Integration

Exercise Set 5.1

1. Endpoints $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1$; using right endpoints,

$$A_n = \left[\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n-1}{n}} + 1\right]\frac{1}{n}.$$

n	2	5	10	50	100
A_n	0.853553	0.749739	0.710509	0.676095	0.671463

3. Endpoints $0, \frac{\pi}{n}, \frac{2\pi}{n}, \ldots, \frac{(n-1)\pi}{n}, \pi$; using right endpoints,

$$A_n = [\sin(\pi/n) + \sin(2\pi/n) + \cdots + \sin(\pi(n-1)/n) + \sin\pi]\,\frac{\pi}{n}.$$

n	2	5	10	50	100
A_n	1.57080	1.93376	1.98352	1.99935	1.99984

5. Endpoints $1, \frac{n+1}{n}, \frac{n+2}{n}, \ldots, \frac{2n-1}{n}, 2$; using right endpoints,

$$A_n = \left[\frac{n}{n+1} + \frac{n}{n+2} + \cdots + \frac{n}{2n-1} + \frac{1}{2}\right]\frac{1}{n}.$$

n	2	5	10	50	100
A_n	0.583333	0.645635	0.668771	0.688172	0.690653

7. Endpoints $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1$; using right endpoints,

$$A_n = \left[\sqrt{1-\left(\frac{1}{n}\right)^2} + \sqrt{1-\left(\frac{2}{n}\right)^2} + \cdots + \sqrt{1-\left(\frac{n-1}{n}\right)^2} + 0\right]\frac{1}{n}.$$

n	2	5	10	50	100
A_n	0.433013	0.659262	0.726130	0.774567	0.780106

9. Endpoints $-1, -1+\frac{2}{n}, -1+\frac{4}{n}, \ldots, 1-\frac{2}{n}, 1$; using right endpoints,

$$A_n = \left[e^{-1+\frac{2}{n}} + e^{-1+\frac{4}{n}} + e^{-1+\frac{6}{n}} + \ldots + e^{1-\frac{2}{n}} + e^{1}\right]\frac{2}{n}.$$

n	2	5	10	50	100
A_n	3.718281	2.851738	2.59327	2.39772	2.37398

11. Endpoints $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1$; using right endpoints,

$$A_n = \left[\sin^{-1}\left(\frac{1}{n}\right) + \sin^{-1}\left(\frac{2}{n}\right) + \ldots + \sin^{-1}\left(\frac{n-1}{n}\right) + \sin^{-1}(1)\right]\frac{1}{n}.$$

n	2	5	10	50	100
A_n	1.04729	0.75089	0.65781	0.58730	0.57894

13. $3(x-1)$.

15. $x(x+2)$.

17. $(x+3)(x-1)$.

19. False; the area is 4π.

21. True.

23. $A(6)$ represents the area between $x=0$ and $x=6$; $A(3)$ represents the area between $x=0$ and $x=3$; their difference $A(6)-A(3)$ represents the area between $x=3$ and $x=6$, and $A(6)-A(3)=\dfrac{1}{3}(6^3-3^3)=63$.

25. B is also the area between the graph of $f(x)=\sqrt{x}$ and the interval $[0,1]$ on the $y-$axis, so $A+B$ is the area of the square.

27. The area which is under the curve lies to the right of $x=2$ (or to the left of $x=-2$). Hence $f(x)=A'(x)=2x$; $0=A(a)=a^2-4$, so take $a=2$.

Exercise Set 5.2

1. **(a)** $\displaystyle\int \frac{x}{\sqrt{1+x^2}}dx=\sqrt{1+x^2}+C.$ **(b)** $\displaystyle\int (x+1)e^x dx = xe^x+C.$

5. $\displaystyle\frac{d}{dx}\left[\sqrt{x^3+5}\right]=\frac{3x^2}{2\sqrt{x^3+5}}$, so $\displaystyle\int\frac{3x^2}{2\sqrt{x^3+5}}dx=\sqrt{x^3+5}+C.$

7. $\displaystyle\frac{d}{dx}\left[\sin\left(2\sqrt{x}\right)\right]=\frac{\cos\left(2\sqrt{x}\right)}{\sqrt{x}}$, so $\displaystyle\int\frac{\cos\left(2\sqrt{x}\right)}{\sqrt{x}}dx=\sin\left(2\sqrt{x}\right)+C.$

9. **(a)** $x^9/9+C.$ **(b)** $\dfrac{7}{12}x^{12/7}+C.$ **(c)** $\dfrac{2}{9}x^{9/2}+C.$

11. $\displaystyle\int\left[5x+\frac{2}{3x^5}\right]dx=\int 5x\,dx+\frac{2}{3}\int\frac{1}{x^5}\,dx=\frac{5}{2}x^2+\frac{2}{3}\left(\frac{-1}{4}\right)\frac{1}{x^4}C=\frac{5}{2}x^2-\frac{1}{6x^4}+C.$

13. $\displaystyle\int\left[x^{-3}-3x^{1/4}+8x^2\right]dx=\int x^{-3}\,dx-3\int x^{1/4}\,dx+8\int x^2\,dx=-\frac{1}{2}x^{-2}-\frac{12}{5}x^{5/4}+\frac{8}{3}x^3+C.$

15. $\displaystyle\int(x+x^4)dx=x^2/2+x^5/5+C.$

17. $\displaystyle\int x^{1/3}(4-4x+x^2)dx=\int(4x^{1/3}-4x^{4/3}+x^{7/3})dx=3x^{4/3}-\frac{12}{7}x^{7/3}+\frac{3}{10}x^{10/3}+C.$

19. $\displaystyle\int(x+2x^{-2}-x^{-4})dx=x^2/2-2/x+1/(3x^3)+C.$

21. $\displaystyle\int\left[\frac{2}{x}+3e^x\right]dx=2\ln|x|+3e^x+C.$

23. $\displaystyle\int [3\sin x - 2\sec^2 x]\, dx = -3\cos x - 2\tan x + C.$

25. $\displaystyle\int (\sec^2 x + \sec x \tan x) dx = \tan x + \sec x + C.$

27. $\displaystyle\int \frac{\sec\theta}{\cos\theta}\, d\theta = \int \sec^2\theta\, d\theta = \tan\theta + C.$

29. $\displaystyle\int \sec x \tan x\, dx = \sec x + C.$

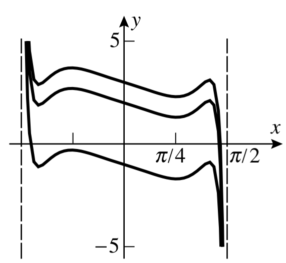
31. $\displaystyle\int (1 + \sin\theta) d\theta = \theta - \cos\theta + C.$

33. $\displaystyle\int \left[\frac{1}{2\sqrt{1-x^2}} - \frac{3}{1+x^2}\right] dx = \frac{1}{2}\sin^{-1} x - 3\tan^{-1} x + C.$

35. $\displaystyle\int \frac{1-\sin x}{1-\sin^2 x} dx = \int \frac{1-\sin x}{\cos^2 x}\, dx = \int \left(\sec^2 x - \sec x \tan x\right) dx = \tan x - \sec x + C.$

37. True.

39. False; $y(0) = 2$.

41. 

43. **(a)** $\displaystyle y(x) = \int x^{1/3} dx = \frac{3}{4}x^{4/3} + C,\ y(1) = \frac{3}{4} + C = 2, C = \frac{5}{4};\ y(x) = \frac{3}{4}x^{4/3} + \frac{5}{4}.$

(b) $\displaystyle y(t) = \int (\sin t + 1)\, dt = -\cos t + t + C,\ y\left(\frac{\pi}{3}\right) = -\frac{1}{2} + \frac{\pi}{3} + C = 1/2,\ C = 1 - \frac{\pi}{3};\ y(t) = -\cos t + t + 1 - \frac{\pi}{3}.$

(c) $\displaystyle y(x) = \int (x^{1/2} + x^{-1/2}) dx = \frac{2}{3}x^{3/2} + 2x^{1/2} + C,\ y(1) = 0 = \frac{8}{3} + C,\ C = -\frac{8}{3},\ y(x) = \frac{2}{3}x^{3/2} + 2x^{1/2} - \frac{8}{3}.$

45. **(a)** $\displaystyle y = \int 4e^x\, dx = 4e^x + C, 1 = y(0) = 4 + C, C = -3, y = 4e^x - 3.$

(b) $\displaystyle y(t) = \int t^{-1} dt = \ln|t| + C,\ y(-1) = C = 5,\ C = 5;\ y(t) = \ln|t| + 5.$

47. $s(t) = 16t^2 + C; s(t) = 16t^2 + 20.$

49. $s(t) = 2t^{3/2} + C; s(t) = 2t^{3/2} - 15.$

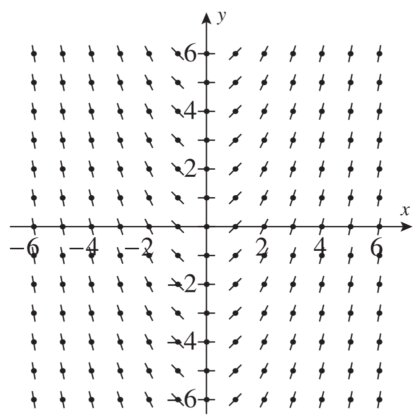
51. $\displaystyle f'(x) = \frac{2}{3}x^{3/2} + C_1;\ f(x) = \frac{4}{15}x^{5/2} + C_1 x + C_2.$

53. $dy/dx = 2x+1, y = \int(2x+1)dx = x^2+x+C$; $y=0$ when $x=-3$, so $(-3)^2+(-3)+C=0$, $C=-6$ thus $y = x^2+x-6$.

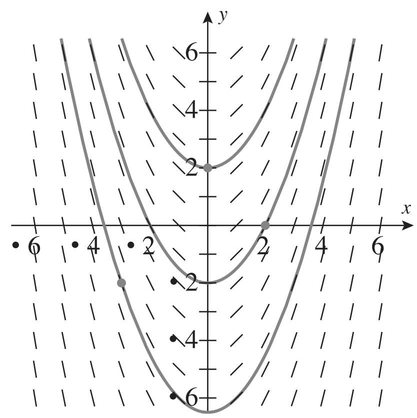
55. $f'(x) = m = -\sin x$, so $f(x) = \int(-\sin x)dx = \cos x + C$; $f(0) = 2 = 1 + C$, so $C = 1$, $f(x) = \cos x + 1$.

57. $dy/dx = \int 6x dx = 3x^2 + C_1$. The slope of the tangent line is -3 so $dy/dx = -3$ when $x = 1$. Thus $3(1)^2 + C_1 = -3$, $C_1 = -6$ so $dy/dx = 3x^2 - 6$, $y = \int(3x^2-6)dx = x^3 - 6x + C_2$. If $x = 1$, then $y = 5 - 3(1) = 2$ so $(1)^2 - 6(1) + C_2 = 2, C_2 = 7$ thus $y = x^3 - 6x + 7$.

59. (a)

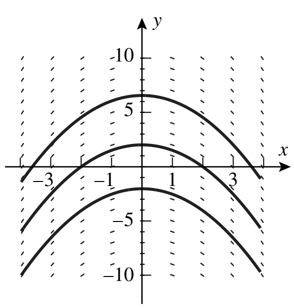


(b)

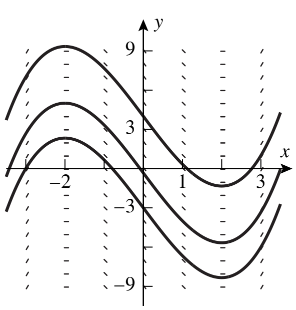


(c) $f(x) = x^2/2 - 1$.

61. This slope field is zero along the y-axis, and so corresponds to (b).



63. This slope field has a negative value along the y-axis, and thus corresponds to (c).



65. (a) $F'(x) = \dfrac{1}{1+x^2}, G'(x) = +\left(\dfrac{1}{x^2}\right)\dfrac{1}{1+1/x^2} = \dfrac{1}{1+x^2} = F'(x)$.

(b) $F(1) = \pi/4; G(1) = -\tan^{-1}(1) = -\pi/4, \tan^{-1} x + \tan^{-1}(1/x) = \pi/2.$

(c) Draw a triangle with sides 1 and x and hypotenuse $\sqrt{1+x^2}$. If α denotes the angle opposite the side of length x and if β denotes its complement, then $\tan\alpha = x$ and $\tan\beta = 1/x$, and $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha = \dfrac{x^2}{1+x^2} + \dfrac{1}{1+x^2} = 1$, and $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \dfrac{x\cdot 1}{1+x^2} - \dfrac{1\cdot x}{1+x^2} = 0$, so the cosine of $\alpha+\beta$ is zero and the sine of $\alpha+\beta$ is 1; consequently $\alpha+\beta = \pi/2$, i.e. $\tan^{-1} x + \tan^{-1}(1/x) = \pi/2$.

67. $\displaystyle\int (\sec^2 x - 1)dx = \tan x - x + C.$

69. (a) $\displaystyle\frac{1}{2}\int (1-\cos x)dx = \frac{1}{2}(x - \sin x) + C.$ **(b)** $\displaystyle\frac{1}{2}\int (1+\cos x)\,dx = \frac{1}{2}(x+\sin x) + C.$

71. $v = \dfrac{1087}{2\sqrt{273}}\displaystyle\int T^{-1/2}\,dT = \dfrac{1087}{\sqrt{273}}T^{1/2} + C$, $v(273) = 1087 = 1087 + C$ so $C = 0$, $v = \dfrac{1087}{\sqrt{273}}T^{1/2}$ ft/s.

73. $dT/dx = C_1$, $T = C_1x + C_2$; $T = 25$ when $x = 0$, so $C_2 = 25$, $T = C_1x + 25$. $T = 85$ when $x = 50$, so $50C_1 + 25 = 85$, $C_1 = 1.2$, $T = 1.2x + 25$.

