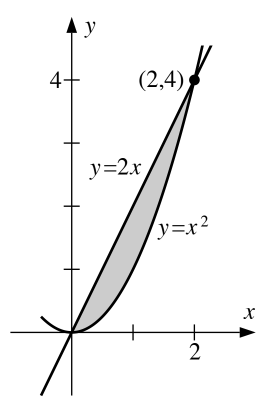
Applications of the Definite Integral in Geometry, Science, and Engineering

Exercise Set 6.1

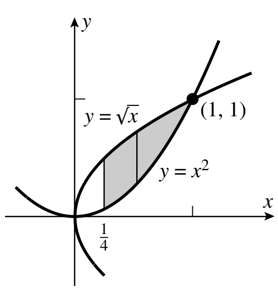
1. $A = \int_{-1}^{2} (x^2 + 1 - x)\, dx = (x^3/3 + x - x^2/2)\Big]_{-1}^{2} = 9/2.$

3. $A = \int_{1}^{2} (y - 1/y^2)\, dy = (y^2/2 + 1/y)\Big]_{1}^{2} = 1.$

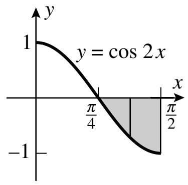
5. **(a)** $A = \int_{0}^{2} (2x - x^2)\, dx = 4/3.$ **(b)** $A = \int_{0}^{4} (\sqrt{y} - y/2)\, dy = 4/3.$



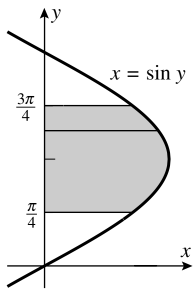
7. $A = \int_{1/4}^{1} (\sqrt{x} - x^2)\, dx = 49/192.$



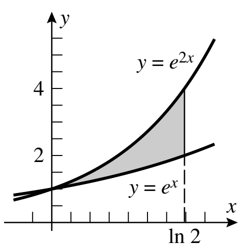
9. $A = \int_{\pi/4}^{\pi/2} (0 - \cos 2x)\, dx = -\int_{\pi/4}^{\pi/2} \cos 2x\, dx = 1/2.$



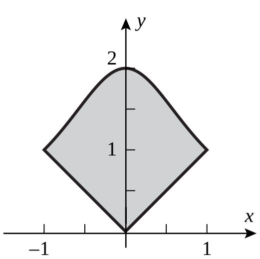
11. $A = \int_{\pi/4}^{3\pi/4} \sin y \, dy = \sqrt{2}$.



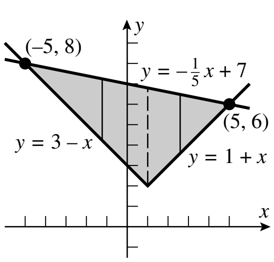
13. $A = \int_0^{\ln 2} \left(e^{2x} - e^x\right) dx = \left(\frac{1}{2}e^{2x} - e^x\right)\Big]_0^{\ln 2} = 1/2$.



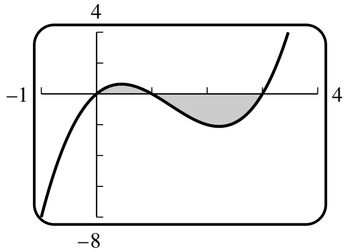
15. $A = \int_{-1}^{1} \left(\frac{2}{1+x^2} - |x|\right) dx = 2\int_0^1 \left(\frac{2}{1+x^2} - x\right) dx = \left[4\tan^{-1} x - x^2\right]_0^1 = \pi - 1$.



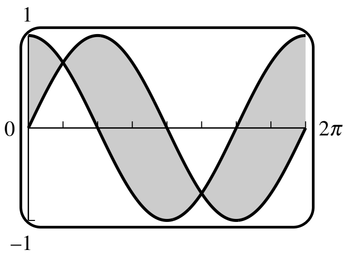
17. $y = 2 + |x-1| = \begin{cases} 3 - x, & x \le 1 \\ 1 + x, & x \ge 1 \end{cases}$, $A = \int_{-5}^{1} \left[\left(-\frac{1}{5}x + 7\right) - (3 - x)\right] dx + \int_1^5 \left[\left(-\frac{1}{5}x + 7\right) - (1 + x)\right] dx =$ $\int_{-5}^{1} \left(\frac{4}{5}x + 4\right) dx + \int_1^5 \left(6 - \frac{6}{5}x\right) dx = 72/5 + 48/5 = 24$.



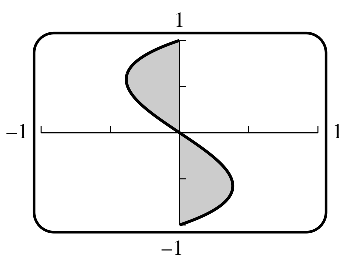
19. $A = \int_0^1 (x^3 - 4x^2 + 3x)\,dx + \int_1^3 [-(x^3 - 4x^2 + 3x)]\,dx = 5/12 + 32/12 = 37/12.$



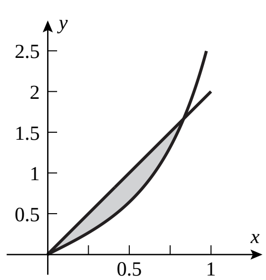
21. From the symmetry of the region $A = 2\int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx = 4\sqrt{2}.$



23. $A = \int_{-1}^{0} (y^3 - y)\,dy + \int_0^1 -(y^3 - y)\,dy = 1/2.$



25. The curves meet when $x = 0$, $\sqrt{\ln 2}$, so $A = \int_0^{\sqrt{\ln 2}} (2x - xe^{x^2})\,dx = \left(x^2 - \frac{1}{2}e^{x^2}\right)\Big]_0^{\sqrt{\ln 2}} = \ln 2 - \frac{1}{2}.$



27. True. If $f(x) - g(x) = c > 0$ then $f(x) > g(x)$ so Formula (1) implies that $A = \int_a^b [f(x) - g(x)]\,dx = \int_a^b c\,dx = c(b-a)$. If $g(x) - f(x) = c > 0$ then $g(x) > f(x)$ so $A = \int_a^b [g(x) - f(x)]\,dx = \int_a^b c\,dx = c(b-a)$.

29. True. Since f and g are distinct, there is some point c in $[a, b]$ for which $f(c) \neq g(c)$. Suppose $f(c) > g(c)$. (The case $f(c) < g(c)$ is similar.) Let $p = f(c) - g(c) > 0$. Since $f - g$ is continuous, there is an interval $[d, e]$ containing c such that $f(x) - g(x) > p/2$ for all x in $[d, e]$. So $\int_d^e [f(x) - g(x)]\,dx \geq \frac{p}{2}(e - d) > 0$. Hence $0 = \int_a^b [f(x) - g(x)]\,dx =$

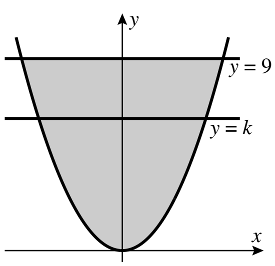
$\displaystyle\int_a^d [f(x)-g(x)]\,dx + \int_d^e [f(x)-g(x)]\,dx + \int_e^b [f(x)-g(x)]\,dx, > \int_a^d [f(x)-g(x)]\,dx + \int_b^e [f(x)-g(x)]\,dx$, so at least one of $\displaystyle\int_a^d [f(x)-g(x)]\,dx$ and $\displaystyle\int_b^e [f(x)-g(x)]\,dx$ is negative. Therefore $f(t)-g(t)<0$ for some point t in one of the intervals $[a,d]$ and $[b,e]$. So the graph of f is above the graph of g at $x=c$ and below it at $x=t$; by the Intermediate Value Theorem, the curves cross somewhere between c and t.

(Note: It is not necessarily true that the curves cross at a point. For example, let $f(x) = \begin{cases} x & \text{if } x<0; \\ 0 & \text{if } 0\le x\le 1; \\ x-1 & \text{if } x>1, \end{cases}$ and $g(x)=0$. Then $\displaystyle\int_{-1}^2 [f(x)-g(x)]\,dx = 0$, and the curves cross between -1 and 2, but there's no single point at which they cross; they coincide for x in $[0,1]$.)

31. The area is given by $\displaystyle\int_0^k (1/\sqrt{1-x^2}-x)\,dx = \sin^{-1}k - k^2/2 = 1$; solve for k to get $k\approx 0.997301$.

33. Solve $3-2x = x^6+2x^5-3x^4+x^2$ to find the real roots $x=-3,1$; from a plot it is seen that the line is above the polynomial when $-3<x<1$, so $A = \displaystyle\int_{-3}^1 (3-2x-(x^6+2x^5-3x^4+x^2))\,dx = 9152/105$.

35. $\displaystyle\int_0^k 2\sqrt{y}\,dy = \int_k^9 2\sqrt{y}\,dy;\ \int_0^k y^{1/2}\,dy = \int_k^9 y^{1/2}\,dy,\ \frac{2}{3}k^{3/2} = \frac{2}{3}(27-k^{3/2}),\ k^{3/2}=27/2,\ k=(27/2)^{2/3} = 9/\sqrt[3]{4}$.



37. (a) $A = \displaystyle\int_0^2 (2x-x^2)\,dx = 4/3$.

(b) $y=mx$ intersects $y=2x-x^2$ where $mx = 2x-x^2, x^2+(m-2)x=0, x(x+m-2)=0$ so $x=0$ or $x=2-m$. The area below the curve and above the line is $\displaystyle\int_0^{2-m}(2x-x^2-mx)\,dx = \int_0^{2-m}[(2-m)x-x^2]\,dx = \left[\frac{1}{2}(2-m)x^2 - \frac{1}{3}x^3\right]_0^{2-m} = \frac{1}{6}(2-m)^3$ so $(2-m)^3/6 = (1/2)(4/3) = 2/3, (2-m)^3 = 4, m = 2-\sqrt[3]{4}$.

39. The curves intersect at $x=0$ and, by Newton's Method, at $x\approx 2.595739080 = b$, so $A\approx \displaystyle\int_0^b (\sin x - 0.2x)\,dx = -\left[\cos x + 0.1x^2\right]_0^b \approx 1.180898334$.

41. By Newton's Method the points of intersection are $x=x_1\approx 0.4814008713$ and $x=x_2\approx 2.363938870$, and $A\approx \displaystyle\int_{x_1}^{x_2}\left(\frac{\ln x}{x}-(x-2)\right)dx \approx 1.189708441$.

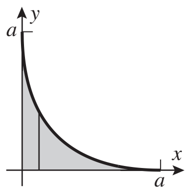
43. The x-coordinates of the points of intersection are $a\approx -0.423028$ and $b\approx 1.725171$; the area is

$A = \displaystyle\int_a^b (2\sin x - x^2+1)\,dx \approx 2.542696$.

45. $\displaystyle\int_0^{60}[v_2(t)-v_1(t)]\,dt = s_2(60)-s_2(0)-[s_1(60)-s_1(0)]$, but they are even at time $t=60$, so $s_2(60)=s_1(60)$. Consequently the integral gives the difference $s_1(0)-s_2(0)$ of their starting points in meters.

47. The area in question is the increase in population from 1960 to 2010.

49. Solve $x^{1/2}+y^{1/2}=a^{1/2}$ for y to get $y=(a^{1/2}-x^{1/2})^2=a-2a^{1/2}x^{1/2}+x$, $A=\displaystyle\int_0^a(a-2a^{1/2}x^{1/2}+x)\,dx=a^2/6$.



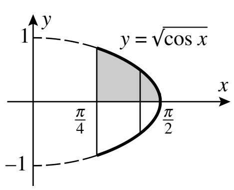
51. First find all solutions of the equation $f(x)=g(x)$ in the interval $[a,b]$; call them $c_1,\cdots,c_n$. Let $c_0=a$ and $c_{n+1}=b$. For $i=0,1,\cdots,n$, $f(x)-g(x)$ has constant sign on $[c_i,c_{i+1}]$, so the area bounded by $x=c_i$ and $x=c_{i+1}$ is either $\displaystyle\int_{c_i}^{c_{i+1}}[f(x)-g(x)]\,dx$ or $\displaystyle\int_{c_i}^{c_{i+1}}[g(x)-f(x)]\,dx$. Compute each of these $n+1$ areas and add them to get the area bounded by $x=a$ and $x=b$.

Exercise Set 6.2

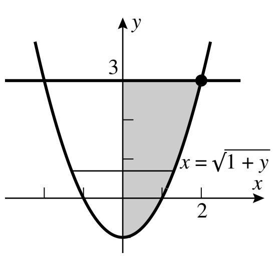
1. $V=\pi\displaystyle\int_{-1}^{3}(3-x)\,dx=8\pi$.

3. $V=\pi\displaystyle\int_0^2\frac{1}{4}(3-y)^2\,dy=13\pi/6$.

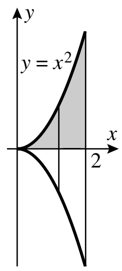
5. $V=\pi\displaystyle\int_{\pi/4}^{\pi/2}\cos x\,dx=(1-\sqrt{2}/2)\pi$.



