

\Rightarrow Identify correctness of Insertion Sort.

\Rightarrow Recurrence

$$c_0 a_x + c_1 a_{x-1} + c_2 a_{x-2} + \dots + c_k a_{x-k} = f(x)$$

is called a recurrence relation of order k . It is also called as k^{th} -order linear recurrence relation.

Q1 Obtain the ^{total} solⁿ of $a_x - 5a_{x-1} + 6a_{x-2} = 0$

Solⁿ The above eqⁿ is the 2nd order linear recurrence relat

i) Characteristic Eqⁿ :-

$$\alpha^2 - 5\alpha + 6 = 0$$

$$\therefore \alpha = 2, 3$$

ii) Homogeneous solⁿ $= (A_1)(2)^x + (A_2)(3)^x$

If equal sol's then $(A_1 x + A_2)(\text{root})^x$

Note If roots are 2, 2, 3 then Homogeneous eqⁿ will be

$$(A_1 x + A_2)(2)^x + (A_3)(3)^x$$

iii) No Particular solⁿ

iv) Total solⁿ

$$a_x = A_1(2)^x + A_2(3)^x$$

Q3 Obtain the solⁿ of

$$a_n = 2a_{n-1} - a_{n-2}$$

[Initial condⁿ] $a_0 = 0, a_1 = 3$

Solⁿ i) Characteristic Eqⁿ

$$\text{Given : } a_n = 2a_{n-1} - a_{n-2}$$

$$a_n - 2a_{n-1} + a_{n-2} = 0$$

$$\therefore \alpha^2 - 2\alpha + 1 = 0$$

$$\alpha = 1, 1$$

Total Solⁿ $\boxed{a_n = A_1 n + A_2}$

$$\text{Also } a_0 = 0, a_1 = 3$$

$$0 = A_1 + A_2 - (i) \quad 3 = 1A_1 + A_2 - (ii)$$

$$\therefore A_1 = 3, A_2 = 0$$

$$\begin{matrix} 2 & 3 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{matrix}$$



→ Some of the recurrence solⁿ's are of form :-

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$a \rightarrow$ No. of subproblems

$\frac{n}{b} \rightarrow$ size of each subproblem

$T(n) \rightarrow$ Time taken to solve each sub-problem

$f(n) \rightarrow$ Funⁿ which denotes the time taken to combine result of sub problems.

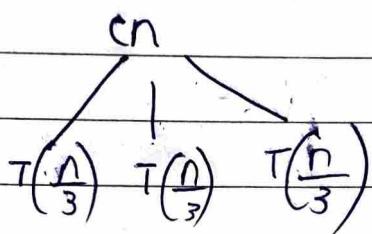
\Rightarrow 3 methods are used

- 1) Recursion Tree method
- 2) Master Method
- 3)

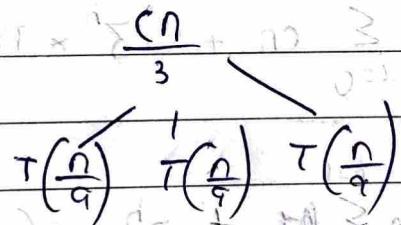
1) Recursion Tree Method

Ex:- $T(n) = 3T\left(\frac{n}{3}\right) + cn$

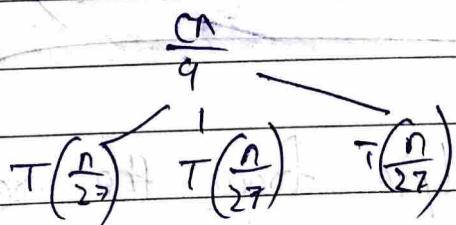
Soln



Now, $T\left(\frac{n}{3}\right) = 3T\left(\frac{n}{9}\right) + \frac{cn}{3}$



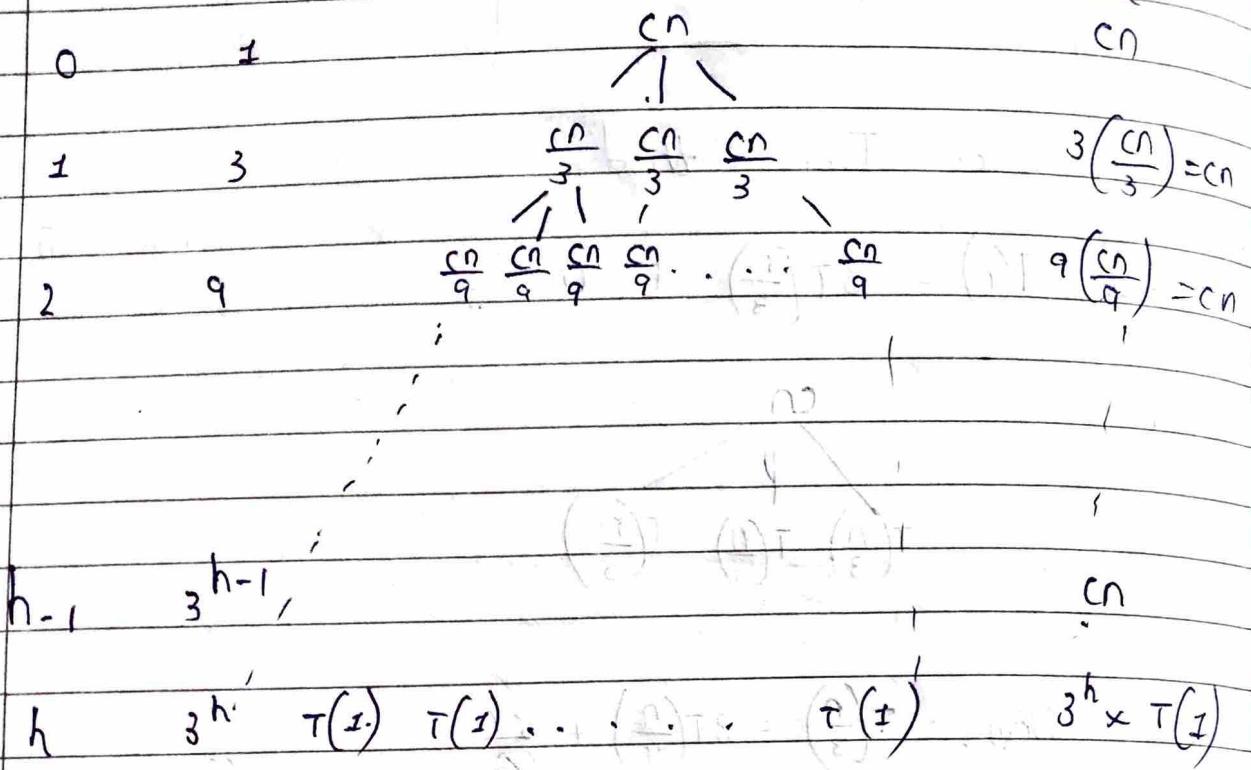
$T\left(\frac{n}{9}\right) = 3\left(T\left(\frac{n}{27}\right)\right) + \frac{cn}{9}$



