Ex#7.8

Improper Integrals

Our main objective in this section is to extend the concept of a definite integral to allow for infinite intervals of integration and integrands with vertical asymptotes within the interval of integration. We will call the vertical asymptotes ***infinite discontinuities***, and we will call integrals with infinite intervals of integration or infinite discontinuities within the interval of integration ***improper integrals***. Here are some examples:

- Improper integrals with infinite intervals of integration:

$$\int_{1}^{+\infty} \frac{dx}{x^2}, \quad \int_{-\infty}^{0} e^x\,dx, \quad \int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$$

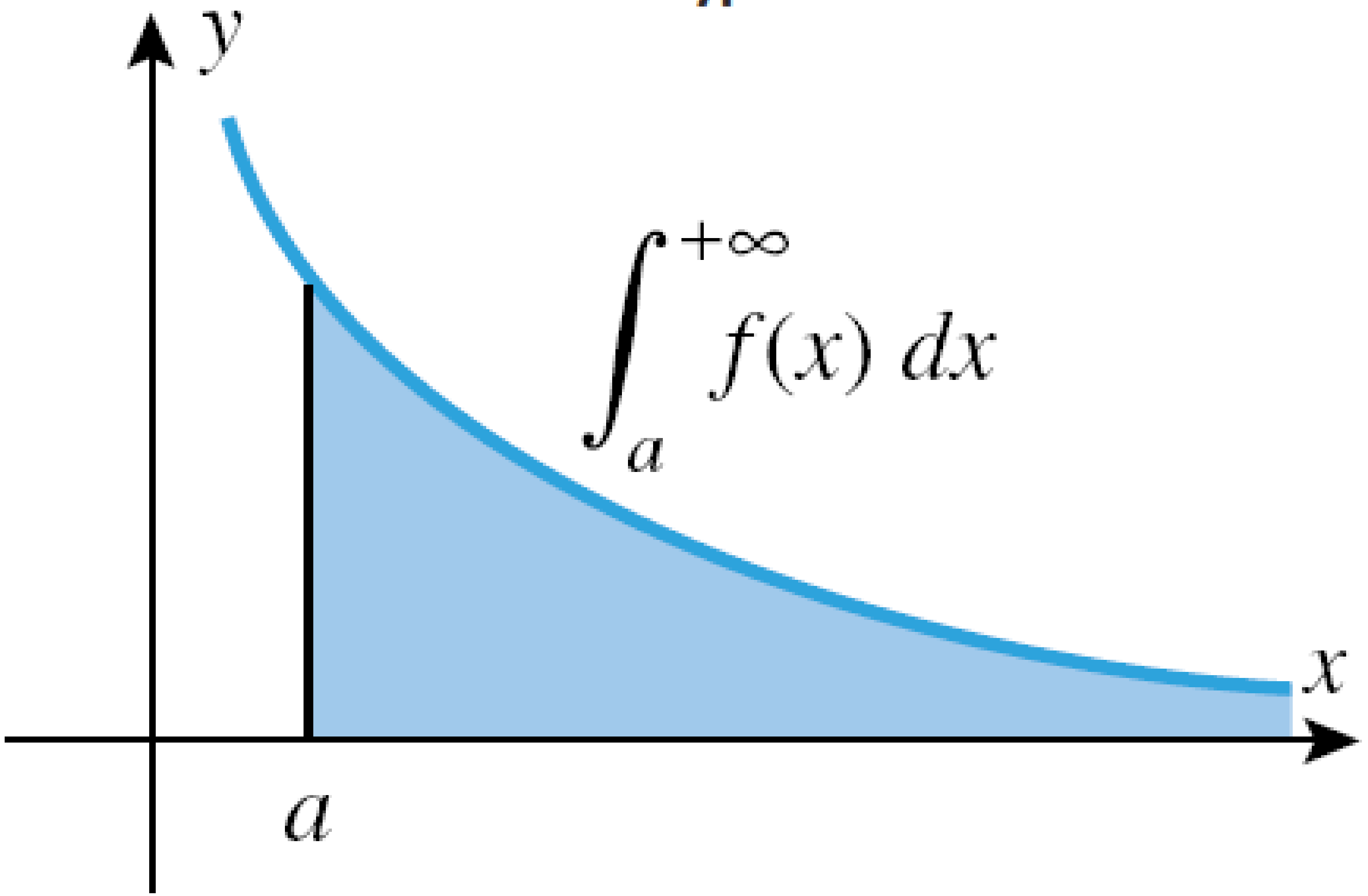
- Improper integrals with infinite discontinuities in the interval of integration:

