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Chapter # 9
Ex# 9.3

Infinite Series

Infinite Series:

Let $\frac{1}{3}$, we know that:

$$\frac{1}{3} = 0.33333333........$$

$$\frac{1}{3} = 0.3 + 0.03 + 0.003 + 0.0003 + \cdots$$

9.3.1 DEFINITION An ***infinite series*** is an expression that can be written in the form

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots$$

The numbers $u_1, u_2, u_3, \ldots$ are called the ***terms*** of the series.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

We use a similar idea to determine whether or not a general series $\boxed{1}$ has a sum. We consider the **partial sums**

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

and, in general,

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

2 **Definition** Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

EXAMPLE 1 Suppose we know that the sum of the first n terms of the series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = a_1 + a_2 + \cdots + a_n = \frac{2n}{3n+5}$$

Then the sum of the series is the limit of the sequence $\{s_n\}$:

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2n}{3n+5} = \lim_{n\to\infty} \frac{2}{3+\frac{5}{n}} = \frac{2}{3}$$

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$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots + ar^k + \cdots \quad (a \neq 0)$$

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Geometric Series:

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots + ar^k + \cdots \quad (a \neq 0) \tag{5}$$

Such series are called ***geometric series***, and the number r is called the ***ratio*** for the series. Here are some examples:

$$1 + 2 + 4 + 8 + \cdots + 2^k + \cdots \qquad \boxed{a = 1, r = 2}$$

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^k} + \cdots \qquad \boxed{a = \tfrac{3}{10}, r = \tfrac{1}{10}}$$

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots + (-1)^{k+1}\frac{1}{2^k} + \cdots \qquad \boxed{a = \tfrac{1}{2}, r = -\tfrac{1}{2}}$$

9.3.3 THEOREM *A geometric series*

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \cdots + ar^k + \cdots \quad (a \neq 0)$$

converges if $|r| < 1$ *and diverges if* $|r| \geq 1$. *If the series converges, then the sum is*

$$\boxed{\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}}$$

► **Example 2** In each part, determine whether the series converges, and if so find its sum.

$$\text{(a)}\ \sum_{k=0}^{\infty} \frac{5}{4^k} \qquad \text{(b)}\ \sum_{k=1}^{\infty} 3^{2k} 5^{1-k}$$

Solution (a). This is a geometric series with $a = 5$ and $r = \frac{1}{4}$. Since $|r| = \frac{1}{4} < 1$, the series converges and the sum is

$$\frac{a}{1-r} = \frac{5}{1-\frac{1}{4}} = \frac{20}{3}$$

(Figure 9.3.3).

Solution (b). This is a geometric series in concealed form, since we can rewrite it as

$$\sum_{k=1}^{\infty} 3^{2k} 5^{1-k} = \sum_{k=1}^{\infty} \frac{9^k}{5^{k-1}} = \sum_{k=1}^{\infty} 9 \left(\frac{9}{5}\right)^{k-1}$$

Since $r = \frac{9}{5} > 1$, the series diverges. ◄

V EXAMPLE 3 Find the sum of the geometric series

$$5 - \tfrac{10}{3} + \tfrac{20}{9} - \tfrac{40}{27} + \cdots$$

SOLUTION The first term is $a = 5$ and the common ratio is $r = -\frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series is convergent by $\boxed{4}$ and its sum is

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{5}{1 - \left(-\frac{2}{3}\right)} = \frac{5}{\frac{5}{3}} = 3$$

Example 5 Determine whether the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots$$

converges or diverges. If it converges, find the sum.

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$$

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

from which we obtain the sum

$$\begin{aligned}
s_n &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{n} + \frac{1}{n}\right) - \frac{1}{n+1} \\
&= 1 - \frac{1}{n+1} \qquad (10)
\end{aligned}$$

Thus,

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n \to +\infty} s_n = \lim_{n \to +\infty} \left(1 - \frac{1}{n+1}\right) = 1 \blacktriangleleft$$

One of the most important of all diverging series is the ***harmonic series***,

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

1–2 In each part, find exact values for the first four partial sums, find a closed form for the nth partial sum, and determine whether the series converges by calculating the limit of the nth partial sum. If the series converges, then state its sum.

1. (a) $2 + \frac{2}{5} + \frac{2}{5^2} + \cdots + \frac{2}{5^{k-1}} + \cdots$

(b) $\frac{1}{4} + \frac{2}{4} + \frac{2^2}{4} + \cdots + \frac{2^{k-1}}{4} + \cdots$

(c) $\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{(k+1)(k+2)} + \cdots$

2. (a) $\sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k$ (b) $\sum_{k=1}^{\infty} 4^{k-1}$ (c) $\sum_{k=1}^{\infty} \left(\frac{1}{k+3} - \frac{1}{k+4}\right)$

Sol:

1. (a) $s_1 = 2$, $s_2 = 12/5$, $s_3 = \frac{62}{25}$, $s_4 = \frac{312}{125}$ $s_n = \frac{2-2(1/5)^n}{1-1/5} = \frac{5}{2} - \frac{5}{2}(1/5)^n$, $\lim_{n\to+\infty} s_n = \frac{5}{2}$, converges.

(b) $s_1 = \frac{1}{4}$, $s_2 = \frac{3}{4}$, $s_3 = \frac{7}{4}$, $s_4 = \frac{15}{4}$ $s_n = \frac{(1/4)-(1/4)2^n}{1-2} = -\frac{1}{4} + \frac{1}{4}(2^n)$, $\lim_{n\to+\infty} s_n = +\infty$, diverges.

