→ Imp to extract features

Tut - 2 (DAA)

Q2

1) $G(n) = n^3$
$f(n) = 5n^3 + n^2 + 3n + 2$

$\cancel{c\,G(n) \le f(n)}$
$0 \le \cancel{c\,f(n) \le G(n}$

$5n^3$

$0 \le f(n) \le c \cdot g(n)$

$5n^3 \le 5n^3$, $n \geq$
$n^2 \le n^3$, n
$3n \le 3n^3$
$2 \le 2n^3$

$c = 11$, $n = 1$

2) $G(n) = \cancel{3n^3 + 4n}\ n^3$
$f(n) = 3n^3 + 4n$

$0 \le f(n) \le c \cdot G(n)$

$3n^3 \le 3n^3$ $n \geq 1$
$4n \le 4n^3$

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$\therefore \quad 3n^3 + 4n \le 3n^3 + 4n^3$
$\le 7n^3$
$\therefore \quad C = 7, \quad n \ge 1$

Q3 $\quad G(n) = n^b$
$f(n) = (n \pm a)^b$

For Big - O

$0 \le f(n) \le C \cdot G(n)$

$f(n) = {}^nC_0 a^b + {}^nC_1 a^{b-1} n^1 + \dots {}^nC_n a^0 n^n$

${}^nC_0 a^b \le n^b$

$C = 2^n, \quad n \ge$

$\Rightarrow$ Note for Big - O

$f(n) = O(g(n)) \Rightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$

If R^+ is our answer then our result is proved for all O, Ω, Θ

$f(n) = \Omega(g(n)) \Longrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$

$f(n) = \Theta(g(n)) \Rightarrow 0 < \lim_{n \to \infty} \frac{f(n)}{g(n)} < \infty$

Using L- Hospital

$$\lim_{n \to \infty} \frac{(n+a)^b}{n^b} = \lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^b = 1$$

$\therefore$ Proved

Q4 $\quad f(n) = 2^{n+1}$
$G(n) = 2^n$

$f(n) = 2^{2n}$
$g(n) = 2^n$

$\lim_{n \to \infty} \frac{f(n)}{G(n)} = 2, \quad \infty$

Yes $\quad$ No

Q5 $n\log n - 2n + 13 = \Omega(n \log n)$

$f(n) = n\log n - 2n + 13$
$g(n) = n\log n$

$$\lim_{n\to\infty} \frac{n\log n - 2n + 13}{n\log n} = 1$$

$0 \leq c.G(n) \leq f(n)$

$c \cdot n\log n \leq n\log n - 2n + 13$

$\cancel{n\log n}$

Tut - 3

Q7 $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 - 2r + 1$

Sol^n i) Characteristic eq^n

$\alpha^2 + 5\alpha + 6 = 0$
$\alpha = -2, -3$

ii) Homogeneous sol^n

$a_r(n) = A_1(-2)^r + A_2(-3)^r$

iii) Particular sol^n

Let the RHS quad eq^n be $p_1 r^2 + p_2 r + p_3$ be the particular eq^n.

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$$P_1 r^2 + P_2 r + P_3 + 5\left[P_1(r-1)^2 + P_2(r-1) + P_3\right]$$