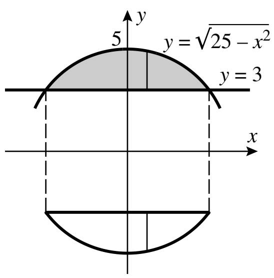
7. $V=\pi\displaystyle\int_{-1}^{3}(1+y)\,dy=8\pi$.



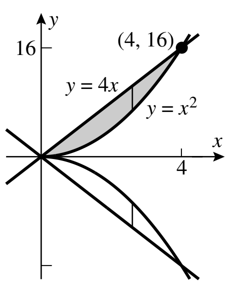
9. $V=\displaystyle\int_0^2 x^4\,dx=32/5$.



11. $V = \pi \int_{-4}^{4} [(25 - x^2) - 9]\, dx = 2\pi \int_0^4 (16 - x^2)\, dx = 256\pi/3.$



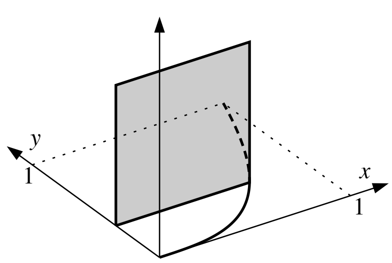
13. $V = \pi \int_0^4 [(4x)^2 - (x^2)^2]\, dx = \pi \int_0^4 (16x^2 - x^4)\, dx = 2048\pi/15.$



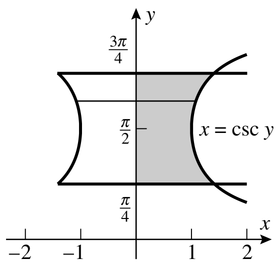
15. $V = \pi \int_0^{\ln 3} e^{2x}\, dx = \left. \frac{\pi}{2} e^{2x} \right]_0^{\ln 3} = 4\pi.$

17. $V = \int_{-2}^{2} \pi \frac{1}{4 + x^2}\, dx = \left. \frac{\pi}{2} \tan^{-1}(x/2) \right]_{-2}^{2} = \pi^2/4.$

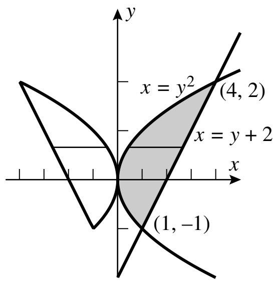
19. $V = \int_0^1 \left(y^{1/3}\right)^2\, dy = \frac{3}{5}.$



21. $V = \pi \int_{\pi/4}^{3\pi/4} \csc^2 y\, dy = 2\pi.$



23. $V = \pi \int_{-1}^{2} [(y+2)^2 - y^4]\, dy = 72\pi/5.$

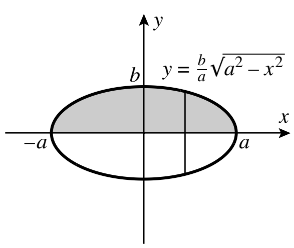


25. $V = \int_0^1 \pi e^{2y}\, dy = \frac{\pi}{2}\left(e^2 - 1\right).$

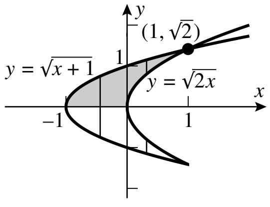
27. False. For example, consider the pyramid in Example 1, with the roles of the x- and y-axes interchanged.

29. False. For example, let S be the solid generated by rotating the region under $y = e^x$ over the interval $[0, 1]$. Then $A(x) = \pi(e^x)^2$.

31. $V = \pi \int_{-a}^{a} \frac{b^2}{a^2}(a^2 - x^2)\, dx = 4\pi ab^2/3.$



33. $V = \pi \int_{-1}^{0} (x+1)\, dx + \pi \int_0^1 [(x+1) - 2x]\, dx = \pi/2 + \pi/2 = \pi.$



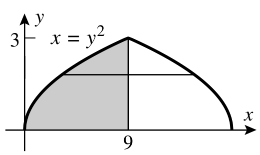
35. Partition the interval $[a, b]$ with $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$. Let x_k^* be an arbitrary point of $[x_{k-1}, x_k]$. The disk in question is obtained by revolving about the line $y = k$ the rectangle for which $x_{k-1} < x < x_k$, and y lies between $y = k$ and $y = f(x)$; the volume of this disk is $\Delta V_k = \pi(f(x_k^*) - k)^2 \Delta x_k$, and the total volume is given by $V = \pi \int_a^b (f(x) - k)^2\, dx$.

37. **(a)** Intuitively, it seems that a line segment which is revolved about a line which is perpendicular to the line segment will generate a larger area, the farther it is from the line. This is because the average point on the line segment will be revolved through a circle with a greater radius, and thus sweeps out a larger circle. Consider the line segment which connects a point (x, y) on the curve $y = \sqrt{3-x}$ to the point $(x, 0)$ beneath it. If this line segment is revolved around the x-axis we generate an area πy^2.

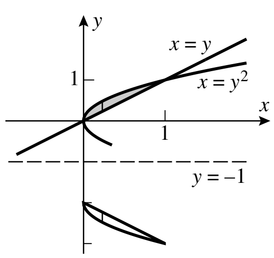
If on the other hand the segment is revolved around the line $y = 2$ then the area of the resulting (infinitely thin) washer is $\pi[2^2 - (2-y)^2]$. So the question can be reduced to asking whether $y^2 \geq [2^2 - (2-y)^2], y^2 \geq 4y - y^2$, or $y \geq 2$. In the present case the curve $y = \sqrt{3-x}$ always satisfies $y \leq 2$, so V_2 has the larger volume.

(b) The volume of the solid generated by revolving the area around the x-axis is $V_1 = \pi \int_{-1}^{3} (3-x)\,dx = 8\pi$, and the volume generated by revolving the area around the line $y = 2$ is $V_2 = \pi \int_{-1}^{3} [2^2 - (2 - \sqrt{3-x})^2]\,dx = \dfrac{40}{3}\pi$.

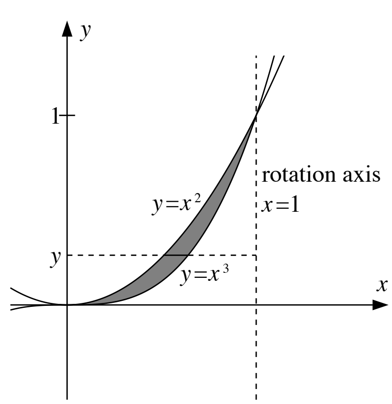
39. $V = \pi \int_0^3 (9 - y^2)^2\,dy = \pi \int_0^3 (81 - 18y^2 + y^4)\,dy = 648\pi/5$.



41. $V = \pi \int_0^1 [(\sqrt{x}+1)^2 - (x+1)^2]\,dx = \pi \int_0^1 (2\sqrt{x} - x - x^2)\,dx = \pi/2$.

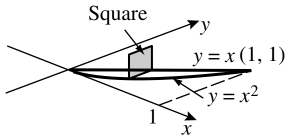


43. The region is given by the inequalities $0 \leq y \leq 1$, $\sqrt{y} \leq x \leq \sqrt[3]{y}$. For each y in the interval $[0, 1]$ the cross-section of the solid perpendicular to the axis $x = 1$ is a washer with outer radius $1 - \sqrt{y}$ and inner radius $1 - \sqrt[3]{y}$. The area of this washer is $A(y) = \pi[(1-\sqrt{y})^2 - (1-\sqrt[3]{y})^2] = \pi(-2y^{1/2} + y + 2y^{1/3} - y^{2/3})$, so the volume is $V = \int_0^1 A(y)\,dy = \pi \int_0^1 (-2y^{1/2} + y + 2y^{1/3} - y^{2/3})\,dy = \pi \left[-\frac{4}{3}y^{3/2} + \frac{1}{2}y^2 + \frac{3}{2}y^{4/3} - \frac{3}{5}y^{5/3}\right]_0^1 = \frac{\pi}{15}$.



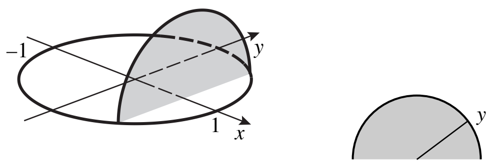
45. $A(x) = \pi(x^2/4)^2 = \pi x^4/16, V = \int_0^{20} (\pi x^4/16)\,dx = 40{,}000\pi$ ft^3.

47. $V = \int_0^1 (x - x^2)^2\,dx = \int_0^1 (x^2 - 2x^3 + x^4)\,dx = 1/30.$

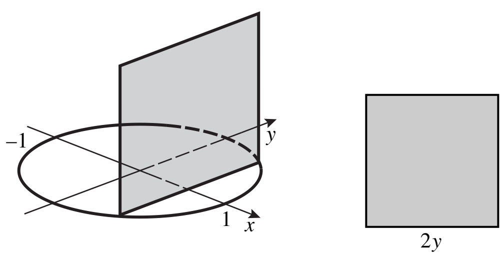


49. On the upper half of the circle, $y = \sqrt{1 - x^2}$, so:

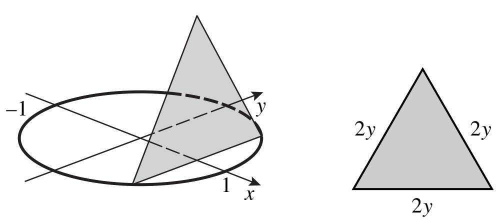
(a) $A(x)$ is the area of a semicircle of radius y, so $A(x) = \pi y^2/2 = \pi(1 - x^2)/2$; $V = \frac{\pi}{2}\int_{-1}^1 (1 - x^2)\,dx = \pi \int_0^1 (1 - x^2)\,dx = 2\pi/3.$



(b) $A(x)$ is the area of a square of side $2y$, so $A(x) = 4y^2 = 4(1 - x^2)$; $V = 4\int_{-1}^1 (1 - x^2)\,dx = 8\int_0^1 (1 - x^2)\,dx = 16/3.$



(c) $A(x)$ is the area of an equilateral triangle with sides $2y$, so $A(x) = \frac{\sqrt{3}}{4}(2y)^2 = \sqrt{3}y^2 = \sqrt{3}(1 - x^2)$; $V = \int_{-1}^1 \sqrt{3}(1 - x^2)\,dx = 2\sqrt{3}\int_0^1 (1 - x^2)\,dx = 4\sqrt{3}/3.$



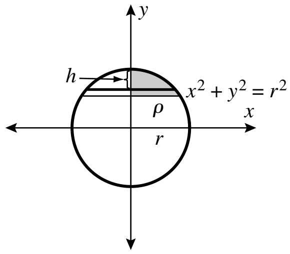
51. The two curves cross at $x = b \approx 1.403288534$, so $V = \pi\int_0^b ((2x/\pi)^2 - \sin^{16} x)\,dx + \pi\int_b^{\pi/2} (\sin^{16} x - (2x/\pi)^2)\,dx \approx 0.710172176.$

53. $V = \pi\int_1^e (1 - (\ln y)^2)\,dy = \pi.$

55. **(a)** $V = \pi\int_{r-h}^r (r^2 - y^2)\,dy = \pi(rh^2 - h^3/3) = \frac{1}{3}\pi h^2(3r - h).$

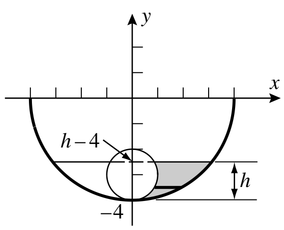
(b) By the Pythagorean Theorem, $r^2 = (r - h)^2 + \rho^2$, $2hr = h^2 + \rho^2$; from part (a), $V = \frac{\pi h}{3}(3hr - h^2) =$

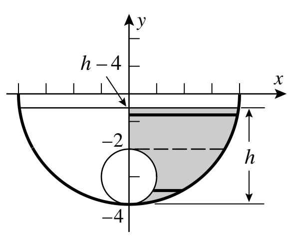
$$\frac{\pi h}{3}\left(\frac{3}{2}(h^2+\rho^2)-h^2\right)=\frac{1}{6}\pi h(3\rho^2+h^2).$$



57. **(a)** The bulb is approximately a sphere of radius 1.25 cm attached to a cylinder of radius 0.625 cm and length 2.5 cm, so its volume is roughly $\frac{4}{3}\pi(1.25)^3+\pi(0.625)^2\cdot 2.5 \approx 11.25$ cm. (Other answers are possible, depending on how we approximate the light bulb using familiar shapes.)

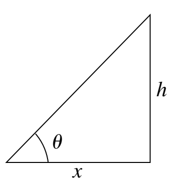
(b) $\Delta x=\frac{5}{10}=0.5$; $\{y_0,y_1,\cdots,y_{10}\}=\{0,2.00,2.45,2.45,2.00,1.46,1.26,1.25,1.25,1.25,1.25\}$; left $=\pi\sum_{i=0}^{9}\left(\frac{y_i}{2}\right)^2\Delta x\approx 11.157$; right $=\pi\sum_{i=1}^{10}\left(\frac{y_i}{2}\right)^2\Delta x\approx 11.771$; $V\approx$ average $=11.464$ cm^3.

