Lecture # 07

Limits

1.1.1 LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x \to a} f(x) = L \tag{6}$$

which is read "the limit of $f(x)$ as x approaches a is L" or "$f(x)$ approaches L as x approaches a." The expression in (6) can also be written as

$$f(x) \to L \quad \text{as} \quad x \to a \tag{7}$$

Example 2 Use numerical evidence to make a conjecture about the value of

$$\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1} \tag{8}$$

Solution. Although the function

$$f(x) = \frac{x-1}{\sqrt{x}-1} \tag{9}$$

is undefined at $x = 1$, this has no bearing on the limit. Table 1.1.1 shows sample x-values approaching 1 from the left side and from the right side. In both cases the corresponding values of $f(x)$, calculated to six decimal places, appear to get closer and closer to 2, and hence we conjecture that

$$\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1} = 2$$

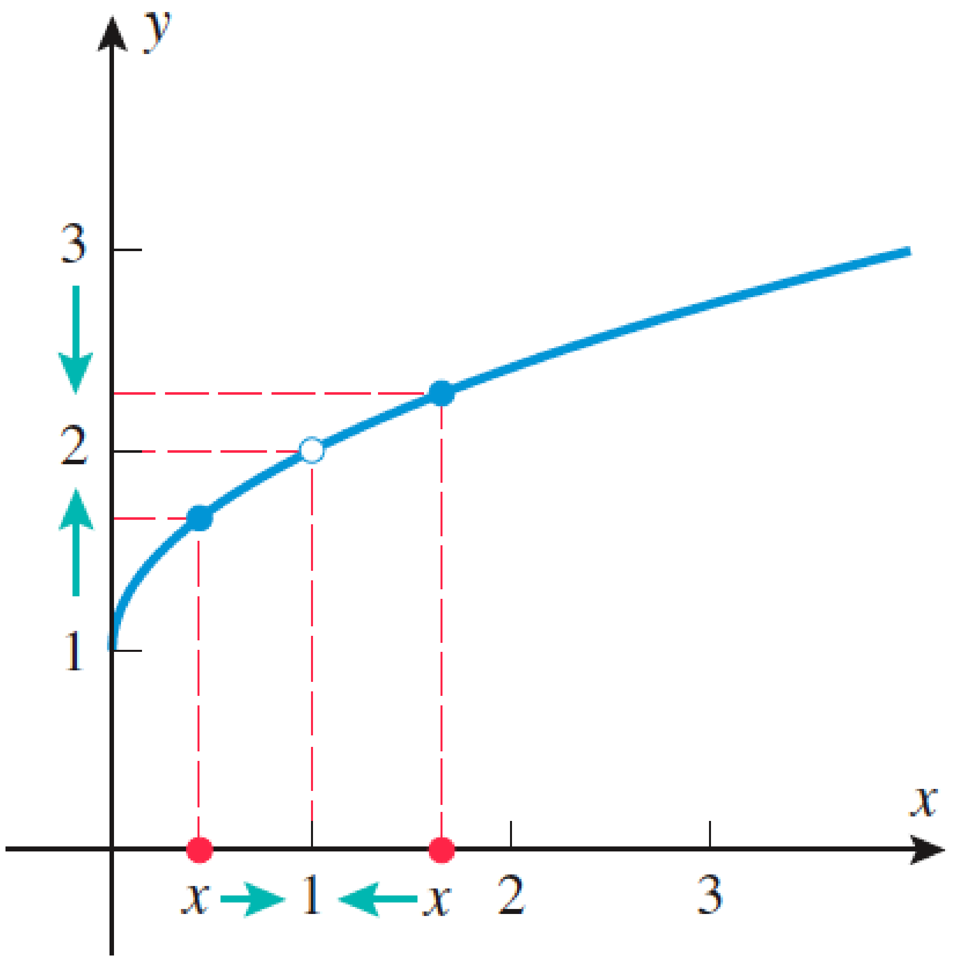
This is consistent with the graph of f shown in Figure 1.1.9. In the next section we will show how to obtain this result algebraically. ◀

TECHNOLOGY MASTERY

Use a graphing utility to generate the graph of the equation $y = f(x)$ for the function in (9). Find a window containing $x = 1$ in which all values of $f(x)$ are within 0.5 of $y = 2$ and one in which all values of $f(x)$ are within 0.1 of $y = 2$.

Table 1.1.1

x	0.99	0.999	0.9999	0.99999		1.00001	1.0001	1.001	1.01
$f(x)$	1.994987	1.999500	1.999950	1.999995		2.000005	2.000050	2.000500	2.004988



1.1.2 ONE-SIDED LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write

$$\lim_{x \to a^+} f(x) = L \tag{14}$$

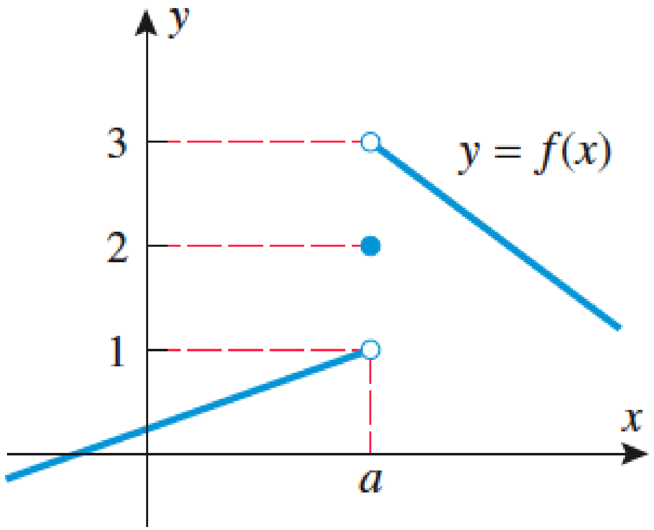
and if the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write

