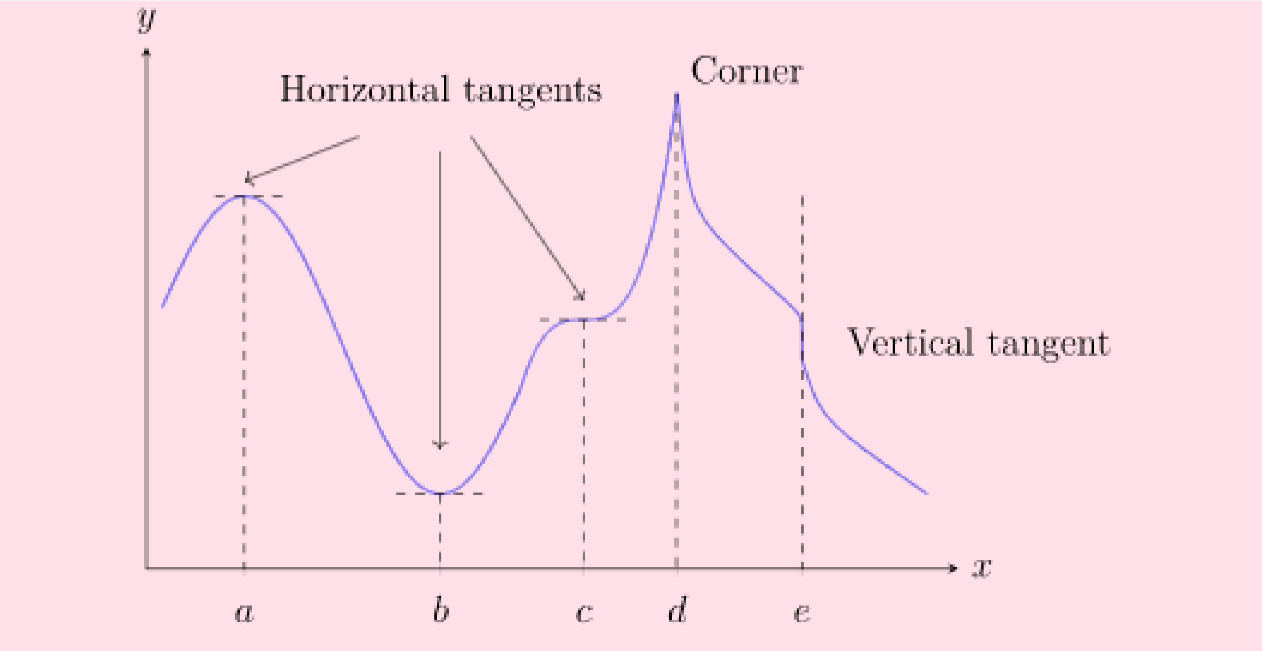
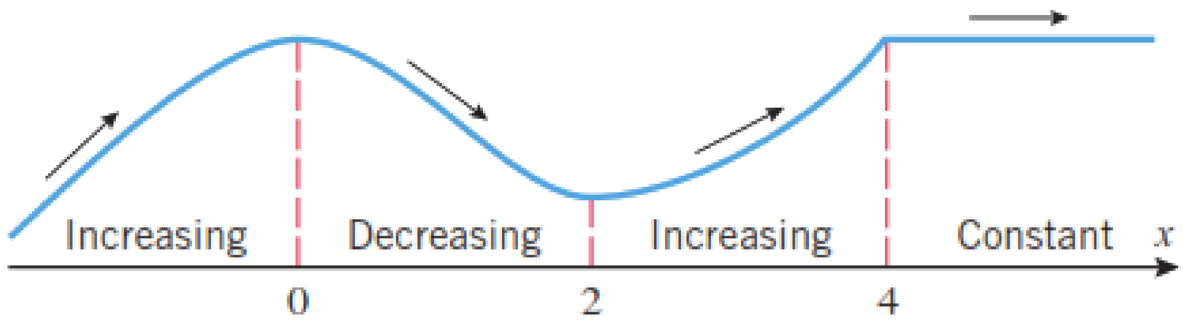
A point c in the domain of a function f(x) is called a **critical point** of f(x), if f ‘(c) = 0 or f ‘(c) does not exist.
This article explains the critical points along with solved examples.
A function f, which is continuous with x in its domain, contains a critical point at point x if the following conditions hold good.
•f ’(x) = 0
•f ’(x) is undefined.
A point of a differentiable function f at which the derivative is zero can be termed a critical point.
The types of critical points are as follows:
•A critical point is a local maximum if the function changes from increasing to decreasing at that point, whereas it is called a local minimum if the function changes from decreasing to increasing at that point.
•A critical point is an inflexion point if the concavity of the function changes at that point.
•If a critical point is neither of the above, then it signifies a vertical tangent in the graph of a function.



Concavity, Increasing and Decreasing.



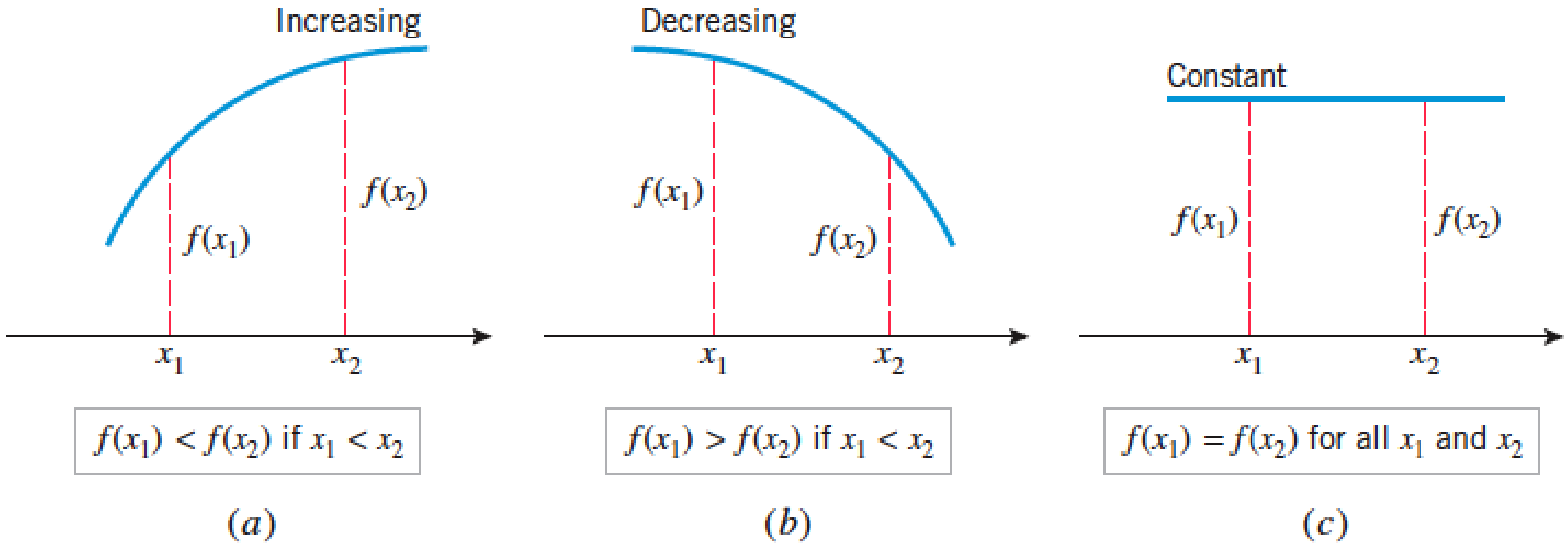
The definitions of "increasing," "decreasing," and "constant" describe the behavior of a function on an *interval* and not at a point. In particular, it is not inconsistent to say that the function in Figure 4.1.1 is decreasing on the interval $[0, 2]$ and increasing on the interval $[2, 4]$.