(c) $\frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2}$, $s_1 = \frac{1}{6}$, $s_2 = \frac{1}{4}$, $s_3 = \frac{3}{10}$, $s_4 = \frac{1}{3}$; $s_n = \frac{1}{2} - \frac{1}{n+2}$, $\lim_{n\to+\infty} s_n = \frac{1}{2}$, converges.

2. (a) $s_1 = 1/4, s_2 = 5/16, s_3 = 21/64, s_4 = 85/256$, $s_n = \frac{1}{4}\left(1 + \frac{1}{4} + \ldots + \left(\frac{1}{4}\right)^{n-1}\right) = \frac{1}{4}\frac{1-(1/4)^n}{1-1/4} = \frac{1}{3}\left(1 - \left(\frac{1}{4}\right)^n\right)$; $\lim_{n\to+\infty} s_n = \frac{1}{3}$.

(b) $s_1 = 1, s_2 = 5, s_3 = 21, s_4 = 85$; $s_n = \frac{4^n - 1}{3}$, diverges.

(c) $s_1 = 1/20, s_2 = 1/12, s_3 = 3/28, s_4 = 1/8$; $s_n = \sum_{k=1}^{n}\left(\frac{1}{k+3} - \frac{1}{k+4}\right) = \frac{1}{4} - \frac{1}{n+4}$, $\lim_{n\to+\infty} s_n = 1/4$.

3–14 Determine whether the series converges, and if so find its sum.

3. $\sum_{k=1}^{\infty}\left(-\frac{3}{4}\right)^{k-1}$

4. $\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k+2}$

5. $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{7}{6^{k-1}}$

6. $\sum_{k=1}^{\infty}\left(-\frac{3}{2}\right)^{k+1}$

7. $\sum_{k=1}^{\infty}\frac{1}{(k+2)(k+3)}$

8. $\sum_{k=1}^{\infty}\left(\frac{1}{2^k}-\frac{1}{2^{k+1}}\right)$

9. $\sum_{k=1}^{\infty}\frac{1}{9k^2+3k-2}$

10. $\sum_{k=2}^{\infty}\frac{1}{k^2-1}$

11. $\sum_{k=3}^{\infty}\frac{1}{k-2}$

12. $\sum_{k=5}^{\infty}\left(\frac{e}{\pi}\right)^{k-1}$

13. $\sum_{k=1}^{\infty}\frac{4^{k+2}}{7^{k-1}}$

14. $\sum_{k=1}^{\infty}5^{3k}7^{1-k}$

Sol:

3. Geometric, $a = 1$, $r = -3/4$, $|r| = 3/4 < 1$, series converges, sum $= \dfrac{1}{1-(-3/4)} = 4/7$.

4. Geometric, $a = (2/3)^3$, $r = 2/3$, $|r| = 2/3 < 1$, series converges, sum $= \dfrac{(2/3)^3}{1-2/3} = 8/9$.

5. Geometric, $a = 7$, $r = -1/6$, $|r| = 1/6 < 1$, series converges, sum $= \dfrac{7}{1+1/6} = 6$.

6. Geometric, $r = -3/2$, $|r| = 3/2 \geq 1$, diverges.

7. $s_n = \displaystyle\sum_{k=1}^{n}\left(\frac{1}{k+2} - \frac{1}{k+3}\right) = \frac{1}{3} - \frac{1}{n+3}$, $\displaystyle\lim_{n\to+\infty} s_n = 1/3$, series converges by definition, sum $= 1/3$.

8. $s_n = \displaystyle\sum_{k=1}^{n}\left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right) = \frac{1}{2} - \frac{1}{2^{n+1}}$, $\displaystyle\lim_{n\to+\infty} s_n = 1/2$, series converges by definition, sum $= 1/2$.

9. $s_n = \displaystyle\sum_{k=1}^{n}\left(\frac{1/3}{3k-1} - \frac{1/3}{3k+2}\right) = \frac{1}{6} - \frac{1/3}{3n+2}$, $\displaystyle\lim_{n\to+\infty} s_n = 1/6$, series converges by definition, sum $= 1/6$.

10. $s_n = \displaystyle\sum_{k=2}^{n+1}\left[\frac{1/2}{k-1} - \frac{1/2}{k+1}\right] = \frac{1}{2}\left[\sum_{k=2}^{n+1}\frac{1}{k-1} - \sum_{k=2}^{n+1}\frac{1}{k+1}\right] = \frac{1}{2}\left[\sum_{k=2}^{n+1}\frac{1}{k-1} - \sum_{k=4}^{n+3}\frac{1}{k-1}\right] =$

$= \dfrac{1}{2}\left[1 + \dfrac{1}{2} - \dfrac{1}{n+1} - \dfrac{1}{n+2}\right]$; $\displaystyle\lim_{n\to+\infty} s_n = \frac{3}{4}$, series converges by definition, sum $= 3/4$.

11. $\displaystyle\sum_{k=3}^{\infty}\frac{1}{k-2} = \sum_{k=1}^{\infty} 1/k$, the harmonic series, so the series diverges.

12. Geometric, $a = (e/\pi)^4$, $r = e/\pi$, $|r| = e/\pi < 1$, series converges, sum $= \dfrac{(e/\pi)^4}{1-e/\pi} = \dfrac{e^4}{\pi^3(\pi - e)}$.

13. $\displaystyle\sum_{k=1}^{\infty}\frac{4^{k+2}}{7^{k-1}} = \sum_{k=1}^{\infty} 64\left(\frac{4}{7}\right)^{k-1}$; geometric, $a = 64$, $r = 4/7$, $|r| = 4/7 < 1$, series converges, sum $= \dfrac{64}{1-4/7} = 448/3$.

14. Geometric, $a = 125$, $r = 125/7$, $|r| = 125/7 \geq 1$, diverges.

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Do Questions (1-14) from Ex # 9.3