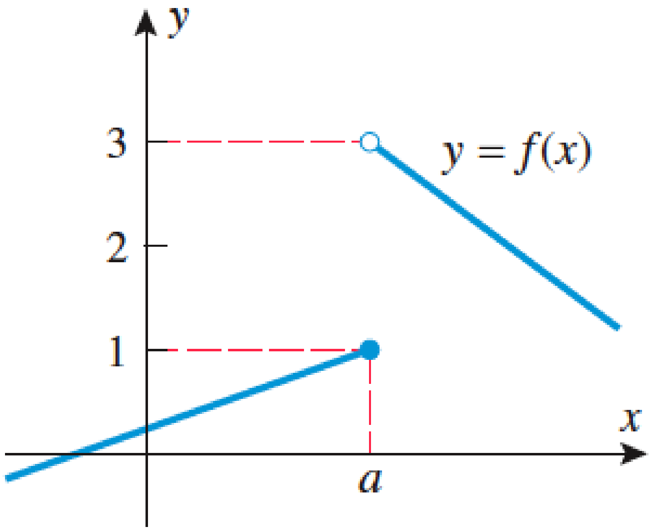
$$\lim_{x \to a^-} f(x) = L \tag{15}$$

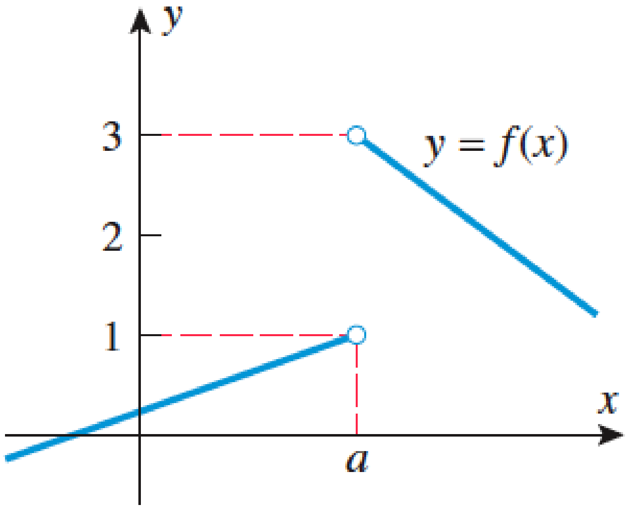
Expression (14) is read "the limit of $f(x)$ as x approaches a from the right is L" or "$f(x)$ approaches L as x approaches a from the right." Similarly, expression (15) is read "the limit of $f(x)$ as x approaches a from the left is L" or "$f(x)$ approaches L as x approaches a from the left."

1.1.3 THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS The two-sided limit of a function $f(x)$ exists at a if and only if both of the one-sided limits exist at a and have the same value; that is,

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$







Solution. The functions in all three figures have the same one-sided limits as $x \to a$, since the functions are identical, except at $x = a$. These limits are

$$\lim_{x \to a^+} f(x) = 3 \quad \text{and} \quad \lim_{x \to a^-} f(x) = 1$$

In all three cases the two-sided limit does not exist as $x \to a$ because the one-sided limits are not equal. ◀

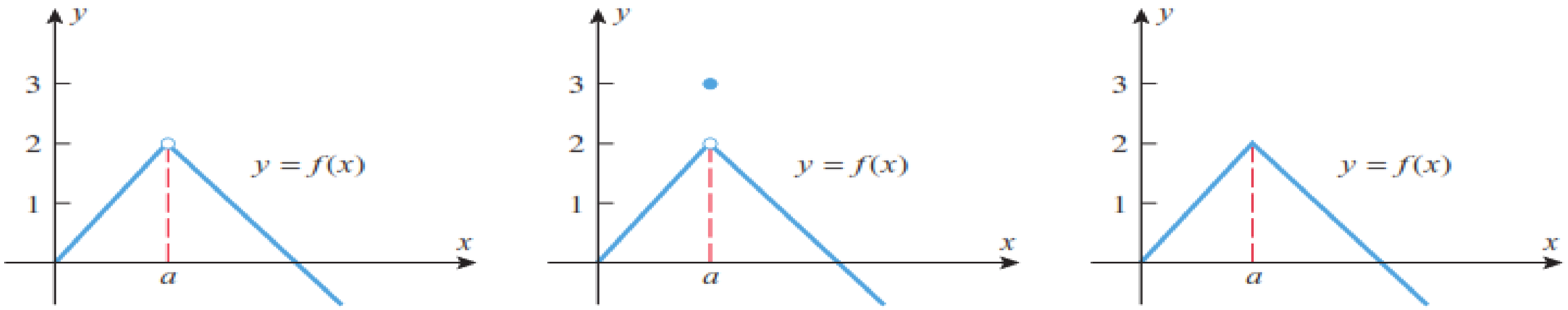
► **Example 6** For the functions in Figure 1.1.14, find the one-sided and two-sided limits at $x = a$ if they exist.

Solution. As in the preceding example, the value of f at $x = a$ has no bearing on the limits as $x \to a$, so in all three cases we have

$$\lim_{x \to a^+} f(x) = 2 \quad \text{and} \quad \lim_{x \to a^-} f(x) = 2$$

Since the one-sided limits are equal, the two-sided limit exists and

$$\lim_{x \to a} f(x) = 2 \quad ◄$$



1.1.4 INFINITE LIMITS (AN INFORMAL VIEW) The expressions

$$\lim_{x\to a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x\to a^+} f(x) = +\infty$$

denote that $f(x)$ increases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