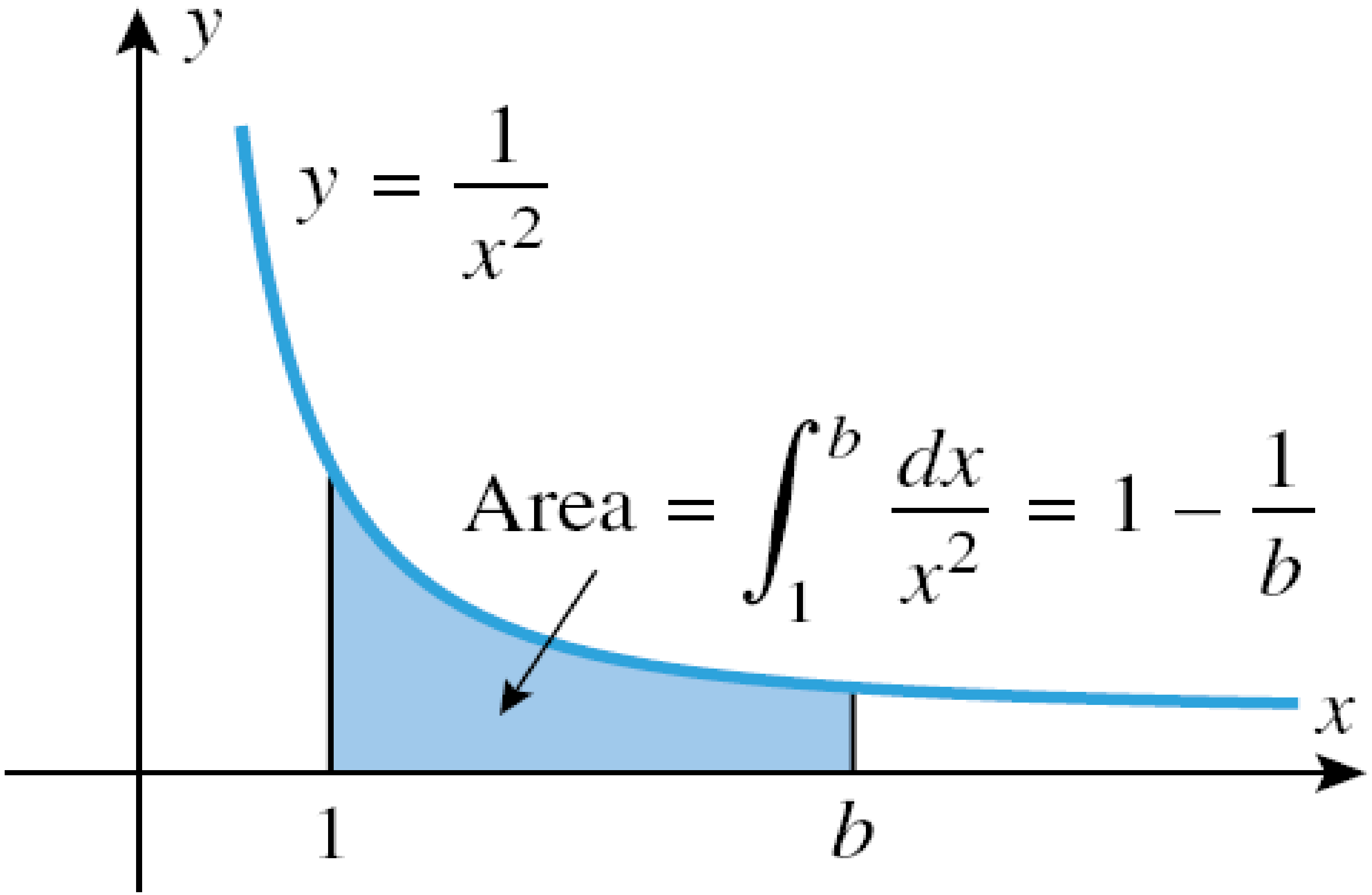
$$\int_{-3}^{3} \frac{dx}{x^2}, \quad \int_{1}^{2} \frac{dx}{x-1}, \quad \int_{0}^{\pi} \tan x\,dx$$

