \mu^2$

$= \mu + \sum_{r=0}^{n} r(r-1) P(r) - \mu^2$

$= \mu + \sum_{r=2}^{n} r(r-1) P(r) - \mu^2$

($\because$ contribution due to $r=0$ and $r=1$ is zero)

$= \mu + \sum_{r=2}^{n} r(r-1)\, {}^nC_r\, q^{n-r} p^r - \mu^2$

$= \mu - \mu^2 + \sum_{r=2}^{n} r(r-1)\, {}^nC_r\, q^{n-r} p^r$

$= \mu - \mu^2 + \{2.1\, {}^nC_2\, q^{n-2} p^2 + 3.2\, {}^nC_3\, q^{n-3}.p^3$

$+ \dots + n(n-1)\, {}^nC_n\, p^n \}$

$$= \mu - \mu^2 + \left\{ 2\cdot1\cdot\frac{n(n-1)}{2!} q^{n-2} p^2 + 3\cdot2\,\frac{n(n-1)(n-2)}{3!} q^{n-3} p^3 + \cdots + n(n-1)p^n \right\}$$

$$= \mu - \mu^2 + \left\{ n(n-1) q^{n-2} p^2 + n(n-1)(n-2) q^{n-3} p^3 + \cdots + n(n-1)p^n \right\}$$

$$= \mu - \mu^2 + n(n-1)p^2 \left\{ q^{n-2} + (n-2) q^{n-3} p + \cdots + p^{n-2} \right\}$$

$$= \mu - \mu^2 + n(n-1)p^2 \left\{ (n-2)C_0\, q^{n-2} + (n-2)C_1\, q^{n-3} p + \cdots + (n-2)C_{(n-2)}\, p^{n-2} \right\}$$

$$= \mu - \mu^2 + n(n-1)p^2 [q+p]^{n-2}$$

$$= \mu - \mu^2 + n(n-1)p^2 \qquad (\because p+q=1)$$

$$= np - n^2p^2 + n(n-1)p^2$$

$$= np\left\{ 1 + (n-1)p - np \right\}$$

$= np\{1-p\}$

$$\boxed{\text{Variance} = npq.}$$

where; Variance $= \sigma^2$

→ standard deviation of the binomial distribution is

$\sqrt{npq} = \sigma.$

$$ {}^nC_r = \frac{n!}{(n-r)!\,r!} $$

Ex-1 Ten unbiased coins are tossed simultaneously. Find the Probability of obtaining.

<1> Exactly 6 heads
<2> Atleast 8 heads
<3> No head
<4> At least one head
<5> Not more than three heads
<6> At least 4 heads.

Solⁿ :→ If p denotes the probability of a head; then

$$ p = q = \frac{1}{2} $$

Here $n = 10$

If the random variable x denotes the number of heads; then by Binomial - Probability law; the Probability of r head is given by :

$$ P(r) = P(x = r) = {}^nC_r\, p^r \cdot q^{n-r} $$

$$ = {}^{10}C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{10-r} $$

$$ = {}^{10}C_r \left(\frac{1}{2}\right)^{10} $$

$$ \therefore P(r) = \frac{1}{1024}\, {}^{10}C_r \quad \text{------} \quad (*) $$

<1> Required Prob. $= P(6)$

$$= \frac{1}{1024} \cdot {}^{10}C_6$$

$$= \frac{\cancel{252}}{1024} \quad 210$$

$$= \cancel{\frac{63}{256}} \quad \frac{105}{512}$$

<2> Required Prob. $= P(8) + P(9) + P(10)$

$P(X \geq 8)$

$$= \frac{1}{1024}\left\{ {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10} \right\}$$

$$= \frac{1}{1024}\left\{ 45 + 10 + 1 \right\}$$

$$= \frac{56}{1024}$$

$$= \frac{7}{128}$$

<3> Req. Prob. $P(X=0) = P(0)$

$$= \frac{1}{1024} \cdot {}^{10}C_0$$

$$= \frac{1}{1024}$$

<4> Req.. Prob. $= P(1) + P(2) + P(3) + \ldots + P(10)$

$= P$ [At least one head]

$= 1 - P[\text{no head}]$

$= 1 - P(0)$

$= 1 - \frac{1}{1024}$

$= \frac{1023}{1024}$

<5> Req. Prob. $P(X \leq 3) = P(0) + P(1) + P(2) + P(3)$

$= \frac{1}{1024}\{{}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3\}$

$= \frac{1}{1024}\{1 + 10 + 45 + 120\}$

$= \frac{176}{1024} = \frac{11}{64}$

<6> Req. Prob $P(X \geq 4) = P(4) + P(5) + \cdots + P(10)$

$= \frac{1}{1024}\{{}^{10}C_4 + {}^{10}C_5 + \cdots + {}^{10}C_{10}\}$

$= \frac{53}{64}$

OR

$= 1 - P(X \leq 3)$

$= 1 - \{P(0) + P(1) + P(2) + P(3)\}$

$= 1 - \frac{11}{64}$ ($\because$ by <5>)

$= \frac{53}{64}$

Ex-2 A pair of dice is thrown 10 times. If getting a doublet (same number on both) is considered a success; find the probability of

<1> 4 successes.

