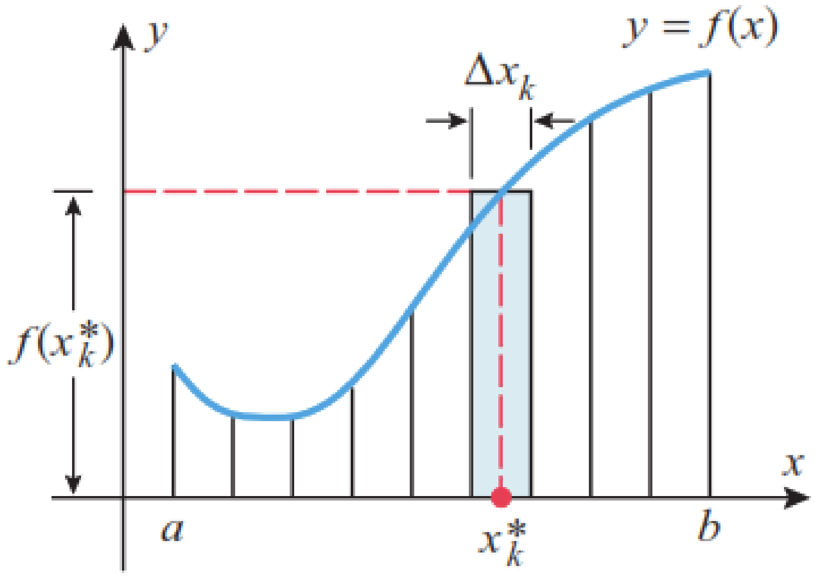
Ex#5.1

Applications of the Definite Integral in Geometry, Science, and Engineering

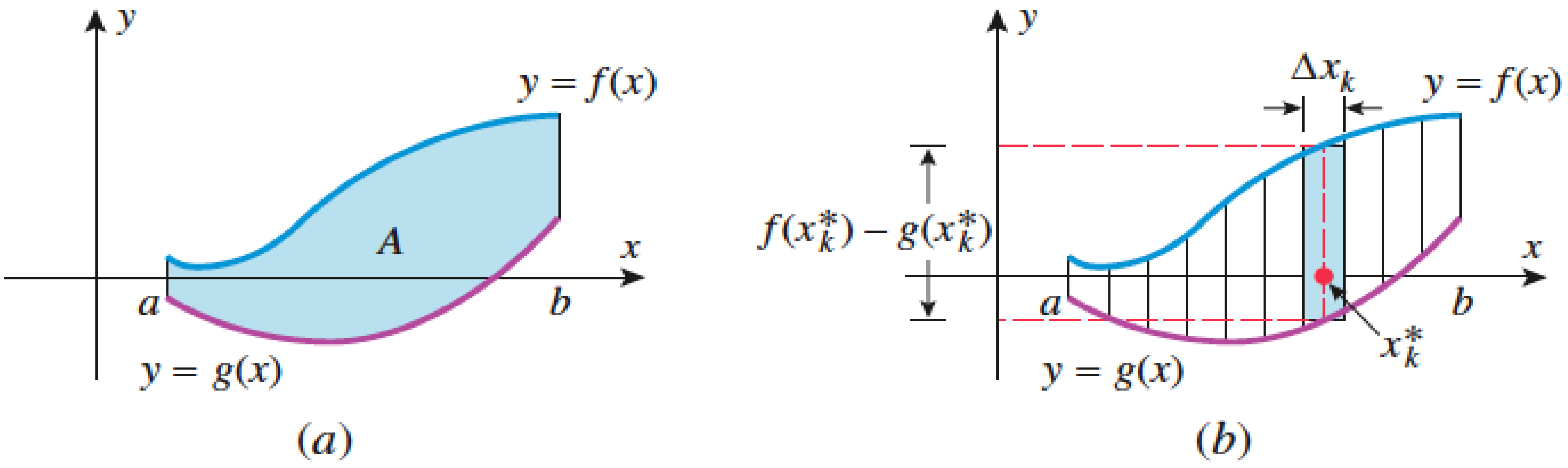
Area Bounded by the Curves

RECALL!



$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \int_a^b f(x)\,dx$$

- The quantity x_k^* in the Riemann sum becomes the variable x in the definite integral.
- The interval width Δx_k in the Riemann sum becomes the dx in the definite integral.
- The interval $[a, b]$, which is the union of the subintervals with widths $\Delta x_1, \Delta x_2, \ldots,$ Δx_n, does not appear explicitly in the Riemann sum but is represented by the upper and lower limits of integration in the definite integral.



