DAA

⇒ Identify correctness of Insertion Sort.

⇒ Recurrence

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \dots C_k a_{r-k} = f(r)$$

is called a recurrence relation of order k. It is also called as k^{th} - order linear recurrence relation.

Q1 Obtain the total sol^n of $a_r - 5a_{r-1} + 6a_{r-2} = 0$

Sol^n The above eq^n is the 2^{nd} order linear recurrence relat

i) Characteristic Eq^n :-

$$x^2 - 5x + 6 = 0$$

$$\therefore x = 2, 3$$

ii) $\therefore$ Homogeneous sol^n $(A_1)(2)^r + (A_2)(3)^r$

If equal sol^{n}s then $(A_1 r + A_2)(\text{root})^r$

Note If roots are 2, 2, 3 then Homogeneous eq^n will be

$$(A_1 r + A_2)(2)^r + (A_3)(3)^r$$

iii) No Particular sol^n

iv) Total sol^n

~~$a_r = A$~~ $a_r = A_1 (2)^r + A_2 (3)^r$

Q3 Obtain the sol^n of

$$a_n = 2a_{n-1} - a_{n-2} \quad [\text{Initial } cond^n \; a_0 = 0, a_1 = 3]$$

Sol^n i) Characteristic Eq^n

Given :- $a_n = 2a_{n-1} - a_{n-2}$

$$a_n - 2a_{n-1} + a_{n-2} = 0$$

$$\therefore \alpha^2 - 2\alpha + 1 = 0$$

$$\alpha = 1, 1$$

Total Sol^n $\boxed{a_n = A_1 n + A_2}$

Also $a_0 = 0$, $a_1 = 3$

$0 = A_2$ —(i) $\quad 3 = A_1 + A_2$ —(ii)

$\therefore A_1 = 3, A_2 = 0$

$\Rightarrow$ Some of the recurrence rel^n's are of form :-

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$a \rightarrow$ No. of subproblems

$\frac{n}{b} \rightarrow$ Size of each subproblem

$T(n) \rightarrow$ Time taken to solve each sub-problem

$f(n) \rightarrow$ Fun^n which denotes the time taken to combine result of sub probems.

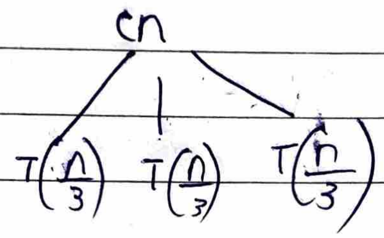
⇒ 3 methods are used

1) Recursion Tree method
2) Master Method
3)

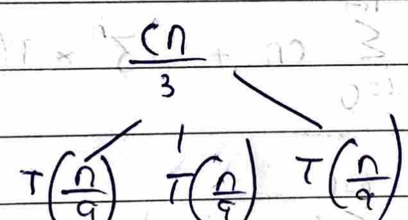
1) Recursion Tree Method.

Ex:- $T(n) = 3T\left(\frac{n}{3}\right) + cn$

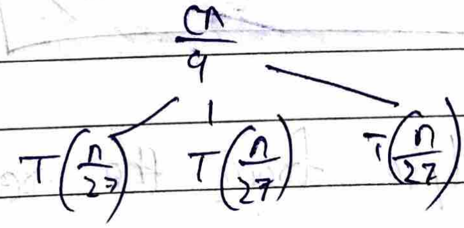
Sol^n



Now, $T\left(\frac{n}{3}\right) = 3T\left(\frac{n}{9}\right) + \frac{cn}{3}$



$T\left(\frac{n}{9}\right) = 3\left(T\left(\frac{n}{27}\right)\right) + \frac{cn}{9}$