Full Recursion Tree

Level No. of nodes

Work done by single node



$$\text{Total work done} = \sum_{i=0}^{h-1} cn + 3^h \times T(1) \quad \rightarrow (1)$$

$$= cn \sum_{i=0}^{h-1} 1 + 3^h \times T(1)$$

$$T(n) = cnh + 3^h \times T(1)$$

$$\rightarrow (2)$$

We can analyse from the recursion tree that

$$\frac{n}{3^h} = 1 \implies n = 3^h$$

$$\therefore h = \log_3 n \quad \rightarrow (3)$$

Putting eqⁿ 3 in eqⁿ 2 we get,

$$\frac{a}{3} = \frac{1}{3}$$

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$$T(n) = cn \log_3 n + n \times T(1)$$

$\therefore T(n) = \Theta(n \log_3 n)$ (As $n \times T(1)$ is non-dominant)

$$2) T(n) = 3(T(\frac{n}{3})) + n^2$$

$$\frac{n^2}{T(\frac{n}{3})} - T(\frac{n}{3})$$

Level	No. of nodes	Tree	Work done
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$$0 \quad 1 \quad n^2 \quad n^2$$

$$1 \quad 3 \quad (\frac{n}{3})^2 (\frac{n}{3})^2 (\frac{n}{3})^2 \quad 3(\frac{n^2}{9}) = \frac{n^2}{3}$$

$$2 \quad 9 \quad (\frac{n}{9})^2 \dots (\frac{n}{9})^2 \quad 9 \cdot \frac{n^2}{9}$$

$$h-1 \quad 3^{h-1} \quad \frac{n^2}{3^{h-1}}$$

$$h \quad 3^h \quad T(1) \dots \quad T(1) \quad 3^h \times T(1)$$

Total work done

$$T(n) = \sum_{i=0}^{h-1} \frac{n^2}{3^i} + 3^h \times T(1)$$

~~$$\frac{3n^2}{2} + 3^h \times T(1)$$~~

$$= n^2 \left[\frac{1}{1 - \frac{1}{3}} \left(1 - \frac{1}{3} \right)^h \right] + 3^h \times T(z)$$

$$= \frac{3n^2}{2} \left[\frac{3^h - 1}{3^h} \right] + 3^h \times T(z)$$

Also $h = \log_3 n$

$$\therefore T(n) = \frac{3}{2} n^2 \left(\frac{n-1}{n} \right) + n(T(\frac{n}{3}))$$

$$\frac{3}{2} n^2 - \frac{3}{2} n + n(T(\frac{n}{3}))$$

Q2 ~~See~~ $T(n) = 24(T(\frac{n}{2})) + n^2$

See photo

Q2 T

Q2 1) $T(n) = 3T(\frac{n}{4}) + n^2$

No.

Level	No. of nodes	Tree	Work done
0	1	n^2	n^2
1	3	$(\frac{n}{4})^2 (\frac{n}{4})^2 (\frac{n}{4})^2$	$\frac{3n^2}{16}$
2	9	$(\frac{n}{16})^2 \cdot \cdot \cdot (\frac{n}{16})^2$	$\frac{9n^2}{256}$
:			
$h-1$	$3^{h-1} \left(\frac{n}{16^{h-1}} \right)^2 \cdot \cdot \cdot \left(\frac{n}{16^{h-1}} \right)^2$	$\frac{3^{h-1} n^2}{16^{h-1}}$	
h	$3^h T(z)$	$3^h T(z)$	

$$3 < 86 \quad a=3, b=4, d=2 \quad \frac{\Theta(n^2)}{4}$$

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$$\therefore T(n) = \text{Also} \sum_{i=0}^{h-1} \frac{3^i}{16^i} n^2 + 3^h \cdot T(1)$$

$$\text{Also } \frac{n^2}{16^h} = 1$$

$$\therefore 2 \log n = 2h \log 4$$

$$\therefore h = \log n$$

$$\log_a^m b^n$$

$$n^2 \left[1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \dots \right] \quad \frac{n \cdot \log b}{m}$$

$$= n^2 \left[\frac{1 - \left(\frac{3}{16}\right)^h}{1 - \frac{3}{16}} \right] + 3^h \cdot T(1)$$

$$= n^2 \left[\frac{16^h - 3^h}{16^{h-1} \cdot 13} \right] + 3^h \cdot T(1)$$

$$= 16n^2 \left[\frac{\frac{1}{16} \log_{16} n + 3^{\log_4 n}}{13 \cdot \frac{1}{16} \log_{16} n} \right] + 3^{\log_4 n} \cdot T(1)$$

$$= 16n^2 \left[\frac{n^{\frac{1}{2}} - 3^{\log_4 n}}{13 \cdot n^{\frac{1}{2}}} \right] + 3^{\log_4 n} \cdot T(1)$$

$\therefore \Theta(n^2)$ is the answer by approximation.