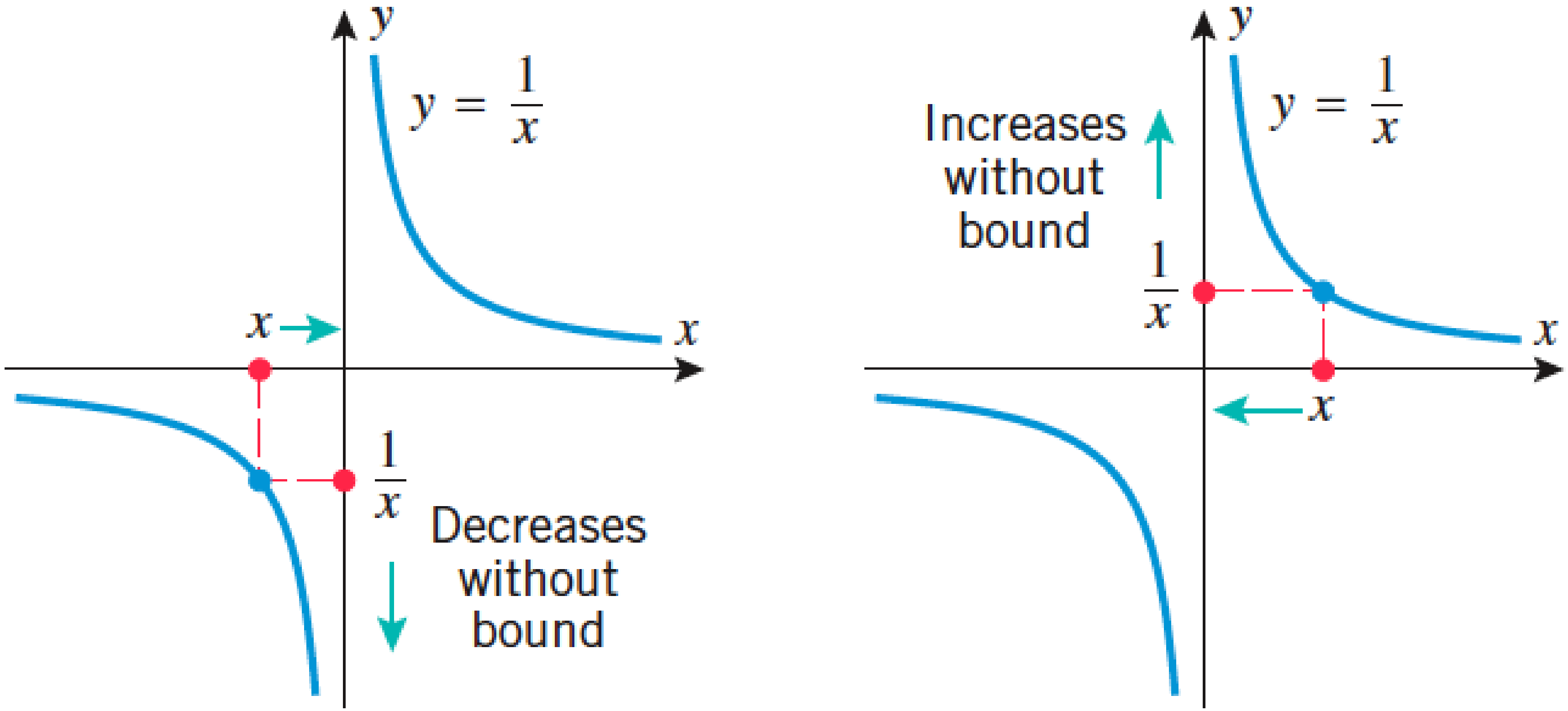
$$\lim_{x\to a} f(x) = +\infty$$

Similarly, the expressions

$$\lim_{x\to a^-} f(x) = -\infty \quad \text{and} \quad \lim_{x\to a^+} f(x) = -\infty$$

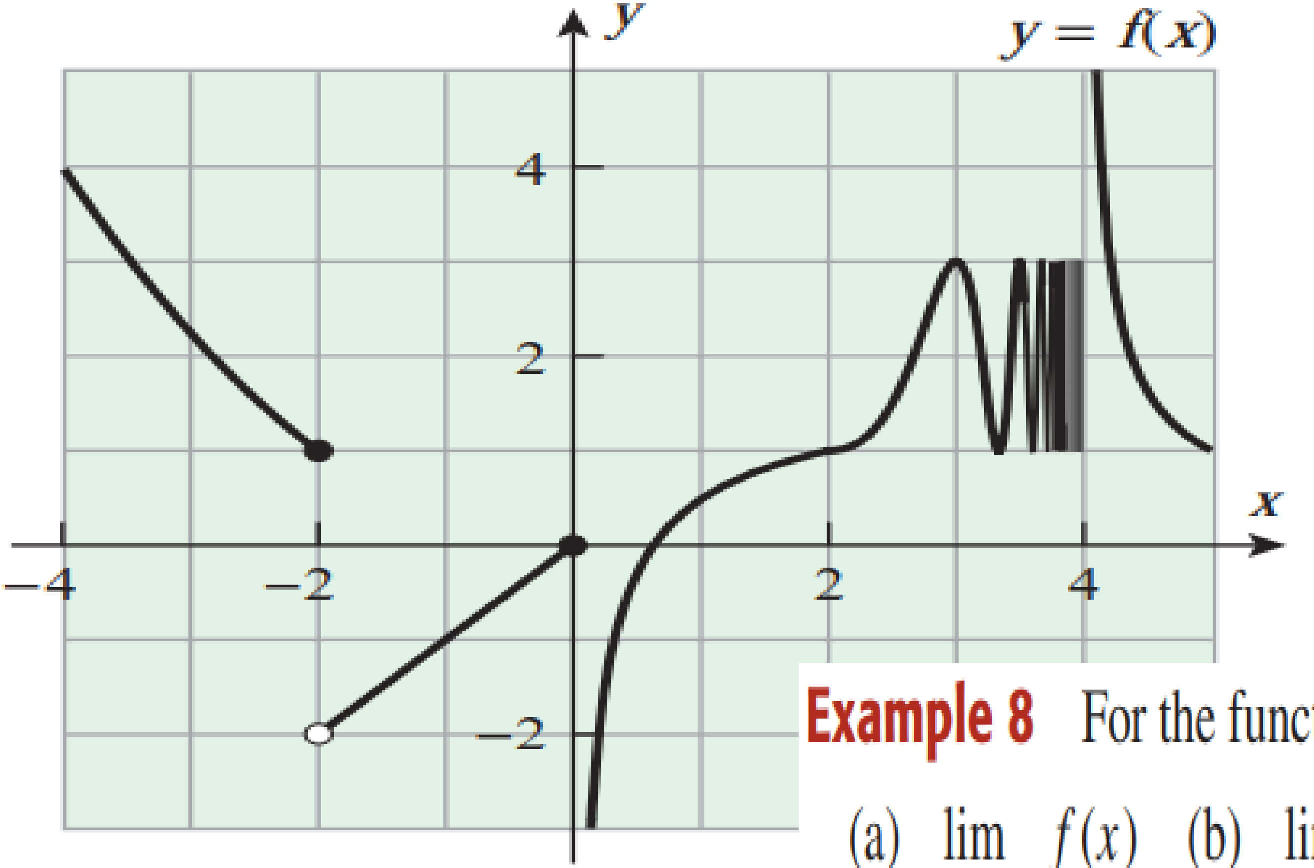
denote that $f(x)$ decreases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x\to a} f(x) = -\infty$$



x	−1	−0.1	−0.01	−0.001	−0.0001	0	0.0001	0.001	0.01	0.1	1
$\frac{1}{x}$	−1	−10	−100	−1000	−10,000		10,000	1000	100	10	1