Full Recursion Tree

Level	No. of nodes		Work done by single node
0	1	cn	cn
1	3	$\frac{cn}{3}\ \frac{cn}{3}\ \frac{cn}{3}$	$3\left(\frac{cn}{3}\right) = cn$
2	9	$\frac{cn}{9}\ \frac{cn}{9}\ \frac{cn}{9}\ \frac{cn}{9} \cdots \frac{cn}{9}$	$9\left(\frac{cn}{9}\right) = cn$
$\vdots$			
$h-1$	3^{h-1}		cn
h	3^h	$T(1)\ T(1) \cdots T(1)$	$3^h \times T(1)$

$$\text{Total work done} = \sum_{i=0}^{h-1} cn + 3^h \times T(1) \longrightarrow (1)$$

$$= cn \sum_{i=0}^{h-1} 1 + 3^h \times T(1)$$

$$\boxed{T(n) = cnh + 3^h \times T(1)} \longleftarrow (2)$$

We can analyse from the recursion tree that

$$\frac{n}{3^h} = 1 \implies n = 3^h$$

$$\therefore h = \log_3 n \longleftarrow (3)$$

Putting eq^n 3 in eq^n 2 we get,

$\frac{a}{1-\frac{1}{3}}$ $\frac{3n^2}{2}$

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$T(n) = cn\log_3 n + n \times T(1)$

$\therefore$ $T(n) = \Theta(n\log_3 n)$ (As $n \times T(1)$ is non-domin)

2) $T(n) = 3\left(T\left(\frac{n}{3}\right)\right) + n^2$

$$n^2 \to T\left(\frac{n}{3}\right) \quad T\left(\frac{n}{3}\right) \quad T\left(\frac{n}{3}\right)$$

Level	No. of nodes	Tree	Work done
0	1	n^2	n^2
1	3	$\left(\frac{n}{3}\right)^2 \left(\frac{n}{3}\right)^2 \left(\frac{n}{3}\right)^2$	$3\left(\frac{n^2}{9}\right) = \frac{n^2}{3}$
2	9	$\left(\frac{n}{9}\right)^2 \dots \left(\frac{n}{9}\right)^2$	$\frac{n^2}{9}$
h-1	3^{h-1}		$\frac{n^2}{3^{h-1}}$
h	3^h	$T(1) \dots T(1)$	$3^h \times T(1)$

Total work done

$$T(n) = \sum_{i=0}^{h-1} \frac{n^2}{3^i} + 3^h \times T(1)$$

$$\cancel{= \frac{3n^2}{2} \sum \cdot 1 + 3^h \times T(1)}$$

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$$= n^2\left[\frac{1\left(1-\left(\frac{1}{3}\right)^h\right)}{1-\frac{1}{3}}\right] + 3^h \times T(1)$$

$$= \frac{3n^2}{2}\left[\frac{3^h-1}{3^h}\right] + 3^h \times T(1)$$

Also $h = \log_3 n$

$$\therefore T(n) = \frac{3}{2}n^2\left(\frac{n-1}{n}\right) + n(T(1))$$

$$= \frac{3}{2}n^2 - \frac{3}{2}n + n(T(1))$$

Q2 ~~Sol~~ $T(n) = 4(T(\frac{n}{2})) + n^2$

See photo

~~Q3~~ T

Q2 1) $T(n) = 3T(\frac{n}{4}) + n^2$

Level	No. of nodes	Tree	Work done
0	1	n^2	n^2
1	3	$\left(\frac{n}{4}\right)^2 \left(\frac{n}{4}\right)^2 \left(\frac{n}{4}\right)^2$	$\frac{3n^2}{16}$
2	9	$\left(\frac{n}{16}\right)^2 \ldots \left(\frac{n}{16}\right)^2$	$\frac{9n^2}{256}$
⋮			
$h-1$	3^{h-1}	$\left(\frac{n}{16^{h-1}}\right)^2 \ldots$	$\frac{3^{h-1}n^2}{16^{h-1}}$
h	3^h	$T(1)$	$3^h \cdot T(1)$