Exercise Set 5.3

1. (a) $\displaystyle\int u^{23}du = u^{24}/24 + C = (x^2+1)^{24}/24 + C.$

(b) $\displaystyle -\int u^3du = -u^4/4 + C = -(\cos^4 x)/4 + C.$

3. (a) $\displaystyle\frac{1}{4}\int \sec^2 u\,du = \frac{1}{4}\tan u + C = \frac{1}{4}\tan(4x+1) + C.$

(b) $\displaystyle\frac{1}{4}\int u^{1/2}du = \frac{1}{6}u^{3/2} + C = \frac{1}{6}(1+2y^2)^{3/2} + C.$

5. (a) $\displaystyle -\int u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\cot^2 x + C.$

(b) $\displaystyle\int u^9du = \frac{1}{10}u^{10} + C = \frac{1}{10}(1+\sin t)^{10} + C.$

7. (a) $\displaystyle\int (u-1)^2u^{1/2}du = \int (u^{5/2} - 2u^{3/2} + u^{1/2})du = \frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C = \frac{2}{7}(1+x)^{7/2} - \frac{4}{5}(1+x)^{5/2} + \frac{2}{3}(1+x)^{3/2} + C.$

(b) $\displaystyle\int \csc^2 u\,du = -\cot u + C = -\cot(\sin x) + C.$

9. (a) $\displaystyle\int \frac{1}{u}du = \ln|u| + C = \ln|\ln x| + C.$

(b) $\displaystyle -\frac{1}{5}\int e^u\,du = -\frac{1}{5}e^u + C = -\frac{1}{5}e^{-5x} + C.$

11. (a) $u = x^3$, $\displaystyle\frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1}(x^3) + C.$

(b) $u = \ln x$, $\int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1}(\ln x) + C$.

15. $u = 4x - 3$, $\frac{1}{4} \int u^9 \, du = \frac{1}{40} u^{10} + C = \frac{1}{40}(4x-3)^{10} + C$.

17. $u = 7x$, $\frac{1}{7} \int \sin u \, du = -\frac{1}{7} \cos u + C = -\frac{1}{7} \cos 7x + C$.

19. $u = 4x$, $du = 4dx$; $\frac{1}{4} \int \sec u \tan u \, du = \frac{1}{4} \sec u + C = \frac{1}{4} \sec 4x + C$.

21. $u = 2x$, $du = 2dx$; $\frac{1}{2} \int e^u \, du = \frac{1}{2} e^u + C = \frac{1}{2} e^{2x} + C$.

23. $u = 2x$, $\frac{1}{2} \int \frac{1}{\sqrt{1-u^2}} du = \frac{1}{2} \sin^{-1}(2x) + C$.

25. $u = 7t^2 + 12$, $du = 14t \, dt$; $\frac{1}{14} \int u^{1/2} du = \frac{1}{21} u^{3/2} + C = \frac{1}{21}(7t^2+12)^{3/2} + C$.

27. $u = 1 - 2x$, $du = -2dx$, $-3 \int \frac{1}{u^3} \, du = (-3)\left(-\frac{1}{2}\right) \frac{1}{u^2} + C = \frac{3}{2} \frac{1}{(1-2x)^2} + C$.

29. $u = 5x^4 + 2$, $du = 20x^3 \, dx$, $\frac{1}{20} \int \frac{du}{u^3} \, du = -\frac{1}{40} \frac{1}{u^2} + C = -\frac{1}{40(5x^4+2)^2} + C$.

31. $u = \sin x$, $du = \cos x \, dx$; $\int e^u \, du = e^u + C = e^{\sin x} + C$.

33. $u = -2x^3$, $du = -6x^2$, $-\frac{1}{6} \int e^u du = -\frac{1}{6} e^u + C = -\frac{1}{6} e^{-2x^3} + C$.

35. $u = e^x$, $\int \frac{1}{1+u^2} du = \tan^{-1}(e^x) + C$.

37. $u = 5/x$, $du = -(5/x^2) dx$; $-\frac{1}{5} \int \sin u \, du = \frac{1}{5} \cos u + C = \frac{1}{5} \cos(5/x) + C$.

39. $u = \cos 3t, du = -3 \sin 3t \, dt$, $-\frac{1}{3} \int u^4 \, du = -\frac{1}{15} u^5 + C = -\frac{1}{15} \cos^5 3t + C$.

41. $u = x^2$, $du = 2x \, dx$; $\frac{1}{2} \int \sec^2 u \, du = \frac{1}{2} \tan u + C = \frac{1}{2} \tan\left(x^2\right) + C$.

43. $u = 2 - \sin 4\theta$, $du = -4 \cos 4\theta \, d\theta$; $-\frac{1}{4} \int u^{1/2} du = -\frac{1}{6} u^{3/2} + C = -\frac{1}{6}(2 - \sin 4\theta)^{3/2} + C$.

45. $u = \tan x$, $\int \frac{1}{\sqrt{1-u^2}} \, du = \sin^{-1}(\tan x) + C$.

47. $u = \sec 2x$, $du = 2 \sec 2x \tan 2x \, dx$; $\frac{1}{2} \int u^2 du = \frac{1}{6} u^3 + C = \frac{1}{6} \sec^3 2x + C$.

49. $\int e^{-x} dx$; $u = -x$, $du = -dx$; $-\int e^u du = -e^u + C = -e^{-x} + C$.

51. $u = 2\sqrt{x},\ du = \dfrac{1}{\sqrt{x}}\,dx;,\ \displaystyle\int \frac{1}{e^u}\,du = -e^{-u} + C = -e^{-2\sqrt{x}} + C.$

53. $u = 2y+1, du = 2dy; \displaystyle\int \frac{1}{4}(u-1)\frac{1}{\sqrt{u}}\,du = \frac{1}{6}u^{3/2} - \frac{1}{2}\sqrt{u} + C = \frac{1}{6}(2y+1)^{3/2} - \frac{1}{2}\sqrt{2y+1} + C.$

55. $\displaystyle\int \sin^2 2\theta \sin 2\theta\,d\theta = \int (1-\cos^2 2\theta)\sin 2\theta\,d\theta;\ u = \cos 2\theta,\ du = -2\sin 2\theta\,d\theta,\ -\frac{1}{2}\int (1-u^2)du = -\frac{1}{2}u + \frac{1}{6}u^3 + C = -\frac{1}{2}\cos 2\theta + \frac{1}{6}\cos^3 2\theta + C.$

57. $\displaystyle\int \left(1 + \frac{1}{t}\right)dt = t + \ln|t| + C.$

59. $\ln(e^x) + \ln(e^{-x}) = \ln(e^x e^{-x}) = \ln 1 = 0$, so $\displaystyle\int [\ln(e^x) + \ln(e^{-x})]dx = C.$

61. **(a)** $\sin^{-1}(x/3) + C.$ **(b)** $(1/\sqrt{5})\tan^{-1}(x/\sqrt{5}) + C.$ **(c)** $(1/\sqrt{\pi})\sec^{-1}|x/\sqrt{\pi}| + C$.

63. $u = a+bx, du = b\,dx, \displaystyle\int (a+bx)^n\,dx = \frac{1}{b}\int u^n du = \frac{(a+bx)^{n+1}}{b(n+1)} + C.$

65. $u = \sin(a+bx),\ du = b\cos(a+bx)dx,\ \displaystyle\frac{1}{b}\int u^n du = \frac{1}{b(n+1)}u^{n+1} + C = \frac{1}{b(n+1)}\sin^{n+1}(a+bx) + C.$

67. **(a)** With $u = \sin x,\ du = \cos x\,dx;\ \displaystyle\int u\,du = \frac{1}{2}u^2 + C_1 = \frac{1}{2}\sin^2 x + C_1;$

with $u = \cos x,\ du = -\sin x\,dx;\ -\displaystyle\int u\,du = -\frac{1}{2}u^2 + C_2 = -\frac{1}{2}\cos^2 x + C_2.$

(b) Because they differ by a constant:

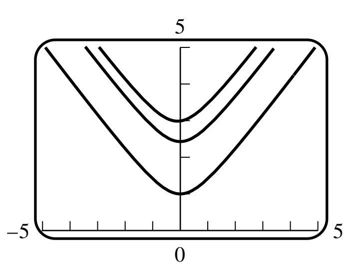
$\left(\dfrac{1}{2}\sin^2 x + C_1\right) - \left(-\dfrac{1}{2}\cos^2 x + C_2\right) = \dfrac{1}{2}(\sin^2 x + \cos^2 x) + C_1 - C_2 = 1/2 + C_1 - C_2.$

69. $y = \displaystyle\int \sqrt{5x+1}\,dx = \frac{2}{15}(5x+1)^{3/2} + C; -2 = y(3) = \frac{2}{15}64 + C$, so $C = -2 - \dfrac{2}{15}64 = -\dfrac{158}{15}$, and $y = \dfrac{2}{15}(5x+1)^{3/2} - \dfrac{158}{15}.$

71. $y = -\displaystyle\int e^{2t}\,dt = -\frac{1}{2}e^{2t} + C, 6 = y(0) = -\frac{1}{2} + C, y = -\frac{1}{2}e^{2t} + \frac{13}{2}.$

73. **(a)** $u = x^2+1, du = 2x\,dx; \displaystyle\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \sqrt{u} + C = \sqrt{x^2+1} + C.$

(b)



75. $f'(x) = m = \sqrt{3x+1}$, $f(x) = \int (3x+1)^{1/2} dx = \frac{2}{9}(3x+1)^{3/2} + C$, $f(0) = 1 = \frac{2}{9} + C$, $C = \frac{7}{9}$, so $f(x) = \frac{2}{9}(3x+1)^{3/2} + \frac{7}{9}$.

77. $y(t) = \int (\ln 2)\, 2^{t/20}\, dt = 20 \cdot 2^{t/20} + C$; $20 = y(0) = 20 + C$, so $C = 0$ and $y(t) = 20 \cdot 2^{t/20}$. This implies that $y(120) = 20 \cdot 2^{120/20} = 1280$ cells.

79. If $u > 0$ then $u = a\sec\theta, du = a\sec\theta\tan\theta\, d\theta$; $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\theta = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$.

Exercise Set 5.4

1. **(a)** $1 + 8 + 27 = 36$. **(b)** $5 + 8 + 11 + 14 + 17 = 55$. **(c)** $20 + 12 + 6 + 2 + 0 + 0 = 40$

(d) $1 + 1 + 1 + 1 + 1 + 1 = 6$. **(e)** $1 - 2 + 4 - 8 + 16 = 11$. **(f)** $0 + 0 + 0 + 0 + 0 + 0 = 0$.

3. $\sum_{k=1}^{10} k$

5. $\sum_{k=1}^{10} 2k$

7. $\sum_{k=1}^{6} (-1)^{k+1}(2k-1)$

9. **(a)** $\sum_{k=1}^{50} 2k$ **(b)** $\sum_{k=1}^{50} (2k-1)$

11. $\frac{1}{2}(100)(100+1) = 5050$.

13. $\frac{1}{6}(20)(21)(41) = 2870$.

15. $\sum_{k=1}^{30} k(k^2-4) = \sum_{k=1}^{30}(k^3-4k) = \sum_{k=1}^{30} k^3 - 4\sum_{k=1}^{30} k = \frac{1}{4}(30)^2(31)^2 - 4 \cdot \frac{1}{2}(30)(31) = 214{,}365$.

17. $\sum_{k=1}^{n} \frac{3k}{n} = \frac{3}{n}\sum_{k=1}^{n} k = \frac{3}{n} \cdot \frac{1}{2}n(n+1) = \frac{3}{2}(n+1)$.

19. $\sum_{k=1}^{n-1} \frac{k^3}{n^2} = \frac{1}{n^2}\sum_{k=1}^{n-1} k^3 = \frac{1}{n^2} \cdot \frac{1}{4}(n-1)^2 n^2 = \frac{1}{4}(n-1)^2$.

21. True.

23. False; if $[a, b]$ consists of positive reals, true; but false on, e.g. $[-2, 1]$.

25. **(a)** $\left(2+\frac{3}{n}\right)^4 \frac{3}{n}, \left(2+\frac{6}{n}\right)^4 \frac{3}{n}, \left(2+\frac{9}{n}\right)^4 \frac{3}{n}, \ldots, \left(2+\frac{3(n-1)}{n}\right)^4 \frac{3}{n}, (2+3)^4\frac{3}{n}$. When $[2, 5]$ is subdivided into n

equal intervals, the endpoints are $2, 2+\dfrac{3}{n}, 2+2\cdot\dfrac{3}{n}, 2+3\cdot\dfrac{3}{n}, \ldots, 2+(n-1)\dfrac{3}{n}, 2+3=5$, and the right endpoint approximation to the area under the curve $y=x^4$ is given by the summands above.

(b) $\displaystyle\sum_{k=0}^{n-1}\left(2+k\cdot\frac{3}{n}\right)^4\frac{3}{n}$ gives the left endpoint approximation.

27. Endpoints $2, 3, 4, 5, 6; \Delta x = 1$;

(a) Left endpoints: $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = 7+10+13+16 = 46.$

(b) Midpoints: $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = 8.5+11.5+14.5+17.5 = 52.$

(c) Right endpoints: $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = 10+13+16+19 = 58.$

29. Endpoints: $0, \pi/4, \pi/2, 3\pi/4, \pi; \Delta x = \pi/4$.

(a) Left endpoints: $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = \left(1+\sqrt{2}/2+0-\sqrt{2}/2\right)(\pi/4) = \pi/4.$

(b) Midpoints: $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = [\cos(\pi/8)+\cos(3\pi/8)+\cos(5\pi/8)+\cos(7\pi/8)](\pi/4) =$

$= [\cos(\pi/8)+\cos(3\pi/8)-\cos(3\pi/8)-\cos(\pi/8)](\pi/4) = 0.$

(c) Right endpoints: $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = \left(\sqrt{2}/2+0-\sqrt{2}/2-1\right)(\pi/4) = -\pi/4.$

31. (a) 0.718771403, 0.705803382, 0.698172179.

(b) 0.692835360, 0.693069098, 0.693134682.

(c) 0.668771403, 0.680803382, 0.688172179.

33. (a) 4.884074734, 5.115572731, 5.248762738.

(b) 5.34707029, 5.338362719, 5.334644416.

(c) 5.684074734, 5.515572731, 5.408762738.