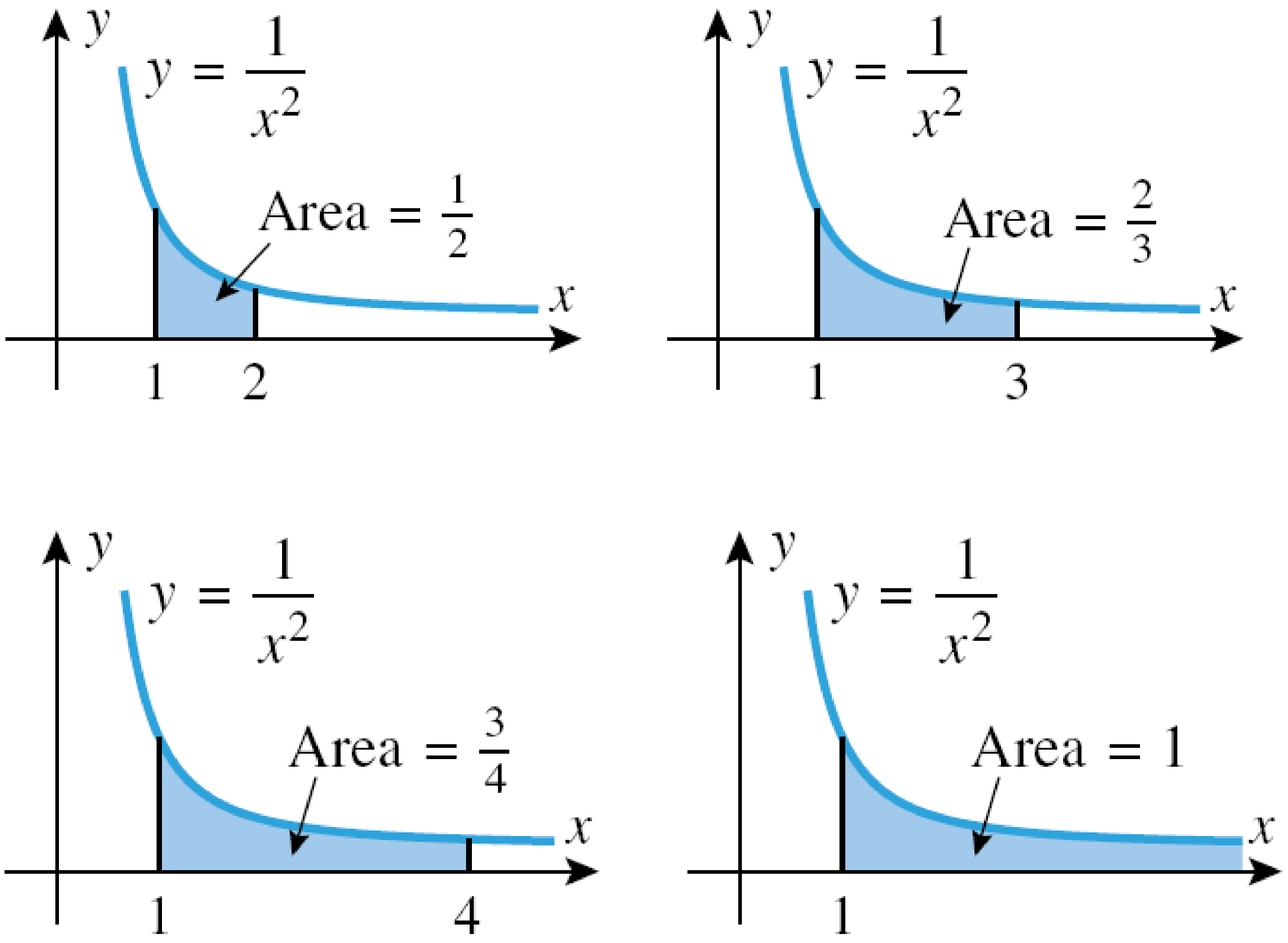
- Improper integrals with infinite discontinuities and infinite intervals of integration:

$$\int_{0}^{+\infty} \frac{dx}{\sqrt{x}}, \quad \int_{-\infty}^{+\infty} \frac{dx}{x^2-9}, \quad \int_{1}^{+\infty} \sec x\,dx$$

Type 1: Infinite Intervals




y
$y = \frac{1}{x^2}$
$\text{Area} = \int_1^b \frac{dx}{x^2} = 1 - \frac{1}{b}$
x
1
b


y
$y = \frac{1}{x^2}$
Area $= \frac{1}{2}$
x
1
2
y
$y = \frac{1}{x^2}$
Area $= \frac{2}{3}$
x
1
3
y
$y = \frac{1}{x^2}$
Area $= \frac{3}{4}$
x
1
4
y
$y = \frac{1}{x^2}$
Area = 1
x
1

7.8.1 DEFINITION The *improper integral of f over the interval* $[a, +\infty)$ is defined to be

$$\int_a^{+\infty} f(x)\,dx = \lim_{b \to +\infty} \int_a^b f(x)\,dx$$

In the case where the limit exists, the improper integral is said to *converge*, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to *diverge*, and it is not assigned a value.