$3 < \cancel{16}$ $\quad a = 3, b = 4, d = 2$ $\quad \frac{\Theta(n^2)}{4 =}$

$\therefore$ ~~T(n)~~ ~~Also~~

$$T(n) = \sum_{i=0}^{h-1} \frac{3^i n^2}{16^i} + 3^h . T(1)$$

Also $\quad \frac{n^2}{16^h} = 1$

$\therefore \quad 2\log n = 2h \log 4$

$\therefore \quad h = \log_4 n$

$\log_{a^m} b^n$

$\frac{n}{m} . \log_a b$

$$n^2 \left[1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \dots \right]$$

$$n^2 \left[\frac{1 - \left(\frac{3}{16}\right)^h}{1 - \frac{3}{16}} \right] + 3^h . T(1)$$

$$= n^2 \left[\frac{16^h - 3^h}{16^{h-1} \cdot 13} \right] + 3^{\log_4 n} . T(1)$$

$$= 16n^2 \left[\frac{16^{\frac{1}{2}\log_{16} n} - 3^{\log_4 n}}{13 . 16^{\frac{1}{2}\log_{16} n}} \right] + 3^{\log_4 n} . T(1)$$

$$= 16n^2 \left[\frac{n^{\cancel{2}} - 3^{\log_4 n}}{13 . n^{\cancel{2}}} \right] + 3^{\log_4 n} . T(1)$$

$\therefore \quad \Theta(n^2)$ is the answer by approximation.

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Q $T(n) = 4\,T\left(\frac{n}{3}\right) + n$

Level	No. of nodes	Tree	Work done
0	1	n	n
1	4	$\frac{n}{3}\ \frac{n}{3}\ \frac{n}{3}\ \frac{n}{3}$	$\frac{4n}{3}$
2	16	$\frac{n}{9} \cdots \frac{n}{9}$	$\frac{4^2 n}{3^2}$
$\vdots$			
$h-1$	4^{h-1}	$\frac{n}{3^{h-1}} \cdots \frac{n}{3^{h-1}}$	$\frac{4^{h-1}}{3^{h-1}} n$
h	4^h	$T(1) \cdots T(1)$	$4^h \cdot T(1)$

$$\frac{n}{3^h} = 1$$

$$\therefore \boxed{\log_3 n = h}$$

$$\therefore T(n) = \sum_{i=0}^{h-1} \frac{4^i n}{3^i} + T(1) \cdot 4^h$$

$$= \left[1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \cdots \left(\frac{4}{3}\right)^{h-1}\right] n$$

$$= \frac{\left(\frac{4}{3}\right)^h - 1}{\frac{1}{3}} \cdot n + T(1) \cdot 4^{\log_3 n}$$

$$= 3\left(\frac{4^h - 3^h}{3^h}\right) n$$

$$= 3\left(\frac{4^{\log_3 n} - n}{n}\right) n$$

$$\Theta\left(4^{\log_3 n}\right) = \Theta\left(n^{\log_3 4}\right)$$

Q $T(n) = 4\,T\left(\frac{n}{2}\right) + n$

Level	No. of node	Tree	Work done
0	1	n	n
1	4	$\frac{n}{2}\ \frac{n}{2}\ \frac{n}{2}\ \frac{n}{2}$	$2n$
2	16	$\frac{n}{4} \cdots \frac{n}{4}$	$4n$
⋮			
$h-1$	4^{h-1}	$\frac{n}{2^{h-1}} \cdots \frac{n}{2^{h-1}}$	$2^{h-1} \cdot n$
h	4^h	$T(1) \cdots T(1)$	$4^h \cdot T(1)$