4.1.1 DEFINITION Let f be defined on an interval, and let x_1 and x_2 denote points in that interval.

(a) f is ***increasing*** on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

(b) f is ***decreasing*** on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

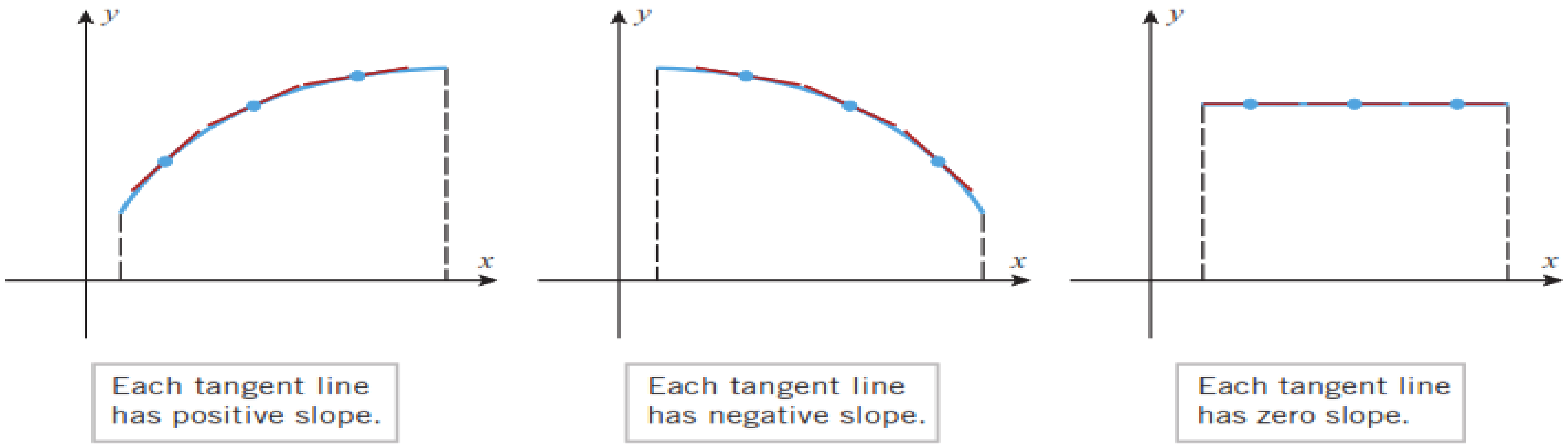
(c) f is ***constant*** on the interval if $f(x_1) = f(x_2)$ for all points x_1 and x_2.



Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.



4.1.2 THEOREM *Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).*

(a) *If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.*

(b) *If $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on $[a, b]$.*

(c) *If $f'(x) = 0$ for every value of x in (a, b), then f is constant on $[a, b]$.*

V EXAMPLE 1 Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Solution:

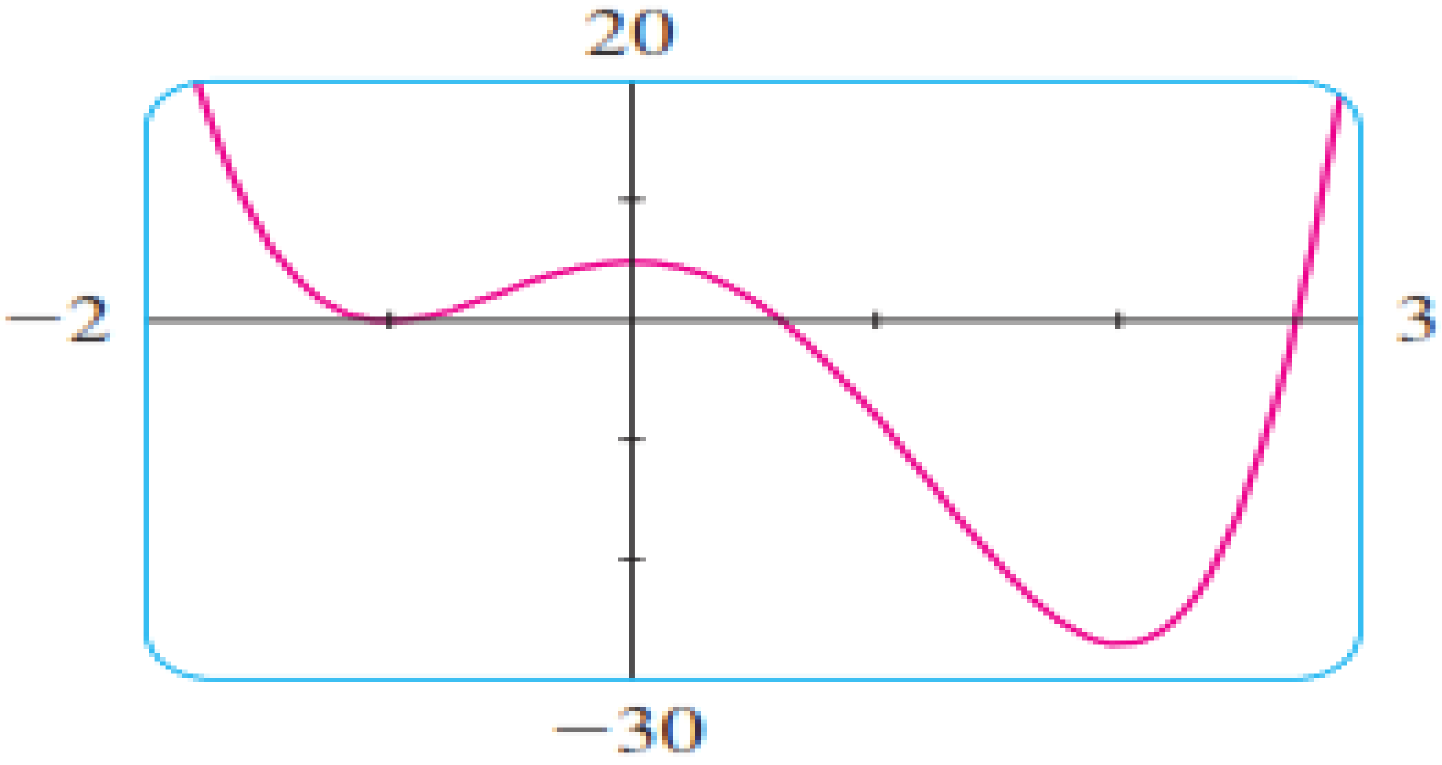
$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$f'(x) = 12x(x^2 - x - 2)$$
$$12x(x^2 - x - 2)=0$$
$$12x = 0 \ \ and \ x^2 - x - 2 = 0$$
$$x = 0 \ and \ x = 2 \ , \mathrm{x} = -1$$

