National University of computer and emerging sciences

Foundation for Advancement of Science and Technology

VOLUMES BY SLICING DISKS AND WASHERS

VOLUME:

Volume is the quantity of 3D space enclosed by a closed surface

VOLUME (DISK & WASHER):

$$V = A \cdot h = [\text{area of a cross section}] \times [\text{height}]$$

The volume of a solid can be obtained by integrating the cross-sectional area from one end of the solid to the other.

6.2.2 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. If, for each x in $[a, b]$, the cross-sectional area of S perpendicular to the x-axis is $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x)\, dx \tag{3}$$

provided $A(x)$ is integrable.

5.2.3 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the y-axis at $y = c$ and $y = d$. If, for each y in $[c, d]$, the cross-sectional area of S perpendicular to the y-axis is $A(y)$, then the volume of the solid is

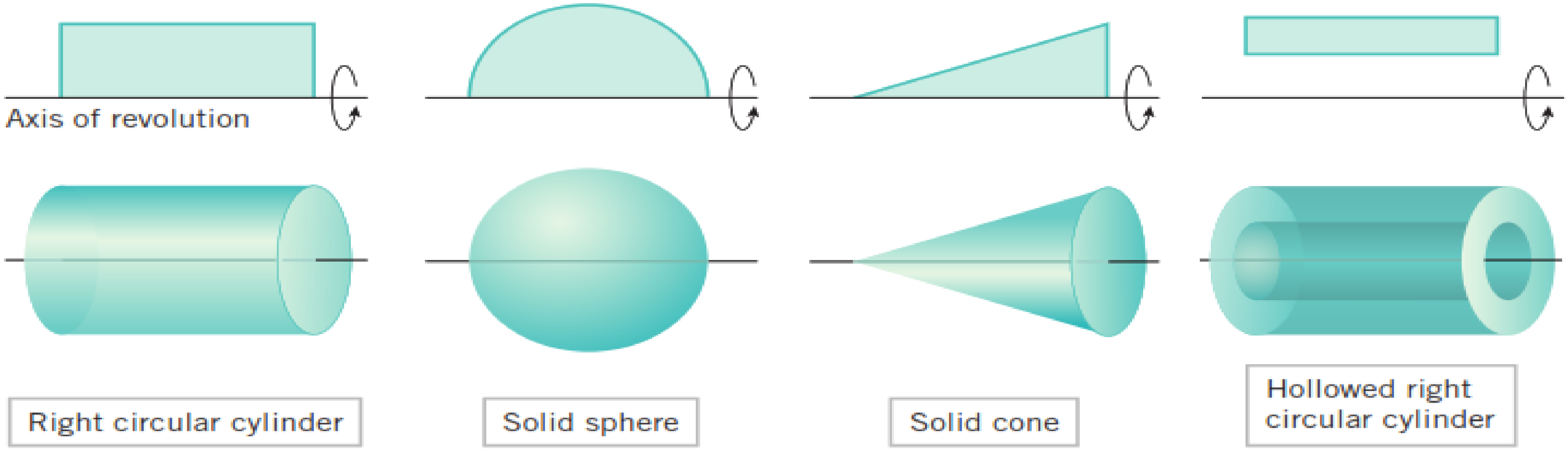
$$V = \int_c^d A(y)\, dy \tag{4}$$

provided $A(y)$ is integrable.

SOLIDS OF REVOLUTION

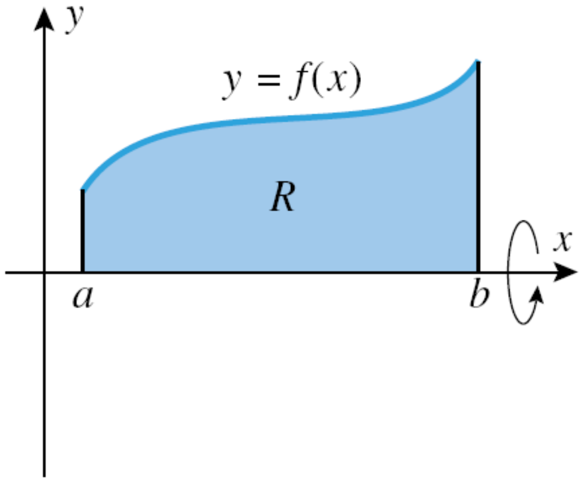
A ***solid of revolution*** is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the ***axis of revolution***. Many familiar solids are of this type (Figure 6.2.8).