59. **(a)**


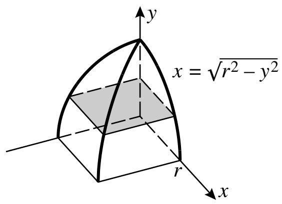
(b)


If the cherry is partially submerged then $0\le h<2$ as shown in Figure (a); if it is totally submerged then $2\le h\le 4$ as shown in Figure (b). The radius of the glass is 4 cm and that of the cherry is 1 cm so points on the sections shown in the figures satisfy the equations $x^2+y^2=16$ and $x^2+(y+3)^2=1$. We will find the volumes of the solids that are generated when the shaded regions are revolved about the y-axis. For $0\le h<2$, $V=\pi\int_{-4}^{h-4}[(16-y^2)-(1-(y+3)^2)]\,dy=6\pi\int_{-4}^{h-4}(y+4)\,dy=3\pi h^2$; for $2\le h\le 4$, $V=\pi\int_{-4}^{-2}[(16-y^2)-(1-(y+3)^2)]\,dy+\pi\int_{-2}^{h-4}(16-y^2)\,dy=6\pi\int_{-4}^{-2}(y+4)\,dy+\pi\int_{-2}^{h-4}(16-y^2)\,dy=12\pi+\frac{1}{3}\pi(12h^2-h^3-40)=\frac{1}{3}\pi(12h^2-h^3-4)$, so $V=\begin{cases}3\pi h^2 & \text{if } 0\le h<2\\ \frac{1}{3}\pi(12h^2-h^3-4) & \text{if } 2\le h\le 4\end{cases}$.

61. $\tan\theta=h/x$ so $h=x\tan\theta$, $A(y)=\frac{1}{2}hx=\frac{1}{2}x^2\tan\theta=\frac{1}{2}(r^2-y^2)\tan\theta$, because $x^2=r^2-y^2$, and this implies that $V=\frac{1}{2}\tan\theta\int_{-r}^{r}(r^2-y^2)\,dy=\tan\theta\int_0^r(r^2-y^2)\,dy=\frac{2}{3}r^3\tan\theta$.



63. Each cross section perpendicular to the y-axis is a square so $A(y)=x^2=r^2-y^2$, $\frac{1}{8}V=\int_0^r(r^2-y^2)\,dy$, so $V=8(2r^3/3)=16r^3/3$.



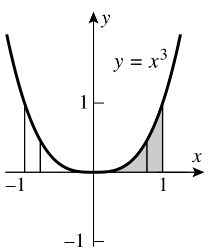
65. Position an x-axis perpendicular to the bases of the solids. Let a be the smallest x-coordinate of any point in either solid, and let b be the largest. Let $A(x)$ be the common area of the cross-sections of the solids at x-coordinate x. By equation (3), each solid has volume $V = \int_a^b A(x)\,dx$, so they are equal.

Exercise Set 6.3

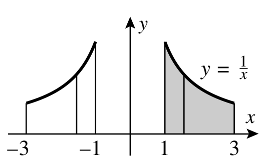
1. $V = \int_1^2 2\pi x(x^2)\,dx = 2\pi \int_1^2 x^3\,dx = 15\pi/2.$

3. $V = \int_0^1 2\pi y(2y - 2y^2)\,dy = 4\pi \int_0^1 (y^2 - y^3)\,dy = \pi/3.$

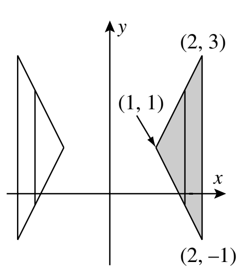
5. $V = \int_0^1 2\pi(x)(x^3)\,dx = 2\pi \int_0^1 x^4\,dx = 2\pi/5.$



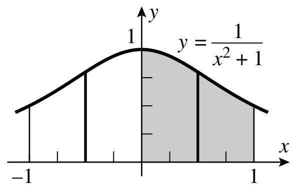
7. $V = \int_1^3 2\pi x(1/x)\,dx = 2\pi \int_1^3 dx = 4\pi.$



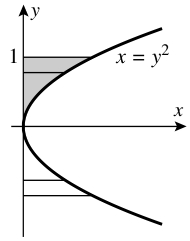
9. $V = \int_1^2 2\pi x[(2x-1) - (-2x+3)]\,dx = 8\pi \int_1^2 (x^2 - x)\,dx = 20\pi/3.$



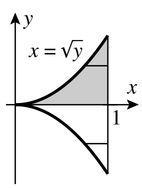
11. $V = 2\pi \int_0^1 \dfrac{x}{x^2+1}\,dx = \pi \ln(x^2+1)\Big]_0^1 = \pi \ln 2.$



13. $V = \displaystyle\int_0^1 2\pi y^3\,dy = \pi/2.$



15. $V = \displaystyle\int_0^1 2\pi y(1-\sqrt{y})\,dy = 2\pi \int_0^1 (y - y^{3/2})\,dy = \pi/5.$



17. True. The surface area of the cylinder is $2\pi \cdot [\text{average radius}] \cdot [\text{height}]$, so by equation (1) the volume equals the thickness times the surface area.

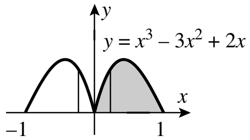
19. True. In 6.3.2 we integrate over an interval on the x-axis, which is perpendicular to the y-axis, which is the axis of revolution.

21. $V = 2\pi \displaystyle\int_1^2 xe^x\,dx = 2\pi(x-1)e^x\Big]_1^2 = 2\pi e^2.$

23. The volume is given by $2\pi \displaystyle\int_0^k x\sin x\,dx = 2\pi(\sin k - k\cos k) = 8$; solve for k to get $k \approx 1.736796$.

25. **(a)** $V = \displaystyle\int_0^1 2\pi x(x^3 - 3x^2 + 2x)\,dx = 7\pi/30.$

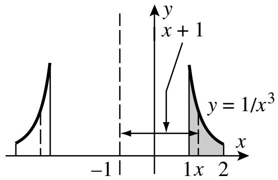
(b) Much easier; the method of slicing would require that x be expressed in terms of y.



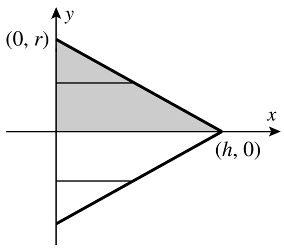
27. **(a)** For x in [0,1], the cross-section with x-coordinate x has length x, and its distance from the axis of revolution is $1-x$, so the volume is $\displaystyle\int_0^1 2\pi(1-x)x\,dx.$

(b) For y in [0,1], the cross-section with y-coordinate y has length $1-y$, and its distance from the axis of revolution is $1+y$, so the volume is $\displaystyle\int_0^1 2\pi(1+y)(1-y)\,dy.$

29. $V = \int_1^2 2\pi(x+1)(1/x^3)\,dx = 2\pi \int_1^2 (x^{-2} + x^{-3})\,dx = 7\pi/4.$



31. $x = \dfrac{h}{r}(r-y)$ is an equation of the line through $(0,r)$ and $(h,0)$, so $V = \int_0^r 2\pi y \left[\dfrac{h}{r}(r-y)\right] dy = \dfrac{2\pi h}{r}\int_0^r (ry - y^2)\,dy = \pi r^2 h/3.$



33. Let the sphere have radius R, the hole radius r. By the Pythagorean Theorem, $r^2 + (L/2)^2 = R^2$. Use cylindrical shells to calculate the volume of the solid obtained by rotating about the y-axis the region $r < x < R$, $-\sqrt{R^2 - x^2} < y < \sqrt{R^2 - x^2}$: $V = \int_r^R (2\pi x)2\sqrt{R^2 - x^2}\,dx = -\dfrac{4}{3}\pi(R^2 - x^2)^{3/2}\Big]_r^R = \dfrac{4}{3}\pi(L/2)^3$, so the volume is independent of R.

35. $V_x = \pi \int_{1/2}^b \dfrac{1}{x^2}\,dx = \pi(2 - 1/b)$, $V_y = 2\pi \int_{1/2}^b dx = \pi(2b-1)$; $V_x = V_y$ if $2 - 1/b = 2b - 1$, $2b^2 - 3b + 1 = 0$, solve to get $b = 1/2$ (reject) or $b = 1$.

37. If the formula for the length of a cross-section perpendicular to the axis of revolution is simpler than the formula for the length of a cross-section parallel to the axis of revolution, then the method of disks/washers is probably easier. Otherwise the method of cylindrical shells probably is.

Exercise Set 6.4

1. By the Theorem of Pythagoras, the length is $\sqrt{(2-1)^2 + (4-2)^2} = \sqrt{1+4} = \sqrt{5}$.

(a) $\dfrac{dy}{dx} = 2$, $L = \int_1^2 \sqrt{1+4}\,dx = \sqrt{5}$.

(b) $\dfrac{dx}{dy} = \dfrac{1}{2}$, $L = \int_2^4 \sqrt{1 + 1/4}\,dy = 2\sqrt{5/4} = \sqrt{5}$.

3. $f'(x) = \dfrac{9}{2}x^{1/2}$, $1 + [f'(x)]^2 = 1 + \dfrac{81}{4}x$, $L = \int_0^1 \sqrt{1 + 81x/4}\,dx = \dfrac{8}{243}\left(1 + \dfrac{81}{4}x\right)^{3/2}\Big]_0^1 = (85\sqrt{85} - 8)/243$.

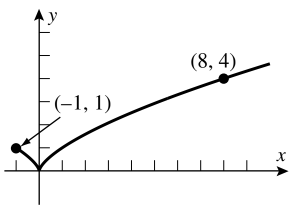
5. $\dfrac{dy}{dx} = \dfrac{2}{3}x^{-1/3}$, $1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{4}{9}x^{-2/3} = \dfrac{9x^{2/3} + 4}{9x^{2/3}}$, $L = \int_1^8 \dfrac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}}\,dx = \dfrac{1}{18}\int_{13}^{40} u^{1/2}du = \dfrac{1}{27}u^{3/2}\Big]_{13}^{40} = \dfrac{1}{27}(40\sqrt{40} - 13\sqrt{13}) = \dfrac{1}{27}(80\sqrt{10} - 13\sqrt{13})$ (we used $u = 9x^{2/3} + 4$); or (alternate solution) $x = y^{3/2}$, $\dfrac{dx}{dy} = \dfrac{3}{2}y^{1/2}$, $1 + \left(\dfrac{dx}{dy}\right)^2 = 1 + \dfrac{9}{4}y = \dfrac{4 + 9y}{4}$, $L = \dfrac{1}{2}\int_1^4 \sqrt{4 + 9y}\,dy = \dfrac{1}{18}\int_{13}^{40} u^{1/2}du = \dfrac{1}{27}(80\sqrt{10} - 13\sqrt{13})$.

7. $x = g(y) = \dfrac{1}{24}y^3 + 2y^{-1}$, $g'(y) = \dfrac{1}{8}y^2 - 2y^{-2}$, $1 + [g'(y)]^2 = 1 + \left(\dfrac{1}{64}y^4 - \dfrac{1}{2} + 4y^{-4}\right) = \dfrac{1}{64}y^4 + \dfrac{1}{2} + 4y^{-4} = \left(\dfrac{1}{8}y^2 + 2y^{-2}\right)^2$, $L = \displaystyle\int_2^4 \left(\frac{1}{8}y^2 + 2y^{-2}\right) dy = 17/6$.

9. False. The derivative $\dfrac{dy}{dx} = -\dfrac{x}{\sqrt{1-x^2}}$ is not defined at $x = \pm 1$, so it is not continuous on $[-1, 1]$.

11. True. If $f(x) = mx + c$ then the approximation equals $\displaystyle\sum_{k=1}^{n} \sqrt{1+m^2}\ \Delta x_k = \sum_{k=1}^{n} \sqrt{1+m^2}\ (x_k - x_{k-1}) = \sqrt{1+m^2}\ (x_n - x_0) = (b-a)\sqrt{1+m^2}$ and the arc length is the distance from $(a, ma+c)$ to $(b, mb+c)$, which equals $\sqrt{(b-a)^2 + [(mb+c)-(ma+c)]^2} = \sqrt{(b-a)^2 + [m(b-a)]^2} = (b-a)\sqrt{1+m^2}$. So each approximation equals the arc length.

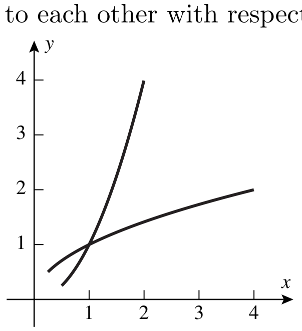
13. $dy/dx = \dfrac{\sec x \tan x}{\sec x} = \tan x$, $\sqrt{1+(y')^2} = \sqrt{1+\tan^2 x} = \sec x$ when $0 < x < \pi/4$, so $L = \displaystyle\int_0^{\pi/4} \sec x\, dx = \ln(1+\sqrt{2})$.

15. (a) 

(b) dy/dx does not exist at $x = 0$.