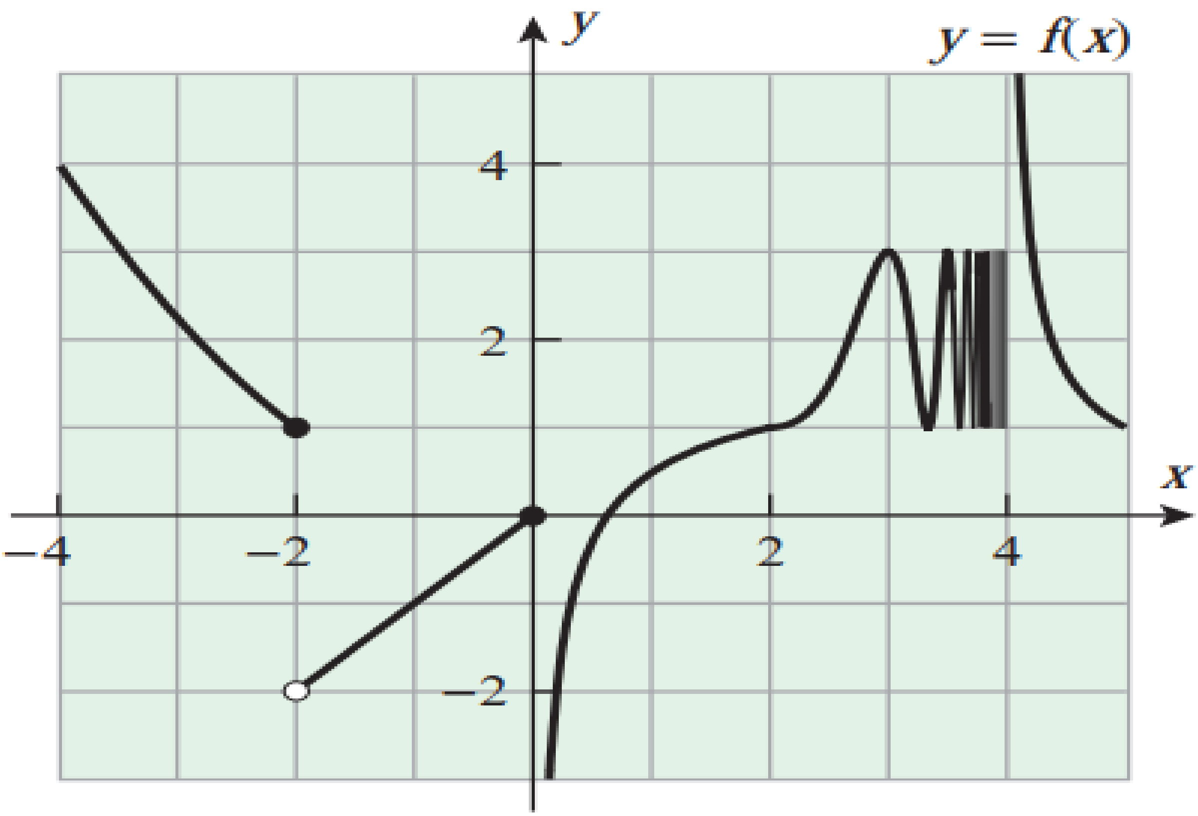
Left side → ← Right side



Example 8 For the function f graphed in Figure 1.1.18, find

(a) $\lim_{x \to -2^-} f(x)$ (b) $\lim_{x \to -2^+} f(x)$ (c) $\lim_{x \to 0^-} f(x)$ (d) $\lim_{x \to 0^+} f(x)$

(e) $\lim_{x \to 4^-} f(x)$ (f) $\lim_{x \to 4^+} f(x)$ (g) the vertical asymptotes of the graph of f.



Solution (a) and (b).

$$\lim_{x \to -2^-} f(x) = 1 = f(-2) \quad \text{and} \quad \lim_{x \to -2^+} f(x) = -2$$

Solution (c) and (d).

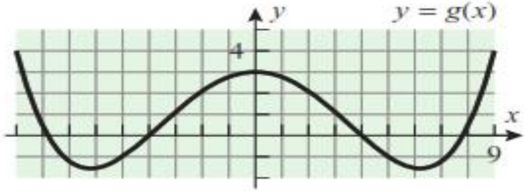
$$\lim_{x \to 0^-} f(x) = 0 = f(0) \quad \text{and} \quad \lim_{x \to 0+} f(x) = -\infty$$

Solution (e) and (f).

$$\lim_{x \to 4^-} f(x) \text{ does not exist due to oscillation} \quad \text{and} \quad \lim_{x \to 4^+} f(x) = +\infty$$

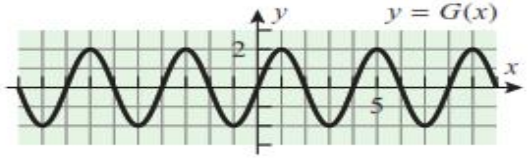
Solution (g). The y-axis and the line $x = 4$ are vertical asymptotes for the graph of f.

1. For the function g graphed in the accompanying figure, find
(a) $\lim_{x\to 0^-} g(x)$ (b) $\lim_{x\to 0^+} g(x)$
(c) $\lim_{x\to 0} g(x)$ (d) $g(0)$.



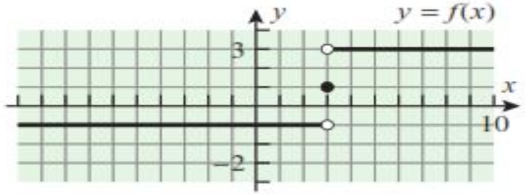
◀ **Figure Ex-1**

2. For the function G graphed in the accompanying figure, find
(a) $\lim_{x\to 0^-} G(x)$ (b) $\lim_{x\to 0^+} G(x)$
(c) $\lim_{x\to 0} G(x)$ (d) $G(0)$.



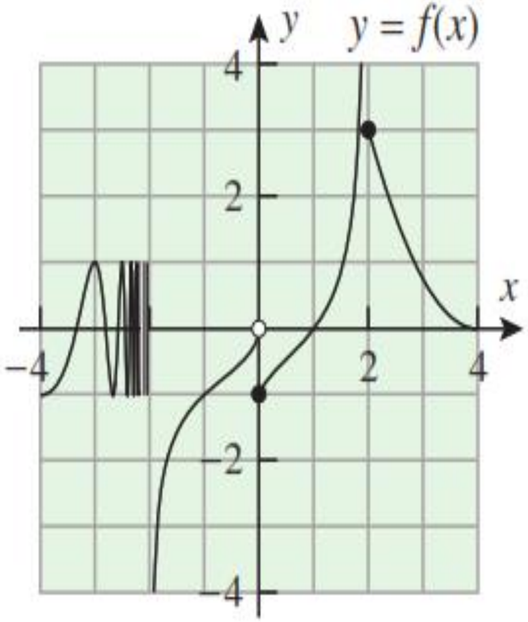
◀ **Figure Ex-2**

3. For the function f graphed in the accompanying figure, find
(a) $\lim_{x\to 3^-} f(x)$ (b) $\lim_{x\to 3^+} f(x)$
(c) $\lim_{x\to 3} f(x)$ (d) $f(3)$.



◀ **Figure Ex-3**

10. For the function f graphed in the accompanying figure, find
(a) $\lim_{x\to -2^-} f(x)$ (b) $\lim_{x\to -2^+} f(x)$ (c) $\lim_{x\to 0^-} f(x)$
(d) $\lim_{x\to 0^+} f(x)$ (e) $\lim_{x\to 2^-} f(x)$ (f) $\lim_{x\to 2^+} f(x)$
(g) the vertical asymptotes of the graph of f.



