Chapter 7
Turing Machines

(Solutions/Hints)

7.1 Design a Turing machine M over $\{0, 1\}$ such that $L(M) = \{w \mid w$ contains equal numbers 0s and 1s$\}$.

Sol.

	0	1	x	β
$\rightarrow q_0$	(q_1, x, R)	(q_2, x, R)	(q_0, x, R)	(q_f, β, L)
q_1	$(q_1, 0, R)$	(q_3, x, L)		
q_2	(q_4, x, L)	$(q_1, 1, R)$		
q_3		$(q_3, 1, L)$	(q_3, x, L)	(q_0, β, R)
q_4	$(q_4, 0, L)$		(q_4, x, L)	(q_0, β, R)
$*q_f$	----------	----------	----------	----------

7.2 Design a Turing machine M to compute $\sum_{k=1}^{n} k$ for a given positive integer n.

Sol.

Input Tape

β	β	β 0	0	0	1	β β	β	β	β	β β	β	β

Output Tape

β	β	β 0	0	0	1	0 0	0	0	0	0 β	β	β

Transition Table

	0	1	β	x
$\rightarrow q_0$	(q_1, x, R)			
q_1	$(q_1, 0, R)$	$(q_2, 1, R)$		
q_2	$(q_2, 0, R)$		$(q_3, 0, L)$	
q_3	$(q_3, 0, L)$	$(q_4, 1, L)$		
q_4	$(q_4, 0, L)$			(q_5, x, R)
q_5 q_6	(q_1, x, R)	$(q_6, 1, L)$	 (q_7, β, R)	 (q_6, x, L)
q_7 q_8		$(q_f, 1, R)$ $(q_9, 1, L)$		(q_8, β, R) $(q_8, 0, R)$
q_9	$(q_9, 0, L)$		$(q_0, 0, R)$	
$*q_f$				

7.3 Design a Turing machine M over $\{0, 1\}$ such that $L(M) = \{0^n1^{2n} \mid n \geq 1\}$.

	0	1	β	x
$\rightarrow q_0$	(q_1, x, R)			
q_1	$(q_1, 0, R)$	$(q_2, 1, R)$		
q_2		$(q_2, 1, R)$	(q_3, β, L)	
q_3		(q_4, β, L)		
q_4		(q_5, β, L)		
q_5	$(q_6, 0, L)$	$(q_5, 1, L)$		(q_f, x, R)
q_6	$(q_6, 0, L)$			(q_0, x, R)
$*q_f$				

7.4 Design a Turing machine M over $\{0, 1\}$ such that $L(M) = \{0^{2n}1^n \mid n \geq 1\}$.

Sol.

	0	1	β	X
$\rightarrow q_0$	(q_1, x, R)			
q_1	(q_2, x, R)			
q_2	$(q_2, 0, R)$	$(q_3, 1, R)$		
q_3		$(q_3, 1, R)$	(q_4, β, L)	
q_4		(q_5, β, L)		
q_5	$(q_6, 0, L)$	$(q_5, 1, L)$		
q_6	$(q_6, 0, L)$		(q_f, β, L)	(q_0, x, R)
$*q_f$				

7.5 Design a Turing machine M over $\{0, 1, 2\}$ such that $L(M) = \{0^n1^{2n}2^n \mid n \geq 1\}$.

Sol.

	0	1	2	B	x
$\rightarrow q_0$	(q_1, β, R)				
q_1	$(q_1, 0, R)$	(q_2, x, R)			
q_2		(q_3, x, R)			
q_3		$(q_4, 1, R)$			
q_4			$(q_5, 2, R)$		
q_5			$(q_5, 2, R)$	(q_6, β, L)	
q_6			(q_7, β, L)		
q_7		$(q_8, 1, L)$	$(q_7, 2, L)$		
q_8		$(q_8, 1, L)$			(q_9, x, L)

q_9	$(q_{10}, 0, L)$				(q_9, x, L)
q_{10}	$(q_{10}, 0, L)$			(q_{11}, β, R)	
q_{11}	(q_1, β, R)				(q_{12}, x, R)
q_{12}				(q_f, β, R)	(q_{12}, x, R)
$*q_f$					

7.6 Design a Turing machine M over $\{0, 1, 2, 3\}$ such that $L(M) = \{0^{2n}1^n2^n3^{2n} \mid n \geq 1\}$.

Sol.