$$+ 6\left[P_1(\cancel{r}-2)^2 + P_2(r-2) + P_3\right] = 3r^2 - 2r + 1$$

$$\therefore (P_1 + 5P_1 + 6P_1) r^2 + (P_2 - 10P_1 + 5P_2 - 24P_1 + 6P_2) r$$

$$+ (P_3 + 5P_1 - 5P_2 + 5P_3 + 24P_1 - 12P_2 + 6P_3) = 3r^2 - 2r + 1$$

$$\therefore \quad 12P_1 = 3 \qquad \therefore P_1 = \frac{1}{4}$$

$$12P_2 - \frac{17}{2} = -2$$

$$12P_2 = \frac{13}{2} \qquad \therefore P_2 = \frac{13}{24}$$

$$P_3 \quad 12P_3 + 29P_1 - 17P_2 = 1$$

$$+\frac{29}{4} - \frac{17.13}{24}$$

$$29.6 -$$

$$12P_3 = 1 - \left(\frac{29.6 - 17.13}{24}\right)$$

$$= \frac{+23}{24} = \frac{71}{12.24} = \frac{71}{288}$$

Total sol^n

$$= A_1(-2)^r + A_2(-3)^r + \frac{1}{4}r^2 + \frac{13}{24}r + \frac{71}{288}$$

Q5 $T(n) = 3T(n-1) + 2n$

Let ~~T(n) =~~

It can be written as

$t_n = 3t_{n-1} + 2n$

$x - 3 = 0$

$\therefore x = 3$

$\therefore$ Homogeneous sol^n

$t_n^{(h)} = A_1 (3)^n$

Let particular sol^n be $= P_1 n + P_2$

~~$P_1 n + P_2 = 2n$~~

$1 (P_1 n + P_2) = 3 (P_1(n-1) + P_2) + 2n$

$(P_1 - 3P_1)n + (P_2 + 3P_1 - 3P_2) = 2n$

$P_1 = -1$

$-6 - 2P_2 = 0$

$P_2 = -\frac{3}{2}$

$\therefore$ Particular sol^n = $A_1(3)^n - \frac{3n}{2}$

$\Rightarrow$ TOH

$2(n-1) + 1$ = Total moves $= 2n-1$

$\therefore$ Recurrence $T_n = 2T_{n-1} + 1$

Q $a_r - 4a_{r-1} + 4a_{r-2} = (r+1)2^r$

Sol^n i) Characteristic eq^n

$\alpha^2 - 4\alpha + 4 = 0$
$(\alpha-2)^2 = 0$
$\alpha = 2, 2$

ii) Homogeneous sol^n $(a_r^{(h)})$

$a_r^{(h)} = (A_1 r + A_2)(2)^r$

iii) Particular Sol^n

Here we have to assume $(P_1 r + P_2)(2)^r r^2$

Here we have to assume r^n degree.

$\therefore (P_1 r + P_2)(2)^r r^2 - 4(P_1(r-1) + P_2)(2)^{r-1}(r-1)^2 + 4(P_1(r-2) + P_2)2^{r-2}(r-2)^2$
$= (r+1)2^r$

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$P_1 r^3 2^r + P_2 2^r r^2$

$\left[4P_1 r \cdot 2^{r-1}(r^2-2r+1) - 4P_1 2^{r-1}(r^2-2r+1) + 4P_2 2^{r-1}(r^2-2r+1)\right]$

$+ 4P_1 r \cdot 2^{r-2}(r^2-4r+4) - 8P_1 2^{r-2}(r^2-4r+4) + 4P_2 2^{r-2}(r^2-4r+4)\Big] = 2^r(r+1)$

$2^r r^3 (P_1 + \cancel{P_2} - 2P_1 + P_1)$

$2^r r^2 (P_2 + \cancel{4P_1} + \cancel{2P_1} - 2P_2 - \cancel{4P_1} - \cancel{2P_1} + P_2 = 0)$

$\boxed{-2P_1 - 4P_1 + 4P_2 + 4P_1 + 8P_1 - 4P_2} = 1$

$-10P_1 = 1$

$\therefore P_1 =$

$\therefore P_1 = \frac{1}{6}$

$+2P_1 - 2P_2 - 8P_1 + 4P_2 = 1$

$6P_2 - 10P_1 = 1$

$1 + \frac{10}{6}$ $\quad \frac{16}{36} = \frac{8}{18}$

$2P_2 - 6P_1 = 1$

$P_2 = 1$

Total sol^n

$(A_1 r + A_2)(2^r) + \left(\frac{1}{6} r + 1\right)(2^r)(r^2)$

Q $T_n = T_{n-1} + 7$

$a_n - a_{n-1} = 7$

$\alpha - 1 = 0$

$\alpha = 1$

$\therefore A_1 (1)^r$

Let ~~character~~ particular sol^n be $P_1 \cdot n + P_2$

$P_1 n = P_1(n-1) + 7$

$\therefore P_1 = 7 \longrightarrow$ This will not satisfy $T_n = T_{n-1} + 7$

$\therefore$ Our assumption is wrong.

