Chapter 2
Mathematical Preliminaries

(Solutions/ Hints)

2.1 Prove by giving a suitable example that if $A \cup B = A \cup C$, then it is not necessary that $B = C$.

Sol. Let A= {1, 2, 3, 4, 5}
B= {3, 4, 5, 6, 7}
C= {1, 2, 4, 5, 6, 7}
$\Rightarrow A \cup B = A \cup C = \{1, 2, 3, 4, 5, 6, 7\}$ but $B \neq C$.

2.2 For the given sets A and B, is there a possibility that $A - B = B - A$? If yes, when?

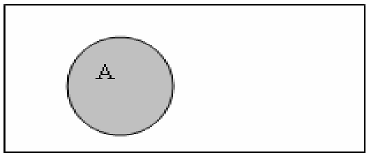
Sol. Yes. When A=B.

2.3 Write any three partition sets for set $A = \{1, 2, 3, 4, 5, 6\}$.

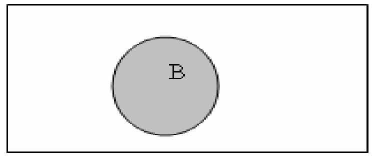
Sol. {{1, 2}, {3, 4, 5}, {6}}, {{1, 2, 3, 4}, {5, 6}}, {{1, 2, 3}, {4, 5, 6}}.

2.4 Prove De Morgan's laws using a Venn diagram.

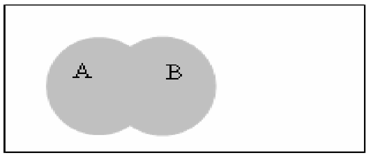
Sol. One part is being solved. Second part is left for the reader.



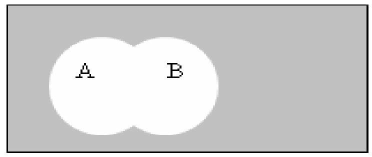
Set A



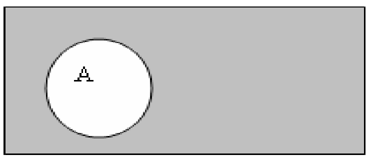
Set B



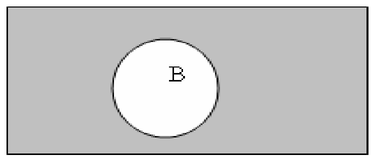
Set A $\cup$ B



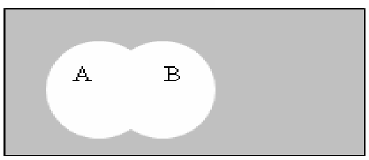
Set (A $\cup$ B)'



Set A'



Set B'



Set (A' $\cap$ B')

2.5 For the two finite sets A and B, is it possible that
(a) $A - B = B$? If yes, when?
(b) $A - B = A$? If yes, when?

Sol. (a) Not possible.
(b) When A $\cap$ B={}.

2.6 Let $A = \{a, b, c\}$ then write $P(A)$, the power set of the set A.
Sol. P(A)={{}, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}.

2.7 Let $A = \{a, b, c\}$ and $B = \{p, q, r\}$, write $A \times B$ and $B \times A$.

Sol. A × B = {(a, p), (a, q), (a, r), (b, p), (b, q), (b, r), (c, p), (c, q), (c, r)}
B × A = {(p, a), (q, a), (r, a), (p, b), (q, b), (r, b), (p, c), (q, c), (r, c)}

2.8 Let $A = \{\{a, b\}, \{c, d\}\}$ then write $P(A)$, the power set of the set A.
Sol. P(A)= {{}, {{a, b}}, {{c, d}}, {{a, b}, {c, d}}}

2.9 Let $A = \{\{a, b\}, \{c, d\}\}$ then write $A \times A$.

Sol. A × A= {({a, b}, {a, b}), ({a, b}, {c, d}),({c, d}, {a, b}),({c, d}, {c, d})}

2.10 Let $A = \{\{\ \}\}$ then write $P(A)$, the power set of the set A.

Sol. {{}, {{}}}.

2.11 Let $A = \{1, 2, 3, 4, 5\}$. Let R be a relation on A such that aRb iff $a + b > 7$. Write R. Check if R is reflexive, symmetric, or transitive.

Sol. Not reflexive – No
Symmetric- Yes
Transitive- No

2.12 Let $\{\{a, b, c\}, \{d, e\}\}$ be a partition set of the set $A = \{a, b, c, d, e\}$. Write the corresponding equivalence relation R.

Sol. R= {(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c), (d, d), (d, e), (e, d), (e, e)}

2.13 Let $A = \{1, 2, 3\}$ and $S = A \times A$. Define a relation R on S such that $(a, b)R(a', b')$ if and only if $ab = a'b'$. Show that R is an equivalence relation.

