Asymptotic Notation

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Asymptotic Notation

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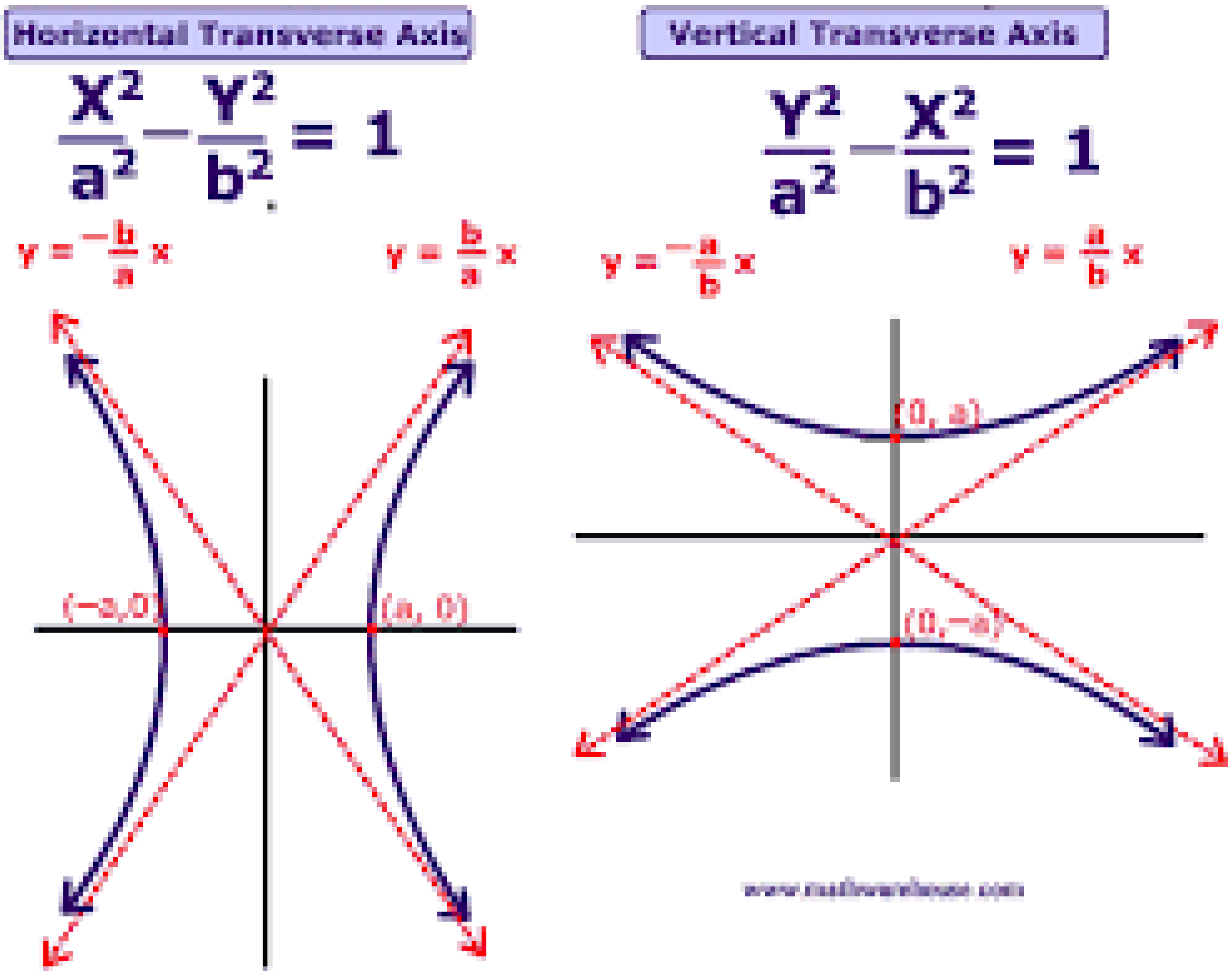

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- Usually, an algorithm that is asymptotically more efficient will be the best choice for all but very small inputs.

- We will use asymptotic notation primarily to describe the running times of algorithms.