35. $\Delta x = \dfrac{3}{n}$, $x_k^* = 1+\dfrac{3}{n}k$; $f(x_k^*)\Delta x = \dfrac{1}{2}x_k^*\Delta x = \dfrac{1}{2}\left(1+\dfrac{3}{n}k\right)\dfrac{3}{n} = \dfrac{3}{2}\left[\dfrac{1}{n}+\dfrac{3}{n^2}k\right]$,

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{3}{2}\left[\sum_{k=1}^{n}\frac{1}{n}+\sum_{k=1}^{n}\frac{3}{n^2}k\right] = \frac{3}{2}\left[1+\frac{3}{n^2}\cdot\frac{1}{2}n(n+1)\right] = \frac{3}{2}\left[1+\frac{3}{2}\frac{n+1}{n}\right],$$

$$A = \lim_{n\to+\infty}\frac{3}{2}\left[1+\frac{3}{2}\left(1+\frac{1}{n}\right)\right] = \frac{3}{2}\left(1+\frac{3}{2}\right) = \frac{15}{4}.$$

37. $\Delta x = \dfrac{3}{n}, x_k^* = 0 + k\dfrac{3}{n}; f(x_k^*)\Delta x = \left(9 - 9\dfrac{k^2}{n^2}\right)\dfrac{3}{n},$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left(9 - 9\frac{k^2}{n^2}\right)\frac{3}{n} = \frac{27}{n}\sum_{k=1}^{n}\left(1 - \frac{k^2}{n^2}\right) = 27 - \frac{27}{n^3}\sum_{k=1}^{n} k^2,$$

$$A = \lim_{n\to+\infty}\left[27 - \frac{27}{n^3}\sum_{k=1}^{n} k^2\right] = 27 - 27\left(\frac{1}{3}\right) = 18.$$

39. $\Delta x = \dfrac{4}{n},\ x_k^* = 2 + k\dfrac{4}{n};\ f(x_k^*)\Delta x = (x_k^*)^3\Delta x = \left[2 + \dfrac{4}{n}k\right]^3\dfrac{4}{n} = \dfrac{32}{n}\left[1 + \dfrac{2}{n}k\right]^3 = \dfrac{32}{n}\left[1 + \dfrac{6}{n}k + \dfrac{12}{n^2}k^2 + \dfrac{8}{n^3}k^3\right],$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{32}{n}\left[\sum_{k=1}^{n} 1 + \frac{6}{n}\sum_{k=1}^{n} k + \frac{12}{n^2}\sum_{k=1}^{n} k^2 + \frac{8}{n^3}\sum_{k=1}^{n} k^3\right] =$$

$$= \frac{32}{n}\left[n + \frac{6}{n}\cdot\frac{1}{2}n(n+1) + \frac{12}{n^2}\cdot\frac{1}{6}n(n+1)(2n+1) + \frac{8}{n^3}\cdot\frac{1}{4}n^2(n+1)^2\right] =$$

$$= 32\left[1 + 3\frac{n+1}{n} + 2\frac{(n+1)(2n+1)}{n^2} + 2\frac{(n+1)^2}{n^2}\right],$$

$$A = \lim_{n\to+\infty} 32\left[1 + 3\left(1 + \frac{1}{n}\right) + 2\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + 2\left(1 + \frac{1}{n}\right)^2\right] = 32[1 + 3(1) + 2(1)(2) + 2(1)^2] = 320.$$

41. $\Delta x = \dfrac{3}{n},\ x_k^* = 1 + (k-1)\dfrac{3}{n};\ f(x_k^*)\Delta x = \dfrac{1}{2}x_k^*\Delta x = \dfrac{1}{2}\left[1 + (k-1)\dfrac{3}{n}\right]\dfrac{3}{n} = \dfrac{1}{2}\left[\dfrac{3}{n} + (k-1)\dfrac{9}{n^2}\right],$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{1}{2}\left[\sum_{k=1}^{n}\frac{3}{n} + \frac{9}{n^2}\sum_{k=1}^{n}(k-1)\right] = \frac{1}{2}\left[3 + \frac{9}{n^2}\cdot\frac{1}{2}(n-1)n\right] = \frac{3}{2} + \frac{9}{4}\frac{n-1}{n},$$

$$A = \lim_{n\to+\infty}\left[\frac{3}{2} + \frac{9}{4}\left(1 - \frac{1}{n}\right)\right] = \frac{3}{2} + \frac{9}{4} = \frac{15}{4}.$$

43. $\Delta x = \dfrac{3}{n}, x_k^* = 0 + (k-1)\dfrac{3}{n}; f(x_k^*)\Delta x = \left[9 - 9\dfrac{(k-1)^2}{n^2}\right]\dfrac{3}{n},$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left[9 - 9\frac{(k-1)^2}{n^2}\right]\frac{3}{n} = \frac{27}{n}\sum_{k=1}^{n}\left(1 - \frac{(k-1)^2}{n^2}\right) = 27 - \frac{27}{n^3}\sum_{k=1}^{n} k^2 + \frac{54}{n^3}\sum_{k=1}^{n} k - \frac{27}{n^2},$$

$$A = \lim_{n\to+\infty} = 27 - 27\left(\frac{1}{3}\right) + 0 + 0 = 18.$$

45. Endpoints $0, \dfrac{4}{n}, \dfrac{8}{n}, \ldots, \dfrac{4(n-1)}{n}, \dfrac{4n}{n} = 4$, and midpoints $\dfrac{2}{n}, \dfrac{6}{n}, \dfrac{10}{n}, \ldots, \dfrac{4n-6}{n}, \dfrac{4n-2}{n}$. Approximate the area with the sum $\displaystyle\sum_{k=1}^{n} 2\left(\frac{4k-2}{n}\right)\frac{4}{n} = \frac{16}{n^2}\left[2\frac{n(n+1)}{2} - n\right] \to 16$ (exact) as $n \to +\infty$.

47. $\Delta x = \dfrac{1}{n}, x_k^* = \dfrac{2k-1}{2n}; f(x_k^*)\Delta x = \dfrac{(2k-1)^2}{(2n)^2}\dfrac{1}{n} = \dfrac{k^2}{n^3} - \dfrac{k}{n^3} + \dfrac{1}{4n^3}, \displaystyle\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{1}{n^3}\sum_{k=1}^{n} k^2 - \frac{1}{n^3}\sum_{k=1}^{n} k + \frac{1}{4n^3}\sum_{k=1}^{n} 1.$ Using Theorem 5.4.4, $A = \displaystyle\lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{1}{3} + 0 + 0 = \frac{1}{3}.$

49. $\Delta x = \dfrac{2}{n}, x_k^* = -1 + \dfrac{2k}{n};\ f(x_k^*)\Delta x = \left(-1 + \dfrac{2k}{n}\right)\dfrac{2}{n} = -\dfrac{2}{n} + 4\dfrac{k}{n^2},\ \displaystyle\sum_{k=1}^{n} f(x_k^*)\Delta x = -2 + \frac{4}{n^2}\sum_{k=1}^{n} k = -2 + \frac{4}{n^2}\frac{n(n+1)}{2} = -2 + 2 + \frac{2}{n},\ A = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = 0.$

The area below the x-axis cancels the area above the x-axis.

51. $\Delta x = \dfrac{2}{n}, x_k^* = \dfrac{2k}{n}; f(x_k^*) = \left[\left(\dfrac{2k}{n}\right)^2 - 1\right]\dfrac{2}{n} = \dfrac{8k^2}{n^3} - \dfrac{2}{n}, \displaystyle\sum_{k=1}^{n} f(x_k^*)\Delta x = \dfrac{8}{n^3}\sum_{k=1}^{n} k^2 - \dfrac{2}{n}\sum_{k=1}^{n} 1 = \dfrac{8}{n^3}\dfrac{n(n+1)(2n+1)}{6} - 2$, $A = \displaystyle\lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \dfrac{16}{6} - 2 = \dfrac{2}{3}$.

53. (a) With x_k^* as the right endpoint, $\Delta x = \dfrac{b}{n}$, $x_k^* = \dfrac{b}{n}k$; $f(x_k^*)\Delta x = (x_k^*)^3\Delta x = \dfrac{b^4}{n^4}k^3$, $\displaystyle\sum_{k=1}^{n} f(x_k^*)\Delta x = \dfrac{b^4}{n^4}\sum_{k=1}^{n} k^3 = \dfrac{b^4}{4}\dfrac{(n+1)^2}{n^2}$, $A = \displaystyle\lim_{n\to+\infty}\dfrac{b^4}{4}\left(1+\dfrac{1}{n}\right)^2 = b^4/4$.

(b) First Method (tedious): $\Delta x = \dfrac{b-a}{n}$, $x_k^* = a + \dfrac{b-a}{n}k$; $f(x_k^*)\Delta x = (x_k^*)^3\Delta x = \left[a + \dfrac{b-a}{n}k\right]^3\dfrac{b-a}{n} = \dfrac{b-a}{n}\left[a^3 + \dfrac{3a^2(b-a)}{n}k + \dfrac{3a(b-a)^2}{n^2}k^2 + \dfrac{(b-a)^3}{n^3}k^3\right]$,

$\displaystyle\sum_{k=1}^{n} f(x_k^*)\Delta x = (b-a)\left[a^3 + \dfrac{3}{2}a^2(b-a)\dfrac{n+1}{n} + \dfrac{1}{2}a(b-a)^2\dfrac{(n+1)(2n+1)}{n^2} + \dfrac{1}{4}(b-a)^3\dfrac{(n+1)^2}{n^2}\right]$,

$A = \displaystyle\lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = (b-a)\left[a^3 + \dfrac{3}{2}a^2(b-a) + a(b-a)^2 + \dfrac{1}{4}(b-a)^3\right] = \dfrac{1}{4}(b^4 - a^4)$.

Alternative method: Apply part (a) of the Exercise to the interval $[0,a]$ and observe that the area under the curve and above that interval is given by $\dfrac{1}{4}a^4$. Apply part (a) again, this time to the interval $[0,b]$ and obtain $\dfrac{1}{4}b^4$. Now subtract to obtain the correct area and the formula $A = \dfrac{1}{4}(b^4 - a^4)$.

55. If $n = 2m$ then $2m + 2(m-1) + \cdots + 2\cdot 2 + 2 = 2\displaystyle\sum_{k=1}^{m} k = 2\cdot\dfrac{m(m+1)}{2} = m(m+1) = \dfrac{n^2+2n}{4}$; if $n = 2m+1$ then $(2m+1) + (2m-1) + \cdots + 5 + 3 + 1 = \displaystyle\sum_{k=1}^{m+1}(2k-1) = 2\sum_{k=1}^{m+1} k - \sum_{k=1}^{m+1} 1 = 2\cdot\dfrac{(m+1)(m+2)}{2} - (m+1) = (m+1)^2 = \dfrac{n^2+2n+1}{4}$.

57. $(3^5 - 3^4) + (3^6 - 3^5) + \cdots + (3^{17} - 3^{16}) = 3^{17} - 3^4$.

59. $\left(\dfrac{1}{2^2} - \dfrac{1}{1^2}\right) + \left(\dfrac{1}{3^2} - \dfrac{1}{2^2}\right) + \cdots + \left(\dfrac{1}{20^2} - \dfrac{1}{19^2}\right) = \dfrac{1}{20^2} - 1 = -\dfrac{399}{400}$.

61. (a) $\displaystyle\sum_{k=1}^{n}\dfrac{1}{(2k-1)(2k+1)} = \dfrac{1}{2}\sum_{k=1}^{n}\left(\dfrac{1}{2k-1} - \dfrac{1}{2k+1}\right) =$

$= \dfrac{1}{2}\left[\left(1 - \dfrac{1}{3}\right) + \left(\dfrac{1}{3} - \dfrac{1}{5}\right) + \left(\dfrac{1}{5} - \dfrac{1}{7}\right) + \cdots + \left(\dfrac{1}{2n-1} - \dfrac{1}{2n+1}\right)\right] = \dfrac{1}{2}\left[1 - \dfrac{1}{2n+1}\right] = \dfrac{n}{2n+1}$.

(b) $\displaystyle\lim_{n\to+\infty}\dfrac{n}{2n+1} = \dfrac{1}{2}$.

63. $\displaystyle\sum_{i=1}^{n}(x_i - \bar{x}) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\bar{x} = \sum_{i=1}^{n} x_i - n\bar{x}$, but $\bar{x} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i$, thus $\displaystyle\sum_{i=1}^{n} x_i = n\bar{x}$, so $\displaystyle\sum_{i=1}^{n}(x_i - \bar{x}) = n\bar{x} - n\bar{x} = 0$.

65. Both are valid.

67. $\sum_{k=1}^{n} ca_k = ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = c\sum_{k=1}^{n} a_k;$

$$\sum_{k=1}^{n}(a_k - b_k) = (a_1 - b_1) + (a_2 - b_2) + \cdots + (a_n - b_n) = (a_1 + a_2 + \cdots + a_n) - (b_1 + b_2 + \cdots + b_n) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k.$$

Exercise Set 5.5

1. **(a)** $(4/3)(1) + (5/2)(1) + (4)(2) = 71/6.$ **(b)** 2.

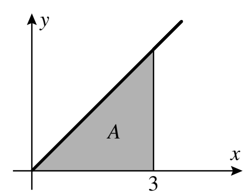
3. **(a)** $(-9/4)(1) + (3)(2) + (63/16)(1) + (-5)(3) = -117/16.$ **(b)** 3.

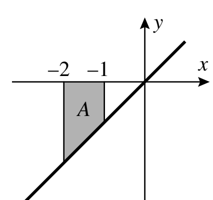
5. $\int_{-1}^{2} x^2\,dx$

7. $\int_{-3}^{3} 4x(1-3x)dx$

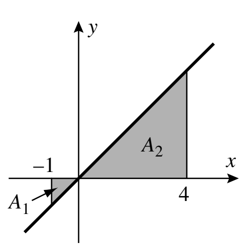
9. **(a)** $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 2x_k^* \Delta x_k;\ a = 1,\ b = 2.$ **(b)** $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \frac{x_k^*}{x_k^* + 1} \Delta x_k;\ a = 0,\ b = 1.$

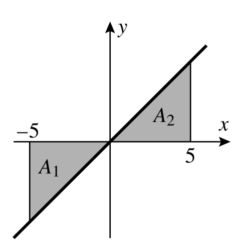
11. Theorem 5.5.4(a) depends on the fact that a constant can move past an integral sign, which by Definition 5.5.1 is possible because a constant can move past a limit and/or a summation sign.

13. **(a)** 

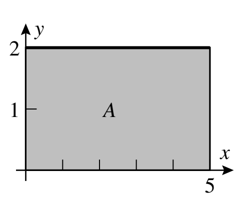
$A = \frac{1}{2}(3)(3) = 9/2.$ **(b)** 

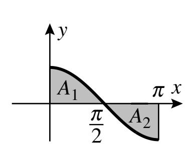
$-A = -\frac{1}{2}(1)(1+2) = -3/2.$

(c) 

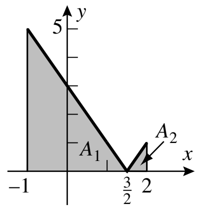
$-A_1 + A_2 = -\frac{1}{2} + 8 = 15/2.$ **(d)** 

$-A_1 + A_2 = 0.$

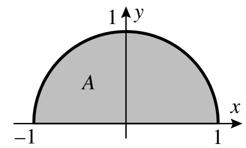
15. **(a)** 

$A = 2(5) = 10.$ **(b)** 

0; $A_1 = A_2$ by symmetry.