6.1.1 FIRST AREA PROBLEM Suppose that f and g are continuous functions on an interval $[a, b]$ and

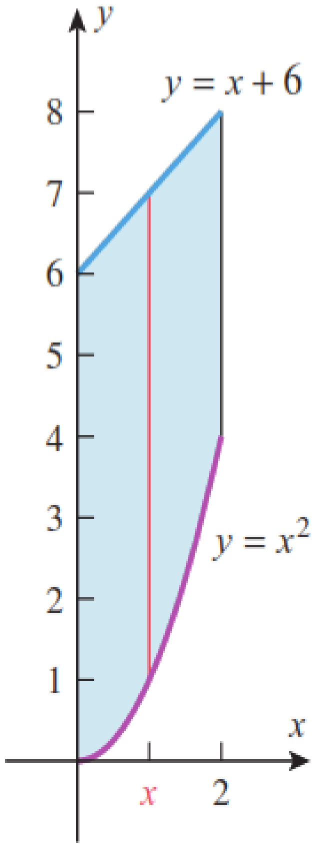
$$f(x) \geq g(x) \quad \text{for} \quad a \leq x \leq b$$

[This means that the curve $y = f(x)$ lies above the curve $y = g(x)$ and that the two can touch but not cross.] Find the area A of the region bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.1.3a).

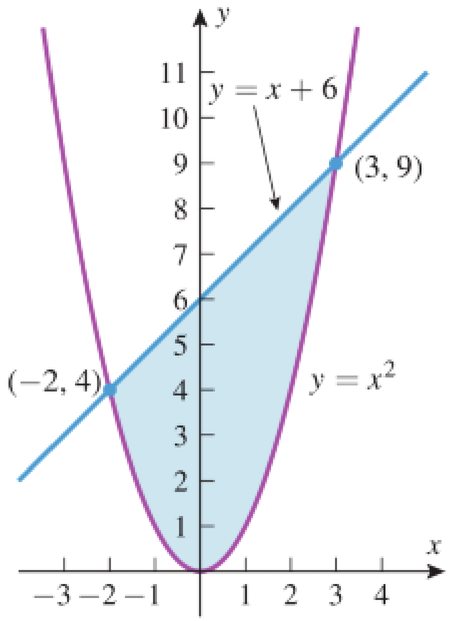
6.1.2 AREA FORMULA If f and g are continuous functions on the interval $[a, b]$, and if $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,dx \tag{1}$$

▶ **Example 1** Find the area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$, and bounded on the sides by the lines $x = 0$ and $x = 2$.



▸ **Example 2** Find the area of the region that is enclosed between the curves $y = x^2$ and $y = x + 6$.



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In the case where f and g are *nonnegative* on the interval $[a, b]$, the formula

$$A = \int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

states that the area A between the curves can be obtained by subtracting the area under $y = g(x)$ from the area under $y = f(x)$ (Figure 6.1.7).

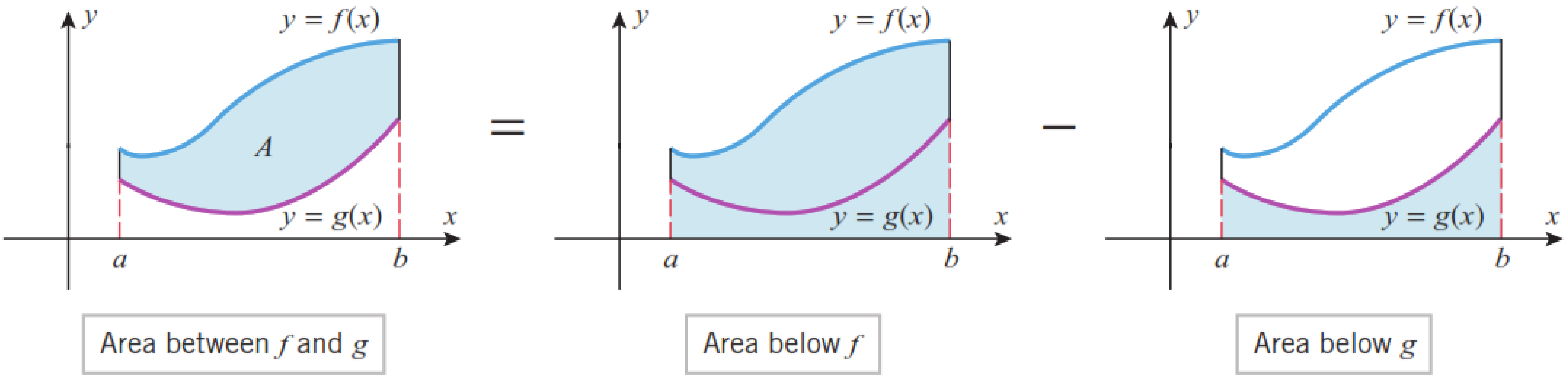


Figure 6.1.7

It is possible that the upper and lower boundaries of a region may intersect at one or both endpoints, in which case the sides of the region will be points, rather than vertical line segments (Figure 6.1.5). When that occurs you will have to determine the points of intersection to obtain the limits of integration.