Sol. Relation R is reflexive, ab= ab.
Relation R is symmetric, ab=ba.
Relation R is transitive, if ab=a'b' and a'b'=a'' b'' then ab = a'' b''.
Hence relation R is equivalence.

2.14 Let $A = \{1, 2, 3, 4, 5\}$. Let R be a relation on A such that aRb iff $a \le b$. Write R. Show that R is a partial order relation.

Sol. Relation R is reflexive, $a \le a$.
Relation R is transitive, whenever $a \le b$ and $b \le c$ then $a \le c$.
Relation R is antisymmetric, whenever $a \le b$ and $b \le a$ then a=b.
Hence relation R is partial order relation.

2.15 Let there be a function $f: N \to N$ such that $f(x) = x^3$. Check f for being injective, surjective, and bijective. In addition, check if f is invertible.

Sol. Injective – Yes.
Surjective – No.
Bijective – No.
Invertible – No.

2.16 Let there be two functions $f: N \to N$ and $g: N \to N$ such that $f(x) = x^3$ and $g(x) = x^2 + 5$. Find $fog(x)$, $gof(x)$, $fof(x)$, and $gog(x)$.

Sol. $fog(x)= f(g(x))= f(x^2+5) = (x^2+5)^3$

$gof(x)= g(f(x))= g(x^3) = x^6+5$
$fof(x)= f(f(x))= f(x^3) = x^9$
$gog(x)= g(g(x))= g(x^2+5) = (x^2+5)^2 + 5$

2.17 Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Does the relation $R = \{(1, a), (2, a), (3, b), (4, b)\}$ qualify as a function?

Sol. Yes.

2.18 Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Does the relation $R = \{(1, a), (1, b), (3, b), (4, c), (2, c)\}$ qualify as a function?

Sol. No. Argument 1 has two outputs.

2.19 Find the number of sequences of each size that can be framed from a character set of size 6.

Sol. Sequences of size 1= 6
Sequences of size 2= 30
Sequences of size 3= 120
Sequences of size 4= 360
Sequences of size 5= 720
Sequences of size 6= 720

2.20 How many permutations can be made from the alphabets in the word ASSOCIATION?

Sol. 11!/ (2! x 2! x 2! x 2!)

2.21 A coin is tossed six times; how many possible sequences of head and tail will be there?

Sol. 2^6

2.22 Show that ${}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$.

Sol. ${}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$
Solving RHS

$$\frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!}$$

$$\frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{n-r+1} + \frac{1}{r}\right]$$

$$\frac{n!}{(r-1)!(n-r)!} \times \frac{n+1}{r(n-r+1)}$$

$$\frac{n+1!}{r!(n-r+1)!}$$

${}^{n+1}C_r$

2.23 How many distinct sequences of size 4 can be made from the alphabets of the word GEETA?

Ans. ${}^4C_4 \times 4! + ({}^1C_1 \times {}^3C_2 \times 4!/ 2!)$

2.24 In how many different ways can 6 cards be drawn from a deck of 52 cards with two red and two black cards?

Sol. Correction: It has to be 4 cards instead of 6.

$^{26}C_2 \text{ X } ^{26}C_2$

2.25 Prove the following equivalences using truth tables:

(a) $p \rightarrow q \equiv (\sim q \rightarrow \sim p)$

(b) $(p \rightarrow q) \wedge (q \rightarrow r) \equiv p \rightarrow r$

(c) $p \rightarrow q \equiv \sim p \vee q$

Sol.

a)

P	Q	p→q	~p	~q	~q→~p	(p→q)↔ (~q→~p)
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	T	F	T	T
F	F	T	T	T	T	T

Since (p→q)↔ (~q→~p) is tautology, hence p→q ≡ (~q→~p)

b) Correction: It is not a tautology.

P	Q	R	p→q	q→r	(p→q) ∧ (q→r)	p→r
F	F	F	T	T	T	T
F	F	T	T	T	T	T
F	T	F	T	F	F	T
F	T	T	T	T	T	T
T	F	F	F	T	F	F
T	F	T	F	T	F	T
T	T	F	T	F	F	F
T	T	T	T	T	T	T

c)

P	q	$p \to q$	$\sim p \vee q$	$(p \to q) \leftrightarrow (\sim p \vee q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

2.26 If $p \to q$ is true, then explain the truth value of $q \vee \sim p \vee (p \to q)$.

Sol. $q \vee \sim p \vee (p \to q) \equiv q \vee \sim p \vee T \equiv T$

2.27 Let p, q, r, s, and t be the following propositions:

p: I am very happy.
q: I am sad.
r: It is sunday today.
s: I will play cricket.
t: I will listen to songs.