Asymptotic Notation

- Which are different asymptotic notations?
  - Θ**-notation**
  - O**-notation**
  - Ω**-notation**
  - o**-notation**
  - ω**-notation**

Asymptotic Notation

- The notations we use to describe the asymptotic running time of an algorithm are **defined in terms of functions whose domains are the set of natural numbers N = {0, 1, 2, …}.**

Asymptotic Notation

- **Θ-notation**

For a given function $g(n)$, we denote by $\Theta(g(n))$ the *set of functions*

$$\Theta(g(n)) = \{ f(n) : \text{ there exist positive constants } c_1, c_2, \text{ and } n_0 \text{ such that } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0 \} .$$[1]

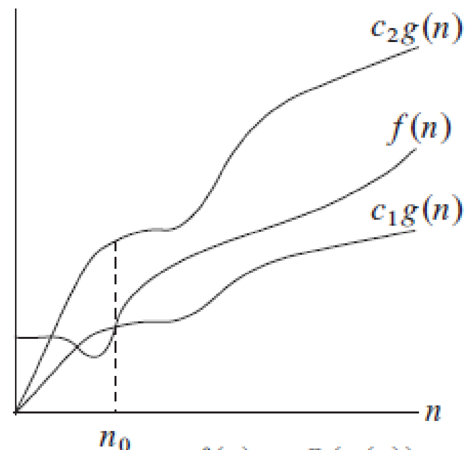

Figure 3.1 Graphic examples of the Θ, O, and Ω notations.

In each part, the value of n_0 shown is the minimum possible value; any greater value would also work. **(a)** Θ-notation bounds a function to within constant factors. We write $f(n) = \Theta(g(n))$ if there exist positive constants n_0, c_1, and c_2 such that at and to the right of n_0, the value of $f(n)$ always lies between $c_1 g(n)$ and $c_2 g(n)$ inclusive.

Asymptotic Notation

- **Θ-notation**
  - A function $f(n)$ belongs to the set $\Theta(g(n))$ if there exist positive constants c_1 and c_2 such that it can be "sandwiched" between $c_1 g(n)$ and $c_2 g(n)$, for sufficiently large n.

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  - Instead, we will usually write "$f(n) = \Theta(g(n))$" to express the same notion.
  - You might be confused because we abuse equality in this way, but we shall see later in this section that doing so has its advantages.

Asymptotic Notation

- **Θ-notation**
  - Figure 3.1(a) gives an intuitive picture of functions f(n) and g(n), where $f(n) = \theta(g(n))$.

Asymptotic Notation

- **Θ-notation**
  - Figure 3.1(a) gives an intuitive picture of functions f(n) and g(n), where $f(n) = \theta(g(n))$.
  - For all values of n at and to the right of n_0, the value of f(n) lies at or above $c_1g(n)$ and at or below $c_2g(n)$. In other words, for all $n \geq n_0$, the function f(n) is equal to g(n) to within a constant factor. We say that g(n) is an asymptotically tight bound for f(n).

Asymptotic Notation

- **Θ-notation**

In Chapter 2, we introduced an informal notion of Θ-notation that amounted to throwing away lower-order terms and ignoring the leading coefficient of the highest-order term. Let us briefly justify this intuition by using the formal definition to show that $\frac{1}{2}n^2 - 3n = \Theta(n^2)$. To do so, we must determine positive constants c_1, c_2, and n_0 such that

$$c_1 n^2 \le \frac{1}{2}n^2 - 3n \le c_2 n^2$$

for all $n \ge n_0$. Dividing by n^2 yields

$$c_1 \le \frac{1}{2} - \frac{3}{n} \le c_2 \,.$$

n	f(n)		n	f(n)
1	-5/2		6	0
2	-1		7	1/14
3	-1/2		8	1/8
4	-1/4		9	1/6
5	-1/10		10	1/5

We can make the right-hand inequality hold for any value of $n \ge 1$ by choosing any constant $c_2 \ge 1/2$. Likewise, we can make the left-hand inequality hold for any value of $n \ge 7$ by choosing any constant $c_1 \le 1/14$. Thus, by choosing $c_1 = 1/14$, $c_2 = 1/2$, and $n_0 = 7$, we can verify that $\frac{1}{2}n^2 - 3n = \Theta(n^2)$. Certainly, other choices for the constants exist, but the important thing is that *some* choice exists. Note that these constants depend on the function $\frac{1}{2}n^2 - 3n$; a different function belonging to $\Theta(n^2)$ would usually require different constants.

