Solving Recurrences

Recurrence relation :- An equation of the form

$$\boxed{c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + \cdots + c_k a_{r-k} = f(r)}$$

is called a recurrence relation of order k.
where, $c_0, c_1, c_2, \ldots, c_k$ are constants. It is also known as k^{th}- order linear recurrence relation.

The Solution of a given recurrence relation is obtained by applying the following steps :-

I) Form characteristic equation and find its roots

Ex:- Consider the recurrence relation

$$a_r - 4a_{r-1} + 4a_{r-2} = 3$$

So, the characteristic equation will be

$$\boxed{\alpha^2 - 4\alpha + 4 = 0}$$

$$\Rightarrow \boxed{\alpha = 2, 2}$$

II) Homogeneous solution $(a_r^{(h)})$

The general formula of homogeneous solution is

$$\left(A_1 r^{m-1} + A_2 r^{m-2} + \cdots + A_k r^{m-k}\right)(\text{root})^r$$

where; $m \rightarrow$ multiplicity of the root (how many times a root is repeated).

Ex:- $a_r - 4a_{r-1} + 4a_{r-2} = 3$ has repeated roots.

i.e. $\boxed{\alpha = 2, 2}$ $(m=2)$

So, the homogeneous solution is given by:-

$$a_r^{(h)} = (A_1 r^{2-1} + A_2 r^{2-2})(2)^r$$

$$\boxed{a_r^{(h)} = (A_1 r + A_2)(2)^r.}$$

III) Particular solution $(a_r^{(P)})$

It exists only when R.H.S. of the recurrence relation is nonzero [i.e. $f(r) \neq 0$].

If $f(r) \neq 0$, then we need to assume the particular solution based on the format of R.H.S.

Ex:- $a_r - 4a_{r-1} + 4a_{r-2} = 3$.

Here, $f(r) = 3 \neq 0$.

So, assume that the particular solution is $a_r^{(P)} = P$ ($\because f(r)$ is constant; so assumption is also const.)

Then, substitute $a_r^{(P)} = P$ in the main recurrence relation $a_r - 4a_{r-1} + 4a_{r-2} = 3$; and find out the value of P.

IV) Total solution (a_r) [Homogeneous Solution + Particular sol^n]

$$\boxed{a_r = a_r^{(h)} + a_r^{(P)}}$$

Q-1 Obtain the total solution of

$$a_r - 5a_{r-1} + 6a_{r-2} = 0.$$

Sol^n :- The above equation is the 2^nd order linear recurrence relation.

(i) Characteristic Equation

$$\alpha^2 - 5\alpha + 6 = 0.$$

$$\Rightarrow \boxed{\alpha = 2, 3}$$

(ii) Homogeneous Solution

$$a_r^{(h)} = A_1(2)^r + A_2(3)^r$$

(iii) Particular solution

As $f(r) = 0$ [R.H.S. = 0], it does not exist.

(iv) Total Solution

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$= A_1(2)^r + A_2(3)^r + 0$$

$$= A_1(2)^r + A_2(3)^r$$

Q-2 Obtain the total solution of

$$a_r + 4a_{r-1} + 4a_{r-2} = 0.$$

Sol^n :- The above equation is the 2^nd order linear recurrence relation.

(i) Characteristic Equation

$$\alpha^2 + 4\alpha + 4 = 0.$$

$$\Rightarrow \boxed{\alpha = -2, -2}$$

(ii) Homogeneous solution

$$a_r^{(h)} = (A_1 r + A_2)(-2)^r$$

(iii) Particular solution

(Does not exist, as $f(r) = 0$).

(iv) Total solution

$$a_r = (A_1 r + A_2)(-2)^r.$$

Q-3 Obtain the solution of

$$a_n = 2a_{n-1} - a_{n-2} \quad \left[\text{Initial conditions :- } a_0 = 0,\ a_1 = 3\right]$$

Sol^n :- (i) Characteristic equation

Given:- $a_n = 2a_{n-1} - a_{n-2}$

$$\Rightarrow a_n - 2a_{n-1} + a_{n-2} = 0.$$

$$\Rightarrow \alpha^2 - 2\alpha + 1 = 0.$$

$$\Rightarrow \boxed{\alpha = 1, 1}$$

(ii) Homogeneous solution

$$a_n^{(h)} = (A_1 n + A_2)(1)^n$$

$$= A_1 n + A_2$$

(iii) Particular solution

(Does not exist, as $f(r) = 0$).