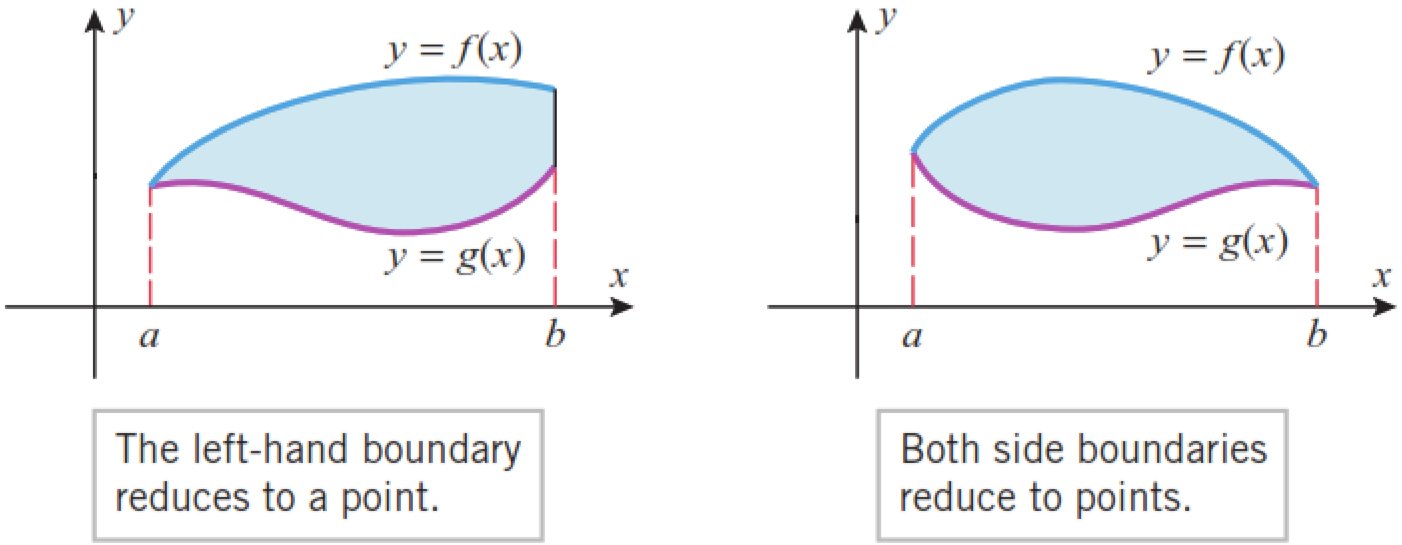
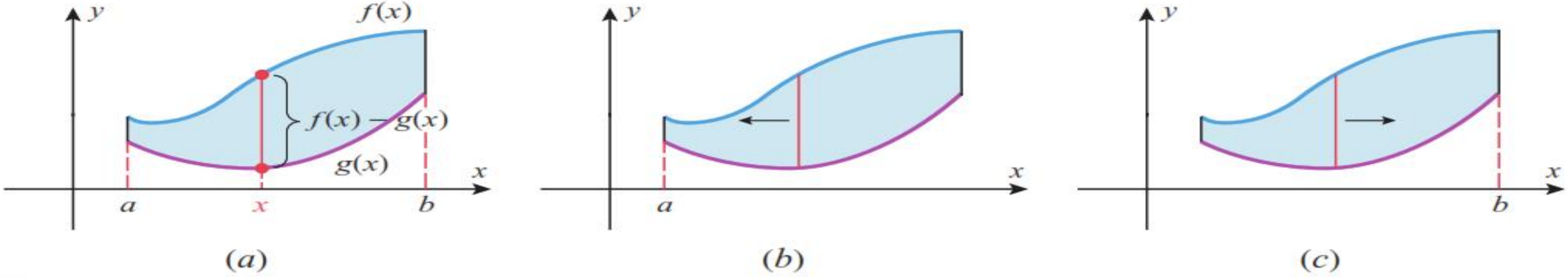


Figure 6.1.5



Finding the Limits of Integration for the Area Between Two Curves

Step 1. Sketch the region and then draw a vertical line segment through the region at an arbitrary point x on the x-axis, connecting the top and bottom boundaries (Figure 6.1.9a).

Step 2. The y-coordinate of the top endpoint of the line segment sketched in Step 1 will be $f(x)$, the bottom one $g(x)$, and the length of the line segment will be $f(x) - g(x)$. This is the integrand in (1).