◀ **Figure Ex-10**

C **13–16** (i) Make a guess at the limit (if it exists) by evaluating the function at the specified x-values. (ii) Confirm your conclusions about the limit by graphing the function over an appropriate interval. (iii) If you have a CAS, then use it to find the limit. [*Note:* For the trigonometric functions, be sure to put your calculating and graphing utilities in radian mode.]

13. (a) $\lim_{x \to 1} \dfrac{x-1}{x^3-1}$; $x = 2, 1.5, 1.1, 1.01, 1.001, 0, 0.5, 0.9, 0.99, 0.999$

(b) $\lim_{x \to 1^+} \dfrac{x+1}{x^3-1}$; $x = 2, 1.5, 1.1, 1.01, 1.001, 1.0001$

(c) $\lim_{x \to 1^-} \dfrac{x+1}{x^3-1}$; $x = 0, 0.5, 0.9, 0.99, 0.999, 0.9999$

14. (a) $\lim_{x \to 0} \dfrac{\sqrt{x+1}-1}{x}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$

(b) $\lim_{x \to 0^+} \dfrac{\sqrt{x+1}+1}{x}$; $x = 0.25, 0.1, 0.001, 0.0001$

Ex # 1.1
Q # 1-16

Lecture # 08

More about Limits

1.2.2 THEOREM *Let a be a real number, and suppose that*

$$\lim_{x \to a} f(x) = L_1 \quad \textit{and} \quad \lim_{x \to a} g(x) = L_2$$

That is, the limits exist and have values L_1 and L_2, respectively. Then:

(a) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L_1 + L_2$

(b) $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L_1 - L_2$

(c) $\lim_{x \to a} [f(x)g(x)] = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right) = L_1 L_2$

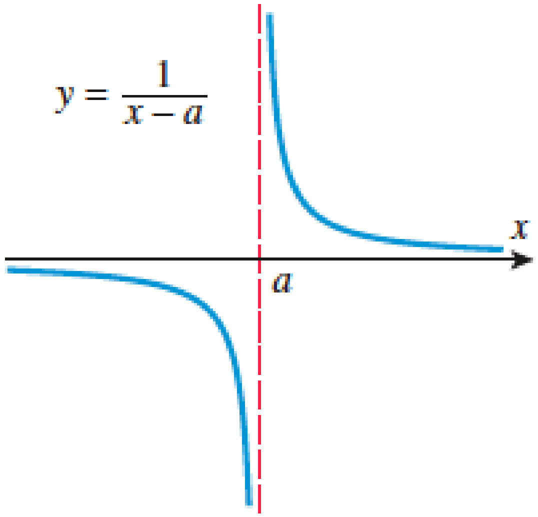
(d) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L_1}{L_2}$, *provided $L_2 \neq 0$*

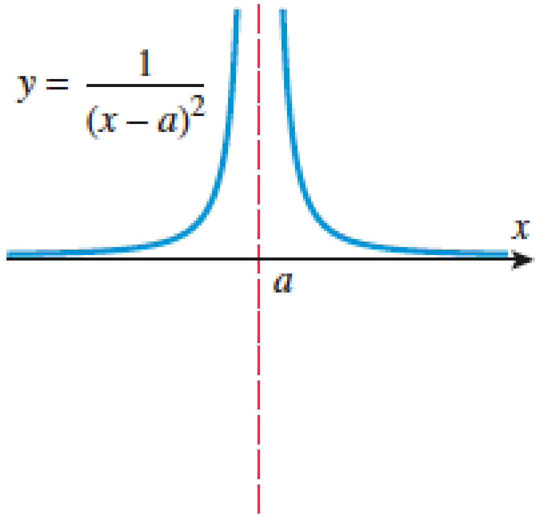
(e) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L_1}$, *provided $L_1 > 0$ if n is even.*

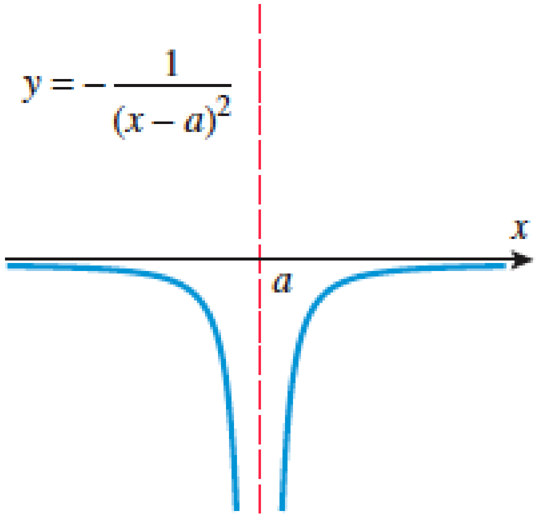
Moreover, these statements are also true for the one-sided limits as $x \to a^-$ or as $x \to a^+$.

1.2.1 THEOREM *Let a and k be real numbers.*

(a) $\lim_{x \to a} k = k$ (b) $\lim_{x \to a} x = a$ (c) $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ (d) $\lim_{x \to 0^+} \frac{1}{x} = +\infty$







$$\lim_{x \to a^+} \frac{1}{x-a} = +\infty$$
$$\lim_{x \to a^-} \frac{1}{x-a} = -\infty$$

$$\lim_{x \to a} \frac{1}{(x-a)^2} = +\infty$$

$$\lim_{x \to a} -\frac{1}{(x-a)^2} = -\infty$$

1.2.4 THEOREM *Let*

$$f(x) = \frac{p(x)}{q(x)}$$

be a rational function, and let a be any real number.

(*a*) *If* $q(a) \neq 0$, *then* $\lim_{x \to a} f(x) = f(a)$.

(*b*) *If* $q(a) = 0$ *but* $p(a) \neq 0$, *then* $\lim_{x \to a} f(x)$ *does not exist.*

Find the limits..

1. $\lim\limits_{x\to 8} 7$ = 7
2. $\lim\limits_{x\to \infty} (-3)$ = -3
3. $\lim\limits_{x\to 0} \pi$ = π
4. $\lim\limits_{y\to 3} 12y$ = 12(3) = 36
5. $\lim\limits_{h\to \infty} (-2h)$ = - ∞
6. $\lim\limits_{x\to 5} \sqrt{x^3 - 3x - 1}$ = $\sqrt{125 - 15 - 1}$ = $\sqrt{109}$
7. $\lim\limits_{x\to 0} (x^4 + 12x + 2)$ = 0 + 0 + 2 = 2
8. $\lim\limits_{x\to 3} \frac{x^2 - 2x}{x+1}$ = $\frac{9-6}{4}$ = $\frac{3}{4}$
9. $\lim\limits_{y\to 3} \frac{(y-1)(y-2)}{y+1}$ = $\frac{2.1}{4}$ $=$ $\frac{1}{2}$
10. $\lim\limits_{x\to 0} \frac{6x-9}{x^3 - 12x + 3}$ = $\frac{-9}{3}$ = -3

1. Given that

$$\lim_{x\to a} f(x) = 2, \quad \lim_{x\to a} g(x) = -4, \quad \lim_{x\to a} h(x) = 0$$

find the limits.