Some Familiar Solids of Revolution

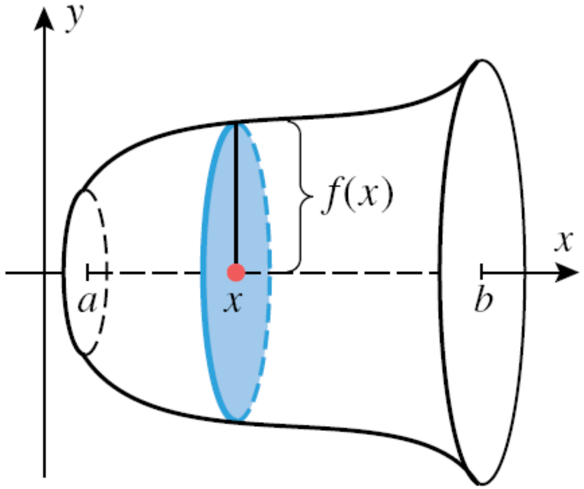


(*a*) (*b*) (*c*) (*d*)

6.2.4 PROBLEM Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.9a). Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis.



(a)

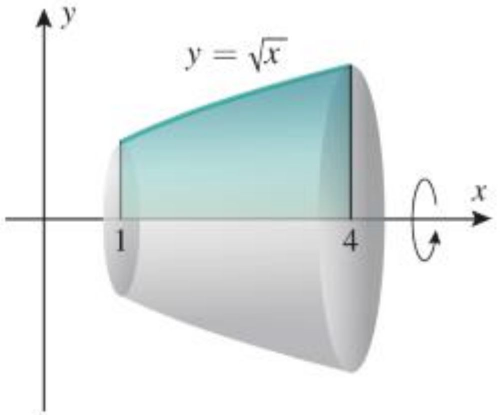


(b)

$$V = \int_a^b \pi[f(x)]^2\,dx$$

Because the cross sections are disk shaped, the application of this formula is called the ***method of disks***.

Example 2 Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the x-axis (Figure 5.2.10).



▲ **Figure 5.2.10**

Solution. From (5), the volume is

$$V = \int_a^b \pi[f(x)]^2\,dx = \int_1^4 \pi x\,dx = \frac{\pi x^2}{2}\bigg]_1^4 = 8\pi - \frac{\pi}{2} = \frac{15\pi}{2} \quad ◀$$