$$T(n) = \sum_{i=0}^{h-1} \cancel{(i+1)^2}\ i^2 \cdot n + 4^h \cdot T(1)$$

Also $\frac{n}{2^h} = 1$

$\therefore \log_2 n = h$

$$\therefore T(n) = \cancel{\sum_{i=0}^{h-1} n \cdot i^2}$$

$$\left[1 + 2 + 4 \ldots + \ldots 2^{h-1}\right] n$$

$$= \left[\frac{2^h - 1}{1}\right] \cdot n + 4^{\log_2 n} \cdot T(1)$$

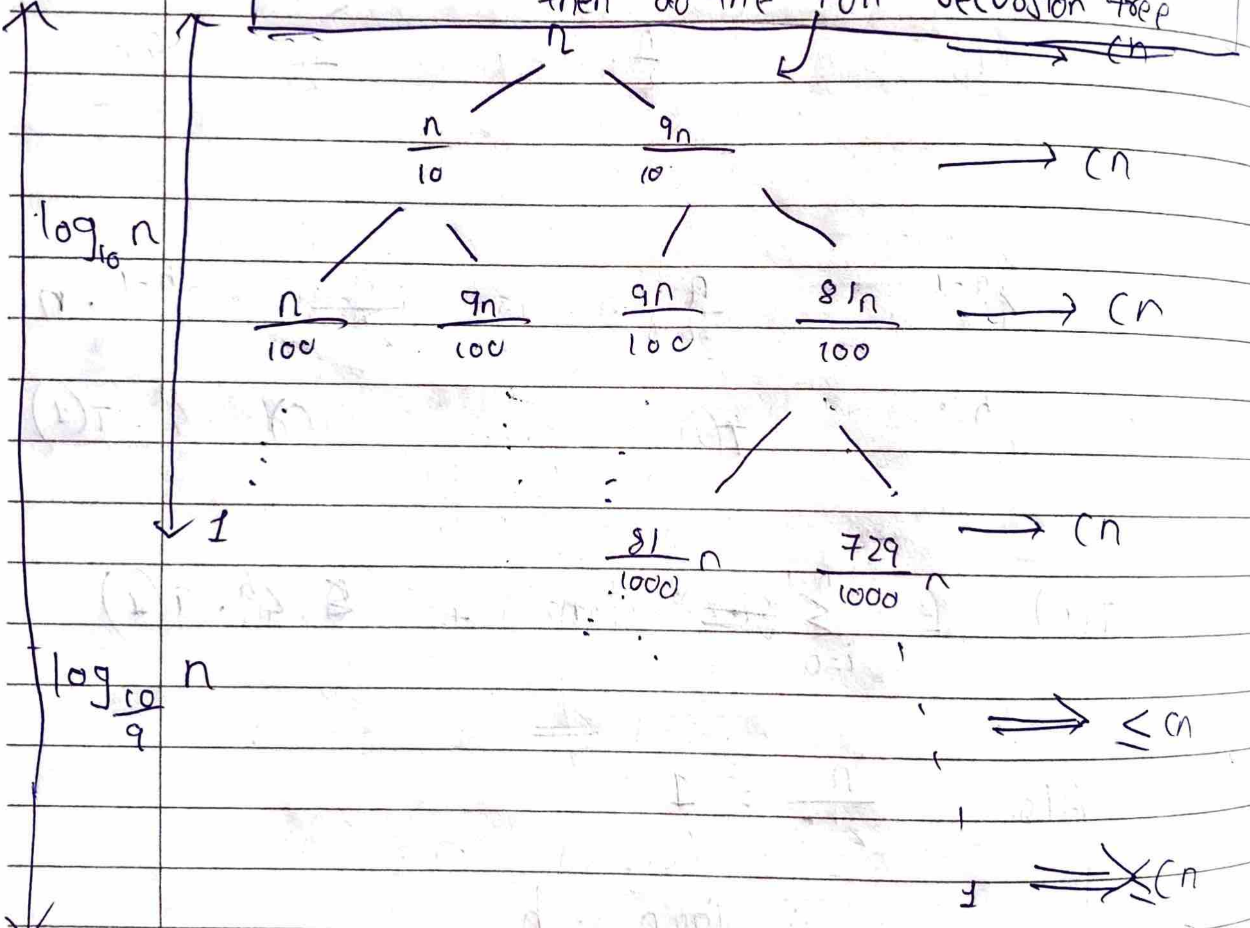
$$= n\left[\frac{n-1}{1}\right] + n^2 \cdot (T(1))$$

$$= n^2 \cdot T(1) + n^2 - n$$

$$\Theta(n^2)$$

Q $T(n) = T\left(\frac{n}{10}\right) + T\left(\frac{9n}{10}\right) + cn$

In exam first do individual $T\left(\frac{n}{10}\right)$ & $T\left(\frac{9n}{10}\right)$ then do the Full recursion tree



$O(n \log n)$

$\therefore$ $cn \cdot \log_{\frac{10}{9}} n$ is the maximum complexity

$\therefore$ $O(n \log n)$

⇒ Change of variable method

Ex:- $T(n) = 2T(\sqrt{n}) + \log(n)$

Let $n = 2^m$, $m = \log_2 n$

So, $T(2^m) = 2T(2^{\frac{m}{2}}) + m$ → As base is not important so directly replace it