▶ **Example 1** Evaluate

$$\text{(a)} \int_1^{+\infty} \frac{dx}{x^3} \qquad \text{(b)} \int_1^{+\infty} \frac{dx}{x}$$

Solution (a). Following the definition, we replace the infinite upper limit by a finite upper limit b, and then take the limit of the resulting integral. This yields

$$\int_1^{+\infty} \frac{dx}{x^3} = \lim_{b \to +\infty} \int_1^b \frac{dx}{x^3} = \lim_{b \to +\infty} \left[-\frac{1}{2x^2}\right]_1^b = \lim_{b \to +\infty} \left(\frac{1}{2} - \frac{1}{2b^2}\right) = \frac{1}{2}$$

Since the limit is finite, the integral converges and its value is $1/2$.

Solution (b).

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{b \to +\infty} \int_1^b \frac{dx}{x} = \lim_{b \to +\infty} [\ln x]_1^b = \lim_{b \to +\infty} \ln b = +\infty$$

In this case the integral diverges and hence has no value. ◀

Example 2 For what values of p does the integral $\int_1^{+\infty} \frac{dx}{x^p}$ converge?

Solution. We know from the preceding example that the integral diverges if $p = 1$, so let us assume that $p \neq 1$. In this case we have

$$\int_1^{+\infty} \frac{dx}{x^p} = \lim_{b \to +\infty} \int_1^b x^{-p}\, dx = \lim_{b \to +\infty} \frac{x^{1-p}}{1-p}\Big]_1^b = \lim_{b \to +\infty} \left[\frac{b^{1-p}}{1-p} - \frac{1}{1-p} \right]$$

If $p > 1$, then the exponent $1 - p$ is negative and $b^{1-p} \to 0$ as $b \to +\infty$; and if $p < 1$, then the exponent $1 - p$ is positive and $b^{1-p} \to +\infty$ as $b \to +\infty$. Thus, the integral converges if $p > 1$ and diverges otherwise. In the convergent case the value of the integral is

$$\int_1^{+\infty} \frac{dx}{x^p} = \left[0 - \frac{1}{1-p} \right] = \frac{1}{p-1} \quad (p > 1)$$

7.8.2 THEOREM

$$\int_1^{+\infty} \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1} & \textit{if} \quad p > 1 \\ \textit{diverges} & \textit{if} \quad p \leq 1 \end{cases}$$

Example 3 Evaluate $\int_0^{+\infty} (1-x)e^{-x}\,dx$.

Solution. We begin by evaluating the indefinite integral using integration by parts. Setting $u = 1 - x$ and $dv = e^{-x}\,dx$ yields

$$\int (1-x)e^{-x}\,dx = -e^{-x}(1-x) - \int e^{-x}\,dx = -e^{-x} + xe^{-x} + e^{-x} + C = xe^{-x} + C$$

Thus,

$$\int_0^{+\infty} (1-x)e^{-x}\,dx = \lim_{b\to+\infty} \int_0^b (1-x)e^{-x}\,dx = \lim_{b\to+\infty} \Big[xe^{-x}\Big]_0^b = \lim_{b\to+\infty} \frac{b}{e^b}$$

The limit is an indeterminate form of type ∞/∞, so we will apply L'Hôpital's rule by differentiating the numerator and denominator with respect to b. This yields

$$\int_0^{+\infty} (1-x)e^{-x}\,dx = \lim_{b\to+\infty} \frac{1}{e^b} = 0$$

We can interpret this to mean that the net signed area between the graph of $y = (1-x)e^{-x}$ and the interval $[0, +\infty)$ is 0 (Figure 7.8.5). ◄

7.8.3 DEFINITION The *improper integral of f over the interval* $(-\infty, b]$ is defined to be

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)\,dx \qquad (2)$$

The integral is said to ***converge*** if the limit exists and ***diverge*** if it does not.

The *improper integral of f over the interval* $(-\infty, +\infty)$ is defined as

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx \qquad (3)$$

where c is any real number. The improper integral is said to ***converge*** if *both* terms converge and ***diverge*** if *either* term diverges.

