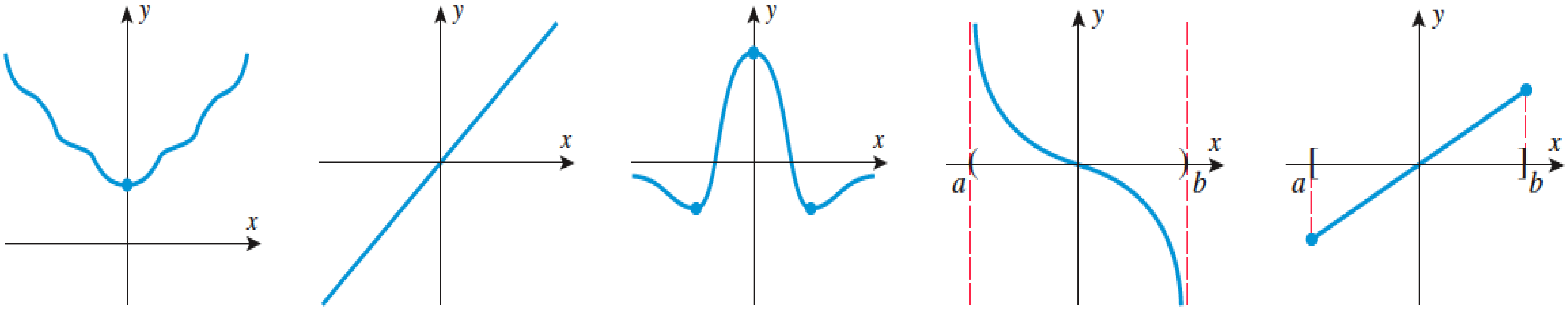
National University
of computer and emerging sciences

Foundation for Advancement
of Science and Technology

ABSOLUTE MAXIMA AND MINIMA

4.4.1 DEFINITION Consider an interval in the domain of a function f and a point x_0 in that interval. We say that f has an ***absolute maximum*** at x_0 if $f(x) \leq f(x_0)$ for all x in the interval, and we say that f has an ***absolute minimum*** at x_0 if $f(x_0) \leq f(x)$ for all x in the interval. We say that f has an ***absolute extremum*** at x_0 if it has either an absolute maximum or an absolute minimum at that point.



f has an absolute minimum but no absolute maximum on $(-\infty, +\infty)$.

(*a*)

f has no absolute extrema on $(-\infty, +\infty)$.

(*b*)

f has an absolute maximum and minimum on $(-\infty, +\infty)$.

(*c*)

f has no absolute extrema on (a, b).

(*d*)

f has an absolute maximum and minimum on $[a, b]$.

(*e*)

4.4.2 THEOREM (*Extreme-Value Theorem*) *If a function f is continuous on a finite closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.*

4.4.3 THEOREM *If f has an absolute extremum on an open interval (a, b), then it must occur at a critical point of f.*

***A Procedure for Finding the Absolute Extrema of a Continuous Function f on a Finite Closed Interval* $[a, b]$**

Step 1. Find the critical points of f in (a, b).

Step 2. Evaluate f at all the critical points and at the endpoints a and b.

Step 3. The largest of the values in Step 2 is the absolute maximum value of f on $[a, b]$ and the smallest value is the absolute minimum.

▸ Example 1 Find the absolute maximum and minimum values of the function $f(x) = 2x^3 - 15x^2 + 36x$ on the interval $[1, 5]$, and determine where these values occur.

Solution. Since f is continuous and differentiable everywhere, the absolute extrema must occur either at endpoints of the interval or at solutions to the equation $f'(x) = 0$ in the open interval $(1, 5)$. The equation $f'(x) = 0$ can be written as

$$6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x - 2)(x - 3) = 0$$

Thus, there are stationary points at $x = 2$ and at $x = 3$. Evaluating f at the endpoints, at $x = 2$, and at $x = 3$ yields

$$f(1) = 2(1)^3 - 15(1)^2 + 36(1) = 23$$
$$f(2) = 2(2)^3 - 15(2)^2 + 36(2) = 28$$
$$f(3) = 2(3)^3 - 15(3)^2 + 36(3) = 27$$
$$f(5) = 2(5)^3 - 15(5)^2 + 36(5) = 55$$

from which we conclude that the absolute minimum of f on $[1, 5]$ is 23, occurring at $x = 1$, and the absolute maximum of f on $[1, 5]$ is 55, occurring at $x = 5$.