Let $T(2^m) = S(m) \implies T(2^{\frac{m}{2}}) = S(\frac{m}{2})$

$\therefore S(m) = 2S(\frac{m}{2}) + m$

Now by master method $a=2, b=2, d=1$

$a = b^d \quad \therefore \Theta(m^{\cancel{\log}} . \log m)$

$\therefore \cancel{\Theta(} \quad S(m) = \Theta(m \log m)$

$\therefore T(2^m) = \Theta(m \log m)$

$\therefore T(n) = \Theta(\log_2 n (\log \log_2 n))$

Q $\cancel{m = 2^n}$ $n = 2^m$ $T(n) = 3T(\sqrt{n}) + \log n$

$\therefore m = \log_2 n$

$T(n) = 3T(\sqrt{n}) + \log n$

$\therefore T(2^m) = 3T(2^{\frac{m}{2}}) + m$

$\therefore S(m) = 3S(\frac{m}{2}) + m$

$\therefore a = 3, b = 2, d = 1$

$a > b^d \quad \Theta(m^{\log_b a})$

$\therefore S(m) = \Theta(m^{\log_2 3})$

$\therefore T(n) = \Theta((\log_2 n)^{\log_2 3})$

Q $T(n) = \sqrt{n}\, T(\sqrt{n}) + n$

let $n = 2^m$ $\therefore$ $m = \log_2 n$

$\therefore S(m) = 2^{\frac{m}{2}} S\left(\frac{m}{2}\right) + 2^m$

$= 2^m + 2^{\frac{m}{2}} S\left(\frac{m}{2}\right)$

$= 2^m + 2^{\frac{m}{2}} \left[2^{\frac{m}{4}} S\left(\frac{m}{4}\right) + 2^{\frac{m}{2}}\right]$

$= 2^m + 2^m + 2^{\frac{m}{2}+\frac{m}{4}} S\left(\frac{m}{4}\right)$

$= 2^m + 2^m + 2^{\frac{m}{2}+\frac{m}{4}} \left[2^{\frac{m}{8}} S\left(\frac{m}{8}\right) + 2^{\frac{m}{4}}\right]$

$= 2^m + 2^m + 2^m + 2^{\frac{m}{2}+\frac{m}{4}+\frac{m}{8}} S\left(\frac{m}{8}\right)$

So after i steps we get

$S(m) = \underbrace{2^m + \ldots\ldots 2^m}_{i \text{ times}} + 2^{\frac{m}{2^1}+\frac{m}{2^2}+\ldots\frac{m}{2^i}} S\left(\frac{m}{2^i}\right)$

Suppose after h steps we get $S(1)$ in the RHS.

So, $\frac{m}{2^h} = 1 \Rightarrow h = \log_2 m$

$S(m) = h \cdot 2^m + S(1)\left[2^{\frac{m}{2}+\frac{m}{2^2}+\ldots\frac{m}{2^h}}\right]$

$= \underbrace{\log_2 m \cdot 2^m}_{\text{dominant term.}} + \underbrace{S(1)\, 2}_{\text{Not dominant term}}$

$S(m) = \Theta(2^m \log_2 m)$

$$T(2^m) = \theta(2^m \log_2 m)$$
$$T(n) = \theta(n \log_2 \log_2 n)$$
$$\boxed{T(n) = \theta(n \log \log n)}$$

Q $T(n) = 12\,T(n^{\frac{1}{3}}) + (\log n)^2$

Hint $n = 3^m$

Ans $\theta(12^{\log_3 \log_3 n}) = \cancel{\theta(n \log}$

⇒ Intelligent Guesswork method. (Rare method, less likely to ask in exam)

1) Calculate the first few values of recurrence
2) Check for regularity
3) Guess a suitable general form

Q $T(n) = 3T(\frac{n}{2}) + n$ (Ans $\theta(n^{\log_2 3})$) (Here given n is exact power of n)