Intervals:

$(-\infty, -1), (-1,0), (0,2) \ and \ (2, \infty)$

Intervals	sing of $f'(x)$	
$(-\infty, -1)$	-ve	Decreasing
$(-1,0)$	+ve	Increasing
$(0,2)$	-ve	Decreasing
$(2, \infty)$	+ve	Increasing

V **EXAMPLE 1** Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.



▶ **Example 2** Find the intervals on which $f(x) = x^3$ is increasing and the intervals on which it is decreasing.

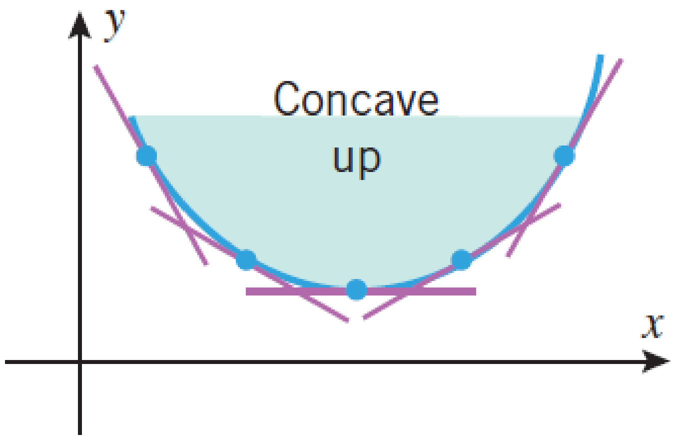
$$f'(x) = 3x^2$$
$$3x^2 = 0$$
$$x = 0$$

Intervals $(-\infty, 0)$ and $(0, \infty)$

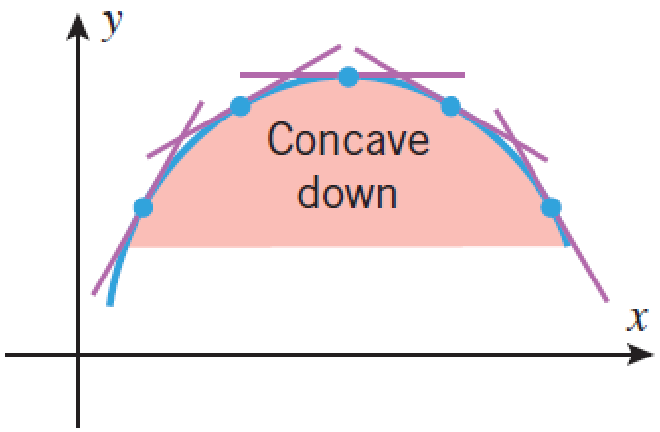
Intervals	sign of $f'(x)$	
$(-\infty, 0)$	+ve	$f(x)$ $is\ incressing$
$(0, \infty)$	+ve	$f(x)$ $is\ incressing$

$$f(x)\ is\ incressing\ \ in\ (-\infty, \infty)$$

4.1.3 DEFINITION If f is differentiable on an open interval, then f is said to be *concave up* on the open interval if f' is increasing on that interval, and f is said to be *concave down* on the open interval if f' is decreasing on that interval.



Increasing slopes



Decreasing slopes

Definition If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

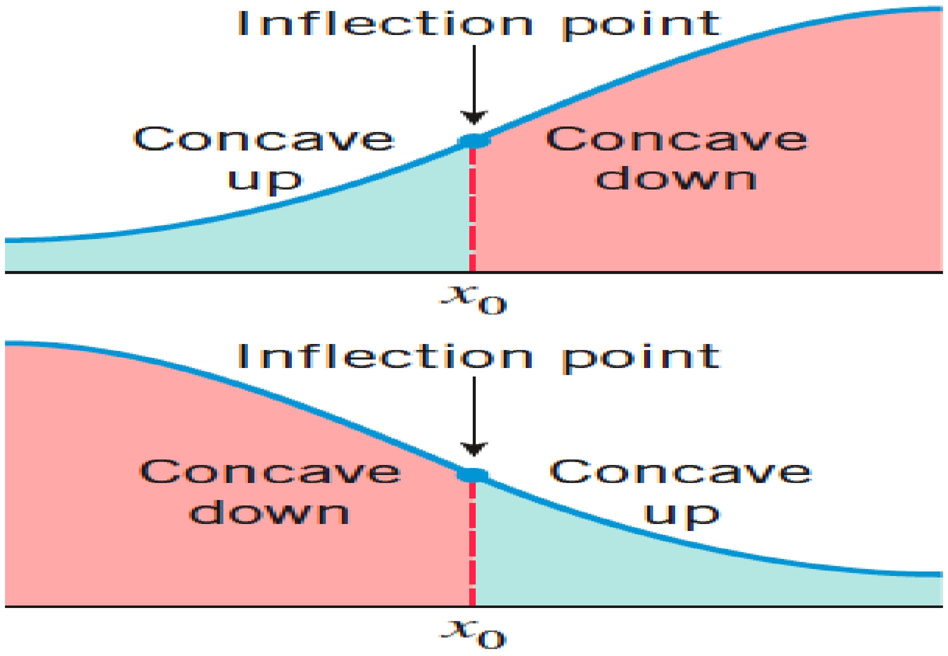
Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

4.1.5 DEFINITION If f is continuous on an open interval containing a value x_0, and if f changes the direction of its concavity at the point $(x_0, f(x_0))$, then we say that f has an ***inflection point at*** $\boldsymbol{x_0}$, and we call the point $(x_0, f(x_0))$ on the graph of f an ***inflection point*** of f (Figure 4.1.9).