Example 4 Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$.

Solution. We will evaluate the integral by choosing $c = 0$ in (3). With this value for c we obtain

$$\int_{0}^{+\infty} \frac{dx}{1+x^2} = \lim_{b \to +\infty} \int_{0}^{b} \frac{dx}{1+x^2} = \lim_{b \to +\infty} \left[\tan^{-1} x\right]_0^b = \lim_{b \to +\infty} (\tan^{-1} b) = \frac{\pi}{2}$$

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{1+x^2} = \lim_{a \to -\infty} \left[\tan^{-1} x\right]_a^0 = \lim_{a \to -\infty} (-\tan^{-1} a) = \frac{\pi}{2}$$

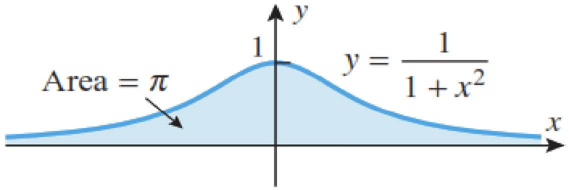


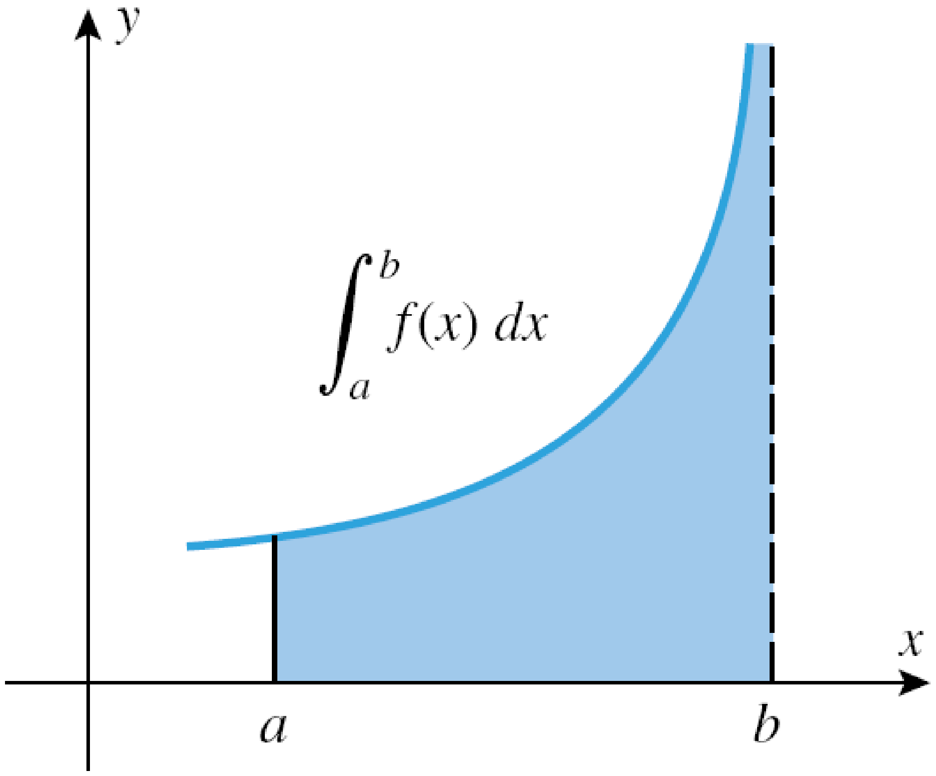
Figure 7.8.6

Thus, the integral converges and its value is

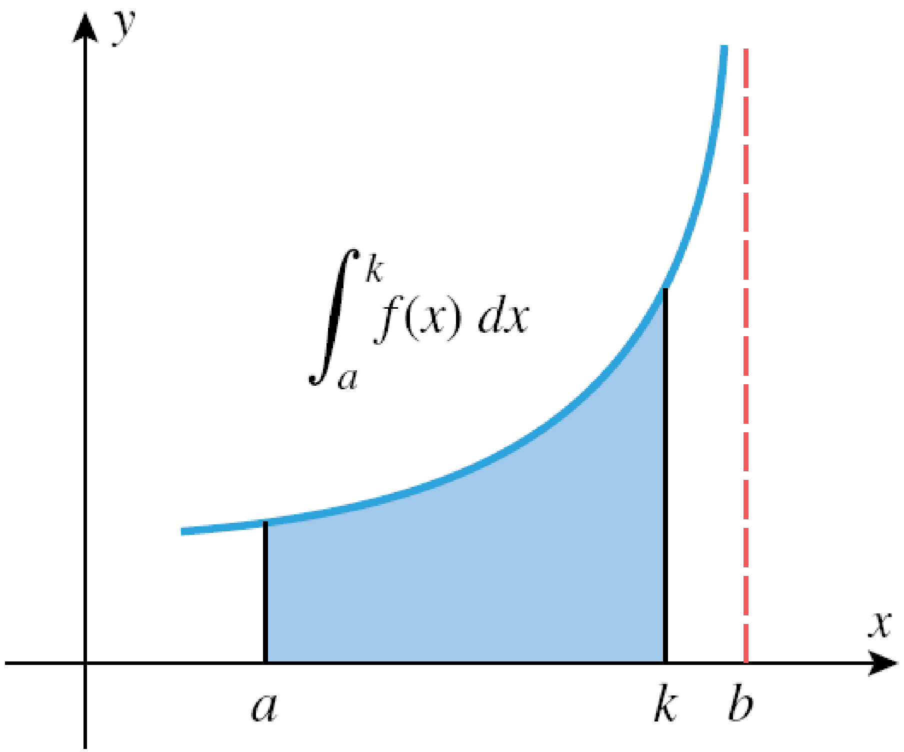
$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Since the integrand is nonnegative on the interval $(-\infty, +\infty)$, the integral represents the area of the region shown in Figure 7.8.6. ◀

INTEGRALS WHOSE INTEGRANDS HAVE INFINITE DISCONTINUITIES



(a)

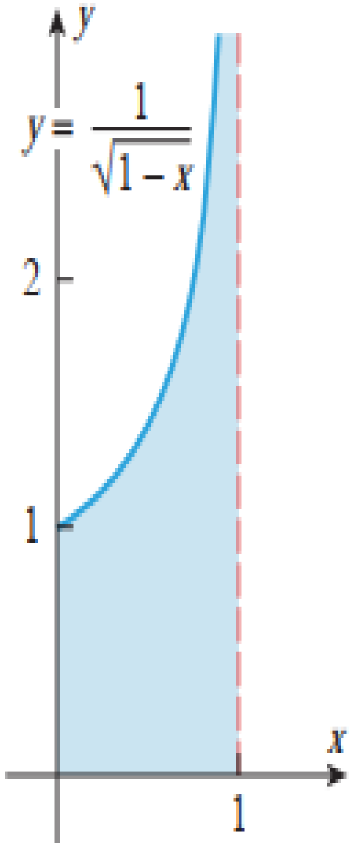


(b)

7.8.4 DEFINITION If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at b, then the ***improper integral of f over the interval*** $[a, b]$ is defined as

$$\int_a^b f(x)\,dx = \lim_{k \to b^-} \int_a^k f(x)\,dx \tag{4}$$

In the case where the indicated limit exists, the improper integral is said to ***converge***, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to ***diverge***, and it is not assigned a value.



▶ **Example 5** Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x}}$.

Solution. The integral is improper because the integrand approaches $+\infty$ as x approaches the upper limit 1 from the left (Figure 7.8.8). From (4),

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{k\to 1^-} \int_0^k \frac{dx}{\sqrt{1-x}} = \lim_{k\to 1^-} \left[-2\sqrt{1-x}\right]_0^k$$

$$= \lim_{k\to 1^-} \left[-2\sqrt{1-k}+2\right] = 2 \blacktriangleleft$$

7.8.5 DEFINITION If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a, then the ***improper integral of f over the interval*** $[a, b]$ is defined as