$$Q \quad T(n) = 4T\left(\frac{n}{3}\right) + n$$

Level	No. of nodes	Tree	Work done
0	1	$\frac{n}{3} \cdot \frac{n}{3} \cdot \frac{n}{3}$	n
1	4	$\frac{4n}{3}$	$\frac{4n}{3}$
2	16	$\frac{n}{9} \cdot \frac{n}{9} \cdot \frac{n}{9}$	$\frac{4^2 n}{3^2}$
:			
$h-1$	4^{h-1}	$\frac{n}{3^{h-1}}$	$\frac{4^{h-1}}{3^{h-1}} n$

$$h \cdot 4^{h-1} \cdot T(1) + T(1) - T(1) = 4^h \cdot T(1)$$

$$\frac{n}{3^h} = 1$$

$$\therefore \boxed{\log_3 n = h}$$

$$T(n) = \sum_{i=0}^{h-1} \frac{4^i n}{3^i} + T(1) \cdot 4^h$$

$$= \left[1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \dots + \left(\frac{4}{3}\right)^{h-1} \right] n$$

$$= \frac{\left(\frac{4}{3}\right)^h - 1}{\frac{1}{3}} \cdot n + T(1) \cdot 4^{\log_3 n}$$

$$= 3 \left(\frac{4^h - 3^h}{3^h} \right) n$$

$$= 3 \left(\frac{4^{\log_3 n} - n}{n} \right) n$$

$$\Theta(4^{\log_3 n}) = \Theta(n^{\log_3 4})$$

$$Q \quad T(n) = 4T\left(\frac{n}{2}\right) + n$$

Level	No. of node	Tree	Work done
0	1	n	n
1	4	$\frac{n}{2} + \frac{n}{2}$	$2n$
2	16	$\frac{n}{4} + \frac{n}{4} + \frac{n}{4} + \frac{n}{4}$	$4n$
3	4^3	$\frac{n}{8} + \frac{n}{8} + \frac{n}{8} + \frac{n}{8}$	$8n$
4	4^4	$\frac{n}{16} + \frac{n}{16} + \frac{n}{16} + \frac{n}{16}$	$16n$
5	4^5	$\frac{n}{32} + \frac{n}{32} + \frac{n}{32} + \frac{n}{32}$	$32n$
6	4^6	$\frac{n}{64} + \frac{n}{64} + \frac{n}{64} + \frac{n}{64}$	$64n$
7	4^7	$\frac{n}{128} + \frac{n}{128} + \frac{n}{128} + \frac{n}{128}$	$128n$
8	4^8	$\frac{n}{256} + \frac{n}{256} + \frac{n}{256} + \frac{n}{256}$	$256n$
9	4^9	$\frac{n}{512} + \frac{n}{512} + \frac{n}{512} + \frac{n}{512}$	$512n$
10	4^{10}	$\frac{n}{1024} + \frac{n}{1024} + \frac{n}{1024} + \frac{n}{1024}$	$1024n$
11	4^{11}	$\frac{n}{2048} + \frac{n}{2048} + \frac{n}{2048} + \frac{n}{2048}$	$2048n$
12	4^{12}	$\frac{n}{4096} + \frac{n}{4096} + \frac{n}{4096} + \frac{n}{4096}$	$4096n$
13	4^{13}	$\frac{n}{8192} + \frac{n}{8192} + \frac{n}{8192} + \frac{n}{8192}$	$8192n$
14	4^{14}	$\frac{n}{16384} + \frac{n}{16384} + \frac{n}{16384} + \frac{n}{16384}$	$16384n$
15	4^{15}	$\frac{n}{32768} + \frac{n}{32768} + \frac{n}{32768} + \frac{n}{32768}$	$32768n$
16	4^{16}	$\frac{n}{65536} + \frac{n}{65536} + \frac{n}{65536} + \frac{n}{65536}$	$65536n$
17	4^{17}	$\frac{n}{131072} + \frac{n}{131072} + \frac{n}{131072} + \frac{n}{131072}$	$131072n$
18	4^{18}	$\frac{n}{262144} + \frac{n}{262144} + \frac{n}{262144} + \frac{n}{262144}$	$262144n$
19	4^{19}	$\frac{n}{524288} + \frac{n}{524288} + \frac{n}{524288} + \frac{n}{524288}$	$524288n$
20	4^{20}	$\frac{n}{1048576} + \frac{n}{1048576} + \frac{n}{1048576} + \frac{n}{1048576}$	$1048576n$
21	4^{21}	$\frac{n}{2097152} + \frac{n}{2097152} + \frac{n}{2097152} + \frac{n}{2097152}$	$2097152n$
22	4^{22}	$\frac{n}{4194304} + \frac{n}{4194304} + \frac{n}{4194304} + \frac{n}{4194304}$	$4194304n$
23	4^{23}	$\frac{n}{8388608} + \frac{n}{8388608} + \frac{n}{8388608} + \frac{n}{8388608}$	$8388608n$
24	4^{24}	$\frac{n}{16777216} + \frac{n}{16777216} + \frac{n}{16777216} + \frac{n}{16777216}$	$16777216n$
25	4^{25}	$\frac{n}{33554432} + \frac{n}{33554432} + \frac{n}{33554432} + \frac{n}{33554432}$	$33554432n$
26	4^{26}	$\frac{n}{67108864} + \frac{n}{67108864} + \frac{n}{67108864} + \frac{n}{67108864}$	$67108864n$
27	4^{27}	$\frac{n}{134217728} + \frac{n}{134217728} + \frac{n}{134217728} + \frac{n}{134217728}$	$134217728n$
28	4^{28}	$\frac{n}{268435456} + \frac{n}{268435456} + \frac{n}{268435456} + \frac{n}{268435456}$	$268435456n$
29	4^{29}	$\frac{n}{536870912} + \frac{n}{536870912} + \frac{n}{536870912} + \frac{n}{536870912}$	$536870912n$
30	4^{30}	$\frac{n}{1073741824} + \frac{n}{1073741824} + \frac{n}{1073741824} + \frac{n}{1073741824}$	$1073741824n$
31	4^{31}	$\frac{n}{2147483648} + \frac{n}{2147483648} + \frac{n}{2147483648} + \frac{n}{2147483648}$	$2147483648n$
32	4^{32}	$\frac{n}{4294967296} + \frac{n}{4294967296} + \frac{n}{4294967296} + \frac{n}{4294967296}$	$4294967296n$
33	4^{33}	$\frac{n}{8589934592} + \frac{n}{8589934592} + \frac{n}{8589934592} + \frac{n}{8589934592}$	$8589934592n$
34	4^{34}	$\frac{n}{17179869184} + \frac{n}{17179869184} + \frac{n}{17179869184} + \frac{n}{17179869184}$	$17179869184n$
35	4^{35}	$\frac{n}{34359738368} + \frac{n}{34359738368} + \frac{n}{34359738368} + \frac{n}{34359738368}$	$34359738368n$
36	4^{36}	$\frac{n}{68719476736} + \frac{n}{68719476736} + \frac{n}{68719476736} + \frac{n}{68719476736}$	$68719476736n$
37	4^{37}	$\frac{n}{137438953472} + \frac{n}{137438953472} + \frac{n}{137438953472} + \frac{n}{137438953472}$	$137438953472n$
38	4^{38}	$\frac{n}{274877906944} + \frac{n}{274877906944} + \frac{n}{274877906944} + \frac{n}{274877906944}$	$274877906944n$
39	4^{39}	$\frac{n}{549755813888} + \frac{n}{549755813888} + \frac{n}{549755813888} + \frac{n}{549755813888}$	$549755813888n$
40	4^{40}	$\frac{n}{1099511627776} + \frac{n}{1099511627776} + \frac{n}{1099511627776} + \frac{n}{1099511627776}$	$1099511627776n$
41	4^{41}	$\frac{n}{2199023255552} + \frac{n}{2199023255552} + \frac{n}{2199023255552} + \frac{n}{2199023255552}$	$2199023255552n$
42	4^{42}	$\frac{n}{4398046511104} + \frac{n}{4398046511104} + \frac{n}{4398046511104} + \frac{n}{4398046511104}$	$4398046511104n$
43	4^{43}	$\frac{n}{8796093022208} + \frac{n}{8796093022208} + \frac{n}{8796093022208} + \frac{n}{8796093022208}$	$8796093022208n$
44	4^{44}	$\frac{n}{17592186044416} + \frac{n}{17592186044416} + \frac{n}{17592186044416} + \frac{n}{17592186044416}$	$17592186044416n$
45	4^{45}	$\frac{n}{35184372088832} + \frac{n}{35184372088832} + \frac{n}{35184372088832} + \frac{n}{35184372088832}$	$35184372088832n$
46	4^{46}	$\frac{n}{70368744177664} + \frac{n}{70368744177664} + \frac{n}{70368744177664} + \frac{n}{70368744177664}$	$70368744177664n$
47	4^{47}	$\frac{n}{140737488355328} + \frac{n}{140737488355328} + \frac{n}{140737488355328} + \frac{n}{140737488355328}$	$140737488355328n$
48	4^{48}	$\frac{n}{281474976710656} + \frac{n}{281474976710656} + \frac{n}{281474976710656} + \frac{n}{281474976710656}$	$281474976710656n$
49	4^{49}	$\frac{n}{562949953421312} + \frac{n}{562949953421312} + \frac{n}{562949953421312} + \frac{n}{562949953421312}$	$562949953421312n$
50	4^{50}	$\frac{n}{1125899906842624} + \frac{n}{1125899906842624} + \frac{n}{1125899906842624} + \frac{n}{1125899906842624}$	$1125899906842624n$