$\therefore T_n = T_{n-1} + 7 \cdot (1)^0$

$\therefore$ Let particulars sol^n be $P_1 n$

$\therefore \quad P_1(n) = P_1(n-1) + 7$

$\therefore \quad P_1 = 7.$

$\therefore$ Particular sol^n $= A_1 + 7n$

$\Rightarrow$ Master method.

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$ (Applicable to only these forms)

where $a \geq 1$

$b > 1$

$f(n) = \theta(n^d \log^p n)$

[$d \geq 0$ & p is a real number]

$\Rightarrow$ Consider the recurrence rel^n

$$T(n) = aT\left(\frac{n}{b}\right) + \theta(n^d)$$

Case I :- If $a < b^d$ then $T(n) = \theta(n^d)$

Case II :- If $a = b^d$, then $T(n) = \theta(n^d \log n)$

Case III :- If $a > b^d$ then $T(n) = \theta\left(n^{\log_b a}\right)$

$\Rightarrow$ If polynomial term ($f(n)$) then follow recursion tree method

$$T(n) = n^d \sum_{i=0}^{h-1} \left(\frac{a}{b^d}\right)^i + a^h \cdot T(i)$$

$\left(\frac{n}{b^h}\right) = 1 \quad \therefore \quad h = \log_b n$

Case I :- $a < b^d \Longrightarrow \frac{a}{b^d} < 1$

where $r = \frac{a}{b^d} < 1$; we have dereasing geometric series & first term of series is dominant term

So $T(n) = n^d[1] + a^{\log_b n} \cdot T(1)$

$T(n) = \Theta(n^d)$

Case II :- $a = b^d$

$n^d[1 + 1^2 + 1^3 + \dots 1^{h-1}] + a^h \, T(1)$

$= n^d \cdot h + a^h T(1)$

$= n^d \log_b n + a^{\log_b n} = n^d \log_b n + n^{\log_b a} \cdot (T(1))$

$T(n) = \Theta(n^d \log n)$

Case III :- $a > b^d$

$\log a > d \log b$

$d < \log_b a$, increasing geo series

$T(n) = n^d \left[1 + \left(\frac{a}{b^d}\right) + \left(\frac{a}{b^d}\right)^2 + \dots \left(\frac{a}{b^d}\right)^{h-1}\right] + a^h \cdot T(1)$

Here $n^d \left[\left(\frac{a}{b^d}\right)^{\log_b n} - 1\right] + \underbrace{a^{\log_b n} \cdot T(1)}_{\text{dominant term}}$

$$T(n) = \Theta\left(n^{\log_b a}\right)$$

⟹ Extended Master Method

Case I :- If $a < b^d$

i) If $p < 0$ then $T(n) = \Theta(n^d)$
ii) If $p >= 0$, then $T(n) = \Theta(n^d \log^p n)$

Case II :- If $a = b^d$

i) If $p < -1$, then $T(n) = \Theta(n^{\log_b a})$
ii) If $p = -1$, then $T(n) = \Theta(n^{\log_b a} \cdot \log^2 n)$
iii) If $p > -1$, then $T(n) = \Theta(n^{\log_b a} \cdot \log^{p+1} n)$
$(p+1)$

Case III :- If $a > b^d$

then $T(n) = \Theta(n^{\log_b a})$

Ex :- $T(n) = 3\left(T\left(\frac{n}{2}\right)\right) + n^2$

Here $a = 3$, $b = 2$, $d = 2$, $p = 0$ → No logarithmic term $\therefore p = 0$

So, $a < b^d$ is true & $p = 0$

$\therefore$ Case I, ii) is applicable

$\therefore$ $T(n) = \Theta(n^2)$

Ex :- $T(n) = 2T\left(\frac{n}{2}\right) + n\log n$

$a = 2$, $b = 2$, $d = 1$, $p = 1$

$a = b^d$ & $p = 1$ $\therefore$ Case II, iii) Applicable

$\therefore \Theta(n^{\log_2^2} \cdot \log^2 n)$

$= \Theta(n \log^2 n)$

Big-O (Notation) (Proof by method)

