10.10 Mathematical Expectation

If X is a discrete random variable having various possible values $x_1, x_2, \ldots, x_n$ and if f(x) is the probability function, the **mathematical expectation** or simply **expectation** of X is defined and denoted by E(x).

$$E(x) = \sum_{i=1}^{n} x_i \, f(x_i) \quad \text{or} \quad \sum x \cdot f(x) \quad \text{or} \quad \sum_{i=1}^{n} x_i p_i$$

where, $\sum_{i=1}^{n} p_i = p_1 + p_2 + \ldots + p_n = 1$

If X is a continuous random variable having probability density function f(x), expectation of X is defined as

$$E(X) = \int_{-\infty}^{\infty} x \, f(x) \, dx$$

E(X) is also called the **mean** value of the probability distribution of x and is denoted by μ.

Properties of Mathematical Expectation

(1) Expected value of constant term is constant that is if c is constant, then

$$E(c) = c$$

(2) If c is constant, then

$$E(cX) = c \cdot E(X)$$

(3) If a and b are constants, then

$$E(aX \pm b) = aE(X) \pm b$$

(4) If a, b and c are constants, then

$$E\left(\frac{aX + b}{c}\right) = \frac{1}{c}[aE(X) + b]$$

(5) If X and Y are two random variables, then

$$E(X + Y) = E(X) + E(Y)$$

(6) If X and Y are two independent random variable, then

$$E(X \cdot Y) = E(X) \cdot E(Y)$$

(7) If g(X) is any function of random variable X and f(x) is proability density function, then

$$E\{g(x)\} = \sum g(x) \cdot f(x)$$

For example, the expected value of $g(x) = x^2 + 2x$ is defined as

$$E\{g(x)\} = \sum (x^2 + 2x) f(x)$$

Remark

The properties of expectation also hold good for continuous variables. We only need to replace summations by integrals.

Variance of a Random Variable

Variance is a characteristic of a random variable X and it is used to measure dispersion (or variation) of X.

If X is a discrete random variable with probability density function f(x), then expected value of $[X - E(X)]^2$ is called the **variance** of X and it is denoted by V(X).

That is $\quad V(X) = E[X - E(X)]^2$

If we put $E(X) = \mu$, then

$$V(X) = E(X - \mu)^2$$

Properties of Variance

(1) $V(c) = 0$, where c is a constant.

(2) $V(cX) = c^2\, V(X)$, where c is a constant.

(3) $V(X + c) = V(X)$, where c is a constant.

(4) If a and b are constants, then

$$V(aX + b) = a^2\, V(X)$$

(5) If X and Y are the independent random variables, then

$$V(X + Y) = V(X) + V(Y)$$

(6) $V(X) = E(X^2) - \mu^2$ or $V(X) = E(X^2) - [E(X)]^2$

Standard Deviation of a Random Variable

The positive square root of V(X) (Variance of X) is called standard deviation of random variable X and is denoted by σ.

i.e. S.D. $\sigma = \sqrt{V(X)}$.

Note : (i) σ^2 is called variance of X.

(ii) It is customary to represent X by x in practice for convenience.

EXAMPLE 1 Find the expected value of X, where the values of X and their corresponding probabilities are given by the following table.

x_i	2	5	9	24
p_i	0.4	0.2	0.3	0.1

SOLUTION $E(X) = 0.4 \times 2 + 0.2 \times 5 + 0.3 \times 9 + 0.1 \times 24$

$= 0.8 + 1.0 + 2.7 + 2.4 = 6.9$

EXAMPLE 2 A tray of electronics components contains nine good components and three defective components. If two components are selected at random, what is the expected number of defective components ?

SOLUTION Let a random variable X be the number of defective components selected. X can have the value 0, 1 or 2. We need the probability of each of those numbers.

P(0) = Probability of no defective (both good)

$$= \frac{9C_2}{12C_2} = \frac{36}{66} = \frac{12}{22}$$

P(1) = Probability of 1 good and 1 defective

$$= \frac{9C_1 \cdot 3C_1}{12C_2} = \frac{27}{66} = \frac{9}{22}$$

$$P(2) = \text{Probability of two defective}$$

$$= \frac{3C_2}{12C_2} = \frac{3}{66} = \frac{1}{22}$$

The expected value is

$$E(X) = \frac{12}{22}(0) + \frac{9}{22}(1) + \frac{1}{22}(2) = \frac{11}{22} = \frac{1}{2}$$

So the expected number of components is $\frac{1}{2}$. The value $\frac{1}{2}$ simply says that if a large number of selections are made, we will *average* one–half each time. We expect to get no defectives a little less than half the time and either one or two the rest of the time, but the average will be one half.