▶ **Example 5** Figure 4.1.10 shows the graph of the function $f(x) = x^3 - 3x^2 + 1$. Use the first and second derivatives of f to determine the intervals on which f is increasing, decreasing, concave up, and concave down. Locate all inflection points and confirm that your conclusions are consistent with the graph.

Solution. Calculating the first two derivatives of f we obtain

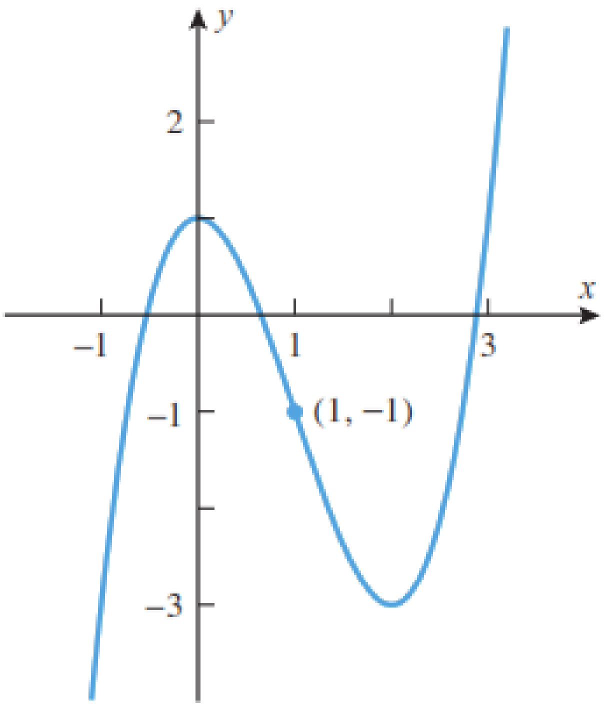
$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

$$f''(x) = 6x - 6 = 6(x-1)$$

The sign analysis of these derivatives is shown in the following tables:

INTERVAL	$(3x)(x-2)$	$f'(x)$	CONCLUSION
$x < 0$	$(-)(-)$	$+$	f is increasing on $(-\infty, 0]$
$0 < x < 2$	$(+)(-)$	$-$	f is decreasing on $[0, 2]$
$x > 2$	$(+)(+)$	$+$	f is increasing on $[2, +\infty)$

INTERVAL	$6(x-1)$	$f''(x)$	CONCLUSION
$x < 1$	$(-)$	$-$	f is concave down on $(-\infty, 1)$
$x > 1$	$(+)$	$+$	f is concave up on $(1, +\infty)$



$f(x) = x^3 - 3x^2 + 1$

15–32 Find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points.

15. $f(x) = x^2 - 3x + 8$

16. $f(x) = 5 - 4x - x^2$

17. $f(x) = (2x + 1)^3$

18. $f(x) = 5 + 12x - x^3$

19. $f(x) = 3x^4 - 4x^3$

20. $f(x) = x^4 - 5x^3 + 9x^2$

21. $f(x) = \dfrac{x - 2}{(x^2 - x + 1)^2}$

22. $f(x) = \dfrac{x}{x^2 + 2}$

23. $f(x) = \sqrt[3]{x^2 + x + 1}$

24. $f(x) = x^{4/3} - x^{1/3}$

25. $f(x) = (x^{2/3} - 1)^2$

26. $f(x) = x^{2/3} - x$

27. $f(x) = e^{-x^2/2}$

28. $f(x) = xe^{x^2}$

29. $f(x) = \ln \sqrt{x^2 + 4}$

30. $f(x) = x^3 \ln x$

15. $f'(x) = 2(x - 3/2)$, $f''(x) = 2$.

(a) $[3/2, +\infty)$ **(b)** $(-\infty, 3/2]$ **(c)** $(-\infty, +\infty)$ **(d)** nowhere **(e)** none

16. $f'(x) = -2(2 + x)$, $f''(x) = -2$.

(a) $(-\infty, -2]$ **(b)** $[-2, +\infty)$ **(c)** nowhere **(d)** $(-\infty, +\infty)$ **(e)** none

17. $f'(x) = 6(2x + 1)^2$, $f''(x) = 24(2x + 1)$.

(a) $(-\infty, +\infty)$ **(b)** nowhere **(c)** $(-1/2, +\infty)$ **(d)** $(-\infty, -1/2)$ **(e)** $-1/2$

18. $f'(x) = 3(4 - x^2)$, $f''(x) = -6x$.

(a) $[-2, 2]$ **(b)** $(-\infty, -2]$, $[2, +\infty)$ **(c)** $(-\infty, 0)$ **(d)** $(0, +\infty)$ **(e)** 0

19. $f'(x) = 12x^2(x - 1)$, $f''(x) = 36x(x - 2/3)$.

(a) $[1, +\infty)$ **(b)** $(-\infty, 1]$ **(c)** $(-\infty, 0)$, $(2/3, +\infty)$ **(d)** $(0, 2/3)$ **(e)** $0, 2/3$

20. $f'(x) = x(4x^2 - 15x + 18)$, $f''(x) = 6(x - 1)(2x - 3)$.

(a) $[0, +\infty)$ **(b)** $(-\infty, 0]$ **(c)** $(-\infty, 1)$, $(3/2, +\infty)$ **(d)** $(1, 3/2)$ **(e)** $1, 3/2$

21. $f'(x) = -\dfrac{3(x^2-3x+1)}{(x^2-x+1)^3}$, $f''(x) = \dfrac{6x(2x^2-8x+5)}{(x^2-x+1)^4}$.

(a) $\left[\dfrac{3-\sqrt{5}}{2}, \dfrac{3+\sqrt{5}}{2}\right]$ **(b)** $\left(-\infty, \dfrac{3-\sqrt{5}}{2}\right], \left[\dfrac{3+\sqrt{5}}{2}, +\infty\right)$ **(c)** $\left(0, 2-\dfrac{\sqrt{6}}{2}\right), \left(2+\dfrac{\sqrt{6}}{2}, +\infty\right)$

(d) $(-\infty, 0), \left(2-\dfrac{\sqrt{6}}{2}, 2+\dfrac{\sqrt{6}}{2}\right)$ **(e)** $0, 2-\dfrac{\sqrt{6}}{2}, 2+\dfrac{\sqrt{6}}{2}$

22. $f'(x) = \dfrac{2-x^2}{(x^2+2)^2}$ $f''(x) = \dfrac{2x(x^2-6)}{(x^2+2)^3}$.

