Master Theorem

Master Theorem / Master method is a popular method to solve the recurrence relations of the form :-

$$\boxed{T(n) = a\,T\left(\frac{n}{b}\right) + f(n)}$$

where; $a \geq 1$

$b > 1$

$f(n) = \Theta(n^d \log^p n)$

[$d \geq 0$ and p is a real number]

* Consider the recurrence relation :-

$$T(n) = a\,T\left(\frac{n}{b}\right) + \Theta(n^d)$$

⇒ The solution of the above recurrence relation is given by :-

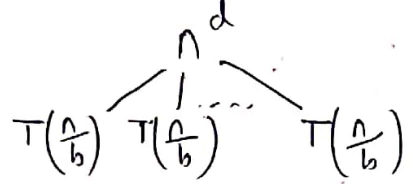
Case(I) :- if $a < b^d$; then $T(n) = \Theta(n^d)$

Case(II) :- if $a = b^d$; then $T(n) = \Theta(n^d \log n)$

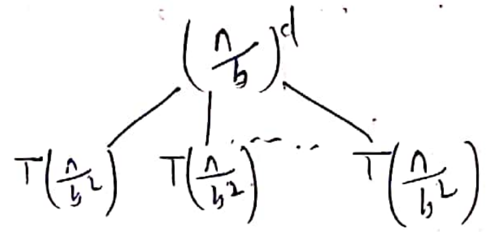
Case(III) :- if $a > b^d$; then $T(n) = \Theta(n^{\log_b a})$.

We'll prove the above results by using the "Recursion Tree" method.

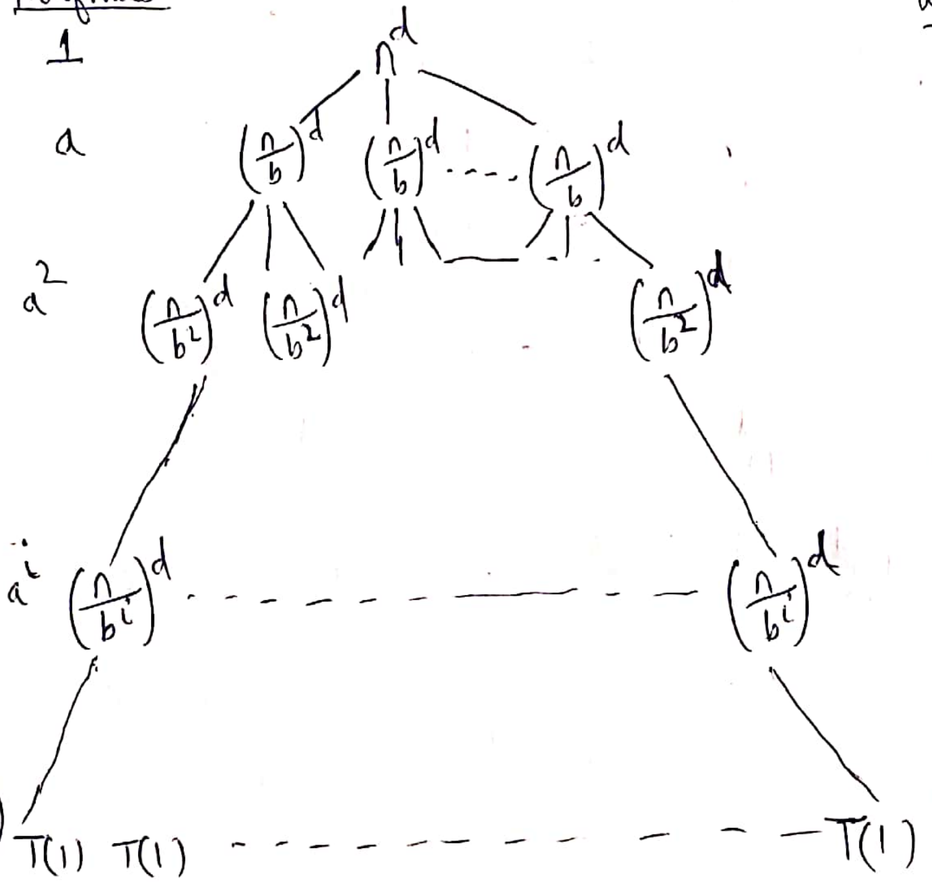
$$T(n) = aT\left(\frac{n}{b}\right) + n^d$$



$$T\left(\frac{n}{b}\right) = aT\left(\frac{n}{b^2}\right) + \left(\frac{n}{b}\right)^d$$



Full Recursion Tree



Level	No. of Nodes	Work Done
0	1	n^d
1	a	$a\left(\frac{n}{b}\right)^d = n^d\left(\frac{a}{b^d}\right)$
2	a^2	$a^2\left(\frac{n}{b^2}\right)^d = n^d\left(\frac{a}{b^d}\right)^2$
i	a^i	$a^i\left(\frac{n}{b^i}\right)^d = n^d\left(\frac{a}{b^d}\right)^i$
h	a^h	$a^h \cdot T(1)$

