Big-O notation

Prove the following:

1) $n^2 + n = O(n^2)$.

Sol^n:- Let $f(n) = n^2 + n$ and $g(n) = n^2$

We need to show that there exists $c > 0$, $n_0 > 0$, such that

$0 \leq f(n) \leq c \cdot g(n)$ [Big-O defintion/condition]

Now, we know that :-

when $n \geq 1$,	$n^2 \leq n^2$	$\longrightarrow$ (1)
and	$n \leq n^2$	$\longrightarrow$ (2)

So, by (1) and (2), we get,

$0 \leq n^2 + n \leq 2 \cdot n^2$

which is of the form

$0 \leq f(n) \leq c \cdot g(n)$

where, $f(n) = n^2 + n$, $g(n) = n^2$ and $c = 2$.

We can take $n_0 = 1$ (as for $n \geq 1$, (1) and (2) are satisfied)

So, for $c = 2$, $n_0 = 1$, we have

$\boxed{n^2 + n = O(n^2)}$ (Hence proved)

2) $n^2+n = O(n^3)$.

Sol^n:- Let $f(n)= n^2+n$ and $g(n)= n^3$.

We need to show that there exists $c>0$, $n_0>0$, such that

$0 \leq f(n) \leq c \cdot g(n)$ [Big-O definition/condition]

Now, we know that :-

when $n \geq 1$,	$n^2 \leq n^3$	$\longrightarrow$ (1)
and	$n \leq n^3$	$\longrightarrow$ (2)

So, by (1) and (2), we get,

$0 \leq n^2+n \leq 2n^3$

which is of the form

$0 \leq f(n) \leq c \cdot g(n)$

where $f(n) = n^2+n$ and $g(n) = n^3$ and $c=2$

We can take $n_0 = 1$ (as for $n \geq 1$, (1) and (2) are satisfied)

So, for $c=2$, $n_0 = 1$, we have

$\boxed{n^2+n = O(n^3)}$ (Hence proved)

Big-Ω notation

Prove the following:

1) $n^3+4n^2 = \Omega(n^2)$

Soln: Let $f(n) = n^3+4n^2$ and $g(n) = n^2$

We need to show that there exists $c>0$ and $n_0>0$, such that

$0 \leq c \cdot g(n) \leq f(n)$ [Big-Ω definition/ condition]

Now, we know that :-

$$\boxed{\begin{array}{ll} \text{when } n \geq 0, \ n^3 \leq n^3+4n^2 & \longrightarrow (1) \\ \text{Also, when } n \geq 1, \ n^2 \leq n^3 & \longrightarrow (2) \end{array}}$$

So, by (1) and (2), we get,

$0 \leq n^2 \leq n^3 \leq n^3+4n^2$

$\Rightarrow 0 \leq 1(n^2) \leq n^3+4n^2$

which is of the form

$0 \leq c \cdot g(n) \leq f(n)$

where, $f(n) = n^3+4n^2$ and $g(n) = n^2$ and $c = 1$

We can take $n_0 = 1$ (as for $n_0 = 1$, (1) and (2) are satisfied)

So, for $c = 1$, $n_0 = 1$, we have

$\boxed{n^3+4n^2 = \Omega(n^2)}$ (Hence proved)

2) $n^2 + n = \Omega(n^2)$

Soln :- Let $f(n) = n^2 + n$ and $g(n) = n^2$

We need to show that there exists $c > 0$ and $n_0 > 0$ such that

$$0 \leq c \cdot g(n) \leq f(n) \quad \text{[Big-}\Omega\text{ definition/condition]}$$

Now, we know that

$$\boxed{\text{when } n \geq 0, \ n^2 \leq n^2 + n \longrightarrow (1)}$$

From (1), it is clear that

$$0 \leq 1(n^2) \leq n^2 + n$$

which is of the form

$$0 \leq c \cdot g(n) \leq f(n)$$

where $f(n) = n^2 + n$, $g(n) = n^2$ and $c = 1$.

We can take $n_0 = 1$ (as for $n \geq 1$, (1) is satisfied)

So, for $c = 1$, $n_0 = 1$, we have

$\boxed{n^2 + n = \Omega(n^2)}$ (Hence proved)

Θ notation

We need to show that there exists

$c_1 > 0$, $c_2 > 0$ and $n_0 > 0$, such that

$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ [Θ definiton/ condition]

Θ notation

2) $n^2 + 5n + 7 = \Theta(n^2)$

Sol^n: Let $f(n) = n^2 + 5n + 7$ and $g(n) = n^2$.

We need to show that :-

I) $n^2 + 5n + 7 = O(n^2)$

when $n \geq 1$; $n^2 \leq n^2 \rightarrow (1)$

when $n \geq 1$; $5n \leq 5n^2 \rightarrow (2)$

when $n \geq 1$; $7 \leq 7n^2 \rightarrow (3)$

So, by (1), (2) and (3), we get,

$n^2 + 5n + 7 \leq 13 \cdot n^2$

$\Rightarrow 0 \leq n^2 + 5n + 7 \leq 13 \cdot n^2$

which is of the form

$0 \leq f(n) \leq c \cdot g(n)$,

where $f(n) = n^2 + 5n + 7$,

$g(n) = n^2$ and $c_2 = 13$

II) $n^2 + 5n + 7 = \Omega(n^2)$

when $n \geq 0$; $n^2 \leq n^2 + 5n + 7 \rightarrow (4)$

$\Rightarrow 0 \leq n^2 \leq n^2 + 5n + 7$

$\Rightarrow 0 \leq 1(n^2) \leq n^2 + 5n + 7$

which is of the form

$0 \leq c \cdot g(n) \leq f(n)$,

where $f(n) = n^2 + 5n + 7$,

$g(n) = n^2$ and $c_1 = 1$

We can take $n_0 = 1$ (as for $n_0 = 1$, (1), (2), (3) and (4) are satisfied)

So, for $c_1 = 1$, $c_2 = 13$ and $n_0 = 1$, we have

$$\boxed{n^2 + 5n + 7 = \Theta(n^2)}$$ (Hence proved).