VOLUME BY DISKS PERPENDICULAR TO THE Y-AXIS

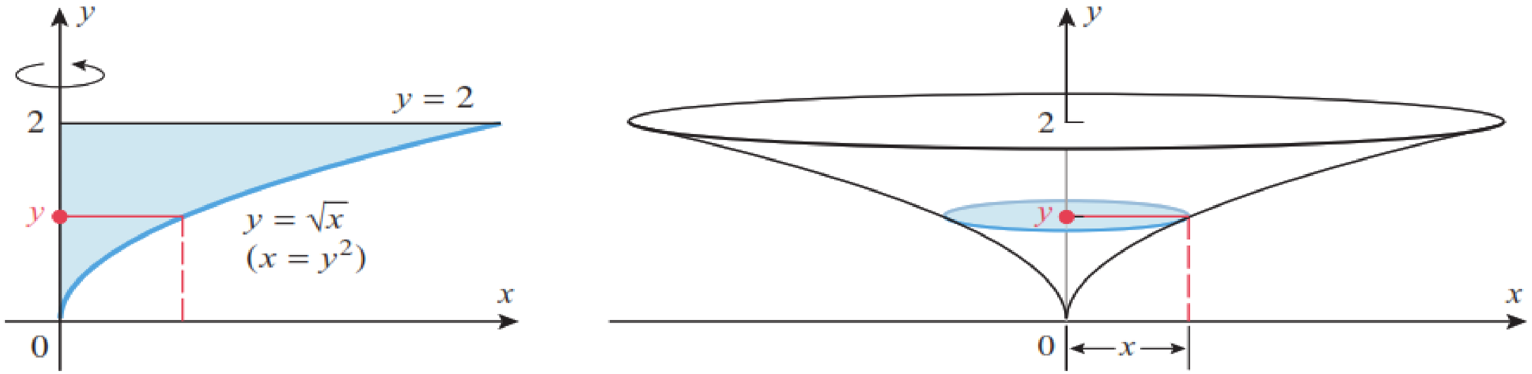
$$V = \int_c^d \pi[u(y)]^2\, dy$$

Disks

► **Example 5** Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is revolved about the y-axis.

Solution. First sketch the region and the solid (Figure 6.2.16). The cross sections taken perpendicular to the y-axis are disks, so we will apply (7). But first we must rewrite $y = \sqrt{x}$ as $x = y^2$. Thus, from (7) with $u(y) = y^2$, the volume is

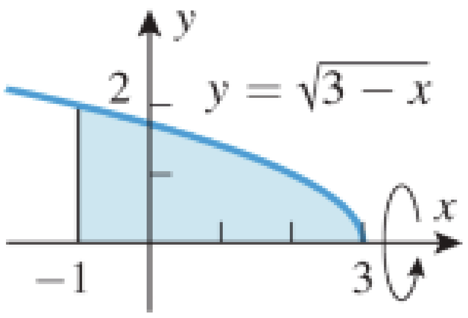
$$V = \int_c^d \pi[u(y)]^2\,dy = \int_0^2 \pi y^4\,dy = \frac{\pi y^5}{5}\bigg]_0^2 = \frac{32\pi}{5}$$ ◄



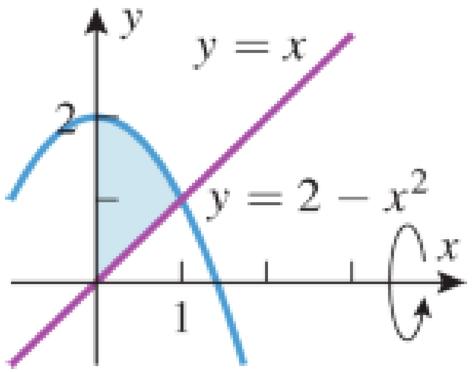
► **Figure 6.2.16**

1–8 Find the volume of the solid that results when the shaded region is revolved about the indicated axis.

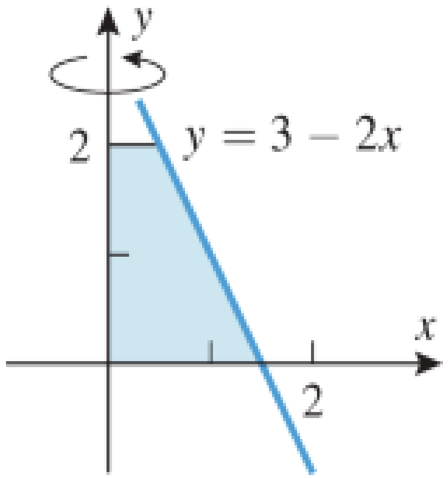
1.



