3) $T(n) = \sqrt{n}\, T(\sqrt{n}) + n.$

$\Rightarrow$ Let $n = 2^m \Rightarrow m = \log_2 n$

$$T(2^m) = 2^{\frac{m}{2}}\, T(2^{\frac{m}{2}}) + 2^m$$

Suppose $T(2^m) = S(m) \Rightarrow T(2^{\frac{m}{2}}) = S\left(\frac{m}{2}\right).$

$$\therefore S(m) = 2^{\frac{m}{2}}\, S\left(\frac{m}{2}\right) + 2^m$$

$$= 2^m + 2^{\frac{m}{2}}\left[S\left(\frac{m}{2}\right)\right]$$

$$= 2^m + 2^{\frac{m}{2}}\left[2^{\frac{m}{4}} S\left(\frac{m}{4}\right) + 2^{\frac{m}{2}}\right]$$

$$= 2^m + 2^{\frac{m}{2}}\left[2^{\frac{m}{2}} + 2^{\frac{m}{4}} S\left(\frac{m}{4}\right)\right]$$

$$= 2^m + 2^m + 2^{\frac{m}{2}+\frac{m}{4}} S\left(\frac{m}{4}\right)$$

$$= 2^m + 2^m + 2^{\frac{m}{2}+\frac{m}{4}}\left[2^{\frac{m}{8}} S\left(\frac{m}{8}\right) + 2^{\frac{m}{4}}\right]$$

$$= 2^m + 2^m + 2^{\frac{m}{2}+\frac{m}{4}}\left[2^{\frac{m}{4}} + 2^{\frac{m}{8}} S\left(\frac{m}{8}\right)\right]$$

$$= 2^m + 2^m + 2^m + 2^{\frac{m}{2}+\frac{m}{4}+\frac{m}{8}}\left[S\left(\frac{m}{8}\right)\right]$$

So, after i steps we get,

$$S(m) = \underbrace{2^m + 2^m + \cdots + 2^m}_{(i \text{ times})} + 2^{\frac{m}{2^1}+\frac{m}{2^2}+\frac{m}{2^3}+\cdots+\frac{m}{2^i}}\left[S\left(\frac{m}{2^i}\right)\right]$$

Suppose after h steps, we get $S(1)$ in the R.H.S.

So, $\frac{m}{2^h} = 1 \Rightarrow \boxed{h = \log_2 m}$

$$\Rightarrow S(m) = h \cdot 2^m + S(1)\left[2^{\frac{m}{2}+\frac{m}{2^2}+\frac{m}{2^3}+\cdots+\frac{m}{2^h}}\right]$$

$$\Downarrow$$

$$= \log_2 m \cdot 2^m + S(1)\, 2^{m-\epsilon} \quad (\epsilon > 0)$$

[If series continues till infinity, then power of 2 would be m) But here it's a finite series so power of 2 will be ($< m$).]

$\Downarrow$

(Dominant Term)

So, $S(m) = \Theta(2^m \log_2 m)$

$$T(2^m) = \Theta(2^m \log_2 m)$$

$$\Rightarrow T(n) = \Theta(n \log_2 \log_2 n)$$

$$\boxed{T(n) = \Theta(n \log\log n)}$$

4) $T(n) = 12T(n^{\frac{1}{3}}) + (\log n)^2$

$\Rightarrow$ Let $n = 3^m \Rightarrow m = \log_3 n$

$$T(3^m) = 12T(3^{\frac{m}{3}}) + (\log(3^m))^2$$

$$= 12T(3^{\frac{m}{3}}) + (m\log 3)^2$$

$$T(3^m) = 12T(3^{\frac{m}{3}}) + m^2(\log 3)^2$$

Let $T(3^m) = S(m)$

~~$\Rightarrow S(m) = 12S\left(\frac{m}{3}\right) + \left(\frac{\log n}{\log 3}\right)^2 (\log 3)^2$~~

$\Rightarrow S(m) = 12S\left(\frac{m}{3}\right) + m^2(\log 3)^2$

The above equation can be solved using Master method :-

$a = 12,\ b = 3,\ d = 2;$

So, $a > b^d$ $(\because 12 > 3^2)$

So, case(iii) is applicable.

$\therefore S(m) = \Theta(m^{\log_b a})$

$= \Theta(m^{\log_3 12})$

$\Rightarrow T(3^m) = \Theta(m^{\log_3 12})$

$\Rightarrow T(n) = \Theta((\log_3 n)^{\log_3 12})$

$$\boxed{T(n) = \Theta\left(12^{\log_3(\log_3 n)}\right)}$$

Scanned with CamScanner

Intelligent Guesswork

The recurrence relation is solved by applying the following steps :-

1) Calculate the first few values of the recurrence.
2) Check for regularity (pattern)
3) Guess a suitable general form.
4) Prove by mathematical induction that the form is correct.

Q Solve the following recurrence :-

$$T(n) = 3T\left(\frac{n}{2}\right) + n \qquad (n \text{ is an exact power of } 2)$$

Sol^n :- Calculating first few values of recurrence we get,

n	1	2	4	8	16	32
$T(n)$	1	5	19	65	211	665

Each term in this table (except the first) is computed from the previous term. For instance,

$$T(16) = 3(T(8)) + 16 = 3 \times 65 + 16 = 211.$$

But, we cannot identify specific pattern from the above table.

Now, writing n as an explicit power of 2;

n	$T(n)$
1	2^0
2	$3 \times 2^0 + 2^1$
4	$3^2 \times 2^0 + 3^1 \times 2^1 + 2^2$
8	$3^3 \times 2^0 + 3^2 \times 2^1 + 3^1 \times 2^2 + 2^3$
16	$3^4 \times 2^0 + 3^3 \times 2^1 + 3^2 \times 2^2 + 3^1 \times 2^3 + 3^0 \times 2^4$

So, the pattern is :-

$$T(n) = T(2^k) = 3^k\,2^0 + 3^{k-1}\,2^1 + \cdots + \cdots + 3^0\,2^k$$

$$= \sum_{i=0}^{k} 3^{k-i}\,2^i$$

$$= \sum_{i=0}^{k} \frac{3^k}{3^i} \cdot 2^i$$

$$= 3^k \sum_{i=0}^{k} \left(\frac{2}{3}\right)^i$$

$$= 3^k \left[\frac{1 - \left(\frac{2}{3}\right)^{k+1}}{1 - \frac{2}{3}} \right]$$

$$T(n) = T(2^k) = 3^{k+1} - 2^{k+1}$$

As, $n = 2^k \Rightarrow k = \log_2 n$

$$T(n) = T\left(2^{\log_2^n}\right) = 3^{\log_2^n + 1} - 2^{\log_2^n + 1}$$

$$= 3^{\log_2^n}(3) - 2^{\log_2^n}(2)$$

$$T(n) = 3n^{\log_2^3} - 2n$$

$$\therefore \boxed{T(n) \in O\left(n^{\log_2^3}\right)} \quad (n \text{ is a power of } 2)$$