(c) $x = g(y) = y^{3/2}$, $g'(y) = \dfrac{3}{2}y^{1/2}$, $L = \displaystyle\int_0^1 \sqrt{1+9y/4}\, dy + \int_0^4 \sqrt{1+9y/4}\, dy = \frac{8}{27}\left(\frac{13}{8}\sqrt{13} - 1\right) + \frac{8}{27}(10\sqrt{10} - 1) = (13\sqrt{13} + 80\sqrt{10} - 16)/27$.

17. (a) The function $y = f(x) = x^2$ is inverse to the function $x = g(y) = \sqrt{y} : f(g(y)) = y$ for $1/4 \le y \le 4$, and $g(f(x)) = x$ for $1/2 \le x \le 2$. Geometrically this means that the graphs of $y = f(x)$ and $x = g(y)$ are symmetric to each other with respect to the line $y = x$ and hence have the same arc length.



(b) $L_1 = \displaystyle\int_{1/2}^{2} \sqrt{1+(2x)^2}\, dx$ and $L_2 = \displaystyle\int_{1/4}^{4} \sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\, dx$. Make the change of variables $x = \sqrt{y}$ in the first integral to obtain $L_1 = \displaystyle\int_{1/4}^{4} \sqrt{1+(2\sqrt{y})^2}\frac{1}{2\sqrt{y}}\, dy = \int_{1/4}^{4} \sqrt{\left(\frac{1}{2\sqrt{y}}\right)^2 + 1}\, dy = L_2$.

(c) $L_1 = \displaystyle\int_{1/4}^{4} \sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\, dy$, $L_2 = \displaystyle\int_{1/2}^{2} \sqrt{1+(2y)^2}\, dy$.

(d) For L_1, $\Delta x = \dfrac{3}{20}$, $x_k = \dfrac{1}{2} + k\dfrac{3}{20} = \dfrac{3k+10}{20}$, and thus

$$L_1 \approx \sum_{k=1}^{10} \sqrt{(\Delta x)^2 + [f(x_k) - f(x_{k-1})]^2} = \sum_{k=1}^{10} \sqrt{\left(\frac{3}{20}\right)^2 + \left(\frac{(3k+10)^2 - (3k+7)^2}{400}\right)^2} \approx 4.072396336.$$

For L_2, $\Delta x = \dfrac{15}{40} = \dfrac{3}{8}$, $x_k = \dfrac{1}{4} + \dfrac{3k}{8} = \dfrac{3k+2}{8}$, and thus

$$L_2 \approx \sum_{k=1}^{10} \sqrt{\left(\frac{3}{8}\right)^2 + \left[\sqrt{\frac{3k+2}{8}} - \sqrt{\frac{3k-1}{8}}\right]^2} \approx 4.071626502.$$

(e) Each polygonal path is shorter than the curve segment, so both approximations in (d) are smaller than the actual length. Hence the larger one, the approximation for L_1, is better.

(f) For L_1, $\Delta x = \dfrac{3}{20}$, the midpoint is $x_k^* = \dfrac{1}{2} + \left(k - \dfrac{1}{2}\right)\dfrac{3}{20} = \dfrac{6k+17}{40}$, and thus

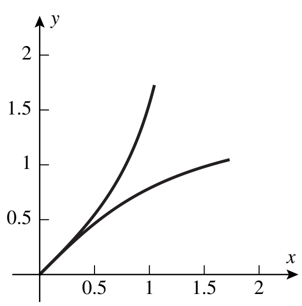
$$L_1 \approx \sum_{k=1}^{10} \frac{3}{20}\sqrt{1 + \left(2\frac{6k+17}{40}\right)^2} \approx 4.072396336.$$

For $L_2, \Delta x = \dfrac{15}{40}$, and the midpoint is $x_k^* = \dfrac{1}{4} + \left(k - \dfrac{1}{2}\right)\dfrac{15}{40} = \dfrac{6k+1}{16}$, and thus

$$L_2 \approx \sum_{k=1}^{10} \frac{15}{40}\sqrt{1 + \left(4\frac{6k+1}{16}\right)^{-1}} \approx 4.066160149.$$

(g) $L_1 = \displaystyle\int_{1/2}^{2} \sqrt{1 + (2x)^2}\, dx \approx 4.0729, L_2 = \int_{1/4}^{4} \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\, dx \approx 4.0729.$

19. **(a)** The function $y = f(x) = \tan x$ is inverse to the function $x = g(y) = \tan^{-1} x : f(g(y)) = y$ for $0 \le y \le \sqrt{3}$, and $g(f(x)) = x$ for $0 \le x \le \pi/3$. Geometrically this means that the graphs of $y = f(x)$ and $x = g(y)$ are symmetric to each other with respect to the line $y = x$.



(b) $L_1 = \displaystyle\int_0^{\pi/3} \sqrt{1 + \sec^4 x}\, dx,\ L_2 = \int_0^{\sqrt{3}} \sqrt{1 + \frac{1}{(1+x^2)^2}}\, dx$. In the expression for L_1 make the change of variable $y = \tan x$ to obtain $L_1 = \displaystyle\int_0^{\sqrt{3}} \sqrt{1 + (\sqrt{1+y^2})^4}\frac{1}{1+y^2}\, dy = \int_0^{\sqrt{3}} \sqrt{\frac{1}{(1+y^2)^2} + 1}\, dy = L_2.$

(c) $L_1 = \displaystyle\int_0^{\sqrt{3}} \sqrt{1 + \frac{1}{(1+y^2)^2}}\, dy,\ L_2 = \int_0^{\pi/3} \sqrt{1 + \sec^4 y}\, dy.$

(d) For L_1, $\Delta x_k = \dfrac{\pi}{30}, x_k = k\dfrac{\pi}{30}$, and thus

$$L_1 \approx \sum_{k=1}^{10} \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2} = \sum_{k=1}^{10} \sqrt{\left(\frac{\pi}{30}\right)^2 + [\tan(k\pi/30) - \tan((k-1)\pi/30)]^2} \approx 2.056603923.$$

For L_2, $\Delta x_k = \dfrac{\sqrt{3}}{10}, x_k = k\dfrac{\sqrt{3}}{10}$, and thus

$$L_2 \approx \sum_{k=1}^{10} \sqrt{\left(\frac{\sqrt{3}}{10}\right)^2 + \left[\tan^{-1}\left(k\frac{\sqrt{3}}{10}\right) - \tan^{-1}\left((k-1)\frac{\sqrt{3}}{10}\right)\right]^2} \approx 2.056724591.$$

(e) Each polygonal path is shorter than the curve segment, so both approximations in (d) are smaller than the actual length. Hence the larger one, the approximation for L_2, is better.

(f) For L_1, $\Delta x_k = \dfrac{\pi}{30}$, the midpoint is $x_k^* = \left(k - \dfrac{1}{2}\right)\dfrac{\pi}{30}$, and thus

$$L_1 \approx \sum_{k=1}^{10} \frac{\pi}{30}\sqrt{1 + \sec^4\left[\left(k - \frac{1}{2}\right)\frac{\pi}{30}\right]} \approx 2.050944217.$$

For $L_2, \Delta x_k = \dfrac{\sqrt{3}}{10}$, and the midpoint is $x_k^* = \left(k - \dfrac{1}{2}\right)\dfrac{\sqrt{3}}{10}$, and thus

$$L_2 \approx \sum_{k=1}^{10} \frac{\sqrt{3}}{10}\sqrt{1 + \frac{1}{((x_k^*)^2+1)^2}} \approx 2.057065139.$$

(g) $L_1 = \displaystyle\int_0^{\pi/3} \sqrt{1+\sec^4 x}\, dx \approx 2.0570$, $L_2 = \displaystyle\int_0^{\sqrt{3}} \sqrt{1 + \frac{1}{(1^2+y^2)^2}}\, dx \approx 2.0570.$

21. $f'(x) = \sec x \tan x$, $0 \le \sec x \tan x \le 2\sqrt{3}$ for $0 \le x \le \pi/3$ so $\dfrac{\pi}{3} \le L \le \dfrac{\pi}{3}\sqrt{13}$.

23. If we model the cable with a parabola $y = ax^2$, then $500 = a \cdot 2100^2$ and then $a = 500/2100^2$. Then the length of the cable is given by $L = \displaystyle\int_{-2100}^{2100} \sqrt{1+(2ax)^2}\, dx \approx 4354$ ft.

25. $y = 0$ at $x = b = 12.54/0.41 \approx 30.585$; distance $= \displaystyle\int_0^b \sqrt{1+(12.54-0.82x)^2}\, dx \approx 196.31$ yd.

27. $(dx/dt)^2 + (dy/dt)^2 = (t^2)^2 + (t)^2 = t^2(t^2+1)$, $L = \displaystyle\int_0^1 t(t^2+1)^{1/2} dt = (2\sqrt{2}-1)/3$.

29. $(dx/dt)^2 + (dy/dt)^2 = (-2\sin 2t)^2 + (2\cos 2t)^2 = 4$, $L = \displaystyle\int_0^{\pi/2} 2\, dt = \pi$.

31. $(dx/dt)^2 + (dy/dt)^2 = [e^t(\cos t - \sin t)]^2 + [e^t(\cos t + \sin t)]^2 = 2e^{2t}$, $L = \displaystyle\int_0^{\pi/2} \sqrt{2}e^t dt = \sqrt{2}(e^{\pi/2} - 1)$.

33. **(a)** $(dx/dt)^2 + (dy/dt)^2 = 4\sin^2 t + \cos^2 t = 4\sin^2 t + (1-\sin^2 t) = 1 + 3\sin^2 t$, $L = \displaystyle\int_0^{2\pi} \sqrt{1+3\sin^2 t}\, dt = 4\displaystyle\int_0^{\pi/2} \sqrt{1+3\sin^2 t}\, dt$.

(b) 9.69 **(c)** Distance traveled $= \displaystyle\int_{1.5}^{4.8} \sqrt{1+3\sin^2 t}\, dt \approx 5.16$ cm.

35. The length of the curve is approximated by the length of a polygon whose vertices lie on the graph of $y = f(x)$. Each term in the sum is the length of one edge of the approximating polygon. By the distance formula, the length of the k'th edge is $\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$, where Δx_k is the change in x along the edge and Δy_k is the change in

y along the edge. We use the Mean Value Theorem to express Δy_k as $f'(x_k^*)\Delta x_k$. Factoring the Δx_k out of the square root yields the k'th term in the sum.

Exercise Set 6.5

1. $S = \int_0^1 2\pi(7x)\sqrt{1+49}\,dx = 70\pi\sqrt{2}\int_0^1 x\ dx = 35\pi\sqrt{2}$.

3. $f'(x) = -x/\sqrt{4-x^2}$, $1+[f'(x)]^2 = 1 + \dfrac{x^2}{4-x^2} = \dfrac{4}{4-x^2}$, $S = \int_{-1}^1 2\pi\sqrt{4-x^2}(2/\sqrt{4-x^2})\,dx = 4\pi\int_{-1}^1 dx = 8\pi$.

5. $S = \int_0^2 2\pi(9y+1)\sqrt{82}\,dy = 2\pi\sqrt{82}\int_0^2 (9y+1)\,dy = 40\pi\sqrt{82}$.

7. $g'(y) = -y/\sqrt{9-y^2}$, $1+[g'(y)]^2 = \dfrac{9}{9-y^2}$, $S = \int_{-2}^2 2\pi\sqrt{9-y^2}\cdot\dfrac{3}{\sqrt{9-y^2}}dy = 6\pi\int_{-2}^2 dy = 24\pi$.

9. $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2}$, $1+[f'(x)]^2 = 1 + \frac{1}{4}x^{-1} - \frac{1}{2} + \frac{1}{4}x = \left(\frac{1}{2}x^{-1/2} + \frac{1}{2}x^{1/2}\right)^2$,

$S = \int_1^3 2\pi\left(x^{1/2} - \frac{1}{3}x^{3/2}\right)\left(\frac{1}{2}x^{-1/2} + \frac{1}{2}x^{1/2}\right)dx = \frac{\pi}{3}\int_1^3 (3+2x-x^2)\,dx = 16\pi/9$.

11. $x = g(y) = \frac{1}{4}y^4 + \frac{1}{8}y^{-2}$, $g'(y) = y^3 - \frac{1}{4}y^{-3}$, $1+[g'(y)]^2 = 1 + \left(y^6 - \frac{1}{2} + \frac{1}{16}y^{-6}\right) = \left(y^3 + \frac{1}{4}y^{-3}\right)^2$,

$S = \int_1^2 2\pi\left(\frac{1}{4}y^4 + \frac{1}{8}y^{-2}\right)\left(y^3 + \frac{1}{4}y^{-3}\right)dy = \frac{\pi}{16}\int_1^2 (8y^7 + 6y + y^{-5})\,dy = 16{,}911\pi/1024$.

13. $f'(x) = \cos x$, $1+[f'(x)]^2 = 1+\cos^2 x$, $S = \int_0^\pi 2\pi\sin x\sqrt{1+\cos^2 x}\,dx = 2\pi(\sqrt{2} + \ln(\sqrt{2}+1)) \approx 14.42$.

15. $f'(x) = e^x$, $1+[f'(x)]^2 = 1+e^{2x}$, $S = \int_0^1 2\pi e^x\sqrt{1+e^{2x}}\,dx \approx 22.94$.