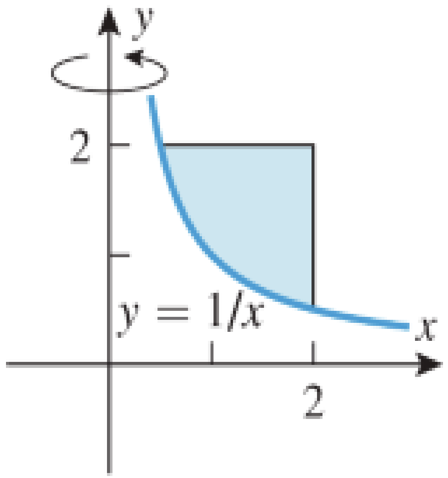
2.



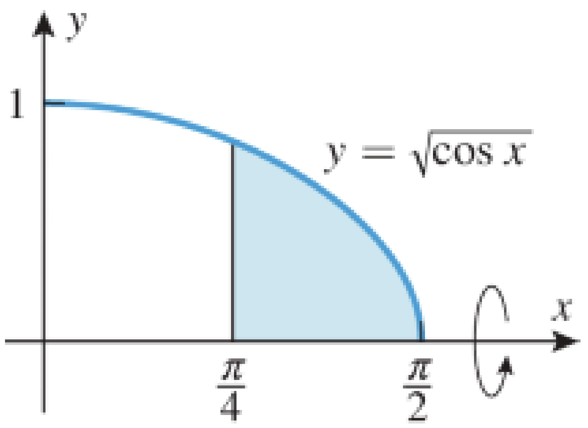
3.



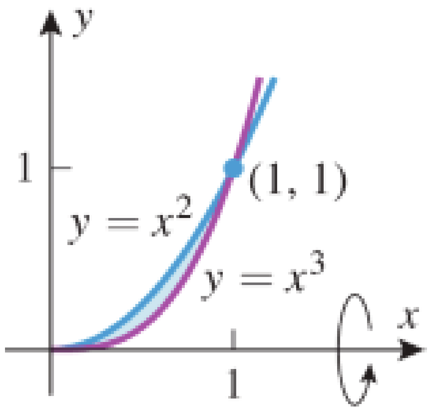
4.



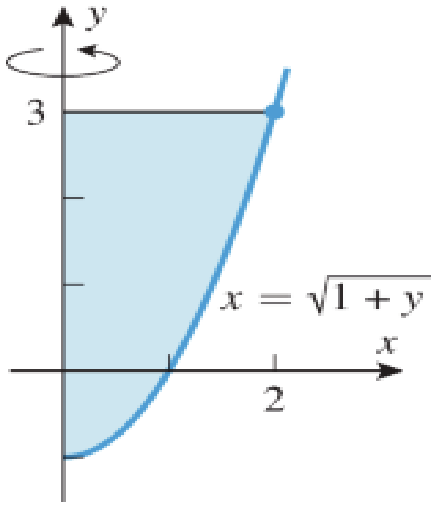
5.


y
1
$y = \sqrt{\cos x}$
x
$\frac{\pi}{4}$
$\frac{\pi}{2}$

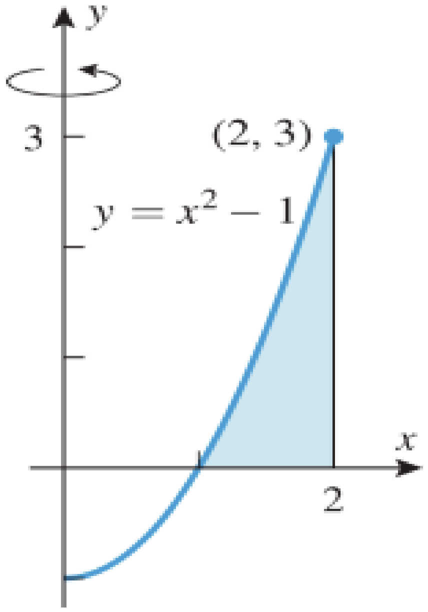
6.


y
1
(1, 1)
$y = x^2$
$y = x^3$
x
1

7.


