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Ex#9.4

Divergence & Integral Test

9.4.1 THEOREM (*The Divergence Test*)

(a) *If* $\lim_{k \to +\infty} u_k \neq 0$, *then the series* $\sum u_k$ *diverges.*

(b) *If* $\lim_{k \to +\infty} u_k = 0$, *then the series* $\sum u_k$ *may either converge or diverge.*

9.4.2 THEOREM *If the series* $\sum u_k$ *converges, then* $\lim_{k \to +\infty} u_k = 0$.

PROOF (*b*) To prove this result, it suffices to produce both a convergent series and a divergent series for which $\lim_{k\to+\infty} u_k = 0$. The following series both have this property:

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} + \cdots \quad \text{and} \quad 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \cdots$$

The first is a convergent geometric series and the second is the divergent harmonic series. ■

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$$\sum_{k=1}^{\infty} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{k}{k+1} + \cdots$$

diverges since

$$\lim_{k \to +\infty} \frac{k}{k+1} = \lim_{k \to +\infty} \frac{1}{1+1/k} = 1 \neq 0 \blacktriangleleft$$

9.4.3 THEOREM

(a) *If* $\sum u_k$ *and* $\sum v_k$ *are convergent series, then* $\sum(u_k + v_k)$ *and* $\sum(u_k - v_k)$ *are convergent series and the sums of these series are related by*

$$\sum_{k=1}^{\infty}(u_k + v_k) = \sum_{k=1}^{\infty} u_k + \sum_{k=1}^{\infty} v_k$$

$$\sum_{k=1}^{\infty}(u_k - v_k) = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^{\infty} v_k$$

(b) *If* c *is a nonzero constant, then the series* $\sum u_k$ *and* $\sum cu_k$ *both converge or both diverge. In the case of convergence, the sums are related by*

$$\sum_{k=1}^{\infty} cu_k = c\sum_{k=1}^{\infty} u_k$$

(c) *Convergence or divergence is unaffected by deleting a finite number of terms from a series; in particular, for any positive integer* K*, the series*

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots$$

$$\sum_{k=K}^{\infty} u_k = u_K + u_{K+1} + u_{K+2} + \cdots$$

both converge or both diverge.

Find the sum of the series

$$\sum_{k=1}^{\infty}\left(\frac{3}{4^k}-\frac{2}{5^{k-1}}\right)$$

Before finding Sum, check whether its converging or diverging

Solution. The series

$$\sum_{k=1}^{\infty} \frac{3}{4^k} = \frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \cdots$$

is a convergent geometric series $\left(a = \frac{3}{4}, r = \frac{1}{4}\right)$, and the series

$$\sum_{k=1}^{\infty} \frac{2}{5^{k-1}} = 2 + \frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} + \cdots$$

is also a convergent geometric series $\left(a = 2, r = \frac{1}{5}\right)$. Thus, from Theorems 9.4.3(*a*) and 9.3.3 the given series converges and

$$\sum_{k=1}^{\infty} \left(\frac{3}{4^k} - \frac{2}{5^{k-1}}\right) = \sum_{k=1}^{\infty} \frac{3}{4^k} - \sum_{k=1}^{\infty} \frac{2}{5^{k-1}}$$

$$= \frac{\frac{3}{4}}{1 - \frac{1}{4}} - \frac{2}{1 - \frac{1}{5}} = -\frac{3}{2} \quad \blacktriangleleft$$

▶ **Example 3** Determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{5}{k} = 5 + \frac{5}{2} + \frac{5}{3} + \cdots + \frac{5}{k} + \cdots$ (b) $\displaystyle\sum_{k=10}^{\infty} \frac{1}{k} = \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots$

▸ **Example 3** Determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{5}{k} = 5 + \frac{5}{2} + \frac{5}{3} + \cdots + \frac{5}{k} + \cdots$ (b) $\displaystyle\sum_{k=10}^{\infty} \frac{1}{k} = \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots$

Solution. The first series is a constant times the divergent harmonic series, and hence diverges by part (b) of Theorem 9.4.3. The second series results by deleting the first nine terms from the divergent harmonic series, and hence diverges by part (c) of Theorem 9.4.3.

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{and} \quad \int_{1}^{+\infty} \frac{1}{x^2}\, dx$$

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9.4.4 THEOREM (*The Integral Test*) *Let $\sum u_k$ be a series with positive terms. If f is a function that is decreasing and continuous on an interval $[a, +\infty)$ and such that $u_k = f(k)$ for all $k \geq a$, then*

$$\sum_{k=1}^{\infty} u_k \quad \text{and} \quad \int_a^{+\infty} f(x)\,dx$$

both converge or both diverge.

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ for convergence or divergence.

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Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \tan^{-1}x\Big]_1^t$$

$$= \lim_{t\to\infty} \left(\tan^{-1}t - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Thus $\int_1^{\infty} 1/(x^2+1)\,dx$ is a convergent integral and so, by the Integral Test, the series $\sum 1/(n^2+1)$ is convergent.

