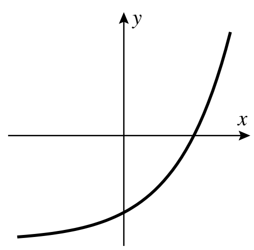
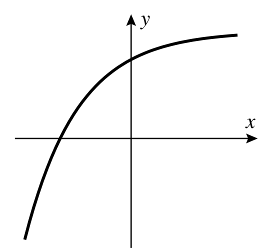
The Derivative in Graphing and Applications

Exercise Set 4.1

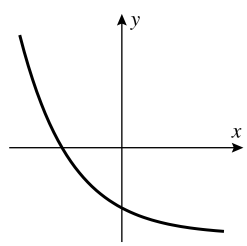
1. **(a)** $f' > 0$ and $f'' > 0$.



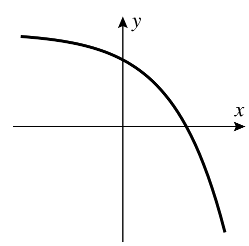
(b) $f' > 0$ and $f'' < 0$.



(c) $f' < 0$ and $f'' > 0$.



(d) $f' < 0$ and $f'' < 0$.



3. A: $dy/dx < 0,\ d^2y/dx^2 > 0,\ B$: $dy/dx > 0,\ d^2y/dx^2 < 0,\ C$: $dy/dx < 0,\ d^2y/dx^2 < 0$.

5. An inflection point occurs when f'' changes sign: at $x = -1, 0, 1$ and 2.

7. **(a)** $[4, 6]$ **(b)** $[1, 4]$ and $[6, 7]$. **(c)** $(1, 2)$ and $(3, 5)$. **(d)** $(2, 3)$ and $(5, 7)$. **(e)** $x = 2, 3, 5$.

9. **(a)** f is increasing on $[1, 3]$.

(b) f is decreasing on $(-\infty, 1], [3, +\infty)$.

(c) f is concave up on $(-\infty, 2), (4, +\infty)$.

(d) f is concave down on $(2, 4)$.

(e) Points of inflection at $x = 2, 4$.

11. True, by Definition 4.1.1(b).

13. False. Let $f(x) = (x - 1)^3$. Then f is increasing on $[0, 2]$, but $f'(1) = 0$.

15. $f'(x) = 2(x - 3/2)$, $f''(x) = 2$.
(a) $[3/2, +\infty)$ **(b)** $(-\infty, 3/2]$ **(c)** $(-\infty, +\infty)$ **(d)** nowhere **(e)** none

17. $f'(x) = 6(2x + 1)^2$, $f''(x) = 24(2x + 1)$.
(a) $(-\infty, +\infty)$ **(b)** nowhere **(c)** $(-1/2, +\infty)$ **(d)** $(-\infty, -1/2)$ **(e)** $-1/2$

19. $f'(x) = 12x^2(x-1)$, $f''(x) = 36x(x-2/3)$.

(a) $[1,+\infty)$ **(b)** $(-\infty,1]$ **(c)** $(-\infty,0)$, $(2/3,+\infty)$ **(d)** $(0,2/3)$ **(e)** $0, 2/3$

21. $f'(x) = -\dfrac{3(x^2-3x+1)}{(x^2-x+1)^3}$, $f''(x) = \dfrac{6x(2x^2-8x+5)}{(x^2-x+1)^4}$.

(a) $\left[\dfrac{3-\sqrt{5}}{2}, \dfrac{3+\sqrt{5}}{2}\right]$ **(b)** $\left(-\infty, \dfrac{3-\sqrt{5}}{2}\right]$, $\left[\dfrac{3+\sqrt{5}}{2}, +\infty\right)$ **(c)** $\left(0, 2-\dfrac{\sqrt{6}}{2}\right)$, $\left(2+\dfrac{\sqrt{6}}{2}, +\infty\right)$

(d) $(-\infty,0)$, $\left(2-\dfrac{\sqrt{6}}{2}, 2+\dfrac{\sqrt{6}}{2}\right)$ **(e)** $0, 2-\dfrac{\sqrt{6}}{2}, 2+\dfrac{\sqrt{6}}{2}$

23. $f'(x) = \dfrac{2x+1}{3(x^2+x+1)^{2/3}}$, $f''(x) = -\dfrac{2(x+2)(x-1)}{9(x^2+x+1)^{5/3}}$.

(a) $[-1/2,+\infty)$ **(b)** $(-\infty,-1/2]$ **(c)** $(-2,1)$ **(d)** $(-\infty,-2), (1,+\infty)$ **(e)** $-2, 1$

25. $f'(x) = \dfrac{4(x^{2/3}-1)}{3x^{1/3}}$, $f''(x) = \dfrac{4(x^{5/3}+x)}{9x^{7/3}}$.

(a) $[-1,0], [1,+\infty)$ **(b)** $(-\infty,-1], [0,1]$ **(c)** $(-\infty,0)$, $(0,+\infty)$ **(d)** nowhere **(e)** none

27. $f'(x) = -xe^{-x^2/2}$, $f''(x) = (-1+x^2)e^{-x^2/2}$.

(a) $(-\infty,0]$ **(b)** $[0,+\infty)$ **(c)** $(-\infty,-1)$, $(1,+\infty)$ **(d)** $(-1,1)$ **(e)** $-1, 1$

29. $f'(x) = \dfrac{x}{x^2+4}$, $f''(x) = -\dfrac{x^2-4}{(x^2+4)^2}$.

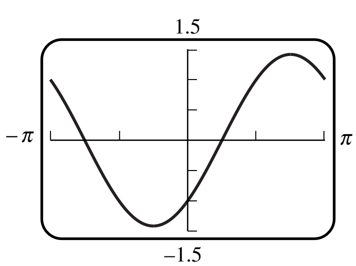
(a) $[0,+\infty)$ **(b)** $(-\infty,0]$ **(c)** $(-2,2)$ **(d)** $(-\infty,-2), (2,+\infty)$ **(e)** $-2, 2$

31. $f'(x) = \dfrac{2x}{1+(x^2-1)^2}$, $f''(x) = -2\dfrac{3x^4-2x^2-2}{[1+(x^2-1)^2]^2}$.

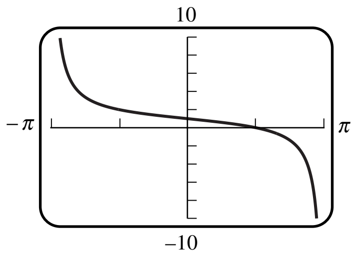
(a) $[0+\infty)$ **(b)** $(-\infty,0]$ **(c)** $\left(-\dfrac{\sqrt{1+\sqrt{7}}}{\sqrt{3}}, \dfrac{\sqrt{1+\sqrt{7}}}{\sqrt{3}}\right)$ **(d)** $\left(-\infty, -\dfrac{\sqrt{1+\sqrt{7}}}{\sqrt{3}}\right)$, $\left(\dfrac{\sqrt{1+\sqrt{7}}}{\sqrt{3}}, +\infty\right)$

(e) $\pm\dfrac{\sqrt{1+\sqrt{7}}}{3}$

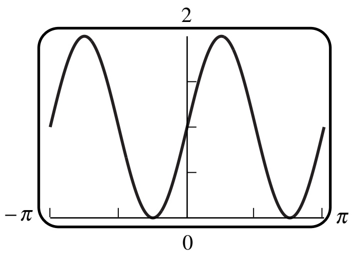
33. $f'(x) = \cos x + \sin x$, $f''(x) = -\sin x + \cos x$, increasing: $[-\pi/4, 3\pi/4]$, decreasing: $(-\pi,-\pi/4], [3\pi/4,\pi)$, concave up: $(-3\pi/4, \pi/4)$, concave down: $(-\pi,-3\pi/4), (\pi/4,\pi)$, inflection points: $-3\pi/4, \pi/4$.



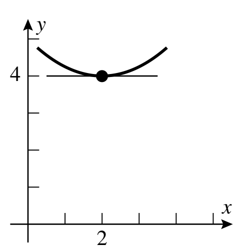
35. $f'(x) = -\dfrac{1}{2}\sec^2(x/2)$, $f''(x) = -\dfrac{1}{2}\tan(x/2)\sec^2(x/2))$, increasing: nowhere, decreasing: $(-\pi,\pi)$, concave up: $(-\pi,0)$, concave down: $(0,\pi)$, inflection point: 0.



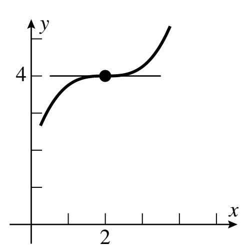
37. $f(x) = 1 + \sin 2x,\ f'(x) = 2\cos 2x,\ f''(x) = -4\sin 2x$, increasing: $[-\pi, -3\pi/4], [-\pi/4, \pi/4], [3\pi/4, \pi]$, decreasing: $[-3\pi/4, -\pi/4], [\pi/4, 3\pi/4]$, concave up: $(-\pi/2, 0), (\pi/2, \pi)$, concave down: $(-\pi, -\pi/2), (0, \pi/2)$, inflection points: $-\pi/2, 0, \pi/2$.



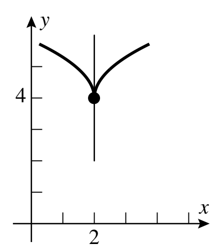
39. **(a)**



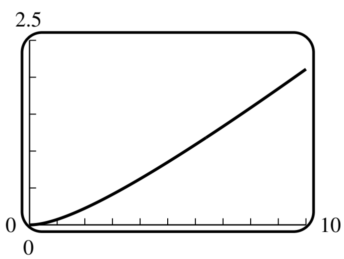
(b)



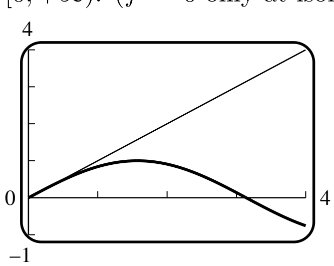
(c)



41. $f'(x) = 1/3 - 1/[3(1+x)^{2/3}]$ so f is increasing on $[0, +\infty)$, thus if $x > 0$, then $f(x) > f(0) = 0$, $1 + x/3 - \sqrt[3]{1+x} > 0$, $\sqrt[3]{1+x} < 1 + x/3$.



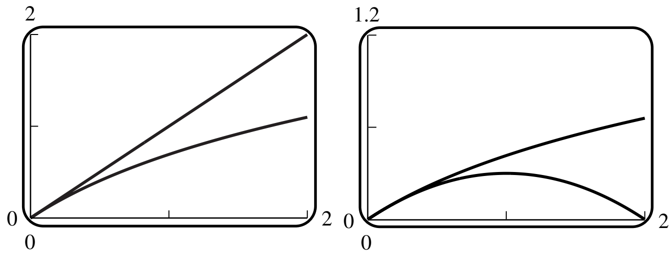
43. $x \geq \sin x$ on $[0, +\infty)$: let $f(x) = x - \sin x$. Then $f(0) = 0$ and $f'(x) = 1 - \cos x \geq 0$, so $f(x)$ is increasing on $[0, +\infty)$. ($f' = 0$ only at isolated points.)



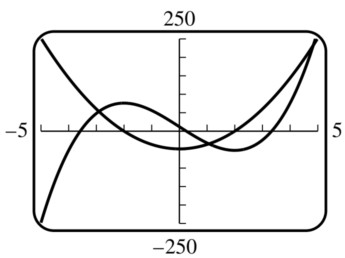
45. **(a)** Let $f(x) = x - \ln(x+1)$ for $x \geq 0$. Then $f(0) = 0$ and $f'(x) = 1 - 1/(x+1) > 0$ for $x > 0$, so f is increasing for $x \geq 0$ and thus $\ln(x+1) \leq x$ for $x \geq 0$.

(b) Let $g(x) = x - \dfrac{1}{2}x^2 - \ln(x+1)$. Then $g(0) = 0$ and $g'(x) = 1 - x - 1/(x+1) < 0$ for $x > 0$ since $1 - x^2 \leq 1$.

Thus g is decreasing and thus $\ln(x+1) \geq x - \dfrac{1}{2}x^2$ for $x \geq 0$.

(c) 

47. Points of inflection at $x = -2, +2$. Concave up on $(-5, -2)$ and $(2, 5)$; concave down on $(-2, 2)$. Increasing on $[-3.5829, 0.2513]$ and $[3.3316, 5]$, and decreasing on $[-5, -3.5829]$ and $[0.2513, 3.3316]$.



49. $f''(x) = 2\dfrac{90x^3 - 81x^2 - 585x + 397}{(3x^2 - 5x + 8)^3}$. The denominator has complex roots, so is always positive; hence the x-coordinates of the points of inflection of $f(x)$ are the roots of the numerator (if it changes sign). A plot of the numerator over $[-5, 5]$ shows roots lying in $[-3, -2]$, $[0, 1]$, and $[2, 3]$. To six decimal places the roots are $x \approx -2.464202, 0.662597, 2.701605$.

51. $f(x_1) - f(x_2) = x_1^2 - x_2^2 = (x_1 + x_2)(x_1 - x_2) < 0$ if $x_1 < x_2$ for x_1, x_2 in $[0, +\infty)$, so $f(x_1) < f(x_2)$ and f is thus increasing.

53. **(a)** True. If $x_1 < x_2$ where x_1 and x_2 are in I, then $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$, so $f(x_1) + g(x_1) < f(x_2) + g(x_2)$, $(f + g)(x_1) < (f + g)(x_2)$. Thus $f + g$ is increasing on I.

(b) False. If $f(x) = g(x) = x$ then f and g are both increasing on $(-\infty, 0)$, but $(f \cdot g)(x) = x^2$ is decreasing there.

55. **(a)** $f(x) = x$, $g(x) = 2x$ **(b)** $f(x) = x$, $g(x) = x + 6$ **(c)** $f(x) = 2x$, $g(x) = x$

57. **(a)** $f''(x) = 6ax + 2b = 6a\left(x + \dfrac{b}{3a}\right)$, $f''(x) = 0$ when $x = -\dfrac{b}{3a}$. f changes its direction of concavity at $x = -\dfrac{b}{3a}$ so $-\dfrac{b}{3a}$ is an inflection point.

(b) If $f(x) = ax^3 + bx^2 + cx + d$ has three x-intercepts, then it has three roots, say x_1, x_2 and x_3, so we can write $f(x) = a(x - x_1)(x - x_2)(x - x_3) = ax^3 + bx^2 + cx + d$, from which it follows that $b = -a(x_1 + x_2 + x_3)$. Thus $-\dfrac{b}{3a} = \dfrac{1}{3}(x_1 + x_2 + x_3)$, which is the average.

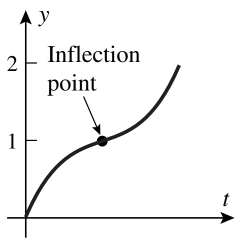
(c) $f(x) = x(x^2 - 3x + 2) = x(x - 1)(x - 2)$ so the intercepts are 0, 1, and 2 and the average is 1. $f''(x) = 6x - 6 = 6(x - 1)$ changes sign at $x = 1$. The inflection point is at (1,0). f is concave up for $x > 1$, concave down for $x < 1$.

59. **(a)** Let $x_1 < x_2$ belong to (a, b). If both belong to $(a, c]$ or both belong to $[c, b)$ then we have $f(x_1) < f(x_2)$ by hypothesis. So assume $x_1 < c < x_2$. We know by hypothesis that $f(x_1) < f(c)$, and $f(c) < f(x_2)$. We conclude that $f(x_1) < f(x_2)$.

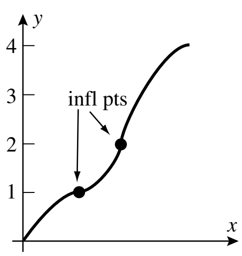
(b) Use the same argument as in part (a), but with inequalities reversed.

61. By Theorem 4.1.2, $f$is decreasing on any interval $[(2n\pi + \pi/2, 2(n+1)\pi + \pi/2]$ $(n = 0, \pm 1, \pm 2, \ldots)$, because $f'(x) = -\sin x + 1 < 0$ on $(2n\pi + \pi/2, 2(n+1)\pi + \pi/2)$. By Exercise 59 (b) we can piece these intervals together to show that $f(x)$ is decreasing on $(-\infty, +\infty)$.

63.



65.

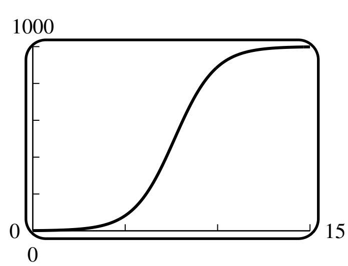


67. (a) $y'(t) = \dfrac{LAke^{-kt}}{(1+Ae^{-kt})^2}S$, so $y'(0) = \dfrac{LAk}{(1+A)^2}$.

(b) The rate of growth increases to its maximum, which occurs when y is halfway between 0 and L, or when $t = \dfrac{1}{k}\ln A$; it then decreases back towards zero.

(c) From (2) one sees that $\dfrac{dy}{dt}$ is maximized when y lies half way between 0 and L, i.e. $y = L/2$. This follows since the right side of (2) is a parabola (with y as independent variable) with y-intercepts $y = 0, L$. The value $y = L/2$ corresponds to $t = \dfrac{1}{k}\ln A$, from (4).

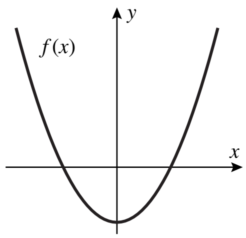
69. $t \approx 7.67$

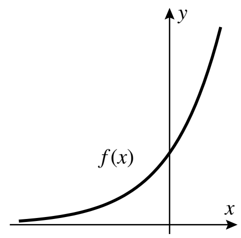


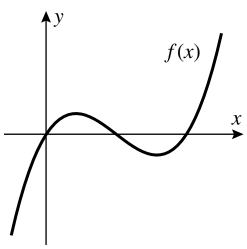
71. Since $0 < y < L$ the right-hand side of (5) of Example 9 can change sign only if the factor $L - 2y$ changes sign, which it does when $y = L/2$, at which point we have $\dfrac{L}{2} = \dfrac{L}{1+Ae^{-kt}}$, $1 = Ae^{-kt}$, $t = \dfrac{1}{k}\ln A$.

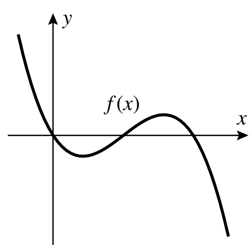
73. Sign analysis of $f'(x)$ tells us where the graph of $y = f(x)$ increases or decreases. Sign analysis of $f''(x)$ tells us where the graph of $y = f(x)$ is concave up or concave down.

Exercise Set 4.2

1. **(a)**


(b)


(c)


(d)


3. **(a)** $f'(x) = 6x - 6$ and $f''(x) = 6$, with $f'(1) = 0$. For the first derivative test, $f' < 0$ for $x < 1$ and $f' > 0$ for $x > 1$. For the second derivative test, $f''(1) > 0$.

(b) $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$. $f'(x) = 0$ at $x = \pm 1$. First derivative test: $f' > 0$ for $x < -1$ and $x > 1$, and $f' < 0$ for $-1 < x < 1$, so there is a relative maximum at $x = -1$, and a relative minimum at $x = 1$. Second derivative test: $f'' < 0$ at $x = -1$, a relative maximum; and $f'' > 0$ at $x = 1$, a relative minimum.

5. **(a)** $f'(x) = 4(x-1)^3$, $g'(x) = 3x^2 - 6x + 3$ so $f'(1) = g'(1) = 0$.

(b) $f''(x) = 12(x-1)^2$, $g''(x) = 6x - 6$, so $f''(1) = g''(1) = 0$, which yields no information.

(c) $f' < 0$ for $x < 1$ and $f' > 0$ for $x > 1$, so there is a relative minimum at $x = 1$; $g'(x) = 3(x-1)^2 > 0$ on both sides of $x = 1$, so there is no relative extremum at $x = 1$.