y
3
$x = \sqrt{1 + y}$
x
2

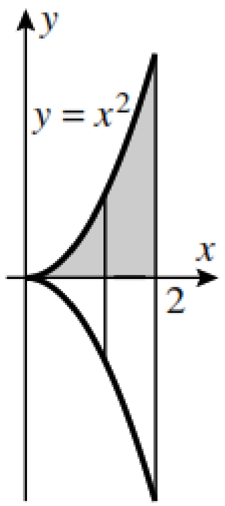
8.


y
3
(2, 3)
$y = x^2 - 1$
x
2

9. Find the volume of the solid whose base is the region bounded between the curve $y = x^2$ and the x-axis from $x = 0$ to $x = 2$ and whose cross sections taken perpendicular to the x-axis are squares.

Sol:

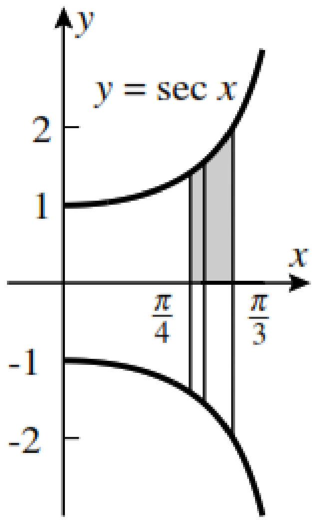
9. $V = \int_0^2 x^4\,dx = 32/5.$



10. Find the volume of the solid whose base is the region bounded between the curve $y = \sec x$ and the x-axis from $x = \pi/4$ to $x = \pi/3$ and whose cross sections taken perpendicular to the x-axis are squares.

Sol:

$$V = \int_{\pi/4}^{\pi/3} \sec^2 x \, dx = \sqrt{3} - 1.$$



washer

- A **washer** is a thin plate (typically disk-shaped) with a hole (typically in the middle)

$$\text{Outer radius:} \qquad R(x)$$
$$\text{Inner radius:} \qquad r(x)$$

The washer's area is

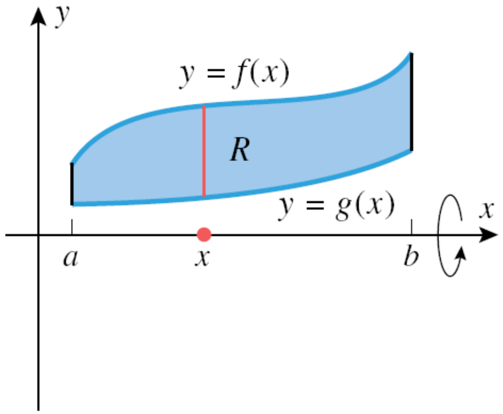
$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi([R(x)]^2 - [r(x)]^2).$$

$$\begin{pmatrix} Area \\ of \\ Outer \end{pmatrix} - \begin{pmatrix} Area \\ of \\ Inner \end{pmatrix}$$

Consequently, the definition of volume gives

$$\boxed{V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx.}$$

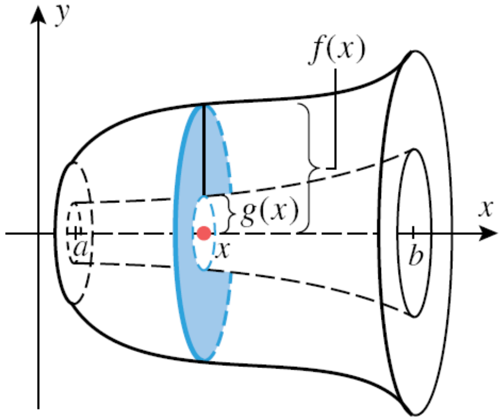
6.2.5 PROBLEM Let f and g be continuous and nonnegative on $[a, b]$, and suppose that $f(x) \geq g(x)$ for all x in the interval $[a, b]$. Let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.12*a*). Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis (Figure 6.2.12*b*).