Sol^n

n	1	2	4	8	16	32
$T(n)$	1	5	19	65	211	665

Assume $T(0) = 0$
$T(0.5) \approx T($

Pattern

n	$T(n)$
1	2^0
2	$3 \times 2^0 + 2^1$
4	$3^2 \times 2^0 + 3^1 \times 2^1 + 2^2$
8	$3^3 \times 2^0 + 3^2 \times 2^1 + 3^1 \times 2^2 + 2^3$
16	$3^4 \times 2^0 + \dots\dots 2^4 \times 3^0$

So the pattern is

$$T(n) = T(2^k) = 3^k 2^0 + \dots 3^0 2^k$$

$$= \sum_{i=0}^{k} 3^{k-i} 2^i$$

$$= \sum_{i=0}^{k} \frac{3^k}{3^i} \cdot 2^i$$

$$= 3^k \sum_{i=0}^{k} \left(\frac{2}{3}\right)^i$$ (This is GP)

$$= 3^k \left[\frac{1 - \left(\frac{2}{3}\right)^{k+1}}{1 - \frac{2}{3}} \right]$$

$$T(n) = T(2^k) = 3^{k+1} - 2^{k+1}$$

As ~~n = log~~ $n = 2^k$

$k = \log_2 n$

$$\therefore T(n) = T(2^{\log_2 n}) = 3^{\log_2 n + 1} - 2^{\log_2 n + 1}$$

⟹ Multiplication by là russe

Ex:-

	Div by 2	Mul by 2	
odd	47	98	98
odd	23	196	+ 196
Odd	11	392	+ 392
odd	5	784	+ 784
	2	1568	
Odd	1	3136	+ 3136
			4606

Just write the answer for odd numbers

See figure 1.3 of book

⇒ Divide & conquer for multiplication (Karaksuba method)

Ex. 981 (No. of digits should be power of 2 so put zeros as per need)
$\times$ 1234

Let $w = 09$, $x = 81$
$y = 12$, $z = 34$

$981 = w \times 10^2 + x$
$1234 = y \times 10^2 + z$

$\therefore (10^2 w + x)(10^2 y + z)$
$= 10^4 wy + 10^2(wz + xy) + xz$

It will be lengthy so another method

$T(n) = 3T\left(\frac{n}{2}\right) + n$

$\therefore O(n^{1.5})$

$p = wy = 09 \times 12 = 108$
$q = xz = 81 \times 34 = 2754$
$r = (w+x)(y+z) = 90 \times 46 = 4140$

$981 \times 1234 = 10^4 p + 10^2(r - p - q) + q$
$= 1080000 + 127800 + 2754 = 1210554$

⇒ For Matrix Multiplication

1) Stressen's Method (for $2^n \times 2^n$ matrices only)

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}_{2\times 2}$

Same level → add^n
different level → subtraction

7 formulas

$P_1 = a(f-h)$ $P_3 = (c+d)e$ $P_5 = (a+d)(e+h)$
$P_2 = (a+b)h$ $P_4 = d(g-e)$ $P_6 = (b-d)(g+h)$
$P_7 = (a-c)(e+f)$

$$C = \begin{bmatrix} P_5 + P_6 + P_4 - P_2 & P_1 + P_2 \\ P_3 + P_4 & P_1 - P_3 + P_5 - P_7 \end{bmatrix}$$

$$T(n) = 7T\left(\frac{n}{2}\right) + n^2$$

Time complexity $= \Theta\left(n^{\log_2 7}\right) = \Theta\left(n^{2.8}\right)$

Originally TC was $\Theta(n^3)$ but now it is $\Theta(n^{2.8})$ so it improved.

⇒

→ Total comparisons : $\frac{3(n-1)}{2}$

Thus in either case total number of comparisons is at most $3\left\lfloor \frac{n}{2} \right\rfloor$

⇒ Find the ith minimum from the array

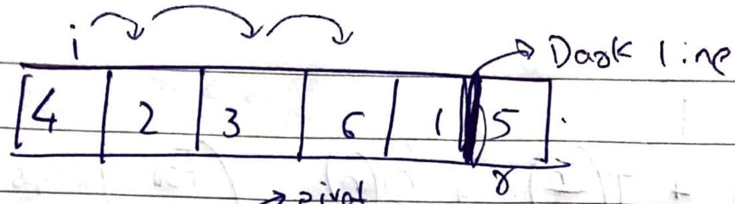
523614

→ Here randomized - partition is the random selection of pivot element & swap it with last & follow same procedure of quick sort.