17. True, by equation (1) with $r_1 = 0$, $r_2 = r$, and $l = \sqrt{r^2+h^2}$.

19. True. If $f(x) = c$ for all x then $f'(x) = 0$ so the approximation is $\sum_{k=1}^n 2\pi c\,\Delta x_k = 2\pi c(b-a)$. Since the surface is the lateral surface of a cylinder of length $b-a$ and radius c, its area is also $2\pi c(b-a)$.

21. $n = 20, a = 0, b = \pi, \Delta x = (b-a)/20 = \pi/20, x_k = k\pi/20$,

$S \approx \pi\sum_{k=1}^{20}[\sin(k-1)\pi/20 + \sin k\pi/20]\sqrt{(\pi/20)^2 + [\sin(k-1)\pi/20 - \sin k\pi/20]^2} \approx 14.39$.

23. $S = \int_a^b 2\pi[f(x)+k]\sqrt{1+[f'(x)]^2}\,dx$.

25. $f(x) = \sqrt{r^2-x^2}$, $f'(x) = -x/\sqrt{r^2-x^2}$, $1+[f'(x)]^2 = r^2/(r^2-x^2)$, $S = \int_{-r}^r 2\pi\sqrt{r^2-x^2}(r/\sqrt{r^2-x^2})\,dx = 2\pi r\int_{-r}^r dx = 4\pi r^2$.

27. Suppose the two planes are $y = y_1$ and $y = y_2$, where $-r \le y_1 \le y_2 \le r$. Then the area of the zone equals the area of a spherical cap of height $r - y_1$ minus the area of a spherical cap of height $r - y_2$. By Exercise 26, this is $2\pi r(r - y_1) - 2\pi r(r - y_2) = 2\pi r(y_2 - y_1)$, which only depends on the radius r and the distance $y_2 - y_1$ between the planes.

29. Note that $1 \le \sec x \le 2$ for $0 \le x \le \pi/3$. Let L be the arc length of the curve $y = \tan x$ for $0 < x < \pi/3$. Then $L = \displaystyle\int_0^{\pi/3} \sqrt{1 + \sec^2 x}\, dx$, and by Exercise 24, and the inequalities above, $2\pi L \le S \le 4\pi L$. But from the inequalities for $\sec x$ above, we can show that $\sqrt{2}\pi/3 \le L \le \sqrt{5}\pi/3$. Hence, combining the two sets of inequalities, $2\pi(\sqrt{2}\pi/3) \le 2\pi L \le S \le 4\pi L \le 4\pi\sqrt{5}\pi/3$. To obtain the inequalities in the text, observe that $\dfrac{2\pi^2}{3} < 2\pi\dfrac{\sqrt{2}\pi}{3} \le 2\pi L \le S \le 4\pi L \le 4\pi\dfrac{\sqrt{5}\pi}{3} < \dfrac{4\pi^2}{3}\sqrt{13}$.

31. Let $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ be a partition of $[a, b]$. Then the lateral area of the frustum of slant height $\ell = \sqrt{\Delta x_k^2 + \Delta y_k^2}$ and radii $y(t_1)$ and $y(t_2)$ is $\pi(y(t_k) + y(t_{k-1}))\ell$. Thus the area of the frustum S_k is given by $S_k = \pi(y(t_{k-1}) + y(t_k))\sqrt{[x(t_k) - x(t_{k-1})]^2 + [y(t_k) - y(t_{k-1})]^2}$ with the limit as $\max \Delta t_k \to 0$ of $S = \displaystyle\int_a^b 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt$.

33. $x' = 2t, y' = 2, (x')^2 + (y')^2 = 4t^2 + 4$, $S = 2\pi \displaystyle\int_0^4 (2t)\sqrt{4t^2 + 4}dt = 8\pi \int_0^4 t\sqrt{t^2 + 1}dt = \frac{8\pi}{3}(17\sqrt{17} - 1)$.

35. $x' = 1$, $y' = 4t$, $(x')^2 + (y')^2 = 1 + 16t^2$, $S = 2\pi \displaystyle\int_0^1 t\sqrt{1 + 16t^2}\, dt = \frac{\pi}{24}(17\sqrt{17} - 1)$.

37. $x' = -r\sin t$, $y' = r\cos t$, $(x')^2 + (y')^2 = r^2$, $S = 2\pi \displaystyle\int_0^{\pi} r\sin t\sqrt{r^2}\, dt = 2\pi r^2 \int_0^{\pi} \sin t\, dt = 4\pi r^2$.

39. Suppose we approximate the k'th frustum by the lateral surface of a cylinder of width Δx_k and radius $f(x_k^*)$, where x_k^* is between x_{k-1} and x_k. The area of this surface is $2\pi f(x_k^*)\ \Delta x_k$. Proceeding as before, we would conclude that $S = \int_a^b 2\pi f(x)\, dx$, which is too small. Basically, when $|f'(x)| > 0$, the area of the frustum is larger than the area of the cylinder, and ignoring this results in an incorrect formula.

Exercise Set 6.6

1. $W = \displaystyle\int_0^3 F(x)\, dx = \int_0^3 (x + 1)\, dx = \left[\frac{1}{2}x^2 + x\right]_0^3 = 7.5$ ft·lb.

3. Since $W = \displaystyle\int_a^b F(x)\, dx =$ the area under the curve, it follows that $d < 2.5$ since the area increases faster under the left part of the curve. In fact, if $d \le 2$, $W_d = \displaystyle\int_0^d F(x)\, dx = 40d$, and $W = \displaystyle\int_0^5 F(x)\, dx = 140$, so $d = 7/4$.

5. Distance traveled $= \displaystyle\int_0^5 v(t)\, dt = \int_0^5 \frac{4t}{5}\, dt = \frac{2}{5}t^2\Big]_0^5 = 10$ ft. The force is a constant 10 lb, so the work done is $10 \cdot 10 = 100$ ft·lb.

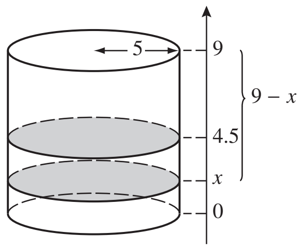
7. $F(x) = kx$, $F(0.2) = 0.2k = 100$, $k = 500$ N/m, $W = \displaystyle\int_0^{0.8} 500x\, dx = 160$ J.

9. $W = \displaystyle\int_0^1 kx\, dx = k/2 = 10$, $k = 20$ lb/ft.

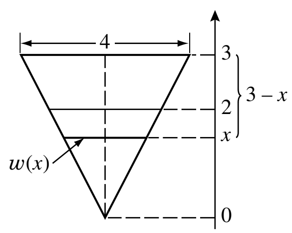
11. False. The work depends on the force and the distance, not on the elapsed time.

13. True. By equation (6), work and energy have the same units in any system of units.

15. $W = \int_0^{9/2} (9-x)62.4(25\pi)\,dx = 1560\pi \int_0^{9/2} (9-x)\,dx = 47{,}385\pi$ ft·lb.

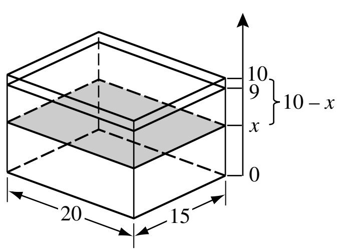


17. $w/4 = x/3, w = 4x/3$, $W = \int_0^2 (3-x)(9810)(4x/3)(6)\,dx = 78480 \int_0^2 (3x-x^2)\,dx = 261,600\,\mathrm{J}$.

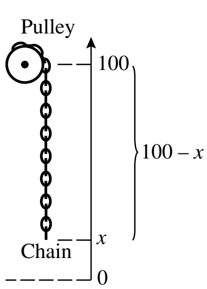


19. **(a)** $W = \int_0^9 (10-x)62.4(300)\,dx = 18{,}720 \int_0^9 (10-x)\,dx = 926{,}640$ ft·lb.

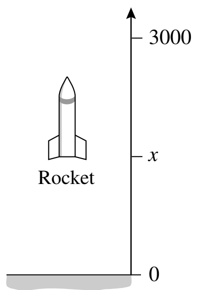
(b) To empty the pool in one hour would require $926{,}640/3600 = 257.4$ ft·lb of work per second so hp of motor $= 257.4/550 = 0.468$.



21. $W = \int_0^{100} 15(100-x)\,dx = 75,000$ ft·lb.



23. When the rocket is x ft above the ground total weight = weight of rocket + weight of fuel $= 3 + [40 - 2(x/1000)] = 43 - x/500$ tons, $W = \int_0^{3000} (43 - x/500)\,dx = 120,000$ ft·tons.



25. **(a)** $150 = k/(4000)^2$, $k = 2.4 \times 10^9$, $w(x) = k/x^2 = 2{,}400{,}000{,}000/x^2$ lb.

(b) $6000 = k/(4000)^2$, $k = 9.6 \times 10^{10}$, $w(x) = \left(9.6 \times 10^{10}\right)/(x + 4000)^2$ lb.

(c) $W = \displaystyle\int_{4000}^{5000} 9.6(10^{10})x^{-2}\,dx = 4{,}800{,}000$ mi·lb $= 2.5344 \times 10^{10}$ ft·lb.

27. $W = \dfrac{1}{2}mv_f^2 - \dfrac{1}{2}mv_i^2 = \dfrac{1}{2}4.00 \times 10^5(v_f^2 - 20^2)$. But $W = F \cdot d = (6.40 \times 10^5) \cdot (3.00 \times 10^3)$, so $19.2 \times 10^8 = 2.00 \times 10^5 v_f^2 - 8.00 \times 10^7, 19200 = 2v_f^2 - 800, v_f = 100$ m/s.

29. **(a)** The kinetic energy would have decreased by $\dfrac{1}{2}mv^2 = \dfrac{1}{2}4 \cdot 10^6(15000)^2 = 4.5 \times 10^{14}$ J.

(b) $(4.5 \times 10^{14})/(4.2 \times 10^{15}) \approx 0.107$. **(c)** $\dfrac{1000}{13}(0.107) \approx 8.24$ bombs.

31. The work-energy relationship involves 4 quantities, the work W, the mass m, and the initial and final velocities v_i and v_f. In any problem in which 3 of these are given, the work-energy relationship can be used to compute the fourth. In cases where the force is constant, we may combine equation (1) with the work-energy relationship to get $Fd = \dfrac{1}{2}mv_f^2 - \dfrac{1}{2}mv_i^2$. In this form there are 5 quantities, the force F, the distance d, the mass m, and the initial and final velocities v_i and v_f. So if any 4 of these are given, the work-energy relationship can be used to compute the fifth.

Exercise Set 6.7

1. **(a)** m_1 and m_3 are equidistant from $x = 5$, but m_3 has a greater mass, so the sum is positive.

(b) Let a be the unknown coordinate of the fulcrum; then the total moment about the fulcrum is $5(0 - a) + 10(5 - a) + 20(10 - a) = 0$ for equilibrium, so $250 - 35a = 0$, $a = 50/7$. The fulcrum should be placed 50/7 units to the right of m_1.

3. By symmetry, the centroid is $(1/2, 1/2)$. We confirm this using Formulas (8) and (9) with $a = 0$, $b = 1$, $f(x) = 1$. The area is 1, so $\overline{x} = \displaystyle\int_0^1 x\,dx = \frac{1}{2}$ and $\overline{y} = \displaystyle\int_0^1 \frac{1}{2}\,dx = \frac{1}{2}$, as expected.

5. By symmetry, the centroid is $(1, 1/2)$. We confirm this using Formulas (8) and (9) with $a = 0$, $b = 2$, $f(x) = 1$. The area is 2, so $\overline{x} = \dfrac{1}{2}\displaystyle\int_0^2 x\,dx = 1$ and $\overline{y} = \dfrac{1}{2}\displaystyle\int_0^2 \frac{1}{2}\,dx = \frac{1}{2}$, as expected.

7. By symmetry, the centroid lies on the line $y = 1 - x$. To find $\overline{x}$ we use Formula (8) with $a = 0$, $b = 1$, $f(x) = x$. The area is $\dfrac{1}{2}$, so $\overline{x} = 2\displaystyle\int_0^1 x^2\,dx = \frac{2}{3}$. Hence $\overline{y} = 1 - \dfrac{2}{3} = \dfrac{1}{3}$ and the centroid is $\left(\dfrac{2}{3}, \dfrac{1}{3}\right)$.

9. We use Formulas (10) and (11) with $a = 0$, $b = 1$, $f(x) = 2 - x^2$, $g(x) = x$. The area is $\int_0^1 (2 - x^2 - x)\,dx = \left[2x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right]_0^1 = \frac{7}{6}$, so $\overline{x} = \frac{6}{7}\int_0^1 x(2 - x^2 - x)\,dx = \frac{6}{7}\left[x^2 - \frac{1}{4}x^4 - \frac{1}{3}x^3\right]_0^1 = \frac{5}{14}$ and $\overline{y} = \frac{6}{7}\int_0^1 \frac{1}{2}[(2 - x^2)^2 - x^2]\,dx = \frac{3}{7}\int_0^1 (4 - 5x^2 + x^4)\,dx = \frac{3}{7}\left[4x - \frac{5}{3}x^3 + \frac{1}{5}x^5\right]_0^1 = \frac{38}{35}$. The centroid is $\left(\frac{5}{14}, \frac{38}{35}\right)$.