7. $f'(x) = 16x^3 - 32x = 16x(x^2 - 2)$, so $x = 0, \pm\sqrt{2}$ are stationary points.

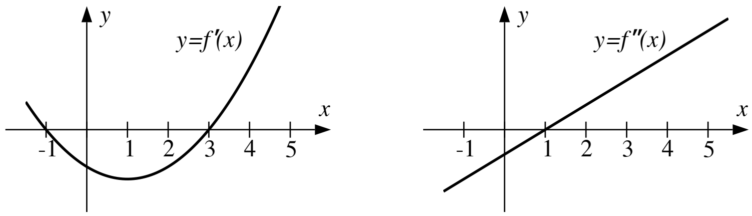
9. $f'(x) = \dfrac{-x^2 - 2x + 3}{(x^2+3)^2}$, so $x = -3, 1$ are the stationary points.

11. $f'(x) = \dfrac{2x}{3(x^2 - 25)^{2/3}}$; so $x = 0$ is the stationary point; $x = \pm 5$ are critical points which are not stationary points.

13. $f(x) = |\sin x| = \begin{cases} \sin x, & \sin x \geq 0 \\ -\sin x, & \sin x < 0 \end{cases}$, so $f'(x) = \begin{cases} \cos x, & \sin x > 0 \\ -\cos x, & \sin x < 0 \end{cases}$ and $f'(x)$ does not exist when $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ (the points where $\sin x = 0$) because $\lim\limits_{x \to n\pi^-} f'(x) \neq \lim\limits_{x \to n\pi^+} f'(x)$ (see Theorem preceding Exercise 65, Section 2.3); these are critical points which are not stationary points. Now $f'(x) = 0$ when $\pm\cos x = 0$ provided $\sin x \neq 0$ so $x = \pi/2 + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ are stationary points.

15. False. Let $f(x) = (x-1)^2(2x-3)$. Then $f'(x) = 2(x-1)(3x-4)$; $f'(x)$ changes sign from $+$ to $-$ at $x = 1$, so f has a relative maximum at $x = 1$. But $f(2) = 1 > 0 = f(1)$.

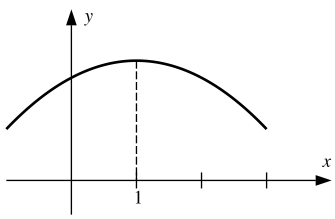
17. False. Let $f(x) = x + (x-1)^2$. Then $f'(x) = 2x - 1$ and $f''(x) = 2$, so $f''(1) > 0$. But $f'(1) = 1 \neq 0$, so f does not have a relative extremum at $x = 1$.

19. 

21. (a) None.

(b) $x = 1$ because f' changes sign from + to − there.

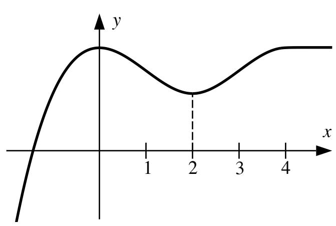
(c) None, because $f'' = 0$ (never changes sign).

(d) 

23. (a) $x = 2$ because $f'(x)$ changes sign from − to + there.

(b) $x = 0$ because $f'(x)$ changes sign from + to − there.

(c) $x = 1, 3$ because $f''(x)$ changes sign at these points.

(d) 

25. f':
− − − 0 − − − 0 + + +
0 $5^{1/3}$
Critical points: $x = 0, 5^{1/3}$; $x = 0$: neither, $x = 5^{1/3}$: relative minimum.

27. f':
− − − ∞ + + + 0 − − −
−2 2/3
Critical points: $x = -2, 2/3$; $x = -2$: relative minimum, $x = 2/3$: relative maximum.

29. f':
− − − 0 + + +
0
Critical point: $x = 0$; $x = 0$: relative minimum.

31. f':
− − − 0 + + + 0 − − −
−1 1
Critical points: $x = -1, 1$; $x = -1$: relative minimum, $x = 1$: relative maximum.

33. $f'(x) = 8 - 6x$: critical point $x = 4/3$, $f''(4/3) = -6 : f$ has a relative maximum of 19/3 at $x = 4/3$.

35. $f'(x) = 2\cos 2x$: critical points at $x = \pi/4, 3\pi/4$, $f''(\pi/4) = -4$: f has a relative maximum of 1 at $x = \pi/4$, $f''(3\pi/4) = 4 : f$ has a relative minimum of -1 at $x = 3\pi/4$.

37. $f'(x) = 4x^3 - 12x^2 + 8x$: $\begin{array}{ccccccc} - - - & 0 & + + + & 0 & - - - & 0 & + + + \\ \hline & 0 & & 1 & & 2 & \end{array}$ Critical points at $x = 0, 1, 2$; relative minimum of 0 at $x = 0$, relative maximum of 1 at $x = 1$, relative minimum of 0 at $x = 2$.

39. $f'(x) = 5x^4 + 8x^3 + 3x^2$: critical points at $x = -3/5, -1, 0$, $f''(-3/5) = 18/25$: f has a relative minimum of $-108/3125$ at $x = -3/5$, $f''(-1) = -2$: f has a relative maximum of 0 at $x = -1$, $f''(0) = 0$: Theorem 4.2.5 with $m = 3$: f has an inflection point at $x = 0$.

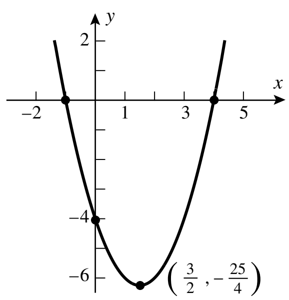
41. $f'(x) = \dfrac{2(x^{1/3}+1)}{x^{1/3}}$: critical point at $x = -1, 0$, $f''(-1) = -\dfrac{2}{3}$: f has a relative maximum of 1 at $x = -1$, f' does not exist at $x = 0$. Using the First Derivative Test, it is a relative minimum of 0.

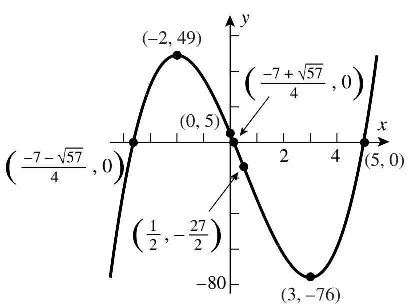
43. $f'(x) = -\dfrac{5}{(x-2)^2}$; no extrema.

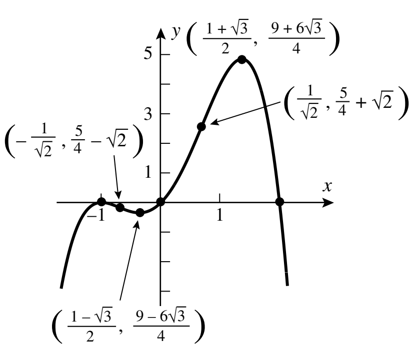
45. $f'(x) = \dfrac{2x}{2+x^2}$; critical point at $x = 0$, $f''(0) = 1$; f has a relative minimum of $\ln 2$ at $x = 0$.

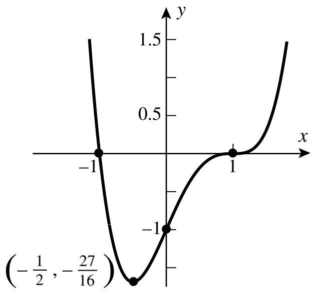
47. $f'(x) = 2e^{2x} - e^x$; critical point $x = -\ln 2$, $f''(-\ln 2) = 1/2$; relative minimum of $-1/4$ at $x = -\ln 2$.

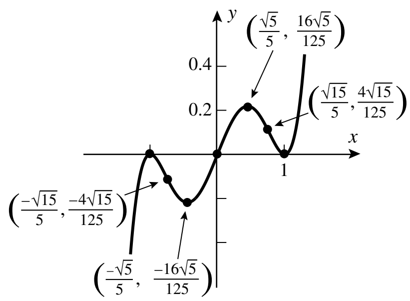
49. $f'(x)$ is undefined at $x = 0, 3$, so these are critical points. Elsewhere, $f'(x) = \begin{cases} 2x - 3 & \text{if} \quad x < 0 \text{ or } x > 3; \\ 3 - 2x & \text{if} \quad 0 < x < 3. \end{cases}$ $f'(x) = 0$ for $x = 3/2$, so this is also a critical point. $f''(3/2) = -2$, so relative maximum of $9/4$ at $x = 3/2$. By the first derivative test, relative minimum of 0 at $x = 0$ and $x = 3$.

51. 

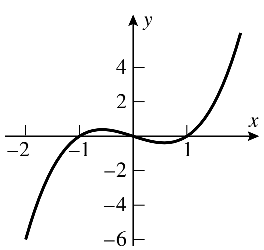
53. 

55. 

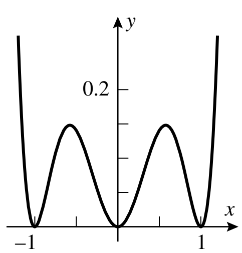
57. 

59. 

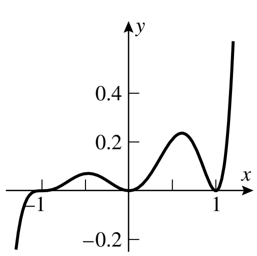
61. **(a)** $\lim_{x\to-\infty} y = -\infty$, $\lim_{x\to+\infty} y = +\infty$; curve crosses x-axis at $x = 0, 1, -1$.



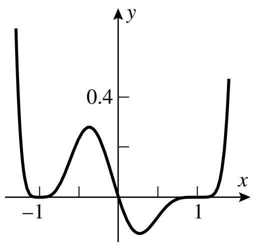
(b) $\lim_{x\to\pm\infty} y = +\infty$; curve never crosses x-axis.



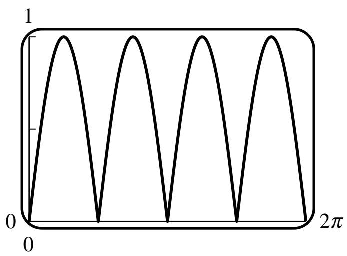
(c) $\lim_{x\to-\infty} y = -\infty$, $\lim_{x\to+\infty} y = +\infty$; curve crosses x-axis at $x = -1$



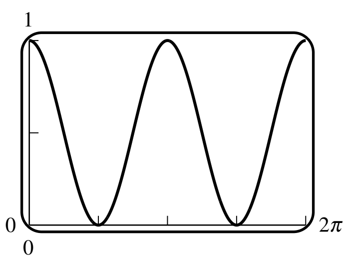
(d) $\lim_{x\to\pm\infty} y = +\infty$; curve crosses x-axis at $x = 0, 1$.



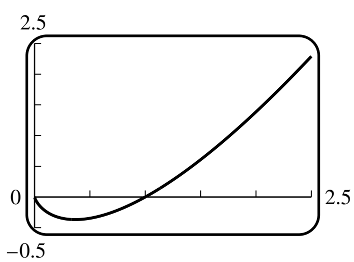
63. $f'(x) = 2\cos 2x$ if $\sin 2x > 0$, $f'(x) = -2\cos 2x$ if $\sin 2x < 0$, $f'(x)$ does not exist when $x = \pi/2, \pi, 3\pi/2$; critical numbers $x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4, \pi/2, \pi, 3\pi/2$, relative minimum of 0 at $x = \pi/2, \pi, 3\pi/2$; relative maximum of 1 at $x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.



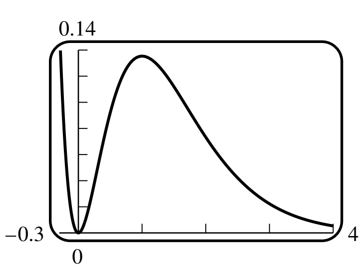
65. $f'(x) = -\sin 2x$; critical numbers $x = \pi/2, \pi, 3\pi/2$, relative minimum of 0 at $x = \pi/2, 3\pi/2$; relative maximum of 1 at $x = \pi$.



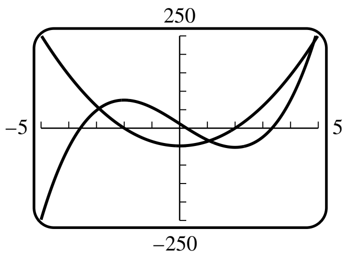
67. $f'(x) = \ln x + 1$, $f''(x) = 1/x$; $f'(1/e) = 0$, $f''(1/e) > 0$; relative minimum of $-1/e$ at $x = 1/e$.



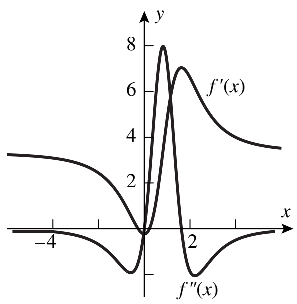
69. $f'(x) = 2x(1-x)e^{-2x} = 0$ at $x = 0, 1$. $f''(x) = (4x^2 - 8x + 2)e^{-2x}$; $f''(0) > 0$ and $f''(1) < 0$, so a relative minimum of 0 at $x = 0$ and a relative maximum of $1/e^2$ at $x = 1$.



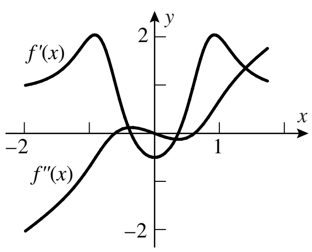
71. Relative minima at $x \approx -3.58, 3.33$; relative maximum at $x \approx 0.25$.



73. Relative maximum at $x \approx -0.272$, relative minimum at $x \approx 0.224$.



75. $f'(x) = \dfrac{4x^3 - \sin 2x}{2\sqrt{x^4 + \cos^2 x}}$, $f''(x) = \dfrac{6x^2 - \cos 2x}{\sqrt{x^4 + \cos^2 x}} - \dfrac{(4x^3 - \sin 2x)(4x^3 - \sin 2x)}{4(x^4 + \cos^2 x)^{3/2}}$. Relative minima at $x \approx \pm 0.618$, relative maximum at $x = 0$.

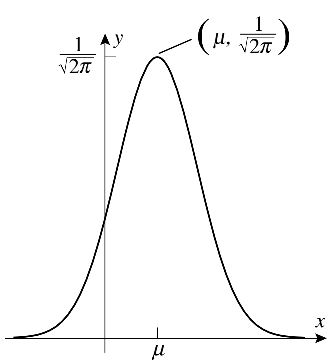


77. **(a)** Let $f(x) = x^2 + \dfrac{k}{x}$, then $f'(x) = 2x - \dfrac{k}{x^2} = \dfrac{2x^3 - k}{x^2}$. f has a relative extremum when $2x^3 - k = 0$, so $k = 2x^3 = 2(3)^3 = 54$.

(b) Let $f(x) = \dfrac{x}{x^2 + k}$, then $f'(x) = \dfrac{k - x^2}{(x^2 + k)^2}$. f has a relative extremum when $k - x^2 = 0$, so $k = x^2 = 3^2 = 9$.

79. **(a)** $f'(x) = -xf(x)$. Since $f(x)$ is always positive, $f'(x) = 0$ at $x = 0$, $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$, so $x = 0$ is a maximum.

(b)



81. **(a)** Because h and g have relative maxima at x_0, $h(x) \le h(x_0)$ for all x in I_1 and $g(x) \le g(x_0)$ for all x in I_2, where I_1 and I_2 are open intervals containing x_0. If x is in both I_1 and I_2 then both inequalities are true and by addition so is $h(x) + g(x) \le h(x_0) + g(x_0)$ which shows that $h + g$ has a relative maximum at x_0.

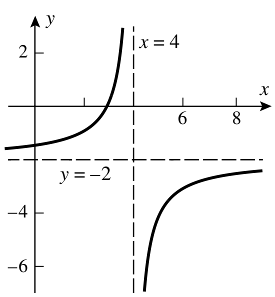
(b) By counterexample; both $h(x) = -x^2$ and $g(x) = -2x^2$ have relative maxima at $x = 0$ but $h(x) - g(x) = x^2$ has a relative minimum at $x = 0$ so in general $h - g$ does not necessarily have a relative maximum at x_0.

83. The first derivative test applies in many cases where the second derivative test does not. For example, it implies that $|x|$ has a relative minimum at $x = 0$, but the second derivative test does not, since $|x|$ is not differentiable there.

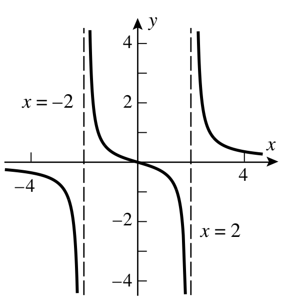
The second derivative test is often easier to apply, since we only need to compute $f'(x_0)$ and $f''(x_0)$, instead of analyzing $f'(x)$ at values of x near x_0. For example, let $f(x) = 10x^3 + (1 - x)e^x$. Then $f'(x) = 30x^2 - xe^x$ and $f''(x) = 60x - (x + 1)e^x$. Since $f'(0) = 0$ and $f''(0) = -1$, the second derivative test tells us that f has a relative maximum at $x = 0$. To prove this using the first derivative test is slightly more difficult, since we need to determine the sign of $f'(x)$ for x near, but not equal to, 0.

Exercise Set 4.3

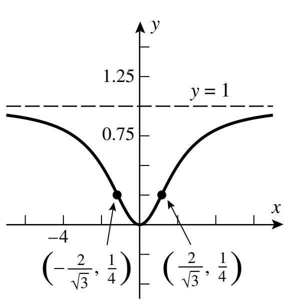
1. Vertical asymptote $x = 4$, horizontal asymptote $y = -2$.



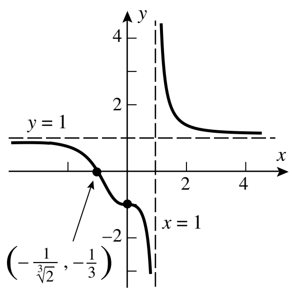
3. Vertical asymptotes $x = \pm 2$, horizontal asymptote $y = 0$.



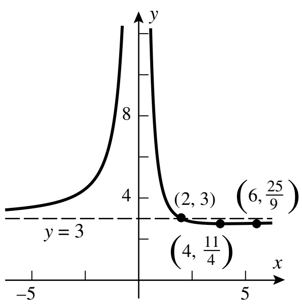
5. No vertical asymptotes, horizontal asymptote $y = 1$.



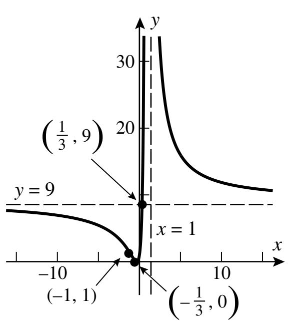
7. Vertical asymptote $x = 1$, horizontal asymptote $y = 1$.



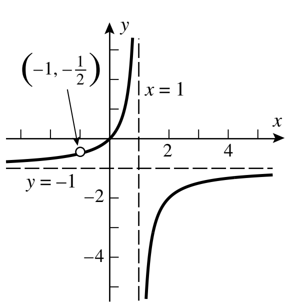
9. Vertical asymptote $x = 0$, horizontal asymptote $y = 3$.



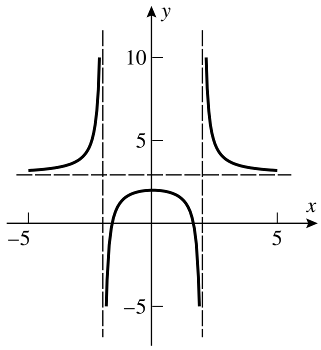
11. Vertical asymptote $x = 1$, horizontal asymptote $y = 9$.



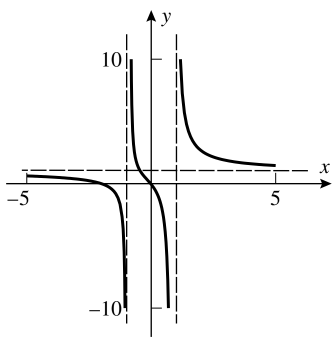
13. Vertical asymptote $x = 1$, horizontal asymptote $y = -1$.