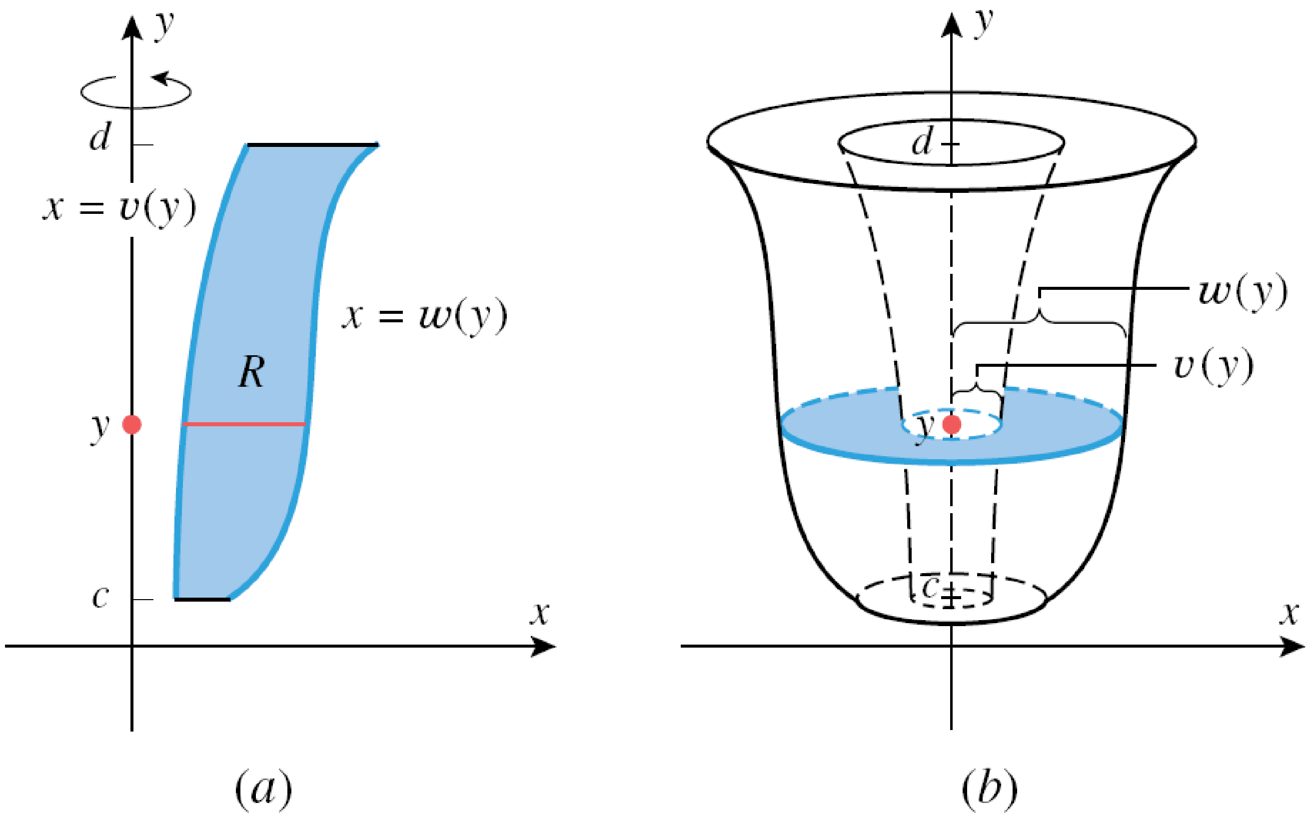
(*a*)

Volume by method of washers

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx$$



(*b*)



$$V = \int_c^d \pi([w(y)]^2 - [v(y)]^2)\, dy$$

Washers

Washers

Example:

The region R is bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = x^2$

between $x = 0$ and $x = 1$. What is the volume of the solid resulting when R is revolved about the x-axis?