Table 4.4.2

ABSOLUTE EXTREMA ON INFINITE INTERVALS

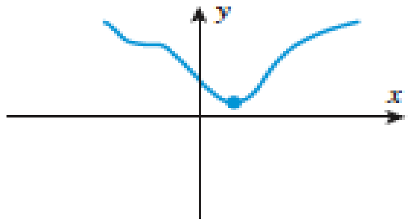
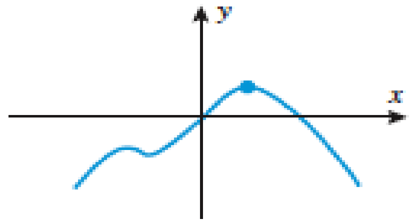
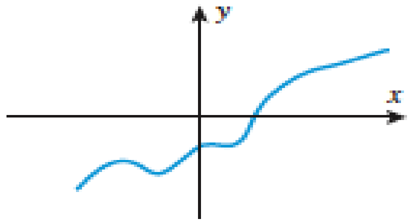
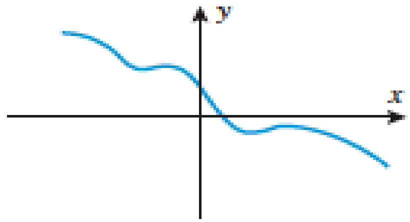
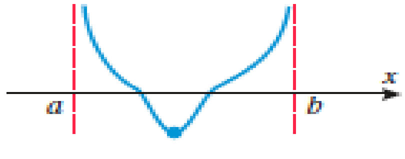
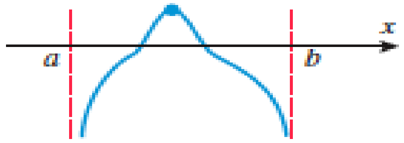
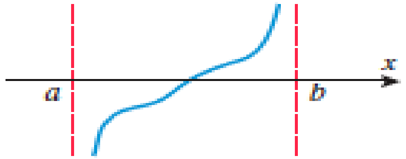
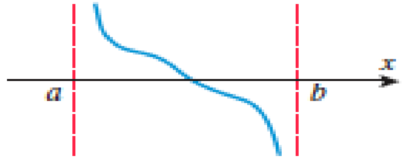
LIMITS	$\lim_{x\to-\infty} f(x) = +\infty$ $\lim_{x\to+\infty} f(x) = +\infty$	$\lim_{x\to-\infty} f(x) = -\infty$ $\lim_{x\to+\infty} f(x) = -\infty$	$\lim_{x\to-\infty} f(x) = -\infty$ $\lim_{x\to+\infty} f(x) = +\infty$	$\lim_{x\to-\infty} f(x) = +\infty$ $\lim_{x\to+\infty} f(x) = -\infty$
CONCLUSION IF *f* IS CONTINUOUS EVERYWHERE	f has an absolute minimum but no absolute maximum on $(-\infty, +\infty)$.	f has an absolute maximum but no absolute minimum on $(-\infty, +\infty)$.	f has neither an absolute maximum nor an absolute minimum on $(-\infty, +\infty)$.	f has neither an absolute maximum nor an absolute minimum on $(-\infty, +\infty)$.
GRAPH				

Table 4.4.3

ABSOLUTE EXTREMA ON OPEN INTERVALS

LIMITS	$\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to b^-} f(x) = +\infty$	$\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to b^-} f(x) = -\infty$	$\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to b^-} f(x) = +\infty$	$\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to b^-} f(x) = -\infty$
CONCLUSION IF *f* IS CONTINUOUS ON (a, b)	f has an absolute minimum but no absolute maximum on (a, b).	f has an absolute maximum but no absolute minimum on (a, b).	f has neither an absolute maximum nor an absolute minimum on (a, b).	f has neither an absolute maximum nor an absolute minimum on (a, b).
GRAPH				

4.4.4 THEOREM *Suppose that f is continuous and has exactly one relative extremum on an interval, say at x_0.*

(a) *If f has a relative minimum at x_0, then $f(x_0)$ is the absolute minimum of f on the interval.*

(b) *If f has a relative maximum at x_0, then $f(x_0)$ is the absolute maximum of f the interval.*

7–16 Find the absolute maximum and minimum values of f on the given closed interval, and state where those values occur.

7. $f(x) = 4x^2 - 12x + 10;\ [1, 2]$

8. $f(x) = 8x - x^2;\ [0, 6]$

9. $f(x) = (x - 2)^3;\ [1, 4]$

10. $f(x) = 2x^3 + 3x^2 - 12x;\ [-3, 2]$

11. $f(x) = \dfrac{3x}{\sqrt{4x^2 + 1}};\ [-1, 1]$

12. $f(x) = (x^2 + x)^{2/3};\ [-2, 3]$

13. $f(x) = x - 2\sin x;\ [-\pi/4, \pi/2]$

14. $f(x) = \sin x - \cos x;\ [0, \pi]$