51	4^{51}	$\frac{n}{2251799813685248} + \frac{n}{2251799813685248} + \frac{n}{2251799813685248} + \frac{n}{2251799813685248}$	$2251799813685248n$
52	4^{52}	$\frac{n}{4503599627370496} + \frac{n}{4503599627370496} + \frac{n}{4503599627370496} + \frac{n}{4503599627370496}$	$4503599627370496n$
53	4^{53}	$\frac{n}{9007199254740992} + \frac{n}{9007199254740992} + \frac{n}{9007199254740992} + \frac{n}{9007199254740992}$	$9007199254740992n$
54	4^{54}	$\frac{n}{18014398509481984} + \frac{n}{18014398509481984} + \frac{n}{18014398509481984} + \frac{n}{18014398509481984}$	$18014398509481984n$
55	4^{55}	$\frac{n}{36028797018963968} + \frac{n}{36028797018963968} + \frac{n}{36028797018963968} + \frac{n}{36028797018963968}$	$36028797018963968n$
56	4^{56}	$\frac{n}{72057594037927936} + \frac{n}{72057594037927936} + \frac{n}{72057594037927936} + \frac{n}{72057594037927936}$	$72057594037927936n$
57	4^{57}	$\frac{n}{144115188075855872} + \frac{n}{144115188075855872} + \frac{n}{144115188075855872} + \frac{n}{144115188075855872}$	$144115188075855872n$
58	4^{58}	$\frac{n}{288230376151711744} + \frac{n}{288230376151711744} + \frac{n}{288230376151711744} + \frac{n}{288230376151711744}$	$288230376151711744n$
59	4^{59}	$\frac{n}{576460752303423488} + \frac{n}{576460752303423488} + \frac{n}{576460752303423488} + \frac{n}{576460752303423488}$	$576460752303423488n$
60	4^{60}	$\frac{n}{1152921504606846976} + \frac{n}{1152921504606846976} + \frac{n}{1152921504606846976} + \frac{n}{1152921504606846976}$	$1152921504606846976n$
61	4^{61}	$\frac{n}{2305843009213693952} + \frac{n}{2305843009213693952} + \frac{n}{2305843009213693952} + \frac{n}{2305843009213693952}$	$2305843009213693952n$
62	4^{62}	$\frac{n}{4611686018427387904} + \frac{n}{4611686018427387904} + \frac{n}{4611686018427387904} + \frac{n}{4611686018427387904}$	$4611686018427387904n$
63	4^{63}	$\frac{n}{9223372036854775808} + \frac{n}{9223372036854775808} + \frac{n}{9223372036854775808} + \frac{n}{9223372036854775808}$	$9223372036854775808n$
64	4^{64}	$\frac{n}{18446744073709551616} + \frac{n}{18446744073709551616} + \frac{n}{18446744073709551616} + \frac{n}{18446744073709551616}$	$18446744073709551616n$
65	4^{65}	$\frac{n}{36893488147419103232} + \frac{n}{36893488147419103232} + \frac{n}{36893488147419103232} + \frac{n}{36893488147419103232}$	$36893488147419103232n$
66	4^{66}	$\frac{n}{73786976294838206464} + \frac{n}{73786976294838206464} + \frac{n}{73786976294838206464} + \frac{n}{73786976294838206464}$	$73786976294838206464n$
67	4^{67}	$\frac{n}{147573952589676412928} + \frac{n}{147573952589676412928} + \frac{n}{147573952589676412928} + \frac{n}{147573952589676412928}$	$147573952589676412928n$
68	4^{68}	$\frac{n}{295147905179352825856} + \frac{n}{295147905179352825856} + \frac{n}{295147905179352825856} + \frac{n}{295147905179352825856}$	$295147905179352825856n$
69	4^{69}	$\frac{n}{590295810358705651712} + \frac{n}{590295810358705651712} + \frac{n}{590295810358705651712} + \frac{n}{590295810358705651712}$	$590295810358705651712n$
70	4^{70}	$\frac{n}{1180591620717411303424} + \frac{n}{1180591620717411303424} + \frac{n}{1180591620717411303424} + \frac{n}{1180591620717411303424}$	$1180591620717411303424n$
71	4^{71}	$\frac{n}{2361183241434822606848} + \frac{n}{2361183241434822606848} + \frac{n}{2361183241434822606848} + \frac{n}{2361183241434822606848}$	$2361183241434822606848n$
72	4^{72}	$\frac{n}{4722366482869645213696} + \frac{n}{4722366482869645213696} + \frac{n}{4722366482869645213696} + \frac{n}{4722366482869645213696}$	$4722366482869645213696n$
73	4^{73}	$\frac{n}{9444732965739290427392} + \frac{n}{9444732965739290427392} + \frac{n}{9444732965739290427392} + \frac{n}{9444732965739290427392}$	$9444732965739290427392n$
74	4^{74}	$\frac{n}{18889465931478580854784} + \frac{n}{18889465931478580854784} + \frac{n}{18889465931478580854784} + \frac{n}{18889465931478580854784}$	$18889465931478580854784n$
75	4^{75}	$\frac{n}{37778931862957161709568} + \frac{n}{37778931862957161709568} + \frac{n}{37778931862957161709568} + \frac{n}{37778931862957161709568}$	$37778931862957161709568n$
76	4^{76}	$\frac{n}{75557863725914323419136} + \frac{n}{75557863725914323419136} + \frac{n}{75557863725914323419136} + \frac{n}{75557863725914323419136}$	$75557863725914323419136n$
77	4^{77}	$\frac{n}{151115727451828646838272} + \frac{n}{151115727451828646838272} + \frac{n}{151115727451828646838272} + \frac{n}{151115727451828646838272}$	$151115727451828646838272n$
78	4^{78}	$\frac{n}{302231454903657293676544} + \frac{n}{302231454903657293676544} + \frac{n}{302231454903657293676544} + \frac{n}{302231454903657293676544}$	$302231454903657293676544n$
79	4^{79}	$\frac{n}{604462909807314587353088} + \frac{n}{604462909807314587353088} + \frac{n}{604462909807314587353088} + \frac{n}{604462909807314587353088}$	$604462909807314587353088n$
80	4^{80}	$\frac{n}{1208925819614629174706176} + \frac{n}{1208925819614629174706176} + \frac{n}{1208925819614629174706176} + \frac{n}{1208925819614629174706176}$	$1208925819614629174706176n$
81	4^{81}	$\frac{n}{2417851639229258349412352} + \frac{n}{2417851639229258349412352} + \frac{n}{2417851639229258349412352} + \frac{n}{2417851639229258349412352}$	$2417851639229258349412352n$
82	4^{82}	$\frac{n}{4835703278458516698824704} + \frac{n}{4835703278458516698824704} + \frac{n}{4835703278458516698824704} + \frac{n}{4835703278458516698824704}$	$4835703278458516698824704n$
83	4^{83}	$\frac{n}{9671406556917033397649408} + \frac{n}{9671406556917033397649408} + \frac{n}{9671406556917033397649408} + \frac{n}{9671406556917033397649408}$	$9671406556917033397649408n$
84	4^{84}	$\frac{n}{19342813113834066795298816} + \frac{n}{19342813113834066795298816} + \frac{n}{19342813113834066795298816} + \frac{n}{19342813113834066795298816}$	$19342813113834066795298816n$
85	4^{85}	$\frac{n}{38685626227668133590597632} + \frac{n}{38685626227668133590597632} + \frac{n}{38685626227668133590597632} + \frac{n}{38685626227668133590597632}$	$38685626227668133590597632n$
86	4^{86}	$\frac{n}{77371252455336267181195264} + \frac{n}{77371252455336267181195264} + \frac{n}{77371252455336267181195264} + \frac{n}{77371252455336267181195264}</$	