Asymptotic Notation

- **Θ-notation**

We can also use the formal definition to verify that $6n^3 \neq \Theta(n^2)$. Suppose for the purpose of contradiction that c_2 and n_0 exist such that $6n^3 \leq c_2 n^2$ for all $n \geq n_0$. But then dividing by n^2 yields $n \leq c_2/6$, which cannot possibly hold for arbitrarily large n, since c_2 is constant.

Asymptotic Notation

- **Θ-notation**

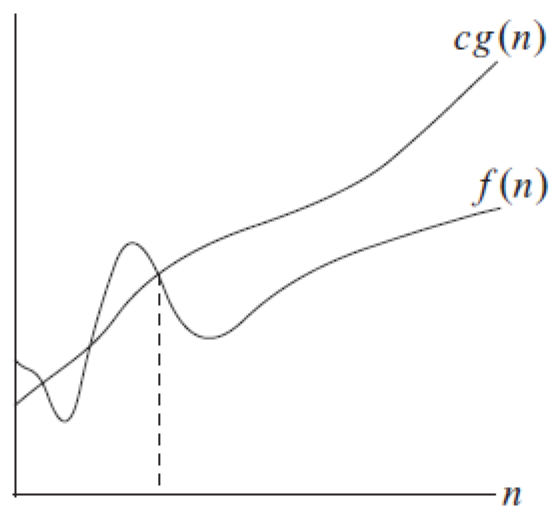
Since any constant is a degree-0 polynomial, we can express any constant function as $\Theta(n^0)$, or $\Theta(1)$. This latter notation is a minor abuse, however, because the expression does not indicate what variable is tending to infinity.[2] We shall often use the notation $\Theta(1)$ to mean either a constant or a constant function with respect to some variable.

Asymptotic Notation

- O**-notation**

The Θ-notation asymptotically bounds a function from above and below. When we have only an ***asymptotic upper bound***, we use O-notation. For a given function $g(n)$, we denote by $O(g(n))$ (pronounced "big-oh of g of n" or sometimes just "oh of g of n") the set of functions

$$O(g(n)) = \{ f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \} .$$



$f(n) = O(g(n))$

Asymptotic Notation

- **O-notation**
  - f(n) = $O(g(n))$ means that a function f(n) belongs to the set $O(g(n))$.

Asymptotic Notation

- ***O*-notation**
  - f(n) = O(g(n)) means that a function f(n) belongs to the set O(g(n)).
  - $f(n) = \theta\big(g(n)\big)$ implies f(n) = O(g(n)), that is $\theta\big(g(n)\big) \subseteq O\big(g(n)\big)$

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  - So, any quadratic function $an^2 + bn + c$, where a > 0 is not only in $\theta(n^2)$ but also in $O(n^2)$.

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  - So, any quadratic function $an^2 + bn + c$, where a > 0 is not only in $\theta(n^2)$ but also in $O(n^2)$.
  - It may be surprising that when a > 0, any linear function an + b is in $O(n^2)$. This can be verified by taking c = a + |b| and n_0 = max (1, -b/a).

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  - So, any quadratic function $an^2 + bn + c$, where $a > 0$ is not only in $\theta(n^2)$ but also in $O(n^2)$.
  - It may be surprising that when $a > 0$, any linear function $an + b$ is in $O(n^2)$. This can be verified by taking $c = a + |b|$ and $n_0 = \max(1, -b/a)$.
  - When we write f(n) = O(g(n)), we are merely claiming that some constant multiple of g(n) is an asymptotic upper bound on f(n), with no claim of how tight an upper bound it is.

Asymptotic Notation

- ***O*-notation**
  - Using O-notation, we can often describe the running time of an algorithm merely by inspecting the algorithm's overall structure.

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  - Using O-notation, we can often describe the running time of an algorithm merely by inspecting the algorithm's overall structure.
  - For example, the doubly nested loop structure of the insertion sort algorithm from Chapter 2 immediately yields an $O(n^2)$ upper bound on the worst-case running time: the cost of each iteration of the inner loop is bounded from above by $O(1)$ (constant), the indices i and j are both at most n, and the inner loop is executed at most once for each of the n^2 pairs of values for i and j .

Asymptotic Notation

- ***O*-notation**

Since O-notation describes an upper bound, when we use it to bound the worst-case running time of an algorithm, we have a bound on the running time of the algorithm on every input—the blanket statement we discussed earlier.

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Thus, the $O(n^2)$ bound on worst-case running time of insertion sort also applies to its running time on every input.