(a) $\lim_{x\to a} [f(x) + 2g(x)]$

(b) $\lim_{x\to a} [h(x) - 3g(x) + 1]$

(c) $\lim_{x\to a} [f(x)g(x)]$ (d) $\lim_{x\to a} [g(x)]^2$

(e) $\lim_{x\to a} \sqrt[3]{6 + f(x)}$ (f) $\lim_{x\to a} \dfrac{2}{g(x)}$

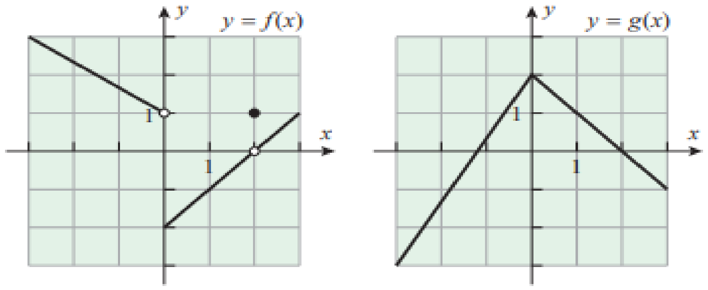
2. Use the graphs of f and g in the accompanying figure to find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x\to 2} [f(x) + g(x)]$ (b) $\lim_{x\to 0} [f(x) + g(x)]$

(c) $\lim_{x\to 0^+} [f(x) + g(x)]$ (d) $\lim_{x\to 0^-} [f(x) + g(x)]$

(e) $\lim_{x\to 2} \dfrac{f(x)}{1 + g(x)}$ (f) $\lim_{x\to 2} \dfrac{1 + g(x)}{f(x)}$

(g) $\lim_{x\to 0^+} \sqrt{f(x)}$ (h) $\lim_{x\to 0^-} \sqrt{f(x)}$



▲ **Figure Ex-2**

3–30 Find the limits.

3. $\lim_{x\to 2} x(x-1)(x+1)$

4. $\lim_{x\to 3} x^3 - 3x^2 + 9x$

5. $\lim_{x\to 3} \dfrac{x^2 - 2x}{x + 1}$

6. $\lim_{x\to 0} \dfrac{6x - 9}{x^3 - 12x + 3}$

7. $\lim_{x\to 1^+} \dfrac{x^4 - 1}{x - 1}$

8. $\lim_{t\to -2} \dfrac{t^3 + 8}{t + 2}$

9. $\lim_{x\to -1} \dfrac{x^2 + 6x + 5}{x^2 - 3x - 4}$

10. $\lim_{x\to 2} \dfrac{x^2 - 4x + 4}{x^2 + x - 6}$

11. $\lim_{x\to -1} \dfrac{2x^2 + x - 1}{x + 1}$

12. $\lim_{x\to 1} \dfrac{3x^2 - x - 2}{2x^2 + x - 3}$

13. $\lim_{t\to 2} \dfrac{t^3 + 3t^2 - 12t + 4}{t^3 - 4t}$

14. $\lim_{t\to 1} \dfrac{t^3 + t^2 - 5t + 3}{t^3 - 3t + 2}$

15. $\lim_{x\to 3^+} \dfrac{x}{x - 3}$

16. $\lim_{x\to 3^-} \dfrac{x}{x - 3}$

17. $\lim_{x\to 3} \dfrac{x}{x - 3}$

18. $\lim_{x\to 2^+} \dfrac{x}{x^2 - 4}$

19. $\lim_{x\to 2^-} \dfrac{x}{x^2 - 4}$

20. $\lim_{x\to 2} \dfrac{x}{x^2 - 4}$

21. $\lim_{y\to 6^+} \dfrac{y + 6}{y^2 - 36}$

22. $\lim_{y\to 6^-} \dfrac{y + 6}{y^2 - 36}$

23. $\lim_{y\to 6} \dfrac{y + 6}{y^2 - 36}$

24. $\lim_{x\to 4^+} \dfrac{3 - x}{x^2 - 2x - 8}$

25. $\lim_{x\to 4^-} \dfrac{3 - x}{x^2 - 2x - 8}$

26. $\lim_{x\to 4} \dfrac{3 - x}{x^2 - 2x - 8}$

27. $\lim_{x\to 2^+} \dfrac{1}{|2 - x|}$

28. $\lim_{x\to 3^-} \dfrac{1}{|x - 3|}$

29. $\lim_{x\to 9} \dfrac{x - 9}{\sqrt{x} - 3}$

30. $\lim_{y\to 4} \dfrac{4 - y}{2 - \sqrt{y}}$

31. Let

$$f(x) = \begin{cases} x - 1, & x \le 3 \\ 3x - 7, & x > 3 \end{cases}$$

(cont.)

32. Let

$$g(t) = \begin{cases} t - 2, & t < 0 \\ t^2, & 0 \leq t \leq 2 \\ 2t, & t > 2 \end{cases}$$

Find

(a) $\lim_{t \to 0} g(t)$ (b) $\lim_{t \to 1} g(t)$ (c) $\lim_{t \to 2} g(t)$.

Ex # 1.2
Q # 1-32

Lecture # 09

Limits & Continuity

1.5.1 DEFINITION A function f is said to be ***continuous at*** $x = c$ provided the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

- The function f is undefined at c (Figure 1.5.1a).
- The limit of $f(x)$ does not exist as x approaches c (Figures 1.5.1b, 1.5.1c).
- The value of the function and the value of the limit at c are different (Figure 1.5.1d).