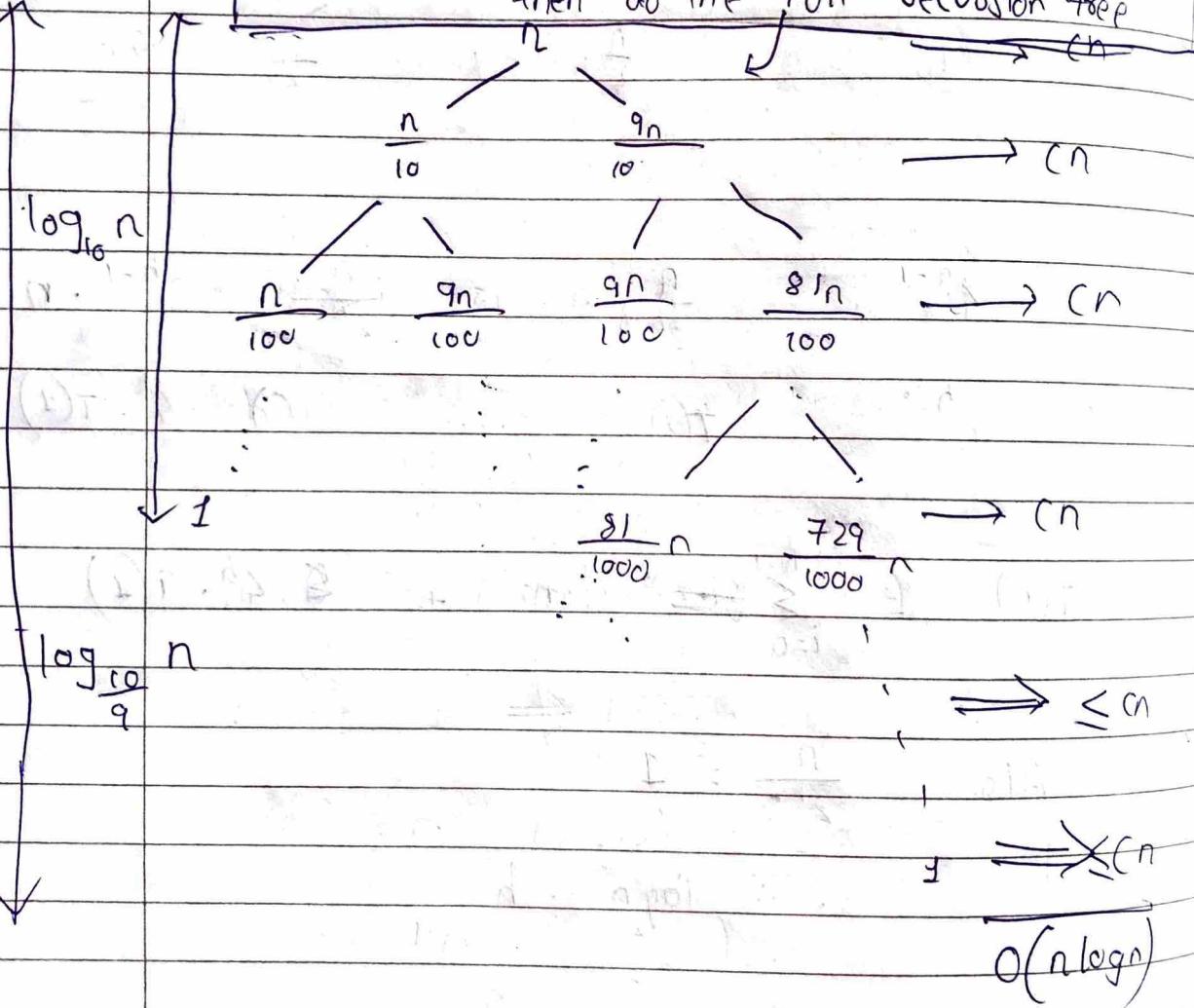
$$= + n \left[\frac{n-1}{2} \right] + n^2 \cdot T(1)$$