(a) $(-\sqrt{2}, \sqrt{2})$ **(b)** $(-\infty, -\sqrt{2}), (\sqrt{2}, +\infty)$ **(c)** $(-\sqrt{6}, 0), (\sqrt{6}, +\infty)$ **(d)** $(-\infty, -\sqrt{6}), (0, \sqrt{6})$ **(e)** $0, \pm\sqrt{6}$

23. $f'(x) = \dfrac{2x+1}{3(x^2+x+1)^{2/3}}$, $f''(x) = -\dfrac{2(x+2)(x-1)}{9(x^2+x+1)^{5/3}}$.

(a) $[-1/2, +\infty)$ **(b)** $(-\infty, -1/2]$ **(c)** $(-2, 1)$ **(d)** $(-\infty, -2), (1, +\infty)$ **(e)** $-2, 1$

24. $f'(x) = \dfrac{4(x-1/4)}{3x^{2/3}}$, $f''(x) = \dfrac{4(x+1/2)}{9x^{5/3}}$.

(a) $[1/4, +\infty)$ **(b)** $(-\infty, 1/4]$ **(c)** $(-\infty, -1/2)$, $(0, +\infty)$ **(d)** $(-1/2, 0)$ **(e)** $-1/2, 0$

25. $f'(x) = \dfrac{4(x^{2/3}-1)}{3x^{1/3}}$, $f''(x) = \dfrac{4(x^{5/3}+x)}{9x^{7/3}}$.

(a) $[-1, 0], [1, +\infty)$ **(b)** $(-\infty, -1], [0, 1]$ **(c)** $(-\infty, 0)$, $(0, +\infty)$ **(d)** nowhere **(e)** none

26. $f'(x) = \frac{2}{3}x^{-1/3} - 1,\ f''(x) = -\frac{2}{9}x^{-4/3}.$

(a) $[0, 8/27]$ **(b)** $(-\infty, 0], [8/27, +\infty)$ **(c)** nowhere **(d)** $(-\infty, 0), (0, +\infty)$ **(e)** none

27. $f'(x) = -xe^{-x^2/2},\ f''(x) = (-1 + x^2)e^{-x^2/2}.$

(a) $(-\infty, 0]$ **(b)** $[0, +\infty)$ **(c)** $(-\infty, -1), (1, +\infty)$ **(d)** $(-1, 1)$ **(e)** $-1, 1$

28. $f'(x) = (2x^2 + 1)e^{x^2},\ f''(x) = 2x(2x^2 + 3)e^{x^2}.$

(a) $(-\infty, +\infty)$ **(b)** none **(c)** $(0, +\infty)$ **(d)** $(-\infty, 0)$ **(e)** 0

29. $f'(x) = \frac{x}{x^2 + 4},\ f''(x) = -\frac{x^2 - 4}{(x^2 + 4)^2}.$

(a) $[0, +\infty)$ **(b)** $(-\infty, 0]$ **(c)** $(-2, 2)$ **(d)** $(-\infty, -2), (2, +\infty)$ **(e)** $-2, 2$

30. $f'(x) = x^2(1 + 3\ln x),\ f''(x) = x(5 + 6\ln x).$

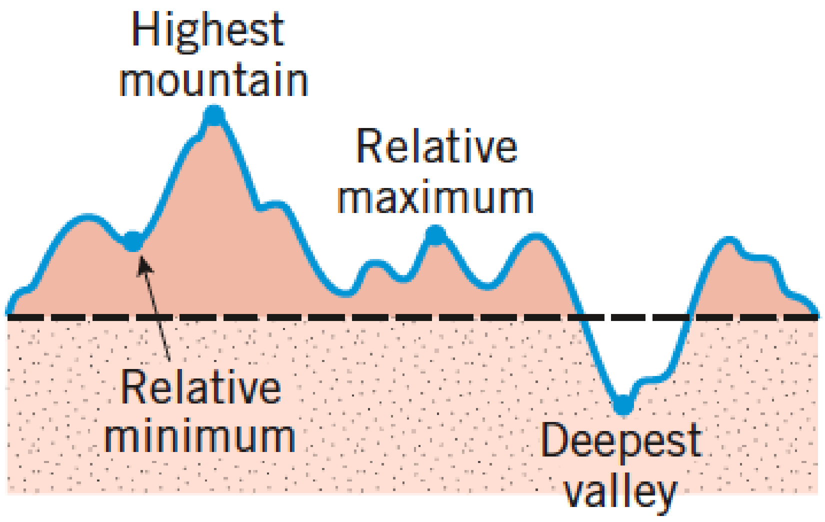
(a) $[e^{-1/3}, +\infty)$ **(b)** $(0, e^{-1/3}]$ **(c)** $(e^{-5/6}, +\infty)$ **(d)** $(0, e^{-5/6})$ **(e)** $e^{-5/6}$

