Principles of Integral Evaluation

Exercise Set 7.1

1. $u = 4 - 2x$, $du = -2dx$, $\quad -\dfrac{1}{2}\displaystyle\int u^3\,du = -\dfrac{1}{8}u^4 + C = -\dfrac{1}{8}(4-2x)^4 + C.$

3. $u = x^2$, $du = 2xdx$, $\quad \dfrac{1}{2}\displaystyle\int \sec^2 u\,du = \dfrac{1}{2}\tan u + C = \dfrac{1}{2}\tan(x^2) + C.$

5. $u = 2 + \cos 3x$, $du = -3\sin 3x dx$, $\quad -\dfrac{1}{3}\displaystyle\int \dfrac{du}{u} = -\dfrac{1}{3}\ln|u| + C = -\dfrac{1}{3}\ln(2+\cos 3x) + C.$

7. $u = e^x$, $du = e^x dx$, $\quad \displaystyle\int \sinh u\,du = \cosh u + C = \cosh e^x + C.$

9. $u = \tan x$, $du = \sec^2 x dx$, $\quad \displaystyle\int e^u\,du = e^u + C = e^{\tan x} + C.$

11. $u = \cos 5x$, $du = -5\sin 5x dx$, $\quad -\dfrac{1}{5}\displaystyle\int u^5\,du = -\dfrac{1}{30}u^6 + C = -\dfrac{1}{30}\cos^6 5x + C.$

13. $u = e^x$, $du = e^x dx$, $\quad \displaystyle\int \dfrac{du}{\sqrt{4+u^2}} = \ln\left(u + \sqrt{u^2+4}\right) + C = \ln\left(e^x + \sqrt{e^{2x}+4}\right) + C.$

15. $u = \sqrt{x-1}$, $du = \dfrac{1}{2\sqrt{x-1}}\,dx$, $\quad 2\displaystyle\int e^u\,du = 2e^u + C = 2e^{\sqrt{x-1}} + C.$

17. $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$, $\quad \displaystyle\int 2\cosh u\,du = 2\sinh u + C = 2\sinh\sqrt{x} + C.$

19. $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$, $\quad \displaystyle\int \dfrac{2\,du}{3^u} = 2\int e^{-u\ln 3}\,du = -\dfrac{2}{\ln 3}e^{-u\ln 3} + C = -\dfrac{2}{\ln 3}3^{-\sqrt{x}} + C.$

21. $u = \dfrac{2}{x}$, $du = -\dfrac{2}{x^2}\,dx$, $\quad -\dfrac{1}{2}\displaystyle\int \operatorname{csch}^2 u\,du = \dfrac{1}{2}\coth u + C = \dfrac{1}{2}\coth\dfrac{2}{x} + C.$

23. $u = e^{-x}$, $du = -e^{-x}dx$, $\quad -\displaystyle\int \dfrac{du}{4-u^2} = -\dfrac{1}{4}\ln\left|\dfrac{2+u}{2-u}\right| + C = -\dfrac{1}{4}\ln\left|\dfrac{2+e^{-x}}{2-e^{-x}}\right| + C.$

25. $u = e^x$, $du = e^x dx$, $\quad \displaystyle\int \dfrac{e^x\,dx}{\sqrt{1-e^{2x}}} = \int \dfrac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}e^x + C.$

27. $u = x^2$, $du = 2xdx$, $\quad \dfrac{1}{2}\displaystyle\int \dfrac{du}{\csc u} = \dfrac{1}{2}\int \sin u\,du = -\dfrac{1}{2}\cos u + C = -\dfrac{1}{2}\cos(x^2) + C.$

29. $4^{-x^2} = e^{-x^2\ln 4}$, $u = -x^2\ln 4$, $du = -2x\ln 4\,dx = -x\ln 16\,dx$, $-\dfrac{1}{\ln 16}\displaystyle\int e^u\,du = -\dfrac{1}{\ln 16}e^u + C = -\dfrac{1}{\ln 16}e^{-x^2\ln 4} + C = -\dfrac{1}{\ln 16}4^{-x^2} + C$.

31. (a) $u = \sin x$, $du = \cos x\,dx$, $\displaystyle\int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$.

(b) $\displaystyle\int \sin x\cos x\,dx = \frac{1}{2}\int \sin 2x\,dx = -\frac{1}{4}\cos 2x + C = -\frac{1}{4}(\cos^2 x - \sin^2 x) + C$.

(c) $-\dfrac{1}{4}(\cos^2 x - \sin^2 x) + C = -\dfrac{1}{4}(1 - \sin^2 x - \sin^2 x) + C = -\dfrac{1}{4} + \dfrac{1}{2}\sin^2 x + C$, and this is the same as the answer in part (a) except for the constants.

33. (a) $\dfrac{\sec^2 x}{\tan x} = \dfrac{1}{\cos^2 x\tan x} = \dfrac{1}{\cos x\sin x}$.

(b) $\csc 2x = \dfrac{1}{\sin 2x} = \dfrac{1}{2\sin x\cos x} = \dfrac{1}{2}\dfrac{\sec^2 x}{\tan x}$, so $\displaystyle\int \csc 2x\,dx = \frac{1}{2}\ln\tan x + C$, then using the substitution $u = 2x$ we obtain that $\displaystyle\int \csc x\,dx = \ln(\tan(x/2)) + C$.

(c) $\sec x = \dfrac{1}{\cos x} = \dfrac{1}{\sin(\pi/2 - x)} = \csc(\pi/2 - x)$, so $\displaystyle\int \sec x\,dx = -\int \csc(\pi/2 - x)\,dx = -\ln\tan(\pi/4 - x/2) + C$.

Exercise Set 7.2

1. $u = x$, $dv = e^{-2x}dx$, $du = dx$, $v = -\dfrac{1}{2}e^{-2x}$; $\displaystyle\int xe^{-2x}dx = -\frac{1}{2}xe^{-2x} + \int \frac{1}{2}e^{-2x}dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$.

3. $u = x^2$, $dv = e^x dx$, $du = 2x\,dx$, $v = e^x$; $\displaystyle\int x^2e^x dx = x^2e^x - 2\int xe^x dx$. For $\displaystyle\int xe^x dx$ use $u = x$, $dv = e^x dx$, $du = dx$, $v = e^x$ to get $\displaystyle\int xe^x dx = xe^x - e^x + C_1$ so $\displaystyle\int x^2e^x dx = x^2e^x - 2xe^x + 2e^x + C$.

5. $u = x$, $dv = \sin 3x\,dx$, $du = dx$, $v = -\dfrac{1}{3}\cos 3x$; $\displaystyle\int x\sin 3x\,dx = -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x\,dx = -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$.

7. $u = x^2$, $dv = \cos x\,dx$, $du = 2x\,dx$, $v = \sin x$; $\displaystyle\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx$. For $\displaystyle\int x\sin x\,dx$ use $u = x$, $dv = \sin x\,dx$ to get $\displaystyle\int x\sin x\,dx = -x\cos x + \sin x + C_1$ so $\displaystyle\int x^2\cos x\,dx = x^2\sin x + 2x\cos x - 2\sin x + C$.

9. $u = \ln x$, $dv = x\,dx$, $du = \dfrac{1}{x}dx$, $v = \dfrac{1}{2}x^2$; $\displaystyle\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$.

11. $u = (\ln x)^2$, $dv = dx$, $du = 2\dfrac{\ln x}{x}dx$, $v = x$; $\displaystyle\int (\ln x)^2 dx = x(\ln x)^2 - 2\int \ln x\,dx$. Use $u = \ln x$, $dv = dx$ to get $\displaystyle\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C_1$ so $\displaystyle\int (\ln x)^2 dx = x(\ln x)^2 - 2x\ln x + 2x + C$.

13. $u = \ln(3x-2)$, $dv = dx$, $du = \dfrac{3}{3x-2}dx$, $v = x$; $\displaystyle\int \ln(3x-2)dx = x\ln(3x-2) - \int \frac{3x}{3x-2}dx$, but $\displaystyle\int \frac{3x}{3x-2}dx = \int\left(1 + \frac{2}{3x-2}\right)dx = x + \frac{2}{3}\ln(3x-2) + C_1$ so $\displaystyle\int \ln(3x-2)dx = x\ln(3x-2) - x - \frac{2}{3}\ln(3x-2) + C$.

15. $u = \sin^{-1} x$, $dv = dx$, $du = 1/\sqrt{1-x^2}dx$, $v = x$; $\int \sin^{-1} x\, dx = x\sin^{-1} x - \int x/\sqrt{1-x^2}dx = x\sin^{-1} x + \sqrt{1-x^2} + C$.

17. $u = \tan^{-1}(3x)$, $dv = dx$, $du = \dfrac{3}{1+9x^2}dx$, $v = x$; $\int \tan^{-1}(3x)dx = x\tan^{-1}(3x) - \int \dfrac{3x}{1+9x^2}dx = x\tan^{-1}(3x) - \dfrac{1}{6}\ln(1+9x^2) + C$.

19. $u = e^x$, $dv = \sin x\, dx$, $du = e^x dx$, $v = -\cos x$; $\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx$. For $\int e^x \cos x\, dx$ use $u = e^x$, $dv = \cos x\, dx$ to get $\int e^x \cos x = e^x \sin x - \int e^x \sin x\, dx$, so $\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx$, $2\int e^x \sin x\, dx = e^x(\sin x - \cos x) + C_1$, $\int e^x \sin x\, dx = \dfrac{1}{2}e^x(\sin x - \cos x) + C$.

21. $u = \sin(\ln x)$, $dv = dx$, $du = \dfrac{\cos(\ln x)}{x}dx$, $v = x$; $\int \sin(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)dx$. Use $u = \cos(\ln x)$, $dv = dx$ to get $\int \cos(\ln x)dx = x\cos(\ln x) + \int \sin(\ln x)dx$ so $\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx$, $\int \sin(\ln x)dx = \dfrac{1}{2}x[\sin(\ln x) - \cos(\ln x)] + C$.

23. $u = x$, $dv = \sec^2 x\, dx$, $du = dx$, $v = \tan x$; $\int x\sec^2 x\, dx = x\tan x - \int \tan x\, dx = x\tan x - \int \dfrac{\sin x}{\cos x}dx = x\tan x + \ln|\cos x| + C$.

25. $u = x^2$, $dv = xe^{x^2}dx$, $du = 2x\, dx$, $v = \dfrac{1}{2}e^{x^2}$; $\int x^3 e^{x^2}dx = \dfrac{1}{2}x^2 e^{x^2} - \int xe^{x^2}dx = \dfrac{1}{2}x^2e^{x^2} - \dfrac{1}{2}e^{x^2} + C$.

27. $u = x$, $dv = e^{2x}dx$, $du = dx$, $v = \dfrac{1}{2}e^{2x}$; $\displaystyle\int_0^2 xe^{2x}dx = \frac{1}{2}xe^{2x}\Big]_0^2 - \frac{1}{2}\int_0^2 e^{2x}dx = e^4 - \frac{1}{4}e^{2x}\Big]_0^2 = e^4 - \frac{1}{4}(e^4-1) = (3e^4+1)/4$.

29. $u = \ln x$, $dv = x^2dx$, $du = \dfrac{1}{x}dx$, $v = \dfrac{1}{3}x^3$; $\displaystyle\int_1^e x^2\ln x\, dx = \frac{1}{3}x^3\ln x\Big]_1^e - \frac{1}{3}\int_1^e x^2dx = \frac{1}{3}e^3 - \frac{1}{9}x^3\Big]_1^e = \frac{1}{3}e^3 - \frac{1}{9}(e^3 - 1) = (2e^3+1)/9$.

31. $u = \ln(x+2)$, $dv = dx$, $du = \dfrac{1}{x+2}dx$, $v = x$; $\displaystyle\int_{-1}^1 \ln(x+2)dx = x\ln(x+2)\Big]_{-1}^1 - \int_{-1}^1 \frac{x}{x+2}dx = \ln 3 + \ln 1 - \int_{-1}^1 \left[1 - \frac{2}{x+2}\right]dx = \ln 3 - [x - 2\ln(x+2)]\Big]_{-1}^1 = \ln 3 - (1 - 2\ln 3) + (-1 - 2\ln 1) = 3\ln 3 - 2$.

33. $u = \sec^{-1}\sqrt{\theta}$, $dv = d\theta$, $du = \dfrac{1}{2\theta\sqrt{\theta-1}}d\theta$, $v = \theta$; $\displaystyle\int_2^4 \sec^{-1}\sqrt{\theta}d\theta = \theta\sec^{-1}\sqrt{\theta}\Big]_2^4 - \frac{1}{2}\int_2^4 \frac{1}{\sqrt{\theta-1}}d\theta = 4\sec^{-1}2 - 2\sec^{-1}\sqrt{2} - \sqrt{\theta-1}\Big]_2^4 = 4\left(\frac{\pi}{3}\right) - 2\left(\frac{\pi}{4}\right) - \sqrt{3} + 1 = \frac{5\pi}{6} - \sqrt{3} + 1$.

35. $u = x$, $dv = \sin 2x\, dx$, $du = dx$, $v = -\dfrac{1}{2}\cos 2x$; $\displaystyle\int_0^\pi x\sin 2x\, dx = -\frac{1}{2}x\cos 2x\Big]_0^\pi + \frac{1}{2}\int_0^\pi \cos 2x\, dx = -\pi/2 + \frac{1}{4}\sin 2x\Big]_0^\pi = -\pi/2$.

37. $u = \tan^{-1}\sqrt{x}$, $dv = \sqrt{x}dx$, $du = \dfrac{1}{2\sqrt{x}(1+x)}dx$, $v = \dfrac{2}{3}x^{3/2}$; $\displaystyle\int_1^3 \sqrt{x}\tan^{-1}\sqrt{x}dx = \frac{2}{3}x^{3/2}\tan^{-1}\sqrt{x}\Big]_1^3 -$

$\dfrac{1}{3}\displaystyle\int_1^3 \frac{x}{1+x}dx = \frac{2}{3}x^{3/2}\tan^{-1}\sqrt{x}\Big]_1^3 - \frac{1}{3}\int_1^3 \left[1 - \frac{1}{1+x}\right] dx = \left[\frac{2}{3}x^{3/2}\tan^{-1}\sqrt{x} - \frac{1}{3}x + \frac{1}{3}\ln|1+x|\right]_1^3 = (2\sqrt{3}\pi - \pi/2 - 2 + \ln 2)/3$.

39. True.

41. False; e^x is not a factor of the integrand.

43. $t = \sqrt{x},\ t^2 = x,\ dx = 2t\,dt,\ \displaystyle\int e^{\sqrt{x}}dx = 2\int te^t\,dt$; $u = t, dv = e^t dt, du = dt, v = e^t$, $\displaystyle\int e^{\sqrt{x}}dx = 2te^t - 2\int e^t\,dt = 2(t-1)e^t + C = 2(\sqrt{x}-1)e^{\sqrt{x}} + C$.