	0	1	2	3	B	x	y
$\rightarrow q_0$	(q_1, β, R)					(q_{12}, x, R)	
q_1	(q_2, β, R)						
q_2	$(q_2, 0, R)$	(q_3, x, R)				(q_2, x, R)	
q_3		$(q_3, 1, R)$	(q_4, y, R)				(q_3, y, R)
q_4			$(q_4, 2, R)$	$(q_5, 3, R)$			
q_5				$(q_5, 3, R)$	(q_6, β, L)		
q_6				(q_7, β, L)			
q_7				(q_8, β, L)			
q_8			$(q_9, 2, L)$	$(q_8, 3, L)$			
q_9		$(q_{10}, 1, L)$	$(q_9, 2, L)$				
q_{10}	$(q_{11}, 0, L)$	$(q_{10}, 1, L)$					
q_{11}	$(q_{11}, 0, L)$				(q_0, β, R)		
q_{12}	(q_{12}, x, R)						(q_{13}, y, R)
q_{13}					(q_f, β, L)		(q_{13}, y, R)
$*q_f$							

7.7 Design a Post machine M over $\{a, b\}$ such that $L(M) = \{0^n1^n0^n \mid n \geq 1\}$.

Sol. The language L accepted by the post machine is
L= {010, 001100, 000111000,…………………………….}

Let the string be 000111000. The initial state of the Post machine is

0	0	0	1	1	1	0	0	0	Z_0							

Queue front

The algorithm for the design of Post machine is as follow :

1. Start in the state q_0 with the r/w head pointing to first symbol of the input string. The first symbol has to be 0. Read it and remove it from the queue front. Change the state to q_1. Add nothing to the rear of queue. The corresponding transition is
$(q_0, 0) \vdash (q_1, \epsilon, \epsilon)$

2. In the state q_1 read subsequent 0s of the input string without changing state. Remove them from queue front and add them to the rear of queue. The corresponding transition is
$(q_1, 0) \vdash (q_1, \epsilon, 0)$

3. In the state q_1 when a symbol 1 is encountered change the state to q_2. Remove 1 from queue front and add nothing to the rear of queue. The corresponding transition is
$(q_1, 1) \vdash (q_2, \epsilon, \epsilon)$

4. In the state q_2 read subsequent 1s of the input string without changing state. Remove them from queue front and add them to the rear of queue. The corresponding transition is
$(q_2, 1) \vdash (q_2, \epsilon, 1)$

5. In the state q_2 when a symbol 0 is encountered change the state to q_3. Remove 0 from queue front and add nothing to the rear of queue. The corresponding transition is
$(q_2, 0) \vdash (q_3, \epsilon, \epsilon)$

6. In the state q_3 read subsequent 0s of the input string without changing state. Remove them from queue front and add them to the rear of queue. The corresponding transition is
$(q_3, 0) \vdash (q_3, \epsilon, 0)$

7. In the state q_3 when a symbol Z_0 is encountered change the state to q_4. Remove Z_0 from queue front and add it to the rear of queue. The corresponding transition is
$(q_3, Z_0) \vdash (q_4, \epsilon, Z_0)$

(This completes one cycle of matching).

8. The behavior of q_4 is same as q_0. The name change has been done to ensure one that null string is not accepted by the machine. If a 0 is encountered in q_4 then new cycle of matching starts. If Z_0 is encountered then the string is accepted and state changes to the final state q_f. Also the queue becomes empty, The corresponding transitions are

$(q_4, 0) \vdash (q_1, \epsilon, \epsilon)$
$(q_4, Z_0) \vdash (q_f, \epsilon, \epsilon)$

7.8 Design a two-track Turing machine M to compute $\sum_{k=1}^{n} k$ for a given positive integer n.

Sol.
Initial tape: k number of 0s followed by 1.