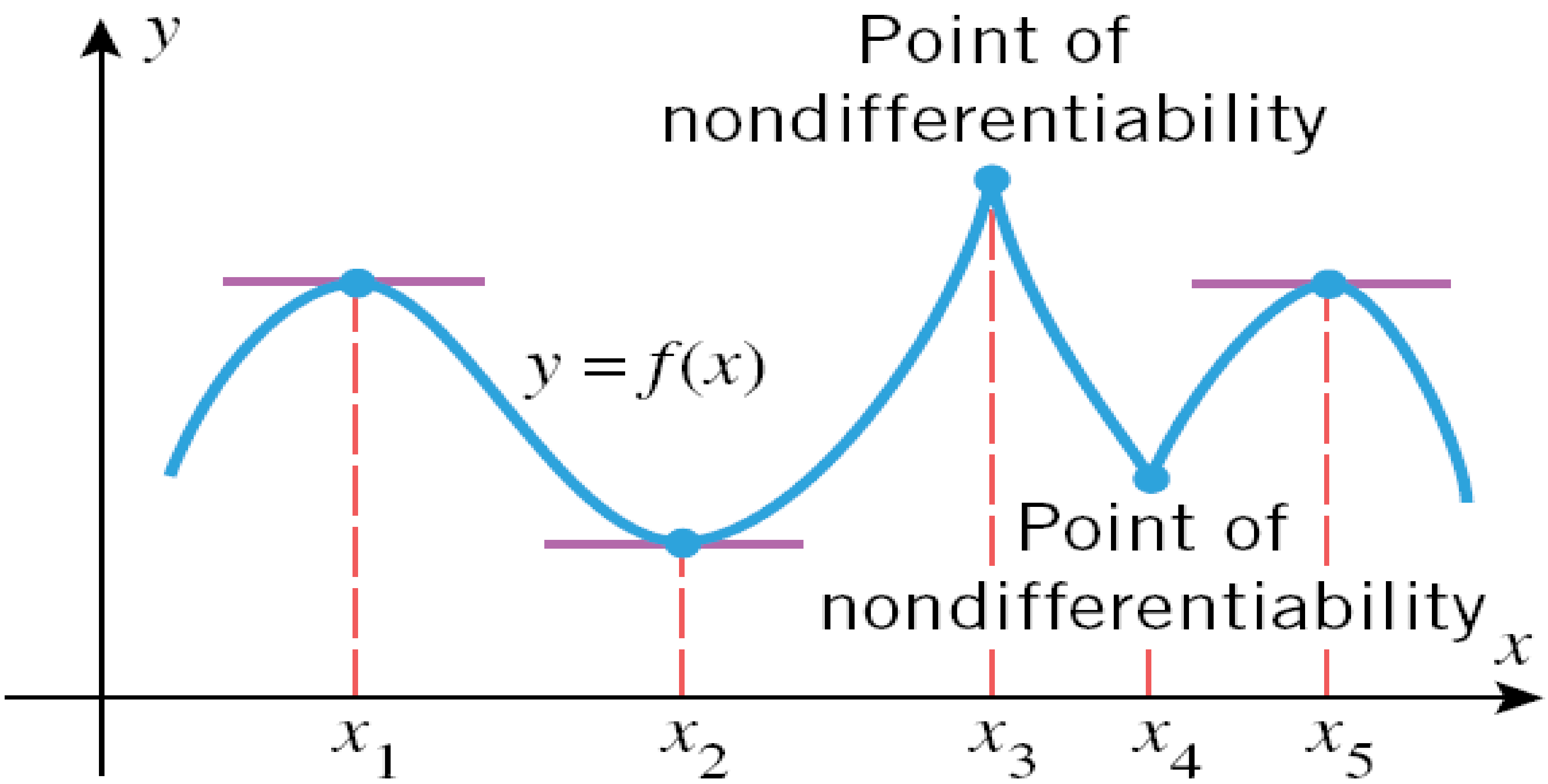
Lecture # 21

ANALYSIS OF FUNCTIONS

RELATIVE EXTREMA; GRAPHING POLYNOMIALS

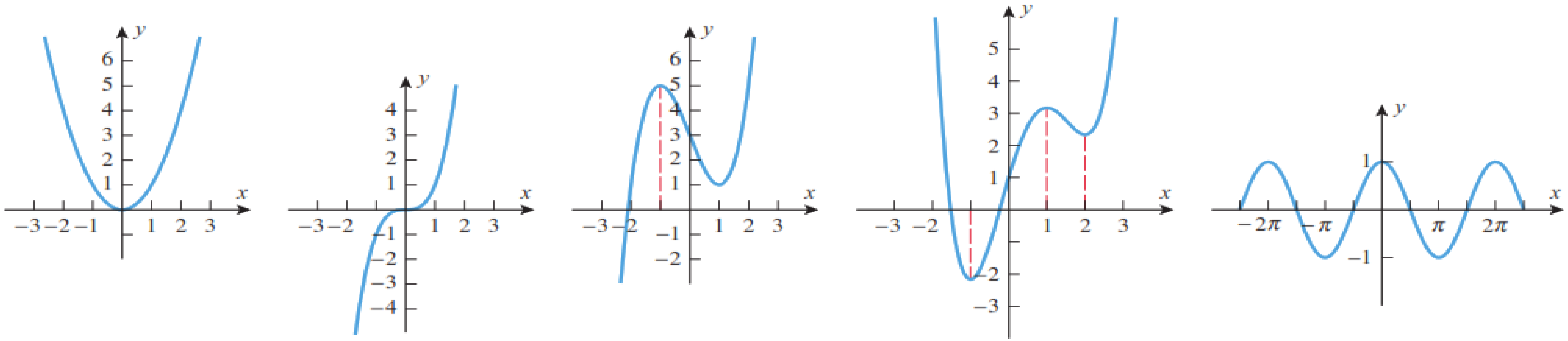
4.2.1 DEFINITION A function f is said to have a ***relative maximum*** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the largest value, that is, $f(x_0) \geq f(x)$ for all x in the interval. Similarly, f is said to have a ***relative minimum*** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the smallest value, that is, $f(x_0) \leq f(x)$ for all x in the interval. If f has either a relative maximum or a relative minimum at x_0, then f is said to have a ***relative extremum*** at x_0.





The points x_1, x_2, x_3, x_4, and x_5 are critical points. Of these, x_1, x_2, and x_5 are stationary points.

- $f(x) = x^2$ has a relative minimum at $x = 0$ but no relative maxima.
- $f(x) = x^3$ has no relative extrema.
- $f(x) = x^3 - 3x + 3$ has a relative maximum at $x = -1$ and a relative minimum at $x = 1$.
- $f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$ has relative minima at $x = -1$ and $x = 2$ and a relative maximum at $x = 1$.
- $f(x) = \cos x$ has relative maxima at all even multiples of π and relative minima at all odd multiples of π. ◄



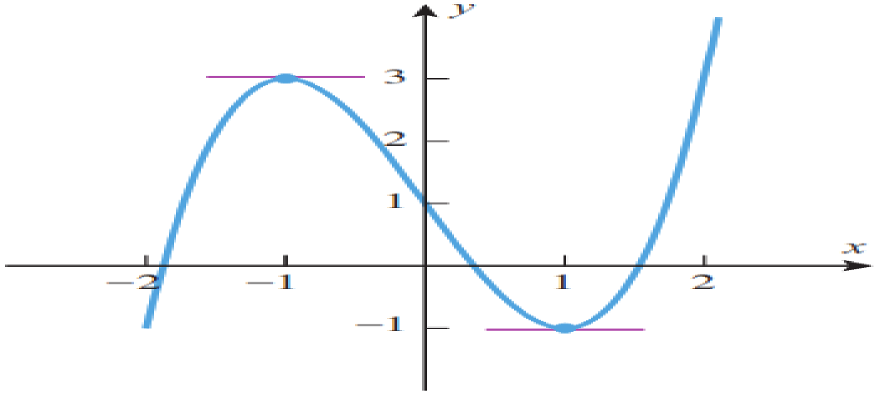
$y = x^2$ $\quad$ $y = x^3$ $\quad$ $y = x^3 - 3x + 3$ $\quad$ $y = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$ $\quad$ $y = \cos x$

▶ **Example 2** Find all critical points of $f(x) = x^3 - 3x + 1$.