11. We use Formulas (8) and (9) with $a = 0$, $b = 2$, $f(x) = 1 - \frac{x}{2}$. The area is 1, so $\overline{x} = \int_0^2 x\left(1 - \frac{x}{2}\right) dx = \left[\frac{1}{2}x^2 - \frac{1}{6}x^3\right]_0^2 = \frac{2}{3}$ and $\overline{y} = \int_0^2 \frac{1}{2}\left(1 - \frac{x}{2}\right)^2 dx = \frac{1}{8}\int_0^2 (4 - 4x + x^2)\,dx = \frac{1}{8}\left[4x - 2x^2 + \frac{1}{3}x^3\right]_0^2 = \frac{1}{3}$. The centroid is $\left(\frac{2}{3}, \frac{1}{3}\right)$.

13. The graphs of $y = x^2$ and $y = 6 - x$ meet when $x^2 = 6 - x$, so $x = -3$ or $x = 2$. We use Formulas (10) and (11) with $a = -3$, $b = 2$, $f(x) = 6 - x$, $g(x) = x^2$. The area is $\int_{-3}^2 (6 - x - x^2)\,dx = \left[6x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-3}^2 = \frac{125}{6}$, so $\overline{x} = \frac{6}{125}\int_{-3}^2 x(6 - x - x^2)\,dx = \frac{6}{125}\left[3x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_{-3}^2 = -\frac{1}{2}$ and $\overline{y} = \frac{6}{125}\int_{-3}^2 \frac{1}{2}[(6 - x)^2 - (x^2)^2]\,dx = \frac{3}{125}\int_{-3}^2 (36 - 12x + x^2 - x^4)\,dx = \frac{3}{125}\left[36x - 6x^2 + \frac{1}{3}x^3 - \frac{1}{5}x^5\right]_{-3}^2 = 4$. The centroid is $\left(-\frac{1}{2}, 4\right)$.

15. The curves meet at $(-1, 1)$ and $(2, 4)$. We use Formulas (10) and (11) with $a = -1$, $b = 2$, $f(x) = x + 2$, $g(x) = x^2$. The area is $\int_{-1}^2 (x + 2 - x^2)\,dx = \left[\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3\right]_{-1}^2 = \frac{9}{2}$, so $\overline{x} = \frac{2}{9}\int_{-1}^2 x(x + 2 - x^2)\,dx = \frac{2}{9}\left[\frac{1}{3}x^3 + x^2 - \frac{1}{4}x^4\right]_{-1}^2 = \frac{1}{2}$ and $\overline{y} = \frac{2}{9}\int_{-1}^2 \frac{1}{2}\left[(x+2)^2 - (x^2)^2\right] dx = \frac{1}{9}\int_{-1}^2 (x^2 + 4x + 4 - x^4)\,dx =$
$= \frac{1}{9}\left[\frac{1}{3}x^3 + 2x^2 + 4x - \frac{1}{5}x^5\right]_{-1}^2 = \frac{8}{5}$. The centroid is $\left(\frac{1}{2}, \frac{8}{5}\right)$.

17. By symmetry, $\overline{y} = \overline{x}$. To find $\overline{x}$ we use Formula (10) with $a = 0$, $b = 1$, $f(x) = \sqrt{x}$, $g(x) = x^2$. The area is $\int_0^1 (\sqrt{x} - x^2)\,dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$, so $\overline{x} = 3\int_0^1 x(\sqrt{x} - x^2)\,dx = 3\left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right]_0^1 = \frac{9}{20}$. The centroid is $\left(\frac{9}{20}, \frac{9}{20}\right)$.

19. We use the analogue of Formulas (10) and (11) with the roles of x and y reversed. The region is described by $1 \le y \le 2$, $y^{-2} \le x \le y$. The area is $\int_1^2 (y - y^{-2})\,dy = \left[\frac{1}{2}y^2 + y^{-1}\right]_1^2 = 1$, so $\overline{x} = \int_1^2 \frac{1}{2}[y^2 - (y^{-2})^2]\,dy = \frac{1}{2}\int_1^2 (y^2 - y^{-4})\,dy = \frac{1}{2}\left[\frac{1}{3}y^3 + \frac{1}{3}y^{-3}\right]_1^2 = \frac{49}{48}$ and $\overline{y} = \int_1^2 y(y - y^{-2})\,dy = \left[\frac{1}{3}y^3 - \ln y\right]_1^2 = \frac{7}{3} - \ln 2$. The centroid is $\left(\frac{49}{48}, \frac{7}{3} - \ln 2\right)$.

21. An isosceles triangle is symmetric across the median to its base. So, if the density is constant, it will balance on a knife-edge under the median. Hence the centroid lies on the median.

23. The region is described by $0 \le x \le 1$, $0 \le y \le \sqrt{x}$. The area is $A = \int_0^1 \sqrt{x}\,dx = \frac{2}{3}$, so the mass is $M = \delta A = 2\cdot\frac{2}{3} = \frac{4}{3}$. By Formulas (8) and (9), $\overline{x} = \frac{3}{2}\int_0^1 x\sqrt{x}\,dx = \frac{3}{2}\left[\frac{2}{5}x^{5/2}\right]_0^1 = \frac{3}{5}$ and $\overline{y} = \frac{3}{2}\int_0^1 \frac{1}{2}(\sqrt{x})^2\,dx = \frac{3}{4}\int_0^1 x\,dx = \frac{3}{8}$.

The center of gravity is $\left(\dfrac{3}{5}, \dfrac{3}{8}\right)$.

25. The region is described by $0 \le y \le 1$, $-y \le x \le y$. The area is $A = 1$, so the mass is $M = \delta A = 3 \cdot 1 = 3$. By symmetry, $\overline{x} = 0$. By the analogue of Formula (10) with the roles of x and y reversed, $\overline{y} = \int_0^1 y[y-(-y)]\,dy = \int_0^1 2y^2\,dy = \dfrac{2}{3}y^3\Big]_0^1 = \dfrac{2}{3}$. The center of gravity is $\left(0, \dfrac{2}{3}\right)$.

27. The region is described by $0 \le x \le \pi$, $0 \le y \le \sin x$. The area is $A = \int_0^\pi \sin x\,dx = 2$, so the mass is $M = \delta A = 4 \cdot 2 = 8$. By symmetry, $\overline{x} = \dfrac{\pi}{2}$. By Formula (9), $\overline{y} = \dfrac{1}{2}\int_0^\pi \dfrac{1}{2}(\sin x)^2\,dx = \dfrac{\pi}{8}$. The center of gravity is $\left(\dfrac{\pi}{2}, \dfrac{\pi}{8}\right)$.

29. The region is described by $1 \le x \le 2$, $0 \le y \le \ln x$. The area is $A = \int_1^2 \ln x\,dx = 2\ln 2 - 1 = \ln 4 - 1$, so the mass is $M = \delta A = \ln 4 - 1$. By Formulas (8) and (9), $\overline{x} = \dfrac{1}{\ln 4 - 1}\int_1^2 x\ln x\,dx = \dfrac{1}{\ln 4 - 1}\left(\ln 4 - \dfrac{3}{4}\right) = \dfrac{4\ln 4 - 3}{4(\ln 4 - 1)}$ and $\overline{y} = \dfrac{1}{\ln 4 - 1}\int_1^2 \dfrac{1}{2}(\ln x)^2\,dx = \dfrac{(\ln 2)^2 - \ln 4 + 1}{\ln 4 - 1}$. The center of gravity is $\left(\dfrac{4\ln 4 - 3}{4(\ln 4 - 1)}, \dfrac{(\ln 2)^2 - \ln 4 + 1}{\ln 4 - 1}\right)$.

31. True, by symmetry.

33. True, by symmetry.

35. By symmetry, $\overline{y} = 0$. We use Formula (10) with a replaced by 0, b replaced by a, $f(x) = \dfrac{bx}{a}$, and $g(x) = -\dfrac{bx}{a}$: The area is ab, so $\overline{x} = \dfrac{1}{ab}\int_0^a x\left(\dfrac{bx}{a} - \left(-\dfrac{bx}{a}\right)\right)dx = \dfrac{2}{a^2}\int_0^a x^2\,dx = \dfrac{2}{a^2}\cdot\dfrac{a^3}{3} = \dfrac{2a}{3}$. The centroid is $\left(\dfrac{2a}{3}, 0\right)$.

37. We will assume that a, b, and c are positive; the other cases are similar. The region is described by $0 \le y \le c$, $-a - \dfrac{b-a}{c}y \le x \le a + \dfrac{b-a}{c}y$. By symmetry, $\overline{x} = 0$. To find $\overline{y}$, we use the analogue of Formula (10) with the roles of x and y reversed. The area is $c(a+b)$, so $\overline{y} = \dfrac{1}{c(a+b)}\int_0^c y\left[\left(a + \dfrac{b-a}{c}y\right) - \left(-a - \dfrac{b-a}{c}y\right)\right]dy = \dfrac{1}{c(a+b)}\int_0^c \left(2ay + \dfrac{2(b-a)}{c}y^2\right)dy = \dfrac{1}{c(a+b)}\left[ay^2 + \dfrac{2(b-a)}{3c}y^3\right]_0^c = \dfrac{c(a+2b)}{3(a+b)}$. The centroid is $\left(0, \dfrac{c(a+2b)}{3(a+b)}\right)$.

39. $\overline{x} = 0$ from the symmetry of the region, $\pi a^2/2$ is the area of the semicircle, $2\pi\overline{y}$ is the distance traveled by the centroid to generate the sphere so $4\pi a^3/3 = (\pi a^2/2)(2\pi\overline{y})$, $\overline{y} = 4a/(3\pi)$.

41. $\overline{x} = k$ so $V = (\pi ab)(2\pi k) = 2\pi^2 abk$.

43. The region generates a cone of volume $\dfrac{1}{3}\pi ab^2$ when it is revolved about the x-axis, the area of the region is $\dfrac{1}{2}ab$ so $\dfrac{1}{3}\pi ab^2 = \left(\dfrac{1}{2}ab\right)(2\pi\overline{y})$, $\overline{y} = b/3$. A cone of volume $\dfrac{1}{3}\pi a^2 b$ is generated when the region is revolved about the y-axis so $\dfrac{1}{3}\pi a^2 b = \left(\dfrac{1}{2}ab\right)(2\pi\overline{x})$, $\overline{x} = a/3$. The centroid is $(a/3, b/3)$.

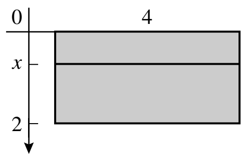
45. The Theorem of Pappus says that $V = 2\pi Ad$, where A is the area of a region in the plane, d is the distance from the region's centroid to an axis of rotation, and V is the volume of the resulting solid of revolution. In any problem in which 2 of these quantities are given, the Theorem of Pappus can be used to compute the third.

Exercise Set 6.8

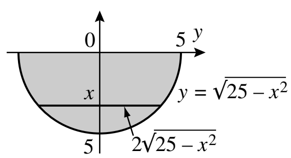
1. (a) $F = \rho hA = 62.4(5)(100) = 31{,}200$ lb, $P = \rho h = 62.4(5) = 312$ lb/ft^2.

(b) $F = \rho hA = 9810(10)(25) = 2{,}452{,}500$ N, $P = \rho h = 9810(10) = 98.1$ kPa.

3. $F = \int_0^2 62.4x(4)\,dx = 249.6\int_0^2 x\,dx = 499.2\,\text{lb}.$

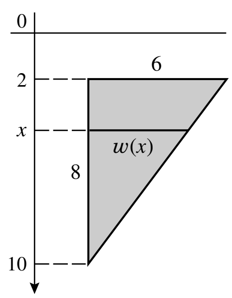


5. $F = \int_0^5 9810x(2\sqrt{25-x^2})\,dx = 19{,}620\int_0^5 x(25-x^2)^{1/2}\,dx = 8.175\times 10^5\,\text{N}.$



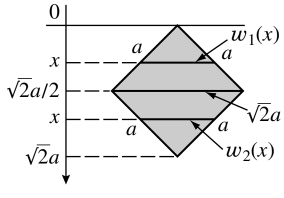
7. By similar triangles, $\dfrac{w(x)}{6} = \dfrac{10-x}{8}$, $w(x) = \dfrac{3}{4}(10-x)$, so $F = \int_2^{10} 9810x\left[\dfrac{3}{4}(10-x)\right]dx =$

$= 7357.5\int_2^{10}(10x - x^2)\,dx = 1{,}098{,}720$ N.



9. Yes: if $\rho_2 = 2\rho_1$ then $F_2 = \int_a^b \rho_2 h(x)w(x)\,dx = \int_a^b 2\rho_1 h(x)w(x)\,dx = 2\int_a^b \rho_1 h(x)w(x)\,dx = 2F_1$.