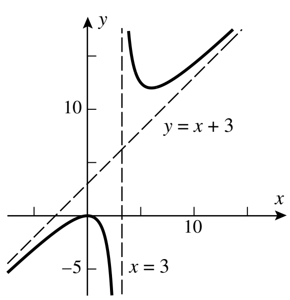
15. (a) Horizontal asymptote $y = 3$ as $x \to \pm\infty$, vertical asymptotes at $x = \pm 2$.



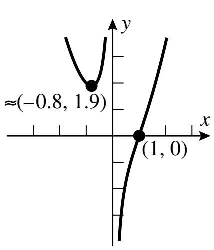
(b) Horizontal asymptote of $y = 1$ as $x \to \pm\infty$, vertical asymptotes at $x = \pm 1$.



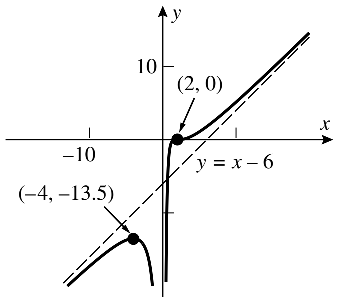
17. $\lim_{x\to\pm\infty}\left|\frac{x^2}{x-3}-(x+3)\right| = \lim_{x\to\pm\infty}\left|\frac{9}{x-3}\right| = 0.$



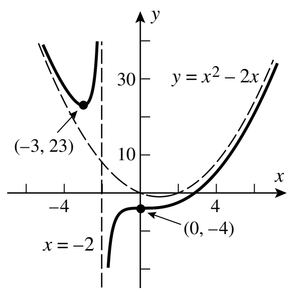
19. $y = x^2 - \frac{1}{x} = \frac{x^3-1}{x}$; y-axis is a vertical asymptote; $y' = \frac{2x^3+1}{x^2}$, $y' = 0$ when $x = -\sqrt[3]{\frac{1}{2}} \approx -0.8$; $y'' = \frac{2(x^3-1)}{x^3}$, curvilinear asymptote $y = x^2$.



21. $y = \frac{(x-2)^3}{x^2} = x - 6 + \frac{12x-8}{x^2}$ so y-axis is a vertical asymptote, $y = x - 6$ is an oblique asymptote; $y' = \frac{(x-2)^2(x+4)}{x^3}$, $y'' = \frac{24(x-2)}{x^4}$.



23. $y = \dfrac{x^3 - 4x - 8}{x + 2} = x^2 - 2x - \dfrac{8}{x+2}$ so $x = -2$ is a vertical asymptote, $y = x^2 - 2x$ is a curvilinear asymptote as $x \to \pm\infty$.

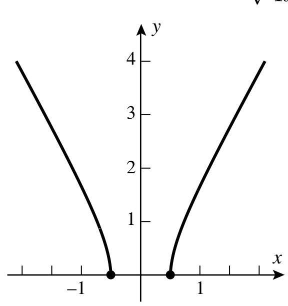


25. **(a)** VI **(b)** I **(c)** III **(d)** V **(e)** IV **(f)** II

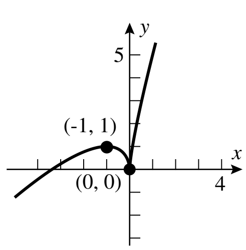
27. True. If the degree of P were larger than the degree of Q, then $\lim\limits_{x\to\pm\infty} f(x)$ would be infinite and the graph would not have a horizontal asymptote. If the degree of P were less than the degree of Q, then $\lim\limits_{x\to\pm\infty} f(x)$ would be zero, so the horizontal asymptote would be $y = 0$, not $y = 5$.

29. False. Let $f(x) = \sqrt[3]{x-1}$. Then f is continuous at $x = 1$, but $\lim\limits_{x\to 1} f'(x) = \lim\limits_{x\to 1} \dfrac{1}{3}(x-1)^{-2/3} = +\infty$, so f' has a vertical asymptote at $x = 1$.

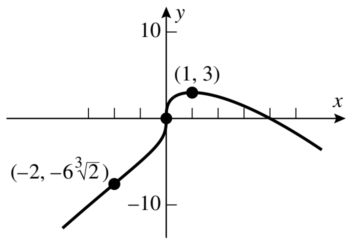
31. $y = \sqrt{4x^2 - 1}$, $y' = \dfrac{4x}{\sqrt{4x^2-1}}$, $y'' = -\dfrac{4}{(4x^2-1)^{3/2}}$ so extrema when $x = \pm\dfrac{1}{2}$, no inflection points.



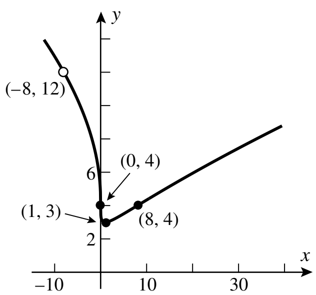
33. $y = 2x + 3x^{2/3}$; $y' = 2 + 2x^{-1/3}$; $y'' = -\dfrac{2}{3}x^{-4/3}$.



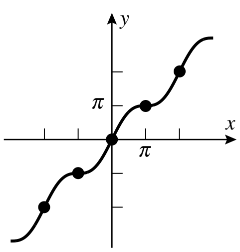
35. $y = x^{1/3}(4-x)$; $y' = \dfrac{4(1-x)}{3x^{2/3}}$; $y'' = -\dfrac{4(x+2)}{9x^{5/3}}$.



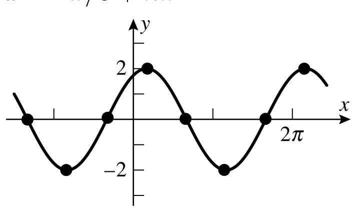
37. $y = x^{2/3} - 2x^{1/3} + 4$; $y' = \dfrac{2(x^{1/3}-1)}{3x^{2/3}}$; $y'' = -\dfrac{2(x^{1/3}-2)}{9x^{5/3}}$.



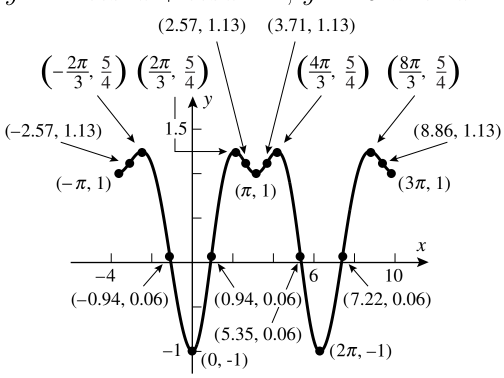
39. $y = x + \sin x$; $y' = 1 + \cos x$, $y' = 0$ when $x = \pi + 2n\pi$; $y'' = -\sin x$; $y'' = 0$ when $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$



41. $y = \sqrt{3}\cos x + \sin x$; $y' = -\sqrt{3}\sin x + \cos x$; $y' = 0$ when $x = \pi/6 + n\pi$; $y'' = -\sqrt{3}\cos x - \sin x$; $y'' = 0$ when $x = 2\pi/3 + n\pi$.

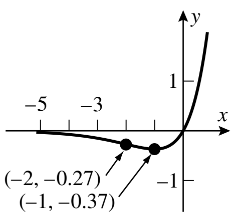


43. $y = \sin^2 x - \cos x$; $y' = \sin x(2\cos x + 1)$; $y' = 0$ when $x = -\pi, 0, \pi, 2\pi, 3\pi$ and when $x = -\dfrac{2}{3}\pi, \dfrac{2}{3}\pi, \dfrac{4}{3}\pi, \dfrac{8}{3}\pi$; $y'' = 4\cos^2 x + \cos x - 2$; $y'' = 0$ when $x \approx \pm 2.57$, $\pm 0.94, 3.71, 5.35, 7.22, 8.86$.



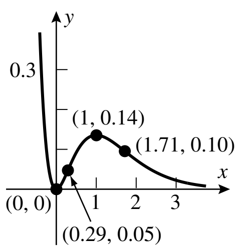
45. **(a)** $\lim_{x\to+\infty} xe^x = +\infty$, $\lim_{x\to-\infty} xe^x = 0$.

(b) $y = xe^x$; $y' = (x+1)e^x$; $y'' = (x+2)e^x$; relative minimum at $(-1, -e^{-1}) \approx (-1, -0.37)$, inflection point at $(-2, -2e^{-2}) \approx (-2, -0.27)$, horizontal asymptote $y = 0$ as $x \to -\infty$.



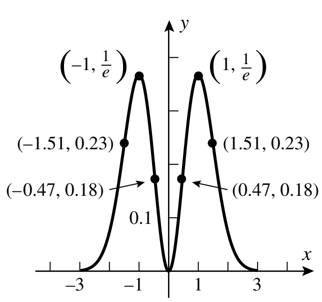
47. **(a)** $\lim_{x\to+\infty} \frac{x^2}{e^{2x}} = 0$, $\lim_{x\to-\infty} \frac{x^2}{e^{2x}} = +\infty$.

(b) $y = x^2/e^{2x} = x^2e^{-2x}$; $y' = 2x(1-x)e^{-2x}$; $y'' = 2(2x^2-4x+1)e^{-2x}$; $y'' = 0$ if $2x^2 - 4x + 1 = 0$, when $x = \frac{4 \pm \sqrt{16-8}}{4} = 1 \pm \sqrt{2}/2 \approx 0.29, 1.71$, horizontal asymptote $y = 0$ as $x \to +\infty$.



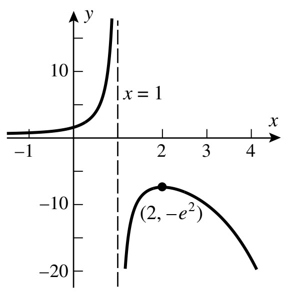
49. **(a)** $\lim_{x\to\pm\infty} x^2e^{-x^2} = 0$.

(b) $y = x^2e^{-x^2}$; $y' = 2x(1-x^2)e^{-x^2}$; $y' = 0$ if $x = 0, \pm 1$; $y'' = 2(1 - 5x^2 + 2x^4)e^{-x^2}$; $y'' = 0$ if $2x^4 - 5x^2 + 1 = 0$, $x^2 = \frac{5 \pm \sqrt{17}}{4}$, $x = \pm\frac{1}{2}\sqrt{5+\sqrt{17}} \approx \pm 1.51$, $x = \pm\frac{1}{2}\sqrt{5-\sqrt{17}} \approx \pm 0.47$, horizontal asymptote $y = 0$ as $x \to \pm\infty$.



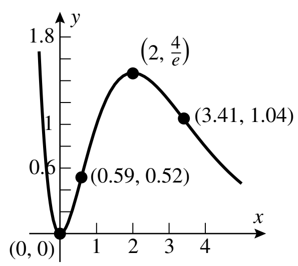
51. **(a)** $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to+\infty} f(x) = -\infty$.

(b) $f'(x) = -\frac{e^x(x-2)}{(x-1)^2}$ so $f'(x) = 0$ when $x = 2$, $f''(x) = -\frac{e^x(x^2-4x+5)}{(x-1)^3}$ so $f''(x) \neq 0$ always, relative maximum when $x = 2$, no point of inflection, vertical asymptote $x = 1$, horizontal asymptote $y = 0$ as $x \to -\infty$.



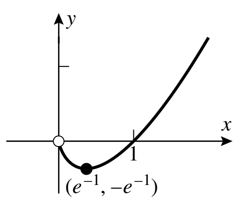
53. (a) $\lim_{x\to+\infty} f(x) = 0$, $\lim_{x\to-\infty} f(x) = +\infty$.

(b) $f'(x) = x(2-x)e^{1-x}$, $f''(x) = (x^2 - 4x + 2)e^{1-x}$, critical points at $x = 0, 2$; relative minimum at $x = 0$, relative maximum at $x = 2$, points of inflection at $x = 2 \pm \sqrt{2}$, horizontal asymptote $y = 0$ as $x \to +\infty$.



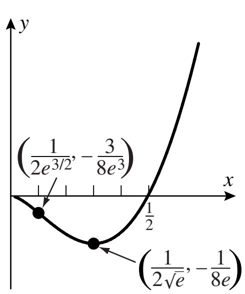
55. (a) $\lim_{x\to 0^+} y = \lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \dfrac{\ln x}{1/x} = \lim_{x\to 0^+} \dfrac{1/x}{-1/x^2} = 0$; $\lim_{x\to+\infty} y = +\infty$.

(b) $y = x \ln x$, $y' = 1 + \ln x$, $y'' = 1/x$, $y' = 0$ when $x = e^{-1}$.



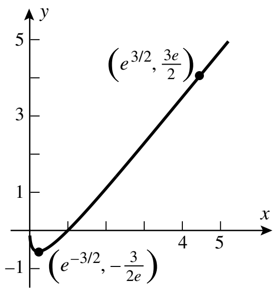
57. (a) $\lim_{x\to 0^+} x^2 \ln(2x) = \lim_{x\to 0^+} (x^2 \ln 2) + \lim_{x\to 0^+} (x^2 \ln x) = 0$ by the rule given, $\lim_{x\to+\infty} x^2 \ln x = +\infty$ by inspection.

(b) $y = x^2 \ln(2x)$, $y' = 2x\ln(2x) + x$, $y'' = 2\ln(2x) + 3$, $y' = 0$ if $x = 1/(2\sqrt{e})$, $y'' = 0$ if $x = 1/(2e^{3/2})$.

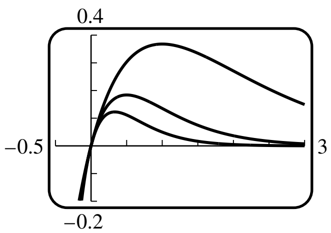


59. (a) $\lim_{x\to+\infty} f(x) = +\infty$, $\lim_{x\to 0+} f(x) = 0$.

(b) $y = x^{2/3} \ln x$, $y' = \dfrac{2\ln x + 3}{3x^{1/3}}$, $y' = 0$ when $\ln x = -\dfrac{3}{2}$, $x = e^{-3/2}$, $y'' = \dfrac{-3 + 2\ln x}{9x^{4/3}}$, $y'' = 0$ when $\ln x = \dfrac{3}{2}$, $x = e^{3/2}$.



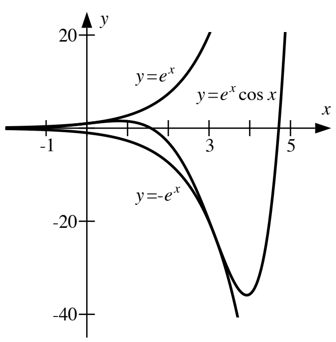
61. **(a)**



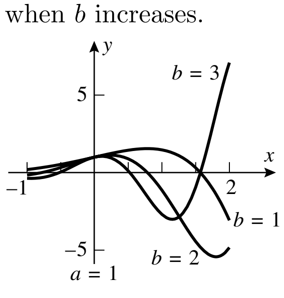
(b) $y' = (1 - bx)e^{-bx}$, $y'' = b^2(x - 2/b)e^{-bx}$; relative maximum at $x = 1/b$, $y = 1/(be)$; point of inflection at $x = 2/b$, $y = 2/(be^2)$. Increasing b moves the relative maximum and the point of inflection to the left and down, i.e. towards the origin.

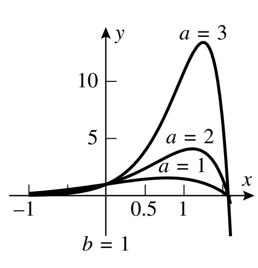
63. **(a)** The oscillations of $e^x \cos x$ about zero increase as $x \to +\infty$ so the limit does not exist, and $\lim_{x\to-\infty} e^x \cos x = 0$.

(b) $y = e^x$ and $y = e^x \cos x$ intersect for $x = 2\pi n$ for any integer n. $y = -e^x$ and $y = e^x \cos x$ intersect for $x = 2\pi n + \pi$ for any integer n. On the graph below, the intersections are at $(0, 1)$ and $(\pi, -e^\pi)$.



(c) The curve $y = e^{ax} \cos bx$ oscillates between $y = e^{ax}$ and $y = -e^{ax}$. The frequency of oscillation increases when b increases.





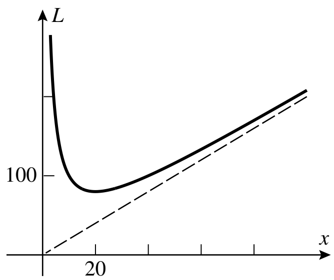
65. **(a)** $x = 1, 2.5, 4$ and $x = 3$, the latter being a cusp.

(b) $(-\infty, 1], [2.5, 3)$.

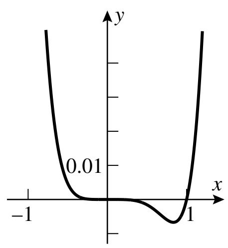
(c) Relative maxima for $x = 1, 3$; relative minima for $x = 2.5$.

(d) $x \approx 0.6, 1.9, 4.$

67. Let y be the length of the other side of the rectangle, then $L = 2x + 2y$ and $xy = 400$ so $y = 400/x$ and hence $L = 2x + 800/x$. $L = 2x$ is an oblique asymptote. $L = 2x + \dfrac{800}{x} = \dfrac{2(x^2+400)}{x}$, $L' = 2 - \dfrac{800}{x^2} = \dfrac{2(x^2-400)}{x^2}$, $L'' = \dfrac{1600}{x^3}$, $L' = 0$ when $x = 20, L = 80$.



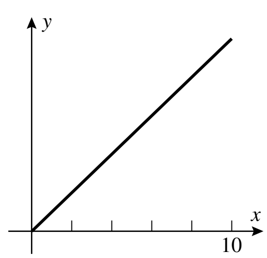
69. $y' = 0.1x^4(6x-5)$; critical numbers: $x = 0$, $x = 5/6$; relative minimum at $x = 5/6$, $y \approx -6.7 \times 10^{-3}$.



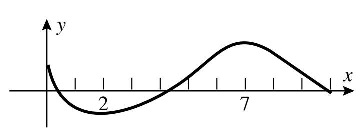
Exercise Set 4.4

1. Relative maxima at $x = 2, 6$; absolute maximum at $x = 6$; relative minimum at $x = 4$; absolute minima at $x = 0, 4$.

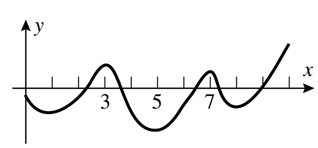
3. **(a)**



(b)



(c)



5. The minimum value is clearly 0; there is no maximum because $\lim_{x\to 1^-} f(x) = \infty$. $x = 1$ is a point of discontinuity of f.

7. $f'(x) = 8x - 12$, $f'(x) = 0$ when $x = 3/2$; $f(1) = 2$, $f(3/2) = 1$, $f(2) = 2$ so the maximum value is 2 at $x = 1, 2$ and the minimum value is 1 at $x = 3/2$.

9. $f'(x) = 3(x-2)^2, f'(x) = 0$ when $x = 2$; $f(1) = -1, f(2) = 0, f(4) = 8$ so the minimum is -1 at $x = 1$ and the maximum is 8 at $x = 4$.

11. $f'(x) = 3/(4x^2+1)^{3/2}$, no critical points; $f(-1) = -3/\sqrt{5}$, $f(1) = 3/\sqrt{5}$ so the maximum value is $3/\sqrt{5}$ at $x = 1$ and the minimum value is $-3/\sqrt{5}$ at $x = -1$.

13. $f'(x) = 1 - 2\cos x, f'(x) = 0$ when $x = \pi/3$; then $f(-\pi/4) = -\pi/4 + \sqrt{2}$; $f(\pi/3) = \pi/3 - \sqrt{3}$; $f(\pi/2) = \pi/2 - 2$, so f has a minimum of $\pi/3 - \sqrt{3}$ at $x = \pi/3$ and a maximum of $-\pi/4 + \sqrt{2}$ at $x = -\pi/4$.