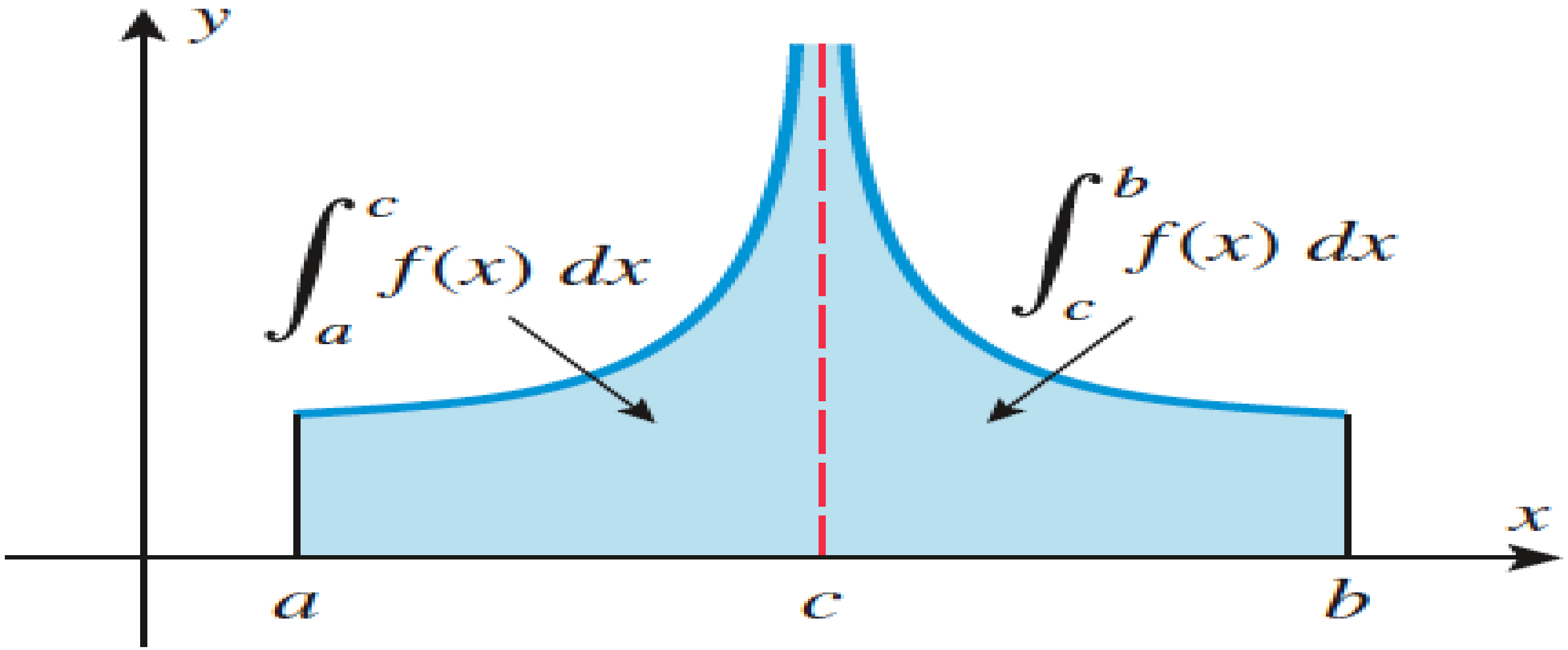
$$\int_a^b f(x)\,dx = \lim_{k \to a^+} \int_k^b f(x)\,dx \tag{5}$$

The integral is said to ***converge*** if the indicated limit exists and ***diverge*** if it does not.

If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a point c in (a, b), then the ***improper integral of f over the interval*** $[a, b]$ is defined as

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{6}$$

where the two integrals on the right side are themselves improper. The improper integral on the left side is said to ***converge*** if *both* terms on the right side converge and ***diverge*** if *either* term on the right side diverges (Figure 7.8.9).



$$\boxed{\int_a^b f(x)\,dx \text{ is improper.}}$$

Example 6 Evaluate

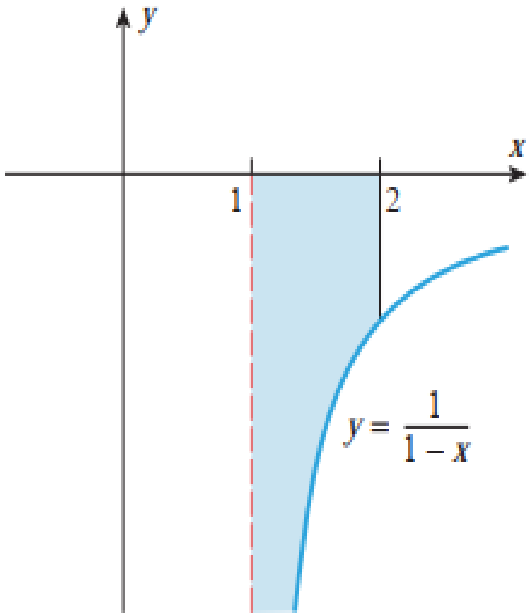
$$\text{(a)} \int_1^2 \frac{dx}{1-x} \qquad \text{(b)} \int_1^4 \frac{dx}{(x-2)^{2/3}}$$

Solution (a). The integral is improper because the integrand approaches $-\infty$ as x approaches the lower limit 1 from the right (Figure 7.8.10). From Definition 7.8.5 we obtain

$$\int_1^2 \frac{dx}{1-x} = \lim_{k \to 1^+} \int_k^2 \frac{dx}{1-x} = \lim_{k \to 1^+} \Big[-\ln|1-x|\Big]_k^2$$

$$= \lim_{k \to 1^+} \Big[-\ln|-1| + \ln|1-k|\Big] = \lim_{k \to 1^+} \ln|1-k| = -\infty$$

so the integral diverges.