45. Let $f_1(x), f_2(x), f_3(x)$ denote successive antiderivatives of $f(x)$, so that $f_3'(x) = f_2(x), f_2'(x) = f_1(x), f_1'(x) = f(x)$. Let $p(x) = ax^2 + bx + c$.

diff.		antidiff.
$ax^2 + bx + c$		$f(x)$
	$\searrow +$	
$2ax + b$		$f_1(x)$
	$\searrow -$	
$2a$		$f_2(x)$
	$\searrow +$	
0		$f_3(x)$

Then $\displaystyle\int p(x)f(x)\,dx = (ax^2+bx+c)f_1(x) - (2ax+b)f_2(x) + 2af_3(x) + C$. Check: $\dfrac{d}{dx}[(ax^2+bx+c)f_1(x) - (2ax+b)f_2(x) + 2af_3(x)] = (2ax+b)f_1(x) + (ax^2+bx+c)f(x) - 2af_2(x) - (2ax+b)f_1(x) + 2af_2(x) = p(x)f(x)$.

47. Let I denote $\displaystyle\int (3x^2 - x + 2)e^{-x}\,dx$. Then

diff.		antidiff.
$3x^2 - x + 2$		e^{-x}
	$\searrow +$	
$6x - 1$		$-e^{-x}$
	$\searrow -$	
6		e^{-x}
	$\searrow +$	
0		$-e^{-x}$

$I = \displaystyle\int (3x^2 - x + 2)e^{-x} = -(3x^2 - x + 2)e^{-x} - (6x-1)e^{-x} - 6e^{-x} + C = -e^{-x}[3x^2 + 5x + 7] + C$.

49. Let I denote $\displaystyle\int 4x^4 \sin 2x\,dx$. Then

diff.		antidiff.
$4x^4$		$\sin 2x$
	$\searrow +$	
$16x^3$		$-\frac{1}{2}\cos 2x$
	$\searrow -$	
$48x^2$		$-\frac{1}{4}\sin 2x$
	$\searrow +$	
$96x$		$\frac{1}{8}\cos 2x$
	$\searrow -$	
96		$\frac{1}{16}\sin 2x$
	$\searrow +$	
0		$-\frac{1}{32}\cos 2x$

$I = \int 4x^4 \sin 2x\, dx = (-2x^4 + 6x^2 - 3)\cos 2x + (4x^3 - 6x)\sin 2x + C.$

51. Let I denote $\int e^{ax}\sin bx\, dx$. Then

diff.		antidiff.
e^{ax}		$\sin bx$
	$\searrow +$	
ae^{ax}		$-\frac{1}{b}\cos bx$
	$\searrow -$	
a^2e^{ax}		$-\frac{1}{b^2}\sin bx$

$I = \int e^{ax}\sin bx\, dx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}I$, so $I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C.$

53. (a) We perform a single integration by parts: $u = \cos x, dv = \sin x\, dx, du = -\sin x\, dx, v = -\cos x$, $\int \sin x\cos x\, dx = -\cos^2 x - \int \sin x \cos x\, dx$. This implies that $2\int \sin x\cos x\, dx = -\cos^2 x + C$, $\int \sin x \cos x\, dx = -\frac{1}{2}\cos^2 x + C$.

Alternatively, $u = \sin x, du = \cos x\, dx$, $\int \sin x\cos x\, dx = \int u\, du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$.

(b) Since $\sin^2 x + \cos^2 x = 1$, they are equal (although the symbol 'C' refers to different constants in the two equations).

55. (a) $A = \int_1^e \ln x\, dx = (x\ln x - x)\Big]_1^e = 1.$

(b) $V = \pi\int_1^e (\ln x)^2 dx = \pi\Big[(x(\ln x)^2 - 2x\ln x + 2x)\Big]_1^e = \pi(e-2).$

57. $V = 2\pi\int_0^\pi x\sin x\, dx = 2\pi(-x\cos x + \sin x)\Big]_0^\pi = 2\pi^2.$

59. Distance $= \int_0^\pi t^3 \sin t dt;$

$$\begin{array}{ccc}
\text{diff.} & & \text{antidiff.} \\
\hline\hline
t^3 & & \sin t \\
& \searrow + & \\
3t^2 & & -\cos t \\
& \searrow - & \\
6t & & -\sin t \\
& \searrow + & \\
6 & & \cos t \\
& \searrow - & \\
0 & & \sin t
\end{array}$$

$\displaystyle\int_0^\pi t^3 \sin t\,dx = [(-t^3\cos t + 3t^2 \sin t + 6t\cos t - 6\sin t)]\Big]_0^\pi = \pi^3 - 6\pi.$

61. **(a)** $u = 1000+5t$, $dv = e^{0.12(20-t)}dt$, $du = 5dt$, $v = -\dfrac{e^{0.12(20-t)}}{0.12}$, $FV = \displaystyle\int_0^{20}(1000+5t)e^{0.12(20-t)}\,dt = -\dfrac{(1000+5t)e^{0.12(20-t)}}{0.12}\Big]_0^{20} + \displaystyle\int_0^{20}\dfrac{5e^{0.12(20-t)}}{0.12}\,dt = \dfrac{25000e^{2.4}-27500}{3} - \dfrac{5e^{0.12(20-t)}}{0.12^2}\Big]_0^{20} \approx 86,173.41.$

(b) $u = 1000+5t$, $dv = e^{-0.12t}dt$, $du = 5dt$, $v = -\dfrac{e^{-0.12t}}{0.12}$, $PV = \displaystyle\int_0^{20}(1000+5t)e^{-0.12t}\,dt = -\dfrac{(1000+5t)e^{-0.12t}}{0.12}\Big]_0^{20} + \displaystyle\int_0^{20}\dfrac{5e^{-0.12t}}{0.12}\,dt = \dfrac{25000-27500e^{-2.4}}{3} - \dfrac{5e^{-0.12t}}{0.12^2}\Big]_0^{20} \approx 7817.48.$

(c) $86,173.41 \approx 7817.48e^{0.12\cdot 20}$

63. **(a)** $u = 20000-200t^2$, $dv = e^{0.05(10-t)}dt$, $du = -400t dt$, $v = -20e^{0.05(10-t)}$, $FV = \displaystyle\int_0^{10}(20000-200t^2)e^{0.05(10-t)}\,dt = -20(20000-200t^2)e^{0.05(10-t)}\Big]_0^{10} - \displaystyle\int_0^{10}8000te^{0.05(10-t)}\,dt = 400000e^{0.5} - 8000\displaystyle\int_0^{10}te^{0.05(10-t)}\,dt$; $u = t$, $dv = e^{0.05(10-t)}dt$, $du = dt$, $v = -20e^{0.05(10-t)}$, $\displaystyle\int_0^{10}te^{0.05(10-t)}\,dt = -20te^{0.05(10-t)}\Big]_0^{10} + \displaystyle\int_0^{10}20e^{0.05(10-t)}\,dt = -200 - 400e^{0.05(10-t)}\Big]_0^{10} = -600+400e^{0.5}$, thus $FV = 400000e^{0.5} - 8000(-600+400e^{0.5}) \approx 183,580.44.$

(b) $u = 20000-200t^2$, $dv = e^{-0.05t}dt$, $du = -400t dt$, $v = -20e^{-0.05t}$, $PV = \displaystyle\int_0^{10}(20000-200t^2)e^{-0.05t}\,dt = -20(20000-200t^2)e^{-0.05t}\Big]_0^{10} - \displaystyle\int_0^{10}8000te^{-0.05t}\,dt = 400000 - 8000\displaystyle\int_0^{10}te^{-0.05t}\,dt$; $u = t$, $dv = e^{-0.05t}dt$, $du = dt$, $v = -20e^{-0.05t}$, $\displaystyle\int_0^{10}te^{-0.05t}\,dt = -20te^{-0.05t}\Big]_0^{10} + \displaystyle\int_0^{10}20e^{-0.05t}\,dt = -200e^{-0.5} - 400e^{-0.05t}\Big]_0^{10} = -600e^{-0.5}+400$, thus $PV = 400000 - 8000(-600e^{-0.5}+400) \approx 111,347.17.$

(c) $183,580.44 \approx 111,347.17e^{0.05\cdot 10}$

65. **(a)** $u = 2000t$, $dv = e^{0.08(10-t)}dt$, $du = 2000dt$, $v = -\dfrac{25}{2}e^{0.08(10-t)}$, $FV = \displaystyle\int_0^{10}(2000t+400e^{-t})e^{0.08(10-t)}\,dt = \displaystyle\int_0^{10}2000te^{0.08(10-t)}\,dt + 400\displaystyle\int_0^{10}e^{0.8-1.08t}\,dt = -\left(\dfrac{25}{2}\right)2000te^{0.08(10-t)}\Big]_0^{10} + \displaystyle\int_0^{10}\left(\dfrac{25}{2}\right)2000e^{0.08(10-t)}\,dt$ $- \dfrac{400}{1.08}e^{0.8-1.08t}\Big]_0^{10} = -250000 - \dfrac{25000}{0.08}e^{0.08(10-t)}\Big]_0^{10} - (\dfrac{400}{1.08}e^{-10} - \dfrac{400}{1.08}e^{0.8}) = -250000 - \dfrac{25000}{0.08}(1-e^{0.8}) - \dfrac{400}{1.08}(e^{-10}-e^{0.8}) \approx 133,805.80.$

(b) $u = 2000t$, $dv = e^{-0.08t}dt$, $du = 2000dt$, $v = -\dfrac{25}{2}e^{-0.08t}$, $PV = \displaystyle\int_0^{10}(2000t + 400e^{-t})e^{-0.08t}\,dt = \int_0^{10} 2000te^{-0.08t}\,dt + 400\int_0^{10} e^{-1.08t}\,dt = -\left(\frac{25}{2}\right)2000te^{-0.08t}\Big]_0^{10} + \int_0^{10}\left(\frac{25}{2}\right)2000e^{-0.08t}\,dt - \frac{400}{1.08}e^{-1.08t}\Big]_0^{10} = -250000e^{-0.8} - \frac{25000}{0.08}e^{-0.08t}\Big]_0^{10} - \left(\frac{400}{1.08}e^{-10.8} - \frac{400}{1.08}\right) = -250000e^{-0.8} - \frac{25000}{0.08}(e^{-0.8} - 1) - \frac{400}{1.08}(e^{-10.8} - 1) \approx 60{,}122.82$.

(c) $133{,}805.80 \approx 60{,}122.82e^{0.08\cdot 10}$

67. (a) The area of a ring of width Δx feet, x feet from the center is given by $A(x) = ((x + \Delta x)^2 - x^2)\pi = (2x\Delta x + (\Delta x)^2)\pi \approx 2x\Delta x\pi$, if we assume Δx is small. Thus the number of ants (in thousands) in this ring is about $2\pi x d(x)\Delta x = 6\pi xe^{-0.25x}\Delta x$.

(b) Using the result of the previous part, the total number of ants within six feet of the center of the colony is given by $\displaystyle\int_0^6 6\pi xe^{-0.25x}\,dx$; using $u = 6\pi x$, $dv = e^{-0.25x}dx$, $du = 6\pi dx$, $v = -4e^{-0.25x}$, we obtain $\displaystyle\int_0^6 6\pi xe^{-0.25x}\,dx = -24\pi xe^{-0.25x}\Big]_0^6 + \int_0^6 24\pi e^{-0.25x}\,dx = -144\pi e^{-3/2} - 96\pi e^{-0.25x}\Big]_0^6 = -240\pi e^{-3/2} + 96\pi \approx 133.357$ (thousand) ants.

69. (a) $\displaystyle\int \sin^4 x\,dx = -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\int \sin^2 x\,dx = -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\left(-\frac{1}{2}\sin x\cos x + \frac{1}{2}x\right) + C = -\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\sin x\cos x + \frac{3}{8}x + C$.

(b) $\displaystyle\int_0^{\pi/2}\sin^5 x\,dx = -\frac{1}{5}\sin^4 x\cos x\Big]_0^{\pi/2} + \frac{4}{5}\int_0^{\pi/2}\sin^3 x\,dx = \frac{4}{5}\left(-\frac{1}{3}\sin^2 x\cos x\Big]_0^{\pi/2} + \frac{2}{3}\int_0^{\pi/2}\sin x\,dx\right)$

$\displaystyle = -\frac{8}{15}\cos x\Big]_0^{\pi/2} = \frac{8}{15}$.

71. $u = \sin^{n-1} x$, $dv = \sin x\,dx$, $du = (n-1)\sin^{n-2} x\cos x\,dx$, $v = -\cos x$; $\displaystyle\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x(1 - \sin^2 x)dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$, so $\displaystyle n\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx$, and $\displaystyle\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$.

73. (a) $\displaystyle\int \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \int \tan^2 x\,dx = \frac{1}{3}\tan^3 x - \tan x + \int dx = \frac{1}{3}\tan^3 x - \tan x + x + C$.

(b) $\displaystyle\int \sec^4 x\,dx = \frac{1}{3}\sec^2 x\tan x + \frac{2}{3}\int \sec^2 x\,dx = \frac{1}{3}\sec^2 x\tan x + \frac{2}{3}\tan x + C$.

(c) $\displaystyle\int x^3e^x dx = x^3e^x - 3\int x^2e^x dx = x^3e^x - 3\left[x^2e^x - 2\int xe^x dx\right] = x^3e^x - 3x^2e^x + 6\left[xe^x - \int e^x dx\right] = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$.

75. $u = x$, $dv = f''(x)dx$, $du = dx$, $v = f'(x)$; $\displaystyle\int_{-1}^1 x f''(x)dx = xf'(x)\Big]_{-1}^1 - \int_{-1}^1 f'(x)dx = f'(1) + f'(-1) - f(x)\Big]_{-1}^1 = f'(1) + f'(-1) - f(1) + f(-1)$.

