I) The Assignment problem

→ The Assignment problem is a fundamental "combinatorial optimization" problem.

→ It is a special type of problem in which the objective function is to find the optimum allocation of a number of jobs (tasks) to an equal number of agents (persons).

→ It is assumed that each person can perform each job (task), but with varying efficiency.

* General-form of an Assignment problem :

The assignment problem can be stated in the form of $n \times n$ matrix ; where each entry $[c_{ij}]$ represents the cost of effectiveness matrix i.e. the cost of assigning i^{th} person to j^{th} job.

Jobs

Persons

$$
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & \cdots & n \\
\hline
1 & c_{11} & c_{12} & c_{13} & & c_{1n} \\
2 & c_{21} & c_{22} & c_{23} & & c_{2n} \\
3 & & & & & \\
\vdots & \cdots & & \cdots & \cdots & \\
\vdots & \cdots & & \cdots & \cdots & \\
n & c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn} \\
\hline
\end{array}
$$

A person can be assigned to n jobs in n! possible ways. One method can be to find all possible n! assignments and evaluate total cost in all cases. Then the assignment with the minimum cost will give the optimal assignment. But this method is extremely infeasible (laborious).

* Mathematical Formulation

$$\text{Minimum } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

where $x_{ij} = \begin{cases} 1; & \text{if } i^{th} \text{ person is assigned to job "j"} \\ 0; & \text{if } i^{th} \text{ person is not assigned to job "j"} \end{cases}$

subject to the conditions:

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, 3, \ldots, n$$

(It means that only one job is done by the i^{th} person $(i = 1, 2, \ldots, n)$)

* Hungarian Assignment Method (Reduced Matrix method)

1) Row reduction (Identify min. element in each row and subtract it from each element of that row).
2) Column reduction (Identify min. element in each column and subtract it from each element of that column).

3) Perform row-scan and column-scan respectively, and allocate the job "i" to the person "j" according to the placement of "0".

Q-1. Find optimal allocation of jobs to persons for the following :-

	1	2	3	4
J1	5	9	3	6
J2	8	7	8	2
J3	6	10	12	7
J4	3	10	8	6

Solⁿ :- 1) Row reduction

	1	2	3	4
J1	2	6	0	3
J2	6	5	6	0
J3	0	4	6	1
J4	0	7	5	3

2) Column reduction

	1	2	3	4
J1	2	2	0	3
J2	6	1	6	0
J3	0	0	6	1
J4	0	3	5	3

3) Row-scan and Column-scan

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline J1 & 2 & 2 & \boxed{0} & 3 \\ J2 & 6 & 1 & 6 & \boxed{0} \\ J3 & \cancel{0} & \boxed{0} & 6 & 1 \\ J4 & \boxed{0} & 3 & 5 & 3 \end{array}$$

Allocation :-

		Cost
J1 → 3		3
J2 → 4		2
J3 → 2		10
J4 → 1		3
		18 Rs.

Q:- 2

	1	2	3	4	5
J1	11	7	10	17	10
J2	13	21	7	11	13
J3	13	13	15	13	14
J4	18	10	13	16	14
J5	12	8	16	19	10

Soln :-

1) Row reduction

	1	2	3	4	5
J1	4	0	3	10	3
J2	6	14	0	4	6
J3	0	0	2	0	1
J4	8	0	3	6	4
J5	4	0	8	11	2

2) Column reduction

	1	2	3	4	5
J1	4	0	3	10	2
J2	6	14	0	4	5
J3	0	0	2	0	0
J4	8	0	3	6	3
J5	4	0	8	11	1

3) Row-scan and column-scan

	1	2	3	4	5
J1	4	$\boxed{0}$	3	10	2
J2	6	14	$\boxed{0}$	4	5
J3	$\boxed{0}$	$\cancel{0}$	2	$\cancel{0}$	$\cancel{0}$
J4	8	$\cancel{0}$	3	6	3
J5	4	$\cancel{0}$	8	11	1

(only 3 assignments have been made)

Procedure :-

1) Tick the unassigned rows. (✓)
2) If the marked row contains a zero, then mark the corresponding column.
3) If the marked column contains an assignment, then mark the corresponding row.
4) Draw lines through <u>unticked rows</u> and <u>ticked columns</u>.

	1	2	3	4	5	
J1	4	$\boxed{0}$	3	10	2	✓
J2	6	14	$\boxed{0}$	4	5	
J3	$\boxed{0}$	$\cancel{0}$	2	$\cancel{0}$	$\cancel{0}$	
J4	8	$\cancel{0}$	3	6	3	✓
J5	4	$\cancel{0}$	8	11	1	✓
		✓				

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline J1 & 3 & \boxed{0} & 2 & 9 & 1 \\ J2 & 6 & 15 & \boxed{0} & 4 & 5 \\ J3 & \boxed{0} & 1 & 2 & \cancel{0} & \cancel{0} \\ J4 & 7 & \cancel{0} & 2 & 5 & 2 \\ J5 & 3 & \cancel{0} & 7 & 10 & \boxed{0} \end{array}$$

$$\begin{bmatrix} 2 & \cancel{0} & 1 & 8 & \cancel{0} \\ 6 & 16 & \boxed{0} & 4 & 5 \\ \boxed{0} & 2 & 2 & \cancel{0} & \cancel{0} \\ 6 & \boxed{0} & 1 & 4 & 1 \\ 3 & 1 & 7 & 10 & \boxed{0} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \cancel{0} & \cancel{0} & 7 & \cancel{0} \\ 6 & 17 & \boxed{0} & 4 & 6 \\ \boxed{0} & 3 & 2 & \cancel{0} & \cancel{0} \\ 5 & \boxed{0} & \cancel{0} & 3 & 1 \\ 2 & 1 & 6 & 9 & \boxed{0} \end{bmatrix}$$

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline J1 & \boxed{0} & \cancel{0} & \cancel{0} & 6 & \cancel{0} \\ J2 & 5 & 17 & \boxed{0} & 3 & 6 \\ J3 & \cancel{0} & 4 & 3 & \boxed{0} & 1 \\ J4 & 4 & \boxed{0} & \cancel{0} & 2 & 1 \\ J5 & 1 & 1 & 6 & 8 & \boxed{0} \end{array}$$

J1 → 1
J2 → 3
J3 → 4
J4 → 2
J5 → 5

<u>Total cost</u> :-

$11 + 7 + 13 + 10 + 10$

$= \underline{51 \text{ Rs.}}$