$$= n^2 \cdot T(1) + n^2 - n$$

$$\Theta(n^2)$$

Q $T(n) = T\left(\frac{n}{10}\right) + T\left(\frac{9n}{10}\right) + cn$

In exam first do individual $T\left(\frac{n}{10}\right)$ & of $T\left(\frac{9n}{10}\right)$
then do the full decoupling tree



$\therefore cn \cdot \log_{\frac{9}{10}} n$ is the maximum complexity
 $\therefore O(n \log n)$

\Rightarrow Change of variable method

$$\text{def: } T(n) = 2T(\sqrt{n}) + \log(n)$$

$$\text{Let } n=2^m, m=\log_2 n$$

$$\text{So, } T(2^m) = 2T\left(2^{\frac{m}{2}}\right) + m \quad \begin{matrix} \text{As base is not important} \\ \text{so directly replace it} \end{matrix}$$

$$\text{Let } T(2^m) = S(m) \Rightarrow T\left(2^{\frac{m}{2}}\right) = S\left(\frac{m}{2}\right)$$

$$\therefore S(m) = 2S\left(\frac{m}{2}\right) + m$$

Now by master method $a=2, b=2, d=1$

$$a=b^d \therefore \Theta(m \cdot \log m)$$

~~$S(m) = \Theta(m \log m)$~~

$$\therefore T(2^m) = \Theta(m \log m)$$

$$\therefore T(n) = \Theta(\log_2 n (\log \log_2 n))$$

Q $\underline{m=2^n} \quad n=2^m \quad T(n) = 3T(\sqrt{n}) + \log n$

$$m = \log_2 n$$

$$T(n) = 3T(\sqrt{n}) + \log n$$

$$\therefore T(2^m) = 3T\left(2^{\frac{m}{2}}\right) + m$$

$$\therefore S(m) = 3S\left(\frac{m}{2}\right) + m$$

$$\therefore a=3, b=2, d=1$$

$$a > b^d \quad \Theta(m^{\log_b a})$$

$$\therefore S(m) = \Theta(m^{\log_2 3})$$

$$\therefore T(n) = \Theta((\log_2 n)^{\log_2 3})$$

$$Q \quad T(n) = \sqrt{n} T(\sqrt{n}) + n$$

let $n=2^m \therefore m=\log n$

$$\therefore T(2^m) = 2^{\frac{m}{2}} T\left(\frac{2^m}{2}\right) + 2^m$$

$$T(2^m) = 2^m + 2^{\frac{m}{2}} T\left(\frac{m}{2}\right)$$

$$T(2^m) = 2^m + 2^{\frac{m}{2}} \left[2^{\frac{m}{4}} T\left(\frac{m}{4}\right) + 2^{\frac{m}{4}} \right]$$

