Pumping Lemma Steps

The following are the steps of pumping lemma required for proving that a given set is not regular.

Step 1 Let us assume that the given language L is regular. Let n be the number of states in the corresponding finite automaton FA.

Step 2 Choose a string z such that $|z| \geq n$. Let us use pumping Lemma to write $z = uvw$, with condition $|uv| \leq n$ and $|v| \geq 1$ or $|v| \neq 0$.

Step 3 Find a suitable integer i such that $uv^iw \notin L$. This contradicts our assumption. Therefore the language L is not regular. (The most important part of the procedure is to find the value of i such that $uv^iw \notin L$).

Example 5.1 Show that the language $L = \{0^n1^n \mid n \geq 1$ is not regular.

Solution

Step 1 Let us suppose L is regular language and if we get a contradiction then L is not regular. Suppose finite automaton has n states that accepts language L

Step 2 Let $z = 0^n1^n$, then $|z| = 2n > n$. By pumping lemma we write $z = uvw$ with $|uv| \leq n$ and $|v| \geq 1$

Step 3 Now we have to find i so that $uv^iw \notin L$ to get a contradiction. The string v can be any one of the three possible forms:

(i) the string v is constructed by using only 0's, it means $v = 0^k$ for some $k \geq 1$

(ii) the string v is constructed by using only 1's it means $v = 1^l$ for some $l \geq 1$

(iii) the string v is constructed by using both symbols 0's and 1's. Then the string v will be of the form $v = 0^m1^p$ for some $m \geq 1$ and $p \geq 1$

Case (i) By pumping lemma we write

$$z = 0^n1^n$$

$$= \underbrace{000000\ldots\ldots0000}_{0^{n-k}}\underbrace{\ldots.00000}_{0^k}\underbrace{11111111\ldots\ldots111111\ldots\ldots11111}_{1^n} = 0^{n-k}0^k1^n$$

we now consider $u = 0^{n-k}$, $v = 0^k$ and $w = 1^n$

By pumping lemma we write

$$z = 0^{n-k}(0^k)^i1^n$$

for $i = 0$ we get

$$z = 0^{n-k}1^n$$

which is a contradiction because $0^{n-k}1^n \notin L$. In $0^{n-k}1^n$, the number of 0's are less than number of 1's as $k \geq 1$.

Case (ii) By pumping lemma we write

$$z = 0^n1^n$$

$$= \underbrace{000000\ldots\ldots0000\ldots\ldots00000}_{0^n}\underbrace{1111111\ldots\ldots111}_{1^l}\underbrace{111\ldots\ldots111111\ldots\ldots11111}_{1^{n-l}} = 0^n0^l1^{n-l}$$

Here we considered $u = 0^n$, $v = 1^l$ and $w = 1^{n-l}$

By pumping lemma we write

$$z = 0^n(1^l)^i1^{n-l}$$

for $i = 0$ we get

$$z = 0^n1^{n-l}$$

which is a contradiction because $0^n1^{n-l} \notin L$. In 0^n1^{n-l} the number of 0's are more than the number of 1's as $l \geq 1$.

Case (iii) By pumping lemma we write

$$z = 0^n1^n$$

$$= \underbrace{000000.....0000}_{0^{n-m}}\underbrace{.....0.....001111}_{0^m1^P}\underbrace{.....111111......11111}_{1^{n-p}} = 0^{n-m}0^m1^P1^{n-p}$$

Here we considered $u = 0^{n-m}$, $v = 0^m1^p$ and $w = 1^{n-p}$

By pumping lemma we write

$$z = 0^{n-m}(0^m1^P)^i1^{n-p}$$

For $i = 0$ we do not get a contradiction in this case, because for $i = 0$, $z = 0^{n-m}1^{n-p}$, here m may be equal to p therefore no contradiction.

For $i = 2$ we get

$$z = 0^{n-m}(0^m1^P)^21^{n-p}$$

$$= 0^{n-m}(0^m1^P)(0^m1^P)1^{n-p}$$

$$= 0^n1^p0^m1^n$$

which is a contradiction because $0^n1^p0^m1^n \notin L$. In $0^n1^p0^m1^n$, m occurrences of 0's occur after 1's which is not supported by 0^n1^n.

All three above cases prove that L is not regular.

Example 5.2 Prove that language $L = \{a^ib^j \mid i \neq j\}$ is not regular.

Solution We will prove that language $L = \{a^ib^j \mid i \neq j\}$ is not regular by using pumping lemma. We apply the following steps:

Step 1 Assume given language $L = \{a^ib^j \mid i \neq j\}$ is regular.

Step 2 By pumping lemma we write

$z = a^ib^j = uvw$, such that $|v| \neq 0$ or $|v| \geq 1$.

Step 3 There are two cases:

(i) $i > j$, in this case $v = a^{i-j}$.

(ii) $j \geq i$, in this case $v = b^{j-i}$.