11. Find the forces on the upper and lower halves and add them: $\dfrac{w_1(x)}{\sqrt{2}a} = \dfrac{x}{\sqrt{2}a/2}$, $w_1(x) = 2x$, $F_1 = \int_0^{\sqrt{2}a/2}\rho x(2x)\,dx = 2\rho\int_0^{\sqrt{2}a/2} x^2\,dx = \sqrt{2}\rho a^3/6$, $\dfrac{w_2(x)}{\sqrt{2}a} = \dfrac{\sqrt{2}a - x}{\sqrt{2}a/2}$, $w_2(x) = 2(\sqrt{2}a - x)$, $F_2 = \int_{\sqrt{2}a/2}^{\sqrt{2}a}\rho x[2(\sqrt{2}a - x)]\,dx = 2\rho\int_{\sqrt{2}a/2}^{\sqrt{2}a}(\sqrt{2}ax - x^2)\,dx = \sqrt{2}\rho a^3/3$, $F = F_1 + F_2 = \sqrt{2}\rho a^3/6 + \sqrt{2}\rho a^3/3 = \rho a^3/\sqrt{2}$ lb.



13. True. By equation (6), the fluid force equals ρhA. For a cylinder, hA is the volume, so ρhA is the weight of the water.

15. False. Let the height of the tank be h, the area of the base be A, and the volume of the tank be V. Then the fluid force on the base is ρhA and the weight of the water is ρV. So if $hA > V$, then the force exceeds the weight. This is true, for example, for a conical tank with its vertex at the top, for which $V = \dfrac{hA}{3}$.

17. Place the x-axis pointing down with its origin at the top of the pool, so that $h(x) = x$ and $w(x) = 10$. The angle between the bottom of the pool and the vertical is $\theta = \tan^{-1}(16/(8-4)) = \tan^{-1} 4$, so $\sec\theta = \sqrt{17}$. Hence $F = \displaystyle\int_4^8 62.4h(x)w(x)\sec\theta\,dx = 624\sqrt{17}\int_4^8 x\,dx = 14976\sqrt{17} \approx 61748$ lb.

19. Place the x-axis starting from the surface, pointing downward. Then using the given formula with $\theta = 30°$, $\sec\theta = 2/\sqrt{3}$, the force is $F = \displaystyle\int_0^{50\sqrt{3}} 9810x(200)(2/\sqrt{3})\,dx = 4{,}905{,}000{,}000\sqrt{3}$ N.

21. **(a)** The force on the window is $F = \displaystyle\int_h^{h+2} \rho_0 x(2)\,dx = 4\rho_0(h+1)$ so (assuming that ρ_0 is constant) $dF/dt = 4\rho_0(dh/dt)$ which is a positive constant if dh/dt is a positive constant.

(b) If $dh/dt = 20$, then $dF/dt = 80\rho_0$ lb/min from part (a).

23. $h = \dfrac{P}{\rho} = \dfrac{14.7\,\text{lb/in}^2}{4.66\times 10^{-5}\,\text{lb/in}^3} \approx 315{,}000\,\text{in} \approx 5\,\text{mi}$. The answer is not reasonable. In fact the atmosphere is thinner at higher altitudes, and it's difficult to define where the "top" of the atmosphere is.

Exercise Set 6.9

1. **(a)** $\sinh 3 \approx 10.0179$. **(b)** $\cosh(-2) \approx 3.7622$. **(c)** $\tanh(\ln 4) = 15/17 \approx 0.8824$.

(d) $\sinh^{-1}(-2) \approx -1.4436$. **(e)** $\cosh^{-1} 3 \approx 1.7627$. **(f)** $\tanh^{-1}\dfrac{3}{4} \approx 0.9730$.

3. **(a)** $\sinh(\ln 3) = \dfrac{1}{2}(e^{\ln 3} - e^{-\ln 3}) = \dfrac{1}{2}\left(3 - \dfrac{1}{3}\right) = \dfrac{4}{3}$.

(b) $\cosh(-\ln 2) = \dfrac{1}{2}(e^{-\ln 2} + e^{\ln 2}) = \dfrac{1}{2}\left(\dfrac{1}{2} + 2\right) = \dfrac{5}{4}$.

(c) $\tanh(2\ln 5) = \dfrac{e^{2\ln 5} - e^{-2\ln 5}}{e^{2\ln 5} + e^{-2\ln 5}} = \dfrac{25 - 1/25}{25 + 1/25} = \dfrac{312}{313}$.

(d) $\sinh(-3\ln 2) = \dfrac{1}{2}(e^{-3\ln 2} - e^{3\ln 2}) = \dfrac{1}{2}\left(\dfrac{1}{8} - 8\right) = -\dfrac{63}{16}$.

5.

	$\sinh x_0$	$\cosh x_0$	$\tanh x_0$	$\coth x_0$	$\text{sech } x_0$	$\text{csch } x_0$
(a)	2	$\sqrt{5}$	$2/\sqrt{5}$	$\sqrt{5}/2$	$1/\sqrt{5}$	1/2
(b)	3/4	5/4	3/5	5/3	4/5	4/3
(c)	4/3	5/3	4/5	5/4	3/5	3/4

(a) $\cosh^2 x_0 = 1 + \sinh^2 x_0 = 1 + (2)^2 = 5$, $\cosh x_0 = \sqrt{5}$.

(b) $\sinh^2 x_0 = \cosh^2 x_0 - 1 = \dfrac{25}{16} - 1 = \dfrac{9}{16}$, $\sinh x_0 = \dfrac{3}{4}$ (because $x_0 > 0$).

(c) $\operatorname{sech}^2 x_0 = 1 - \tanh^2 x_0 = 1 - \left(\dfrac{4}{5}\right)^2 = 1 - \dfrac{16}{25} = \dfrac{9}{25}$, $\operatorname{sech} x_0 = \dfrac{3}{5}$, $\cosh x_0 = \dfrac{1}{\operatorname{sech} x_0} = \dfrac{5}{3}$, from $\dfrac{\sinh x_0}{\cosh x_0} = \tanh x_0$ we get $\sinh x_0 = \left(\dfrac{5}{3}\right)\left(\dfrac{4}{5}\right) = \dfrac{4}{3}$.

7. $\dfrac{d}{dx}\cosh^{-1} x = \dfrac{d}{dx}\ln(x+\sqrt{x^2-1}) = \dfrac{1}{x+\sqrt{x^2-1}}\left(1+\dfrac{2x}{2\sqrt{x^2-1}}\right) = \dfrac{1}{x+\sqrt{x^2-1}}\dfrac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}} = \dfrac{1}{\sqrt{x^2-1}}$.

$\dfrac{d}{dx}\tanh^{-1} x = \dfrac{d}{dx}\left[\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)\right] = \dfrac{1}{2}\cdot\dfrac{1}{\frac{1+x}{1-x}}\cdot\dfrac{(1-x)\cdot 1-(1+x)(-1)}{(1-x)^2} = \dfrac{2}{2(1+x)(1-x)} = \dfrac{1}{1-x^2}$.

9. $\dfrac{dy}{dx} = 4\cosh(4x-8)$.

11. $\dfrac{dy}{dx} = -\dfrac{1}{x}\operatorname{csch}^2(\ln x)$.

13. $\dfrac{dy}{dx} = \dfrac{1}{x^2}\operatorname{csch}(1/x)\coth(1/x)$.

15. $\dfrac{dy}{dx} = \dfrac{2+5\cosh(5x)\sinh(5x)}{\sqrt{4x+\cosh^2(5x)}}$.

17. $\dfrac{dy}{dx} = x^{5/2}\tanh(\sqrt{x})\operatorname{sech}^2(\sqrt{x}) + 3x^2\tanh^2(\sqrt{x})$.

19. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+x^2/9}}\left(\dfrac{1}{3}\right) = 1/\sqrt{9+x^2}$.

21. $\dfrac{dy}{dx} = 1/\left[(\cosh^{-1} x)\sqrt{x^2-1}\right]$.

23. $\dfrac{dy}{dx} = -(\tanh^{-1} x)^{-2}/(1-x^2)$.

25. $\dfrac{dy}{dx} = \dfrac{\sinh x}{\sqrt{\cosh^2 x - 1}} = \dfrac{\sinh x}{|\sinh x|} = \begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases}$.

27. $\dfrac{dy}{dx} = -\dfrac{e^x}{2x\sqrt{1-x}} + e^x \operatorname{sech}^{-1}\sqrt{x}$.

29. $u = \sinh x$, $\displaystyle\int u^6\,du = \frac{1}{7}\sinh^7 x + C$.

31. $u = \tanh x$, $\displaystyle\int \sqrt{u}\,du = \frac{2}{3}(\tanh x)^{3/2} + C$.

33. $u = \cosh 2x$, $\displaystyle\int \frac{1}{2}\frac{1}{u}\,du = \frac{1}{2}\ln(\cosh 2x) + C$.

35. $-\dfrac{1}{3}\operatorname{sech}^3 x\Big]_{\ln 2}^{\ln 3} = 37/375$.

37. $u = 3x$, $\displaystyle\frac{1}{3}\int \frac{1}{\sqrt{1+u^2}}du = \frac{1}{3}\sinh^{-1} 3x + C$.

39. $u = e^x$, $\displaystyle\int \frac{1}{u\sqrt{1-u^2}}du = -\operatorname{sech}^{-1}(e^x) + C$.

41. $u = 2x$, $\displaystyle\int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + C = -\operatorname{csch}^{-1}|2x| + C$.

43. $\displaystyle\tanh^{-1} x\Big]_0^{1/2} = \tanh^{-1}(1/2) - \tanh^{-1}(0) = \frac{1}{2}\ln\frac{1+1/2}{1-1/2} = \frac{1}{2}\ln 3$.

45. True. $\displaystyle\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = e^{-x}$ is positive for all x.

47. True. Only $\sinh x$ has this property.

49. $\displaystyle A = \int_0^{\ln 3} \sinh 2x\,dx = \frac{1}{2}\cosh 2x\Big]_0^{\ln 3} = \frac{1}{2}[\cosh(2\ln 3) - 1]$, but $\displaystyle\cosh(2\ln 3) = \cosh(\ln 9) = \frac{1}{2}(e^{\ln 9} + e^{-\ln 9}) = \frac{1}{2}(9 + 1/9) = 41/9$ so $\displaystyle A = \frac{1}{2}[41/9 - 1] = 16/9$.

51. $\displaystyle V = \pi\int_0^5 (\cosh^2 2x - \sinh^2 2x)\,dx = \pi\int_0^5 dx = 5\pi$.

53. $y' = \sinh x$, $1 + (y')^2 = 1 + \sinh^2 x = \cosh^2 x$, $\displaystyle L = \int_0^{\ln 2} \cosh x\,dx = \sinh x\Big]_0^{\ln 2} = \sinh(\ln 2) = \frac{1}{2}(e^{\ln 2} - e^{-\ln 2}) = \frac{1}{2}\left(2 - \frac{1}{2}\right) = \frac{3}{4}$.

55. (a) $\displaystyle\lim_{x\to+\infty} \sinh x = \lim_{x\to+\infty} \frac{1}{2}(e^x - e^{-x}) = +\infty - 0 = +\infty$.

(b) $\displaystyle\lim_{x\to-\infty} \sinh x = \lim_{x\to-\infty} \frac{1}{2}(e^x - e^{-x}) = 0 - \infty = -\infty$.

(c) $\displaystyle\lim_{x\to+\infty} \tanh x = \lim_{x\to+\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to+\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1$.

(d) $\displaystyle\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to-\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = -1$.

(e) $\displaystyle\lim_{x\to+\infty} \sinh^{-1} x = \lim_{x\to+\infty} \ln(x + \sqrt{x^2+1}) = +\infty$.

(f) $\displaystyle\lim_{x\to 1^-} \tanh^{-1} x = \lim_{x\to 1^-} \frac{1}{2}[\ln(1+x) - \ln(1-x)] = +\infty$.

57. $\displaystyle\sinh(-x) = \frac{1}{2}(e^{-x} - e^x) = -\frac{1}{2}(e^x - e^{-x}) = -\sinh x$, $\displaystyle\cosh(-x) = \frac{1}{2}(e^{-x} + e^x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$.

59. (a) Divide $\cosh^2 x - \sinh^2 x = 1$ by $\cosh^2 x$.

(b) $$\tanh(x+y) = \frac{\sinh x\cosh y + \cosh x\sinh y}{\cosh x\cosh y + \sinh x\sinh y} = \frac{\dfrac{\sinh x}{\cosh x} + \dfrac{\sinh y}{\cosh y}}{1 + \dfrac{\sinh x\sinh y}{\cosh x\cosh y}} = \frac{\tanh x + \tanh y}{1 + \tanh x\tanh y}.$$

(c) Let $y = x$ in part (b).

61. **(a)** $\dfrac{d}{dx}(\cosh^{-1} x) = \dfrac{1 + x/\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = 1/\sqrt{x^2-1}$.

(b) $\dfrac{d}{dx}(\tanh^{-1} x) = \dfrac{d}{dx}\left[\dfrac{1}{2}(\ln(1+x) - \ln(1-x))\right] = \dfrac{1}{2}\left(\dfrac{1}{1+x} + \dfrac{1}{1-x}\right) = 1/(1-x^2)$.