1) $n^2+n = O(n^2)$

Solⁿ Let $f(n) = n^2+n$ & $g(n) = n^2$

We need to show that there exists $c>0$, $n_0 \geq 0$, such that $0 \leq f(n) \leq c \cdot g(n)$ [Big-O defⁿ]

Now we know that :-

When $n \geq 1$, $n^2 \leq n^2 \longrightarrow (1)$
& $n \leq n^2 \longrightarrow (2)$

So by (1) & (2), we get

$0 \leq n^2+n \leq 2n^2$
which is of the form
$0 \leq f(n) \leq c \cdot g(n)$

here $c=2$ & $n_0 = 1$ (as for $n \geq 1$, (1) & (2) are satisfied)

So for $c=2$, $n_0 = 1$, we have

$n^2+n = O(n^2)$ (Hence proved).

Big-Ω Notation

1) Prove $n^3 + 4n^2 = \Omega(n^2)$

Let $f(n) = n^3 + 4n^2$
$g(n) = n^2$

We need to show that there exists $c > 0$ & $n_0 > 0$, such that

$0 \leq c \cdot g(n) \leq f(n)$ [Big-Ω def^n]

Now when $n \geq 0$, $n^3 \leq n^3 + 4n^2$ — (1)

~~$4n^2 \leq n^3 + 4n^2$ — (2)~~

Also when $n \geq 1$, $n^2 \leq n^3$ — (2)

So by (1) & (2), we get

$0 \leq n^2 \leq n^3 \leq n^3 + 4n^2$

$\therefore$ $0 \leq 1 \cdot (n^2) \leq n^3 + 4n^2$

We can take $n_0 = 1$ (as for $n_0 = 1$, (1) & (2) are satisfied)

So for $c = 1$, $n_0 = 1$, we have
$n^3 + 4n^2 = \Omega(n^2)$ (Hence proved.

⇒ Limit Rules

Rule 1 :- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} \in R^+$ then $f(n) \in O(g(n))$ & $g(n) \in O(f(n))$
$[f(n) \in \Theta(g(n))$ & $g(n) \in \Theta(f(n))]$

Rule 2 :- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then $f(n) \in O(g(n))$ but $g(n) \notin O(f(n))$

Rule 3 :- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = +\infty$, then $g(n) \in O(f(n))$ but $f(n) \notin O(g(n))$

Range ~~Brain's~~

⇒ ~~Brain's~~ Range Transformation

→ Degree of T can be greater than 1, ~~as~~ these method will only be useful.

Ex :- $T(n) = \begin{cases} \frac{1}{3} & \text{if } n = 1 \\ n\,T^2\left(\frac{n}{2}\right) & \text{otherwise.} \end{cases}$

Soln. First step is change of variable

$$n = 2^i$$

$$\therefore \quad T(2^i) = 2^i\,T^2(2^{i-1})$$

Now consider

$$T(2^i) = \cancel{t(i)}\ t_i$$

$$\therefore \quad \cancel{t(i) = 2^i\,t^2(i_2)}$$

$$\therefore \quad t_i = 2^i\,t_{i-1}^2$$

$$\log_2 t_i = \log_2 (2^i\,t_{i-1}^2) \qquad (\because \text{Applying } \log_2 \text{ in both sides})$$

$$\therefore \quad \log_2 t_i = \log_2 2^i + \log_2 t_{i-1}^2$$

$$\therefore \quad \log_2 t_i = i + 2\log_2 t_{i-1}$$

Assume $\log_2 t_i = u(i)$

$$\therefore \quad u(i) = i + 2u(i-1)$$

$$\therefore \quad u(i) - 2u(i-1) = i$$

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$\therefore \quad u(i) - 2u(i-1) = i$