Step 3. To determine the limits of integration, imagine moving the line segment left and then right. The leftmost position at which the line segment intersects the region is $x = a$ and the rightmost is $x = b$ (Figures 6.1.9b and 6.1.9c).

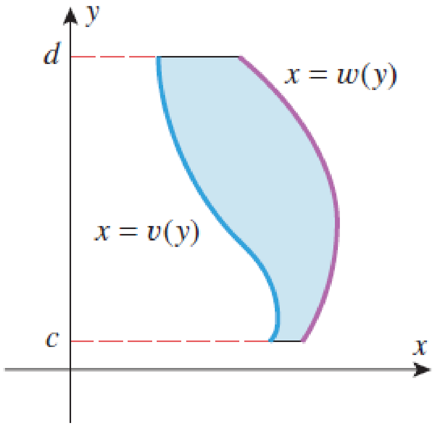
6.1.3 SECOND AREA PROBLEM Suppose that w and v are continuous functions of y on an interval $[c, d]$ and that

$$w(y) \geq v(y) \quad \text{for} \quad c \leq y \leq d$$

[This means that the curve $x = w(y)$ lies to the right of the curve $x = v(y)$ and that the two can touch but not cross.] Find the area A of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, and above and below by the lines $y = d$ and $y = c$ (Figure 6.1.11).

6.1.4 AREA FORMULA If w and v are continuous functions and if $w(y) \geq v(y)$ for all y in $[c, d]$, then the area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below by $y = c$, and above by $y = d$ is

$$A = \int_c^d [w(y) - v(y)]\, dy \tag{4}$$



▶ Example 4 Find the area of the region enclosed by $x = y^2$ and $y = x - 2$.

Example 4 Find the area of the region enclosed by $x = y^2$ and $y = x - 2$.

intersection are $(1, -1)$ and $(4, 2)$

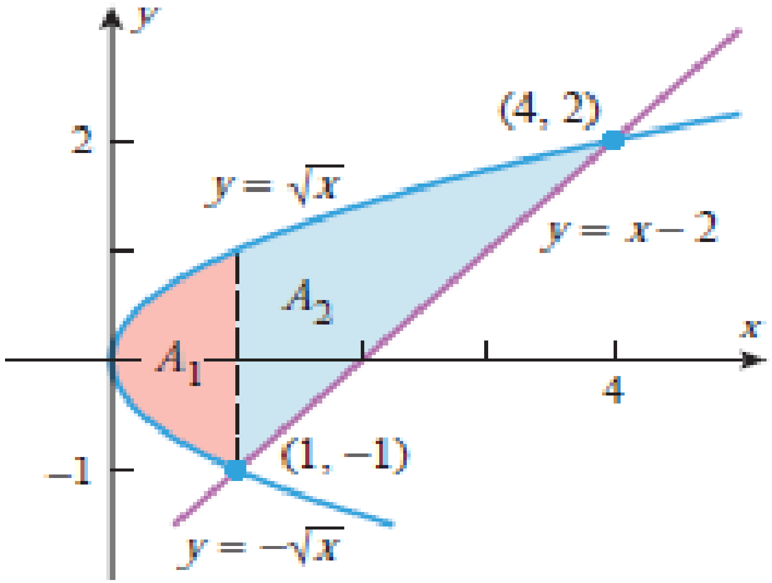
$$A_1 = \int_0^1 [\sqrt{x} - (-\sqrt{x})]\,dx$$

$$= 2\int_0^1 \sqrt{x}\,dx = 2\left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{4}{3} - 0 = \frac{4}{3}$$

$$A_2 = \int_1^4 [\sqrt{x} - (x-2)]\,dx = \int_1^4 (\sqrt{x} - x + 2)\,dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right]_1^4 = \frac{19}{6}$$

$$\boxed{A = A_1 + A_2 = \frac{4}{3} + \frac{19}{6} = \frac{9}{2}}$$



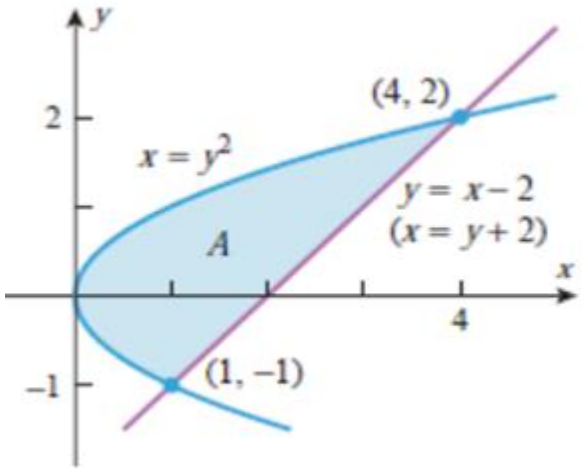
▸ **Example 5** Find the area of the region enclosed by $x = y^2$ and $y = x - 2$, integrating with respect to y.