63. If $|u| < 1$ then, by Theorem 6.9.6, $\displaystyle\int \frac{du}{1-u^2} = \tanh^{-1} u + C$. For $|u| > 1$, $\displaystyle\int \frac{du}{1-u^2} = \coth^{-1} u + C = \tanh^{-1}(1/u) + C$.

65. **(a)** $\displaystyle\lim_{x\to+\infty}(\cosh^{-1} x - \ln x) = \lim_{x\to+\infty}[\ln(x+\sqrt{x^2-1}) - \ln x] = \lim_{x\to+\infty} \ln\frac{x+\sqrt{x^2-1}}{x} = \lim_{x\to+\infty}\ln(1+\sqrt{1-1/x^2}) = \ln 2$.

(b) $\displaystyle\lim_{x\to+\infty}\frac{\cosh x}{e^x} = \lim_{x\to+\infty}\frac{e^x+e^{-x}}{2e^x} = \lim_{x\to+\infty}\frac{1}{2}(1+e^{-2x}) = 1/2$.

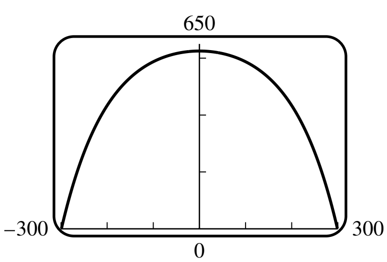
67. Let $x = -u/a$, $\displaystyle\int \frac{1}{\sqrt{u^2-a^2}}\,du = -\int \frac{a}{a\sqrt{x^2-1}}\,dx = -\cosh^{-1} x + C = -\cosh^{-1}(-u/a) + C$.

$-\cosh^{-1}(-u/a) = -\ln(-u/a + \sqrt{u^2/a^2-1}) = \ln\left[\dfrac{a}{-u+\sqrt{u^2-a^2}}\dfrac{u+\sqrt{u^2-a^2}}{u+\sqrt{u^2-a^2}}\right] = \ln\left|u+\sqrt{u^2-a^2}\right| - \ln a = \ln|u+\sqrt{u^2-a^2}| + C_1$, so $\displaystyle\int \frac{1}{\sqrt{u^2-a^2}}\,du = \ln\left|u+\sqrt{u^2-a^2}\right| + C_2$.

69. $\displaystyle\int_{-a}^{a} e^{tx}\,dx = \frac{1}{t}e^{tx}\Big]_{-a}^{a} = \frac{1}{t}(e^{at} - e^{-at}) = \frac{2\sinh at}{t}$ for $t \neq 0$.

71. From part (b) of Exercise 70, $S = a\cosh(b/a) - a$ so $30 = a\cosh(200/a) - a$. Let $u = 200/a$, then $a = 200/u$ so $30 = (200/u)[\cosh u - 1]$, $\cosh u - 1 = 0.15u$. If $f(u) = \cosh u - 0.15u - 1$, then $u_{n+1} = u_n - \dfrac{\cosh u_n - 0.15u_n - 1}{\sinh u_n - 0.15}$; $u_1 = 0.3, \ldots, u_4 \approx u_5 \approx 0.297792782 \approx 200/a$ so $a \approx 671.6079505$. From part (a), $L = 2a\sinh(b/a) \approx 2(671.6079505)\sinh(0.297792782) \approx 405.9$ ft.

73. Set $a = 68.7672$, $b = 0.0100333$, $c = 693.8597$, $d = 299.2239$.

(a) 

(b) $L = 2\displaystyle\int_0^d \sqrt{1+a^2b^2\sinh^2 bx}\,dx \approx 1480.2798$ ft.

(c) $x \approx \pm 283.6249$ ft. **(d)** $82°$

75. **(a)** When the bow of the boat is at the point (x, y) and the person has walked a distance D, then the person is located at the point $(0, D)$, the line segment connecting $(0, D)$ and (x, y) has length a; thus $a^2 = x^2 + (D-y)^2$, $D = y + \sqrt{a^2-x^2} = a\,\mathrm{sech}^{-1}(x/a)$.

(b) Find D when $a = 15$, $x = 10$: $D = 15\,\mathrm{sech}^{-1}(10/15) = 15\ln\left(\dfrac{1+\sqrt{5/9}}{2/3}\right) \approx 14.44$ m.

(c) $dy/dx = -\dfrac{a^2}{x\sqrt{a^2-x^2}} + \dfrac{x}{\sqrt{a^2-x^2}} = \dfrac{1}{\sqrt{a^2-x^2}}\left[-\dfrac{a^2}{x}+x\right] = -\dfrac{1}{x}\sqrt{a^2-x^2}$, $1+[y']^2 = 1+\dfrac{a^2-x^2}{x^2} = \dfrac{a^2}{x^2}$; with $a = 15$, $L = \displaystyle\int_5^{15}\sqrt{\frac{225}{x^2}}\,dx = \int_5^{15}\frac{15}{x}\,dx = 15\ln x\Big]_5^{15} = 15\ln 3 \approx 16.48$ m.

77. First we would need to show that the line segment from the origin to P meets the right branch of the hyperbola only at P, so that the shaded region in Figure 6.9.3b is well-defined. (This is easy.)

Next we'd need to show that the area of the shaded region approaches $+\infty$ as the point P moves upward and to the right along the curve, so that $\cosh t$ and $\sinh t$ will be defined for all $t > 0$ (and hence, by symmetry, for all t.) (This is not quite as easy.)

Chapter 6 Review Exercises

7. (a) $A = \displaystyle\int_a^b (f(x)-g(x))\,dx + \int_b^c (g(x)-f(x))\,dx + \int_c^d (f(x)-g(x))\,dx.$

(b) $A = \displaystyle\int_{-1}^0 (x^3-x)\,dx + \int_0^1 (x-x^3)\,dx + \int_1^2 (x^3-x)\,dx = \frac{1}{4}+\frac{1}{4}+\frac{9}{4} = \frac{11}{4}.$

9. Find where the curves cross: set $x^3 = x^2+4$; by observation $x = 2$ is a solution. Then $V = \pi\displaystyle\int_0^2 [(x^2+4)^2-(x^3)^2]\,dx = \frac{4352}{105}\pi.$

11. $V = \displaystyle\int_1^4 \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx = 2\ln 2 + \frac{3}{2}.$

13. By implicit differentiation $\dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^{1/3}$, so $1+\left(\dfrac{dy}{dx}\right)^2 = 1+\left(\dfrac{y}{x}\right)^{2/3} = \dfrac{x^{2/3}+y^{2/3}}{x^{2/3}} = \dfrac{4}{x^{2/3}}$, $L = \displaystyle\int_{-8}^{-1}\frac{2}{(-x)^{1/3}}\,dx = 9.$

15. $A = \displaystyle\int_9^{16} 2\pi\sqrt{25-x}\sqrt{1+\left(\frac{-1}{2\sqrt{25-x}}\right)^2}\,dx = \pi\int_9^{16}\sqrt{101-4x}\,dx = \frac{\pi}{6}\left(65^{3/2}-37^{3/2}\right).$

17. A cross section of the solid, perpendicular to the x-axis, has area equal to $\pi(\sec x)^2$, and the average of these cross sectional areas is given by $A_{\text{ave}} = \dfrac{1}{\pi/3}\displaystyle\int_0^{\pi/3}\pi(\sec x)^2\,dx = \frac{3}{\pi}\pi\tan x\Big]_0^{\pi/3} = 3\sqrt{3}.$

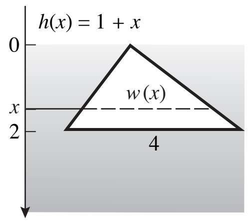
19. (a) $F = kx$, $\dfrac{1}{2} = k\dfrac{1}{4}$, $k = 2$, $W = \displaystyle\int_0^{1/4} kx\,dx = 1/16$ J. **(b)** $25 = \displaystyle\int_0^L kx\,dx = kL^2/2$, $L = 5$ m.

21. The region is described by $-4 \le y \le 4$, $\dfrac{y^2}{4} \le x \le 2+\dfrac{y^2}{8}$. By symmetry, $\overline{y} = 0$. To find $\overline{x}$, we use the analogue of Formula (11) in Section 6.7. The area is $A = \displaystyle\int_{-4}^4\left(2+\frac{y^2}{8}-\frac{y^2}{4}\right)dy = \int_{-4}^4\left(2-\frac{y^2}{8}\right)dy = \left[2y-\frac{y^3}{24}\right]_{-4}^4 = \frac{32}{3}.$ So $\overline{x} = \dfrac{3}{32}\displaystyle\int_{-4}^4\frac{1}{2}\left[\left(2+\frac{y^2}{8}\right)^2-\left(\frac{y^2}{4}\right)^2\right]dy = \frac{3}{64}\int_{-4}^4\left(4+\frac{y^2}{2}-\frac{3y^4}{64}\right)dy = \frac{3}{64}\left[4y+\frac{y^3}{6}-\frac{3y^5}{320}\right]_{-4}^4 = \frac{8}{5}.$ The centroid is $\left(\dfrac{8}{5},0\right)$.

23. (a) $F = \displaystyle\int_0^1 \rho x 3\,dx$ N.

(b) By similar triangles, $\dfrac{w(x)}{4} = \dfrac{x}{2}$, $w(x) = 2x$, so $F = \displaystyle\int_0^2 \rho(1+x)2x\,dx$ lb/ft^2.

(c) A formula for the parabola is $y = \dfrac{8}{125}x^2 - 10$, so $F = \displaystyle\int_{-10}^0 9810|y|2\sqrt{\frac{125}{8}(y+10)}\,dy$ N.



25. (a) $\cosh 3x = \cosh(2x + x) = \cosh 2x \cosh x + \sinh 2x \sinh x = (2\cosh^2 x - 1)\cosh x + (2\sinh x \cosh x)\sinh x = 2\cosh^3 x - \cosh x + 2\sinh^2 x \cosh x = 2\cosh^3 x - \cosh x + 2(\cosh^2 x - 1)\cosh x = 4\cosh^3 x - 3\cosh x$.

(b) From Theorem 6.9.2 with x replaced by $\dfrac{x}{2}$: $\cosh x = 2\cosh^2 \dfrac{x}{2} - 1$, $2\cosh^2 \dfrac{x}{2} = \cosh x + 1$, $\cosh^2 \dfrac{x}{2} = \dfrac{1}{2}(\cosh x + 1)$, $\cosh \dfrac{x}{2} = \sqrt{\dfrac{1}{2}(\cosh x + 1)}$ (because $\cosh \dfrac{x}{2} > 0$).

(c) From Theorem 6.9.2 with x replaced by $\dfrac{x}{2}$: $\cosh x = 2\sinh^2 \dfrac{x}{2} + 1$, $2\sinh^2 \dfrac{x}{2} = \cosh x - 1$, $\sinh^2 \dfrac{x}{2} = \dfrac{1}{2}(\cosh x - 1)$, $\sinh \dfrac{x}{2} = \pm\sqrt{\dfrac{1}{2}(\cosh x - 1)}$.

Chapter 6 Making Connections

1. (a) By equation (2) of Section 6.3, the volume is $V = \displaystyle\int_0^1 2\pi x f(x^2)\,dx$. Making the substitution $u = x^2$, $du = 2x\,dx$ gives $V = \displaystyle\int_0^1 2\pi f(u) \cdot \frac{1}{2}\,du = \pi \int_0^1 f(u)\,du = \pi A_1$.

(b) By the Theorem of Pappus, the volume in (a) equals $2\pi A_2 \overline{x}$, where $\overline{x} = a$ is the x-coordinate of the centroid of R. Hence $a = \dfrac{\pi A_1}{2\pi A_2} = \dfrac{A_1}{2A_2}$.

3. The area of the annulus with inner radius r and outer radius $r + \Delta r$ is $\pi(r+\Delta r)^2 - \pi r^2 \approx 2\pi r \Delta r$, so its mass is approximately $2\pi r f(r)\Delta r$. Hence the total mass of the lamina is $\displaystyle\int_0^a 2\pi r f(r)\,dr$.

5. (a) Consider any solid obtained by sliding a horizontal region, of any shape, some distance vertically. Thus the top and bottom faces, and every horizontal cross-section in between, are all congruent. This includes all of the cases described in part (a) of the problem.

Suppose such a solid, whose base has area A, is floating in a fluid so that its base is a distance h below the surface. The pressure at the base is ρh, so the fluid exerts an upward force on the base of magnitude $\rho h A$. The fluid also exerts forces on the sides of the solid, but those are horizontal, so they don't contribute to the buoyancy. Hence the buoyant force equals $\rho h A$. Since the part of the solid which is below the surface has volume hA, the buoyant force equals the weight of fluid which would fill that volume; i.e. the weight of the fluid displaced by the solid.

(b) Now consider a solid which is the union of finitely many solids of the type described above. The buoyant force on such a solid is the sum of the buoyant forces on its constituents, which equals the sum of the weights of the fluid displaced by them, which equals the weight of the fluid displaced by the whole solid. So the Archimedes Principle applies to the union.

Any solid can be approximated by such unions, so it is plausible that the Archimedes Principle applies to all solids.