(c) 

$$A_1 + A_2 = \frac{1}{2}(5)\frac{5}{2} + \frac{1}{2}(1)\frac{1}{2} = \frac{13}{2}.$$

(d) 

$$A = \frac{1}{2}[\pi(1)^2] = \pi/2.$$

17. **(a)** $\displaystyle\int_{-2}^{0} f(x)\,dx = \int_{-2}^{0}(x+2)\,dx.$

Triangle of height 2 and width 2, above x-axis, so answer is 2.

(b) $\displaystyle\int_{-2}^{2} f(x)\,dx = \int_{-2}^{0}(x+2)\,dx + \int_{2}^{0}(2-x)\,dx.$

Two triangles of height 2 and base 2; answer is 4.

(c) $\displaystyle\int_{0}^{6} |x-2|\,dx = \int_{0}^{2}(2-x)\,dx + \int_{2}^{6}(x-2)\,dx.$

Triangle of height 2 and base 2 together with a triangle of height 4 and base 4, so $2+8=10$.

(d) $\displaystyle\int_{-4}^{6} f(x)\,dx = \int_{-4}^{-2}(x+2)\,dx + \int_{-2}^{0}(x+2)\,dx + \int_{0}^{2}(2-x)\,dx + \int_{2}^{6}(x-2)\,dx.$

Triangle of height 2 and base 2, below axis, plus a triangle of height 2, base 2 above axis, another of height 2 and base 2 above axis, and a triangle of height 4 and base 4, above axis. Thus $\int f(x) = -2+2+2+8 = 10$.

19. **(a)** 0.8 **(b)** -2.6 **(c)** -1.8 **(d)** -0.3

21. $\displaystyle\int_{-1}^{2} f(x)dx + 2\int_{-1}^{2} g(x)dx = 5 + 2(-3) = -1.$

23. $\displaystyle\int_{1}^{5} f(x)dx = \int_{0}^{5} f(x)dx - \int_{0}^{1} f(x)dx = 1 - (-2) = 3.$

25. $\displaystyle 4\int_{-1}^{3} dx - 5\int_{-1}^{3} xdx = 4\cdot 4 - 5(-1/2 + (3\cdot 3)/2) = -4.$

27. $\displaystyle\int_{0}^{1} xdx + 2\int_{0}^{1}\sqrt{1-x^2}dx = 1/2 + 2(\pi/4) = (1+\pi)/2.$

29. False; e.g. $f(x) = 1$ if $x > 0$, $f(x) = 0$ otherwise, then f is integrable on $[-1,1]$ but not continuous.

31. False; e.g. $f(x) = x$ on $[-2,+1]$.

33. **(a)** $\sqrt{x} > 0,\ 1 - x < 0$ on $[2,3]$ so the integral is negative.

(b) $3 - \cos x > 0$ for all x and $x^2 \geq 0$ for all x and $x^2 > 0$ for all $x > 0$ so the integral is positive.

35. If f is continuous on $[a,b]$ then f is integrable on $[a,b]$, and, considering Definition 5.5.1, for every partition and choice of $f(x^*)$ we have $\displaystyle\sum_{k=1}^{n} m\Delta x_k \leq \sum_{k=1}^{n} f(x_k^*)\Delta x_k \leq \sum_{k=1}^{n} M\Delta x_k$. This is equivalent to $\displaystyle m(b-a) \leq \sum_{k=1}^{n} f(x_k^*)\Delta x_k \leq M(b-a)$, and, taking the limit over $\max\Delta x_k \to 0$ we obtain the result.

37. $\int_0^{10} \sqrt{25-(x-5)^2}dx = \pi(5)^2/2 = 25\pi/2.$

39. $\int_0^1 (3x+1)dx = 5/2.$

41. **(a)** The graph of the integrand is the horizontal line $y = C$. At first, assume that $C > 0$. Then the region is a rectangle of height C whose base extends from $x = a$ to $x = b$. Thus $\int_a^b C\,dx =$ (area of rectangle) $= C(b-a)$. If $C \le 0$ then the rectangle lies below the axis and its integral is the negative area, i.e. $-|C|(b-a) = C(b-a)$.

(b) Since $f(x) = C$, the Riemann sum becomes $\lim_{\max\,\Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{\max\,\Delta x_k \to 0} \sum_{k=1}^{n} C\Delta x_k =$

$= \lim_{\max\,\Delta x_k \to 0} C(b-a) = C(b-a)$. By Definition 5.5.1, $\int_a^b f(x)\,dx = C(b-a)$.

43. Each subinterval of a partition of $[a,b]$ contains both rational and irrational numbers. If all x_k^* are chosen to be rational then $\sum_{k=1}^{n} f(x_k^*)\Delta x_k = \sum_{k=1}^{n} (1)\Delta x_k = \sum_{k=1}^{n} \Delta x_k = b-a$ so $\lim_{\max\,\Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = b-a$. If all x_k^* are irrational then $\lim_{\max\,\Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = 0$. Thus f is not integrable on $[a,b]$ because the preceding limits are not equal.

45. **(a)** f is continuous on $[-1,1]$ so f is integrable there by Theorem 5.5.2.

(b) $|f(x)| \le 1$ so f is bounded on $[-1,1]$, and f has one point of discontinuity, so by part (a) of Theorem 5.5.8 f is integrable on $[-1,1]$.

(c) f is not bounded on [-1,1] because $\lim_{x\to 0} f(x) = +\infty$, so f is not integrable on [0,1].

(d) $f(x)$ is discontinuous at the point $x = 0$ because $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist. f is continuous elsewhere. $-1 \le f(x) \le 1$ for x in $[-1,1]$ so f is bounded there. By part (a), Theorem 5.5.8, f is integrable on $[-1,1]$.

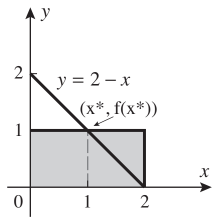
Exercise Set 5.6

1. **(a)** $\int_0^2 (2-x)dx = (2x - x^2/2)\Big]_0^2 = 4 - 4/2 = 2.$

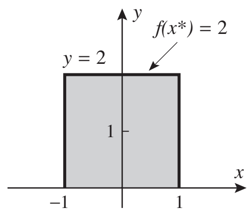
(b) $\int_{-1}^1 2dx = 2x\Big]_{-1}^1 = 2(1) - 2(-1) = 4.$

(c) $\int_1^3 (x+1)dx = (x^2/2 + x)\Big]_1^3 = 9/2 + 3 - (1/2 + 1) = 6.$

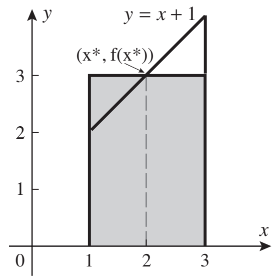
3. (a)



(b)



(c)



5. $\displaystyle\int_2^3 x^3 dx = x^4/4\Big]_2^3 = 81/4 - 16/4 = 65/4.$

7. $\displaystyle\int_1^4 3\sqrt{x}\, dx = 2x^{3/2}\Big]_1^4 = 16 - 2 = 14.$

9. $\displaystyle\int_0^{\ln 2} e^{2x}\, dx = \frac{1}{2}e^{2x}\Big]_0^{\ln 2} = \frac{1}{2}(4-1) = \frac{3}{2}.$

11. (a) $\displaystyle\int_0^3 \sqrt{x}\, dx = \frac{2}{3}x^{3/2}\Big]_0^3 = 2\sqrt{3} = f(x^*)(3-0)$, so $f(x^*) = \dfrac{2}{\sqrt{3}}, x^* = \dfrac{4}{3}.$

(b) $\displaystyle\int_{-12}^0 (x^2+x)\, dx = \frac{1}{3}x^3 + \frac{1}{2}x^2\Big]_{-12}^0 = 504$, so $f(x^*)(0-(-12)) = 504, (x^*)^2 + x^* = 42,\ x^* = 6, -7$ but only -7 lies in the interval. $f(-7) = 49 - 7 = 42$, so the area is that of a rectangle 12 wide and 42 high.

13. $\displaystyle\int_{-2}^1 (x^2 - 6x + 12)\, dx = \frac{1}{3}x^3 - 3x^2 + 12x\Big]_{-2}^1 = \frac{1}{3} - 3 + 12 - \left(-\frac{8}{3} - 12 - 24\right) = 48.$

15. $\displaystyle\int_1^4 \frac{4}{x^2}\, dx = -4x^{-1}\Big]_1^4 = -1 + 4 = 3.$

17. $\displaystyle\frac{4}{5}x^{5/2}\Big]_4^9 = 844/5.$

19. $-\cos\theta]_{-\pi/2}^{\pi/2} = 0.$

21. $\sin x]_{-\pi/4}^{\pi/4} = \sqrt{2}.$

23. $5e^x]_{\ln 2}^3 = 5e^3 - 5(2) = 5e^3 - 10.$

25. $\displaystyle\sin^{-1} x\Big]_0^{1/\sqrt{2}} = \sin^{-1}(1/\sqrt{2}) - \sin^{-1} 0 = \pi/4.$

27. $\displaystyle\sec^{-1} x\Big]_{\sqrt{2}}^2 = \sec^{-1} 2 - \sec^{-1}\sqrt{2} = \pi/3 - \pi/4 = \pi/12.$

29. $\displaystyle\left(2\sqrt{t} - 2t^{3/2}\right)\Big]_1^4 = -12.$

31. (a) $\displaystyle\int_{-1}^1 |2x-1|\, dx = \int_{-1}^{1/2}(1-2x)\, dx + \int_{1/2}^1 (2x-1)\, dx = (x - x^2)\Big]_{-1}^{1/2} + (x^2 - x)\Big]_{1/2}^1 = \frac{5}{2}.$

(b) $\displaystyle\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{3\pi/4} (-\cos x)dx = \sin x\Big]_0^{\pi/2} - \sin x\Big]_{\pi/2}^{3\pi/4} = 2 - \sqrt{2}/2.$

33. **(a)** $\displaystyle\int_{-1}^{0} (1-e^x)dx + \int_0^1 (e^x - 1)dx = (x - e^x)\Big]_{-1}^{0} + (e^x - x)\Big]_0^1 = -1 - (-1 - e^{-1}) + e - 1 - 1 = e + 1/e - 2.$

(b) $\displaystyle\int_1^2 \frac{2-x}{x}\,dx + \int_2^4 \frac{x-2}{x}\,dx = 2\ln x\Big]_1^2 - 1 + 2 - 2\ln x\Big]_2^4 = 2\ln 2 + 1 - 2\ln 4 + 2\ln 2 = 1.$

35. **(a)** 17/6 **(b)** $F(x) = \begin{cases} \dfrac{1}{2}x^2, & x \le 1 \\ \dfrac{1}{3}x^3 + \dfrac{1}{6}, & x > 1 \end{cases}$

37. False; consider $F(x) = x^2/2$ if $x \ge 0$ and $F(x) = -x^2/2$ if $x \le 0$.

39. True.

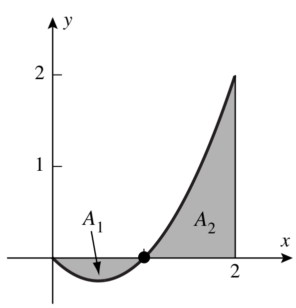
41. 0.665867079; $\displaystyle\int_1^3 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^3 = 2/3.$

43. 3.106017890; $\displaystyle\int_{-1}^{1} \sec^2 x\,dx = \tan x\Big]_{-1}^{1} = 2\tan 1 \approx 3.114815450.$

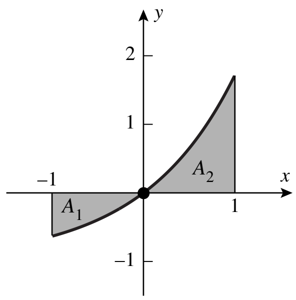
45. $\displaystyle A = \int_0^3 (x^2+1)dx = \left(\frac{1}{3}x^3 + x\right)\Big]_0^3 = 12.$

47. $\displaystyle A = \int_0^{2\pi/3} 3\sin x\,dx = -3\cos x\Big]_0^{2\pi/3} = 9/2.$

49. Area $\displaystyle = -\int_0^1 (x^2 - x)\,dx + \int_1^2 (x^2 - x)\,dx = 5/6 + 1/6 = 1.$



51. Area $\displaystyle = -\int_{-1}^{0} (e^x - 1)\,dx + \int_0^1 (e^x - 1)\,dx = 1/e + e - 2.$



53. **(a)** $A = \int_0^{0.8} \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x \Big]_0^{0.8} = \sin^{-1}(0.8)$.

(b) The calculator was in degree mode instead of radian mode; the correct answer is 0.93.

55. **(a)** The increase in height in inches, during the first ten years.

(b) The change in the radius in centimeters, during the time interval $t = 1$ to $t = 2$ seconds.

(c) The change in the speed of sound in ft/s, during an increase in temperature from $t = 32°\text{F}$ to $t = 100°\text{F}$.

(d) The displacement of the particle in cm, during the time interval $t = t_1$ to $t = t_2$ hours.

57. **(a)** $F'(x) = 3x^2 - 3$. **(b)** $\int_1^x (3t^2 - 3)\,dt = (t^3 - 3t)\Big]_1^x = x^3 - 3x + 2$, and $\frac{d}{dx}(x^3 - 3x + 2) = 3x^2 - 3$.

59. **(a)** $\sin x^2$ **(b)** $e^{\sqrt{x}}$

61. $-\frac{x}{\cos x}$

63. $F'(x) = \sqrt{x^2 + 9}$, $F''(x) = \frac{x}{\sqrt{x^2 + 9}}$. **(a)** 0 **(b)** 5 **(c)** $\frac{4}{5}$

65. **(a)** $F'(x) = \frac{x - 3}{x^2 + 7} = 0$ when $x = 3$, which is a relative minimum, and hence the absolute minimum, by the first derivative test.

(b) Increasing on $[3, +\infty)$, decreasing on $(-\infty, 3]$.

(c) $F''(x) = \frac{7 + 6x - x^2}{(x^2 + 7)^2} = \frac{(7 - x)(1 + x)}{(x^2 + 7)^2}$; concave up on $(-1, 7)$, concave down on $(-\infty, -1)$ and on $(7, +\infty)$.

67. **(a)** $(-\pi/2, \pi/2)$ because f is continuous there and 0 is in $(-\pi/2, \pi/2)$.