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx \quad = \int_0^1 \pi\left(x - x^4\right)dx$$

$$= \pi\left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{3\pi}{10}. \quad =0.942$$

Example 4 Find the volume of the solid generated when the region between the graphs of the equations $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x-axis.

Solution. First sketch the region (Figure 6.2.13*a*); then imagine revolving it about the x-axis (Figure 6.2.13*b*). From (6) the volume is

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx = \int_0^2 \pi\left(\left[\tfrac{1}{2} + x^2\right]^2 - x^2\right)dx$$

$$= \int_0^2 \pi\left(\frac{1}{4} + x^4\right)dx = \pi\left[\frac{x}{4} + \frac{x^5}{5}\right]_0^2 = \frac{69\pi}{10}$$

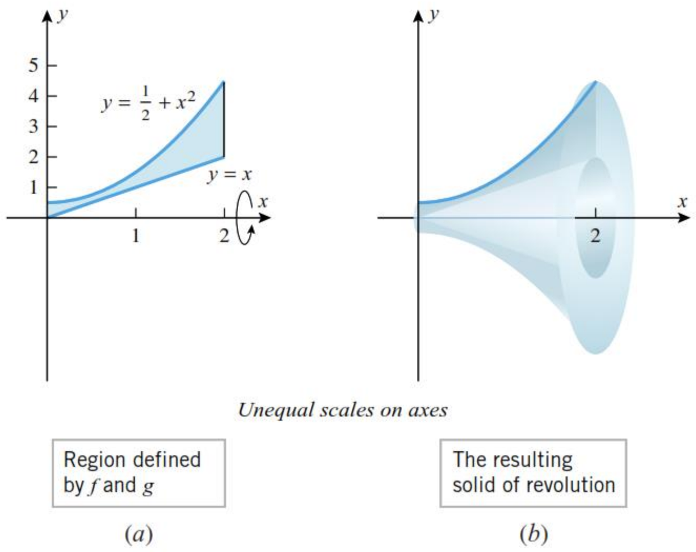


Figure 6.2.13

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx$$

Examples: Find the volume(Disk/Washer) when

$$V = \int_c^d \pi[u(y)]^2\,dy$$

The region is bounded by the curves $y = 3 + 2x - x^2$ and $x + y = 3$ and is rotated about the x-axis.

$$3 - x = 3 + 2x - x^2 \Rightarrow x = 0, 3$$

Therefore, $r_{out} = 3 + 2x - x^2$ and $r_{in} = 3 - x$,

$$\pi\int_0^3\left[\left(3+2x-x^2\right)^2-\left(3-x\right)^2\right]dx$$

$$= \pi\int_0^3\left[\left(x^4-4x^3-2x^2+12x+9\right)-\left(9-6x+x^2\right)\right]dx$$

$$= \pi\int_0^3\left(x^4-4x^3-3x^2+18x\right)dx = \frac{108\pi}{5}$$

Washer method

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx$$

The region is bounded by the curves $x = y^{2/3}$, $x = 0$, and $y = 8$ and is rotated about the y-axis.

With the limits of integration

$y = 0$ and $y = 8$ and with $g(y) = y^{2/3}$,

$$\pi\int_0^8\left(y^{2/3}\right)^2 dy = \pi\int_0^8 y^{4/3}dy$$

$$= \pi\frac{3}{7}y^{7/3}\Big|_0^8$$

$$= \frac{3\pi}{7}\left(8^{7/3}-0\right)$$

$$= \frac{384\pi}{7}$$

Disk Method

$$V = \int_c^d \pi[u(y)]^2\,dy$$

Example:(washer method)