15. $f(x) = 1 + |9 - x^2| = \begin{cases} 10 - x^2, & |x| \le 3 \\ -8 + x^2, & |x| > 3 \end{cases}$, $f'(x) = \begin{cases} -2x, & |x| < 3 \\ 2x, & |x| > 3 \end{cases}$, thus $f'(x) = 0$ when $x = 0$, $f'(x)$ does not exist for x in $(-5, 1)$ when $x = -3$ because $\lim_{x \to -3^-} f'(x) \neq \lim_{x \to -3^+} f'(x)$ (see Theorem preceding Exercise 65, Section 2.3); $f(-5) = 17$, $f(-3) = 1$, $f(0) = 10$, $f(1) = 9$ so the maximum value is 17 at $x = -5$ and the minimum value is 1 at $x = -3$.

17. True, by Theorem 4.4.2.

19. True, by Theorem 4.4.3.

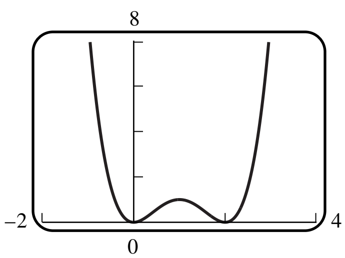
21. $f'(x) = 2x - 1$, $f'(x) = 0$ when $x = 1/2$; $f(1/2) = -9/4$ and $\lim_{x \to \pm\infty} f(x) = +\infty$. Thus f has a minimum of $-9/4$ at $x = 1/2$ and no maximum.

23. $f'(x) = 12x^2(1 - x)$; critical points $x = 0, 1$. Maximum value $f(1) = 1$, no minimum because $\lim_{x \to +\infty} f(x) = -\infty$.

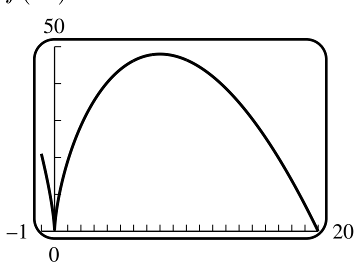
25. No maximum or minimum because $\lim_{x \to +\infty} f(x) = +\infty$ and $\lim_{x \to -\infty} f(x) = -\infty$.

27. $\lim_{x \to -1^-} f(x) = -\infty$, so there is no absolute minimum on the interval; $f'(x) = \dfrac{x^2 + 2x - 1}{(x + 1)^2} = 0$ at $x = -1 - \sqrt{2}$, for which $y = -2 - 2\sqrt{2} \approx -4.828$. Also $f(-5) = -13/2$, so the absolute maximum of f on the interval is $y = -2 - 2\sqrt{2}$, taken at $x = -1 - \sqrt{2}$.

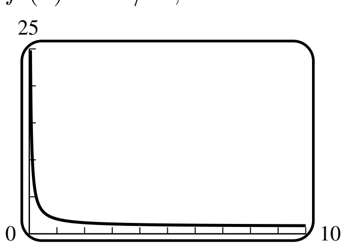
29. $\lim_{x \to \pm\infty} = +\infty$ so there is no absolute maximum. $f'(x) = 4x(x - 2)(x - 1)$, $f'(x) = 0$ when $x = 0, 1, 2$, and $f(0) = 0, f(1) = 1, f(2) = 0$ so f has an absolute minimum of 0 at $x = 0, 2$.



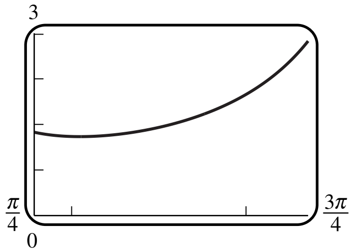
31. $f'(x) = \dfrac{5(8 - x)}{3x^{1/3}}$, $f'(x) = 0$ when $x = 8$ and $f'(x)$ does not exist when $x = 0$; $f(-1) = 21$, $f(0) = 0$, $f(8) = 48$, $f(20) = 0$ so the maximum value is 48 at $x = 8$ and the minimum value is 0 at $x = 0, 20$.



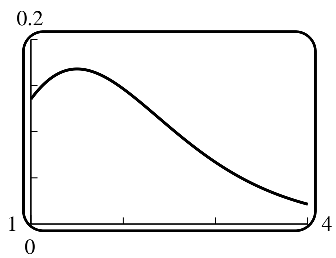
33. $f'(x) = -1/x^2$; no maximum or minimum because there are no critical points in $(0, +\infty)$.



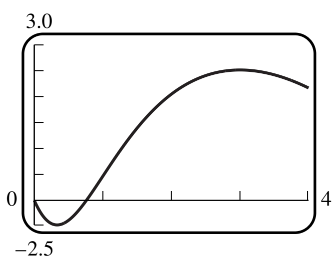
35. $f'(x) = \dfrac{1 - 2\cos x}{\sin^2 x}$; $f'(x) = 0$ on $[\pi/4, 3\pi/4]$ only when $x = \pi/3$. Then $f(\pi/4) = 2\sqrt{2} - 1$, $f(\pi/3) = \sqrt{3}$ and $f(3\pi/4) = 2\sqrt{2} + 1$, so f has an absolute maximum value of $2\sqrt{2} + 1$ at $x = 3\pi/4$ and an absolute minimum value of $\sqrt{3}$ at $x = \pi/3$.



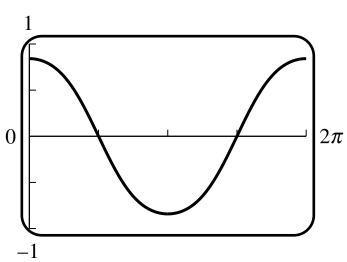
37. $f'(x) = x^2(3 - 2x)e^{-2x}$, $f'(x) = 0$ for x in $[1, 4]$ when $x = 3/2$; if $x = 1, 3/2, 4$, then $f(x) = e^{-2}, \dfrac{27}{8}e^{-3}, 64e^{-8}$; critical point at $x = 3/2$; absolute maximum of $\dfrac{27}{8}e^{-3}$ at $x = 3/2$, absolute minimum of $64e^{-8}$ at $x = 4$.



39. $f'(x) = -\dfrac{3x^2 - 10x + 3}{x^2 + 1}$, $f'(x) = 0$ when $x = \dfrac{1}{3}, 3$. $f(0) = 0$, $f\left(\dfrac{1}{3}\right) = 5\ln\left(\dfrac{10}{9}\right) - 1$, $f(3) = 5\ln 10 - 9$, $f(4) = 5\ln 17 - 12$ and thus f has an absolute minimum of $5(\ln 10 - \ln 9) - 1$ at $x = 1/3$ and an absolute maximum of $5\ln 10 - 9$ at $x = 3$.



41. $f'(x) = -[\cos(\cos x)]\sin x$; $f'(x) = 0$ if $\sin x = 0$ or if $\cos(\cos x) = 0$. If $\sin x = 0$, then $x = \pi$ is the critical point in $(0, 2\pi)$; $\cos(\cos x) = 0$ has no solutions because $-1 \le \cos x \le 1$. Thus $f(0) = \sin(1)$, $f(\pi) = \sin(-1) = -\sin(1)$, and $f(2\pi) = \sin(1)$ so the maximum value is $\sin(1) \approx 0.84147$ and the minimum value is $-\sin(1) \approx -0.84147$.



43. $f'(x) = \begin{cases} 4, & x < 1 \\ 2x - 5, & x > 1 \end{cases}$ so $f'(x) = 0$ when $x = 5/2$, and $f'(x)$ does not exist when $x = 1$ because $\lim\limits_{x \to 1^-} f'(x) \neq \lim\limits_{x \to 1^+} f'(x)$ (see Theorem preceding Exercise 65, Section 2.3); $f(1/2) = 0$, $f(1) = 2$, $f(5/2) = -1/4$, $f(7/2) = 3/4$ so the maximum value is 2 and the minimum value is $-1/4$.

45. The period of $f(x)$ is 2π, so check $f(0) = 3, f(2\pi) = 3$ and the critical points. $f'(x) = -2\sin x - 2\sin 2x = -2\sin x(1+2\cos x) = 0$ on $[0, 2\pi]$ at $x = 0, \pi, 2\pi$ and $x = 2\pi/3, 4\pi/3$. Check $f(\pi) = -1, f(2\pi/3) = -3/2, f(4\pi/3) = -3/2$. Thus f has an absolute maximum on $(-\infty, +\infty)$ of 3 at $x = 2k\pi, k = 0, \pm1, \pm2, \ldots$ and an absolute minimum of $-3/2$ at $x = 2k\pi \pm 2\pi/3, k = 0, \pm1, \pm2, \ldots$.

47. Let $f(x) = x - \sin x$, then $f'(x) = 1 - \cos x$ and so $f'(x) = 0$ when $\cos x = 1$ which has no solution for $0 < x < 2\pi$ thus the minimum value of f must occur at 0 or 2π. $f(0) = 0$, $f(2\pi) = 2\pi$ so 0 is the minimum value on $[0, 2\pi]$ thus $x - \sin x \geq 0$, $\sin x \leq x$ for all x in $[0, 2\pi]$.

49. Let $m =$ slope at x, then $m = f'(x) = 3x^2 - 6x + 5$, $dm/dx = 6x - 6$; critical point for m is $x = 1$, minimum value of m is $f'(1) = 2$.

51. $\lim_{x\to+\infty} f(x) = +\infty$, $\lim_{x\to 8^+} f(x) = +\infty$, so there is no absolute maximum value of f for $x > 8$. By Table 4.4.3 there must be a minimum. Since $f'(x) = \dfrac{2x(-520 + 192x - 24x^2 + x^3)}{(x-8)^3}$, we must solve a quartic equation to find the critical points. But it is easy to see that $x = 0$ and $x = 10$ are real roots, and the other two are complex. Since $x = 0$ is not in the interval in question, we must have an absolute minimum of f on $(8, +\infty)$ of 125 at $x = 10$.

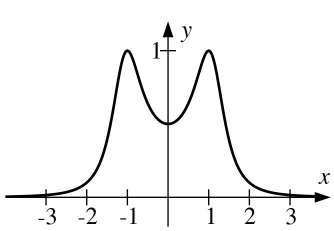
53. $dA/dx = -7 + \dfrac{2800}{x^2}$; thus $dA/dx = 0$ when $x = 20$ in (clearly $x > 0$, thus $x = -20$ is not feasible). The derivative changes sign from positive to negative at $x = 20$, and we found the location of the maximum printing area. The value of $A(x)$ at $x = 20$ in is $A(20) = 448$ in^2, thus the other side of the poster is $448/20 = 22.4$ in.

55. $dy/dx = (-8/x^2)\sqrt{x^2-1} + (8/x - 1)x/\sqrt{x^2-1} = ((-8/x^2)(x^2-1) + (8-x))/\sqrt{x^2-1} = (8/x^2 - x)/\sqrt{x^2-1}$; thus $dy/dx = 0$ when $x = 2$ ft. The derivative changes sign from positive to negative at $x = 2$, and we found the location of the maximum distance. The value of y at $x = 2$ is $y(2) = 3\sqrt{3}$ ft.

57. The absolute extrema of $y(t)$ can occur at the endpoints $t = 0, 12$ or when $dy/dt = 2\sin t = 0$, i.e. $t = 0, 12, k\pi$, $k = 1, 2, 3$; the absolute maximum is $y = 4$ at $t = \pi, 3\pi$; the absolute minimum is $y = 0$ at $t = 0, 2\pi$.

59. $f'(x) = 2ax + b$; critical point is $x = -\dfrac{b}{2a}$. $f''(x) = 2a > 0$ so $f\left(-\dfrac{b}{2a}\right)$ is the minimum value of f, but $f\left(-\dfrac{b}{2a}\right) = a\left(-\dfrac{b}{2a}\right)^2 + b\left(-\dfrac{b}{2a}\right) + c = \dfrac{-b^2 + 4ac}{4a}$ thus $f(x) \geq 0$ if and only if $f\left(-\dfrac{b}{2a}\right) \geq 0$, $\dfrac{-b^2 + 4ac}{4a} \geq 0$, $-b^2 + 4ac \geq 0$, $b^2 - 4ac \leq 0$.

61. If f has an absolute minimum, say at $x = a$, then, for all x, $f(x) \geq f(a) > 0$. But since $\lim_{x\to+\infty} f(x) = 0$, there is some x such that $f(x) < f(a)$. This contradiction shows that f cannot have an absolute minimum. On the other hand, let $f(x) = \dfrac{1}{(x^2-1)^2 + 1}$. Then $f(x) > 0$ for all x. Also, $\lim_{x\to+\infty} f(x) = 0$ so the x-axis is an asymptote, both as $x \to -\infty$ and as $x \to +\infty$. But since $f(0) = \frac{1}{2} < 1 = f(1) = f(-1)$, the absolute minimum of f on $[-1, 1]$ does not occur at $x = 1$ or $x = -1$, so it is a relative minimum. (In fact it occurs at $x = 0$.)



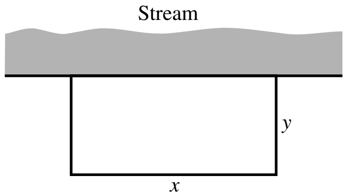
Exercise Set 4.5

1. If $y = x + 1/x$ for $1/2 \leq x \leq 3/2$, then $dy/dx = 1 - 1/x^2 = (x^2-1)/x^2$, $dy/dx = 0$ when $x = 1$. If $x = 1/2, 1, 3/2$, then $y = 5/2, 2, 13/6$ so

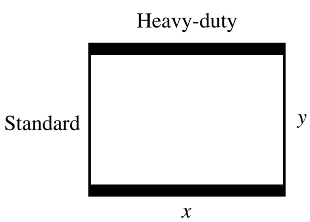
(a) y is as small as possible when $x = 1$.

(b) y is as large as possible when $x = 1/2$.

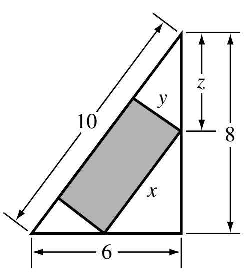
3. $A = xy$ where $x + 2y = 1000$ so $y = 500 - x/2$ and $A = 500x - x^2/2$ for x in $[0, 1000]$; $dA/dx = 500 - x$, $dA/dx = 0$ when $x = 500$. If $x = 0$ or 1000 then $A = 0$, if $x = 500$ then $A = 125,000$ so the area is maximum when $x = 500$ ft and $y = 500 - 500/2 = 250$ ft.



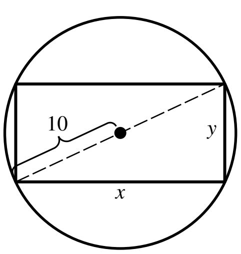
5. Let x and y be the dimensions shown in the figure and A the area, then $A = xy$ subject to the cost condition $3(2x) + 2(2y) = 6000$, or $y = 1500 - 3x/2$. Thus $A = x(1500 - 3x/2) = 1500x - 3x^2/2$ for x in $[0, 1000]$. $dA/dx = 1500 - 3x$, $dA/dx = 0$ when $x = 500$. If $x = 0$ or 1000 then $A = 0$, if $x = 500$ then $A = 375,000$ so the area is greatest when $x = 500$ ft and (from $y = 1500 - 3x/2$) when $y = 750$ ft.



7. Let x, y, and z be as shown in the figure and A the area of the rectangle, then $A = xy$ and, by similar triangles, $z/10 = y/6$, $z = 5y/3$; also $x/10 = (8 - z)/8 = (8 - 5y/3)/8$ thus $y = 24/5 - 12x/25$ so $A = x(24/5 - 12x/25) = 24x/5 - 12x^2/25$ for x in $[0, 10]$. $dA/dx = 24/5 - 24x/25$, $dA/dx = 0$ when $x = 5$. If $x = 0, 5, 10$ then $A = 0, 12, 0$ so the area is greatest when $x = 5$ in and $y = 12/5$ in.



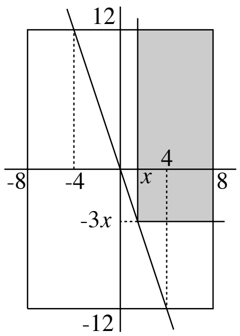
9. $A = xy$ where $x^2 + y^2 = 20^2 = 400$ so $y = \sqrt{400 - x^2}$ and $A = x\sqrt{400 - x^2}$ for $0 \le x \le 20$; $dA/dx = 2(200 - x^2)/\sqrt{400 - x^2}$, $dA/dx = 0$ when $x = \sqrt{200} = 10\sqrt{2}$. If $x = 0, 10\sqrt{2}, 20$ then $A = 0, 200, 0$ so the area is maximum when $x = 10\sqrt{2}$ and $y = \sqrt{400 - 200} = 10\sqrt{2}$.



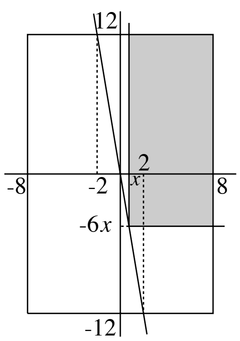
11. Let x = length of each side that uses the \$1 per foot fencing, y = length of each side that uses the \$2 per foot fencing. The cost is $C = (1)(2x) + (2)(2y) = 2x + 4y$, but $A = xy = 3200$ thus $y = 3200/x$ so $C = 2x + 12800/x$ for $x > 0$, $dC/dx = 2 - 12800/x^2$, $dC/dx = 0$ when $x = 80$, $d^2C/dx^2 > 0$ so C is least when $x = 80$, $y = 40$.

13. Let x and y be the dimensions of a rectangle; the perimeter is $p = 2x + 2y$. But $A = xy$ thus $y = A/x$ so $p = 2x + 2A/x$ for $x > 0$, $dp/dx = 2 - 2A/x^2 = 2(x^2 - A)/x^2$, $dp/dx = 0$ when $x = \sqrt{A}$, $d^2p/dx^2 = 4A/x^3 > 0$ if $x > 0$ so p is a minimum when $x = \sqrt{A}$ and $y = \sqrt{A}$ and thus the rectangle is a square.

15. Suppose that the lower left corner of S is at $(x, -3x)$. From the figure it's clear that the maximum area of the intersection of R and S occurs for some x in $[-4, 4]$, and the area is $A(x) = (8 - x)(12 + 3x) = 96 + 12x - 3x^2$. Since $A'(x) = 12 - 6x = 6(2 - x)$ is positive for $x < 2$ and negative for $x > 2$, $A(x)$ is increasing for x in $[-4, 2]$ and decreasing for x in $[2, 4]$. So the maximum area is $A(2) = 108$.



17. Suppose that the lower left corner of S is at $(x, -6x)$. From the figure it's clear that the maximum area of the intersection of R and S occurs for some x in $[-2, 2]$, and the area is $A(x) = (8 - x)(12 + 6x) = 96 + 36x - 6x^2$. Since $A'(x) = 36 - 12x = 12(3 - x)$ is positive for $x < 2$, $A(x)$ is increasing for x in $[-2, 2]$. So the maximum area is $A(2) = 144$.



19. Let the box have dimensions x, x, y, with $y \geq x$. The constraint is $4x + y \leq 108$, and the volume $V = x^2y$. If we take $y = 108 - 4x$ then $V = x^2(108 - 4x)$ and $dV/dx = 12x(-x + 18)$ with roots $x = 0, 18$. The maximum value of V occurs at $x = 18, y = 36$ with $V = 11{,}664$ in^3. The First Derivative Test shows this is indeed a maximum.

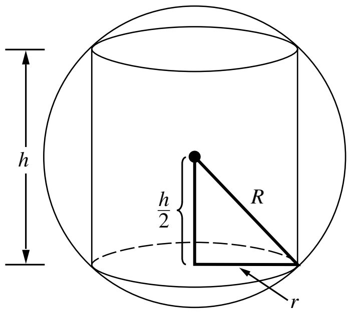
21. Let x be the length of each side of a square, then $V = x(3 - 2x)(8 - 2x) = 4x^3 - 22x^2 + 24x$ for $0 \leq x \leq 3/2$; $dV/dx = 12x^2 - 44x + 24 = 4(3x - 2)(x - 3)$, $dV/dx = 0$ when $x = 2/3$ for $0 < x < 3/2$. If $x = 0, 2/3, 3/2$ then $V = 0, 200/27, 0$ so the maximum volume is $200/27$ ft^3.