Solution. The function f, being a polynomial, is differentiable everywhere, so its critical points are all stationary points. To find these points we must solve the equation $f'(x) = 0$. Since

$$f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$$

we conclude that the critical points occur at $x = -1$ and $x = 1$. This is consistent with the graph of f in Figure 4.2.4. ◀



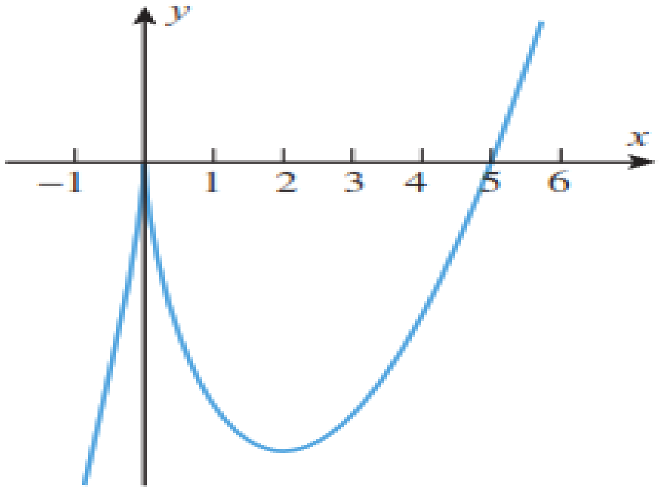
$y = x^3 - 3x + 1$

Example 3 Find all critical points of $f(x) = 3x^{5/3} - 15x^{2/3}$.

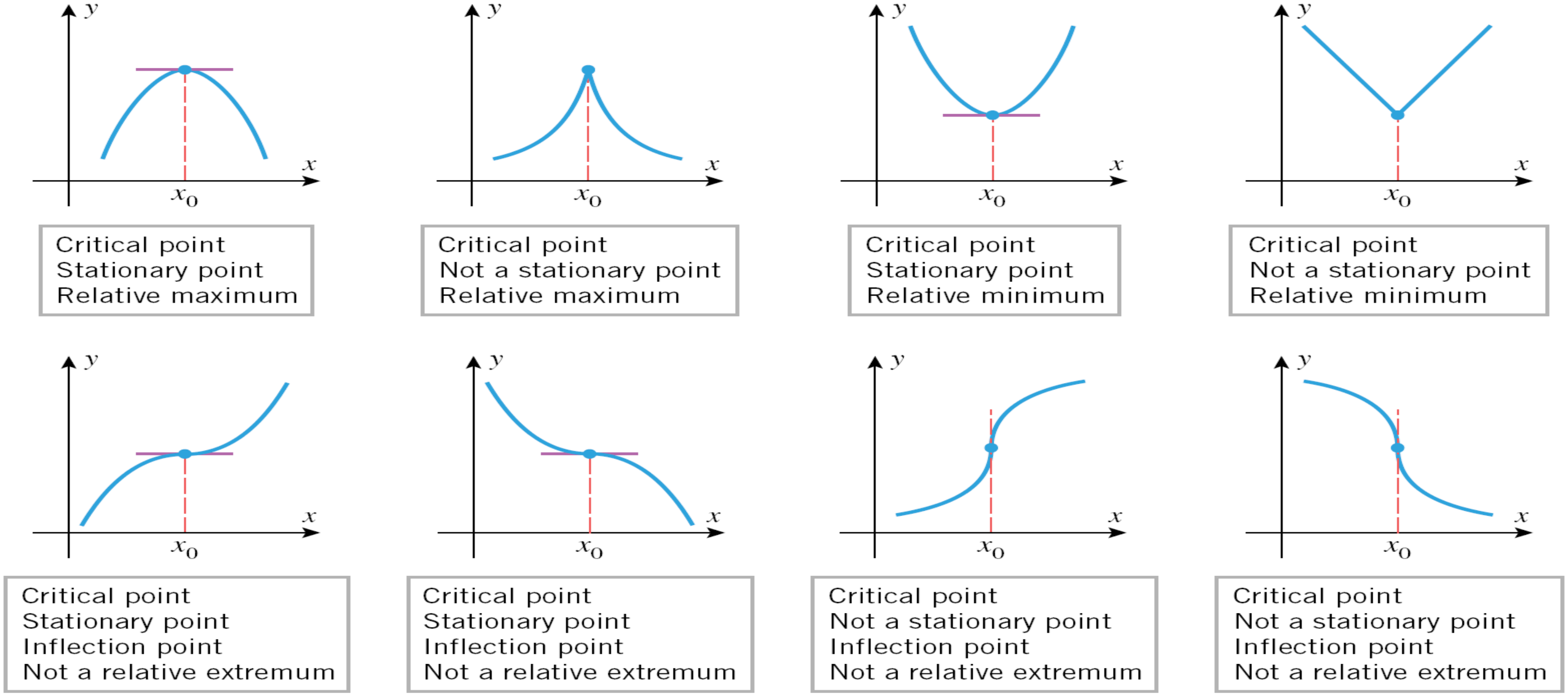
Solution. The function f is continuous everywhere and its derivative is

$$f'(x) = 5x^{2/3} - 10x^{-1/3} = 5x^{-1/3}(x-2) = \frac{5(x-2)}{x^{1/3}}$$

We see from this that $f'(x) = 0$ if $x = 2$ and $f'(x)$ is undefined if $x = 0$. Thus $x = 0$ and $x = 2$ are critical points and $x = 2$ is a stationary point. This is consistent with the graph of f shown in Figure 4.2.5. ◀



$y = 3x^{5/3} - 15x^{2/3}$



A function f has a relative extremum at those critical points where f' changes sign.

4.2.3 THEOREM (First Derivative Test) *Suppose that f is continuous at a critical point x_0.*

(a) *If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0, then f has a relative maximum at x_0.*

(b) *If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0, then f has a relative minimum at x_0.*

(c) *If $f'(x)$ has the same sign on an open interval extending left from x_0 as it does on an open interval extending right from x_0, then f does not have a relative extremum at x_0.*

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

▶ **Example 4** We showed in Example 3 that the function $f(x) = 3x^{5/3} - 15x^{2/3}$ has critical points at $x = 0$ and $x = 2$. Figure 4.2.5 suggests that f has a relative maximum at $x = 0$ and a relative minimum at $x = 2$. Confirm this using the first derivative test.