$$x = y^2 \quad \text{and} \quad x = y + 2$$

$$y^2 = y + 2 \quad \text{or} \quad y^2 - y - 2 = 0 \quad \text{or} \quad (y+1)(y-2) = 0$$

$$A = \int_{-1}^{2} [(y+2) - y^2]\, dy$$

$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$



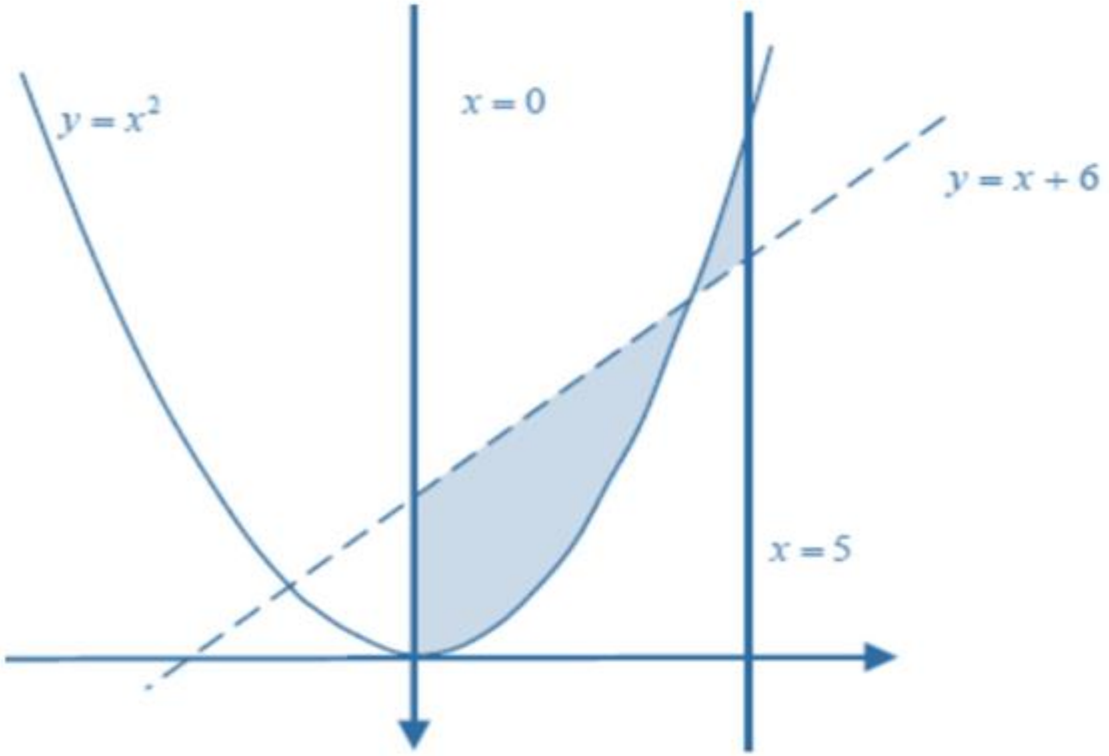
Find the area of the region enclosed by the following curves: $y = x^2$, $y = x + 6$, $x = 0$ and $x = 5$

$$A = \int_a^b [f(x) - g(x)]\,dx$$

A = A1 + A2

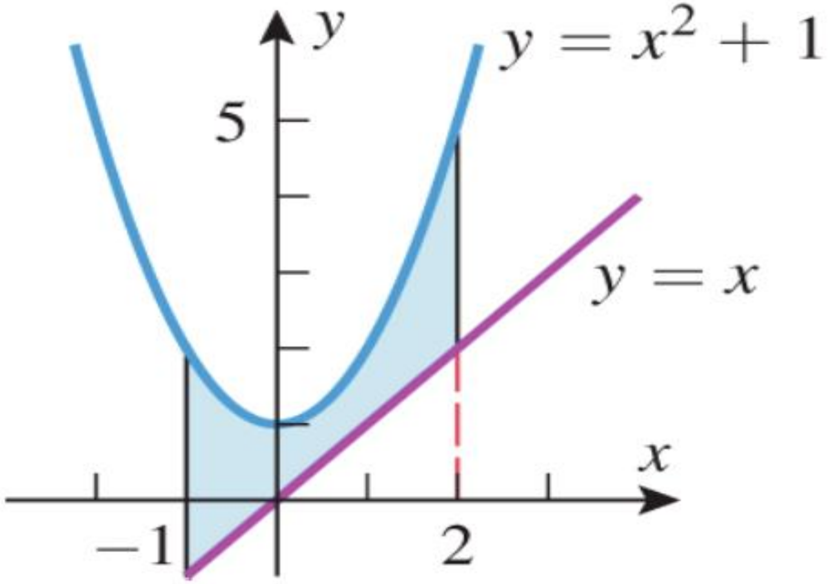
$$A = \int_0^3 \left(x + 6 - x^2\right)dx + \int_3^5 \left(x^2 - (x + 6)\right)dx$$

$= \dfrac{157}{6}$ square units.