23. Let x = length of each edge of base, y = height, $k = \$/\text{cm}^2$ for the sides. The cost is $C = (2k)(2x^2) + (k)(4xy) = 4k(x^2 + xy)$, but $V = x^2y = 2000$ thus $y = 2000/x^2$ so $C = 4k(x^2 + 2000/x)$ for $x > 0$, $dC/dx = 4k(2x - 2000/x^2)$, $dC/dx = 0$ when $x = \sqrt[3]{1000} = 10$, $d^2C/dx^2 > 0$ so C is least when $x = 10$, $y = 20$.

25. Let x = height and width, y = length. The surface area is $S = 2x^2 + 3xy$ where $x^2y = V$, so $y = V/x^2$ and $S = 2x^2 + 3V/x$ for $x > 0$; $dS/dx = 4x - 3V/x^2$, $dS/dx = 0$ when $x = \sqrt[3]{3V/4}$, $d^2S/dx^2 > 0$ so S is minimum when $x = \sqrt[3]{\dfrac{3V}{4}}$, $y = \dfrac{4}{3}\sqrt[3]{\dfrac{3V}{4}}$.

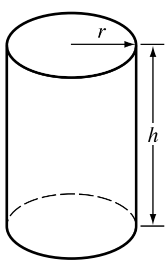
27. Let r and h be the dimensions shown in the figure, then the volume of the inscribed cylinder is $V = \pi r^2 h$. But

$r^2 + \left(\dfrac{h}{2}\right)^2 = R^2$ so $r^2 = R^2 - \dfrac{h^2}{4}$. Hence $V = \pi\left(R^2 - \dfrac{h^2}{4}\right)h = \pi\left(R^2 h - \dfrac{h^3}{4}\right)$ for $0 \le h \le 2R$. $\dfrac{dV}{dh} = \pi\left(R^2 - \dfrac{3}{4}h^2\right)$, $\dfrac{dV}{dh} = 0$ when $h = 2R/\sqrt{3}$. If $h = 0,\ 2R/\sqrt{3}, 2R$ then $V = 0, \dfrac{4\pi}{3\sqrt{3}}R^3, 0$ so the volume is largest when $h = 2R/\sqrt{3}$ and $r = \sqrt{2/3}R$.



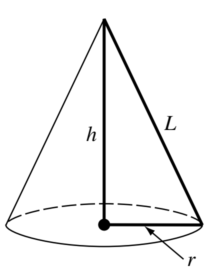
29. From (13), $S = 2\pi r^2 + 2\pi rh$. But $V = \pi r^2 h$ thus $h = V/(\pi r^2)$ and so $S = 2\pi r^2 + 2V/r$ for $r > 0$. $dS/dr = 4\pi r - 2V/r^2, dS/dr = 0$ if $r = \sqrt[3]{V/(2\pi)}$. Since $d^2S/dr^2 = 4\pi + 4V/r^3 > 0$, the minimum surface area is achieved when $r = \sqrt[3]{V/2\pi}$ and so $h = V/(\pi r^2) = [V/(\pi r^3)]r = 2r$.

31. The surface area is $S = \pi r^2 + 2\pi rh$ where $V = \pi r^2 h = 500$ so $h = 500/(\pi r^2)$ and $S = \pi r^2 + 1000/r$ for $r > 0$; $dS/dr = 2\pi r - 1000/r^2 = (2\pi r^3 - 1000)/r^2$, $dS/dr = 0$ when $r = \sqrt[3]{500/\pi}$, $d^2S/dr^2 > 0$ for $r > 0$ so S is minimum when $r = \sqrt[3]{500/\pi}$ cm and $h = \dfrac{500}{\pi r^2} = \dfrac{500}{\pi}\left(\dfrac{\pi}{500}\right)^{2/3} = \sqrt[3]{500/\pi}$ cm.



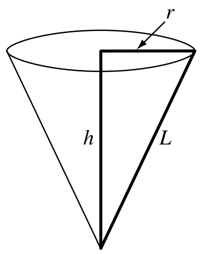
33. Let x be the length of each side of the squares and y the height of the frame, then the volume is $V = x^2 y$. The total length of the wire is L thus $8x + 4y = L$, $y = (L - 8x)/4$ so $V = x^2(L - 8x)/4 = (Lx^2 - 8x^3)/4$ for $0 \le x \le L/8$. $dV/dx = (2Lx - 24x^2)/4$, $dV/dx = 0$ for $0 < x < L/8$ when $x = L/12$. If $x = 0, L/12, L/8$ then $V = 0, L^3/1728, 0$ so the volume is greatest when $x = L/12$ and $y = L/12$.

35. Let h and r be the dimensions shown in the figure, then the volume is $V = \dfrac{1}{3}\pi r^2 h$. But $r^2 + h^2 = L^2$ thus $r^2 = L^2 - h^2$ so $V = \dfrac{1}{3}\pi(L^2 - h^2)h = \dfrac{1}{3}\pi(L^2 h - h^3)$ for $0 \le h \le L$. $\dfrac{dV}{dh} = \dfrac{1}{3}\pi(L^2 - 3h^2)$. $\dfrac{dV}{dh} = 0$ when $h = L/\sqrt{3}$. If $h = 0, L/\sqrt{3}, 0$ then $V = 0, \dfrac{2\pi}{9\sqrt{3}}L^3, 0$ so the volume is as large as possible when $h = L/\sqrt{3}$ and $r = \sqrt{2/3}L$.

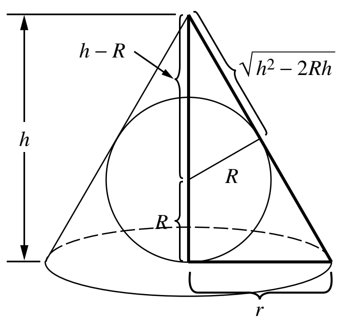


37. The area of the paper is $A = \pi rL = \pi r\sqrt{r^2 + h^2}$, but $V = \dfrac{1}{3}\pi r^2 h = 100$ so $h = 300/(\pi r^2)$ and $A =$

$\pi r\sqrt{r^2+90000/(\pi^2 r^4)}$. To simplify the computations let $S = A^2$, $S = \pi^2 r^2\left(r^2 + \dfrac{90000}{\pi^2 r^4}\right) = \pi^2 r^4 + \dfrac{90000}{r^2}$ for $r > 0$, $\dfrac{dS}{dr} = 4\pi^2 r^3 - \dfrac{180000}{r^3} = \dfrac{4(\pi^2 r^6 - 45000)}{r^3}$, $dS/dr = 0$ when $r = \sqrt[6]{45000/\pi^2}$, $d^2S/dr^2 > 0$, so S and hence A is least when $r = \sqrt[6]{45000/\pi^2} = \sqrt{2}\sqrt[3]{75/\pi}$ cm, $h = \dfrac{300}{\pi}\sqrt[3]{\pi^2/45000} = 2\sqrt[3]{75/\pi}$ cm.



39. The volume of the cone is $V = \dfrac{1}{3}\pi r^2 h$. By similar triangles (see figure) $\dfrac{r}{h} = \dfrac{R}{\sqrt{h^2-2Rh}}$, $r = \dfrac{Rh}{\sqrt{h^2-2Rh}}$ so $V = \dfrac{1}{3}\pi R^2\dfrac{h^3}{h^2-2Rh} = \dfrac{1}{3}\pi R^2\dfrac{h^2}{h-2R}$ for $h > 2R$, $\dfrac{dV}{dh} = \dfrac{1}{3}\pi R^2\dfrac{h(h-4R)}{(h-2R)^2}$, $\dfrac{dV}{dh} = 0$ for $h > 2R$ when $h = 4R$, by the first derivative test V is minimum when $h = 4R$. If $h = 4R$ then $r = \sqrt{2}R$.



41. The revenue is $R(x) = x(225 - 0.25x) = 225x - 0.25x^2$. The marginal revenue is $R'(x) = 225 - 0.5x = \dfrac{1}{2}(450 - x)$. Since $R'(x) > 0$ for $x < 450$ and $R'(x) < 0$ for $x > 450$, the maximum revenue occurs when the company mines 450 tons of ore.

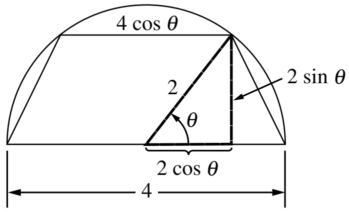
43. **(a)** The daily profit is $P =$ (revenue) $-$ (production cost) $= 100x - (100,000 + 50x + 0.0025x^2) = -100,000 + 50x - 0.0025x^2$ for $0 \le x \le 7000$, so $dP/dx = 50 - 0.005x$ and $dP/dx = 0$ when $x = 10,000$. Because 10,000 is not in the interval $[0, 7000]$, the maximum profit must occur at an endpoint. When $x = 0$, $P = -100,000$; when $x = 7000$, $P = 127,500$ so 7000 units should be manufactured and sold daily.

(b) Yes, because $dP/dx > 0$ when $x = 7000$ so profit is increasing at this production level.

(c) $dP/dx = 15$ when $x = 7000$, so $P(7001) - P(7000) \approx 15$, and the marginal profit is \$15.

45. The profit is $P =$ (profit on nondefective) $-$ (loss on defective) $= 100(x - y) - 20y = 100x - 120y$ but $y = 0.01x + 0.00003x^2$, so $P = 100x - 120(0.01x + 0.00003x^2) = 98.8x - 0.0036x^2$ for $x > 0$, $dP/dx = 98.8 - 0.0072x$, $dP/dx = 0$ when $x = 98.8/0.0072 \approx 13,722$, $d^2P/dx^2 < 0$ so the profit is maximum at a production level of about 13,722 pounds.

47. The area is (see figure) $A = \dfrac{1}{2}(2\sin\theta)(4 + 4\cos\theta) = 4(\sin\theta + \sin\theta\cos\theta)$ for $0 \le \theta \le \pi/2$; $dA/d\theta = 4(\cos\theta - \sin^2\theta + \cos^2\theta) = 4(\cos\theta - [1 - \cos^2\theta] + \cos^2\theta) = 4(2\cos^2\theta + \cos\theta - 1) = 4(2\cos\theta - 1)(\cos\theta + 1)$. $dA/d\theta = 0$ when $\theta = \pi/3$ for $0 < \theta < \pi/2$. If $\theta = 0, \pi/3, \pi/2$ then $A = 0, 3\sqrt{3}, 4$ so the maximum area is $3\sqrt{3}$.



49. $I = k\dfrac{\cos\phi}{\ell^2}$, k the constant of proportionality. If h is the height of the lamp above the table then $\cos\phi = h/\ell$ and $\ell = \sqrt{h^2+r^2}$ so $I = k\dfrac{h}{\ell^3} = k\dfrac{h}{(h^2+r^2)^{3/2}}$ for $h>0$, $\dfrac{dI}{dh} = k\dfrac{r^2-2h^2}{(h^2+r^2)^{5/2}}$, $\dfrac{dI}{dh} = 0$ when $h = r/\sqrt{2}$, by the first derivative test I is maximum when $h = r/\sqrt{2}$.

51. The distance between the particles is $D = \sqrt{(1-t-t)^2+(t-2t)^2} = \sqrt{5t^2-4t+1}$ for $t \geq 0$. For convenience, we minimize D^2 instead, so $D^2 = 5t^2-4t+1$, $dD^2/dt = 10t-4$, which is 0 when $t = 2/5$. $d^2D^2/dt^2 > 0$ so D^2 hence D is minimum when $t = 2/5$. The minimum distance is $D = 1/\sqrt{5}$.

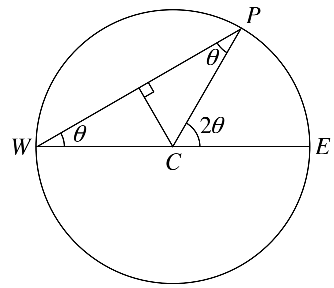
53. If $P(x_0, y_0)$ is on the curve $y = 1/x^2$, then $y_0 = 1/x_0^2$. At P the slope of the tangent line is $-2/x_0^3$ so its equation is $y - \dfrac{1}{x_0^2} = -\dfrac{2}{x_0^3}(x-x_0)$, or $y = -\dfrac{2}{x_0^3}x + \dfrac{3}{x_0^2}$. The tangent line crosses the y-axis at $\dfrac{3}{x_0^2}$, and the x-axis at $\dfrac{3}{2}x_0$. The length of the segment then is $L = \sqrt{\dfrac{9}{x_0^4}+\dfrac{9}{4}x_0^2}$ for $x_0 > 0$. For convenience, we minimize L^2 instead, so $L^2 = \dfrac{9}{x_0^4}+\dfrac{9}{4}x_0^2$, $\dfrac{dL^2}{dx_0} = -\dfrac{36}{x_0^5}+\dfrac{9}{2}x_0 = \dfrac{9(x_0^6-8)}{2x_0^5}$, which is 0 when $x_0^6 = 8$, $x_0 = \sqrt{2}$. $\dfrac{d^2L^2}{dx_0^2} > 0$ so L^2 and hence L is minimum when $x_0 = \sqrt{2}$, $y_0 = 1/2$.

55. At each point (x,y) on the curve the slope of the tangent line is $m = \dfrac{dy}{dx} = -\dfrac{2x}{(1+x^2)^2}$ for any x, $\dfrac{dm}{dx} = \dfrac{2(3x^2-1)}{(1+x^2)^3}$, $\dfrac{dm}{dx} = 0$ when $x = \pm 1/\sqrt{3}$, by the first derivative test the only relative maximum occurs at $x = -1/\sqrt{3}$, which is the absolute maximum because $\lim\limits_{x\to\pm\infty} m = 0$. The tangent line has greatest slope at the point $(-1/\sqrt{3}, 3/4)$.

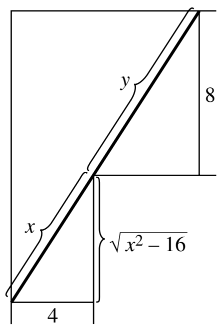
57. Let C be the center of the circle and let θ be the angle $\angle PWE$. Then $\angle PCE = 2\theta$, so the distance along the shore from E to P is 2θ miles. Also, the distance from P to W is $2\cos\theta$ miles. So Nancy takes $t(\theta) = \dfrac{2\theta}{8} + \dfrac{2\cos\theta}{2} = \dfrac{\theta}{4} + \cos\theta$ hours for her training routine; we wish to find the extrema of this for θ in $[0, \dfrac{\pi}{2}]$. We have $t'(\theta) = \dfrac{1}{4} - \sin\theta$, so the only critical point in $[0, \dfrac{\pi}{2}]$ is $\theta = \sin^{-1}(\dfrac{1}{4})$. So we compute $t(0) = 1$, $t(\sin^{-1}(\dfrac{1}{4})) = \dfrac{1}{4}\sin^{-1}(\dfrac{1}{4}) + \dfrac{\sqrt{15}}{4} \approx 1.0314$, and $t(\dfrac{\pi}{2}) = \dfrac{\pi}{8} \approx 0.3927$.

(a) The minimum is $t(\dfrac{\pi}{2}) = \dfrac{\pi}{8} \approx 0.3927$. To minimize the time, Nancy should choose $P = W$; i.e. she should jog all the way from E to W, π miles.

(b) The maximum is $t(\sin^{-1}(\dfrac{1}{4})) = \dfrac{1}{4}\sin^{-1}(\dfrac{1}{4}) + \dfrac{\sqrt{15}}{4} \approx 1.0314$. To maximize the time, she should jog $2\sin^{-1}(\dfrac{1}{4}) \approx 0.5054$ miles.

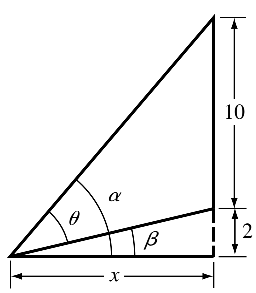


59. With x and y as shown in the figure, the maximum length of pipe will be the smallest value of $L = x + y$. By similar triangles $\dfrac{y}{8} = \dfrac{x}{\sqrt{x^2-16}}$, $y = \dfrac{8x}{\sqrt{x^2-16}}$ so $L = x + \dfrac{8x}{\sqrt{x^2-16}}$ for $x > 4$, $\dfrac{dL}{dx} = 1 - \dfrac{128}{(x^2-16)^{3/2}}$, $\dfrac{dL}{dx} = 0$ when $(x^2-16)^{3/2} = 128$, $x^2 - 16 = 128^{2/3} = 16(2^{2/3})$, $x^2 = 16(1+2^{2/3})$, $x = 4(1+2^{2/3})^{1/2}$, $d^2L/dx^2 = 384x/(x^2-16)^{5/2} > 0$ if $x > 4$ so L is smallest when $x = 4(1+2^{2/3})^{1/2}$. For this value of x, $L = 4(1+2^{2/3})^{3/2}$ ft.



61. Let x = distance from the weaker light source, I = the intensity at that point, and k the constant of proportionality. Then $I = \dfrac{kS}{x^2} + \dfrac{8kS}{(90-x)^2}$ if $0 < x < 90$; $\dfrac{dI}{dx} = -\dfrac{2kS}{x^3} + \dfrac{16kS}{(90-x)^3} = \dfrac{2kS[8x^3 - (90-x)^3]}{x^3(90-x)^3} = 18\dfrac{kS(x-30)(x^2+2700)}{x^3(x-90)^3}$, which is 0 when $x = 30$; $\dfrac{dI}{dx} < 0$ if $x < 30$, and $\dfrac{dI}{dx} > 0$ if $x > 30$, so the intensity is minimum at a distance of 30 cm from the weaker source.

63. $\theta = \alpha - \beta = \cot^{-1}(x/12) - \cot^{-1}(x/2)$, $\dfrac{d\theta}{dx} = -\dfrac{12}{144+x^2} + \dfrac{2}{4+x^2} = \dfrac{10(24-x^2)}{(144+x^2)(4+x^2)}$, $d\theta/dx = 0$ when $x = \sqrt{24} = 2\sqrt{6}$ feet, by the first derivative test θ is maximum there.



65. The total time required for the light to travel from A to P to B is $t =$ (time from A to P)+ (time from P to B) $= \dfrac{\sqrt{x^2+a^2}}{v_1} + \dfrac{\sqrt{(c-x)^2+b^2}}{v_2}$, $\dfrac{dt}{dx} = \dfrac{x}{v_1\sqrt{x^2+a^2}} - \dfrac{c-x}{v_2\sqrt{(c-x)^2+b^2}}$ but $x/\sqrt{x^2+a^2} = \sin\theta_1$ and $(c-x)/\sqrt{(c-x)^2+b^2} = \sin\theta_2$ thus $\dfrac{dt}{dx} = \dfrac{\sin\theta_1}{v_1} - \dfrac{\sin\theta_2}{v_2}$ so $\dfrac{dt}{dx} = 0$ when $\dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$.

67. $s = (x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2$, $ds/d\bar{x} = -2(x_1 - \bar{x}) - 2(x_2 - \bar{x}) - \cdots - 2(x_n - \bar{x})$, $ds/d\bar{x} = 0$ when $(x_1 - \bar{x}) + (x_2 - \bar{x}) + \cdots + (x_n - \bar{x}) = 0$, $(x_1 + x_2 + \cdots + x_n) - n\bar{x} = 0$, $\bar{x} = \dfrac{1}{n}(x_1 + x_2 + \cdots + x_n)$, $d^2s/d\bar{x}^2 = 2 + 2 + \cdots + 2 = 2n > 0$, so s is minimum when $\bar{x} = \dfrac{1}{n}(x_1 + x_2 + \cdots + x_n)$.