(b) At $x = 0$ because $F(0) = 0$.

69. **(a)** Amount of water = (rate of flow)(time) = $4t$ gal, total amount = $4(30) = 120$ gal.

(b) Amount of water = $\int_0^{60} (4 + t/10)dt = 420$ gal.

(c) Amount of water = $\int_0^{120} (10 + \sqrt{t})dt = 1200 + 160\sqrt{30} \approx 2076.36$ gal.

71. $\int_0^4 (2.7+0.4t)\,dt = 2.7t+0.2t^2\Big]_0^4 = 14$ students.

73. $\sum_{k=1}^{n} \frac{\pi}{4n}\sec^2\left(\frac{\pi k}{4n}\right) = \sum_{k=1}^{n} f(x_k^*)\Delta x$ where $f(x) = \sec^2 x$, $x_k^* = \frac{\pi k}{4n}$ and $\Delta x = \frac{\pi}{4n}$ for $0 \le x \le \frac{\pi}{4}$. Thus $\lim_{n\to+\infty}\sum_{k=1}^{n} \frac{\pi}{4n}\sec^2\left(\frac{\pi k}{4n}\right) = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \int_0^{\pi/4}\sec^2 x\,dx = \tan x\Big]_0^{\pi/4} = 1.$

75. Let f be continuous on a closed interval $[a,b]$ and let F be an antiderivative of f on $[a,b]$. By Theorem 5.7.2, $\frac{F(b)-F(a)}{b-a} = F'(x^*)$ for some x^* in (a,b). By Theorem 5.6.1, $\int_a^b f(x)\,dx = F(b)-F(a)$, i.e. $\int_a^b f(x)\,dx = F'(x^*)(b-a) = f(x^*)(b-a)$.

Exercise Set 5.7

1. (a) displ $= s(3)-s(0) = \int_0^3 dt = 3$; dist $= \int_0^3 dt = 3$.

(b) displ $= s(3)-s(0) = -\int_0^3 dt = -3$; dist $= \int_0^3 |v(t)|\,dt = 3$.

(c) displ $= s(3)-s(0) = \int_0^3 v(t)dt = \int_0^2 (1-t)dt + \int_2^3 (t-3)dt = (t-t^2/2)\Big]_0^2 + (t^2/2-3t)\Big]_2^3 = -1/2$; dist $= \int_0^3 |v(t)|dt = (t-t^2/2)\Big]_0^1 + (t^2/2-t)\Big]_1^2 - (t^2/2-3t)\Big]_2^3 = 3/2$.

(d) displ $= s(3)-s(0) = \int_0^3 v(t)dt = \int_0^1 tdt + \int_1^2 dt + \int_2^3 (5-2t)dt = t^2/2\Big]_0^1 + t\Big]_1^2 + (5t-t^2)\Big]_2^3 = 3/2$; dist $= \int_0^1 tdt + \int_1^2 dt + \int_2^{5/2}(5-2t)dt + \int_{5/2}^3 (2t-5)dt = t^2/2\Big]_0^1 + t\Big]_1^2 + (5t-t^2)\Big]_2^{5/2} + (t^2-5t)\Big]_{5/2}^3 = 2$.

3. (a) $v(t) = 20 + \int_0^t a(u)du$; add areas of the small blocks to get $v(4) \approx 20+1.4+3.0+4.7+6.2 = 35.3$ m/s.

(b) $v(6) = v(4) + \int_4^6 a(u)du \approx 35.3+7.5+8.6 = 51.4$ m/s.

5. (a) $s(t) = t^3-t^2+C; 1 = s(0) = C$, so $s(t) = t^3-t^2+1$.

(b) $v(t) = -\cos 3t + C_1; 3 = v(0) = -1+C_1, C_1 = 4$, so $v(t) = -\cos 3t + 4$. Then $s(t) = -\frac{1}{3}\sin 3t + 4t + C_2$; $3 = s(0) = C_2$, so $s(t) = -\frac{1}{3}\sin 3t + 4t + 3$.

7. (a) $s(t) = \frac{3}{2}t^2 + t + C; 4 = s(2) = 6+2+C, C = -4$ and $s(t) = \frac{3}{2}t^2+t-4$.

(b) $v(t) = -t^{-1}+C_1, 0 = v(1) = -1+C_1, C_1 = 1$ and $v(t) = -t^{-1}+1$ so $s(t) = -\ln t + t + C_2, 2 = s(1) = 1+C_2, C_2 = 1$ and $s(t) = -\ln t + t + 1$.

9. (a) displacement $= s(\pi/2)-s(0) = \int_0^{\pi/2}\sin t dt = -\cos t\Big]_0^{\pi/2} = 1$ m; distance $= \int_0^{\pi/2}|\sin t|dt = 1$ m.

(b) displacement $= s(2\pi)-s(\pi/2) = \int_{\pi/2}^{2\pi} \cos t dt = \sin t \Big]_{\pi/2}^{2\pi} = -1$ m; distance $= \int_{\pi/2}^{2\pi} |\cos t| dt = -\int_{\pi/2}^{3\pi/2} \cos t dt + \int_{3\pi/2}^{2\pi} \cos t dt = 3$ m.

11. **(a)** $v(t) = t^3 - 3t^2 + 2t = t(t-1)(t-2)$, displacement $= \int_0^3 (t^3 - 3t^2 + 2t) dt = 9/4$ m; distance $= \int_0^3 |v(t)| dt = \int_0^1 v(t) dt + \int_1^2 -v(t) dt + \int_2^3 v(t) dt = 11/4$ m.

(b) displacement $= \int_0^3 (\sqrt{t} - 2) dt = 2\sqrt{3} - 6$ m; distance $= \int_0^3 |v(t)| dt = -\int_0^3 v(t) dt = 6 - 2\sqrt{3}$ m.

13. $v = 3t - 1$, displacement $= \int_0^2 (3t-1)\, dt = 4$ m; distance $= \int_0^2 |3t-1|\, dt = \dfrac{13}{3}$ m.

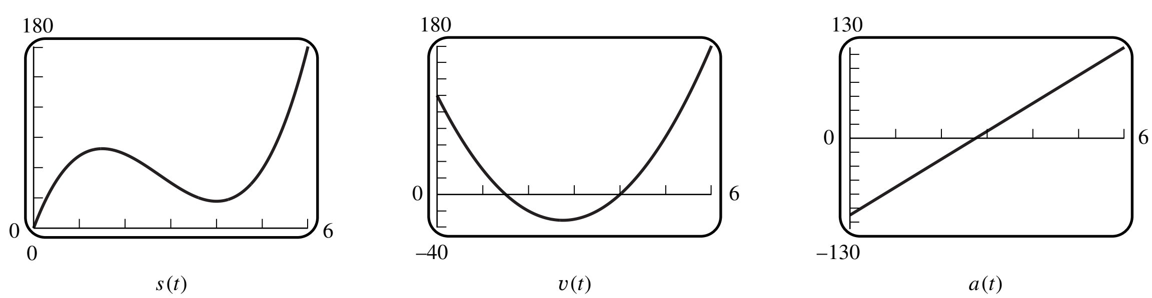
15. $v = \int 1/\sqrt{3t+1}\, dt = \dfrac{2}{3}\sqrt{3t+1} + C$; $v(0) = 4/3$ so $C = 2/3$, $v = \dfrac{2}{3}\sqrt{3t+1} + 2/3$, displacement $= \int_1^5 \left(\dfrac{2}{3}\sqrt{3t+1} + \dfrac{2}{3}\right) dt = \dfrac{296}{27}$ m; distance $= \int_1^5 \left(\dfrac{2}{3}\sqrt{3t+1} + \dfrac{2}{3}\right) dt = \dfrac{296}{27}$ m.

17. **(a)** $s = \int \sin \dfrac{1}{2}\pi t\, dt = -\dfrac{2}{\pi}\cos \dfrac{1}{2}\pi t + C$, $s = 0$ when $t = 0$ which gives $C = \dfrac{2}{\pi}$ so $s = -\dfrac{2}{\pi}\cos\dfrac{1}{2}\pi t + \dfrac{2}{\pi}$. $a = \dfrac{dv}{dt} = \dfrac{\pi}{2}\cos\dfrac{1}{2}\pi t$. When $t = 1: s = 2/\pi$, $v = 1$, $|v| = 1$, $a = 0$.

(b) $v = -3\int t\, dt = -\dfrac{3}{2}t^2 + C_1$, $v = 0$ when $t = 0$ which gives $C_1 = 0$ so $v = -\dfrac{3}{2}t^2$. $s = -\dfrac{3}{2}\int t^2 dt = -\dfrac{1}{2}t^3 + C_2$, $s = 1$ when $t = 0$ which gives $C_2 = 1$ so $s = -\dfrac{1}{2}t^3 + 1$. When $t = 1: s = 1/2$, $v = -3/2$, $|v| = 3/2$, $a = -3$.

19. By inspection the velocity is positive for $t > 0$, and during the first second the ant is at most 5/2 cm from the starting position. For $T > 1$ the displacement of the ant during the time interval $[0, T]$ is given by $\int_0^T v(t)\, dt = 5/2 + \int_1^T (6\sqrt{t} - 1/t)\, dt = 5/2 + (4t^{3/2} - \ln t)\Big]_1^T = -3/2 + 4T^{3/2} - \ln T$, and the displacement equals 4 cm if $4T^{3/2} - \ln T = 11/2$, $T \approx 1.272$ s.

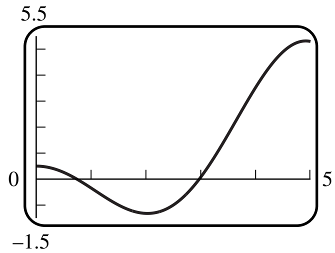
21. $s(t) = \int (20t^2 - 110t + 120)\, dt = \dfrac{20}{3}t^3 - 55t^2 + 120t + C$. But $s = 0$ when $t = 0$, so $C = 0$ and $s = \dfrac{20}{3}t^3 - 55t^2 + 120t$. Moreover, $a(t) = \dfrac{d}{dt}v(t) = 40t - 110$.



23. True; if $a(t) = a_0$ then $v(t) = a_0 t + v_0$.

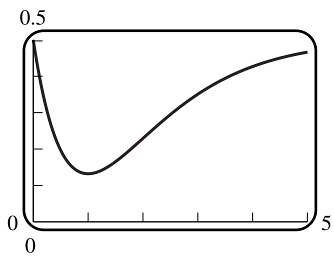
25. False; consider $v(t) = t$ on $[-1, 1]$.

27. **(a)** The displacement is positive on $(0, 5)$.



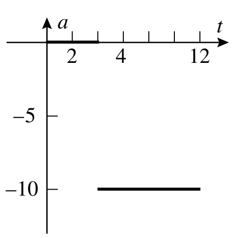
(b) The displacement is $\frac{5}{2} - \sin 5 + 5\cos 5 \approx 4.877$.

29. **(a)** The displacement is positive on $(0, 5)$.

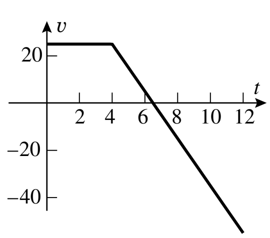


(b) The displacement is $\frac{3}{2} + 6e^{-5}$.

31. **(a)** $a(t) = \begin{cases} 0, & t < 4 \\ -10, & t > 4 \end{cases}$



(b) $v(t) = \begin{cases} 25, & t < 4 \\ 65 - 10t, & t > 4 \end{cases}$



(c) $x(t) = \begin{cases} 25t, & t < 4 \\ 65t - 5t^2 - 80, & t > 4 \end{cases}$, so $x(8) = 120$, $x(12) = -20$. **(d)** $x(6.5) = 131.25$.

33. $a = a_0$ ft/s^2, $v = a_0t + v_0 = a_0t + 132$ ft/s, $s = a_0t^2/2 + 132t + s_0 = a_0t^2/2 + 132t$ ft; $s = 200$ ft when $v = 88$ ft/s. Solve $88 = a_0t + 132$ and $200 = a_0t^2/2 + 132t$ to get $a_0 = -\dfrac{121}{5}$ when $t = \dfrac{20}{11}$, so $s = -12.1t^2 + 132t$, $v = -\dfrac{121}{5}t + 132$.

(a) $a_0 = -\dfrac{121}{5}$ ft/s^2. **(b)** $v = 55$ mi/h $= \dfrac{242}{3}$ ft/s when $t = \dfrac{70}{33}$ s. **(c)** $v = 0$ when $t = \dfrac{60}{11}$ s.

35. Suppose $s = s_0 = 0$, $v = v_0 = 0$ at $t = t_0 = 0$; $s = s_1 = 120$, $v = v_1$ at $t = t_1$; and $s = s_2$, $v = v_2 = 12$ at $t = t_2$. From formulas (10) and (11), we get that in the case of constant acceleration, $a = \dfrac{v^2 - v_0^2}{2(s - s_0)}$. This implies that $2.6 = a = \dfrac{v_1^2 - v_0^2}{2(s_1 - s_0)}$, $v_1^2 = 2as_1 = 5.2(120) = 624$. Applying the formula again, $-1.5 = a = \dfrac{v_2^2 - v_1^2}{2(s_2 - s_1)}$, $v_2^2 = v_1^2 - 3(s_2 - s_1)$, so $s_2 = s_1 - (v_2^2 - v_1^2)/3 = 120 - (144 - 624)/3 = 280$ m.

37. The truck's velocity is $v_T = 50$ and its position is $s_T = 50t + 2500$. The car's acceleration is $a_C = 4$ ft/s^2, so $v_C = 4t$, $s_C = 2t^2$ (initial position and initial velocity of the car are both zero). $s_T = s_C$ when $50t + 2500 = 2t^2$, $2t^2 - 50t - 2500 = 2(t + 25)(t - 50) = 0$, $t = 50$ s and $s_C = s_T = 2t^2 = 5000$ ft.

39. $s = 0$ and $v = 112$ when $t = 0$ so $v(t) = -32t + 112$, $s(t) = -16t^2 + 112t$.

(a) $v(3) = 16$ ft/s, $v(5) = -48$ ft/s.

(b) $v = 0$ when the projectile is at its maximum height so $-32t + 112 = 0$, $t = 7/2$ s, $s(7/2) = -16(7/2)^2 + 112(7/2) = 196$ ft.

(c) $s = 0$ when it reaches the ground so $-16t^2 + 112t = 0$, $-16t(t - 7) = 0$, $t = 0, 7$ of which $t = 7$ is when it is at ground level on its way down. $v(7) = -112$, $|v| = 112$ ft/s.