Self Practice

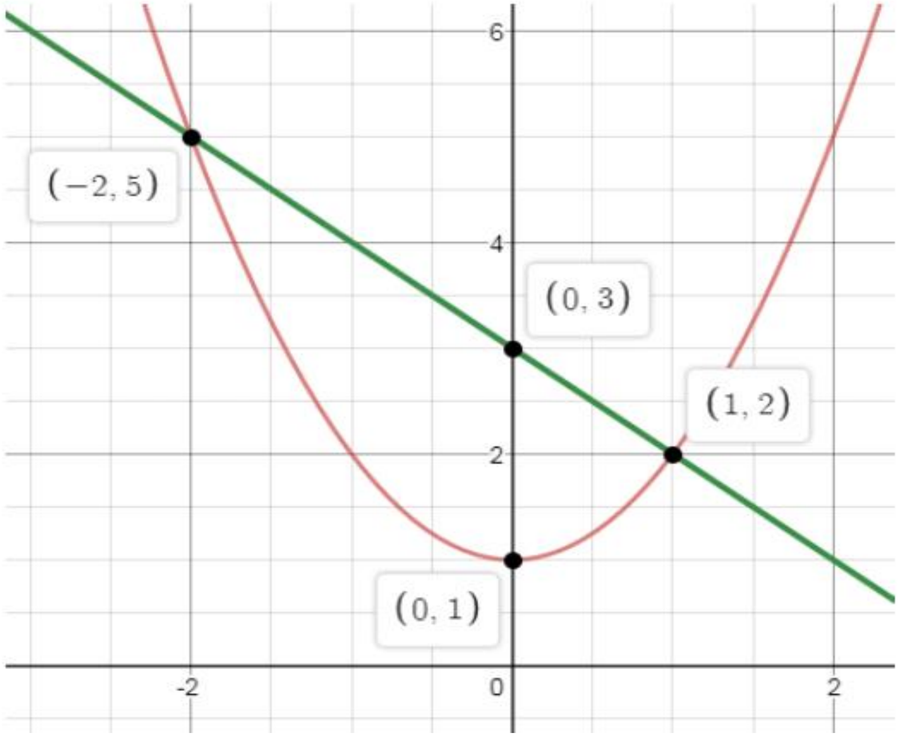
EXAMPLE 9 A Washer Cross-Section (Rotation About the x-Axis)

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

EXAMPLE 9 A Washer Cross-Section (Rotation About the x-Axis)

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

1. Draw the region



2. Find the outer and inner radii of the washer

Outer radius: $R(x) = -x + 3$

Inner radius: $r(x) = x^2 + 1$

3. Find the limits of integration

$$x^2 + 1 = -x + 3$$
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2, \quad x = 1$$

4-Established the formula and evaluate

$$V = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx \quad \Longleftrightarrow \quad V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx$$

$$= \int_{-2}^{1} \pi((-x+3)^2 - (x^2+1)^2)\,dx$$

$$= \int_{-2}^{1} \pi(8 - 6x - x^2 - x^4)\,dx$$

$$= \pi\left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5} \quad =73.51$$

11–14 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the x-axis.

11. $y = \sqrt{25 - x^2}$, $y = 3$

12. $y = 9 - x^2$, $y = 0$

13. $x = \sqrt{y}$, $x = y/4$

14. $y = \sin x$, $y = \cos x$, $x = 0$, $x = \pi/4$
[*Hint:* Use the identity $\cos 2x = \cos^2 x - \sin^2 x$.]

15. Find the volume of the solid whose base is the region bounded between the curve $y = x^3$ and the y-axis from $y = 0$ to $y = 1$ and whose cross sections taken perpendicular to the y-axis are squares.

16. Find the volume of the solid whose base is the region enclosed between the curve $x = 1 - y^2$ and the y-axis and whose cross sections taken perpendicular to the y-axis are squares.

17–20 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the y-axis.

17. $x = \csc y,\ y = \pi/4,\ y = 3\pi/4,\ x = 0$

18. $y = x^2,\ x = y^2$

19. $x = y^2,\ x = y + 2$

20. $x = 1 - y^2,\ x = 2 + y^2,\ y = -1,\ y = 1$

Do Questions (1-20) from Ex # 5.2