$$= 2^m + 2^m + 2^{\frac{m}{2}} \cdot 2^{\frac{m}{4}} T\left(\frac{m}{4}\right)$$

$$= 2^m + 2^m + 2^{\frac{m+m}{4}} \left[2^{\frac{m}{8}} T\left(\frac{m}{8}\right) + 2^{\frac{m}{8}} \right]$$

$$T(2^m) = 2^m + 2^m + 2^m + 2^{\frac{m+m+m}{8}} T\left(\frac{m}{8}\right)$$

So after i steps we get

$$S(m) = 2^m + \dots + 2^m + 2^{\frac{m}{2^1}} + 2^{\frac{m}{2^2}} + \dots + 2^{\frac{m}{2^i}} S\left(\frac{m}{2^i}\right)$$

$\underbrace{\quad \quad \quad}_{i \text{ times}}$

Suppose after h steps we get $S(1)$ in the RHS.

$$\text{So, } \frac{m}{2^h} = 1 \Rightarrow h = \log_2 m$$

$$S(m) = h \cdot 2^m + S(1) \left[2^{\frac{m}{2} + \frac{m}{2^2} + \dots + \frac{m}{2^h}} \right]$$

$$= \underbrace{\log_2 m \cdot 2^m}_{\text{dominant term.}} + S(1) \underbrace{2}_{\text{Not dominant term}}$$

$\underbrace{\quad \quad \quad}_{\text{dominant term.}}$

$\underbrace{\quad \quad \quad}_{\text{Not dominant term}}$

$$S(m) = \Theta(2^m \log_2 m)$$

$$T(2^m) = \Theta(2^m \log m)$$

$$T(n) = \Theta(n \log_2 \log n)$$

$$\boxed{T(n) = \Theta(n \log \log n)}$$

Q $T(n) = 12 T(n^{\frac{1}{3}}) + (\log n)^2$

Hint $n = 3^m$

Ans $\Theta(12^{\log_3 \log_3 n}) = \Theta(\log n)$

\Rightarrow Intelligent guesswork method. (Rare method, less likely to ask in exam)

- 1) Calculate the first few values of recurrence
- 2) Check for regularity
- 3) Guess a suitable general form

Q $T(n) = 3T(\frac{n}{3}) + n$ (Ans $\Theta(n^{\log_3 3})$) (Here given n is exact power of 3)

Soln	n	1	2	4	8	16	32	Assume $T(0) = 0$
	$T(n)$	1	5	19	65	211	665	$T(0.5) \approx T(0)$

Pattern	n	$T(n)$
	1	2^0
	2	$3 \times 2^0 + 2^1$
	4	$3^2 \times 2^0 + 3^1 \times 2^1 + 2^2$
	8	$3^3 \times 2^0 + 3^2 \times 2^1 + 3^1 \times 2^2 + 2^3$
	16	$3^4 \times 2^0 + \dots + 2^4 \times 3^0$

So the pattern is

$$T(n) = T(2^k) = 3^k 2^0 + \dots + 3^0 2^k$$

$$= \sum_{i=0}^k 3^{k-i} 2^i$$

$$= \sum_{i=0}^k \frac{3^k}{3^i} \cdot 2^i$$

$$= 3^k \sum_{i=0}^k \left(\frac{2}{3}\right)^i \quad (\text{This is GP}).$$

$$= 3^k \left[\frac{1 - \left(\frac{2}{3}\right)^{k+1}}{1 - \frac{2}{3}} \right]$$

$$T(n) = T(2^k) = 3^{k+1} - 2^{k+1}$$

$$\text{As } n = \log_2 n = 2^k$$

$$k = \log_2 n$$

$$\therefore T(n) = T(2^{\log_2 n}) = 3^{2\log_2 n + 1} - 2^{2\log_2 n + 1}$$

\Rightarrow Multiplication by 12 russe.

	Div by 2	Mul by 2
odd	47	98
odd	23	196
odd	11	392
odd	5	784
2	1568	
odd	1	3136

Just write the
answers for odd
numbers

4606

See figure 1.3 of
book

→ Divide & Conquer for multiplication (Karatsuba method)

Ex. $\begin{array}{r} 981 \\ \times 1234 \end{array}$ (No. of digits should be power of 2 so put zeros as per need)

Let $w = 09, x = 81$

$y = 12, z = 34$

$$981 = w \times 10^2 + x$$

$$1234 = y \times 10^2 + z$$

$$\begin{aligned} & \therefore (10^2w + x)(10^2y + z) \\ &= 10^4wy + 10^2(wz + xy) + (xz) \end{aligned}$$

It will be lengthy so another method

$$T(n) = 3T\left(\frac{n}{2}\right) + n$$

$$p = wy = 09 \times 12 = 108$$

$$q = xz = 81 \times 34 = 2754 \quad \therefore O(n^{\log_2 3})$$

$$r = (w+x)(y+z) = 90 \times 46 = 4140$$

$$\begin{aligned} 981 \times 1234 &= 10^4 p + 10^2(r-p-q) + q \\ &= 1080000 + 127800 + 2754 = 1210554 \end{aligned}$$

→ For Matrix Multiplication

i) Strassen's Method (for $2^n \times 2^n$ matrices only)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2 \times 2}, \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}_{2 \times 2}$$