41. (a) $s(t) = 0$ when it hits the ground, $s(t) = -16t^2 + 16t = -16t(t - 1) = 0$ when $t = 1$ s.

(b) The projectile moves upward until it gets to its highest point where $v(t) = 0$, $v(t) = -32t + 16 = 0$ when $t = 1/2$ s.

43. $s(t) = s_0 + v_0t - \frac{1}{2}gt^2 = 60t - 4.9t^2$ m and $v(t) = v_0 - gt = 60 - 9.8t$ m/s.

(a) $v(t) = 0$ when $t = 60/9.8 \approx 6.12$ s.

(b) $s(60/9.8) \approx 183.67$ m.

(c) Another 6.12 s; solve for t in $s(t) = 0$ to get this result, or use the symmetry of the parabola $s = 60t - 4.9t^2$ about the line $t = 6.12$ in the t-s plane.

(d) Also 60 m/s, as seen from the symmetry of the parabola (or compute $v(6.12)$).

45. $s(t) = -4.9t^2 + 49t + 150$ and $v(t) = -9.8t + 49$.

(a) The model rocket reaches its maximum height when $v(t) = 0$, $-9.8t + 49 = 0$, $t = 5$ s.

(b) $s(5) = -4.9(5)^2 + 49(5) + 150 = 272.5$ m.

(c) The model rocket reaches its starting point when $s(t) = 150$, $-4.9t^2 + 49t + 150 = 150$, $-4.9t(t - 10) = 0$, $t = 10$ s.

(d) $v(10) = -9.8(10) + 49 = -49$ m/s.

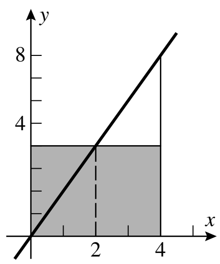
(e) $s(t) = 0$ when the model rocket hits the ground, $-4.9t^2 + 49t + 150 = 0$ when (use the quadratic formula) $t \approx 12.46$ s.

(f) $v(12.46) = -9.8(12.46) + 49 \approx -73.1$, the speed at impact is about 73.1 m/s.

Exercise Set 5.8

1. **(a)** $f_{\text{ave}} = \dfrac{1}{4-0}\displaystyle\int_0^4 2x\,dx = 4.$ **(b)** $2x^* = 4$, $x^* = 2$.

(c)



3. $f_{\text{ave}} = \dfrac{1}{3-1}\displaystyle\int_1^3 3x\,dx = \dfrac{3}{4}x^2\Big]_1^3 = 6.$

5. $f_{\text{ave}} = \dfrac{1}{\pi}\displaystyle\int_0^\pi \sin x\,dx = -\dfrac{1}{\pi}\cos x\Big]_0^\pi = \dfrac{2}{\pi}.$

7. $f_{\text{ave}} = \dfrac{1}{e-1}\displaystyle\int_1^e \dfrac{1}{x}\,dx = \dfrac{1}{e-1}(\ln e - \ln 1) = \dfrac{1}{e-1}$

9. $f_{\text{ave}} = \dfrac{1}{\sqrt{3}-1}\displaystyle\int_1^{\sqrt{3}} \dfrac{dx}{1+x^2} = \dfrac{1}{\sqrt{3}-1}\tan^{-1}x\Big]_1^{\sqrt{3}} = \dfrac{1}{\sqrt{3}-1}\left(\dfrac{\pi}{3}-\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{3}-1}\dfrac{\pi}{12}.$

11. $f_{\text{ave}} = \dfrac{1}{4}\displaystyle\int_0^4 e^{-2x}\,dx = -\dfrac{1}{8}e^{-2x}\Big]_0^4 = \dfrac{1}{8}(1-e^{-8}).$

13. **(a)** $\dfrac{1}{5}[f(0.4)+f(0.8)+f(1.2)+f(1.6)+f(2.0)] = \dfrac{1}{5}[0.48+1.92+4.32+7.68+12.00] = 5.28.$

(b) $\dfrac{1}{20}3[(0.1)^2+(0.2)^2+\ldots+(1.9)^2+(2.0)^2] = \dfrac{861}{200} = 4.305.$

(c) $f_{\text{ave}} = \dfrac{1}{2}\displaystyle\int_0^2 3x^2\,dx = \dfrac{1}{2}x^3\Big]_0^2 = 4.$

(d) Parts (a) and (b) can be interpreted as being two Riemann sums (n = 5, n = 20) for the average, using right endpoints. Since f is increasing, these sums overestimate the integral.

15. **(a)** $\displaystyle\int_0^3 v(t)\,dt = \int_0^2 (1-t)\,dt + \int_2^3 (t-3)\,dt = -\dfrac{1}{2}$, so $v_{\text{ave}} = -\dfrac{1}{6}.$

(b) $\displaystyle\int_0^3 v(t)\,dt = \int_0^1 t\,dt + \int_1^2 dt + \int_2^3 (-2t+5)\,dt = \dfrac{1}{2}+1+0 = \dfrac{3}{2}$, so $v_{\text{ave}} = \dfrac{1}{2}.$

17. Linear means $f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2)$, so $f\left(\dfrac{a+b}{2}\right) = \dfrac{1}{2}f(a) + \dfrac{1}{2}f(b) = \dfrac{f(a)+f(b)}{2}.$

19. False; $f(x) = x, g(x) = -1/2$ on $[-1,1]$.

21. True; Theorem 5.5.4(b).

23. **(a)** $v_{\text{ave}} = \dfrac{1}{4-1}\displaystyle\int_1^4 (3t^3+2)dt = \dfrac{1}{3}\dfrac{789}{4} = \dfrac{263}{4}$.

(b) $v_{\text{ave}} = \dfrac{s(4)-s(1)}{4-1} = \dfrac{100-7}{3} = 31$.

25. Time to fill tank = (volume of tank)/(rate of filling) = $[\pi(3)^2 5]/(1) = 45\pi$, weight of water in tank at time $t = (62.4)$ (rate of filling)(time) $= 62.4t$, weight$_{\text{ave}} = \dfrac{1}{45\pi}\displaystyle\int_0^{45\pi} 62.4t\,dt = 1404\pi = 4410.8$ lb.

27. $\displaystyle\int_0^{30} 100(1-0.0001t^2)dt = 2910$ cars, so an average of $\dfrac{2910}{30} = 97$ cars/min.

29. From the chart we read $\dfrac{dV}{dt} = f(t) = \begin{cases} 40t, & 0 \le t \le 1 \\ 40, & 1 \le t \le 3 \\ -20t+100, & 3 \le t \le 5 \end{cases}$.

It follows that (constants of integration are chosen to ensure that $V(0) = 0$ and that $V(t)$ is continuous)

$$V(t) = \begin{cases} 20t^2, & 0 \le t \le 1 \\ 40t-20, & 1 \le t \le 3 \\ -10t^2+100t-110, & 3 \le t \le 5 \end{cases}.$$

Now the average rate of change of the volume of juice in the glass during these 5 seconds refers to the quantity $\dfrac{1}{5}(V(5)-V(0)) = \dfrac{1}{5}140 = 28$, and the average value of the flow rate is

$$f_{\text{ave}} = \frac{1}{5}\int_0^1 f(t)\,dt = \frac{1}{5}\left[\int_0^1 40t\,dt + \int_1^3 40\,dt + \int_3^5 (-20t+100)\,dt\right] = \frac{1}{5}[20+80-160+200] = 28.$$

31. The average is $f_{\text{ave}} = \dfrac{1}{11}\displaystyle\int_{2000}^{2011} P(t)\,dt = \dfrac{1}{11}\int_{2000}^{2011} 248e^{0.105(t-2000)}\,dt = \dfrac{1}{11}\cdot\dfrac{248}{0.105}e^{0.105(t-2000)}\Big]_{2000}^{2011} \approx 466.8$ wolves.

Exercise Set 5.9

1. **(a)** $\dfrac{1}{2}\displaystyle\int_1^5 u^3\,du$ **(b)** $\dfrac{3}{2}\displaystyle\int_9^{25} \sqrt{u}\,du$ **(c)** $\dfrac{1}{\pi}\displaystyle\int_{-\pi/2}^{\pi/2} \cos u\,du$ **(d)** $\displaystyle\int_1^2 (u+1)u^5\,du$

3. **(a)** $\dfrac{1}{2}\displaystyle\int_{-1}^1 e^u\,du$ **(b)** $\displaystyle\int_1^2 u\,du$

5. $u = 2x+1$, $\dfrac{1}{2}\displaystyle\int_1^3 u^3\,du = \dfrac{1}{8}u^4\Big]_1^3 = 10$, or $\dfrac{1}{8}(2x+1)^4\Big]_0^1 = 10$.

7. $u = 2x-1$, $\dfrac{1}{2}\displaystyle\int_{-1}^1 u^3 du = 0$, because u^3 is odd on $[-1,1]$.

9. $u = 1+x$, $\displaystyle\int_1^9 (u-1)u^{1/2}du = \int_1^9 (u^{3/2}-u^{1/2})du = \dfrac{2}{5}u^{5/2} - \dfrac{2}{3}u^{3/2}\Big]_1^9 = 1192/15$, or $\dfrac{2}{5}(1+x)^{5/2} - \dfrac{2}{3}(1+x)^{3/2}\Big]_0^8 = 1192/15$.

11. $u = x/2$, $8\int_0^{\pi/4} \sin u\, du = -8\cos u\Big]_0^{\pi/4} = 8 - 4\sqrt{2}$, or $-8\cos(x/2)\Big]_0^{\pi/2} = 8 - 4\sqrt{2}$.

13. $u = x^2 + 2$, $\dfrac{1}{2}\int_6^3 u^{-3} du = -\dfrac{1}{4u^2}\Big]_6^3 = -1/48$, or $-\dfrac{1}{4}\dfrac{1}{(x^2+2)^2}\Big]_{-2}^{-1} = -1/48$.

15. $u = e^x + 4$, $du = e^x dx$, $u = e^{-\ln 3} + 4 = \dfrac{1}{3} + 4 = \dfrac{13}{3}$ when $x = -\ln 3$, $u = e^{\ln 3} + 4 = 3 + 4 = 7$ when $x = \ln 3$; $\int_{13/3}^{7} \dfrac{1}{u} du = \ln u\Big]_{13/3}^{7} = \ln(7) - \ln(13/3) = \ln(21/13)$, or $\ln(e^x + 4)\Big]_{-\ln 3}^{\ln 3} = \ln 7 - \ln(13/3) = \ln(21/13)$.

17. $u = \sqrt{x}$, $2\int_1^{\sqrt{3}} \dfrac{1}{u^2+1} du = 2\tan^{-1} u\Big]_1^{\sqrt{3}} = 2(\tan^{-1}\sqrt{3} - \tan^{-1} 1) = 2(\pi/3 - \pi/4) = \pi/6$, or $2\tan^{-1}\sqrt{x}\Big]_1^3 = \pi/6$.

19. $\dfrac{1}{3}\int_{-5}^{5} \sqrt{25 - u^2}\, du = \dfrac{1}{3}\left[\dfrac{1}{2}\pi(5)^2\right] = \dfrac{25}{6}\pi$.

21. $-\dfrac{1}{2}\int_1^0 \sqrt{1 - u^2}\, du = \dfrac{1}{2}\int_0^1 \sqrt{1 - u^2}\, du = \dfrac{1}{2}\cdot\dfrac{1}{4}[\pi(1)^2] = \pi/8$.

23. $\int_0^1 \sin \pi x dx = -\dfrac{1}{\pi}\cos \pi x\Big]_0^1 = -\dfrac{1}{\pi}(-1 - 1) = 2/\pi$ m.

25. $A = \int_{-1}^{1} \dfrac{9}{(x+2)^2} dx = -9(x+2)^{-1}\Big]_{-1}^{1} = -9\left[\dfrac{1}{3} - 1\right] = 6$.

27. $A = \int_0^{1/6} \dfrac{1}{\sqrt{1 - 9x^2}} dx = \dfrac{1}{3}\int_0^{1/2} \dfrac{1}{\sqrt{1 - u^2}} du = \dfrac{1}{3}\sin^{-1} u\Big]_0^{1/2} = \pi/18$.

29. $f_{\text{ave}} = \dfrac{1}{2-0}\int_0^2 \dfrac{x}{(5x^2+1)^2}\, dx = -\dfrac{1}{2}\dfrac{1}{10}\dfrac{1}{5x^2+1}\Big]_0^2 = \dfrac{1}{21}$.

31. $u = 2x - 1$, $\dfrac{1}{2}\int_1^9 \dfrac{1}{\sqrt{u}}\, du = \sqrt{u}\Big]_1^9 = 2$.

33. $\dfrac{2}{3}(x^3 + 9)^{1/2}\Big]_{-1}^{1} = \dfrac{2}{3}(\sqrt{10} - 2\sqrt{2})$.

35. $u = x^2 + 4x + 7$, $\dfrac{1}{2}\int_{12}^{28} u^{-1/2} du = u^{1/2}\Big]_{12}^{28} = \sqrt{28} - \sqrt{12} = 2(\sqrt{7} - \sqrt{3})$.

37. $2\sin^2 x\Big]_0^{\pi/4} = 1$.

39. $\dfrac{5}{2}\sin(x^2)\Big]_0^{\sqrt{\pi}} = 0$.

41. $u = 3\theta$, $\dfrac{1}{3}\int_{\pi/4}^{\pi/3} \sec^2 u\, du = \dfrac{1}{3}\tan u\Big]_{\pi/4}^{\pi/3} = (\sqrt{3} - 1)/3$.