77. $u = \ln(x+1), dv = dx, du = \dfrac{dx}{x+1}, v = x+1; \displaystyle\int \ln(x+1)\,dx = \int u\,dv = uv - \int v\,du = (x+1)\ln(x+1) - \int dx = (x+1)\ln(x+1) - x + C.$

79. $u = \tan^{-1} x, dv = x\,dx, du = \dfrac{1}{1+x^2}\,dx, v = \dfrac{1}{2}(x^2+1)$ $\displaystyle\int x\tan^{-1} x\,dx = \int u\,dv = uv - \int v\,du = \frac{1}{2}(x^2+1)\tan^{-1} x - \frac{1}{2}\int dx = \frac{1}{2}(x^2+1)\tan^{-1} x - \frac{1}{2}x + C.$

Exercise Set 7.3

1. $u = \cos x,\ \displaystyle -\int u^3 du = -\frac{1}{4}\cos^4 x + C.$

3. $\displaystyle\int \sin^2 5\theta = \frac{1}{2}\int (1-\cos 10\theta)\,d\theta = \frac{1}{2}\theta - \frac{1}{20}\sin 10\theta + C.$

5. $\displaystyle\int \sin^3 a\theta\,d\theta = \int \sin a\theta(1-\cos^2 a\theta)\,d\theta = -\frac{1}{a}\cos a\theta + \frac{1}{3a}\cos^3 a\theta + C. \quad (a \neq 0)$

7. $u = \sin ax, \quad \displaystyle\frac{1}{a}\int u\,du = \frac{1}{2a}\sin^2 ax + C. \quad (a \neq 0)$

9. $\displaystyle\int \sin^2 t\cos^3 t\,dt = \int \sin^2 t(1-\sin^2 t)\cos t\,dt = \int (\sin^2 t - \sin^4 t)\cos t\,dt = \frac{1}{3}\sin^3 t - \frac{1}{5}\sin^5 t + C.$

11. $\displaystyle\int \sin^2 x\cos^2 x\,dx = \frac{1}{4}\int \sin^2 2x\,dx = \frac{1}{8}\int (1-\cos 4x)dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + C.$

13. $\displaystyle\int \sin 2x\cos 3x\,dx = \frac{1}{2}\int (\sin 5x - \sin x)dx = -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + C.$

15. $\displaystyle\int \sin x\cos(x/2)dx = \frac{1}{2}\int [\sin(3x/2) + \sin(x/2)]dx = -\frac{1}{3}\cos(3x/2) - \cos(x/2) + C.$

17. $\displaystyle\int_0^{\pi/2} \cos^3 x\,dx = \int_0^{\pi/2} (1-\sin^2 x)\cos x\,dx = \left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{\pi/2} = \frac{2}{3}.$

19. $\displaystyle\int_0^{\pi/3} \sin^4 3x\cos^3 3x\,dx = \int_0^{\pi/3} \sin^4 3x(1-\sin^2 3x)\cos 3x\,dx = \left[\frac{1}{15}\sin^5 3x - \frac{1}{21}\sin^7 3x\right]_0^{\pi/3} = 0.$

21. $\displaystyle\int_0^{\pi/6} \sin 4x\cos 2x\,dx = \frac{1}{2}\int_0^{\pi/6} (\sin 2x + \sin 6x)dx = \left[-\frac{1}{4}\cos 2x - \frac{1}{12}\cos 6x\right]_0^{\pi/6} = [(-1/4)(1/2) - (1/12)(-1)] - [-1/4 - 1/12] = 7/24.$

23. $u = 2x-1,\ du = 2dx,\ \displaystyle\frac{1}{2}\int \sec^2 u\,du = \frac{1}{2}\tan(2x-1) + C.$

25. $u = e^{-x}, du = -e^{-x}\,dx;\ \displaystyle -\int \tan u\,du = \ln|\cos u| + C = \ln|\cos(e^{-x})| + C.$

27. $u = 4x,\ du = 4dx,\ \displaystyle\frac{1}{4}\int \sec u\,du = \frac{1}{4}\ln|\sec 4x + \tan 4x| + C.$

29. $u = \tan x,\ \displaystyle\int u^2 du = \frac{1}{3}\tan^3 x + C.$

31. $\displaystyle\int \tan 4x(1+\tan^2 4x)\sec^2 4x\,dx = \int(\tan 4x+\tan^3 4x)\sec^2 4x\,dx = \frac{1}{8}\tan^2 4x + \frac{1}{16}\tan^4 4x + C.$

33. $\displaystyle\int \sec^4 x(\sec^2 x-1)\sec x\tan x\,dx = \int(\sec^6 x-\sec^4 x)\sec x\tan x\,dx = \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C.$

35. $\displaystyle\int(\sec^2 x-1)^2\sec x\,dx = \int(\sec^5 x-2\sec^3 x+\sec x)dx = \int \sec^5 x\,dx - 2\int\sec^3 x\,dx + \int \sec x\,dx = \frac{1}{4}\sec^3 x\tan x + \frac{3}{4}\int\sec^3 x\,dx - 2\int\sec^3 x\,dx + \ln|\sec x+\tan x| = \frac{1}{4}\sec^3 x\tan x - \frac{5}{4}\left[\frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x+\tan x|\right] + \ln|\sec x + \tan x| + C = \frac{1}{4}\sec^3 x\tan x - \frac{5}{8}\sec x\tan x + \frac{3}{8}\ln|\sec x+\tan x| + C.$

37. $\displaystyle\int \sec^2 t(\sec t\tan t)dt = \frac{1}{3}\sec^3 t + C.$

39. $\displaystyle\int \sec^4 x\,dx = \int(1+\tan^2 x)\sec^2 x\,dx = \int(\sec^2 x+\tan^2 x\sec^2 x)dx = \tan x + \frac{1}{3}\tan^3 x + C.$

41. $u=4x$, use equation (19) to get $\displaystyle\frac{1}{4}\int\tan^3 u\,du = \frac{1}{4}\left[\frac{1}{2}\tan^2 u + \ln|\cos u|\right] + C = \frac{1}{8}\tan^2 4x + \frac{1}{4}\ln|\cos 4x| + C.$

43. $\displaystyle\int\sqrt{\tan x}(1+\tan^2 x)\sec^2 x\,dx = \frac{2}{3}\tan^{3/2} x + \frac{2}{7}\tan^{7/2} x + C.$

45. $\displaystyle\int_0^{\pi/8}(\sec^2 2x - 1)dx = \left[\frac{1}{2}\tan 2x - x\right]_0^{\pi/8} = 1/2 - \pi/8.$

47. $u=x/2$, $\displaystyle 2\int_0^{\pi/4}\tan^5 u\,du = \left[\frac{1}{2}\tan^4 u - \tan^2 u - 2\ln|\cos u|\right]_0^{\pi/4} = 1/2 - 1 - 2\ln(1/\sqrt{2}) = -1/2 + \ln 2.$

49. $\displaystyle\int(\csc^2 x-1)\csc^2 x(\csc x\cot x)dx = \int(\csc^4 x - \csc^2 x)(\csc x\cot x)dx = -\frac{1}{5}\csc^5 x + \frac{1}{3}\csc^3 x + C.$

51. $\displaystyle\int(\csc^2 x - 1)\cot x\,dx = \int \csc x(\csc x\cot x)dx - \int\frac{\cos x}{\sin x}dx = -\frac{1}{2}\csc^2 x - \ln|\sin x| + C.$

53. True.

55. False.

57. **(a)** $\displaystyle\int_0^{2\pi}\sin mx\cos nx\,dx = \frac{1}{2}\int_0^{2\pi}[\sin(m+n)x + \sin(m-n)x]dx = \left[-\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)}\right]_0^{2\pi}$, but we know that $\displaystyle\cos(m+n)x\Big]_0^{2\pi} = 0,\ \cos(m-n)x\Big]_0^{2\pi} = 0.$

(b) $\displaystyle\int_0^{2\pi}\cos mx\cos nx\,dx = \frac{1}{2}\int_0^{2\pi}[\cos(m+n)x + \cos(m-n)x]dx$; since $m \neq n$, evaluate sine at integer multiples of 2π to get 0.

(c) $\displaystyle\int_0^{2\pi}\sin mx\sin nx\,dx = \frac{1}{2}\int_0^{2\pi}[\cos(m-n)x - \cos(m+n)x]\ dx$; since $m \neq n$, evaluate sine at integer multiples of 2π to get 0.

59. $y' = \tan x$, $1 + (y')^2 = 1 + \tan^2 x = \sec^2 x$, $L = \displaystyle\int_0^{\pi/4} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx = \ln|\sec x + \tan x|\Big]_0^{\pi/4} =$ $\ln(\sqrt{2} + 1)$.

61. $V = \pi \displaystyle\int_0^{\pi/4} (\cos^2 x - \sin^2 x)dx = \pi \int_0^{\pi/4} \cos 2x\, dx = \frac{1}{2}\pi \sin 2x \Big]_0^{\pi/4} = \pi/2$.

63. With $0 < \alpha < \beta$, $D = D_\beta - D_\alpha = \dfrac{L}{2\pi} \displaystyle\int_{\alpha\pi/180}^{\beta\pi/180} \sec x\, dx = \frac{L}{2\pi} \ln|\sec x + \tan x| \Big]_{\alpha\pi/180}^{\beta\pi/180} = \frac{L}{2\pi} \ln\left|\frac{\sec\beta^\circ + \tan\beta^\circ}{\sec\alpha^\circ + \tan\alpha^\circ}\right|$.

65. **(a)** $\displaystyle\int \csc x\, dx = \int \sec(\pi/2 - x)dx = -\ln|\sec(\pi/2 - x) + \tan(\pi/2 - x)| + C = -\ln|\csc x + \cot x| + C$.

(b) $-\ln|\csc x + \cot x| = \ln \dfrac{1}{|\csc x + \cot x|} = \ln \dfrac{|\csc x - \cot x|}{|\csc^2 x - \cot^2 x|} = \ln|\csc x - \cot x|$, $-\ln|\csc x + \cot x| =$ $-\ln\left|\dfrac{1}{\sin x} + \dfrac{\cos x}{\sin x}\right| = \ln\left|\dfrac{\sin x}{1 + \cos x}\right| = \ln\left|\dfrac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\right| = \ln|\tan(x/2)|$.

67. $a \sin x + b \cos x = \sqrt{a^2 + b^2}\left[\dfrac{a}{\sqrt{a^2 + b^2}} \sin x + \dfrac{b}{\sqrt{a^2 + b^2}} \cos x\right] = \sqrt{a^2 + b^2}(\sin x \cos\theta + \cos x \sin\theta)$, where $\cos\theta =$ $a/\sqrt{a^2 + b^2}$ and $\sin\theta = b/\sqrt{a^2 + b^2}$, so $a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin(x + \theta)$ and then we obtain that

$$\int \frac{dx}{a \sin x + b \cos x} = \frac{1}{\sqrt{a^2 + b^2}} \int \csc(x + \theta)dx = -\frac{1}{\sqrt{a^2 + b^2}} \ln|\csc(x + \theta) + \cot(x + \theta)| + C =$$
$$= -\frac{1}{\sqrt{a^2 + b^2}} \ln\left|\frac{\sqrt{a^2 + b^2} + a\cos x - b \sin x}{a \sin x + b \cos x}\right| + C.$$

69. **(a)** $\displaystyle\int_0^{\pi/2} \sin^3 x\, dx = \frac{2}{3}$. **(b)** $\displaystyle\int_0^{\pi/2} \sin^4 x\, dx = \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\pi}{2} = 3\pi/16$.

(c) $\displaystyle\int_0^{\pi/2} \sin^5 x\, dx = \frac{2 \cdot 4}{3 \cdot 5} = 8/15$. **(d)** $\displaystyle\int_0^{\pi/2} \sin^6 x\, dx = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{\pi}{2} = 5\pi/32$.

Exercise Set 7.4

1. $x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$, $4\displaystyle\int \cos^2\theta\, d\theta = 2\int (1 + \cos 2\theta)d\theta = 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C =$ $2\sin^{-1}(x/2) + \dfrac{1}{2}x\sqrt{4 - x^2} + C$.

3. $x = 4\sin\theta$, $dx = 4\cos\theta\, d\theta$, $16\displaystyle\int \sin^2\theta\, d\theta = 8\int (1 - \cos 2\theta)d\theta = 8\theta - 4\sin 2\theta + C = 8\theta - 8\sin\theta\cos\theta + C =$ $8\sin^{-1}(x/4) - \dfrac{1}{2}x\sqrt{16 - x^2} + C$.

5. $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$, $\dfrac{1}{8}\displaystyle\int \frac{1}{\sec^2\theta} d\theta = \frac{1}{8}\int \cos^2\theta\, d\theta = \frac{1}{16}\int (1 + \cos 2\theta)d\theta = \frac{1}{16}\theta + \frac{1}{32}\sin 2\theta + C =$ $\dfrac{1}{16}\theta + \dfrac{1}{16}\sin\theta\cos\theta + C = \dfrac{1}{16}\tan^{-1}\dfrac{x}{2} + \dfrac{x}{8(4 + x^2)} + C$.

7. $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\, d\theta$, $3\displaystyle\int \tan^2\theta\, d\theta = 3\int (\sec^2\theta - 1)d\theta = 3\tan\theta - 3\theta + C = \sqrt{x^2 - 9} - 3\sec^{-1}\frac{x}{3} + C$.

9. $x = \sin\theta,\ dx = \cos\theta\, d\theta,\ 3\displaystyle\int \sin^3\theta\, d\theta = 3\int \left[1-\cos^2\theta\right]\sin\theta\, d\theta = 3\left(-\cos\theta + \cos^3\theta\right) + C = -3\sqrt{1-x^2} + (1-x^2)^{3/2} + C.$

11. $x = \dfrac{2}{3}\sec\theta,\ dx = \dfrac{2}{3}\sec\theta\tan\theta\, d\theta,\ \dfrac{3}{4}\displaystyle\int \frac{1}{\sec\theta}d\theta = \frac{3}{4}\int \cos\theta\, d\theta = \frac{3}{4}\sin\theta + C = \frac{1}{4x}\sqrt{9x^2-4} + C.$

13. $x = \sin\theta,\ dx = \cos\theta\, d\theta,\ \displaystyle\int \frac{1}{\cos^2\theta}d\theta = \int \sec^2\theta\, d\theta = \tan\theta + C = x/\sqrt{1-x^2} + C.$

15. $x = 3\sec\theta,\ dx = 3\sec\theta\tan\theta\, d\theta,\ \displaystyle\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{1}{3}x + \frac{1}{3}\sqrt{x^2-9}\right| + C.$

17. $x = \dfrac{3}{2}\sec\theta,\ dx = \dfrac{3}{2}\sec\theta\tan\theta\, d\theta,\ \dfrac{3}{2}\displaystyle\int \frac{\sec\theta\tan\theta\, d\theta}{27\tan^3\theta} = \frac{1}{18}\int \frac{\cos\theta}{\sin^2\theta}d\theta = -\frac{1}{18}\frac{1}{\sin\theta} + C = -\frac{1}{18}\csc\theta + C = -\frac{x}{9\sqrt{4x^2-9}} + C.$

19. $e^x = \sin\theta,\ e^x dx = \cos\theta\, d\theta,\ \displaystyle\int \cos^2\theta\, d\theta = \frac{1}{2}\int (1+\cos 2\theta)d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C = \frac{1}{2}\sin^{-1}(e^x) + \frac{1}{2}e^x\sqrt{1-e^{2x}} + C.$

21. $x = \sin\theta,\ dx = \cos\theta\, d\theta,\ 5\displaystyle\int_0^1 \sin^3\theta\cos^2\theta\, d\theta = 5\left[-\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta\right]_0^{\pi/2} = 5(1/3 - 1/5) = 2/3.$

23. $x = \sec\theta,\ dx = \sec\theta\tan\theta\, d\theta,\ \displaystyle\int_{\pi/4}^{\pi/3} \frac{1}{\sec\theta}d\theta = \int_{\pi/4}^{\pi/3} \cos\theta\, d\theta = \sin\theta\Big]_{\pi/4}^{\pi/3} = (\sqrt{3} - \sqrt{2})/2.$

25. $x = \sqrt{3}\tan\theta,\ dx = \sqrt{3}\sec^2\theta\, d\theta,\ \dfrac{1}{9}\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sec\theta}{\tan^4\theta}d\theta = \frac{1}{9}\int_{\pi/6}^{\pi/3} \frac{\cos^3\theta}{\sin^4\theta}d\theta = \frac{1}{9}\int_{\pi/6}^{\pi/3} \frac{1-\sin^2\theta}{\sin^4\theta}\cos\theta\, d\theta$

$= \dfrac{1}{9}\displaystyle\int_{1/2}^{\sqrt{3}/2} \frac{1-u^2}{u^4}du$ (with $u = \sin\theta$) $= \dfrac{1}{9}\displaystyle\int_{1/2}^{\sqrt{3}/2} (u^{-4} - u^{-2})du = \frac{1}{9}\left[-\frac{1}{3u^3} + \frac{1}{u}\right]_{1/2}^{\sqrt{3}/2} = \frac{10\sqrt{3}+18}{243}.$

27. True.

29. False; $x = a\sec\theta$.

31. $u = x^2+4,\ du = 2x\, dx,\ \dfrac{1}{2}\displaystyle\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+4) + C$; or $x = 2\tan\theta,\ dx = 2\sec^2\theta\, d\theta,\ \displaystyle\int \tan\theta\, d\theta = \ln|\sec\theta| + C_1 = \ln\frac{\sqrt{x^2+4}}{2} + C_1 = \ln(x^2+4)^{1/2} - \ln 2 + C_1 = \frac{1}{2}\ln(x^2+4) + C$ with $C = C_1 - \ln 2$.