The $\Theta(n^2)$ bound on the worst-case running time of insertion sort, however, does not imply a $\Theta(n^2)$ bound on the running time of insertion sort on *every* input. For example, we saw in Chapter 2 that when the input is already sorted, insertion sort runs in $\Theta(n)$ time.

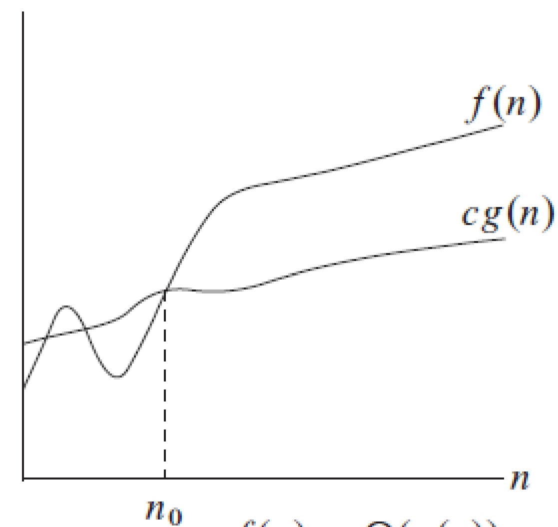
Asymptotic Notation

- **Ω-notation**

Just as O-notation provides an asymptotic *upper* bound on a function, Ω-notation provides an ***asymptotic lower bound***. For a given function $g(n)$, we denote by $\Omega(g(n))$ (pronounced "big-omega of g of n" or sometimes just "omega of g of n") the set of functions

$$\Omega(g(n)) = \{ f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0 \} .$$

Figure 3.1(c) shows the intuition behind Ω-notation. For all values n at or to the right of n_0, the value of $f(n)$ is on or above $cg(n)$.



$f(n) = \Omega(g(n))$

Asymptotic Notations

- ***Theorem 3.1***

  For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. ■

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  For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. ■

- When best case running time of some algorithm is $\Omega(g(n))$, it means that running time of that algorithm for every input is also $\Omega(g(n))$

Asymptotic Notations

- Asymptotic notation in equations and inequalities

We have already seen how asymptotic notation can be used within mathematical formulas. For example, in introducing O-notation, we wrote "$n = O(n^2)$." We might also write $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$. How do we interpret such formulas?

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When the asymptotic notation stands alone (that is, not within a larger formula) on the right-hand side of an equation (or inequality), as in $n = O(n^2)$, we have already defined the equal sign to mean set membership: $n \in O(n^2)$. In general, however, when asymptotic notation appears in a formula, we interpret it as standing for some anonymous function that we do not care to name. For example, the formula $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$ means that $2n^2 + 3n + 1 = 2n^2 + f(n)$, where $f(n)$ is some function in the set $\Theta(n)$. In this case, we let $f(n) = 3n + 1$, which indeed is in $\Theta(n)$.

Asymptotic Notations

- ***o*-notation**

The asymptotic upper bound provided by O-notation may or may not be asymptotically tight. The bound $2n^2 = O(n^2)$ is asymptotically tight, but the bound $2n = O(n^2)$ is not. We use o-notation to denote an upper bound that is not asymptotically tight. We formally define $o(g(n))$ ("little-oh of g of n") as the set

$$o(g(n)) = \{ f(n) : \text{for any positive constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \leq f(n) < cg(n) \text{ for all } n \geq n_0 \} .$$

For example, $2n = o(n^2)$, but $2n^2 \neq o(n^2)$.

Asymptotic Notations

- **ω-notation**

By analogy, ω-notation is to Ω-notation as o-notation is to O-notation. We use ω-notation to denote a lower bound that is not asymptotically tight. One way to define it is by

$f(n) \in \omega(g(n))$ if and only if $g(n) \in o(f(n))$.

Formally, however, we define $\omega(g(n))$ ("little-omega of g of n") as the set

$$\omega(g(n)) = \{ f(n) : \text{for any positive constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le cg(n) < f(n) \text{ for all } n \ge n_0 \} .$$

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For example, $n^2/2 = \omega(n)$, but $n^2/2 \ne \omega(n^2)$. The relation $f(n) = \omega(g(n))$ implies that

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty ,$$

if the limit exists. That is, $f(n)$ becomes arbitrarily large relative to $g(n)$ as n approaches infinity.