43. $u = 4 - 3y$, $y = \frac{1}{3}(4-u)$, $dy = -\frac{1}{3}du$, $-\frac{1}{27}\int_4^1 \frac{16-8u+u^2}{u^{1/2}}du = \frac{1}{27}\int_1^4 (16u^{-1/2} - 8u^{1/2} + u^{3/2})du = \frac{1}{27}\left[32u^{1/2} - \frac{16}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right]_1^4 = 106/405.$

45. $\frac{1}{2}\ln(2x+e)\Big]_0^e = \frac{1}{2}(\ln(3e) - \ln e) = \frac{\ln 3}{2}.$

47. $u = \sqrt{3}x^2$, $\frac{1}{2\sqrt{3}}\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-u^2}}du = \frac{1}{2\sqrt{3}}\sin^{-1}\frac{u}{2}\Big]_0^{\sqrt{3}} = \frac{1}{2\sqrt{3}}\left(\frac{\pi}{3}\right) = \frac{\pi}{6\sqrt{3}}.$

49. $u = 3x$, $\frac{1}{3}\int_0^{\sqrt{3}} \frac{1}{1+u^2}du = \frac{1}{3}\tan^{-1}u\Big]_0^{\sqrt{3}} = \frac{1}{3}\frac{\pi}{3} = \frac{\pi}{9}.$

51. **(b)** $\int_0^{\pi/6} \sin^4 x(1-\sin^2 x)\cos x\,dx = \left(\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x\right)\Big]_0^{\pi/6} = \frac{1}{160} - \frac{1}{896} = \frac{23}{4480}.$

53. **(a)** $u = 3x+1$, $\frac{1}{3}\int_1^4 f(u)du = 5/3.$

(b) $u = 3x$, $\frac{1}{3}\int_0^9 f(u)du = 5/3.$

(c) $u = x^2$, $1/2\int_4^0 f(u)du = -1/2\int_0^4 f(u)du = -1/2.$

55. $\sin x = \cos(\pi/2 - x)$, $\int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\cos^n(\pi/2-x)dx = -\int_{\pi/2}^0 \cos^n u\,du$ (with $u = \pi/2 - x$) $= \int_0^{\pi/2}\cos^n u\,du = \int_0^{\pi/2}\cos^n x\,dx$, by replacing u by x.

57. Method 1: $\int_0^4 5(e^{-0.2t} - e^{-t})\,dt = 5\frac{1}{-0.2}e^{-0.2t} + 5e^{-t}\Big]_0^4 \approx 8.85835,$

Method 2: $\int_0^4 4(e^{-0.2t} - e^{-3t})\,dt = 4\frac{1}{-0.2}e^{-0.2t} + \frac{4}{3}e^{-3t}\Big]_0^4 \approx 9.6801$, so Method 2 provides the greater availability.

59. Method 1: $\int_0^4 5.78(e^{-0.4t} - e^{-1.3t})\,dt = 5.78\frac{1}{-0.4}e^{-0.4t} + 5.78\frac{1}{1.3}e^{-1.3t}\Big]_0^4 \approx 7.11097,$

Method 2: $\int_0^4 4.15(e^{-0.4t} - e^{-3t})\,dt = 4.15\frac{1}{-0.4}e^{-0.4t} + \frac{4.15}{3}e^{-3t}\Big]_0^4 \approx 6.897$, so Method 1 provides the greater availability.

61. $y(t) = (802.137)\int e^{1.528t}dt = 524.959e^{1.528t} + C$; $y(0) = 750 = 524.959 + C$, $C = 225.041$, $y(t) = 524959e^{1.528t} + 225.041$, $y(12) \approx 48{,}233{,}500{,}000.$

63. **(a)** $\frac{1}{7}[0.74 + 0.65 + 0.56 + 0.45 + 0.35 + 0.25 + 0.16] = 0.4514285714.$

(b) $\frac{1}{7}\int_0^7 [0.5 + 0.5\sin(0.213x + 2.481)\,dx = 0.4614.$

65. $\int_0^k e^{2x}dx = 3,\ \frac{1}{2}e^{2x}\Big]_0^k = 3,\ \frac{1}{2}(e^{2k}-1) = 3, e^{2k} = 7, k = \frac{1}{2}\ln 7.$

67. **(a)** $\int_0^1 \sin \pi x dx = 2/\pi.$

69. **(a)** Let $u = -x$, then $\int_{-a}^{a} f(x)dx = -\int_{a}^{-a} f(-u)du = \int_{-a}^{a} f(-u)du = -\int_{-a}^{a} f(u)du$, so, replacing u by x in the latter integral, $\int_{-a}^{a} f(x)dx = -\int_{-a}^{a} f(x)dx$, $2\int_{-a}^{a} f(x)dx = 0$, $\int_{-a}^{a} f(x)dx = 0$. The graph of f is symmetric about the origin, so $\int_{-a}^{0} f(x)dx$ is the negative of $\int_0^a f(x)dx$ thus $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)dx = 0.$

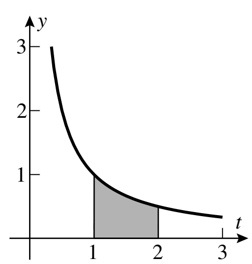
(b) $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$, let $u = -x$ in $\int_{-a}^{0} f(x)dx$ to get $\int_{-a}^{0} f(x)dx = -\int_{a}^{0} f(-u)du = \int_0^a f(-u)du = \int_0^a f(u)du = \int_0^a f(x)dx$, so $\int_{-a}^{a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx$. The graph of $f(x)$ is symmetric about the y-axis so there is as much signed area to the left of the y-axis as there is to the right.

71. **(a)** $I = -\int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}du = \int_0^a \frac{f(a-u)+f(u)-f(u)}{f(a-u)+f(u)}du = \int_0^a du - \int_0^a \frac{f(u)}{f(a-u)+f(u)}du, I = a - I$, so $2I = a, I = a/2$.

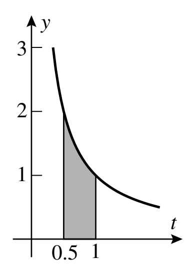
(b) $3/2$ **(c)** $\pi/4$

Exercise Set 5.10

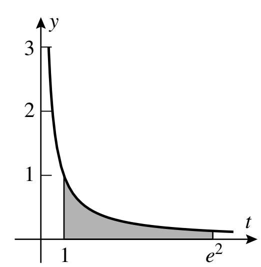
1. **(a)**



(b)



(c)



3. **(a)** $\ln t\Big]_1^{ac} = \ln(ac) = \ln a + \ln c = 7.$ **(b)** $\ln t\Big]_1^{1/c} = \ln(1/c) = -5.$

(c) $\ln t\Big]_1^{a/c} = \ln(a/c) = 2 - 5 = -3.$ **(d)** $\ln t\Big]_1^{a^3} = \ln a^3 = 3\ln a = 6.$

5. $\ln 5$ midpoint rule approximation: 1.603210678; $\ln 5 \approx 1.609437912$; magnitude of error is < 0.0063.

7. **(a)** $x^{-1}, x > 0.$ **(b)** $x^2, x \neq 0.$ **(c)** $-x^2, -\infty < x < +\infty.$ **(d)** $-x, -\infty < x < +\infty.$

(e) $x^3, x > 0.$ **(f)** $\ln x + x, x > 0.$ **(g)** $x - \sqrt[3]{x}, -\infty < x < +\infty.$ **(h)** $\frac{e^x}{x}, x > 0.$

9. **(a)** $3^\pi = e^{\pi \ln 3}.$ **(b)** $2^{\sqrt{2}} = e^{\sqrt{2}\ln 2}.$

11. **(a)** $y = 2x, \lim_{x\to+\infty}\left(1+\frac{1}{2x}\right)^x = \lim_{x\to+\infty}\left[\left(1+\frac{1}{2x}\right)^{2x}\right]^{1/2} = \lim_{y\to+\infty}\left[\left(1+\frac{1}{y}\right)^{y}\right]^{1/2} = e^{1/2}.$

(b) $y = 2x$, $\lim_{y\to 0}(1+y)^{2/y} = \lim_{y\to 0}\left[(1+y)^{1/y}\right]^2 = e^2$.

13. $g'(x) = x^2 - x$.

15. **(a)** $\dfrac{1}{x^3}(3x^2) = \dfrac{3}{x}$. **(b)** $e^{\ln x}\dfrac{1}{x} = 1$.

17. $F'(x) = \dfrac{\sin x}{x^2+1}$, $F''(x) = \dfrac{(x^2+1)\cos\ x - 2x\sin x}{(x^2+1)^2}$.

(a) 0 **(b)** 0 **(c)** 1

19. True; both integrals are equal to $-\ln a$.

21. False; the integral does not exist.

23. **(a)** $\dfrac{d}{dx}\displaystyle\int_1^{x^2} t\sqrt{1+t}dt = x^2\sqrt{1+x^2}(2x) = 2x^3\sqrt{1+x^2}$.

(b) $\displaystyle\int_1^{x^2} t\sqrt{1+t}dt = -\frac{2}{3}(x^2+1)^{3/2} + \frac{2}{5}(x^2+1)^{5/2} - \frac{4\sqrt{2}}{15}$.

25. **(a)** $-\cos x^3$ **(b)** $-\dfrac{\tan^2 x}{1+\tan^2 x}\sec^2 x = -\tan^2 x$.

27. $-3\dfrac{3x-1}{9x^2+1} + 2x\dfrac{x^2-1}{x^4+1}$.

29. **(a)** $\sin^2(x^3)(3x^2) - \sin^2(x^2)(2x) = 3x^2\sin^2(x^3) - 2x\sin^2(x^2)$.

(b) $\dfrac{1}{1+x}(1) - \dfrac{1}{1-x}(-1) = \dfrac{2}{1-x^2}$ (for $-1 < x < 1$).

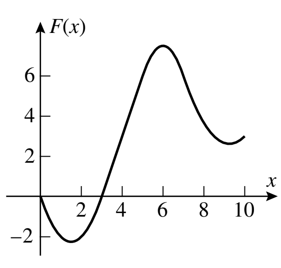
31. From geometry, $\displaystyle\int_0^3 f(t)dt = 0$, $\displaystyle\int_3^5 f(t)dt = 6$, $\displaystyle\int_5^7 f(t)dt = 0$; and $\displaystyle\int_7^{10} f(t)dt = \int_7^{10}(4t-37)/3dt = -3$.

(a) $F(0) = 0$, $F(3) = 0$, $F(5) = 6$, $F(7) = 6$, $F(10) = 3$.

(b) F is increasing where $F' = f$ is positive, so on $[3/2, 6]$ and $[37/4, 10]$, decreasing on $[0, 3/2]$ and $[6, 37/4]$.

(c) Critical points when $F'(x) = f(x) = 0$, so $x = 3/2, 6, 37/4$; maximum $15/2$ at $x = 6$, minimum $-9/4$ at $x = 3/2$. (Endpoints: $F(0) = 0$ and $F(10) = 3$.)

(d)



33. $x < 0 : F(x) = \displaystyle\int_{-1}^{x}(-t)dt = -\frac{1}{2}t^2\Big]_{-1}^{x} = \frac{1}{2}(1-x^2)$,

$x \geq 0: F(x) = \int_{-1}^{0} (-t)dt + \int_{0}^{x} t\,dt = \frac{1}{2} + \frac{1}{2}x^2$; $F(x) = \begin{cases} (1-x^2)/2, & x < 0 \\ (1+x^2)/2, & x \geq 0 \end{cases}$

35. $y(x) = 2 + \int_1^x \frac{2t^2+1}{t} dt = 2 + (t^2 + \ln t)\Big]_1^x = x^2 + \ln x + 1.$

37. $y(x) = 1 + \int_{\pi/4}^{x} (\sec^2 t - \sin t)dt = \tan x + \cos x - \sqrt{2}/2.$

39. $P(x) = P_0 + \int_0^x r(t)dt$ individuals.

41. II has a minimum at $x = 12$, and I has a zero there, so I could be the derivative of II; on the other hand I has a minimum near $x = 1/3$, but II is not zero there, so II could not be the derivative of I, so I is the graph of $f(x)$ and II is the graph of $\int_0^x f(t)\,dt$.

43. **(a)** Where $f(t) = 0$; by the First Derivative Test, at $t = 3$.

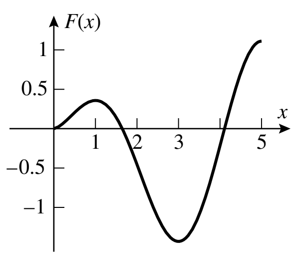
(b) Where $f(t) = 0$; by the First Derivative Test, at $t = 1, 5$.

(c) At $t = 0, 1$ or 5; from the graph it is evident that it is at $t = 5$.

(d) At $t = 0, 3$ or 5; from the graph it is evident that it is at $t = 3$.

(e) F is concave up when $F'' = f'$ is positive, i.e. where f is increasing, so on $(0, 1/2)$ and $(2, 4)$; it is concave down on $(1/2, 2)$ and $(4, 5)$.

(f)



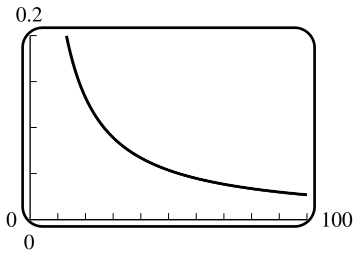
45. $C'(x) = \cos(\pi x^2/2)$, $C''(x) = -\pi x \sin(\pi x^2/2)$.

(a) $\cos t$ goes from negative to positive at $2k\pi - \pi/2$, and from positive to negative at $t = 2k\pi + \pi/2$, so $C(x)$ has relative minima when $\pi x^2/2 = 2k\pi - \pi/2$, $x = \pm\sqrt{4k-1}$, $k = 1, 2, \ldots$, and $C(x)$ has relative maxima when $\pi x^2/2 = (4k+1)\pi/2$, $x = \pm\sqrt{4k+1}$, $k = 0, 1, \ldots$.

(b) $\sin t$ changes sign at $t = k\pi$, so $C(x)$ has inflection points at $\pi x^2/2 = k\pi$, $x = \pm\sqrt{2k}$, $k = 1, 2, \ldots$; the case $k = 0$ is distinct due to the factor of x in $C''(x)$, but x changes sign at $x = 0$ and $\sin(\pi x^2/2)$ does not, so there is also a point of inflection at $x = 0$.

47. Differentiate: $f(x) = 2e^{2x}$, so $4 + \int_a^x f(t)dt = 4 + \int_a^x 2e^{2t}dt = 4 + e^{2t}\Big]_a^x = 4 + e^{2x} - e^{2a} = e^{2x}$ provided $e^{2a} = 4$, $a = (\ln 4)/2 = \ln 2$.

49. From Exercise 48(d) $\left| e - \left(1 + \frac{1}{50}\right)^{50} \right| < y(50)$, and from the graph $y(50) < 0.06$.



Chapter 5 Review Exercises

1. $-\dfrac{1}{4x^2}+\dfrac{8}{3}x^{3/2}+C$.

3. $-4\cos x+2\sin x+C$.

5. $3x^{1/3}-5e^x+C$.

7. $\tan^{-1}x+2\sin^{-1}x+C$.

9. **(a)** $y(x)=2\sqrt{x}-\dfrac{2}{3}x^{3/2}+C$; $y(1)=0$, so $C=-\dfrac{4}{3}$, $y(x)=2\sqrt{x}-\dfrac{2}{3}x^{3/2}-\dfrac{4}{3}$.