Table 4.2.1

INTERVAL	$5(x-2)/x^{1/3}$	$f'(x)$
$x < 0$	$(-)/(-)$	$+$
$0 < x < 2$	$(-)/(+)$	$-$
$x > 2$	$(+)/(+)$	$+$

Solution. We showed in Example 3 that

$$f'(x) = \frac{5(x-2)}{x^{1/3}}$$

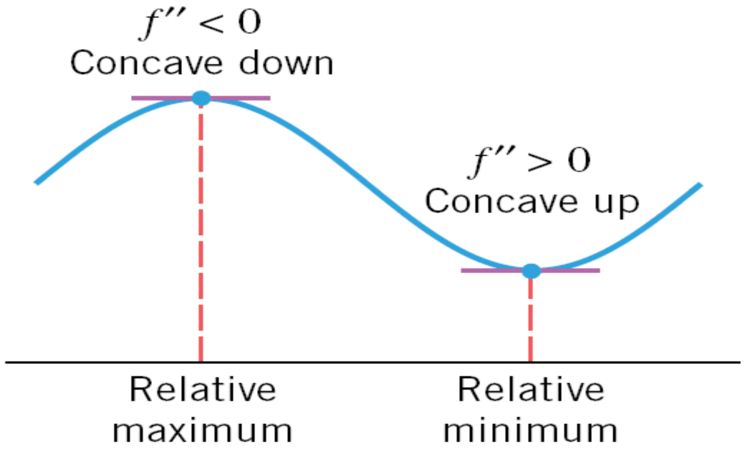
A sign analysis of this derivative is shown in Table 4.2.1. The sign of f' changes from $+$ to $-$ at $x = 0$, so there is a relative maximum at that point. The sign changes from $-$ to $+$ at $x = 2$, so there is a relative minimum at that point. ◀

4.2.4 THEOREM (*Second Derivative Test*) *Suppose that f is twice differentiable at the point x_0.*

(a) *If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a relative minimum at x_0.*

(b) *If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a relative maximum at x_0.*

(c) *If $f'(x_0) = 0$ and $f''(x_0) = 0$, then the test is inconclusive; that is, f may have a relative maximum, a relative minimum, or neither at x_0.*



▶ **Example 5** Find the relative extrema of $f(x) = 3x^5 - 5x^3$.

Solution. We have

$$f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1) = 15x^2(x + 1)(x - 1)$$
$$f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$$

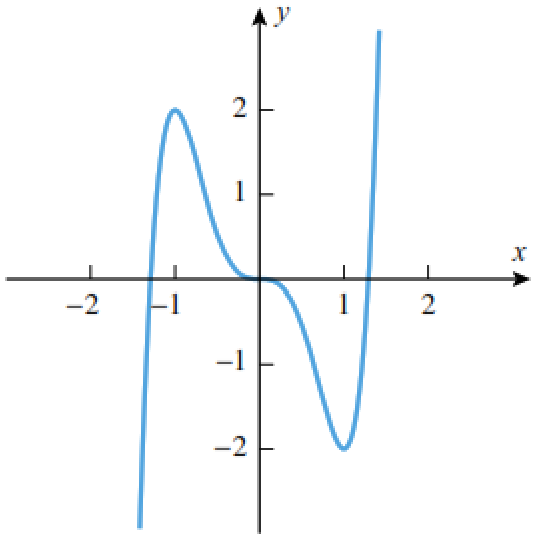
Solving $f'(x) = 0$ yields the stationary points $x = 0$, $x = -1$, and $x = 1$. As shown in the following table, we can conclude from the second derivative test that f has a relative maximum at $x = -1$ and a relative minimum at $x = 1$.

STATIONARY POINT	$30x(2x^2 - 1)$	$f''(x)$	SECOND DERIVATIVE TEST
$x = -1$	-30	$-$	f has a relative maximum
$x = 0$	0	0	Inconclusive
$x = 1$	30	$+$	f has a relative minimum

The test is inconclusive at $x = 0$, so we will try the first derivative test at that point. A sign analysis of f' is given in the following table:

INTERVAL	$15x^2(x+1)(x-1)$	$f'(x)$
$-1 < x < 0$	$(+)(+)(-)$	$-$
$0 < x < 1$	$(+)(+)(-)$	$-$

Since there is no sign change in f' at $x = 0$, there is neither a relative maximum nor a relative minimum at that point. All of this is consistent with the graph of f shown in Figure 4.2.8. ◀



$y = 3x^5 - 5x^3$

Find the relative extrema of the following:

(i) f(x) = $x^3 - 5x$ + 2

(ii) f(x) = x^4 - 2$x^2 + 7$

(i) f(x) = $x^3 - 5x + 2$ (1)

Solution:

Differentiate w.r.t 'x' on both side of (1)

$\frac{dy}{dx} = \frac{d}{dx}\ (x^3 - 5x + 2)$

$f'(x) = 3\,x^2 - 5$ (2)

For stationary point take $f'(x) = 0$

$3\,x^2 - 5 = 0$

$x^2 = \frac{5}{3}$

x = $\pm\sqrt{\frac{5}{3}}$ (critical points)

For relative extrema again differentiate (2)

$f''(x) = 6x$

At x = $\sqrt{\frac{5}{3}}$, $f''\left(\sqrt{\frac{5}{3}}\right) = 6(\sqrt{\frac{5}{3}}) = \frac{6\sqrt{5}}{\sqrt{3}} > 0$ (relative minima)

At x = $-\sqrt{\frac{5}{3}}$, $f''\left(-\sqrt{\frac{5}{3}}\right) = 6(-\sqrt{\frac{5}{3}}) = \frac{-6\sqrt{5}}{\sqrt{3}} < 0$ (relative maxima)

7–14 Locate the critical points and identify which critical points are stationary points.

7. $f(x) = 4x^4 - 16x^2 + 17$
8. $f(x) = 3x^4 + 12x$
9. $f(x) = \dfrac{x+1}{x^2+3}$
10. $f(x) = \dfrac{x^2}{x^3+8}$
11. $f(x) = \sqrt[3]{x^2 - 25}$
12. $f(x) = x^2(x-1)^{2/3}$
13. $f(x) = |\sin x|$
14. $f(x) = \sin|x|$

25–32 Use the given derivative to find all critical points of f, and at each critical point determine whether a relative maximum, relative minimum, or neither occurs. Assume in each case that f is continuous everywhere.

25. $f'(x) = x^2(x^3 - 5)$
26. $f'(x) = 4x^3 - 9x$
27. $f'(x) = \dfrac{2-3x}{\sqrt[3]{x+2}}$
28. $f'(x) = \dfrac{x^2-7}{\sqrt[3]{x^2+4}}$
29. $f'(x) = xe^{1-x^2}$
30. $f'(x) = x^4(e^x - 3)$
31. $f'(x) = \ln\left(\dfrac{2}{1+x^2}\right)$
32. $f'(x) = e^{2x} - 5e^x + 6$

33–36 Find the relative extrema using both first and second derivative tests.

33. $f(x) = 1 + 8x - 3x^2$
34. $f(x) = x^4 - 12x^3$
35. $f(x) = \sin 2x, \quad 0 < x < \pi$
36. $f(x) = (x-3)e^x$