Same level → add

Different level → subtraction

7 formulas

$$P_1 = a(f-h)$$

$$P_2 = (a+b)h$$

$$P_3 = (c+d)e$$

$$P_4 = d(g-c)$$

$$P_5 = (a+d)(e+h)$$

$$P_6 = (b-d)(g+h)$$

$$P_7 = (a-c)(e+f)$$

$$C = \begin{bmatrix} P_5 + P_6 + P_4 - P_2 & P_1 + P_2 \\ P_3 + P_4 & P_1 - P_3 + P_5 - P_7 \end{bmatrix}$$

$$T(n) = 7T\left(\frac{n}{2}\right) + n^2$$

$$\text{Time complexity} = \Theta(n^{\log_2 7}) = \Theta(n^{2.8})$$

Originally TC was $\Theta(n^3)$ but now it is $\Theta(n^{2.8})$ so it improved.



$$\rightarrow \text{Total comparisons} = \frac{3(n-1)}{2}$$

Thus in either case total number of

comparisons is at most $3\left[\frac{n}{2}\right]$

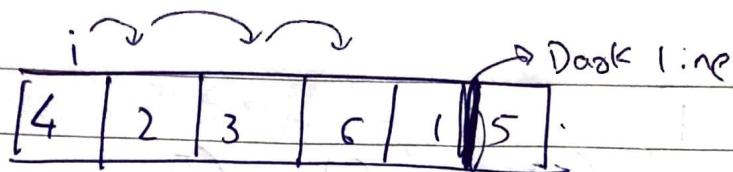
\Rightarrow Find the i th minimum from the array

~~5 3 6 1 4~~ → Here randomized - partition is the ^{random} selection of pivot element & swap it with last & follow same procedure of quick sort.

e.g. 5 2 3 6 1 4, find 3rd minimum

) 4 & 5 will be swapped

$\therefore 4 \ 2 \ 3 \ 6 \ 1 \ 5 \rightarrow \text{pivot}, 8$



Now $4 < 5$, increment i

Now $2 < 5$, increment i

Now $3 < 5$,

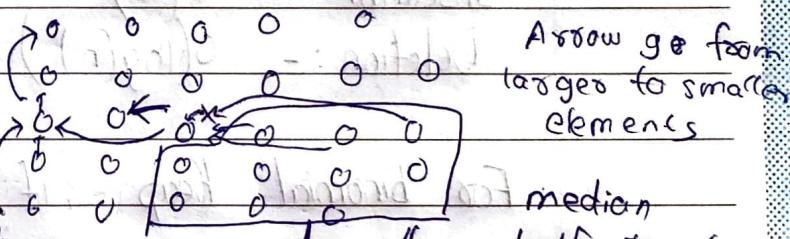
j
 $[4 | 2 | 3 | 6 | 1 | 5]$ Now $6 > 5$, j will be incremented $\therefore 6$ will be marked

$[4 | 2 | 3 | 6 | 1 | 5]$ Here $i < 5 \therefore$ Masked element will be replaced with 1. $\therefore [4 | 2 | 3 | 1 | 6 | 5]$

→ After partition pivot will come to correct position

→ See last page for one more example,

⇒ Median of medians



→ At least half of the medians are greater than ~~half~~ of the medians.

→ At least half of the groups contribute at least 3 elements which are greater than x , except the gap which is incomplete & the group which contains medians of median

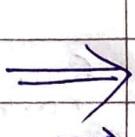
$$T(n) + T\left(\frac{2n}{3}\right) + O(n)$$

(For groups of 3)

Same as $T\left(\frac{n}{2}\right) + T\left(\frac{3n}{4}\right)$ and ans is $O(n^{\log_3 2})$

→ Time complexity won't be upper bounded by linear factors

So more than groups of 5 will give linear complexity.



BINOMIAL HEAPS

→ To merge 2 binary heap then Time complexity will be $O(n)$

Binomial Heap

Merge :- $O(\log(n))$

Insertion :- $O(1)$

Deletion :- $O(\log(n))$

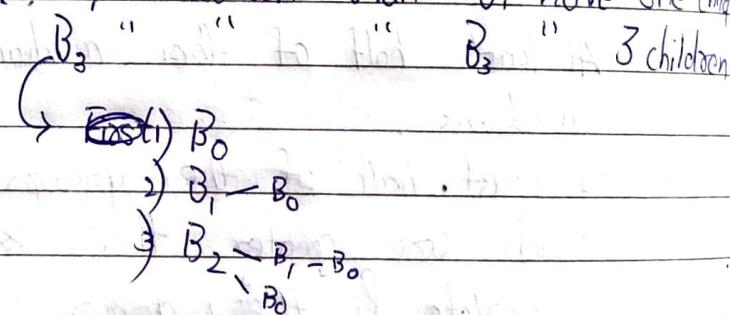
Binary Heap

Merge :- $O(n)$

Insertion :- $O(\log n)$

Deletion :- $O(\log n)$

For binomial heap, if B_i is written then B_i have one child

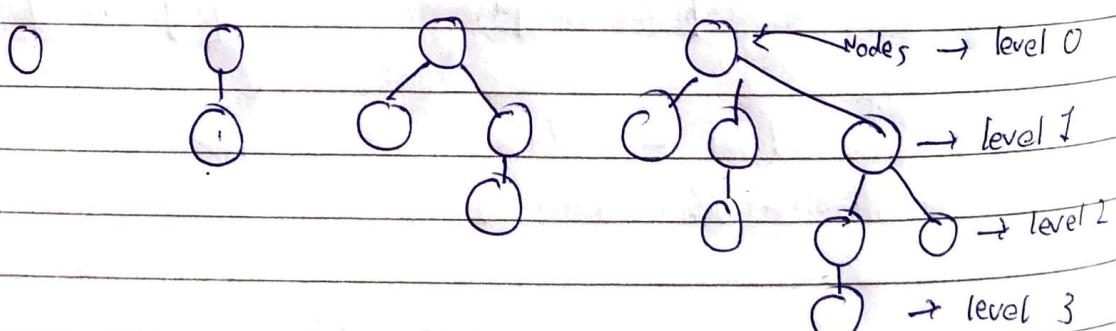


B_0

B_1

B_2

B_3



For B_i we have total of 2^i nodes.

For B_i at k^{th} level we have ${}^i C_k$ number of nodes.

→ Binomial heaps is a collection of binary tree.