(b) $y(x)=\sin x-5e^x+C$, $y(0)=0=-5+C$, $C=5$, $y(x)=\sin x-5e^x+5$.

(c) $y(x)=2+\displaystyle\int_1^x t^{1/3}dt=2+\dfrac{3}{4}t^{4/3}\Big]_1^x=\dfrac{5}{4}+\dfrac{3}{4}x^{4/3}$.

(d) $y(x)=\displaystyle\int_0^x te^{t^2}dt=\dfrac{1}{2}e^{x^2}-\dfrac{1}{2}$.

11. **(a)** If $u=\sec x$, $du=\sec x\tan x dx$, $\displaystyle\int\sec^2 x\tan x dx=\int u du=u^2/2+C_1=(\sec^2 x)/2+C_1$; if $u=\tan x$, $du=\sec^2 x dx$, $\displaystyle\int\sec^2 x\tan x dx=\int u du=u^2/2+C_2=(\tan^2 x)/2+C_2$.

(b) They are equal only if $\sec^2 x$ and $\tan^2 x$ differ by a constant, which is true.

13. $u=x^2-1, du=2x\,dx, \dfrac{1}{2}\displaystyle\int\frac{du}{u\sqrt{u^2-1}}=\frac{1}{2}\sec^{-1}|u|+C=\frac{1}{2}\sec^{-1}|x^2-1|+C$.

15. $u=5+2\sin 3x$, $du=6\cos 3x dx$; $\displaystyle\int\frac{1}{6\sqrt{u}}du=\frac{1}{3}u^{1/2}+C=\frac{1}{3}\sqrt{5+2\sin 3x}+C$.

17. $u=ax^3+b$, $du=3ax^2dx$; $\displaystyle\int\frac{1}{3au^2}du=-\frac{1}{3au}+C=-\frac{1}{3a^2x^3+3ab}+C$.

19. **(a)** $\displaystyle\sum_{k=0}^{14}(k+4)(k+1)$ **(b)** $\displaystyle\sum_{k=5}^{19}(k-1)(k-4)$

21. $\displaystyle\lim_{n\to+\infty}\sum_{k=1}^{n}\left[4\frac{4k}{n}-\left(\frac{4k}{n}\right)^2\right]\frac{4}{n}=\lim_{n\to+\infty}\frac{64}{n^3}\sum_{k=1}^{n}(kn-k^2)=\lim_{n\to+\infty}\frac{64}{n^3}\left[\frac{n^2(n+1)}{2}-\frac{n(n+1)(2n+1)}{6}\right]=$ $\displaystyle\lim_{n\to+\infty}\frac{64}{6n^3}[n^3-n]=\frac{32}{3}$.

23. $0.351220577, 0.420535296, 0.386502483$.

27. **(a)** $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$. **(b)** $-1-\frac{1}{2}=-\frac{3}{2}$. **(c)** $5\left(-1-\frac{3}{4}\right)=-\frac{35}{4}$. **(d)** -2

(e) Not enough information. **(f)** Not enough information.

29. **(a)** $\int_{-1}^{1} dx+\int_{-1}^{1}\sqrt{1-x^2}\,dx=2(1)+\pi(1)^2/2=2+\pi/2$.

(b) $\frac{1}{3}(x^2+1)^{3/2}\Big]_0^3-\pi(3)^2/4=\frac{1}{3}(10^{3/2}-1)-9\pi/4$.

(c) $u=x^2$, $du=2xdx$; $\frac{1}{2}\int_0^1\sqrt{1-u^2}du=\frac{1}{2}\pi(1)^2/4=\pi/8$.

31. $\left(\frac{1}{3}x^3-2x^2+7x\right)\Big]_{-3}^{0}=48$.

33. $\int_1^3 x^{-2}dx=-\frac{1}{x}\Big]_1^3=2/3$.

35. $\left(\frac{1}{2}x^2-\sec x\right)\Big]_0^1=3/2-\sec(1)$.

37. $\int_0^{3/2}(3-2x)dx+\int_{3/2}^{2}(2x-3)dx=(3x-x^2)\big]_0^{3/2}+(x^2-3x)\big]_{3/2}^{2}=9/4+1/4=5/2$.

39. $\int_1^9\sqrt{x}dx=\frac{2}{3}x^{3/2}\Big]_1^9=\frac{2}{3}(27-1)=52/3$.

41. $\int_1^3 e^x dx=e^x\Big]_1^3=e^3-e$.

43. $A=\int_1^2(-x^2+3x-2)dx=\left(-\frac{1}{3}x^3+\frac{3}{2}x^2-2x\right)\Big]_1^2=1/6$.

45. $A=A_1+A_2=\int_0^1(1-x^2)dx+\int_1^3(x^2-1)dx=2/3+20/3=22/3$.

47. **(a)** x^3+1 **(b)** $F(x)=\left(\frac{1}{4}t^4+t\right)\Big]_1^x=\frac{1}{4}x^4+x-\frac{5}{4}$; $F'(x)=x^3+1$.

49. e^{x^2}

51. $|x-1|$

53. $\dfrac{\cos x}{1+\sin^3 x}$

57. **(a)** $F'(x)=\frac{1}{1+x^2}+\frac{1}{1+(1/x)^2}(-1/x^2)=0$ so F is constant on $(0,+\infty)$.

(b) $F(1) = \int_0^1 \frac{1}{1+t^2}\,dt + \int_0^1 \frac{1}{1+t^2}\,dt = 2\tan^{-1} 1 = \pi/2$, so $F(x) = \tan^{-1} x + \tan^{-1}(1/x) = \pi/2$.

59. **(a)** The domain is $(-\infty, +\infty)$; $F(x)$ is 0 if $x = 1$, positive if $x > 1$, and negative if $x < 1$, because the integrand is positive, so the sign of the integral depends on the orientation (forwards or backwards).

(b) The domain is $[-2, 2]$; $F(x)$ is 0 if $x = -1$, positive if $-1 < x \le 2$, and negative if $-2 \le x < -1$; same reasons as in part (a).

61. **(a)** $f_{\text{ave}} = \frac{1}{3}\int_0^3 x^{1/2}dx = 2\sqrt{3}/3$; $\sqrt{x^*} = 2\sqrt{3}/3$, $x^* = \frac{4}{3}$.

(b) $f_{\text{ave}} = \frac{1}{e-1}\int_1^e \frac{1}{x}dx = \frac{1}{e-1}\ln x\Big]_1^e = \frac{1}{e-1}$; $\frac{1}{x^*} = \frac{1}{e-1}$, $x^* = e - 1$.

63. For $0 < x < 3$ the area between the curve and the x-axis consists of two triangles of equal area but of opposite signs, hence 0. For $3 < x < 5$ the area is a rectangle of width 2 and height 3. For $5 < x < 7$ the area consists of two triangles of equal area but opposite sign, hence 0; and for $7 < x < 10$ the curve is given by $y = (4t - 37)/3$ and $\int_7^{10} (4t - 37)/3\,dt = -3$. Thus the desired average is $\frac{1}{10}(0 + 6 + 0 - 3) = 0.3$.

65. If the acceleration $a =$ const, then $v(t) = at + v_0$, $s(t) = \frac{1}{2}at^2 + v_0 t + s_0$.

67. $s(t) = \int (t^3 - 2t^2 + 1)dt = \frac{1}{4}t^4 - \frac{2}{3}t^3 + t + C$, $s(0) = \frac{1}{4}(0)^4 - \frac{2}{3}(0)^3 + 0 + C = 1$, $C = 1$, $s(t) = \frac{1}{4}t^4 - \frac{2}{3}t^3 + t + 1$.

69. $s(t) = \int (2t - 3)dt = t^2 - 3t + C$, $s(1) = (1)^2 - 3(1) + C = 5$, $C = 7$, $s(t) = t^2 - 3t + 7$.

71. displacement $= s(6) - s(0) = \int_0^6 (2t - 4)dt = (t^2 - 4t)\Big]_0^6 = 12$ m.

distance $= \int_0^6 |2t - 4|dt = \int_0^2 (4 - 2t)dt + \int_2^6 (2t - 4)dt = (4t - t^2)\Big]_0^2 + (t^2 - 4t)\Big]_2^6 = 20$ m.

73. displacement $= \int_1^3 \left(\frac{1}{2} - \frac{1}{t^2}\right) dt = 1/3$ m.

distance $= \int_1^3 |v(t)|dt = -\int_1^{\sqrt{2}} v(t)dt + \int_{\sqrt{2}}^3 v(t)dt = 10/3 - 2\sqrt{2}$ m.

75. $v(t) = -2t + 3$;

displacement $= \int_1^4 (-2t + 3)dt = -6$ m.

distance $= \int_1^4 |-2t + 3|dt = \int_1^{3/2} (-2t + 3)dt + \int_{3/2}^4 (2t - 3)dt = 13/2$ m.

77. Take $t = 0$ when deceleration begins, then $a = -10$ so $v = -10t + C_1$, but $v = 88$ when $t = 0$ which gives $C_1 = 88$ thus $v = -10t + 88$, $t \ge 0$.

(a) $v = 45$ mi/h $= 66$ ft/s, $66 = -10t + 88$, $t = 2.2$ s.

(b) $v = 0$ (the car is stopped) when $t = 8.8$ s, $s = \int v\,dt = \int (-10t + 88)dt = -5t^2 + 88t + C_2$, and taking $s = 0$ when $t = 0$, $C_2 = 0$ so $s = -5t^2 + 88t$. At $t = 8.8$, $s = 387.2$. The car travels 387.2 ft before coming to a stop.

79. From the free-fall model $s = -\dfrac{1}{2}gt^2 + v_0t + s_0$ the ball is caught when $s_0 = -\dfrac{1}{2}gt_1^2 + v_0t_1 + s_0$ with the positive root $t_1 = 2v_0/g$ so the average speed of the ball while it is up in the air is average speed $= \dfrac{1}{t_1}\displaystyle\int_0^{t_1} |v_0 - gt|\, dt =$ $\dfrac{g}{2v_0}\left[\displaystyle\int_0^{v_0/g} (v_0 - gt)\, gt + \int_{v_0/g}^{2v_0/g} (gt - v_0)\, dt\right] = v_0/2.$

81. $u = 2x+1,\ \dfrac{1}{2}\displaystyle\int_1^3 u^4 du = \dfrac{1}{10}u^5\Big]_1^3 = 121/5$, or $\dfrac{1}{10}(2x+1)^5\Big]_0^1 = 121/5.$

83. $\dfrac{2}{3}(3x+1)^{1/2}\Big]_0^1 = 2/3.$

85. $\dfrac{1}{3\pi}\sin^3 \pi x\Big]_0^1 = 0.$

87. $\displaystyle\int_0^1 e^{-x/2}dx = 2(1 - 1/\sqrt{e}).$

89. **(a)** $\displaystyle\lim_{x\to+\infty}\left[\left(1+\frac{1}{x}\right)^x\right]^2 = \left[\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x\right]^2 = e^2.$

(b) $y = 3x,\ \displaystyle\lim_{y\to 0}\left(1+\frac{1}{y}\right)^{y/3} = \lim_{y\to 0}\left[\left(1+\frac{1}{y}\right)^y\right]^{1/3} = e^{1/3}.$

Chapter 5 Making Connections

1. **(a)** $\displaystyle\sum_{k=1}^{n} 2x_k^*\Delta x_k = \sum_{k=1}^{n}(x_k + x_{k-1})(x_k - x_{k-1}) = \sum_{k=1}^{n}(x_k^2 - x_{k-1}^2) = \sum_{k=1}^{n} x_k^2 - \sum_{k=0}^{n-1} x_k^2 = b^2 - a^2.$

(b) By Theorem 5.5.2, f is integrable on $[a,b]$. Using part (a) of Definition 5.5.1, in which we choose any partition and use the midpoints $x_k^* = (x_k + x_{k-1})/2$, we see from part (a) of this exercise that the Riemann sum is equal to $x_n^2 - x_0^2 = b^2 - a^2$. Since the right side of this equation does not depend on partitions, the limit of the Riemann sums as $\max(\Delta x_k) \to 0$ is equal to $b^2 - a^2$.

3. Use the partition $0 < 8(1)^3/n^3 < 8(2)^3/n^3 < \ldots < 8(n-1)^3/n^3 < 8$ with x_k^* as the right endpoint of the k-th interval, $x_k^* = 8k^3/n^3$. Then $\displaystyle\sum_{k=1}^{n} f(x_k^*)\Delta x_k = \sum_{k=1}^{n} \sqrt[3]{8k^3/n^3}\left(\frac{8k^3}{n^3} - \frac{8(k-1)^3}{n^3}\right) = \sum_{k=1}^{n}\frac{16}{n^4}(k^4 - k(k-1)^3) =$ $\dfrac{16}{n^4}\dfrac{3n^4 + 2n^3 - n^2}{4} \to 16\dfrac{3}{4} = 12$ as $n \to \infty$.

5. **(a)** $\displaystyle\sum_{k=1}^{n} g(x_k^*)\Delta x_k = \sum_{k=1}^{n} 2x_k^* f((x_k^*)^2)\Delta x_k = \sum_{k=1}^{n}(x_k + x_{k-1})f((x_k^*)^2)(x_k - x_{k-1}) = \sum_{k=1}^{n} f((x_k^*)^2)(x_k^2 - x_{k-1}^2) =$ $\displaystyle\sum_{k=1}^{n} f(u_k^*)\Delta u_k$. The two Riemann sums are equal.

(b) In part (a) note that $\Delta u_k = \Delta x_k^2 = x_k^2 - x_{k-1}^2 = (x_k + x_{k-1})\Delta x_k$, and since $2 \le x_k \le 3$, $4\Delta x_k \le \Delta u_k$ and $\Delta u_k \le 6\Delta x_k$, so that $\max\{u_k\}$ tends to zero iff $\max\{x_k\}$ tends to zero. $\displaystyle\int_2^3 g(x)\, dx = \lim_{\max(\Delta x_k)\to 0}\sum_{k=1}^{n} g(x_k^*)\Delta x_k =$ $\displaystyle\lim_{\max(\Delta u_k)\to 0}\sum_{k=1}^{n} f(u_k^*)\Delta u_k = \int_4^9 f(u)\, du.$

(c) Since the symbol g is already in use, we shall use γ to denote the mapping $u = \gamma(x) = x^2$ of Theorem 5.9.1. Applying the Theorem, $\displaystyle\int_4^9 f(u)\,du = \int_2^3 f(\gamma(x))\gamma'(x)\,dx = \int_2^3 f(x^2)2x\,dx = \int_2^3 g(x)\,dx.$