3) $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log n}$

$\Rightarrow$ Here, $a=2$, $b=2$, $d=1$, $p=-1$

So, $\underline{a = b^d}$ $(\because 2 = 2^1)$

So, case(II)(ii) is applicable.

Thus, $T(n) = \Theta(n^{\log_b a} \log^2 n)$

$= \Theta(n^{\log_2 2} \log^2 n)$

$= \underline{\Theta(n \log^2 n)}$.

4) $T(n) = \sqrt{2}\, T\left(\frac{n}{2}\right) + \log n$

$\Rightarrow$ Here, $a = \sqrt{2}$, $b=2$, $d=0$, $p=1$.

So, $a > b^d$ $(\sqrt{2} > 2^0)$

So, case(III) is applicable.

Thus, $T(n) = \Theta(n^{\log_b a})$

$= \Theta(n^{\log_2 \sqrt{2}})$

$= \Theta(n^{\log_2 2^{\frac{1}{2}}})$

$= \underline{\Theta(\sqrt{n})}$

<u>NOTE:-</u> The following equations <u>cannot be solved</u> using Master method

1) $T(n) = 2^n T\left(\frac{n}{2}\right) + n^n$

(a is not a constant; the no. of subproblems should be fixed)

2) $T(n) = 0.5\,T\left(\frac{n}{2}\right) + n$

($a < 1$; cannot have less than one subproblem)

3) $T(n) = 64\,T\left(\frac{n}{8}\right) - n^2 \log n$

($f(n)$ cannot be negative)

4) $T(n) = T\left(\frac{n}{2}\right) + n(2 - \cos n)$

(trigonometric functions cannot be part of $f(n)$).

Change of Variable method

1) $T(n) = 2T(\sqrt{n}) + \log n$

$\Rightarrow$ Let $n = 2^m \Rightarrow m = \log_2 n$.

So, $T(2^m) = 2T\left(2^{\frac{m}{2}}\right) + m \longrightarrow (1)$

Let $T(2^m) = S(m) \Rightarrow T\left(2^{\frac{m}{2}}\right) = S\left(\frac{m}{2}\right)$

So, eqn (1) becomes ;

$S(m) = 2\,S\left(\frac{m}{2}\right) + m$

We can apply master method to the above recurrence.

$a = 2,\ b = 2,\ d = 1,\ p = 0.$

So, $a = b^d$ (case (II) is applicable)

$\therefore\ S(m) = \Theta(m^d \log m)$

$S(m) = \Theta(m \log m)$

$\Rightarrow T(2^m) = \Theta(m \log m)$

$\Rightarrow \boxed{T(n) = \Theta(\log_2 n \log \log_2 n)}$ (Putting $m = \log_2 n$)

2) $T(n) = 3T(\sqrt{n}) + \log n$

$\rightarrow$ Let $n = 2^m \Rightarrow m = \log_2 n$.

So, $T(2^m) = 3T(2^{\frac{m}{2}}) + m$

Consider $T(2^m) = S(m) \Rightarrow T(2^{\frac{m}{2}}) = S(\frac{m}{2})$.

$\rightarrow S(m) = 3S(\frac{m}{2}) + m$.

So, we can apply master method to the above recurrence;

$a = 3,\ b = 2,\ d = 1,\ p = 0$.

Here, $a > b^d$ $(3 > 2^1)$

So, $S(m) = \Theta(m^{\log_2 3})$

$\Rightarrow T(2^m) = \Theta(m^{\log_2 3})$.

$\Rightarrow \boxed{T(n) = \Theta((\log_2 n)^{\log_2 3})}$