1–4 Find the area of the shaded region.

1.



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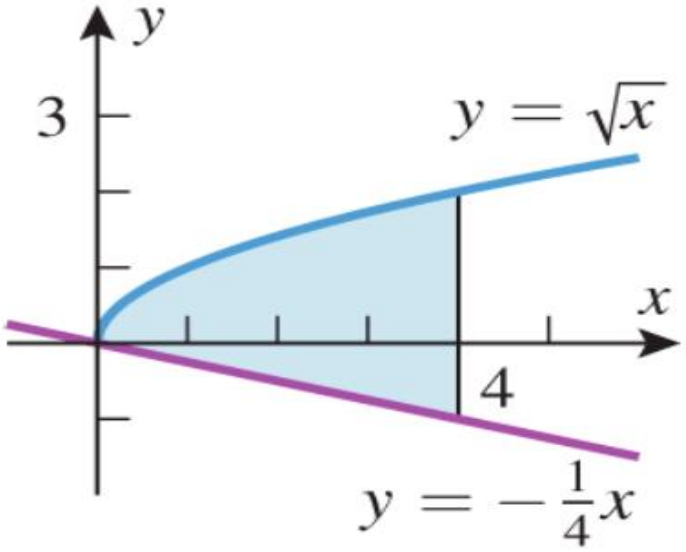

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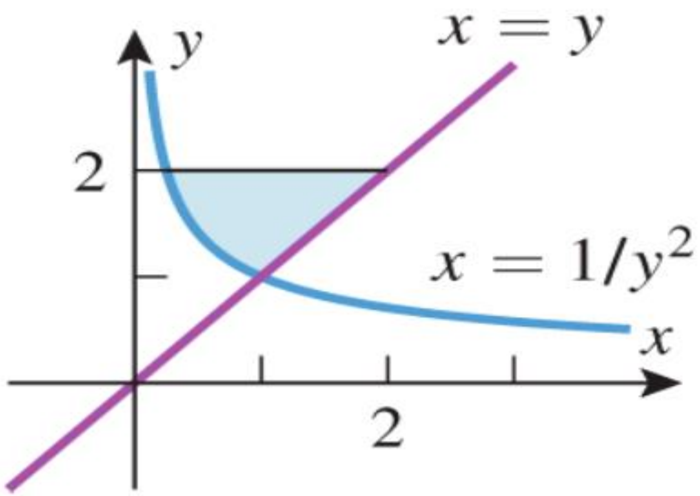