33. $y' = \dfrac{1}{x},\ 1 + (y')^2 = 1 + \dfrac{1}{x^2} = \dfrac{x^2+1}{x^2},\ L = \displaystyle\int_1^2 \sqrt{\frac{x^2+1}{x^2}}dx;\ x = \tan\theta,\ dx = \sec^2\theta\, d\theta,\ L = \int_{\pi/4}^{\tan^{-1}(2)} \frac{\sec^3\theta}{\tan\theta}d\theta = \int_{\pi/4}^{\tan^{-1}(2)} \frac{\tan^2\theta+1}{\tan\theta}\sec\theta\, d\theta = \int_{\pi/4}^{\tan^{-1}(2)} (\sec\theta\tan\theta + \csc\theta)d\theta = \Big[\sec\theta + \ln|\csc\theta - \cot\theta|\Big]_{\pi/4}^{\tan^{-1}(2)}$

$= \sqrt{5} + \ln\left(\dfrac{\sqrt{5}}{2} - \dfrac{1}{2}\right) - \left[\sqrt{2} + \ln|\sqrt{2}-1|\right] = \sqrt{5} - \sqrt{2} + \ln\dfrac{2+2\sqrt{2}}{1+\sqrt{5}}.$

35. $y' = 2x,\ 1+(y')^2 = 1+4x^2,\ S = 2\pi\displaystyle\int_0^1 x^2\sqrt{1+4x^2}dx;\ x = \frac{1}{2}\tan\theta,\ dx = \frac{1}{2}\sec^2\theta\, d\theta,\ S = \frac{\pi}{4}\int_0^{\tan^{-1}2} \tan^2\theta\sec^3\theta\, d\theta = \frac{\pi}{4}\int_0^{\tan^{-1}2} (\sec^2\theta - 1)\sec^3\theta\, d\theta = \frac{\pi}{4}\int_0^{\tan^{-1}2} (\sec^5\theta - \sec^3\theta)d\theta =$

$= \frac{\pi}{4}\left[\frac{1}{4}\sec^3\theta\tan\theta - \frac{1}{8}\sec\theta\tan\theta - \frac{1}{8}\ln|\sec\theta+\tan\theta|\right]_0^{\tan^{-1}2} = \frac{\pi}{32}[18\sqrt{5} - \ln(2+\sqrt{5})]$.

37. $\int \frac{1}{(x-2)^2+1}dx = \tan^{-1}(x-2) + C$.

39. $\int \frac{1}{\sqrt{4-(x-1)^2}}dx = \sin^{-1}\left(\frac{x-1}{2}\right) + C$

41. $\int \frac{1}{\sqrt{(x-3)^2+1}}dx = \ln\left(x-3+\sqrt{(x-3)^2+1}\right) + C$.

43. $\int \sqrt{4-(x+1)^2}\,dx$, let $x+1 = 2\sin\theta$, $\int 4\cos^2\theta\,d\theta = \int 2(1+\cos 2\theta)\,d\theta = 2\theta + \sin 2\theta + C = 2\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{1}{2}(x+1)\sqrt{3-2x-x^2} + C$.

45. $\int \frac{1}{2(x+1)^2+5}dx = \frac{1}{2}\int \frac{1}{(x+1)^2+5/2}dx = \frac{1}{\sqrt{10}}\tan^{-1}\sqrt{2/5}(x+1) + C$.

47. $\int_1^2 \frac{1}{\sqrt{4x-x^2}}\,dx = \int_1^2 \frac{1}{\sqrt{4-(x-2)^2}}\,dx = \sin^{-1}\frac{x-2}{2}\Big]_1^2 = \pi/6$.

49. $u = \sin^2 x$, $du = 2\sin x\cos x\,dx$; $\frac{1}{2}\int \sqrt{1-u^2}\,du = \frac{1}{4}\left[u\sqrt{1-u^2} + \sin^{-1}u\right] + C = \frac{1}{4}\left[\sin^2 x\sqrt{1-\sin^4 x} + \sin^{-1}(\sin^2 x)\right] + C$.

51. **(a)** $x = 3\sinh u$, $dx = 3\cosh u\,du$, $\int du = u + C = \sinh^{-1}(x/3) + C$.

(b) $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$, $\int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C = \ln\left(\sqrt{x^2+9}/3 + x/3\right) + C$, but $\sinh^{-1}(x/3) = \ln\left(x/3 + \sqrt{x^2/9+1}\right) = \ln\left(x/3 + \sqrt{x^2+9}/3\right)$, so the results agree.

Exercise Set 7.5

1. $\frac{3x-1}{(x-3)(x+4)} = \frac{A}{(x-3)} + \frac{B}{(x+4)}$.

3. $\frac{2x-3}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$.

5. $\frac{1-x^2}{x^3(x^2+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+2}$.

7. $\frac{4x^3-x}{(x^2+5)^2} = \frac{Ax+B}{x^2+5} + \frac{Cx+D}{(x^2+5)^2}$.

9. $\frac{1}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1}$; $A = \frac{1}{5}$, $B = -\frac{1}{5}$, so $\frac{1}{5}\int \frac{1}{x-4}dx - \frac{1}{5}\int \frac{1}{x+1}dx = \frac{1}{5}\ln|x-4| - \frac{1}{5}\ln|x+1| + C = \frac{1}{5}\ln\left|\frac{x-4}{x+1}\right| + C$.

11. $\dfrac{11x+17}{(2x-1)(x+4)} = \dfrac{A}{2x-1} + \dfrac{B}{x+4}$; $A=5$, $B=3$, so $5\displaystyle\int \dfrac{1}{2x-1}dx + 3\int \dfrac{1}{x+4}dx = \dfrac{5}{2}\ln|2x-1| + 3\ln|x+4| + C$.

13. $\dfrac{2x^2-9x-9}{x(x+3)(x-3)} = \dfrac{A}{x} + \dfrac{B}{x+3} + \dfrac{C}{x-3}$; $A=1$, $B=2$, $C=-1$, so $\displaystyle\int \frac{1}{x}\,dx + 2\int \frac{1}{x+3}\,dx - \int \frac{1}{x-3}\,dx =$ $\ln|x| + 2\ln|x+3| - \ln|x-3| + C = \ln\left|\dfrac{x(x+3)^2}{x-3}\right| + C$. Note that the symbol C has been recycled; to save space this recycling is usually not mentioned.

15. $\dfrac{x^2-8}{x+3} = x-3+\dfrac{1}{x+3}$, $\displaystyle\int \left(x-3+\frac{1}{x+3}\right)dx = \frac{1}{2}x^2 - 3x + \ln|x+3| + C$.

17. $\dfrac{3x^2-10}{x^2-4x+4} = 3 + \dfrac{12x-22}{x^2-4x+4}$, $\dfrac{12x-22}{(x-2)^2} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2}$; $A=12$, $B=2$, so $\displaystyle\int 3dx + 12\int \frac{1}{x-2}dx +$ $2\displaystyle\int \frac{1}{(x-2)^2}dx = 3x + 12\ln|x-2| - 2/(x-2) + C$.

19. $u = x^2-3x-10, du = (2x-3)\,dx$, $\displaystyle\int \frac{du}{u} = \ln|u| + C = \ln|x^2-3x-10| + C$.

21. $\dfrac{x^5+x^2+2}{x^3-x} = x^2+1+\dfrac{x^2+x+2}{x^3-x}$, $\dfrac{x^2+x+2}{x(x+1)(x-1)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$; $A=-2$, $B=1$, $C=2$, so $\displaystyle\int (x^2+1)dx -$ $\displaystyle\int \frac{2}{x}dx + \int \frac{1}{x+1}dx + \int \frac{2}{x-1}dx = \frac{1}{3}x^3 + x - 2\ln|x| + \ln|x+1| + 2\ln|x-1| + C = \frac{1}{3}x^3 + x + \ln\left|\frac{(x+1)(x-1)^2}{x^2}\right| + C$.

23. $\dfrac{2x^2+3}{x(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$; $A=3$, $B=-1$, $C=5$, so $3\displaystyle\int \frac{1}{x}\,dx - \int \frac{1}{x-1}\,dx + 5\int \frac{1}{(x-1)^2}dx =$ $3\ln|x| - \ln|x-1| - 5/(x-1) + C$.

25. $\dfrac{2x^2-10x+4}{(x+1)(x-3)^2} = \dfrac{A}{x+1} + \dfrac{B}{x-3} + \dfrac{C}{(x-3)^2}$; $A=1$, $B=1$, $C=-2$, so $\displaystyle\int \frac{1}{x+1}\,dx + \int \frac{1}{x-3}\,dx - \int \frac{2}{(x-3)^2}\,dx =$ $\ln|x+1| + \ln|x-3| + \dfrac{2}{x-3} + C_1$.

27. $\dfrac{x^2}{(x+1)^3} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3}$; $A=1$, $B=-2$, $C=1$, so $\displaystyle\int \frac{1}{x+1}\,dx - \int \frac{2}{(x+1)^2}\,dx + \int \frac{1}{(x+1)^3}\,dx =$ $\ln|x+1| + \dfrac{2}{x+1} - \dfrac{1}{2(x+1)^2} + C$.

29. $\dfrac{2x^2-1}{(4x-1)(x^2+1)} = \dfrac{A}{4x-1} + \dfrac{Bx+C}{x^2+1}$; $A=-14/17$, $B=12/17$, $C=3/17$, so $\displaystyle\int \frac{2x^2-1}{(4x-1)(x^2+1)}dx =$ $-\dfrac{7}{34}\ln|4x-1| + \dfrac{6}{17}\ln(x^2+1) + \dfrac{3}{17}\tan^{-1}x + C$.

31. $\dfrac{x^3+3x^2+x+9}{(x^2+1)(x^2+3)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+3}$; $A=0$, $B=3$, $C=1$, $D=0$, so $\displaystyle\int \frac{x^3+3x^2+x+9}{(x^2+1)(x^2+3)}\,dx = 3\tan^{-1}x +$ $\dfrac{1}{2}\ln(x^2+3) + C$.

33. $\dfrac{x^3-2x^2+2x-2}{x^2+1} = x-2+\dfrac{x}{x^2+1}$, so $\displaystyle\int \frac{x^3-3x^2+2x-3}{x^2+1}\,dx = \frac{1}{2}x^2 - 2x + \frac{1}{2}\ln(x^2+1) + C$.

35. True.

37. True.

39. Let $x = \sin\theta$ to get $\displaystyle\int \frac{1}{x^2+4x-5}\,dx$, and $\displaystyle\frac{1}{(x+5)(x-1)} = \frac{A}{x+5} + \frac{B}{x-1}$; $A = -1/6$, $B = 1/6$, so we get $\displaystyle -\frac{1}{6}\int \frac{1}{x+5}\,dx + \frac{1}{6}\int \frac{1}{x-1}\,dx = \frac{1}{6}\ln\left|\frac{x-1}{x+5}\right| + C = \frac{1}{6}\ln\left(\frac{1-\sin\theta}{5+\sin\theta}\right) + C$.

41. $\displaystyle u = e^x, du = e^x\,dx, \int \frac{e^{3x}}{e^{2x}+4}\,dx = \int \frac{u^2}{u^2+4}\,du = u - 2\tan^{-1}\frac{u}{2} + C = e^x - 2\tan^{-1}(e^x/2) + C$.

43. $\displaystyle V = \pi\int_0^2 \frac{x^4}{(9-x^2)^2}\,dx$, $\displaystyle\frac{x^4}{x^4-18x^2+81} = 1 + \frac{18x^2-81}{x^4-18x^2+81}$, $\displaystyle\frac{18x^2-81}{(9-x^2)^2} = \frac{18x^2-81}{(x+3)^2(x-3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x-3} + \frac{D}{(x-3)^2}$; $A = -\frac{9}{4}$, $B = \frac{9}{4}$, $C = \frac{9}{4}$, $D = \frac{9}{4}$, so $\displaystyle V = \pi\left[x - \frac{9}{4}\ln|x+3| - \frac{9/4}{x+3} + \frac{9}{4}\ln|x-3| - \frac{9/4}{x-3}\right]_0^2 = \pi\left(\frac{19}{5} - \frac{9}{4}\ln 5\right)$.

45. $\displaystyle\frac{x^2+1}{(x^2+2x+3)^2} = \frac{Ax+B}{x^2+2x+3} + \frac{Cx+D}{(x^2+2x+3)^2}$; $A = 0$, $B = 1$, $C = D = -2$, so $\displaystyle\int \frac{x^2+1}{(x^2+2x+3)^2}dx = \int \frac{1}{(x+1)^2+2}dx - \int \frac{2x+2}{(x^2+2x+3)^2}dx = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}} + 1/(x^2+2x+3) + C$.

47. $x^4 - 3x^3 - 7x^2 + 27x - 18 = (x-1)(x-2)(x-3)(x+3)$, $\displaystyle\frac{1}{(x-1)(x-2)(x-3)(x+3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} + \frac{D}{x+3}$; $A = 1/8$, $B = -1/5$, $C = 1/12$, $D = -1/120$, so $\displaystyle\int \frac{dx}{x^4-3x^3-7x^2+27x-18} = \frac{1}{8}\ln|x-1| - \frac{1}{5}\ln|x-2| + \frac{1}{12}\ln|x-3| - \frac{1}{120}\ln|x+3| + C$.

49. Let $\displaystyle u = x^2, du = 2x\,dx, \int_0^1 \frac{x}{x^4+1}\,dx = \frac{1}{2}\int_0^1 \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1} u\Big]_0^1 = \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{8}$.

51. If the polynomial has distinct roots $r_1, r_2, r_1 \neq r_2$, then the partial fraction decomposition will contain terms of the form $\displaystyle\frac{A}{x-r_1}, \frac{B}{x-r_2}$, and they will give logarithms and no inverse tangents. If there are two roots not distinct, say $x = r$, then the terms $\displaystyle\frac{A}{x-r}, \frac{B}{(x-r)^2}$ will appear, and neither will give an inverse tangent term. The only other possibility is no real roots, and the integrand can be written in the form $\displaystyle\frac{1}{a\left(x+\frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}}$, which will yield an inverse tangent, specifically of the form $\displaystyle\tan^{-1}\left[A\left(x + \frac{b}{2a}\right)\right]$ for some constant A.