69. If we ignored the interval of possible values of the variables, we might find an extremum that is not physically meaningful, or conclude that there is no extremum. For instance, in Example 2, if we didn't restrict x to the interval $[0, 8]$, there would be no maximum value of V, since $\lim_{x \to +\infty} (480x - 92x^2 + 4x^3) = +\infty$.

Exercise Set 4.6

1. **(a)** Positive, negative, slowing down.

(b) Positive, positive, speeding up.

(c) Negative, positive, slowing down.

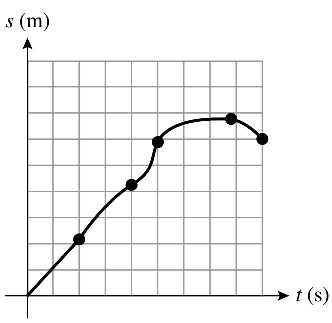
3. **(a)** Left because $v = ds/dt < 0$ at t_0.

(b) Negative because $a = d^2s/dt^2$ and the curve is concave down at $t_0 (d^2s/dt^2 < 0)$.

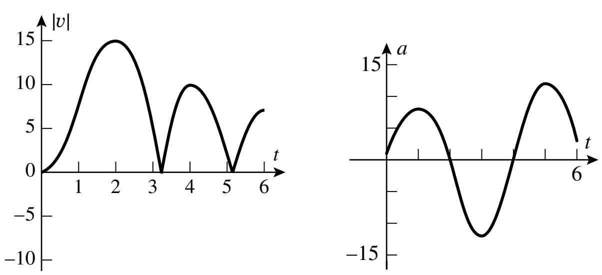
(c) Speeding up because v and a have the same sign.

(d) $v < 0$ and $a > 0$ at t_1 so the particle is slowing down because v and a have opposite signs.

5.



7.



9. False. A particle is speeding up when its speed versus time curve is increasing. When the position versus time graph is increasing, the particle is moving in the positive direction along the s-axis.

11. False. Acceleration is the derivative of velocity.

13. **(a)** At 60 mi/h the tangent line seems to pass through the points $(5, 42)$ and $(10, 63)$. Thus the acceleration would be $\dfrac{v_1 - v_0}{t_1 - t_0} \cdot \dfrac{5280}{60^2} = \dfrac{63 - 42}{10 - 5} \cdot \dfrac{5280}{60^2} \approx 6.2 \text{ ft/s}^2$.

(b) The maximum acceleration occurs at maximum slope, so when $t = 0$.

15. **(a)**

t	1	2	3	4	5
s	0.71	1.00	0.71	0.00	−0.71
v	0.56	0.00	−0.56	−0.79	−0.56
a	−0.44	−0.62	−0.44	0.00	0.44

(b) To the right at $t = 1$, stopped at $t = 2$, otherwise to the left.

(c) Speeding up at $t = 3$; slowing down at $t = 1, 5$; neither at $t = 2, 4$.

17. **(a)** $v(t) = 3t^2 - 6t$, $a(t) = 6t - 6$.

(b) $s(1) = -2$ ft, $v(1) = -3$ ft/s, speed = 3 ft/s, $a(1) = 0$ ft/s^2.

(c) $v = 0$ at $t = 0, 2$.

(d) For $t \geq 0$, $v(t)$ changes sign at $t = 2$, and $a(t)$ changes sign at $t = 1$; so the particle is speeding up for $0 < t < 1$ and $2 < t$ and is slowing down for $1 < t < 2$.

(e) Total distance $= |s(2) - s(0)| + |s(5) - s(2)| = |-4 - 0| + |50 - (-4)| = 58$ ft.

19. **(a)** $s(t) = 9 - 9\cos(\pi t/3)$, $v(t) = 3\pi \sin(\pi t/3)$, $a(t) = \pi^2 \cos(\pi t/3)$.

(b) $s(1) = 9/2$ ft, $v(1) = 3\pi\sqrt{3}/2$ ft/s, speed $= 3\pi\sqrt{3}/2$ ft/s, $a(1) = \pi^2/2$ ft/s^2.

(c) $v = 0$ at $t = 0, 3$.

(d) For $0 < t < 5$, $v(t)$ changes sign at $t = 3$ and $a(t)$ changes sign at $t = 3/2, 9/2$; so the particle is speeding up for $0 < t < 3/2$ and $3 < t < 9/2$ and slowing down for $3/2 < t < 3$ and $9/2 < t < 5$.

(e) Total distance $= |s(3) - s(0)| + |s(5) - s(3)| = |18 - 0| + |9/2 - 18| = 18 + 27/2 = 63/2$ ft.

21. **(a)** $s(t) = (t^2 + 8)e^{-t/3}$ ft, $v(t) = \left(-\frac{1}{3}t^2 + 2t - \frac{8}{3}\right) e^{-t/3}$ ft/s, $a(t) = \left(\frac{1}{9}t^2 - \frac{4}{3}t + \frac{26}{9}\right) e^{-t/3}$ ft/s^2.

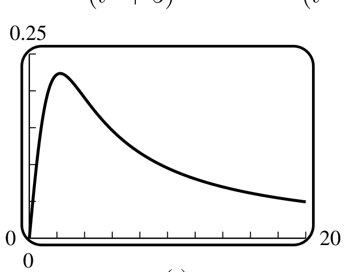
(b) $s(1) = 9e^{-1/3}$ ft, $v(1) = -e^{-1/3}$ ft/s, speed$= e^{-1/3}$ ft/s, $a(1) = \frac{5}{3}e^{-1/3}$ ft/s^2.

(c) $v = 0$ for $t = 2, 4$.

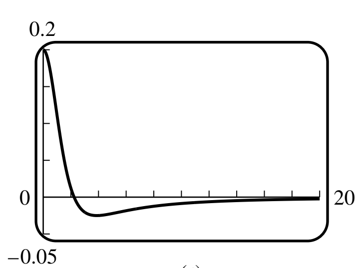
(d) v changes sign at $t = 2, 4$ and a changes sign at $t = 6 \pm \sqrt{10}$, so the particle is speeding up for $2 < t < 6 - \sqrt{10}$ and $4 < t < 6 + \sqrt{10}$, and slowing down for $0 < t < 2$, $6 - \sqrt{10} < t < 4$ and $t > 6 + \sqrt{10}$.

(e) Total distance $= |s(2) - s(0)| + |s(4) - s(2)| + |s(5) - s(4)| = |12e^{-2/3} - 8| + |24e^{-4/3} - 12e^{-2/3}| + |33e^{-5/3} - 24e^{-4/3}|$ $= (8 - 12e^{-2/3}) + (24e^{-4/3} - 12e^{-2/3}) + (24e^{-4/3} - 33e^{-5/3}) = 8 - 24e^{-2/3} + 48e^{-4/3} - 33e^{-5/3} \approx 2.098$ ft.

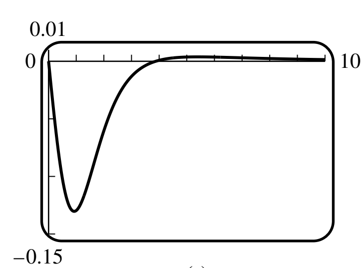
23. $v(t) = \dfrac{5 - t^2}{(t^2 + 5)^2}$, $a(t) = \dfrac{2t(t^2 - 15)}{(t^2 + 5)^3}$.



$s(t)$



$v(t)$



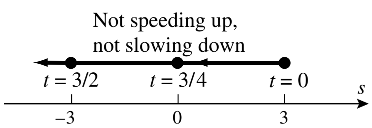
$a(t)$

(a) $v = 0$ at $t = \sqrt{5}$.

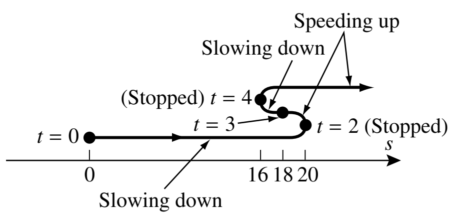
(b) $s = \sqrt{5}/10$ at $t = \sqrt{5}$.

(c) a changes sign at $t = \sqrt{15}$, so the particle is speeding up for $\sqrt{5} < t < \sqrt{15}$ and slowing down for $0 < t < \sqrt{5}$ and $\sqrt{15} < t$.

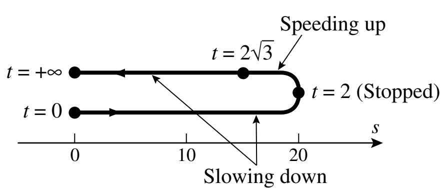
25. $s = -4t + 3$, $v = -4$, $a = 0$.



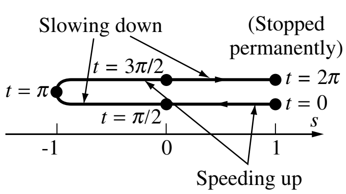
27. $s = t^3 - 9t^2 + 24t$, $v = 3(t-2)(t-4)$, $a = 6(t-3)$.



29. $s = 16te^{-t^2/8}$, $v = (-4t^2 + 16)e^{-t^2/8}$, $a = t(-12 + t^2)e^{-t^2/8}$.



31. $s = \begin{cases} \cos t, & 0 \le t \le 2\pi \\ 1, & t > 2\pi \end{cases}$, $v = \begin{cases} -\sin t, & 0 \le t \le 2\pi \\ 0, & t > 2\pi \end{cases}$, $a = \begin{cases} -\cos t, & 0 \le t < 2\pi \\ 0, & t > 2\pi \end{cases}$.



33. **(a)** $v = 10t - 22$, speed $= |v| = |10t - 22|$. $d|v|/dt$ does not exist at $t = 2.2$ which is the only critical point. If $t = 1, 2.2, 3$ then $|v| = 12, 0, 8$. The maximum speed is 12 ft/s.

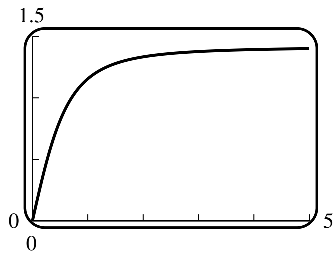
(b) The distance from the origin is $|s| = |5t^2 - 22t| = |t(5t - 22)|$, but $t(5t-22) < 0$ for $1 \le t \le 3$ so $|s| = -(5t^2 - 22t) = 22t - 5t^2$, $d|s|/dt = 22 - 10t$, thus the only critical point is $t = 2.2$. $d^2|s|/dt^2 < 0$ so the particle is farthest from the origin when $t = 2.2$ s. Its position is $s = 5(2.2)^2 - 22(2.2) = -24.2$ ft.

35. $s = \ln(3t^2 - 12t + 13)$, $v = \dfrac{6t - 12}{3t^2 - 12t + 13}$, $a = -\dfrac{6(3t^2 - 12t + 11)}{(3t^2 - 12t + 13)^2}$.

(a) $a = 0$ when $t = 2 \pm \sqrt{3}/3$; $s(2 - \sqrt{3}/3) = \ln 2$; $s(2 + \sqrt{3}/3) = \ln 2$; $v(2 - \sqrt{3}/3) = -\sqrt{3}$; $v(2 + \sqrt{3}/3) = \sqrt{3}$.

(b) $v = 0$ when $t = 2$; $s(2) = 0$; $a(2) = 6$.

37. (a)



(b) $v = \dfrac{2t}{\sqrt{2t^2+1}}$, $\lim\limits_{t\to+\infty} v = \dfrac{2}{\sqrt{2}} = \sqrt{2}$.

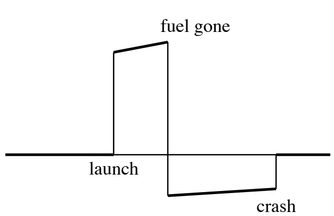
39. (a) $s_1 = s_2$ if they collide, so $\dfrac{1}{2}t^2 - t + 3 = -\dfrac{1}{4}t^2 + t + 1$, $\dfrac{3}{4}t^2 - 2t + 2 = 0$ which has no real solution.

(b) Find the minimum value of $D = |s_1 - s_2| = \left|\dfrac{3}{4}t^2 - 2t + 2\right|$. From part (a), $\dfrac{3}{4}t^2 - 2t + 2$ is never zero, and for $t = 0$ it is positive, hence it is always positive, so $D = \dfrac{3}{4}t^2 - 2t + 2$. $\dfrac{dD}{dt} = \dfrac{3}{2}t - 2 = 0$ when $t = \dfrac{4}{3}$. $\dfrac{d^2D}{dt^2} > 0$ so D is minimum when $t = \dfrac{4}{3}$, $D = \dfrac{2}{3}$.

(c) $v_1 = t - 1$, $v_2 = -\dfrac{1}{2}t + 1$. $v_1 < 0$ if $0 \le t < 1$, $v_1 > 0$ if $t > 1$; $v_2 < 0$ if $t > 2$, $v_2 > 0$ if $0 \le t < 2$. They are moving in opposite directions during the intervals $0 \le t < 1$ and $t > 2$.

41. $r(t) = \sqrt{v^2(t)}$, $r'(t) = 2v(t)v'(t)/[2\sqrt{v^2(t)}] = v(t)a(t)/|v(t)|$ so $r'(t) > 0$ (speed is increasing) if v and a have the same sign, and $r'(t) < 0$ (speed is decreasing) if v and a have opposite signs.

43. While the fuel is burning, the acceleration is positive and the rocket is speeding up. After the fuel is gone, the acceleration (due to gravity) is negative and the rocket slows down until it reaches the highest point of its flight. Then the acceleration is still negative, and the rocket speeds up as it falls, until it hits the ground. After that the acceleration is zero, and the rocket neither speeds up nor slows down. During the powered part of the flight, the acceleration is not constant, and it's hard to say whether it will be increasing or decreasing. First, the power output of the engine may not be constant. Even if it is, the mass of the rocket decreases as the fuel is used up, which tends to increase the acceleration. But as the rocket moves faster, it encounters more air resistance, which tends to decrease the acceleration. Air resistance also acts during the free-fall part of the flight. While the rocket is still rising, air resistance increases the deceleration due to gravity; while the rocket is falling, air resistance decreases the deceleration.



Exercise Set 4.7

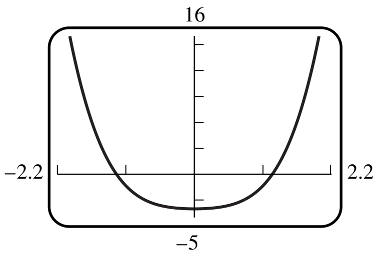
1. $f(x) = x^2 - 2$, $f'(x) = 2x$, $x_{n+1} = x_n - \dfrac{x_n^2 - 2}{2x_n}$; $x_1 = 1$, $x_2 = 1.5$, $x_3 \approx 1.416666667, \ldots, x_5 \approx x_6 \approx 1.414213562$.

3. $f(x) = x^3 - 6$, $f'(x) = 3x^2$, $x_{n+1} = x_n - \dfrac{x_n^3 - 6}{3x_n^2}$; $x_1 = 2$, $x_2 \approx 1.833333333$, $x_3 \approx 1.817263545, \ldots, x_5 \approx x_6 \approx 1.817120593$.

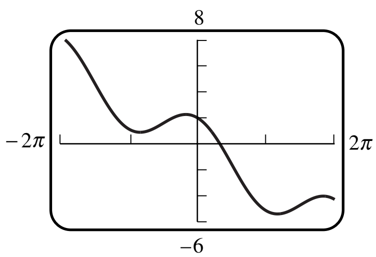
5. $f(x) = x^3 - 2x - 2, f'(x) = 3x^2 - 2, x_{n+1} = x_n - \dfrac{x_n^3 - 2x_n - 2}{3x_n^2 - 2}$; $x_1 = 2, x_2 = 1.8, x_3 \approx 1.7699481865, x_4 \approx 1.7692926629, x_5 \approx x_6 \approx 1.7692923542$.

7. $f(x) = x^5 + x^4 - 5$, $f'(x) = 5x^4 + 4x^3$, $x_{n+1} = x_n - \dfrac{x_n^5 + x_n^4 - 5}{5x_n^4 + 4x_n^3}$; $x_1 = 1$, $x_2 \approx 1.333333333$, $x_3 \approx 1.239420573, \ldots, x_6 \approx x_7 \approx 1.224439550$.

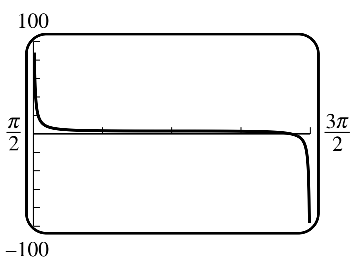
9. There are 2 solutions. $f(x) = x^4 + x^2 - 4$, $f'(x) = 4x^3 + 2x$, $x_{n+1} = x_n - \dfrac{x_n^4 + x_n^2 - 4}{4x_n^3 + 2x_n}$; $x_1 = -1, x_2 \approx -1.3333, x_3 \approx -1.2561, x_4 \approx -1.24966, \ldots,\ x_7 \approx x_8 \approx -1.249621068$.



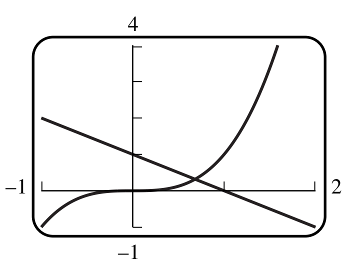
11. There is 1 solution. $f(x) = 2\cos x - x$, $f'(x) = -2\sin x - 1$, $x_{n+1} = x_n - \dfrac{2\cos x - x}{-2\sin x - 1}$; $x_1 = 1, x_2 \approx 1.03004337, x_3 \approx 1.02986654$, $x_4 \approx x_5 \approx 1.02986653$.



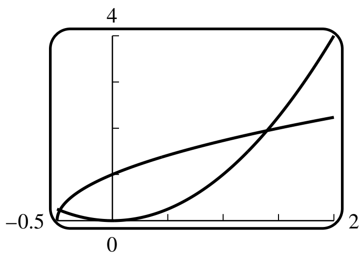
13. There are infinitely many solutions. $f(x) = x - \tan x$, $f'(x) = 1 - \sec^2 x = -\tan^2 x$, $x_{n+1} = x_n + \dfrac{x_n - \tan x_n}{\tan^2 x_n}$; $x_1 = 4.5$, $x_2 \approx 4.493613903$, $x_3 \approx 4.493409655$, $x_4 \approx x_5 \approx 4.493409458$.



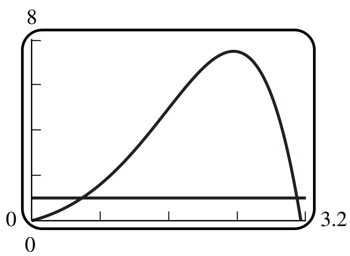
15. The graphs of $y = x^3$ and $y = 1 - x$ intersect once, near $x = 0.7$. Let $f(x) = x^3 + x - 1$, so that $f'(x) = 3x^2 + 1$, and $x_{n+1} = x_n - \dfrac{x_n^3 + x_n - 1}{3x_n^2 + 1}$. If $x_1 = 0.7$ then $x_2 \approx 0.68259109$, $x_3 \approx 0.68232786$, $x_4 \approx x_5 \approx 0.68232780$.



17. The graphs of $y = x^2$ and $y = \sqrt{2x+1}$ intersect twice, near $x = -0.5$ and $x = 1.4$. $x^2 = \sqrt{2x+1}$, $x^4 - 2x - 1 = 0$. Let $f(x) = x^4 - 2x - 1$, then $f'(x) = 4x^3 - 2$ so $x_{n+1} = x_n - \dfrac{x_n^4 - 2x_n - 1}{4x_n^3 - 2}$. If $x_1 = -0.5$, then $x_2 = -0.475$, $x_3 \approx -0.474626695$, $x_4 \approx x_5 \approx -0.474626618$; if $x_1 = 1$, then $x_2 = 2$, $x_3 \approx 1.633333333, \ldots, x_8 \approx x_9 \approx 1.395336994$.