(iv) Total solution

$$a_n = A_1 n + A_2$$

Now, using initial conditions $(a_0 = 0, a_1 = 3)$ we get,

$a_0 = A_1(0) + A_2 = 0$

$\Rightarrow a_0 = 0 + A_2 = 0.$

$\Rightarrow \boxed{A_2 = 0}$

$a_1 = A_1(1) + A_2 = 3.$

$\Rightarrow A_1 + 0 = 3.$

$\Rightarrow \boxed{A_1 = 3}$

So, the total solution is $a_n = A_1 n + A_2$

$= 3n + 0.$

$\therefore \boxed{a_n = 3n}$

Q-4 Solve the following recurrence relation :-

$$a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 - 2r + 1.$$

Soln :- It's a second order linear recurrence relation.

(i) Characteristic equation

$$\alpha^2 + 5\alpha + 6 = 0.$$

$$\boxed{\alpha = -2, -3}$$

(ii) Homogeneous solution

$$a_r^{(h)} = A_1(-2)^r + A_2(-3)^r.$$

(iii) Particular solution

Let $a_r^{(p)} = P_1 r^2 + P_2 r + P_3$ be the particular solution.

$$\Rightarrow P_1 r^2 + P_2 r + P_3 + 5\left[P_1(r-1)^2 + P_2(r-1) + P_3\right] + 6\left[P_1(r-2)^2 + P_2(r-2) + P_3\right] = 3r^2 - 2r + 1$$

$$\Rightarrow P_1r^2+P_2r+P_3+5\left[P_1r^2-2P_1r+P_1+P_2r-P_2+P_3\right]$$
$$+6\left[P_1r^2-4P_1r+4P_1+P_2r-2P_2+P_3\right]=3r^2-2r+1$$

$$\Rightarrow\left(P_1+5P_1+6P_1\right)r^2+\left(P_2-10P_1+5P_2-24P_1+6P_2\right)r$$
$$+\left(P_3+5P_1-5P_2+5P_3+24P_1-12P_2+6P_3\right)=3r^2-2r+1$$

Comparing terms of the same power on both sides we get,

$$P_1=\frac{1}{4},\quad P_2=\frac{13}{24}\quad \text{and}\quad P_3=\frac{71}{288}.$$

(iv) Total Solution

$$a_r=a_r^{(h)}+a_r^{(p)}$$
$$=A_1(-2)^r+A_2(-3)^r+\frac{1}{4}r^2+\frac{13}{24}r+\frac{71}{288}$$

Q-5 Solve the following :-

$$T(n)=3T(n-1)+2n.$$

Sol:- It can be written as

$$t_n=3t_{n-1}+2n \qquad [\text{Let } T(n)=t_n].$$

$$\Rightarrow t_n-3t_{n-1}=2n.$$

(i) Characteristic equation

$$\alpha-3=0.$$
$$\Rightarrow \boxed{\alpha=3}.$$

(ii) Homogeneous solution

$$t_n^{(h)} = A_1 (3)^n$$

(iii) Particular solution

Let $t_n^{(P)} = P_1 n + P_2$ be the particular solution.

$\Rightarrow P_1 n + P_2 - 3[P_1(n-1) + P_2] = 2n.$

$\Rightarrow P_1 n + P_2 - 3P_1 n + 3P_1 - 3P_2 = 2n$

$\Rightarrow -2P_1 n + 3P_1 - 2P_2 = 2n.$

$\Rightarrow -2P_1 = 2$ and $3P_1 - 2P_2 = 0.$

$\Rightarrow \boxed{P_1 = -1}$ and $-3 - 2P_2 = 0.$

$\Rightarrow \boxed{P_2 = -\frac{3}{2}}$

(iv) Total solution

$$t_n = t_n^{(h)} + t_n^{(P)}$$

$$\boxed{t_n = A_1(3)^n - n - \frac{3}{2}}$$