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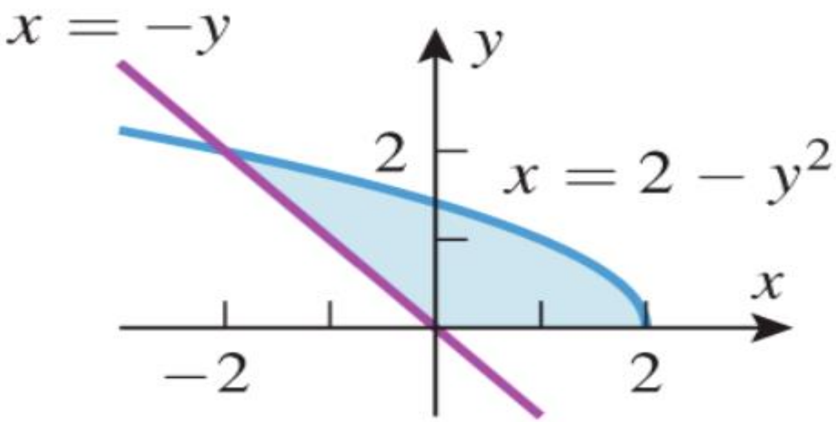
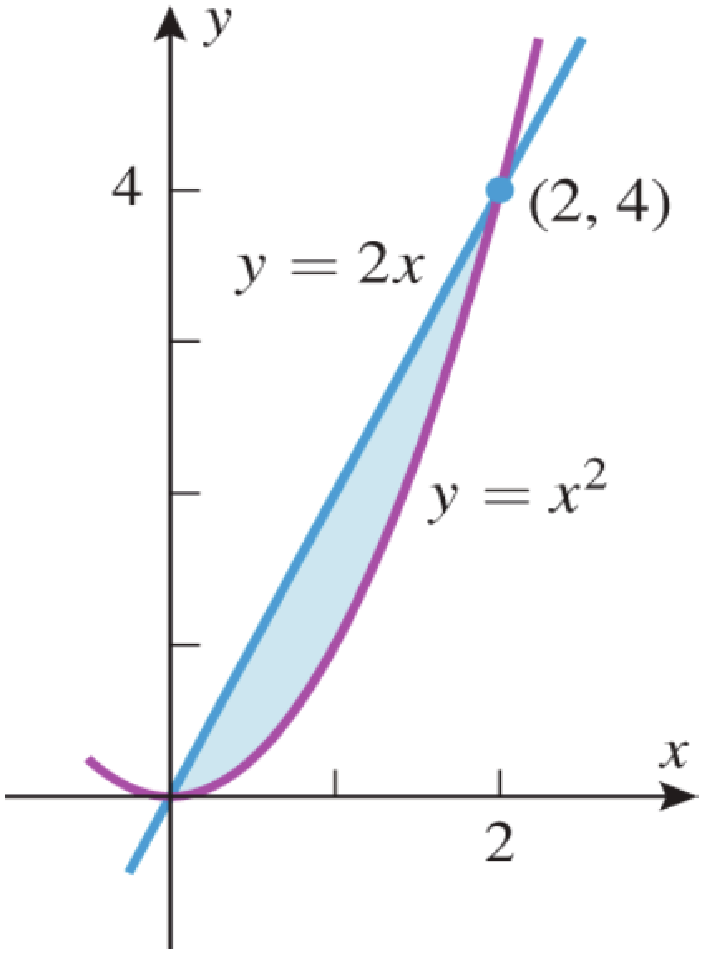
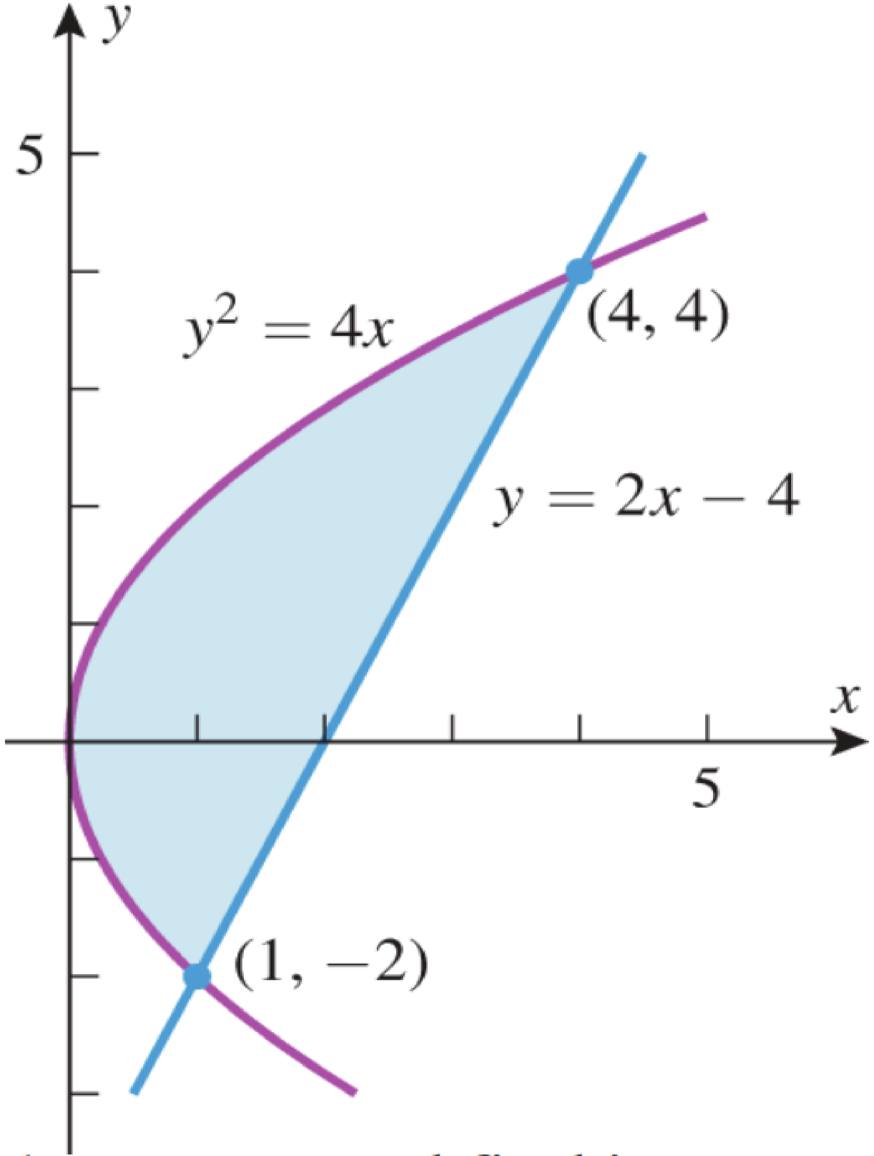


5–6 Find the area of the shaded region by (a) integrating with respect to x and (b) integrating with respect to y.