19. Between $x = 0$ and $x = \pi$, the graphs of $y = 1$ and $y = e^x \sin x$ intersect twice, near $x = 1$ and $x = 3$. Let $f(x) = 1 - e^x \sin x$, $f'(x) = -e^x(\cos x + \sin x)$, and $x_{n+1} = x_n + \dfrac{1 - e_n^x \sin x_n}{e_n^x(\cos x_n + \sin x_n)}$. If $x_1 = 1$ then $x_2 \approx 0.65725814, x_3 \approx 0.59118311, \ldots, x_5 \approx x_6 \approx 0.58853274$, and if $x_1 = 3$ then $x_2 \approx 3.10759324, x_3 \approx 3.09649396, \ldots, x_5 \approx x_6 \approx 3.09636393$.



21. True. See the discussion before equation (1).

23. False. The function $f(x) = x^3 - x^2 - 110x$ has 3 roots: $x = -10$, $x = 0$, and $x = 11$. Newton's method in this case gives $x_{n+1} = x_n - \dfrac{x_n^3 - x_n^2 - 110x_n}{3x_n^2 - 2x_n - 110} = \dfrac{2x_n^3 - x_n^2}{3x_n^2 - 2x_n - 110}$. Starting from $x_1 = 5$, we find $x_2 = -5$, $x_3 = x_4 = x_5 = \cdots = 11$. So the method converges to the root $x = 11$, although the root closest to x_1 is $x = 0$.

25. (a) $f(x) = x^2 - a$, $f'(x) = 2x$, $x_{n+1} = x_n - \dfrac{x_n^2 - a}{2x_n} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$.

(b) $a = 10$; $x_1 = 3$, $x_2 \approx 3.166666667$, $x_3 \approx 3.162280702$, $x_4 \approx x_5 \approx 3.162277660$.

27. $f'(x) = x^3 + 2x - 5$; solve $f'(x) = 0$ to find the critical points. Graph $y = x^3$ and $y = -2x + 5$ to see that they intersect at a point near $x = 1.25$; $f''(x) = 3x^2 + 2$ so $x_{n+1} = x_n - \dfrac{x_n^3 + 2x_n - 5}{3x_n^2 + 2}$. $x_1 = 1.25, x_2 \approx 1.3317757009, x_3 \approx 1.3282755613, x_4 \approx 1.3282688557, x_5 \approx 1.3282688557$ so the minimum value of $f(x)$ occurs at $x \approx 1.3282688557$ because $f''(x) > 0$; its value is approximately -4.098859132.

29. A graphing utility shows that there are two inflection points at $x \approx 0.25, -1.25$. These points are the zeros of $f''(x) = (x^4 + 4x^3 + 8x^2 + 4x - 1)\dfrac{e^{-x}}{(x^2+1)^3}$. It is equivalent to find the zeros of $g(x) = x^4 + 4x^3 + 8x^2 + 4x - 1$. One root is $x = -1$ by inspection. Since $g'(x) = 4x^3 + 12x^2 + 16x + 4$, Newton's Method becomes $x_{n+1} = x_n - \dfrac{x_n^4 + 4x_n^3 + 8x_n^2 + 4x_n - 1}{4x_n^3 + 12x_n^2 + 16x_n + 4}$ With $x_0 = 0.25$, $x_1 \approx 0.18572695, x_2 \approx 0.179563312, x_3 \approx 0.179509029, x_4 \approx x_5 \approx 0.179509025$. So the points of inflection are at $x \approx 0.17951, x = -1$.

31. Let $f(x)$ be the square of the distance between $(1, 0)$ and any point (x, x^2) on the parabola, then $f(x) = (x-1)^2 + (x^2 - 0)^2 = x^4 + x^2 - 2x + 1$ and $f'(x) = 4x^3 + 2x - 2$. Solve $f'(x) = 0$ to find the critical points; $f''(x) = 12x^2 + 2$

so $x_{n+1} = x_n - \dfrac{4x_n^3 + 2x_n - 2}{12x_n^2 + 2} = x_n - \dfrac{2x_n^3 + x_n - 1}{6x_n^2 + 1}$. $x_1 = 1$, $x_2 \approx 0.714285714$, $x_3 \approx 0.605168701, \ldots, x_6 \approx x_7 \approx 0.589754512$; the coordinates are approximately $(0.589754512, 0.347810385)$.

33. (a) Let s be the arc length, and L the length of the chord, then $s = 1.5L$. But $s = r\theta$ and $L = 2r\sin(\theta/2)$ so $r\theta = 3r\sin(\theta/2)$, $\theta - 3\sin(\theta/2) = 0$.

(b) Let $f(\theta) = \theta - 3\sin(\theta/2)$, then $f'(\theta) = 1 - 1.5\cos(\theta/2)$ so $\theta_{n+1} = \theta_n - \dfrac{\theta_n - 3\sin(\theta_n/2)}{1 - 1.5\cos(\theta_n/2)}$. $\theta_1 = 3$, $\theta_2 \approx 2.991592920$, $\theta_3 \approx 2.991563137$, $\theta_4 \approx \theta_5 \approx 2.991563136$ rad so $\theta \approx 171°$.

35. The point $(2, 1)$ is inside the ellipse $4x^2 + 9y^2 = 36$. It is clear that the minimum distance will be reached somewhere on the upper half of the ellipse, so we can use the equation $y = \sqrt{36 - 4x^2}/3$. The minimum of the square of the distance is located at the same point as the minimum of the distance, thus it is sufficient to find the (x, y) points on the ellipse for which $(x-2)^2 + (y-1)^2$ is minimal. Using the value of y computed above, we have to find the minimum of the function $d(x) = (x-2)^2 + (\sqrt{36 - 4x^2}/3 - 1)^2$. The minimum is located at the point where $0 = d'(x) = 2(x-2) + 2(\sqrt{36 - 4x^2}/3 - 1)(1/3)(1/2)(-8x)/\sqrt{36 - 4x^2} = \dfrac{10}{9}x - 4 + \dfrac{8x}{3\sqrt{36 - 4x^2}}$. Using this equation, Newton's Method gives us the formula $x_{n+1} = x_n - \dfrac{10x_n/9 - 4 + 8x_n/3\sqrt{36 - 4x_n^2}}{10/9 + 12/\sqrt{(9 - x_n^2)^3}}$. Using $x_1 = 2$, we obtain $x_2 \approx 2.267900490$, $x_3 \approx 2.245763757$, $x_4 \approx 2.245510252$, $x_5 \approx 2.24551022$, after which the last 10 digits do not change. Thus we conclude that $x \approx 2.24551022$, which gives that $y \approx 1.32626189$, and the minimum distance is about 40.832 meters.

37. If $x = 1$, then $y^4 + y = 1$, $y^4 + y - 1 = 0$. Graph $z = y^4$ and $z = 1 - y$ to see that they intersect near $y = -1$ and $y = 1$. Let $f(y) = y^4 + y - 1$, then $f'(y) = 4y^3 + 1$ so $y_{n+1} = y_n - \dfrac{y_n^4 + y_n - 1}{4y_n^3 + 1}$. If $y_1 = -1$, then $y_2 \approx -1.333333333$, $y_3 \approx -1.235807860, \ldots, y_6 \approx y_7 \approx -1.220744085$; if $y_1 = 1$, then $y_2 = 0.8$, $y_3 \approx 0.731233596, \ldots, y_6 \approx y_7 \approx 0.724491959$.

39. $S(25) = 250{,}000 = \dfrac{5000}{i}\left[(1+i)^{25} - 1\right]$; set $f(i) = 50i - (1+i)^{25} + 1$, $f'(i) = 50 - 25(1+i)^{24}$; solve $f(i) = 0$. Set $i_0 = .06$ and $i_{k+1} = i_k - \left[50i - (1+i)^{25} + 1\right] / \left[50 - 25(1+i)^{24}\right]$. Then $i_1 \approx 0.05430$, $i_2 \approx 0.05338$, $i_3 \approx 0.05336$, $\ldots$, $i \approx 0.053362$.

41. (a)

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
0.5000	−0.7500	0.2917	−1.5685	−0.4654	0.8415	−0.1734	2.7970	1.2197	0.1999

(b) The sequence x_n must diverge, since if it did converge then $f(x) = x^2 + 1 = 0$ would have a solution. It seems the x_n are oscillating back and forth in a quasi-cyclical fashion.

43. Suppose we know an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs. Here are some differences between the two methods:

The Intermediate-Value method is guaranteed to converge to a root in $[a, b]$; Newton's Method starting from some x_1 in the interval might not converge, or might converge to some root outside of the interval.

If the starting approximation x_1 is close enough to the actual root, then Newton's Method converges much faster than the Intermediate-Value method.

Newton's Method can only be used if f is differentiable and we have a way to compute $f'(x)$ for any x. For the Intermediate-Value method we only need to be able to compute $f(x)$.

Exercise Set 4.8

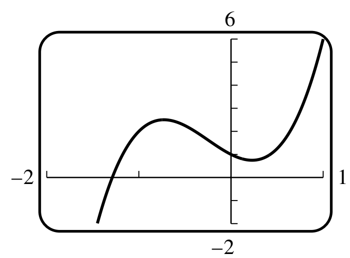
1. f is continuous on $[3, 5]$ and differentiable on $(3, 5)$, $f(3) = f(5) = 0$; $f'(x) = 2x - 8$, $2c - 8 = 0$, $c = 4$, $f'(4) = 0$.

3. f is continuous on $[\pi/2, 3\pi/2]$ and differentiable on $(\pi/2, 3\pi/2)$, $f(\pi/2) = f(3\pi/2) = 0$, $f'(x) = -\sin x$, $-\sin c = 0$, $c = \pi$.

5. f is continuous on $[-3, 5]$ and differentiable on $(-3, 5)$, $(f(5) - f(-3))/(5 - (-3)) = 1$; $f'(x) = 2x - 1$; $2c - 1 = 1$, $c = 1$.

7. f is continuous on $[-5, 3]$ and differentiable on $(-5, 3)$, $(f(3) - f(-5))/(3 - (-5)) = 1/2$; $f'(x) = -\dfrac{x}{\sqrt{25 - x^2}}$; $-\dfrac{c}{\sqrt{25 - c^2}} = 1/2$, $c = -\sqrt{5}$.

9. **(a)** $f(-2) = f(1) = 0$. The interval is $[-2, 1]$.



(b) $c \approx -1.29$.

(c) $x_0 = -1$, $x_1 = -1.5$, $x_2 = -1.328125$, $x_3 \approx -1.2903686$, $x_4 \approx -1.2885882$, $x_5 \approx x_6 \approx -1.2885843$.

11. False. Rolle's Theorem only applies to the case in which f is differentiable on (a, b) and the common value of $f(a)$ and $f(b)$ is zero.

13. False. The Constant Difference Theorem states that if the <u>derivatives</u> are equal, then the <u>functions</u> differ by a constant.

15. **(a)** $f'(x) = \sec^2 x$, $\sec^2 c = 0$ has no solution. **(b)** $\tan x$ is not continuous on $[0, \pi]$.

17. **(a)** Two x-intercepts of f determine two solutions a and b of $f(x) = 0$; by Rolle's Theorem there exists a point c between a and b such that $f'(c) = 0$, i.e. c is an x-intercept for f'.

(b) $f(x) = \sin x = 0$ at $x = n\pi$, and $f'(x) = \cos x = 0$ at $x = n\pi + \pi/2$, which lies between $n\pi$ and $(n + 1)\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$

19. Let $s(t)$ be the position function of the automobile for $0 \le t \le 5$, then by the Mean-Value Theorem there is at least one point c in $(0, 5)$ where $s'(c) = v(c) = [s(5) - s(0)]/(5 - 0) = 4/5 = 0.8$ mi/min $= 48$ mi/h.

21. Let $f(t)$ and $g(t)$ denote the distances from the first and second runners to the starting point, and let $h(t) = f(t) - g(t)$. Since they start (at $t = 0$) and finish (at $t = t_1$) at the same time, $h(0) = h(t_1) = 0$, so by Rolle's Theorem there is a time t_2 for which $h'(t_2) = 0$, i.e. $f'(t_2) = g'(t_2)$; so they have the same velocity at time t_2.

23. **(a)** By the Constant Difference Theorem $f(x) - g(x) = k$ for some k; since $f(x_0) = g(x_0)$, $k = 0$, so $f(x) = g(x)$ for all x.

(b) Set $f(x) = \sin^2 x + \cos^2 x$, $g(x) = 1$; then $f'(x) = 2\sin x \cos x - 2\cos x \sin x = 0 = g'(x)$. Since $f(0) = 1 = g(0)$, $f(x) = g(x)$ for all x.

25. By the Constant Difference Theorem it follows that $f(x) = g(x) + c$; since $g(1) = 0$ and $f(1) = 2$ we get $c = 2$; $f(x) = xe^x - e^x + 2$.

27. **(a)** If x, y belong to I and $x < y$ then for some c in I, $\dfrac{f(y) - f(x)}{y - x} = f'(c)$, so $|f(x) - f(y)| = |f'(c)||x - y| \le M|x - y|$; if $x > y$ exchange x and y; if $x = y$ the inequality also holds.

(b) $f(x) = \sin x$, $f'(x) = \cos x$, $|f'(x)| \le 1 = M$, so $|f(x) - f(y)| \le |x - y|$ or $|\sin x - \sin y| \le |x - y|$.

29. (a) Let $f(x) = \sqrt{x}$. By the Mean-Value Theorem there is a number c between x and y such that $\dfrac{\sqrt{y} - \sqrt{x}}{y - x} = \dfrac{1}{2\sqrt{c}} < \dfrac{1}{2\sqrt{x}}$ for c in (x, y), thus $\sqrt{y} - \sqrt{x} < \dfrac{y - x}{2\sqrt{x}}$.

(b) Multiply through and rearrange to get $\sqrt{xy} < \dfrac{1}{2}(x + y)$.

31. (a) If $f(x) = x^3 + 4x - 1$ then $f'(x) = 3x^2 + 4$ is never zero, so by Exercise 30 f has at most one real root; since f is a cubic polynomial it has at least one real root, so it has exactly one real root.

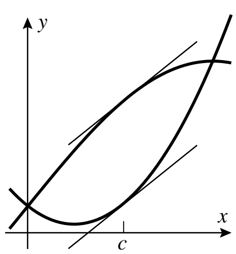
(b) Let $f(x) = ax^3 + bx^2 + cx + d$. If $f(x) = 0$ has at least two distinct real solutions r_1 and r_2, then $f(r_1) = f(r_2) = 0$ and by Rolle's Theorem there is at least one number between r_1 and r_2 where $f'(x) = 0$. But $f'(x) = 3ax^2 + 2bx + c = 0$ for $x = (-2b \pm \sqrt{4b^2 - 12ac})/(6a) = (-b \pm \sqrt{b^2 - 3ac})/(3a)$, which are not real if $b^2 - 3ac < 0$ so $f(x) = 0$ must have fewer than two distinct real solutions.

33. By the Mean-Value Theorem on the interval $[0, x]$, $\dfrac{\tan^{-1} x - \tan^{-1} 0}{x - 0} = \dfrac{\tan^{-1} x}{x} = \dfrac{1}{1 + c^2}$ for c in $(0, x)$, but $\dfrac{1}{1 + x^2} < \dfrac{1}{1 + c^2} < 1$ for c in $(0, x)$, so $\dfrac{1}{1 + x^2} < \dfrac{\tan^{-1} x}{x} < 1$, $\dfrac{x}{1 + x^2} < \tan^{-1} x < x$.

35. (a) $\dfrac{d}{dx}[f^2(x) + g^2(x)] = 2f(x)f'(x) + 2g(x)g'(x) = 2f(x)g(x) + 2g(x)[-f(x)] = 0$, so $f^2(x) + g^2(x)$ is constant.

(b) $f(x) = \sin x$ and $g(x) = \cos x$.

37.



39. (a) Similar to the proof of part (a) with $f'(c) < 0$.

(b) Similar to the proof of part (a) with $f'(c) = 0$.

41. If f is differentiable at $x = 1$, then f is continuous there; $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1) = 3$, $a + b = 3$; $\lim_{x \to 1^+} f'(x) = a$ and $\lim_{x \to 1^-} f'(x) = 6$ so $a = 6$ and $b = 3 - 6 = -3$.

43. From Section 2.2 a function has a vertical tangent line at a point of its graph if the slopes of secant lines through the point approach $+\infty$ or $-\infty$. Suppose f is continuous at $x = x_0$ and $\lim_{x \to x_0^+} f(x) = +\infty$. Then a secant line through $(x_1, f(x_1))$ and $(x_0, f(x_0))$, assuming $x_1 > x_0$, will have slope $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$. By the Mean Value Theorem, this quotient is equal to $f'(c)$ for some c between x_0 and x_1. But as x_1 approaches x_0, c must also approach x_0, and it is given that $\lim_{c \to x_0^+} f'(c) = +\infty$, so the slope of the secant line approaches $+\infty$. The argument can be altered appropriately for $x_1 < x_0$, and/or for $f'(c)$ approaching $-\infty$.

45. If an object travels s miles in t hours, then at some time during the trip its instantaneous speed is exactly s/t miles per hour.

Chapter 4 Review Exercises

3. $f'(x) = 2x - 5$, $f''(x) = 2$.

(a) $[5/2, +\infty)$ **(b)** $(-\infty, 5/2]$ **(c)** $(-\infty, +\infty)$ **(d)** none **(e)** none

5. $f'(x) = \dfrac{4x}{(x^2+2)^2}$, $f''(x) = -4\dfrac{3x^2-2}{(x^2+2)^3}$.

(a) $[0, +\infty)$ **(b)** $(-\infty, 0]$ **(c)** $(-\sqrt{2/3}, \sqrt{2/3})$ **(d)** $(-\infty, -\sqrt{2/3})$, $(\sqrt{2/3}, +\infty)$ **(e)** $-\sqrt{2/3}, \sqrt{2/3}$

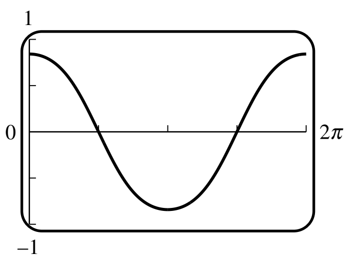
7. $f'(x) = \dfrac{4(x+1)}{3x^{2/3}}$, $f''(x) = \dfrac{4(x-2)}{9x^{5/3}}$.

(a) $[-1, +\infty)$ **(b)** $(-\infty, -1]$ **(c)** $(-\infty, 0)$, $(2, +\infty)$ **(d)** $(0, 2)$ **(e)** $0, 2$

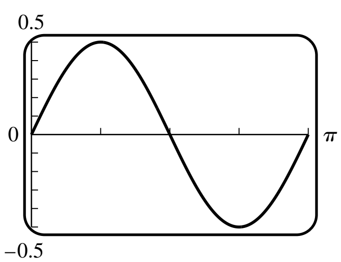
9. $f'(x) = -\dfrac{2x}{e^{x^2}}$, $f''(x) = \dfrac{2(2x^2-1)}{e^{x^2}}$.

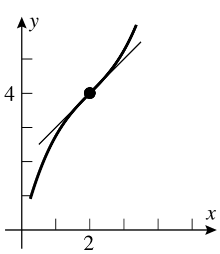
(a) $(-\infty, 0]$ **(b)** $[0, +\infty)$ **(c)** $(-\infty, -\sqrt{2}/2)$, $(\sqrt{2}/2, +\infty)$ **(d)** $(-\sqrt{2}/2, \sqrt{2}/2)$ **(e)** $-\sqrt{2}/2, \sqrt{2}/2$

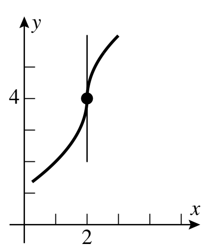
11. $f'(x) = -\sin x$, $f''(x) = -\cos x$, increasing: $[\pi, 2\pi]$, decreasing: $[0, \pi]$, concave up: $(\pi/2, 3\pi/2)$, concave down: $(0, \pi/2)$, $(3\pi/2, 2\pi)$, inflection points: $\pi/2$, $3\pi/2$.