eg. 5 2 3 6 1 4 , find 3rd minimum

4 & 5 will be swapped

∴ 4 2 3 6 1 5 → pivot, r



→ pivot

Now 4 < 5, increment i

→ pivot

Now 2 < 5, increment i

Now 3 < 5, " "

j

| 4 | 2 | 3 | 6 | 1 | 5 | Now 6 > 5, j will be incremented ∴ 6 will be marked

i j

| 4 | 2 | 3 | 6 | 1 | 5 | Here 1 < 5 ∴ Marked element will be replaced with 1.

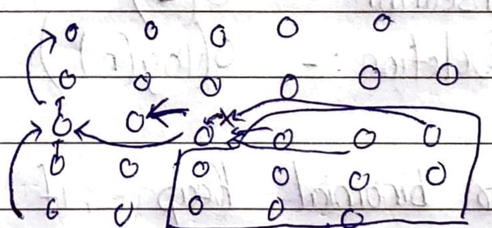
∴ | 4 2 3 1 6 5 |

Now marked element will be swapped with pivot.

→ After partition pivot will come to correct position

→ See last page for one more example.

⇒ Median of medians.



Arrow go from larger to smaller elements

median

→ At least half of the medians are greater than ~~half of~~ the medians.

→ At least half of the groups contribute at least 3 elements which are greater than X, except the grp which is incomplete & the group which contains median of median

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Same as $T\left(\frac{n}{10}\right) + T\left(\frac{9n}{10}\right)$ and ans is $O(n \log n)$

$$T(n) + T\left(\frac{2n}{3}\right) + O(n) \quad \text{(For groups of 3)}$$

→ Time complexity won't be upper bounded by linear factors
So more than groups of 5 will give linear complexity.

⇒ BINOMIAL HEAPS

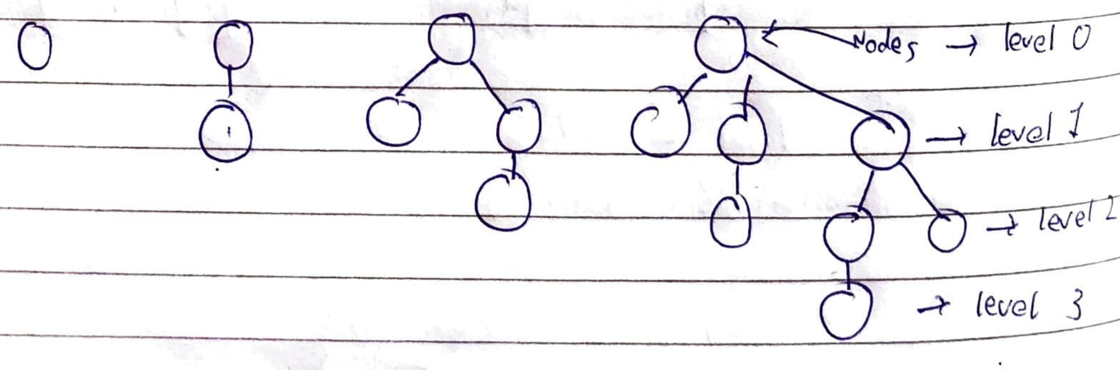
→ To merge 2 binary heap then Time complexity will be $O(n)$

Binomial Heap		Binary Heap	
Merge :-	$O(\log(n))$	Merge :-	$O(n)$
Insertion :-	$O(1)$	Insertion :-	$O(\log n)$
Deletion :-	$O(\log(n))$	Deletion :-	$O(\log n)$

For binomial heap, if B_1 is written than B_1 have one child
" B_3 " " " B_3 " 3 children

→ ~~First~~ 1) B_0
2) B_1 — B_0
3) B_2 — B_1 — B_0
B_0

B_0 B_1 B_2 B_3



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For B_i we have total of 2^i nodes.

For B_i at k^{th} level we have ${}^iC_k$ number of nodes.

→ Binomial heaps is a collection of binary tree.