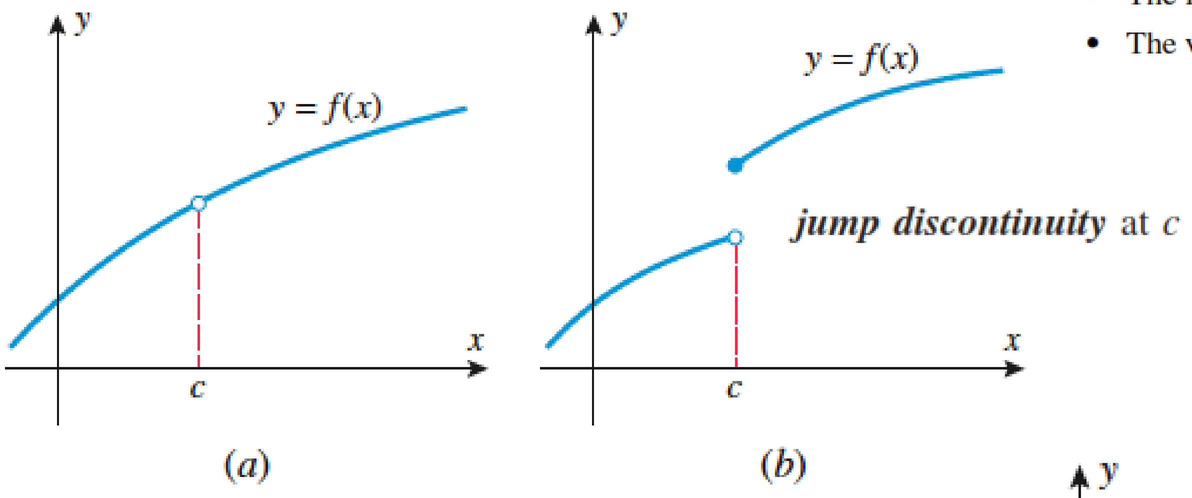
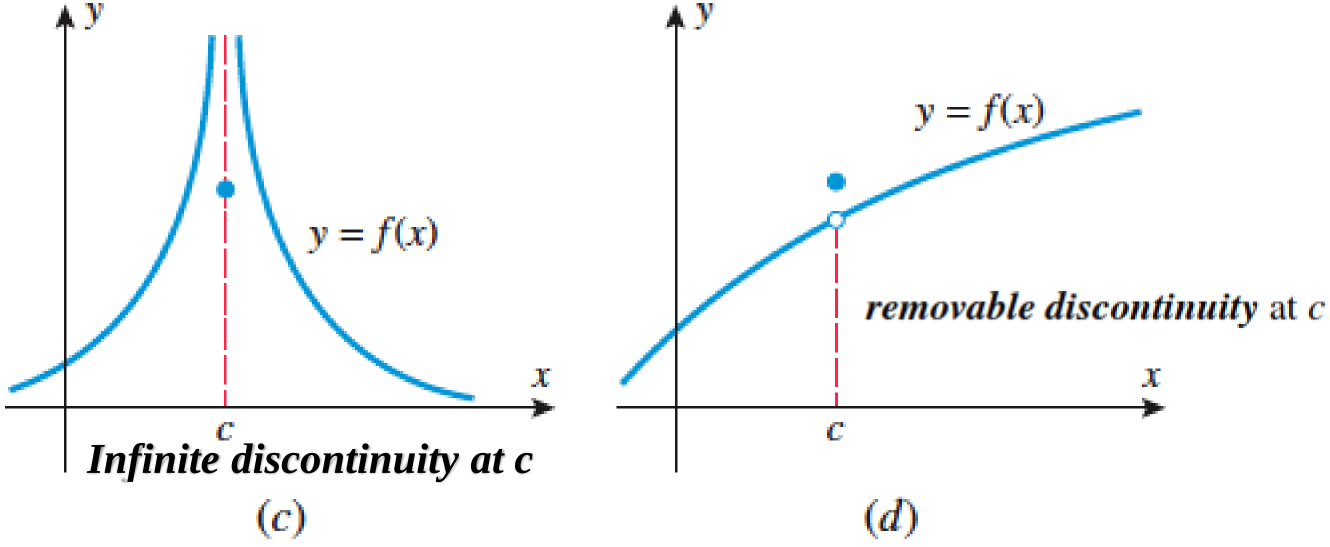


Figure 1.5.1



1.5.2 DEFINITION A function f is said to be ***continuous on a closed interval*** $[a, b]$ if the following conditions are satisfied:

1. f is continuous on (a, b).
2. f is continuous from the right at a.
3. f is continuous from the left at b.

1.5.3 THEOREM *If the functions f and g are continuous at c, then*

(a) $f + g$ *is continuous at* c.

(b) $f - g$ *is continuous at* c.

(c) fg *is continuous at* c.

(d) f/g *is continuous at* c *if* $g(c) \neq 0$ *and has a discontinuity at* c *if* $g(c) = 0$.

National University of computer and emerging sciences

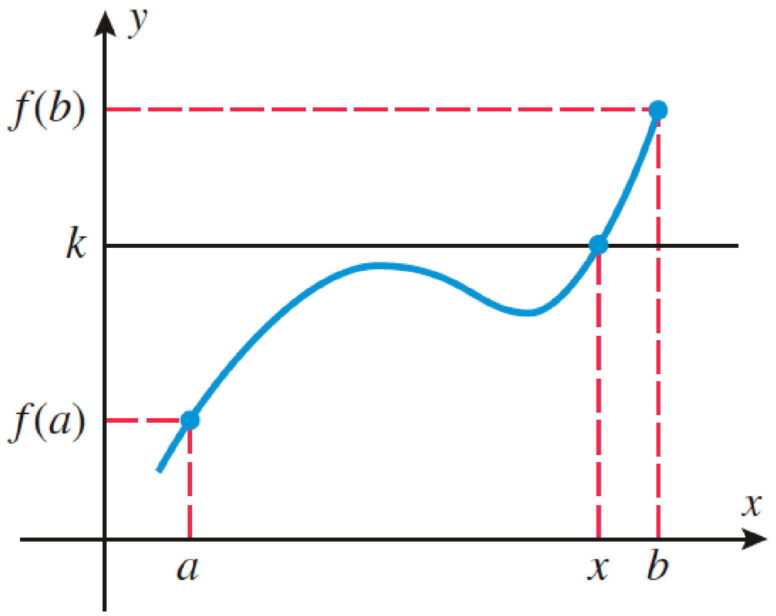
Foundation for Advancement of Science and Technology

1.5.5 THEOREM *If* $\lim_{x \to c} g(x) = L$ *and if the function* f *is continuous at* L, *then* $\lim_{x \to c} f(g(x)) = f(L)$. *That is,*