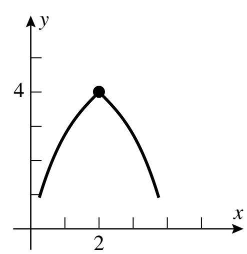


13. $f'(x) = \cos 2x$, $f''(x) = -2\sin 2x$, increasing: $[0, \pi/4]$, $[3\pi/4, \pi]$, decreasing: $[\pi/4, 3\pi/4]$, concave up: $(\pi/2, \pi)$, concave down: $(0, \pi/2)$, inflection point: $\pi/2$.



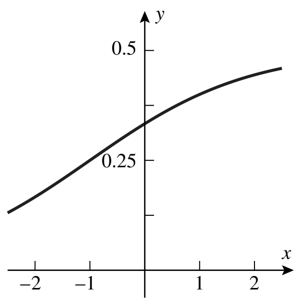
15. **(a)**


(b)


(c)


17. $f'(x) = 2ax + b$; $f'(x) > 0$ or $f'(x) < 0$ on $[0, +\infty)$ if $f'(x) = 0$ has no positive solution, so the polynomial is always increasing or always decreasing on $[0, +\infty)$ provided $-b/2a \leq 0$.

19. The maximum increase in y seems to occur near $x = -1$, $y = 1/4$.



25. **(a)** $f'(x) = (2 - x^2)/(x^2+2)^2$, $f'(x) = 0$ when $x = \pm\sqrt{2}$ (stationary points).

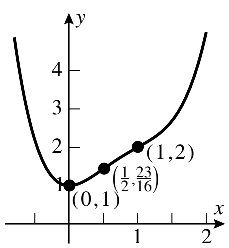
(b) $f'(x) = 8x/(x^2+1)^2$, $f'(x) = 0$ when $x = 0$ (stationary point).

27. **(a)** $f'(x) = \dfrac{7(x-7)(x-1)}{3x^{2/3}}$; critical numbers at $x = 0, 1, 7$; neither at $x = 0$, relative maximum at $x = 1$, relative minimum at $x = 7$ (First Derivative Test).

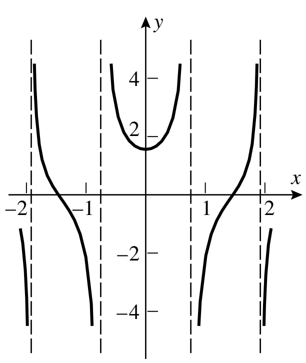
(b) $f'(x) = 2\cos x(1+2\sin x)$; critical numbers at $x = \pi/2, 3\pi/2, 7\pi/6, 11\pi/6$; relative maximum at $x = \pi/2, 3\pi/2$, relative minimum at $x = 7\pi/6, 11\pi/6$.

(c) $f'(x) = 3 - \dfrac{3\sqrt{x-1}}{2}$; critical number at $x = 5$; relative maximum at $x = 5$.

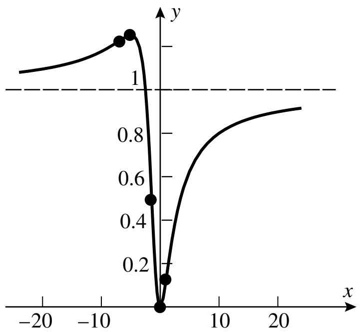
29. $\lim\limits_{x\to-\infty} f(x) = +\infty$, $\lim\limits_{x\to+\infty} f(x) = +\infty$, $f'(x) = x(4x^2 - 9x + 6)$, $f''(x) = 6(2x-1)(x-1)$, relative minimum at $x = 0$, points of inflection when $x = 1/2, 1$, no asymptotes.



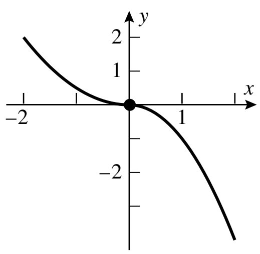
31. $\lim\limits_{x\to\pm\infty} f(x)$ doesn't exist, $f'(x) = 2x\sec^2(x^2+1)$, $f''(x) = 2\sec^2(x^2+1)\left[1 + 4x^2\tan(x^2+1)\right]$, critical number at $x = 0$; relative minimum at $x = 0$, point of inflection when $1 + 4x^2\tan(x^2+1) = 0$, vertical asymptotes at $x = \pm\sqrt{\pi(n+\frac{1}{2}) - 1}$, $n = 0, 1, 2, \ldots$



33. $f'(x) = 2\dfrac{x(x+5)}{(x^2+2x+5)^2}$, $f''(x) = -2\dfrac{2x^3+15x^2-25}{(x^2+2x+5)^3}$, critical numbers at $x = -5, 0$; relative maximum at $x = -5$, relative minimum at $x = 0$, points of inflection at $x \approx -7.26, -1.44, 1.20$, horizontal asymptote $y = 1$ as $x \to \pm\infty$.



35. $\lim_{x\to-\infty} f(x) = +\infty$, $\lim_{x\to+\infty} f(x) = -\infty$, $f'(x) = \begin{cases} x, & x \le 0 \\ -2x, & x > 0 \end{cases}$, critical number at $x = 0$, no extrema, inflection point at $x = 0$ (f changes concavity), no asymptotes.



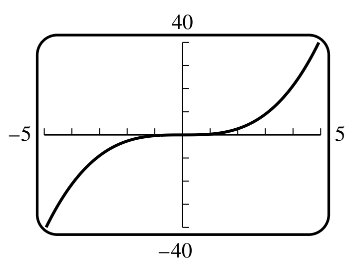
37. $f'(x) = 3x^2 + 5$; no relative extrema because there are no critical numbers.

39. $f'(x) = \dfrac{4}{5}x^{-1/5}$; critical number $x = 0$; relative minimum of 0 at $x = 0$ (first derivative test).

41. $f'(x) = 2x/(x^2 + 1)^2$; critical number $x = 0$; relative minimum of 0 at $x = 0$.

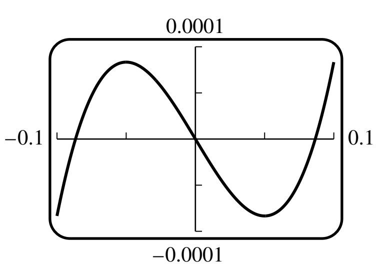
43. $f'(x) = 2x/(1 + x^2)$; critical point at $x = 0$; relative minimum of 0 at $x = 0$ (first derivative test).

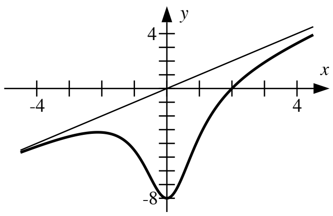
45. **(a)**



(b) $f'(x) = x^2 - \dfrac{1}{400}$, $f''(x) = 2x$, critical points at $x = \pm\dfrac{1}{20}$; relative maximum at $x = -\dfrac{1}{20}$, relative minimum at $x = \dfrac{1}{20}$.

(c) The finer details can be seen when graphing over a much smaller x-window.

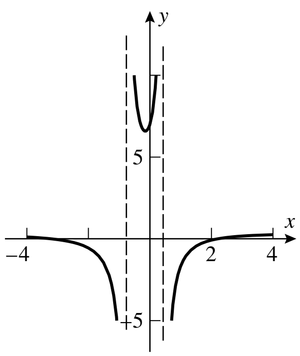


47. (a) 

$y = x$ appears to be an asymptote for $y = (x^3 - 8)/(x^2 + 1)$.

(b) $\dfrac{x^3 - 8}{x^2 + 1} = x - \dfrac{x + 8}{x^2 + 1}$. Since the limit of $\dfrac{x + 8}{x^2 + 1}$ as $x \to \pm\infty$ is 0, $y = x$ is an asymptote for $y = \dfrac{x^3 - 8}{x^2 + 1}$.

49. $f(x) = \dfrac{(2x - 1)(x^2 + x - 7)}{(2x - 1)(3x^2 + x - 1)} = \dfrac{x^2 + x - 7}{3x^2 + x - 1}$, $x \neq 1/2$, horizontal asymptote: $y = 1/3$, vertical asymptotes: $x = (-1 \pm \sqrt{13})/6$.



51. (a) $f(x) \leq f(x_0)$ for all x in I.

(b) $f(x) \geq f(x_0)$ for all x in I.

53. (a) True. If f has an absolute extremum at a point of (a, b) then it must, by Theorem 4.4.3, be at a critical point of f; since f is differentiable on (a, b) the critical point is a stationary point.

(b) False. It could occur at a critical point which is not a stationary point: for example, $f(x) = |x|$ on $[-1, 1]$ has an absolute minimum at $x = 0$ but is not differentiable there.

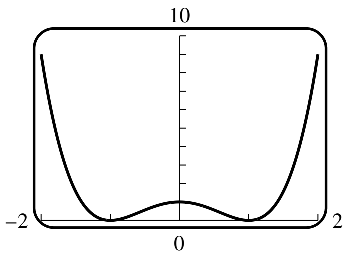
55. (a) $f'(x) = 2x - 3$; critical point $x = 3/2$. Minimum value $f(3/2) = -13/4$, no maximum.

(b) No maximum or minimum because $\lim\limits_{x \to +\infty} f(x) = +\infty$ and $\lim\limits_{x \to -\infty} f(x) = -\infty$.

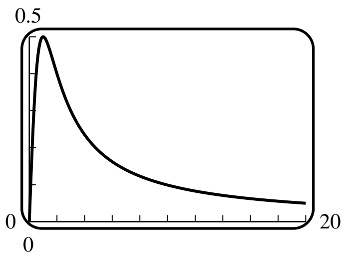
(c) $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to +\infty} f(x) = +\infty$ and $f'(x) = \dfrac{e^x(x - 2)}{x^3}$, stationary point at $x = 2$; by Theorem 4.4.4 $f(x)$ has absolute minimum value $e^2/4$ at $x = 2$; no maximum value.

(d) $f'(x) = (1 + \ln x)x^x$, critical point at $x = 1/e$; $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} e^{x \ln x} = 1$, $\lim\limits_{x \to +\infty} f(x) = +\infty$; no absolute maximum, absolute minimum $m = e^{-1/e}$ at $x = 1/e$.

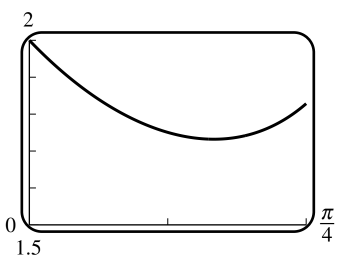
57. (a) $(x^2 - 1)^2$ can never be less than zero because it is the square of $x^2 - 1$; the minimum value is 0 for $x = \pm 1$, no maximum because $\lim\limits_{x \to +\infty} f(x) = +\infty$.



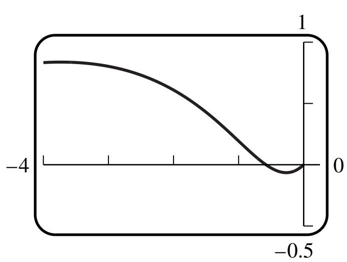
(b) $f'(x) = (1 - x^2)/(x^2+1)^2$; critical point $x = 1$. Maximum value $f(1) = 1/2$, minimum value 0 because $f(x)$ is never less than zero on $[0, +\infty)$ and $f(0) = 0$.



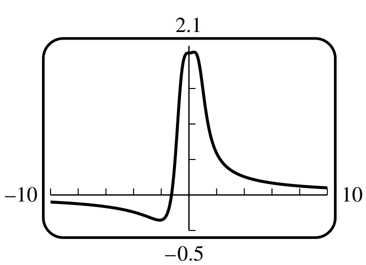
(c) $f'(x) = 2\sec x \tan x - \sec^2 x = (2\sin x - 1)/\cos^2 x$, $f'(x) = 0$ for x in $(0, \pi/4)$ when $x = \pi/6$; $f(0) = 2$, $f(\pi/6) = \sqrt{3}$, $f(\pi/4) = 2\sqrt{2} - 1$ so the maximum value is 2 at $x = 0$ and the minimum value is $\sqrt{3}$ at $x = \pi/6$.



(d) $f'(x) = 1/2 + 2x/(x^2+1)$, $f'(x) = 0$ on $[-4, 0]$ for $x = -2 \pm \sqrt{3}$; if $x = -2 - \sqrt{3}, -2 + \sqrt{3}$, then $f(x) = -1 - \sqrt{3}/2 + \ln 4 + \ln(2 + \sqrt{3}) \approx 0.84, -1 + \sqrt{3}/2 + \ln 4 + \ln(2 - \sqrt{3}) \approx -0.06$, absolute maximum at $x = -2 - \sqrt{3}$, absolute minimum at $x = -2 + \sqrt{3}$.



59. (a)

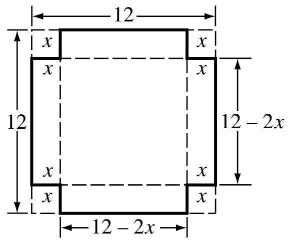


(b) Minimum: $(-2.111985,\ -0.355116)$, maximum: $(0.372591,\ 2.012931)$.

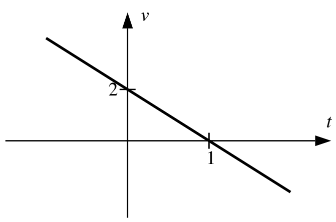
61. If one corner of the rectangle is at (x, y) with $x > 0$, $y > 0$, then $A = 4xy$, $y = 3\sqrt{1 - (x/4)^2}$, $A =$

$12x\sqrt{1-(x/4)^2} = 3x\sqrt{16-x^2}$, $\dfrac{dA}{dx} = 6\dfrac{8-x^2}{\sqrt{16-x^2}}$, critical point at $x = 2\sqrt{2}$. Since $A = 0$ when $x = 0, 4$ and $A > 0$ otherwise, there is an absolute maximum $A = 24$ at $x = 2\sqrt{2}$. The rectangle has width $2x = 4\sqrt{2}$ and height $2y = A/(2x) = 3\sqrt{2}$.

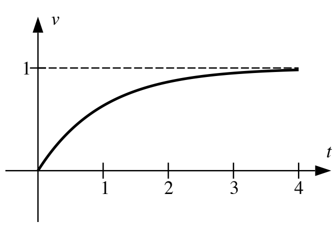
63. $V = x(12-2x)^2$ for $0 \le x \le 6$; $dV/dx = 12(x-2)(x-6)$, $dV/dx = 0$ when $x = 2$ for $0 < x < 6$. If $x = 0, 2, 6$ then $V = 0, 128, 0$ so the volume is largest when $x = 2$ in.



65. **(a)** Yes. If $s = 2t - t^2$ then $v = ds/dt = 2 - 2t$ and $a = dv/dt = -2$ is constant. The velocity changes sign at $t = 1$, so the particle reverses direction then.

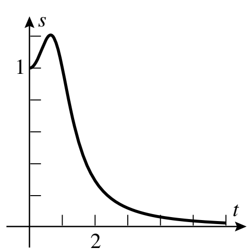


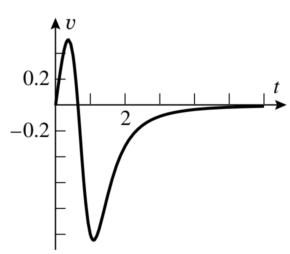
(b) Yes. If $s = t + e^{-t}$ then $v = ds/dt = 1 - e^{-t}$ and $a = dv/dt = e^{-t}$. For $t > 0$, $v > 0$ and $a > 0$, so the particle is speeding up. But $da/dt = -e^{-t} < 0$, so the acceleration is decreasing.

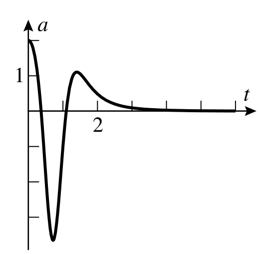


67. **(a)** $v = -2\dfrac{t(t^4+2t^2-1)}{(t^4+1)^2}$, $a = 2\dfrac{3t^8+10t^6-12t^4-6t^2+1}{(t^4+1)^3}$.

(b)







(c) It is farthest from the origin at $t \approx 0.64$ (when $v = 0$) and $s \approx 1.2$.

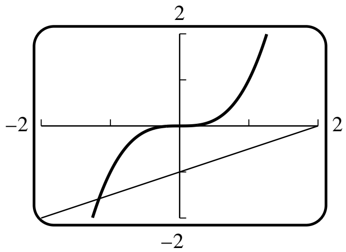
(d) Find t so that the velocity $v = ds/dt > 0$. The particle is moving in the positive direction for $0 \le t \le 0.64$, approximately.

(e) It is speeding up when $a, v > 0$ or $a, v < 0$, so for $0 \le t < 0.36$ and $0.64 < t < 1.1$ approximately, otherwise it is slowing down.

(f) Find the maximum value of $|v|$ to obtain: maximum speed ≈ 1.05 when $t \approx 1.10$.

69. $x \approx -2.11491, 0.25410, 1.86081$.

71. At the point of intersection, $x^3 = 0.5x - 1$, $x^3 - 0.5x + 1 = 0$. Let $f(x) = x^3 - 0.5x + 1$. By graphing $y = x^3$ and $y = 0.5x - 1$ it is evident that there is only one point of intersection and it occurs in the interval $[-2, -1]$; note that $f(-2) < 0$ and $f(-1) > 0$. $f'(x) = 3x^2 - 0.5$ so $x_{n+1} = x_n - \dfrac{x_n^3 - 0.5x_n + 1}{3x_n^2 - 0.5}$; $x_1 = -1$, $x_2 = -1.2$, $x_3 \approx -1.166492147, \ldots, x_5 \approx x_6 \approx -1.165373043$.



73. Solve $\phi - 0.0934 \sin \phi = 2\pi(1)/1.88$ to get $\phi \approx 3.325078$ so $r = 228 \times 10^6(1 - 0.0934 \cos \phi) \approx 248.938 \times 10^6$ km.

75. **(a)** Yes; $f'(0) = 0$.

(b) No, f is not differentiable on $(-1, 1)$.

(c) Yes, $f'(\sqrt{\pi/2}) = 0$.

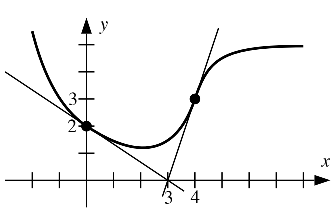
77. $f(x) = x^6 - 2x^2 + x$ satisfies $f(0) = f(1) = 0$, so by Rolle's Theorem $f'(c) = 0$ for some c in $(0, 1)$.

Chapter 4 Making Connections

1. **(a)** $g(x)$ has no zeros. Since $g(x)$ is concave up for $x < 3$, its graph lies on or above the line $y = 2 - \dfrac{2}{3}x$, which is the tangent line at $(0, 2)$. So for $x < 3$, $g(x) \geq 2 - \dfrac{2}{3}x > 0$. Since $g(x)$ is concave up for $3 \leq x < 4$, its graph lies above the line $y = 3x - 9$, which is the tangent line at $(4, 3)$. So for $3 \leq x < 4$, $g(x) > 3x - 9 \geq 0$. Finally, if $x \geq 4$, $g(x)$ could only have a zero if $g'(a)$ were negative for some $a > 4$. But then the graph would lie below the tangent line at $(a, g(a))$, which crosses the line $y = -10$ for some $x > a$. So $g(x)$ would be less than -10 for some x.

(b) One, between 0 and 4.

(c) Since $g(x)$ is concave down for $x > 4$ and $g'(4) = 3$, $g'(x) < 3$ for all $x > 4$. Hence the limit can't be 5. If it were -5 then the graph of $g(x)$ would cross the line $y = -10$ at some point. So the limit must be 0.



3. $g''(x) = 1 - r'(x)$, so $g(x)$ has an inflection point where the graph of $y = r'(x)$ crosses the line $y = 1$; i.e. at $x = -4$ and $x = 5$.