<2> No. success.

Sol^n:→ Here $n = 10$

A doublet can be obtained when a pair of dice is thrown in

$(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)$

i.e in 6 ways.

$\therefore\ p = 6/36 = 1/6$

$\therefore\ q = 1 - p = 1 - 1/6 = 5/6$

since ; $P(X = r) = {}^nC_r\, p^r . q^{n-r}$

<1> $P(4 \text{ success}) = {}^{10}C_4 \left(\frac{5}{6}\right)^6 \left(\frac{1}{6}\right)^4$

$$= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} . \frac{(5)^6}{(6)^{10}}$$

$$= 210 \times \frac{(5)^6}{(6)^{10}}$$

$$= 7 \times \frac{(5)^7}{(6)^9}$$

$$= \frac{7}{36} \left(\frac{5}{6}\right)^7 = 0.054$$

$\langle 2 \rangle$ $P(\text{no success}) = P(0)$

$$= {}^{10}C_0 \left(\frac{5}{6}\right)^{10}$$

$$= \left(\frac{5}{6}\right)^{10} = 0.1615$$

Ex-3 The probability of a man hitting a target is $1/4$. If he fires 7 times; what is the probability of his hitting the target at least twice.

Solⁿ:→ Here $p = 1/4$

$\therefore q = 1 - 1/4 = 3/4$ $\Big\}$ $n = 7$

The probability of hitting the target twice

$$= P(X \geq 2)$$

$$= P(2) + P(3) + P(4) + \cdots + P(7)$$

$$= 1 - P(0) - P(1)$$

$$= 1 - {}^nC_0\, p^0 q^7 - {}^nC_1\, p^1 q^6$$

$$= 1 - {}^7C_0 (1/4)^0 (3/4)^7 - {}^7C_1 (1/4)^1 (3/4)^6$$

$$= 1 - (3/4)^7 - 7 \cdot \frac{1}{4} (3/4)^6$$

$$= 1 - \frac{3^6}{4^7}(3 + 7)$$

$$= 1 - \frac{7290}{16384} = \frac{9094}{16384} = 0.5551$$

Ex-4 Eight coins are thrown simultaneously. Find the chance of obtaining at least six heads.

Sol^n:→ when a coin is thrown;

Let p = the Prob. of getting head }
q = " " " tail }

∴ $p = q = \frac{1}{2}$ and $n = 8$

∴ P(at least 6 heads) $= P(x \geq 6)$

$= P(6 \text{ heads}) + P(7 \text{ heads}) + P(8 \text{ heads})$

$= {}^8C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^2 + {}^8C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right) + {}^8C_8 \left(\frac{1}{2}\right)^8$

$= \left(\frac{1}{2}\right)^8 \left\{ {}^8C_6 + {}^8C_7 + {}^8C_8 \right\}$

$= \frac{1}{256} \left\{ \frac{8 \times 7}{2 \times 1} + \frac{8}{1} + 1 \right\}$

$= \frac{1}{256} \left\{ 28 + 8 + 1 \right\}$

$= \frac{37}{256}$

$= 0.144$

Ex-5 If the Probability that a man aged 60 will live to be 70 is 0.65 what is the Probability that out of 10 men now 60 at least 7 will live to be 70?

Sol:→ Let p = Prob. of living up to 70

$= 0.65$

$= \frac{65}{100} = \frac{13}{20}$

$\therefore q$ = Prob. of dying $= 1 - \frac{13}{20} = \frac{7}{20}$

Here n = total no. of men $= 10$

and at least 7 will live to be 70

∴ There are following four possibilities.

Req. Prob. $P(x \geq 7) = P(7) + P(8) + P(9) + P(10)$

$= P_1 + P_2 + P_3 + P_4$

⟨1⟩ P_1 = The prob. of 7 living and 3 dying

$= {}^{10}C_7 \, p^7 q^3$

$= \frac{10!}{7!\,3!} \left(\frac{13}{20}\right)^7 \left(\frac{7}{20}\right)^3$

$= \frac{120\,(13)^7\,(7)^3}{(20)^{10}} = 0.25222$

<2> P_2 = The Prob. of 8 living and 2 dying

$$= {}^{10}C_8 \; p^8 q^2$$

$$= \frac{10!}{8!\,3!} \left(\frac{13}{20}\right)^8 \left(\frac{7}{20}\right)^2$$

$$= \frac{45\,(13)^8\,(7)^2}{(20)^{10}} = 0.17565$$

<3> P_3 = The Prob. of 9 living and 1 dying

$$= {}^{10}C_9 \; p^9 q^1$$

$$= \frac{10!}{9!\,1!} \left(\frac{13}{20}\right)^9 \left(\frac{7}{20}\right)^1$$

$$= \frac{10\,(13)^9\,(7)^1}{(20)^{10}} = 0.072492$$

<4> P_4 = The Prob. of 10 living

$$= {}^{10}C_{10} \; p^{10}$$

$$= 1 \cdot \left(\frac{13}{20}\right)^{10} = 0.013463$$

Hence Required Probability $= P_1 + P_2 + P_3 + P_4$

$=$

$= 0.5139.$