53. Yes, for instance the integrand $\displaystyle\frac{1}{x^2+1}$, whose integral is precisely $\tan^{-1} x + C$.

Exercise Set 7.6

1. Formula (60): $\displaystyle\frac{4}{9}\Big[3x + \ln|-1+3x|\Big] + C$.

3. Formula (65): $\displaystyle\frac{1}{5}\ln\left|\frac{x}{5+2x}\right| + C$.

5. Formula (102): $\displaystyle\frac{1}{5}(x-1)(2x+3)^{3/2} + C$.

7. Formula (108): $\displaystyle\frac{1}{2}\ln\left|\frac{\sqrt{4-3x}-2}{\sqrt{4-3x}+2}\right| + C$.

9. Formula (69): $\dfrac{1}{8}\ln\left|\dfrac{x+4}{x-4}\right|+C$.

11. Formula (73): $\dfrac{x}{2}\sqrt{x^2-3}-\dfrac{3}{2}\ln\left|x+\sqrt{x^2-3}\right|+C$.

13. Formula (95): $\dfrac{x}{2}\sqrt{x^2+4}-2\ln(x+\sqrt{x^2+4})+C$.

15. Formula (74): $\dfrac{x}{2}\sqrt{9-x^2}+\dfrac{9}{2}\sin^{-1}\dfrac{x}{3}+C$.

17. Formula (79): $\sqrt{4-x^2}-2\ln\left|\dfrac{2+\sqrt{4-x^2}}{x}\right|+C$.

19. Formula (38): $-\dfrac{1}{14}\sin(7x)+\dfrac{1}{2}\sin x+C$.

21. Formula (50): $\dfrac{x^4}{16}\left[4\ln x-1\right]+C$.

23. Formula (42): $\dfrac{e^{-2x}}{13}(-2\sin(3x)-3\cos(3x))+C$.

25. $u=e^{2x}, du=2e^{2x}dx$, Formula (62): $\dfrac{1}{2}\displaystyle\int\frac{u\,du}{(4-3u)^2}=\frac{1}{18}\left[\frac{4}{4-3e^{2x}}+\ln\left|4-3e^{2x}\right|\right]+C$.

27. $u=3\sqrt{x}, du=\dfrac{3}{2\sqrt{x}}dx$, Formula (68): $\dfrac{2}{3}\displaystyle\int\frac{du}{u^2+4}=\frac{1}{3}\tan^{-1}\frac{3\sqrt{x}}{2}+C$.

29. $u=2x, du=2dx$, Formula (76): $\dfrac{1}{2}\displaystyle\int\frac{du}{\sqrt{u^2-9}}=\frac{1}{2}\ln\left|2x+\sqrt{4x^2-9}\right|+C$.

31. $u=2x^2, du=4xdx, u^2du=16x^5\,dx$, Formula (81): $\dfrac{1}{4}\displaystyle\int\frac{u^2\,du}{\sqrt{2-u^2}}=-\frac{x^2}{4}\sqrt{2-4x^4}+\frac{1}{4}\sin^{-1}(\sqrt{2}x^2)+C$.

33. $u=\ln x, du=dx/x$, Formula (26): $\displaystyle\int\sin^2 u\,du=\frac{1}{2}\ln x-\frac{1}{4}\sin(2\ln x)+C$.

35. $u=-2x, du=-2dx$, Formula (51): $\dfrac{1}{4}\displaystyle\int ue^u\,du=\frac{1}{4}(-2x-1)e^{-2x}+C$.

37. $u=\sin 3x, du=3\cos 3x\,dx$, Formula (67): $\dfrac{1}{3}\displaystyle\int\frac{du}{u(u+1)^2}=\frac{1}{3}\left[\frac{1}{1+\sin 3x}+\ln\left|\frac{\sin 3x}{1+\sin 3x}\right|\right]+C$.

39. $u=4x^2, du=8xdx$, Formula (70): $\dfrac{1}{8}\displaystyle\int\frac{du}{u^2-1}=\frac{1}{16}\ln\left|\frac{4x^2-1}{4x^2+1}\right|+C$.

41. $u=2e^x, du=2e^x dx$, Formula (74): $\dfrac{1}{2}\displaystyle\int\sqrt{3-u^2}\,du=\frac{1}{4}u\sqrt{3-u^2}+\frac{3}{4}\sin^{-1}(u/\sqrt{3})+C=\frac{1}{2}e^x\sqrt{3-4e^{2x}}+\frac{3}{4}\sin^{-1}(2e^x/\sqrt{3})+C$.

43. $u=3x, du=3dx$, Formula (112): $\dfrac{1}{3}\displaystyle\int\sqrt{\frac{5}{3}u-u^2}\,du=\frac{1}{6}\left(u-\frac{5}{6}\right)\sqrt{\frac{5}{3}u-u^2}+\frac{25}{216}\sin^{-1}\left(\frac{6u-5}{5}\right)+C=\frac{18x-5}{36}\sqrt{5x-9x^2}+\frac{25}{216}\sin^{-1}\left(\frac{18x-5}{5}\right)+C$.

45. $u = 2x, du = 2dx$, Formula (44): $\displaystyle\int u \sin u \, du = (\sin u - u \cos u) + C = \sin 2x - 2x \cos 2x + C$.

47. $u = -\sqrt{x}, u^2 = x, 2udu = dx$, Formula (51): $2\displaystyle\int ue^u du = -2(\sqrt{x}+1)e^{-\sqrt{x}} + C$.

49. $x^2 + 6x - 7 = (x+3)^2 - 16; u = x+3, du = dx$, Formula (70): $\displaystyle\int \frac{du}{u^2-16} = \frac{1}{8}\ln\left|\frac{u-4}{u+4}\right| + C = \frac{1}{8}\ln\left|\frac{x-1}{x+7}\right| + C$.

51. $x^2 - 4x - 5 = (x-2)^2 - 9, u = x-2, du = dx$, Formula (77): $\displaystyle\int \frac{u+2}{\sqrt{9-u^2}}\,du = \int \frac{u\,du}{\sqrt{9-u^2}} + 2\int \frac{du}{\sqrt{9-u^2}} =$ $-\sqrt{9-u^2} + 2\sin^{-1}\dfrac{u}{3} + C = -\sqrt{5+4x-x^2} + 2\sin^{-1}\left(\dfrac{x-2}{3}\right) + C$.

53. $u = \sqrt{x-2}$, $x = u^2+2$, $dx = 2u\,du$; $\displaystyle\int 2u^2(u^2+2)du = 2\int (u^4+2u^2)du = \frac{2}{5}u^5 + \frac{4}{3}u^3 + C = \frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2} + C$.

55. $u = \sqrt{x^3+1}$, $x^3 = u^2-1$, $3x^2dx = 2u\,du$; $\dfrac{2}{3}\displaystyle\int u^2(u^2-1)du = \frac{2}{3}\int (u^4-u^2)du = \frac{2}{15}u^5 - \frac{2}{9}u^3 + C = \frac{2}{15}(x^3+1)^{5/2} - \frac{2}{9}(x^3+1)^{3/2} + C$.

57. $u = x^{1/3}$, $x = u^3$, $dx = 3u^2\,du$; $\displaystyle\int \frac{3u^2}{u^3-u}du = 3\int \frac{u}{u^2-1}du = 3\int \left[\frac{1}{2(u+1)} + \frac{1}{2(u-1)}\right] du = \frac{3}{2}\ln|x^{1/3}+1| + \frac{3}{2}\ln|x^{1/3}-1| + C$.

59. $u = x^{1/4}$, $x = u^4$, $dx = 4u^3du$; $4\displaystyle\int \frac{1}{u(1-u)}du = 4\int \left[\frac{1}{u} + \frac{1}{1-u}\right] du = 4\ln\frac{x^{1/4}}{|1-x^{1/4}|} + C$.

61. $u = x^{1/6}$, $x = u^6$, $dx = 6u^5du$; $6\displaystyle\int \frac{u^3}{u-1}du = 6\int \left[u^2+u+1+\frac{1}{u-1}\right] du = 2x^{1/2} + 3x^{1/3} + 6x^{1/6} + 6\ \ln|x^{1/6}-1| + C$.

63. $u = \sqrt{1+x^2}$, $x^2 = u^2-1$, $2x\,dx = 2u\,du$, $x\,dx = u\,du$; $\displaystyle\int (u^2-1)du = \frac{1}{3}(1+x^2)^{3/2} - (1+x^2)^{1/2} + C$.

65. $u = \tan(x/2)$, $\displaystyle\int \frac{1}{1+\dfrac{2u}{1+u^2}+\dfrac{1-u^2}{1+u^2}}\,\frac{2}{1+u^2}du = \int \frac{1}{u+1}du = \ln|\tan(x/2)+1| + C$.

67. $u = \tan(\theta/2)$, $\displaystyle\int \frac{d\theta}{1-\cos\theta} = \int \frac{1}{u^2}du = -\frac{1}{u} + C = -\cot(\theta/2) + C$.

69. $u = \tan(x/2)$, $\dfrac{1}{2}\displaystyle\int \frac{1-u^2}{u}du = \frac{1}{2}\int (1/u - u)du = \frac{1}{2}\ln|\tan(x/2)| - \frac{1}{4}\tan^2(x/2) + C$.

71. $\displaystyle\int_2^x \frac{1}{t(4-t)}\,dt = \frac{1}{4}\ln\frac{t}{4-t}\bigg]_2^x$ (Formula (65), $a = 4, b = -1$) $= \dfrac{1}{4}\left[\ln\dfrac{x}{4-x} - \ln 1\right] = \dfrac{1}{4}\ln\dfrac{x}{4-x}$, $\dfrac{1}{4}\ln\dfrac{x}{4-x} = 0.5$, $\ln\dfrac{x}{4-x} = 2$, $\dfrac{x}{4-x} = e^2$, $x = 4e^2 - e^2x$, $x(1+e^2) = 4e^2$, $x = 4e^2/(1+e^2) \approx 3.523188312$.

73. $A = \displaystyle\int_0^4 \sqrt{25-x^2}\,dx = \left(\frac{1}{2}x\sqrt{25-x^2} + \frac{25}{2}\sin^{-1}\frac{x}{5}\right)\bigg]_0^4$ (Formula (74), $a = 5$) $= 6 + \dfrac{25}{2}\sin^{-1}\dfrac{4}{5} \approx 17.59119023$.

75. $A = \int_0^1 \frac{1}{25-16x^2}\,dx$; $u = 4x$, $A = \frac{1}{4}\int_0^4 \frac{1}{25-u^2}\,du = \frac{1}{40}\ln\left|\frac{u+5}{u-5}\right|\Big]_0^4 = \frac{1}{40}\ln 9 \approx 0.054930614$. (Formula (69), $a = 5$)

77. $V = 2\pi\int_0^{\pi/2} x\cos x\,dx = 2\pi(\cos x + x\sin x)\Big]_0^{\pi/2} = \pi(\pi-2) \approx 3.586419094$. (Formula (45))

79. $V = 2\pi\int_0^3 xe^{-x}dx$; $u = -x$, $V = 2\pi\int_0^{-3} ue^u du = 2\pi e^u(u-1)\Big]_0^{-3} = 2\pi(1-4e^{-3}) \approx 5.031899801$. (Formula (51))

81. $L = \int_0^2 \sqrt{1+16x^2}\,dx$; $u = 4x$, $L = \frac{1}{4}\int_0^8 \sqrt{1+u^2}\,du = \frac{1}{4}\left(\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right)\Big]_0^8$

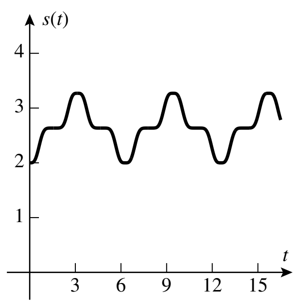
(Formula (72), $a^2 = 1$) $= \sqrt{65} + \frac{1}{8}\ln(8+\sqrt{65}) \approx 8.409316783$.

83. $S = 2\pi\int_0^{\pi}(\sin x)\sqrt{1+\cos^2 x}\,dx$; $u = \cos x, a^2 = 1$, $S = -2\pi\int_1^{-1}\sqrt{1+u^2}\,du = 4\pi\int_0^1\sqrt{1+u^2}\,du$

$= 4\pi\left(\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right)\Big]_0^1 = 2\pi\left[\sqrt{2}+\ln(1+\sqrt{2})\right] \approx 14.42359945$. (Formula (72))

85. (a) $s(t) = 2 + \int_0^t 20\cos^6 u\sin^3 u\,du = -\frac{20}{9}\sin^2 t\cos^7 t - \frac{40}{63}\cos^7 t + \frac{166}{63}$.

(b)



87. (a) $\int \sec x\,dx = \int \frac{1}{\cos x}dx = \int \frac{2}{1-u^2}du = \ln\left|\frac{1+u}{1-u}\right| + C = \ln\left|\frac{1+\tan(x/2)}{1-\tan(x/2)}\right| + C =$

$= \ln\left\{\left|\frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)-\sin(x/2)}\right|\left|\frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)+\sin(x/2)}\right|\right\} + C = \ln\left|\frac{1+\sin x}{\cos x}\right| + C = \ln|\sec x + \tan x| + C$.

(b) $\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \frac{\tan\frac{\pi}{4}+\tan\frac{x}{2}}{1-\tan\frac{\pi}{4}\tan\frac{x}{2}} = \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$.

89. Let $u = \tanh(x/2)$ then $\cosh(x/2) = 1/\operatorname{sech}(x/2) = 1/\sqrt{1-\tanh^2(x/2)} = 1/\sqrt{1-u^2}$, $\sinh(x/2) = \tanh(x/2)\cosh(x/2) = u/\sqrt{1-u^2}$, so $\sinh x = 2\sinh(x/2)\cosh(x/2) = 2u/(1-u^2)$, $\cosh x = \cosh^2(x/2) + \sinh^2(x/2) = (1+u^2)/(1-u^2)$, $x = 2\tanh^{-1}u, dx = [2/(1-u^2)]du$; $\int \frac{dx}{2\cosh x + \sinh x} = \int \frac{1}{u^2+u+1}du = \frac{2}{\sqrt{3}}\tan^{-1}\frac{2u+1}{\sqrt{3}} + C = \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\tanh(x/2)+1}{\sqrt{3}} + C$.

91. $\int(\cos^{32}x\sin^{30}x - \cos^{30}x\sin^{32}x)dx = \int\cos^{30}x\sin^{30}x(\cos^2 x - \sin^2 x)dx = \frac{1}{2^{30}}\int\sin^{30}2x\cos 2x\,dx =$

$= \dfrac{\sin^{31} 2x}{31(2^{31})} + C.$

93. $\displaystyle\int \frac{1}{x^{10}(1+x^{-9})}dx = -\frac{1}{9}\int \frac{1}{u}du = -\frac{1}{9}\ln|u| + C = -\frac{1}{9}\ln|1+x^{-9}| + C.$

Exercise Set 7.7

1. Exact value $= 14/3 \approx 4.666666667$.

(a) 4.667600662, $|E_M| \approx 0.000933995$. **(b)** 4.664795676, $|E_T| \approx 0.001870991$. **(c)** 4.666666602, $|E_S| \approx 9.9 \cdot 10^{-7}$.

3. Exact value $= 1$.

(a) 1.001028824, $|E_M| \approx 0.001028824$. **(b)** 0.997942986, $|E_T| \approx 0.002057013$. **(c)** 1.000000013, $|E_S| \approx 2.12 \cdot 10^{-7}$.

5. Exact value $= \dfrac{1}{2}(e^{-2} - e^{-6}) \approx 0.06642826551$.

(a) 0.065987468, $|E_M| \approx 0.000440797$. **(b)** 0.067311623, $|E_T| \approx 0.000883357$. **(c)** 0.066428302, $|E_S| \approx 5.88 \cdot 10^{-7}$.