B	β	β	β	β	β	β	β	β	β	β	β	β	β	β	β	β
B	β	β	0	0	0	1	β	β	β	β	β	β	β	β	β	β

Final Tape:

B	β	β	√	√	√	β	β	β	β	β	β	β	β	β	β	β
B	β	β	0	0	0	1	0	0	0	0	0	0	β	β	β	β

Put * mark on the 0s on LHS of 1 (one by one) and copy them on the RHS of 1. Corresponding transitions are

$([q_0, \beta], [\beta, 0]) \vdash ([q_1, \beta], [*, 0], R)$
$([q_1, \beta], [\beta, 0]) \vdash ([q_1, \beta], [\beta, 0], R)$
$([q_1, \beta], [\beta, 1]) \vdash ([q_2, \beta], [\beta, 1], R)$
$([q_2, \beta], [\beta, \beta]) \vdash ([q_3, \beta], [\beta, 0], L)$
$([q_2, \beta], [\beta, 0]) \vdash ([q_2, \beta], [\beta, 0], R)$
$([q_3, \beta], [\beta, 0]) \vdash ([q_3, \beta], [\beta, 0], L)$
$([q_3, \beta], [\beta, 1]) \vdash ([q_4, \beta], [\beta, 1], L)$
$([q_4, \beta], [\beta, 0]) \vdash ([q_5, \beta], [\beta, 0], L)$
$([q_5, \beta], [\beta, 0]) \vdash ([q_5, \beta], [\beta, 0], L)$
$([q_5, \beta], [*, 0]) \vdash ([q_6, \beta], [*, 0], R)$
$([q_6, \beta], [\beta, 0]) \vdash ([q_1, \beta], [*, 0], R)$

This copies number k on RHS of 1. Put a check mark √ above the left most 0 to reduce k to k-1. Repeat the above process to copy k-1.

$([q_4, \beta], [*, 0]) \vdash ([q_7, \beta], [*, 0], L)$
$([q_7, \beta], [*, 0]) \vdash ([q_7, \beta], [*, 0], L)$
$([q_7, \beta], [\beta, \beta]) \vdash ([q_8, \beta], [\beta, \beta], R)$
$([q_8, \beta], [*, 0]) \vdash ([q_9, \beta], [\surd, 0], R)$
$([q_7, \beta], [\surd, 0]) \vdash ([q_8, \beta], [\surd, 0], R)$

$([q_9, \beta], [*, 0]) \vdash ([q_9, \beta], [\beta, 0], R)$
$([q_9, \beta], [\beta, 1]) \vdash ([q_{10}, \beta], [\beta, 1], L)$
$([q_{10}, \beta], [\beta, 0]) \vdash ([q_{10}, \beta], [\beta, 0], L)$
$([q_{10}, \beta], [\surd, 0]) \vdash ([q_0, \beta], [\surd, 0], R)$

A check mark is encountered just next to left 1 when all numbers are copied to RHS of 1.

$([q_4, \beta], [[\surd, 0]) \vdash ([q_f, \beta], [\surd, 0], L)$

7.9 Design a Turing machine *M* to find the predecessor of a positive integer.

Sol.

	0	β
→q_0	(q_1, β , R)	
*q_1		

7.10 Design a Turing machine *M* to find the successor of a positive integer.

Sol.

	0	β
→q_0	(q_0, 0, R)	(q_1, 0, R)
*q_1		

7.11 Design a Turing machine *M* over $\{a, b\}$ such that $L(M) = \{ x \mid \text{length of } x \text{ is odd}\}$.

Sol.

	a	b	β
→q_0	(q_1, a, R)	(q_1, b, R)	
*q_1	(q_0, a, R)	(q_0, b, R)	