$\therefore \quad \alpha - 2 = 0 \qquad$ (By characteristic eqⁿ

$\therefore \quad \alpha = 2$

$\therefore \quad t_n = A_1(2^i)$

$\therefore$ Let particlular solⁿ be $P_1 i + P_2$

$\therefore \quad P_1 i + P_2 - 2P_1(i-1) - 2P_2 = i$

$\therefore \quad (P_1 - 2P_1) = 1$

$\cancel{P_2 - 2P_2 = 0} \quad P_2 + 2P_1 - 2P_2 = 0$

$\therefore \quad P_2 = -2$

$\therefore \quad P_1 = -1$

$P_2 = -2$

$\therefore$ Final solⁿ $= A_1 \cdot 2^i + P_1 i + P_2$

$u(i) = A_1 \cdot 2^i - i - 2$

$\log_2 t_i = A_1 n - \log n - 2$

$\therefore \quad \cancel{t_i = 2}$

$t_i = 2^{A_1 2^i - i - 2}$

$\therefore \quad \cancel{T(2^i) = 2^{A_i}}$

$\therefore \quad \cancel{T(n) = A_1 n}$

$T(2^i) = 2^{A_1 \cdot 2^i - i - 2}$

$\therefore \quad T(n) = 2^{A_1 \cdot n - \log n - 2}$

$n \times \left(n + \frac{n}{\quad}\right)$

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Also $T(1) = \frac{1}{3}$

$2^{A_1 - 2} = \frac{1}{3}$

$A_1 = -\log_2 3 + 2$

$\therefore \quad T(n) = 2^{(2 - \log_2 3)n - \log n - 2}$

$= \frac{2^{2n}}{2^{n\log_2 3} \cdot 2^{\log n} \cdot 2^2}$

$= \cancel{\frac{2^{2n}}{3 \cdot 4 \cdot n \cdot 2^n}} \quad \frac{2^{2n}}{3^n \cdot n \cdot 4}$

$= \cancel{\frac{2^n}{2^{2n}}}$

⇒ Tut - 4

13) N logn complexity coz pivot element is not first or last.

⇒ Heapsort

→ An algorithm which requires constant number of elements to be ~~shown~~ stored at any time is called heap sort.

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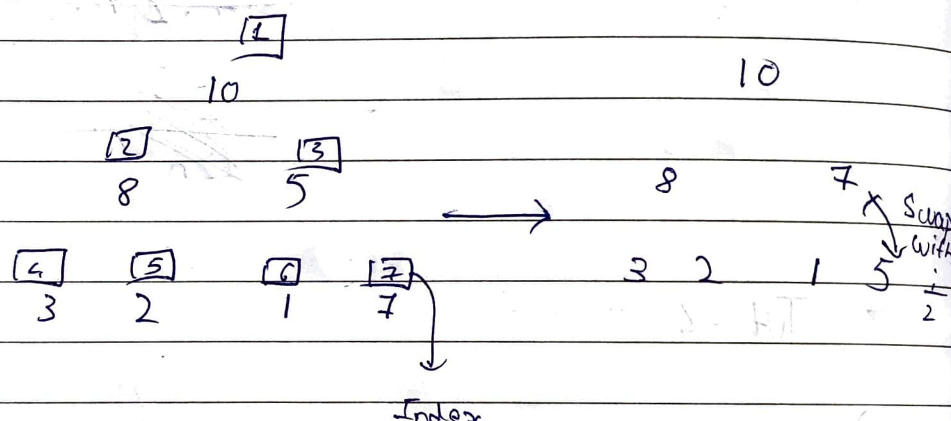
left - child = $2i$
Right - child = $2i + 1$

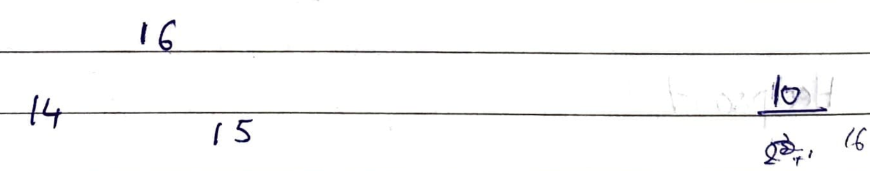
Min^m no. of nodes in a heap of level $h = 2^h$
Max^m " " " " $= 2^{h+1} - 1$

Q What is the tightest upperbound of Build Max-Heap Algorithm?

Ex 6 ~~32~~ ~~15~~ 10 8 5 3 2

2 Elements inserted 1 & 7.





$\Rightarrow$ $\left\lceil \dfrac{n}{2^{h+1}} \right\rceil$ $\longrightarrow$ At max how many nodes present at height h.