7. $f(x) = \sqrt{x+1}$, $f''(x) = -\dfrac{1}{4}(x+1)^{-3/2}$, $f^{(4)}(x) = -\dfrac{15}{16}(x+1)^{-7/2}$; $K_2 = 1/4$, $K_4 = 15/16$.

(a) $|E_M| \le \dfrac{27}{2400}(1/4) = 0.002812500$. **(b)** $|E_T| \le \dfrac{27}{1200}(1/4) = 0.00562500$.

(c) $|E_S| \le \dfrac{81}{10240000} \approx 0.000007910156250$.

9. $f(x) = \cos x$, $f''(x) = -\cos x$, $f^{(4)}(x) = \cos x$; $K_2 = K_4 = 1$.

(a) $|E_M| \le \dfrac{\pi^3/8}{2400}(1) \approx 0.00161491$. **(b)** $|E_T| \le \dfrac{\pi^3/8}{1200}(1) \approx 0.003229820488$.

(c) $|E_S| \le \dfrac{\pi^5/32}{180 \times 20^4}(1) \approx 3.320526095 \cdot 10^{-7}$.

11. $f(x) = e^{-2x}$, $f''(x) = 4e^{-2x}$; $f^{(4)}(x) = 16e^{-2x}$; $K_2 = 4e^{-2}$; $K_4 = 16e^{-2}$.

(a) $|E_M| \le \dfrac{8}{2400}(4e^{-2}) \approx 0.0018044704$. **(b)** $|E_T| \le \dfrac{8}{1200}(4e^{-2}) \approx 0.0036089409$.

(c) $|E_S| \le \dfrac{32}{180 \times 20^4}(16e^{-2}) \approx 0.00000240596$.

13. **(a)** $n > \left[\dfrac{(27)(1/4)}{(24)(5 \times 10^{-4})}\right]^{1/2} \approx 23.7$; $n = 24$. **(b)** $n > \left[\dfrac{(27)(1/4)}{(12)(5 \times 10^{-4})}\right]^{1/2} \approx 33.5$; $n = 34$.

(c) $n > \left[\dfrac{(243)(15/16)}{(180)(5 \times 10^{-4})}\right]^{1/4} \approx 7.1$; $n = 8$.

15. **(a)** $n > \left[\dfrac{(\pi^3/8)(1)}{(24)(10^{-3})}\right]^{1/2} \approx 12.7$; $n = 13$. **(b)** $n > \left[\dfrac{(\pi^3/8)(1)}{(12)(10^{-3})}\right]^{1/2} \approx 17.97$; $n = 18$.

(c) $n > \left[\dfrac{(\pi^5/32)(1)}{(180)(10^{-3})}\right]^{1/4} \approx 2.7$; $n = 4$.

17. **(a)** $n > \left[\dfrac{(8)(4e^{-2})}{(24)(10^{-6})}\right]^{1/2} \approx 42.5$; $n = 43$. **(b)** $n > \left[\dfrac{(8)(4e^{-2})}{(12)(10^{-6})}\right]^{1/2} \approx 60.2$; $n = 61$.

(c) $n > \left[\dfrac{(32)(16e^{-2})}{(180)(10^{-6})}\right]^{1/4} \approx 7.9$; $n = 8$.

19. False; T_n is the average of L_n and R_n.

21. False, it is the weighted average of M_{25} and T_{25}.

23. $g(X_0) = aX_0^2 + bX_0 + c = 4a + 2b + c = f(X_0) = 1/X_0 = 1/2$; similarly $9a + 3b + c = 1/3, 16a + 4b + c = 1/4$. Three equations in three unknowns, with solution $a = 1/24, b = -3/8, c = 13/12, g(x) = x^2/24 - 3x/8 + 13/12$. $\displaystyle\int_2^4 g(x)\,dx = \int_2^4 \left(\frac{x^2}{24} - \frac{3x}{8} + \frac{13}{12}\right) dx = \frac{25}{36}$, $\dfrac{\Delta x}{3}[f(X_0) + 4f(X_1) + f(X_2)] = \dfrac{1}{3}\left[\dfrac{1}{2} + \dfrac{4}{3} + \dfrac{1}{4}\right] = \dfrac{25}{36}$.

25. 1.49367411, 1.493648266.

27. 3.806779393, 3.805537256.

29. 0.9045242448, 0.9045242380.

31. Exact value $= 4\tan^{-1}(x/2)\Big]_0^2 = \pi$.

(a) 3.142425985, $|E_M| \approx 0.000833331$. **(b)** 3.139925989, $|E_T| \approx 0.001666665$.

(c) 3.141592654, $|E_S| \approx 6.2 \times 10^{-10}$.

33. $S_{14} = 0.693147984$, $|E_S| \approx 0.000000803 = 8.03 \times 10^{-7}$; the method used in Example 6 results in a value of n which ensures that the magnitude of the error will be less than 10^{-6}, this is not necessarily the *smallest* value of n.

35. $f(x) = x\sin x$, $f''(x) = 2\cos x - x\sin x$, $|f''(x)| \le 2|\cos x| + |x|\,|\sin x| \le 2 + 2 = 4$, so $K_2 \le 4$, $n > \left[\dfrac{(8)(4)}{(24)(10^{-4})}\right]^{1/2} \approx 115.5$; $n = 116$ (a smaller n might suffice).

37. $f(x) = x\sqrt{x}$, $f''(x) = \dfrac{3}{4\sqrt{x}}$, $\displaystyle\lim_{x\to 0^+} |f''(x)| = +\infty$.

39. $\displaystyle s(x) = \int_0^x \sqrt{1 + (y'(t))^2}\,dt = \int_0^x \sqrt{1 + \cos^2 t}\,dt$, $\displaystyle \ell = \int_0^\pi \sqrt{1 + \cos^2 t}\,dt \approx 3.820187624$.

41. $\displaystyle\int_0^{30} v\,dt \approx \frac{30}{(3)(6)}\frac{22}{15}[0 + 4(60) + 2(90) + 4(110) + 2(126) + 4(138) + 146] \approx 4424$ ft.

43. $\displaystyle\int_0^{180} v\,dt \approx \frac{180}{(3)(6)}[0.00 + 4(0.03) + 2(0.08) + 4(0.16) + 2(0.27) + 4(0.42) + 0.65] = 37.9$ mi.

45. $\displaystyle V = \int_0^{16} \pi r^2 dy = \pi \int_0^{16} r^2 dy \approx \pi \frac{16}{(3)(4)}[(8.5)^2 + 4(11.5)^2 + 2(13.8)^2 + 4(15.4)^2 + (16.8)^2] \approx 9270 \text{ cm}^3 \approx 9.3$ L.

47. **(a)** The maximum value of $|f''(x)|$ is approximately 3.8442. **(b)** $n = 18$. **(c)** 0.9047406684.

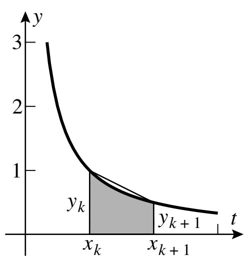
49. **(a)** $K_4 = \displaystyle\max_{0\le x\le 1} |f^{(4)}(x)| \approx 12.4282$.

(b) $\dfrac{(b-a)^5 K_4}{180n^4} < 10^{-4}$ provided $n^4 > \dfrac{10^4 K_4}{180}, n > 5.12$, so $n \geq 6$.

(c) $\dfrac{K_4}{180} \cdot 6^4 \approx 0.0000531$ with $S_6 \approx 0.983347$.

51. **(a)** Left endpoint approximation $\approx \dfrac{b-a}{n}[y_0+y_1+\ldots+y_{n-2}+y_{n-1}]$. Right endpoint approximation $\approx \dfrac{b-a}{n}[y_1+y_2+\ldots+y_{n-1}+y_n]$. Average of the two $= \dfrac{b-a}{n}\dfrac{1}{2}[y_0+2y_1+2y_2+\ldots+2y_{n-2}+2y_{n-1}+y_n]$.

(b) Area of trapezoid $= (x_{k+1}-x_k)\dfrac{y_k+y_{k+1}}{2}$. If we sum from $k=0$ to $k=n-1$ then we get the right hand side of (2).



53. Given $g(x) = Ax^2 + Bx + C$, suppose $\Delta x = 1$ and $m = 0$. Then set $Y_0 = g(-1), Y_1 = g(0), Y_2 = g(1)$. Also $Y_0 = g(-1) = A - B + C, Y_1 = g(0) = C, Y_2 = g(1) = A + B + C$, with solution $C = Y_1, B = \frac{1}{2}(Y_2 - Y_0)$, and $A = \frac{1}{2}(Y_0+Y_2)-Y_1$. Then $\displaystyle\int_{-1}^{1} g(x)\,dx = 2\int_0^1 (Ax^2+C)\,dx = \frac{2}{3}A+2C = \frac{1}{3}(Y_0+Y_2)-\frac{2}{3}Y_1+2Y_1 = \frac{1}{3}(Y_0+4Y_1+Y_2)$, which is exactly what one gets applying the Simpson's Rule. The general case with the interval $(m-\Delta x, m+\Delta x)$ and values Y_0, Y_1, Y_2, can be converted by the change of variables $z = \dfrac{x-m}{\Delta x}$. Set $g(x) = h(z) = h((x-m)/\Delta x)$ to get $dx = \Delta x\, dz$ and $\displaystyle\Delta x \int_{m-\Delta x}^{m+\Delta x} h(z)\,dz = \int_{-1}^{1} g(x)\,dx$. Finally, $Y_0 = g(m-\Delta x) = h(-1), Y_1 = g(m) = h(0), Y_2 = g(m+\Delta x) = h(1)$.

Exercise Set 7.8

1. **(a)** Improper; infinite discontinuity at $x = 3$.

(b) Continuous integrand, not improper.

(c) Improper; infinite discontinuity at $x = 0$.

(d) Improper; infinite interval of integration.

(e) Improper; infinite interval of integration and infinite discontinuity at $x = 1$.

(f) Continuous integrand, not improper.

3. $\displaystyle\lim_{\ell\to+\infty}\left(-\frac{1}{2}e^{-2x}\right)\Big]_0^{\ell} = \frac{1}{2}\lim_{\ell\to+\infty}(-e^{-2\ell}+1) = \frac{1}{2}$.

5. $\displaystyle\lim_{\ell\to+\infty} -2\coth^{-1}x\Big]_3^{\ell} = \lim_{\ell\to+\infty}\left(2\coth^{-1}3 - 2\coth^{-1}\ell\right) = 2\coth^{-1}3$.

7. $\displaystyle\lim_{\ell\to+\infty} -\frac{1}{2\ln^2 x}\Big]_e^{\ell} = \lim_{\ell\to+\infty}\left[-\frac{1}{2\ln^2\ell}+\frac{1}{2}\right] = \frac{1}{2}$.

9. $\displaystyle\lim_{\ell\to-\infty} -\frac{1}{4(2x-1)^2}\Big]_{\ell}^{0} = \lim_{\ell\to-\infty}\frac{1}{4}[-1+1/(2\ell-1)^2] = -1/4$.

11. $\lim_{\ell\to-\infty} \frac{1}{3}e^{3x}\Big]_{\ell}^{0} = \lim_{\ell\to-\infty}\left[\frac{1}{3}-\frac{1}{3}e^{3\ell}\right] = \frac{1}{3}$.

13. $\int_{-\infty}^{+\infty} x\,dx$ converges if $\int_{-\infty}^{0} x\,dx$ and $\int_{0}^{+\infty} x\,dx$ both converge; it diverges if either (or both) diverges. $\int_{0}^{+\infty} x\,dx = \lim_{\ell\to+\infty}\frac{1}{2}x^2\Big]_0^{\ell} = \lim_{\ell\to+\infty}\frac{1}{2}\ell^2 = +\infty$, so $\int_{-\infty}^{+\infty} x\,dx$ is divergent.

15. $\int_0^{+\infty}\frac{x}{(x^2+3)^2}dx = \lim_{\ell\to+\infty} -\frac{1}{2(x^2+3)}\Big]_0^{\ell} = \lim_{\ell\to+\infty}\frac{1}{2}[-1/(\ell^2+3)+1/3] = \frac{1}{6}$, similarly $\int_{-\infty}^{0}\frac{x}{(x^2+3)^2}dx = -1/6$, so $\int_{-\infty}^{\infty}\frac{x}{(x^2+3)^2}dx = 1/6+(-1/6) = 0$.

17. $\lim_{\ell\to4^-} -\frac{1}{x-4}\Big]_0^{\ell} = \lim_{\ell\to4^-}\left[-\frac{1}{\ell-4}-\frac{1}{4}\right] = +\infty$, divergent.

19. $\lim_{\ell\to\pi/2^-} -\ln(\cos x)\Big]_0^{\ell} = \lim_{\ell\to\pi/2^-} -\ln(\cos\ell) = +\infty$, divergent.

21. $\lim_{\ell\to1^-}\sin^{-1}x\Big]_0^{\ell} = \lim_{\ell\to1^-}\sin^{-1}\ell = \pi/2$.

23. $\lim_{\ell\to\pi/3^+}\sqrt{1-2\cos x}\Big]_{\ell}^{\pi/2} = \lim_{\ell\to\pi/3^+}(1-\sqrt{1-2\cos\ell}) = 1$.

25. $\int_0^2\frac{dx}{x-2} = \lim_{\ell\to2^-}\ln|x-2|\Big]_0^{\ell} = \lim_{\ell\to2^-}(\ln|\ell-2|-\ln 2) = -\infty$, so $\int_0^3\frac{dx}{x-2}$ is divergent.

27. $\int_0^8 x^{-1/3}dx = \lim_{\ell\to0^+}\frac{3}{2}x^{2/3}\Big]_{\ell}^{8} = \lim_{\ell\to0^+}\frac{3}{2}(4-\ell^{2/3}) = 6$, $\int_{-1}^{0}x^{-1/3}dx = \lim_{\ell\to0^-}\frac{3}{2}x^{2/3}\Big]_{-1}^{\ell} = \lim_{\ell\to0^-}\frac{3}{2}(\ell^{2/3}-1) = -3/2$, so $\int_{-1}^{8}x^{-1/3}dx = 6+(-3/2) = 9/2$.

29. $\int_0^{+\infty}\frac{1}{x^2}dx = \int_0^a\frac{1}{x^2}dx + \int_a^{+\infty}\frac{1}{x^2}dx$ where $a>0$; take $a=1$ for convenience, $\int_0^1\frac{1}{x^2}dx = \lim_{\ell\to0^+}(-1/x)\Big]_{\ell}^{1} = \lim_{\ell\to0^+}(1/\ell-1) = +\infty$ so $\int_0^{+\infty}\frac{1}{x^2}dx$ is divergent.

31. Let $u=\sqrt{x}, x=u^2, dx=2u\,du$. Then $\int\frac{dx}{\sqrt{x}(x+1)} = \int 2\frac{du}{u^2+1} = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x}+C$ and $\int_0^1\frac{dx}{\sqrt{x}(x+1)} = 2\lim_{\epsilon\to0^+}\tan^{-1}\sqrt{x}\Big]_{\epsilon}^{1} = 2\lim_{\epsilon\to0^+}(\pi/4-\tan^{-1}\sqrt{\epsilon}) = \pi/2$.