Write the English sentences corresponding to the following statements:

a) $p \to s$
b) $\sim r \wedge q$
c) $(q \wedge r) \to t$
d) $(p \vee r) \to s$

Sol. $p \to s$: If I am very happy then I will play cricket.
$\sim r \wedge q$: It is not Sunday and I am sad.
$(q \wedge r) \to t$: If I am not sad and it is Sunday today then I will listen to songs.
$(p \vee r) \to s$: If I am very happy or it is Sunday today then I will play cricket.

2.28 Write the converse and contrapositive for the following statements.

(a) If tomorrow is a holiday then I will go to picnic.
(b) If it is exam tomorrow then I will study the whole night.

Sol. a) Converse: If I go to picnic then tomorrow is a holiday.
Contrapositive: If I don't go to picnic then tomorrow is not a holiday.

b) **Converse:** If I study the whole night then it is exam tomorrow.
Contrapositive: If I don't study the whole night then it is not exam tomorrow.

2.29 Prove by mathematical induction:

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = n^2(n + 1)^2/4 \text{ for } n \geq 1.$$

Sol. Basis Step
$1^3 = 1^2(1+1)^2/4 = 4/4 = 1$
LHS=RHS

Hence the equation is true for n=1.

Induction Step

Let the statement be true for $n=k$. Thus we have

$1^3+2^3+3^3+\text{-------------}+k^3= k^2(k+1)^2/4$
Now, for $n=k+1$ we must have
$1^3+2^3+3^3+\text{-------------}+k^3 + (k+1)^3 = k^2(k+1)^2/4 + (k+1)^3$
$= (k+1)^2[\,k^2/4 + (k+1)]$
$= (k+1)^2[\,k^2 + 4(k+1)]\,/4$
$= (k+1)^2(\,k+2)^2\,/4$
$= (k+1)^2(\,k+1+1)^2\,/4$
Hence the statement is true for $n=k+1$.
Hence the statement $1^3+2^3+3^3+\text{---------------------}+n^3=n^2(n+1)^2/\,4$ is true.

2.30 Prove by mathematical induction:

$$1 + r + r^2 + r^3 + \ldots + r^n = (r^n - 1)/(r - 1).$$

Sol. Basis Step
At n=1
$1+r = (r^2-1)/(r-1) = r+1$
LHS=RHS

Hence the equation is true for n=1.

Induction Step

Let the statement be true for $n=k$. Thus we have

$1+r+r^2+r^3+\text{------------}+r^k = (r^{k+1}-1)/(r-1)$

Now, for $n=k+1$ we must have
$1+r+r^2+r^3+\text{------------}+r^k+r^{k+1} = (r^{k+1}-1)/(r-1) + r^{k+1}$
$= (r^{k+1}-1 + r^{k+2} - r^{k+1})\ /(r-1)$
$=(\,r^{k+2} -1)\,/(r-1)$
$= (\,r^{k+1+1} -1)\ /(r-1)$
Hence the statement is true for $n=k+1$.
Hence the statement $1+r+r^2+r^3+\text{------------------}+r^n = (r^{n+1}-1)/(r-1)$ is true.

2.31 Prove that n^2 is odd if and only if n is odd.

Sol. Let n be odd. Let n=2k+1 for some k.
$n^2= (2k+1)^2 =4k^2+4k+1=2(2k^2+2k)+1 \rightarrow$ odd number.
If n is odd then n^2 is odd.

Let n^2 be odd then n^2=2k+1 for some k.

Let us assume n to be even and let n=2p
$n^2=(2p)^2= 4p^2$ →even number
If n is even then n^2 has to be even. Hence our assumption is wrong. n can not be even.
If n has to be odd.
If n^2 is odd then n is odd

2.32 Prove that the sum of five consecutive numbers is divisible by 5.

Sol. Let 5 consecutive numbers be k, k+1, k+2, k+3, k+4.

Since sum is multiple of 5, it is divisible by 5.

2.33 Let A and B be two sets. Prove that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Sol. If A=B then

Every element of A is contained in B that is $A \subseteq B$ and

Every element of B is contained in A that is $B \subseteq A$.

if $A \subseteq B$ and $B \subseteq A$ then

Every element of A is contained in B and

Every element of B is contained in A.

Hence A and B are same. Therefore, A=B.

Since both conditions imply each other therefore if and only if implication of two statements is proved

2.34 Prove that n^2 is even if and only if n is even.

Sol. On same lines as in case of Sol 2.31 given above.

2.35 When is $A \times B = B \times A$ possible?

Sol. When A=B.