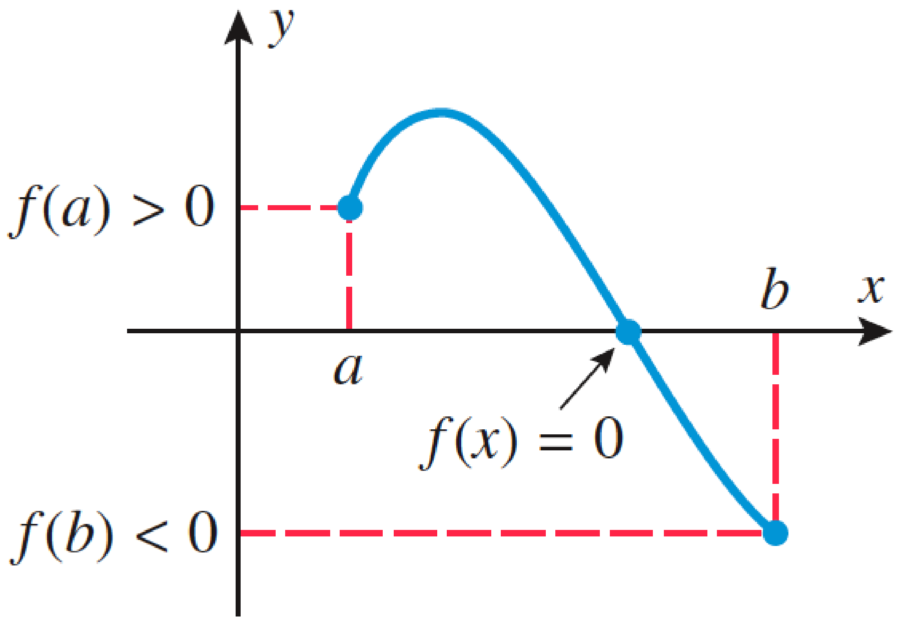
$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right)$$

This equality remains valid if $\lim_{x \to c}$ *is replaced everywhere by one of* $\lim_{x \to c^+}$, $\lim_{x \to c^-}$, $\lim_{x \to +\infty}$, *or* $\lim_{x \to -\infty}$.

1.5.7 THEOREM (Intermediate-Value Theorem) *If f is continuous on a closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, inclusive, then there is at least one number x in the interval $[a, b]$ such that $f(x) = k$.*



1.5.8 THEOREM *If f is continuous on $[a, b]$, and if $f(a)$ and $f(b)$ are nonzero and have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b).*

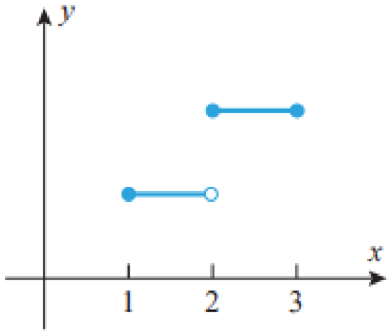


1–4 Let f be the function whose graph is shown. On which of the following intervals, if any, is f continuous?

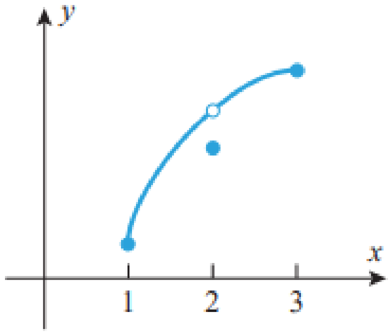
(a) $[1, 3]$ (b) $(1, 3)$ (c) $[1, 2]$
(d) $(1, 2)$ (e) $[2, 3]$ (f) $(2, 3)$

For each interval on which f is not continuous, indicate which conditions for the continuity of f do not hold.

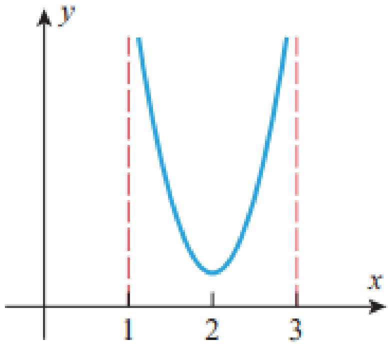
1.



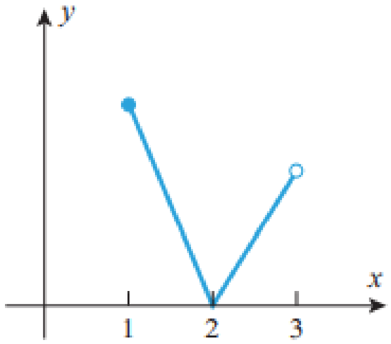
2.



3.



4.



11–22 Find values of x, if any, at which f is not continuous.

11. $f(x) = 5x^4 - 3x + 7$

12. $f(x) = \sqrt[3]{x - 8}$

13. $f(x) = \dfrac{x+2}{x^2+4}$

14. $f(x) = \dfrac{x+2}{x^2-4}$

15. $f(x) = \dfrac{x}{2x^2+x}$

16. $f(x) = \dfrac{2x+1}{4x^2+4x+5}$

17. $f(x) = \dfrac{3}{x} + \dfrac{x-1}{x^2-1}$

18. $f(x) = \dfrac{5}{x} + \dfrac{2x}{x+4}$

19. $f(x) = \dfrac{x^2+6x+9}{|x|+3}$

20. $f(x) = \left|4 - \dfrac{8}{x^4+x}\right|$

21. $f(x) = \begin{cases} 2x+3, & x \le 4 \\ 7 + \dfrac{16}{x}, & x > 4 \end{cases}$

22. $f(x) = \begin{cases} \dfrac{3}{x-1}, & x \ne 1 \\ 3, & x = 1 \end{cases}$

35. (a) $f(x) = \dfrac{|x|}{x}$ (b) $f(x) = \dfrac{x^2+3x}{x+3}$

(c) $f(x) = \dfrac{x-2}{|x|-2}$

29–30 Find a value of the constant k, if possible, that will make the function continuous everywhere.

29. (a) $f(x) = \begin{cases} 7x - 2, & x \leq 1 \\ kx^2, & x > 1 \end{cases}$

(b) $f(x) = \begin{cases} kx^2, & x \leq 2 \\ 2x + k, & x > 2 \end{cases}$

30. (a) $f(x) = \begin{cases} 9 - x^2, & x \geq -3 \\ k/x^2, & x < -3 \end{cases}$

(b) $f(x) = \begin{cases} 9 - x^2, & x \geq 0 \\ k/x^2, & x < 0 \end{cases}$

31. Find values of the constants k and m, if possible, that will make the function f continuous everywhere.

$$f(x) = \begin{cases} x^2 + 5, & x > 2 \\ m(x + 1) + k, & -1 < x \leq 2 \\ 2x^3 + x + 7, & x \leq -1 \end{cases}$$

35–36 Find the values of x (if any) at which f is not continuous, and determine whether each such value is a removable discontinuity.

35. (a) $f(x) = \dfrac{|x|}{x}$ (b) $f(x) = \dfrac{x^2 + 3x}{x + 3}$

(c) $f(x) = \dfrac{x - 2}{|x| - 2}$

36. (a) $f(x) = \dfrac{x^2 - 4}{x^3 - 8}$ (b) $f(x) = \begin{cases} 2x - 3, & x \le 2 \\ x^2, & x > 2 \end{cases}$

(c) $f(x) = \begin{cases} 3x^2 + 5, & x \ne 1 \\ 6, & x = 1 \end{cases}$

Do Ex # 1.5

Q # 1-6 , 11-22, 29, 30, 35 & 36