33. True, Theorem 7.8.2.

35. False, neither 0 nor 3 lies in $[1,2]$, so the integrand is continuous.

37. $\int_0^{+\infty}\frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = 2\int_0^{+\infty}e^{-u}du = 2\lim_{\ell\to+\infty}\left(-e^{-u}\right)\Big]_0^{\ell} = 2\lim_{\ell\to+\infty}\left(1-e^{-\ell}\right) = 2$.

39. $\int_0^{+\infty}\frac{e^{-x}}{\sqrt{1-e^{-x}}}dx = \int_0^1\frac{du}{\sqrt{u}} = \lim_{\ell\to0^+}2\sqrt{u}\Big]_{\ell}^{1} = \lim_{\ell\to0^+}2(1-\sqrt{\ell}) = 2$.

41. $\lim_{\ell\to+\infty}\int_0^\ell e^{-x}\cos x\,dx = \lim_{\ell\to+\infty}\frac{1}{2}e^{-x}(\sin x-\cos x)\Big]_0^\ell = 1/2.$

43. **(a)** 2.726585 **(b)** 2.804364 **(c)** 0.219384 **(d)** 0.504067

45. $1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{4-x^{2/3}}{x^{2/3}} = \frac{4}{x^{2/3}}$; the arc length is $\int_0^8 \frac{2}{x^{1/3}}\,dx = 3x^{2/3}\Big]_0^8 = 12.$

47. $\int \ln x\,dx = x\ln x - x + C$, $\int_0^1 \ln x\,dx = \lim_{\ell\to 0^+}\int_\ell^1 \ln x\,dx = \lim_{\ell\to 0^+}(x\ln x - x)\Big]_\ell^1 = \lim_{\ell\to 0^+}(-1-\ell\ln\ell+\ell)$, but $\lim_{\ell\to 0^+}\ell\ln\ell = \lim_{\ell\to 0^+}\frac{\ln\ell}{1/\ell} = \lim_{\ell\to 0^+}(-\ell) = 0$, so $\int_0^1 \ln x\,dx = -1.$

49. $\int_0^\infty e^{-3x}\,dx = \lim_{\ell\to+\infty}\int_0^\ell e^{-3x}\,dx = \lim_{\ell\to+\infty} -\frac{1}{3}e^{-3x}\Big]_0^\ell = \frac{1}{3}.$

51. **(a)** $V=\pi\int_0^{+\infty} e^{-2x}dx = -\frac{\pi}{2}\lim_{\ell\to+\infty} e^{-2x}\Big]_0^\ell = \pi/2.$

(b) $S=\pi+2\pi\int_0^{+\infty} e^{-x}\sqrt{1+e^{-2x}}dx$, let $u=e^{-x}$ to get

$S=\pi-2\pi\int_1^0\sqrt{1+u^2}du = \pi+2\pi\left[\frac{u}{2}\sqrt{1+u^2}+\frac{1}{2}\ln\left|u+\sqrt{1+u^2}\right|\right]_0^1 = \pi+\pi\left[\sqrt{2}+\ln(1+\sqrt{2})\right].$

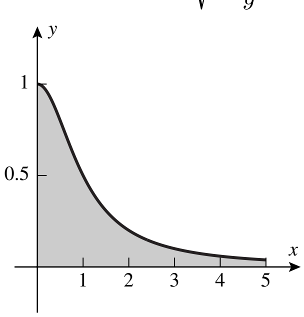
53. **(a)** For $x\geq 1, x^2\geq x, e^{-x^2}\leq e^{-x}$.

(b) $\int_1^{+\infty} e^{-x}\,dx = \lim_{\ell\to+\infty}\int_1^\ell e^{-x}\,dx = \lim_{\ell\to+\infty} -e^{-x}\Big]_1^\ell = \lim_{\ell\to+\infty}(e^{-1}-e^{-\ell}) = 1/e.$

(c) By parts (a) and (b) and Exercise 52(b), $\int_1^{+\infty} e^{-x^2}\,dx$ is convergent and is $\leq 1/e$.

55. $V=\lim_{\ell\to+\infty}\int_1^\ell(\pi/x^2)\,dx = \lim_{\ell\to+\infty} -(\pi/x)\Big]_1^\ell = \lim_{\ell\to+\infty}(\pi-\pi/\ell) = \pi$, $A=\pi+\lim_{\ell\to+\infty}\int_1^\ell 2\pi(1/x)\sqrt{1+1/x^4}\,dx$; use Exercise 52(a) with $f(x)=2\pi/x$, $g(x)=(2\pi/x)\sqrt{1+1/x^4}$ and $a=1$ to see that the area is infinite.

57. The area under the curve $y=\frac{1}{1+x^2}$, above the x-axis, and to the right of the y-axis is given by $\int_0^\infty \frac{1}{1+x^2}$. Solving for $x=\sqrt{\frac{1-y}{y}}$, the area is also given by the improper integral $\int_0^1\sqrt{\frac{1-y}{y}}\,dy$.



59. Let $x=r\tan\theta$ to get $\int\frac{dx}{(r^2+x^2)^{3/2}} = \frac{1}{r^2}\int\cos\theta\,d\theta = \frac{1}{r^2}\sin\theta + C = \frac{x}{r^2\sqrt{r^2+x^2}}+C$, so

$$u = \frac{2\pi NIr}{k} \lim_{\ell\to+\infty} \frac{x}{r^2\sqrt{r^2+x^2}}\Bigg]_a^\ell = \frac{2\pi NI}{kr} \lim_{\ell\to+\infty} (\ell/\sqrt{r^2+\ell^2} - a/\sqrt{r^2+a^2})] = \frac{2\pi NI}{kr}(1 - a/\sqrt{r^2+a^2}).$$

61. $\int_0^{+\infty} 5(e^{-0.2t} - e^{-t})\,dt = \lim_{\ell\to+\infty} -25e^{-0.2t} + 5e^{-t}\Big]_0^\ell = 20$; $\int_0^{+\infty} 4(e^{-0.2t} - e^{-3t})\,dt = \lim_{\ell\to+\infty} -20e^{-0.2t} + \frac{4}{3}e^{-3t}\Bigg]_0^\ell = \frac{56}{3}$, so Method 1 provides greater availability.

63. **(a)** Satellite's weight $= w(x) = k/x^2$ lb when $x =$ distance from center of Earth; $w(4000) = 6000$, so $k = 9.6\times 10^{10}$ and $W = \int_{4000}^{4000+b} 9.6\times 10^{10} x^{-2} dx$ mi·lb.

(b) $\int_{4000}^{+\infty} 9.6\times 10^{10} x^{-2} dx = \lim_{\ell\to+\infty} -9.6\times 10^{10}/x\Bigg]_{4000}^\ell = 2.4\times 10^7$ mi·lb.

65. **(a)** $\mathcal{L}\{f(t)\} = \int_0^{+\infty} te^{-st}\,dt = \lim_{\ell\to+\infty} -(t/s + 1/s^2)e^{-st}\Bigg]_0^\ell = \frac{1}{s^2}$.

(b) $\mathcal{L}\{f(t)\} = \int_0^{+\infty} t^2e^{-st}\,dt = \lim_{\ell\to+\infty} -(t^2/s + 2t/s^2 + 2/s^3)e^{-st}\Bigg]_0^\ell = \frac{2}{s^3}$.

(c) $\mathcal{L}\{f(t)\} = \int_3^{+\infty} e^{-st}dt = \lim_{\ell\to+\infty} -\frac{1}{s}e^{-st}\Bigg]_3^\ell = \frac{e^{-3s}}{s}$.

67. **(a)** $u = \sqrt{a}x, du = \sqrt{a}\,dx,\ 2\int_0^{+\infty} e^{-ax^2}\,dx = \frac{2}{\sqrt{a}}\int_0^{+\infty} e^{-u^2}\,du = \sqrt{\pi/a}$.

(b) $x = \sqrt{2}\sigma u, dx = \sqrt{2}\sigma\,du,\ \frac{2}{\sqrt{2\pi}\sigma}\int_0^{+\infty} e^{-x^2/2\sigma^2}\,dx = \frac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-u^2}\,du = 1$.

69. **(a)** $\int_0^4 \frac{1}{x^6+1}dx \approx 1.047$; $\pi/3 \approx 1.047$

(b) $\int_0^{+\infty} \frac{1}{x^6+1}dx = \int_0^4 \frac{1}{x^6+1}dx + \int_4^{+\infty} \frac{1}{x^6+1}dx$, so $E = \int_4^{+\infty} \frac{1}{x^6+1}dx < \int_4^{+\infty} \frac{1}{x^6}dx = \frac{1}{5(4)^5} < 2\times 10^{-4}$.

71. If $p = 1$, then $\int_0^1 \frac{dx}{x} = \lim_{\ell\to 0^+} \ln x\Bigg]_\ell^1 = +\infty$; if $p \neq 1$, then $\int_0^1 \frac{dx}{x^p} = \lim_{\ell\to 0^+} \frac{x^{1-p}}{1-p}\Bigg]_\ell^1 = \lim_{\ell\to 0^+}[(1-\ell^{1-p})/(1-p)] = \begin{cases} 1/(1-p), & p < 1 \\ +\infty, & p > 1 \end{cases}$.

73. $2\int_0^1 \cos(u^2)du \approx 1.809$.

Chapter 7 Review Exercises

1. $u = 4+9x, du = 9\,dx,\ \frac{1}{9}\int u^{1/2}\,du = \frac{2}{27}(4+9x)^{3/2} + C$.

3. $u = \cos\theta,\ -\int u^{1/2}du = -\frac{2}{3}\cos^{3/2}\theta + C$.

5. $u = \tan(x^2),\ \frac{1}{2}\int u^2du = \frac{1}{6}\tan^3(x^2) + C$.

7. **(a)** With $u = \sqrt{x}$: $\displaystyle\int \frac{1}{\sqrt{x}\sqrt{2-x}}\,dx = 2\int \frac{1}{\sqrt{2-u^2}}\,du = 2\sin^{-1}(u/\sqrt{2}) + C = 2\sin^{-1}(\sqrt{x/2}) + C$; with $u = \sqrt{2-x}$: $\displaystyle\int \frac{1}{\sqrt{x}\sqrt{2-x}}\,dx = -2\int \frac{1}{\sqrt{2-u^2}}\,du = -2\sin^{-1}(u/\sqrt{2}) + C = -2\sin^{-1}(\sqrt{2-x}/\sqrt{2}) + C_1$; completing the square: $\displaystyle\int \frac{1}{\sqrt{1-(x-1)^2}}\,dx = \sin^{-1}(x-1) + C$.

(b) In the three results in part (a) the antiderivatives differ by a constant, in particular $2\sin^{-1}(\sqrt{x/2}) = \pi - 2\sin^{-1}(\sqrt{2-x}/\sqrt{2}) = \pi/2 + \sin^{-1}(x-1)$.

9. $u = x$, $dv = e^{-x}dx$, $du = dx$, $v = -e^{-x}$; $\displaystyle\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C$.

11. $u = \ln(2x+3)$, $dv = dx$, $du = \dfrac{2}{2x+3}dx$, $v = x$; $\displaystyle\int \ln(2x+3)dx = x\ln(2x+3) - \int \frac{2x}{2x+3}dx$, but $\displaystyle\int \frac{2x}{2x+3}dx = \int \left(1 - \frac{3}{2x+3}\right)dx = x - \frac{3}{2}\ln(2x+3) + C_1$, so $\displaystyle\int \ln(2x+3)dx = x\ln(2x+3) - x + \frac{3}{2}\ln(2x+3) + C$.

13. Let I denote $\displaystyle\int 8x^4\cos 2x\,dx$. Then

diff.		antidiff.
$8x^4$		$\cos 2x$
	$\searrow +$	
$32x^3$		$\frac{1}{2}\sin 2x$
	$\searrow -$	
$96x^2$		$-\frac{1}{4}\cos 2x$
	$\searrow +$	
$192x$		$-\frac{1}{8}\sin 2x$
	$\searrow -$	
192		$\frac{1}{16}\cos 2x$
	$\searrow +$	
0		$\frac{1}{32}\sin 2x$

$I = \displaystyle\int 8x^4\cos 2x\,dx = (4x^4 - 12x^2 + 6)\sin 2x + (8x^3 - 12x)\cos 2x + C$.

15. $\displaystyle\int \sin^2 5\theta\,d\theta = \frac{1}{2}\int(1 - \cos 10\theta)d\theta = \frac{1}{2}\theta - \frac{1}{20}\sin 10\theta + C$.

17. $\displaystyle\int \sin x\cos 2x\,dx = \frac{1}{2}\int(\sin 3x - \sin x)dx = -\frac{1}{6}\cos 3x + \frac{1}{2}\cos x + C$.

19. $u = 2x$, $\displaystyle\int \sin^4 2x\,dx = \frac{1}{2}\int \sin^4 u\,du = \frac{1}{2}\left[-\frac{1}{4}\sin^3 u\cos u + \frac{3}{4}\int \sin^2 u\,du\right] = -\frac{1}{8}\sin^3 u\cos u + {}$ $\displaystyle+\frac{3}{8}\left[-\frac{1}{2}\sin u\cos u + \frac{1}{2}\int du\right] = -\frac{1}{8}\sin^3 u\cos u - \frac{3}{16}\sin u\cos u + \frac{3}{16}u + C = -\frac{1}{8}\sin^3 2x\cos 2x - \frac{3}{16}\sin 2x\cos 2x + \frac{3}{8}x + C$.

21. $x = 3\sin\theta$, $dx = 3\cos\theta\,d\theta$, $\displaystyle 9\int \sin^2\theta\,d\theta = \frac{9}{2}\int(1 - \cos 2\theta)d\theta = \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + C = \frac{9}{2}\theta - \frac{9}{2}\sin\theta\cos\theta + C$ $\displaystyle = \frac{9}{2}\sin^{-1}(x/3) - \frac{1}{2}x\sqrt{9-x^2} + C$.

23. $x = \sec\theta$, $dx = \sec\theta\tan\theta\, d\theta$, $\displaystyle\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|x + \sqrt{x^2-1}\right| + C$.

25. $x = 3\tan\theta$, $dx = 3\sec^2\theta\, d\theta$, $9\displaystyle\int \tan^2\theta\sec\theta\, d\theta = 9\int \sec^3\theta\, d\theta - 9\int \sec\theta\, d\theta = \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta + \tan\theta| + C$
$= \dfrac{1}{2}x\sqrt{9+x^2} - \dfrac{9}{2}\ln\left|\dfrac{1}{3}\sqrt{9+x^2} + \dfrac{1}{3}x\right| + C$.

27. $\dfrac{1}{(x+4)(x-1)} = \dfrac{A}{x+4} + \dfrac{B}{x-1}$; $A = -\dfrac{1}{5}$, $B = \dfrac{1}{5}$, so $-\dfrac{1}{5}\displaystyle\int \frac{1}{x+4}dx + \frac{1}{5}\int \frac{1}{x-1}dx = -\frac{1}{5}\ln|x+4| + \frac{1}{5}\ln|x-1| + C = \frac{1}{5}\ln\left|\frac{x-1}{x+4}\right| + C$.

29. $\dfrac{x^2+2}{x+2} = x - 2 + \dfrac{6}{x+2}$, $\displaystyle\int\left(x - 2 + \frac{6}{x+2}\right)dx = \frac{1}{2}x^2 - 2x + 6\ \ln|x+2| + C$.

