Limit Rules

If $f(n)$ and $g(n)$ are asymptotically increasing functions, then the following three rules hold true:

Rule 1: if $\lim_{n\to\infty} \frac{f(n)}{g(n)} \in R^+$, then $f(n) \in O(g(n))$ and $g(n) \in O(f(n))$

$[f(n) \in \Theta(g(n))$ and $g(n) \in \Theta(f(n))]$

Rule 2: if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then $f(n) \in O(g(n))$ but $g(n) \notin O(f(n))$

Rule 3: if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = +\infty$, then $g(n) \in O(f(n))$ but $f(n) \notin O(g(n))$

Proof:- We know that, by the definition of "Limit",

$$\left[\text{if } \lim_{x\to a} f(x) = l \implies |f(x) - l| \leq \underset{(\text{epsilon})}{\varepsilon} \; (\varepsilon > 0) \right.$$

(OR)

$$\left. |f(x) - l| \leq \delta \; (\delta > 0) \right]$$

So, we need to apply this definition of limit;

For Rule 1:- Given: $\lim_{n\to\infty} \frac{f(n)}{g(n)} \in R^+$

$$\Rightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = l \text{ (say)} \in R^+$$

$$\Rightarrow \left|\frac{f(n)}{g(n)} - l\right| \leq \delta \; (\delta > 0; \text{ very small value of } \delta)$$

(By definition of Limit)

$$\Rightarrow \frac{f(n)}{g(n)} - l \leq \delta$$ (as $f(n)$ and $g(n)$ are increasing functions).

$$\Rightarrow \frac{f(n)}{g(n)} \leq l + \delta$$

$$\Rightarrow \frac{f(n)}{g(n)} \leq l + l$$ (Let $\delta = l$; for simplicity).

$$\Rightarrow \frac{f(n)}{g(n)} \leq 2l$$

$$\Rightarrow f(n) \leq 2l \cdot g(n)$$

$$\Rightarrow f(n) \leq c \cdot g(n)$$ (Let $2l = c$)

$$\Rightarrow \boxed{f(n) \in O(g(n))}$$

Similarly, if we consider $\lim_{n \to \infty} \frac{g(n)}{f(n)} \in R^+$

i.e. $\lim_{n \to \infty} \frac{g(n)}{f(n)} = \frac{1}{l}$ (say) $\in R^+$

$$\Rightarrow \left| \frac{g(n)}{f(n)} - \frac{1}{l} \right| \leq \delta \; (\delta > 0)$$

$$\Rightarrow \frac{g(n)}{f(n)} \leq \frac{1}{l} + \delta$$

$$\Rightarrow \frac{g(n)}{f(m)} \leq \frac{1}{l} + \frac{1}{l}$$ (Let $\delta = \frac{1}{l}$)

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$$\Rightarrow \frac{g(n)}{f(n)} \leq \frac{2}{l}$$

$$\Rightarrow \frac{g(n)}{f(n)} \leq c. \quad \left(\text{let } c = \frac{2}{l}\right)$$

$$\Rightarrow g(n) \leq c \cdot f(n)$$

$$\Rightarrow \boxed{g(n) \in O(f(n))}$$

<u>For Rule 2</u>: <u>Given</u>:- $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$

$$\Rightarrow \frac{f(n)}{g(n)} - 0 \leq \delta \quad (\delta > 0)$$

$$\Rightarrow \frac{f(n)}{g(n)} \leq \delta$$

$$\Rightarrow \frac{f(n)}{g(n)} \leq c. \quad (\text{Let } \delta = c, \text{ for simplicity})$$

$$\Rightarrow f(n) \leq c \cdot g(n)$$

$$\Rightarrow \boxed{f(n) \in O(g(n))}$$

Now, we need to prove $g(n) \notin O(f(n))$.

Let's assume $g(n) \in O(f(n))$ [<u>Proof by Contradiction</u>]

$$\Rightarrow g(n) \leq c \cdot f(n) \quad (c > 0)$$

$$\Rightarrow \frac{1}{c} \leq \frac{f(n)}{g(n)}$$

Applying $\lim_{n \to \infty}$ on both sides, we get,

$$\lim_{n\to\infty} \frac{1}{c} \leq \lim_{n\to\infty} \frac{f(n)}{g(n)}$$

$$\Rightarrow \boxed{\frac{1}{c} \leq 0} \quad \left[\text{Since, } \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \text{ (given)}\right]$$

(Impossible)

So, our assumption is FALSE;

Hence, $\boxed{g(n) \notin O(f(n))}$.

For Rule 3 :- Apply similar procedure as for Rule 2.

Start with, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = +\infty$

$$\Rightarrow \boxed{\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0}$$

Follow same steps as for Rule 2 (Proof)

Ex:- P.T.:- $\sqrt{n}$ grows asymptotically faster than $\log n$.

(OR)

$$\boxed{\sqrt{n} \notin O(\log n)}$$

(OR)

$$\boxed{\log n \in O(\sqrt{n})}$$

Proof:- Let $f(n) = \sqrt{n}$ and $g(n) = \log n$.

Now, $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{\sqrt{n}}{\log n}$

$= \lim_{n \to \infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}}$ (L'Hospital's rule)

$= \lim_{n \to \infty} \frac{\frac{1}{2}}{\frac{1}{\sqrt{n}}}$

$= \lim_{n \to \infty} \frac{\sqrt{n}}{2}$

$= +\infty$ (Rule(3) is applicable).

By Rule 3, $f(n) \notin O(g(n))$

$\Rightarrow \boxed{\sqrt{n} \notin O(\log n)}$