Case (i) $z = uv^kw$

$$z = a^ib^j = a^j(a^{i-j})^kb^j$$

for $k = 0$, we have $z = a^jb^j$, which is a contradiction as we can see that number a's are equal to number of b's, therefore the language $L = \{a^ib^j \mid i \neq j\}$ is not regular.

Case (ii) $z = uv^kw$

$$z = a^ib^j = a^i(b^{j-i})^kb^i$$

for $k = 0$, we have $z = a^ib^i$, which is a contradiction as we can see that number a's are equal to number of b's, therefore the language $L = \{a^ib^j \mid i \neq j\}$ is not regular.

Example 5.3

Show that the language $L = \{0^p \mid p$ is a prime number$\}$ is not regular.

Solution

Step 1 Let us suppose L is regular language and if get a contradiction then L is not regular. Let n be number of states in the finite automaton accepting language L.

Step 2 Let p be the prime number greater than n. By pumping lemma, we have $z = 0^p = uvw$ and $|z| = |uvw| = |0^p| = p$.

By using pumping lemma, we can write $z = uvw$, with $|uv| \leq n$ and $|v| \geq 1$. u, v, w are the strings of 0's. Therefore, $v = 0^m$ for some $n \geq m \geq 1$. So $|v| = m$.

Step 3 Let $i = p + 1$. Then,

$$\begin{aligned} |uv^iw| &= |uvw| + |v^{i-1}| \\ &= p + (i-1)\,m \\ &= p + (i+1-1)\,m \qquad \text{by } i = p+1 \\ &= p + pm \\ &= p(1+m). \end{aligned}$$

By pumping lemma $uv^iw \notin L$ with $i = p + 1$, because $|uv^iw| = p(1 + m)$, and $p(1 + m)$ is not a prime number, since it is divisible by p and $(1 + m)$ where $|(1 + m)| \geq 2$. Here we get a contradiction to say that L is not regular.

Example 5.4

By using pumping lemma show that the language $L = \{a^{n^2} \mid n > 1\}$ is not regular.

Solution

Step 1 Let us assume that $L = \{a^{n^2} \mid n > 1\}$ is regular. If the language $L = \{a^{n^2} \mid n > 1\}$ is regular then by mathematical induction $L = \{a^{(m-1)^2} \mid m > 0\}$ is also regular. Let us assume $n = m + 1$

Step 2 Let $z = a^{(m+1)^2}$ then $|z| = (m + 1)^2$

By pumping lemma we write

$z = uvw$ such that $|uv| \leq (m + 1)^2$ and $|v| \geq 1$.

Step 3 As the string $a^{(m+1)^2}$ contains occurrences of a's we have

$$z = a^{(m+1)^2} = a^{m^2+2m+1} = a^{m^2}\, a^{2m}\, a$$

Let $u = a^{m^2}$, $v = a^{2m}$, and $w = a$, then by pumping lemma, we have

$$z = a^{m^2}\, (a^{2m})^i\, a$$

for $i = 0$

$$z = a^{m^2}\, a = a^{m^2+1}$$

which is a contradiction, because $m^2 + 1$ can never be complete square for $m > 0$. In other words

$$m^2 < m^2 + 1 < (m + 1)^2$$

Therefore, L is not regular.

Example 5.5 By using pumping lemma prove that the language $L = \{a^n b^k \mid n > k \geq 0\}$.

Solution

Step 1 Let us assume that $L = \{a^n b^k \mid n > k \geq 0\}$ is regular.

Step 2 By pumping lemma we write

$z = a^n b^k$

$z = uvw$ such that $|uv| \leq n + k$, and $|v| \geq 1$.

Step 3 The part 'v' may contain

(i) only a's such that $v = a^{n-k}$ with $n - k \neq 0$.

(ii) Both a's and b's such that $v = a^m b^l$, with $m, l \neq 0$.

Case (i) By pumping lemma we write

$$z = a^n b^k$$
$$= a^k a^{n-k} b^k$$

Here we assume $u = a^k$, $v = a^{n-k}$, and $w = b^k$, then by pumping lemma, we have

$$z = a^k (a^{n-k})^i b^k$$

for $i = 0$

$$z = a^k b^k$$

which is a contradiction because $a^k b^k \notin L$ for any k. In $a^k b^k$ the number of a's and b's are same.

Case (ii) By pumping lemma we write

$$z = a^n b^k$$
$$= a^{n-m} a^m b^l b^{k-l}$$

Here we assume $u = a^{n-m}$, $v = a^m b^l$, and $w = b^{k-l}$, then by pumping lemma, we have

$$z = a^{n-m}(a^m b^l)^i b^{k-l}$$

For $i = 2$ we get

$$z = a^{n-m}(a^m b^l)^2 b^{k-l}$$
$$= a^{n-m}(a^m b^l)(a^m b^l) b^{k-l}$$
$$= a^n b^l a^m b^k$$

Which is a contradiction because $a^n b^l a^m b^k \notin L$. In $a^n b^l a^m b^k$, some occurrences of b's are followed by a's, which is not the property of language L.