Do by your own

For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

$\int_1^{\infty} \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges if $p \leqslant 1$

9.4.5 THEOREM (*Convergence of p-Series*)

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{k^p} + \cdots$$

converges if $p > 1$ *and diverges if* $0 < p \leq 1$.

Example 5

$$1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \cdots + \frac{1}{\sqrt[3]{k}} + \cdots$$

diverges since it is a p-series with $p = \frac{1}{3} < 1$.

3–4 For each given p-series, identify p and determine whether the series converges.

3. (a) $\sum_{k=1}^{\infty} \frac{1}{k^3}$ (b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ (c) $\sum_{k=1}^{\infty} k^{-1}$ (d) $\sum_{k=1}^{\infty} k^{-2/3}$

4. (a) $\sum_{k=1}^{\infty} k^{-4/3}$ (b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k}}$ (c) $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^5}}$ (d) $\sum_{k=1}^{\infty} \frac{1}{k^{\pi}}$

Sol:

3. **(a)** $p=3>1$, converges. **(b)** $p=1/2\leq 1$, diverges. **(c)** $p=1\leq 1$, diverges. **(d)** $p=2/3\leq 1$, diverges.

4. **(a)** $p=4/3>1$, converges. **(b)** $p=1/4\leq 1$, diverges. **(c)** $p=5/3>1$, converges. **(d)** $p=\pi>1$, converges.

5–6 Apply the divergence test and state what it tells you about the series.

5. (a) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2+k+3}{2k^2+1}$ (b) $\displaystyle\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^k$

(c) $\displaystyle\sum_{k=1}^{\infty} \cos k\pi$ (d) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k!}$

6. (a) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{e^k}$ (b) $\displaystyle\sum_{k=1}^{\infty} \ln k$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ (d) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt{k}+3}$

5. **(a)** $\displaystyle\lim_{k\to+\infty} \frac{k^2+k+3}{2k^2+1} = \frac{1}{2} \neq 0$; the series diverges. **(b)** $\displaystyle\lim_{k\to+\infty} \left(1+\frac{1}{k}\right)^k = e \neq 0$; the series diverges.

(c) $\displaystyle\lim_{k\to+\infty} \cos k\pi$ does not exist; the series diverges. **(d)** $\displaystyle\lim_{k\to+\infty} \frac{1}{k!} = 0$; no information.

6. **(a)** $\displaystyle\lim_{k\to+\infty} \frac{k}{e^k} = 0$; no information. **(b)** $\displaystyle\lim_{k\to+\infty} \ln k = +\infty \neq 0$; the series diverges.

(c) $\displaystyle\lim_{k\to+\infty} \frac{1}{\sqrt{k}} = 0$; no information. **(d)** $\displaystyle\lim_{k\to+\infty} \frac{\sqrt{k}}{\sqrt{k}+3} = 1 \neq 0$; the series diverges.

7–8 Confirm that the integral test is applicable and use it to determine whether the series converges.

7. (a) $\sum_{k=1}^{\infty} \frac{1}{5k+2}$ (b) $\sum_{k=1}^{\infty} \frac{1}{1+9k^2}$

8. (a) $\sum_{k=1}^{\infty} \frac{k}{1+k^2}$ (b) $\sum_{k=1}^{\infty} \frac{1}{(4+2k)^{3/2}}$

7. (a) $\int_1^{+\infty} \frac{1}{5x+2} = \lim_{\ell\to+\infty} \frac{1}{5}\ln(5x+2)\Big]_1^{\ell} = +\infty$, the series diverges by the Integral Test (which can be applied, because the series has positive terms, and f is decreasing and continuous).

(b) $\int_1^{+\infty} \frac{1}{1+9x^2}dx = \lim_{\ell\to+\infty} \frac{1}{3}\tan^{-1}3x\Big]_1^{\ell} = \frac{1}{3}(\pi/2 - \tan^{-1}3)$, the series converges by the Integral Test (which can be applied, because the series has positive terms, and f is decreasing and continuous).

8. (a) $\int_1^{+\infty} \frac{x}{1+x^2}dx = \lim_{\ell\to+\infty} \frac{1}{2}\ln(1+x^2)\Big]_1^{\ell} = +\infty$, the series diverges by the Integral Test (which can be applied, because the series has positive terms, and f is decreasing and continuous).

(b) $\int_1^{+\infty} (4+2x)^{-3/2}dx = \lim_{\ell\to+\infty} -1/\sqrt{4+2x}\Big]_1^{\ell} = 1/\sqrt{6}$, the series converges by the Integral Test (which can be applied, because the series has positive terms, and f is decreasing and continuous).

9–24 Determine whether the series converges.

9. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k+6}$

10. $\displaystyle\sum_{k=1}^{\infty} \frac{3}{5k}$

11. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+5}}$

12. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{e}}$

13. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{2k-1}}$

14. $\displaystyle\sum_{k=3}^{\infty} \frac{\ln k}{k}$

15. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\ln(k+1)}$

16. $\displaystyle\sum_{k=1}^{\infty} ke^{-k^2}$

17. $\displaystyle\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^{-k}$

18. $\displaystyle\sum_{k=1}^{\infty} \frac{k^2+1}{k^2+3}$

19. $\displaystyle\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{1+k^2}$

20. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}$

21. $\displaystyle\sum_{k=1}^{\infty} k^2 \sin^2\left(\frac{1}{k}\right)$

22. $\displaystyle\sum_{k=1}^{\infty} k^2 e^{-k^3}$

23. $\displaystyle\sum_{k=5}^{\infty} 7k^{-1.01}$

24. $\displaystyle\sum_{k=1}^{\infty} \operatorname{sech}^2 k$

9. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k+6}=\sum_{k=7}^{\infty}\frac{1}{k}$, diverges because the harmonic series diverges.

10. $\displaystyle\sum_{k=1}^{\infty}\frac{3}{5k}=\sum_{k=1}^{\infty}\frac{3}{5}\left(\frac{1}{k}\right)$, diverges because the harmonic series diverges.

11. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\sqrt{k+5}}=\sum_{k=6}^{\infty}\frac{1}{\sqrt{k}}$, diverges because the p-series with $p=1/2\le 1$ diverges.

12. $\displaystyle\lim_{k\to+\infty}\frac{1}{e^{1/k}}=1$, the series diverges by the Divergence Test, because $\displaystyle\lim_{k\to+\infty}u_k=1\neq 0$.

13. $\displaystyle\int_1^{+\infty}(2x-1)^{-1/3}dx=\lim_{\ell\to+\infty}\frac{3}{4}(2x-1)^{2/3}\Big]_1^{\ell}=+\infty$, the series diverges by the Integral Test (which can be applied, because the series has positive terms, and f is decreasing and continuous).

14. $\dfrac{\ln x}{x}$ is decreasing for $x\ge e$, and $\displaystyle\int_3^{+\infty}\frac{\ln x}{x}=\lim_{\ell\to+\infty}\frac{1}{2}(\ln x)^2\Big]_3^{\ell}=+\infty$, so the series diverges by the Integral Test (which can be applied, because the series has positive terms, and f is decreasing and continuous).

15. $\displaystyle\lim_{k\to+\infty}\frac{k}{\ln(k+1)}=\lim_{k\to+\infty}\frac{1}{1/(k+1)}=+\infty$, the series diverges by the Divergence Test, because $\displaystyle\lim_{k\to+\infty}u_k\neq 0$.

16. $\displaystyle\int_1^{+\infty}xe^{-x^2}dx=\lim_{\ell\to+\infty}-\frac{1}{2}e^{-x^2}\Big]_1^{\ell}=e^{-1}/2$, the series converges by the Integral Test (which can be applied, because the series has positive terms, and f is decreasing and continuous).

17. $\lim_{k\to+\infty}(1+1/k)^{-k} = 1/e \neq 0$, the series diverges by the Divergence Test.

18. $\lim_{k\to+\infty}\dfrac{k^2+1}{k^2+3} = 1 \neq 0$, the series diverges by the Divergence Test.

19. $\displaystyle\int_1^{+\infty}\frac{\tan^{-1}x}{1+x^2}dx = \lim_{\ell\to+\infty}\frac{1}{2}\left(\tan^{-1}x\right)^2\Big]_1^{\ell} = 3\pi^2/32$, the series converges by the Integral Test (which can be applied, because the series has positive terms, and f is decreasing and continuous), since $\dfrac{d}{dx}\dfrac{\tan^{-1}x}{1+x^2} = \dfrac{1-2x\tan^{-1}x}{(1+x^2)^2} < 0$ for $x \geq 1$.

20. $\displaystyle\int_1^{+\infty}\frac{1}{\sqrt{x^2+1}}dx = \lim_{\ell\to+\infty}\sinh^{-1}x\Big]_1^{\ell} = +\infty$, the series diverges by the Integral Test (which can be applied, because the series has positive terms, and f is decreasing and continuous).

21. $\lim_{k\to+\infty}k^2\sin^2(1/k) = 1 \neq 0$, the series diverges by the Divergence Test.

22. $\displaystyle\int_1^{+\infty}x^2e^{-x^3}dx = \lim_{\ell\to+\infty}-\frac{1}{3}e^{-x^3}\Big]_1^{\ell} = e^{-1}/3$, the series converges by the Integral Test (which can be applied, because $x^2e^{-x^3}$ is decreasing for $x \geq 1$, it is continuous and the series has positive terms).

23. $7\displaystyle\sum_{k=5}^{\infty}k^{-1.01}$, p-series with $p = 1.01 > 1$, converges.

24. $\displaystyle\int_1^{+\infty}\operatorname{sech}^2 x\,dx = \lim_{\ell\to+\infty}\tanh x\Big]_1^{\ell} = 1 - \tanh(1)$, the series converges by the Integral Test (which can be applied, because the series has positive terms, and f is decreasing and continuous).

Do Questions (3-24) from Ex # 9.4