Solution (b). The integral is improper because the integrand approaches $+\infty$ at $x = 2$, which is inside the interval of integration. From Definition 7.8.5 we obtain

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}} \tag{7}$$

and we must investigate the convergence of both improper integrals on the right. Since

$$\int_1^2 \frac{dx}{(x-2)^{2/3}} = \lim_{k \to 2^-} \int_1^k \frac{dx}{(x-2)^{2/3}} = \lim_{k \to 2^-} \left[3(k-2)^{1/3} - 3(1-2)^{1/3}\right] = 3$$

$$\int_2^4 \frac{dx}{(x-2)^{2/3}} = \lim_{k \to 2^+} \int_k^4 \frac{dx}{(x-2)^{2/3}} = \lim_{k \to 2^+} \left[3(4-2)^{1/3} - 3(k-2)^{1/3}\right] = 3\sqrt[3]{2}$$

we have from (7) that

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = 3 + 3\sqrt[3]{2}$$

3–32 Evaluate the integrals that converge.

3. $\displaystyle\int_0^{+\infty} e^{-2x}\,dx$

4. $\displaystyle\int_{-1}^{+\infty} \frac{x}{1+x^2}\,dx$

5. $\displaystyle\int_3^{+\infty} \frac{2}{x^2-1}\,dx$

6. $\displaystyle\int_0^{+\infty} xe^{-x^2}\,dx$

7. $\displaystyle\int_e^{+\infty} \frac{1}{x\ln^3 x}\,dx$

8. $\displaystyle\int_2^{+\infty} \frac{1}{x\sqrt{\ln x}}\,dx$

9. $\displaystyle\int_{-\infty}^{0} \frac{dx}{(2x-1)^3}$

10. $\displaystyle\int_{-\infty}^{3} \frac{dx}{x^2+9}$

11. $\displaystyle\int_{-\infty}^{0} e^{3x}\,dx$

12. $\displaystyle\int_{-\infty}^{0} \frac{e^x\,dx}{3-2e^x}$

13. $\displaystyle\int_{-\infty}^{+\infty} x\,dx$

14. $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{\sqrt{x^2+2}}\,dx$

15. $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{(x^2+3)^2}\,dx$

16. $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{-t}}{1+e^{-2t}}\,dt$

17. $\displaystyle\int_{0}^{4} \frac{dx}{(x-4)^2}$

18. $\displaystyle\int_{0}^{8} \frac{dx}{\sqrt[3]{x}}$

19. $\displaystyle\int_{0}^{\pi/2} \tan x\,dx$

20. $\displaystyle\int_{0}^{4} \frac{dx}{\sqrt{4-x}}$

21. $\displaystyle\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}$

22. $\displaystyle\int_{-3}^{1} \frac{x\,dx}{\sqrt{9-x^2}}$

23. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{\sin x}{\sqrt{1-2\cos x}}\,dx$

24. $\displaystyle\int_{0}^{\pi/4} \frac{\sec^2 x}{1-\tan x}\,dx$

25. $\displaystyle\int_{0}^{3} \frac{dx}{x-2}$

26. $\displaystyle\int_{-2}^{2} \frac{dx}{x^2}$

27. $\displaystyle\int_{-1}^{8} x^{-1/3}\,dx$

28. $\displaystyle\int_{0}^{1} \frac{dx}{(x-1)^{2/3}}$

29. $\displaystyle\int_{0}^{+\infty} \frac{1}{x^2}\,dx$

30. $\displaystyle\int_{1}^{+\infty} \frac{dx}{x\sqrt{x^2-1}}$

31. $\displaystyle\int_{0}^{1} \frac{dx}{\sqrt{x}(x+1)}$

32. $\displaystyle\int_{0}^{+\infty} \frac{dx}{\sqrt{x}(x+1)}$

Do Questions (3-32) from Ex # 7.8