Total work done

$$T(n) = \sum_{i=0}^{h-1} a^i\left(\frac{n}{b^i}\right)^d + a^h \cdot T(1)$$

$$= \sum_{i=0}^{h-1} n^d\left(\frac{a}{b^d}\right)^i + a^h \cdot T(1)$$

$$T(n) = n^d \sum_{i=0}^{h-1} \left(\frac{a}{b^d}\right)^i + a^h \cdot T(1)$$

$$= n^d \left[1 + \frac{a}{b^d} + \left(\frac{a}{b^d}\right)^2 + \cdots + \left(\frac{a}{b^d}\right)^{h-1}\right] + a^h T(1)$$

We can analyse from the recursion tree that,

$$\boxed{\frac{n}{b^h} = 1} \Rightarrow n = b^h$$

$$\Rightarrow \boxed{h = \log_b n}$$

Case(I) :- $a < b^d \Rightarrow \frac{a}{b^d} < 1$

when $r = \frac{a}{b^d} < 1$; we have decreasing geometric series and first term of the series is the dominant term.

[For Ex:- 64, 32, 16, 8, 4, 2, 1]

So, $T(n) = n^d [1] + a^{\log_b n} \cdot T(1)$

$$= n^d + n^{\log_b a} \cdot T(1)$$

$$\boxed{T(n) = \Theta(n^d)}$$

Case(II) :- $a = b^d$

$$T(n) = n^d \left[1 + 1 + 1^2 + \cdots + 1^{h-1}\right] + a^h T(1)$$

$$= n^d [1 + 1 + \cdots + 1] + a^h T(1)$$

$$= n^d \cdot h + a^h T(1)$$

$$= n^d \log_b n + a^{\log_b n} \cdot T(1) = n^d \log_b n + n^{\log_b a} \cdot T(1)$$

So, $\boxed{T(n) = \Theta(n^d \log n)}$

Case(III) :- $a > b^d \Rightarrow \frac{a}{b^d} > 1 \Rightarrow \boxed{d < \log_b a}$ ~~20/10/2020 14:02:16~~

when $r = \frac{a}{b^d} > 1$; we have increasing geometric series and last term of the series is the dominant term.

[For ex:- 1, 10, 100, 1000, 10000]

$$T(n) = n^d\left(1 + \left(\frac{a}{b^d}\right) + \left(\frac{a}{b^d}\right)^2 + \cdots + \left(\frac{a}{b^d}\right)^{h-1}\right) + a^h T(1)$$

$$= n^d\left[\left(\frac{a}{b^d}\right)^{h-1}\right] + a^h T(1)$$

$$= n^d\left[\left(\frac{a}{b^d}\right)^{\log_b n - 1}\right] + \underbrace{a^{\log_b n} \cdot T(1)}_{\text{(dominant term)}}$$

because

$$= n^d\left[\left(\frac{a}{b^d}\right)^{\log_b n - 1}\right] + n^{\log_b a} \cdot T(1)$$

$$= \Theta\left(n^{\log_b a} \cdot T(1)\right) \quad [\text{As, } d < \log_b a]$$

$$\boxed{T(n) = \Theta\left(n^{\log_b a}\right)}$$

Master Theorem

→ Master Theorem or Master Method is a popular method for solving the recurrence relations of the form :-

$$\boxed{T(n) = aT\left(\frac{n}{b}\right) + f(n)}$$

where, $a \geq 1$

$b > 1$

$f(n) = \Theta(n^d \log^p n)$

[$d >= 0$ and p is a real number]

→ To solve recurrence relations using Master's theorem, we compare a with b^d. [Extended Master Theorem]

Case (I) :- If $a < b^d$

(i) if $p < 0$, then $T(n) = \Theta(n^d)$

(ii) if $p \geq 0$, then $T(n) = \Theta(n^d \log^p n)$

Case (II) :- If $a = b^d$

(i) if $p < -1$, then $T(n) = \Theta(n^{\log_b a})$

(ii) if $p = -1$, then $T(n) = \Theta(n^{\log_b a} \log^2 n)$

(iii) if $p > -1$, then $T(n) = \Theta(n^{\log_b a} \log^{p+1} n)$

Case (III) :- If $a > b^d$

then $T(n) = \Theta(n^{\log_b a})$

Example 1 :- $T(n) = 3T(n/2) + n^2$

Here; $a = 3$, $b = 2$, $d = 2$, $p = 0$

So, $a < b^d$ is true $(\because 3 < 2^2)$.

and $p = 0$

So, case (I) (ii) is applicable.

Hence, $T(n) = \Theta(n^d \log^p n)$

$= \Theta(n^2 \log^0 n)$

$\boxed{T(n) = \Theta(n^2)}$

Example 2 :- $T(n) = 2T(n/2) + n\log n$.

Here, $a = 2$, $b = 2$, $d = 1$, $p = 1$.

So, $a = b^d$ is true $(\because 2 = 2^1)$.

and $p = 1$

So, case (II) (iii) is applicable.

Hence, $T(n) = \Theta(n^{\log_b a} \log^{p+1} n)$

$= \Theta(n^{\log_2 2} \log^{1+1} n) = \Theta(n \log^2 n)$