5.



6.



7–14 Sketch the region enclosed by the curves and find its area.

7. $y = x^2,\ y = \sqrt{x},\ x = \frac{1}{4},\ x = 1$

8. $y = x^3 - 4x,\ y = 0,\ x = 0,\ x = 2$

9. $y = \cos 2x,\ y = 0,\ x = \pi/4,\ x = \pi/2$

10. $y = \sec^2 x,\ y = 2,\ x = -\pi/4,\ x = \pi/4$

11. $x = \sin y,\ x = 0,\ y = \pi/4,\ y = 3\pi/4$

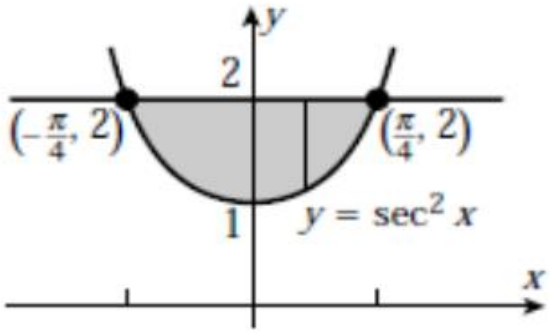
12. $x^2 = y,\ x = y - 2$

13. $y = 2 + |x - 1|,\ y = -\frac{1}{5}x + 7$

14. $y = x,\ y = 4x,\ y = -x + 2$

7–18 Sketch the region enclosed by the curves and find its area.

10. $y = \sec^2 x,\ \ y = 2,\ \ x = -\pi/4,\ \ x = \pi/4$

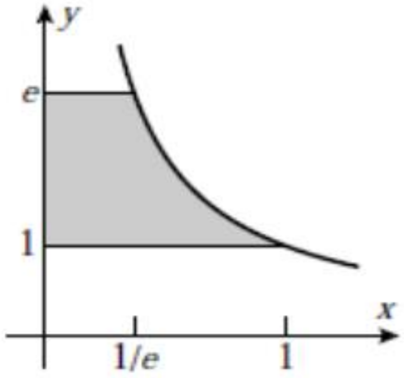


$$\sec^2 x = 2,$$

$$\sec x = \pm\sqrt{2},\ x = \pm\pi/4.$$

$$A = \int_{-\pi/4}^{\pi/4} (2 - \sec^2 x)\, dx = \pi - 2.$$

14. $x = 1/y,\ \ x = 0,\ \ y = 1,\ \ y = e$



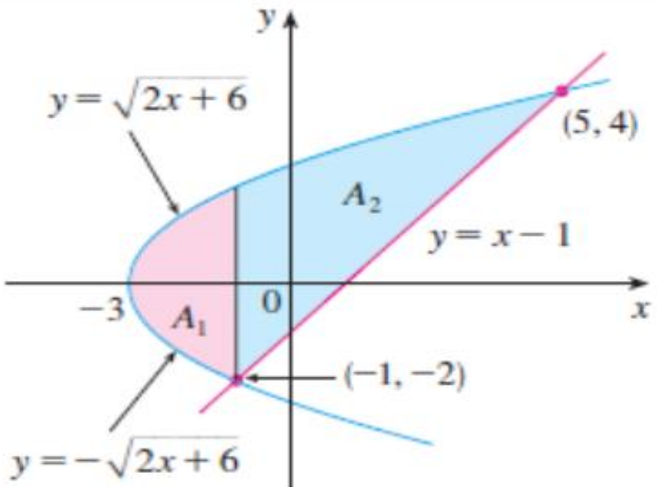
$$A = \int_1^e \frac{dy}{y} = \ln y\Big]_1^e = 1.$$

HOME ACTIVITY

EXAMPLE 2 Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

$$A = \int_0^1 (2x - 2x^2)\, dx$$

EXAMPLE 6 Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.



$$= \int_{-2}^{4} \left[(y + 1) - \left(\tfrac{1}{2}y^2 - 3\right)\right] dy$$

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Do Questions (1-14) from Ex # 5.1