EXAMPLE 3 Probability distribution of random variable X is given below.

x	−1	0	1	2	3
p(x)	0.1	0.2	0.5	0.6	0.4

Is this possible ?

SOLUTION Here, p(x) = 1 but the last value of p(x) is negative which is not possible. Therefore given distribution is not possible.

EXAMPLE 4 The probability density function of a random variable X is defined as $p(x) = \frac{x}{k}$, where x = 1, 2, 3, 4, 5. Find k.

SOLUTION For different values of X, p(x) is given in the following table.

X	1	2	3	4	5
$p(x) = \frac{x}{k}$	$\frac{1}{k}$	$\frac{2}{k}$	$\frac{3}{k}$	$\frac{4}{k}$	$\frac{5}{k}$

$$\therefore \quad \sum p(x) = \frac{1}{k} + \frac{2}{k} + \frac{3}{k} + \frac{4}{k} + \frac{5}{k} = \frac{15}{k}$$

But, $\sum p(x) = 1$

$$\therefore \quad \frac{15}{k} = 1$$

$$\therefore \quad k = 15$$

EXAMPLE 5 For a random variable X, $p(x) = \frac{x}{x+1}$, where x = 1, 2, 3. Is p(x) a probability density function ?

SOLUTION Here, $p(x) = \frac{x}{x+1}$

$\therefore$ For x = 1, 2, 3, p(x) will be $\frac{1}{2}, \frac{2}{3}$ and $\frac{3}{4}$.

$$\therefore \quad \sum p(x) = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \frac{23}{12} > 1$$

Since $\sum p(x) > 1$

$p(x) = \frac{x}{x+1}$ can not be the probability density function of x.

EXAMPLE 6 The probability function of a random variable X is $p(x) = \frac{2x+1}{48}$, where x = 1, 2, 3, 4, 5, 6. Verify whether p(x) is probability function ?

SOLUTION For different values of x, p(x) is given in the following table.

x	1	2	3	4	5	6
$p(x) = \frac{2x+1}{48}$	$\frac{3}{48}$	$\frac{5}{48}$	$\frac{7}{48}$	$\frac{9}{48}$	$\frac{11}{48}$	$\frac{13}{48}$

$$\therefore \quad \sum p(x) = \frac{1}{48}[3 + 5 + 7 + 9 + 11 + 13]$$

$$= \frac{1}{48}$$

$$= 1$$

$$\therefore \quad \sum p(x) = 1 \text{ and } p(x) > 0 \text{ for all } x$$

$\therefore$ p(x) is probability function.

EXAMPLE 7 There are 3 white and 2 black balls in a box. If 2 balls are selected at random, find the expected number of black balls.

SOLUTION Let x denote the number of black balls. Probabilities and different values of x are given for 2 balls in the following table.

Black balls, x	p(x)	xp(x)
0	$\frac{2C_0 \times 3C_2}{5C_2} = \frac{3}{10}$	0
1	$\frac{2C_1 \times 3C_1}{5C_2} = \frac{6}{10}$	$\frac{6}{10}$
2	$\frac{2C_2 \times 3C_0}{5C_2} = \frac{1}{10}$	$\frac{2}{10}$
	$\sum p(x) = 1$	$\sum xp(x) = \frac{8}{10}$

Now, $E(x) = \sum xp(x) = \frac{8}{10} = \frac{4}{5}$

$E(x) = \frac{4}{5}$ is the expected number of black balls.

EXAMPLE 8 Two unbiased coins are tossed. Find expected value of number of heads.

SOLUTION In the experiment of tossing two coins, the sample space U will be

$$U = \{TT, TH, HT, HH\} \qquad \therefore n = 4$$

Let x denote number of heads. Therefore x will take the values 0, 1, 2. Their probabilities are given below.

Outcome	x	p(x)	xp(x)
TT	0	$\frac{1}{4}$	0
TH, HT	1	$\frac{2}{4}$	$\frac{2}{4}$
HH	2	$\frac{1}{4}$	$\frac{2}{4}$

$\therefore \quad E(x) = \sum xp(x) = 0 + \frac{2}{4} + \frac{2}{4} = 1$

$\therefore \quad E(x) = 1$

EXAMPLE 9 a bag contains 5 white and 7 black balls. Find the expectation man who is allowed to draw two balls from the bag and who is to receive