31. $\dfrac{x^2}{(x+2)^3} = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2} + \dfrac{C}{(x+2)^3}$; $A = 1$, $B = -4$, $C = 4$, so $\displaystyle\int \frac{1}{x+2}\,dx - 4\int \frac{1}{(x+2)^2}\,dx + 4\int \frac{1}{(x+2)^3}\,dx = \ln|x+2| + \frac{4}{x+2} - \frac{2}{(x+2)^2} + C$.

33. **(a)** With $x = \sec\theta$: $\displaystyle\int \frac{1}{x^3 - x}dx = \int \cot\theta\ d\theta = \ln|\sin\theta| + C = \ln\frac{\sqrt{x^2-1}}{|x|} + C$; valid for $|x| > 1$.

(b) With $x = \sin\theta$: $\displaystyle\int \frac{1}{x^3-x}dx = -\int \frac{1}{\sin\theta\cos\theta}d\theta = -\int 2\csc 2\theta\ d\theta = -\ln|\csc 2\theta - \cot 2\theta| + C = \ln|\cot\theta| + C = \ln\frac{\sqrt{1-x^2}}{|x|} + C$, $0 < |x| < 1$.

(c) $\dfrac{1}{x^3-x} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+1} = -\dfrac{1}{x} + \dfrac{1}{2(x-1)} + \dfrac{1}{2(x+1)}$; $\displaystyle\int \frac{1}{x^3-x}\,dx = -\ln|x| + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C$, valid on any interval not containing the numbers $x = 0, \pm 1$.

35. Formula (40); $\dfrac{1}{4}\cos 2x - \dfrac{1}{32}\cos 16x + C$.

37. Formula (113); $\dfrac{1}{24}(8x^2 - 2x - 3)\sqrt{x - x^2} + \dfrac{1}{16}\sin^{-1}(2x-1) + C$.

39. Formula (28); $\dfrac{1}{2}\tan 2x - x + C$.

41. Exact value $= 4 - 2\sqrt{2} \approx 1.17157$.

(a) 1.17138, $|E_M| \approx 0.000190169$. **(b)** 1.17195, $|E_T| \approx 0.000380588$. **(c)** 1.17157, $|E_S| \approx 8.35 \times 10^{-8}$.

43. $f(x) = \dfrac{1}{\sqrt{x+1}}$, $f''(x) = \dfrac{3}{4(x+1)^{5/2}}$, $f^{(4)}(x) = \dfrac{105}{16(x+1)^{9/2}}(x+1)^{-7/2}$; $K_2 = \dfrac{3}{2^4\sqrt{2}}$, $K_4 = \dfrac{105}{2^8\sqrt{2}}$.

(a) $|E_M| \le \dfrac{2^3}{2400}\dfrac{3}{2^4\sqrt{2}} = \dfrac{1}{10^2 2^4\sqrt{2}} \approx 4.419417 \times 10^{-4}$.

(b) $|E_T| \le \dfrac{2^3}{1200}\dfrac{3}{2^4\sqrt{2}} = 8.838834 \times 10^{-4}$.

(c) $|E_S| \le \dfrac{2^5}{180 \times 20^4}\dfrac{105}{2^8\sqrt{2}} = \dfrac{7}{3 \cdot 10^4 \cdot 2^9\sqrt{2}} \approx 3.2224918 \times 10^{-7}$.

45. **(a)** $n^2 \geq 10^4 \dfrac{8\cdot 3}{24\times 2^4\sqrt{2}}$, so $n \geq \dfrac{10^2}{2^2 2^{1/4}} \approx 21.02, n \geq 22$.

(b) $n^2 \geq \dfrac{10^4}{2^3\sqrt{2}}$, so $n \geq \dfrac{10^2}{2\cdot 2^{3/4}} \approx 29.73, n \geq 30$.

(c) Let $n = 2k$, then want $\dfrac{2^5 K_4}{180(2k)^4} \leq 10^{-4}$, or $k^4 \geq 10^4 \dfrac{2^5}{180}\dfrac{105}{2^4\cdot 2^8\sqrt{2}} = 10^4\dfrac{7}{2^9\cdot 3\sqrt{2}}$, so $k \geq 10\left(\dfrac{7}{3\cdot 2^9\sqrt{2}}\right)^{1/4} \approx$ 2.38; so $k \geq 3$, $n \geq 6$

47. $\lim\limits_{\ell\to+\infty}(-e^{-x})\Big]_0^{\ell} = \lim\limits_{\ell\to+\infty}(-e^{-\ell}+1) = 1$.

49. $\lim\limits_{\ell\to 9^-} -2\sqrt{9-x}\Big]_0^{\ell} = \lim\limits_{\ell\to 9^-} 2(-\sqrt{9-\ell}+3) = 6$.

51. $A = \displaystyle\int_e^{+\infty}\frac{\ln x - 1}{x^2}\,dx = \lim_{\ell\to+\infty} c - \frac{\ln x}{x}\Big]_e^{\ell} = 1/e$.

53. $\displaystyle\int_0^{+\infty}\frac{dx}{x^2+a^2} = \lim_{\ell\to+\infty}\frac{1}{a}\tan^{-1}(x/a)\Big]_0^{\ell} = \lim_{\ell\to+\infty}\frac{1}{a}\tan^{-1}(\ell/a) = \frac{\pi}{2a} = 1, a = \pi/2$.

55. $x = \sqrt{3}\tan\theta$, $dx = \sqrt{3}\sec^2\theta\,d\theta$, $\dfrac{1}{3}\displaystyle\int\frac{1}{\sec\theta}d\theta = \frac{1}{3}\int\cos\theta\,d\theta = \frac{1}{3}\sin\theta + C = \frac{x}{3\sqrt{3+x^2}} + C$.

57. Use Endpaper Formula (31) to get $\displaystyle\int_0^{\pi/4}\tan^7\theta\,d\theta = \frac{1}{6}\tan^6\theta\Big]_0^{\pi/4} - \frac{1}{4}\tan^4\theta\Big]_0^{\pi/4} + \frac{1}{2}\tan^2\theta\Big]_0^{\pi/4} + \ln|\cos\theta|\Big]_0^{\pi/4} =$ $\dfrac{1}{6} - \dfrac{1}{4} + \dfrac{1}{2} - \ln\sqrt{2} = \dfrac{5}{12} - \ln\sqrt{2}$.

59. $\displaystyle\int\sin^2 2x\cos^3 2x\,dx = \int\sin^2 2x(1-\sin^2 2x)\cos 2x\,dx = \int(\sin^2 2x - \sin^4 2x)\cos 2x\,dx = \frac{1}{6}\sin^3 2x - \frac{1}{10}\sin^5 2x + C$.

61. $u = e^{2x}$, $dv = \cos 3x\,dx$, $du = 2e^{2x}dx$, $v = \dfrac{1}{3}\sin 3x$; $\displaystyle\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x\,dx$. Use $u = e^{2x}$, $dv = \sin 3x\,dx$ to get $\displaystyle\int e^{2x}\sin 3x\,dx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\,dx$, so $\displaystyle\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x +$ $\dfrac{2}{9}e^{2x}\cos 3x - \dfrac{4}{9}\displaystyle\int e^{2x}\cos 3x\,dx$, $\dfrac{13}{9}\displaystyle\int e^{2x}\cos 3x\,dx = \frac{1}{9}e^{2x}(3\sin 3x + 2\cos 3x) + C_1$, $\displaystyle\int e^{2x}\cos 3x\,dx = \frac{1}{13}e^{2x}(3\sin 3x +$ $2\cos 3x) + C$.

63. $\dfrac{1}{(x-1)(x+2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{x-3}$; $A = -\dfrac{1}{6}$, $B = \dfrac{1}{15}$, $C = \dfrac{1}{10}$, so $-\dfrac{1}{6}\displaystyle\int\frac{1}{x-1}dx + \frac{1}{15}\int\frac{1}{x+2}dx +$ $\dfrac{1}{10}\displaystyle\int\frac{1}{x-3}dx = -\frac{1}{6}\ln|x-1| + \frac{1}{15}\ln|x+2| + \frac{1}{10}\ln|x-3| + C$.

65. $u = \sqrt{x-4}$, $x = u^2+4$, $dx = 2u\,du$, $\displaystyle\int_0^2\frac{2u^2}{u^2+4}du = 2\int_0^2\left[1 - \frac{4}{u^2+4}\right]du = \left[2u - 4\tan^{-1}(u/2)\right]_0^2 = 4 - \pi$.

67. $u = \sqrt{e^x+1}$, $e^x = u^2-1$, $x = \ln(u^2-1)$, $dx = \dfrac{2u}{u^2-1}du$, $\displaystyle\int\frac{2}{u^2-1}du = \int\left[\frac{1}{u-1} - \frac{1}{u+1}\right]du = \ln|u-1| -$ $\ln|u+1| + C = \ln\dfrac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1} + C$.

69. $u = \sin^{-1} x$, $dv = dx$, $du = \dfrac{1}{\sqrt{1-x^2}}dx$, $v = x$; $\displaystyle\int_0^{1/2} \sin^{-1} x\,dx = x\sin^{-1} x\Big]_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1-x^2}}dx = \frac{1}{2}\sin^{-1}\frac{1}{2} + \sqrt{1-x^2}\Big]_0^{1/2} = \frac{1}{2}\left(\frac{\pi}{6}\right) + \sqrt{\frac{3}{4}} - 1 = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$

71. $\displaystyle\int \frac{x+3}{\sqrt{(x+1)^2+1}}dx$, let $u = x+1$, $\displaystyle\int \frac{u+2}{\sqrt{u^2+1}}du = \int \left[u(u^2+1)^{-1/2} + \frac{2}{\sqrt{u^2+1}}\right]du = \sqrt{u^2+1} + 2\sinh^{-1} u + C = \sqrt{x^2+2x+2} + 2\sinh^{-1}(x+1) + C.$

Alternate solution: let $x+1 = \tan\theta$, $\displaystyle\int (\tan\theta + 2)\sec\theta\,d\theta = \int \sec\theta\tan\theta\,d\theta + 2\int \sec\theta\,d\theta = \sec\theta + 2\ln|\sec\theta + \tan\theta| + C = \sqrt{x^2+2x+2} + 2\ln(\sqrt{x^2+2x+2} + x + 1) + C.$

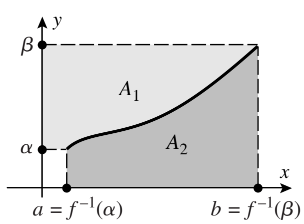
73. $\displaystyle\lim_{\ell\to+\infty} -\frac{1}{2(x^2+1)}\Big]_a^\ell = \lim_{\ell\to+\infty}\left[-\frac{1}{2(\ell^2+1)} + \frac{1}{2(a^2+1)}\right] = \frac{1}{2(a^2+1)}.$

Chapter 7 Making Connections

1. **(a)** $u = f(x), dv = dx, du = f'(x), v = x$; $\displaystyle\int_a^b f(x)\,dx = xf(x)\Big]_a^b - \int_a^b xf'(x)\,dx = bf(b) - af(a) - \int_a^b xf'(x)\,dx.$

(b) Substitute $y = f(x), dy = f'(x)\,dx, x = a$ when $y = f(a)$, $x = b$ when $y = f(b)$, $\displaystyle\int_a^b xf'(x)\,dx = \int_{f(a)}^{f(b)} x\,dy = \int_{f(a)}^{f(b)} f^{-1}(y)\,dy.$

(c) From $a = f^{-1}(\alpha)$ and $b = f^{-1}(\beta)$, we get $bf(b) - af(a) = \beta f^{-1}(\beta) - \alpha f^{-1}(\alpha)$; then $\displaystyle\int_\alpha^\beta f^{-1}(x)\,dx = \int_\alpha^\beta f^{-1}(y)\,dy = \int_{f(a)}^{f(b)} f^{-1}(y)\,dy$, which, by part (b), yields $\displaystyle\int_\alpha^\beta f^{-1}(x)\,dx = bf(b) - af(a) - \int_a^b f(x)\,dx = \beta f^{-1}(\beta) - \alpha f^{-1}(\alpha) - \int_{f^{-1}(\alpha)}^{f^{-1}(\beta)} f(x)\,dx$. Note from the figure that $A_1 = \displaystyle\int_\alpha^\beta f^{-1}(x)\,dx, A_2 = \int_{f^{-1}(\alpha)}^{f^{-1}(\beta)} f(x)\,dx$, and $A_1 + A_2 = \beta f^{-1}(\beta) - \alpha f^{-1}(\alpha)$, a "picture proof".



3. **(a)** $\Gamma(1) = \displaystyle\int_0^{+\infty} e^{-t}dt = \lim_{\ell\to+\infty} -e^{-t}\Big]_0^\ell = \lim_{\ell\to+\infty}(-e^{-\ell} + 1) = 1.$

(b) $\Gamma(x+1) = \displaystyle\int_0^{+\infty} t^x e^{-t}dt$; let $u = t^x$, $dv = e^{-t}dt$ to get $\Gamma(x+1) = -t^x e^{-t}\Big]_0^{+\infty} + x\displaystyle\int_0^{+\infty} t^{x-1}e^{-t}dt = -t^x e^{-t}\Big]_0^{+\infty} + x\Gamma(x)$, $\displaystyle\lim_{t\to+\infty} t^x e^{-t} = \lim_{t\to+\infty}\frac{t^x}{e^t} = 0$ (by multiple applications of L'Hôpital's rule), so $\Gamma(x+1) = x\Gamma(x)$.

(c) $\Gamma(2) = (1)\Gamma(1) = (1)(1) = 1$, $\Gamma(3) = 2\Gamma(2) = (2)(1) = 2$, $\Gamma(4) = 3\Gamma(3) = (3)(2) = 6$. Thus $\Gamma(n) = (n-1)!$ if n is a positive integer.

(d) $\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-1/2} e^{-t} dt = 2\int_0^{+\infty} e^{-u^2} du$ (with $u = \sqrt{t}$) $= 2(\sqrt{\pi}/2) = \sqrt{\pi}$.

(e) $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$, $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{4}\sqrt{\pi}$.

5. (a) $\sqrt{\cos\theta - \cos\theta_0} = \sqrt{2\left[\sin^2(\theta_0/2) - \sin^2(\theta/2)\right]} = \sqrt{2(k^2 - k^2\sin^2\phi)} = \sqrt{2k^2\cos^2\phi} = \sqrt{2}\,k\cos\phi$; $k\sin\phi = \sin(\theta/2)$, so $k\cos\phi\,d\phi = \frac{1}{2}\cos(\theta/2)\,d\theta = \frac{1}{2}\sqrt{1-\sin^2(\theta/2)}\,d\theta = \frac{1}{2}\sqrt{1-k^2\sin^2\phi}\,d\theta$, thus $d\theta = \frac{2k\cos\phi}{\sqrt{1-k^2\sin^2\phi}}\,d\phi$ and hence $T = \sqrt{\frac{8L}{g}}\int_0^{\pi/2} \frac{1}{\sqrt{2}k\cos\phi}\cdot\frac{2k\cos\phi}{\sqrt{1-k^2\sin^2\phi}}\,d\phi = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{1}{\sqrt{1-k^2\sin^2\phi}}\,d\phi$.

(b) If $L = 1.5$ ft and $\theta_0 = (\pi/180)(20) = \pi/9$, then $T = \frac{\sqrt{3}}{2}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-\sin^2(\pi/18)\sin^2\phi}} \approx 1.37$ s.