National University
of computer and emerging sciences

Foundation for Advancement
of Science and Technology

Lecture # 23

Integration

Basics

Table 5.2.1

INTEGRATION FORMULAS

DIFFERENTIATION FORMULA	INTEGRATION FORMULA	DIFFERENTIATION FORMULA	INTEGRATION FORMULA
1. $\frac{d}{dx}[x] = 1$	$\int dx = x + C$	8. $\frac{d}{dx}[-\csc x] = \csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + C$
2. $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r \quad (r \neq -1)$	$\int x^r \, dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$	9. $\frac{d}{dx}[e^x] = e^x$	$\int e^x \, dx = e^x + C$
3. $\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x dx = \sin x + C$	10. $\frac{d}{dx}\left[\frac{b^x}{\ln b}\right] = b^x \quad (0 < b, b \neq 1)$	$\int b^x \, dx = \frac{b^x}{\ln b} + C \quad (0 < b, b \neq 1)$
4. $\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x \, dx = -\cos x + C$	11. $\frac{d}{dx}[\ln \lvert x \rvert] = \frac{1}{x}$	$\int \frac{1}{x} \, dx = \ln\lvert x \rvert + C$
5. $\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + C$	12. $\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$
6. $\frac{d}{dx}[-\cot x] = \csc^2 x$	$\int \csc^2 x \, dx = -\cot x + C$	13. $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$
7. $\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + C$	14. $\frac{d}{dx}[\sec^{-1}\lvert x \rvert] = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1}\lvert x \rvert + C$

5.2.3 THEOREM *Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively, and that c is a constant. Then:*

(*a*) *A constant factor can be moved through an integral sign; that is,*

$$\int cf(x)\,dx = cF(x) + C$$

(*b*) *An antiderivative of a sum is the sum of the antiderivatives; that is,*

$$\int [f(x) + g(x)]\,dx = F(x) + G(x) + C$$

(*c*) *An antiderivative of a difference is the difference of the antiderivatives; that is,*

$$\int [f(x) - g(x)]\,dx = F(x) - G(x) + C$$

Evaluate the integrals.

1. $\int x^8\, dx$
2. $\int \frac{1}{x^6}\, dx$
3. $\int x^{5/7}\, dx$
4. $\int \sqrt[3]{x^2}\, dx$
5. $\int \frac{4}{\sqrt{t}}\, dt$
6. $\int x^3 \sqrt{x}\, dx$
7. $\int (u^3 - 2u + 7)\, du$
8. $\int (\frac{7}{y^{\frac{3}{4}}} - \sqrt[3]{y} + 4\sqrt{y})\, dy$
9. $\int (2 + y^2)^2 dy$
10. $\int \frac{x^5+2x^2-1}{x^4}\, dx$
11. $\int \left[\frac{1}{t^2} - \cos t\right] dt$
12. $\int [4sec^2 x + \csc x \cot x] dx$
13. $\int \sec x\, (\sec x + \tan x)\, dx$
14. $\int \sec x (\tan x + \cos x) dx$
15. $\int \frac{\sin x}{cos^2 x} dx$
16. $\int \frac{\sin 2x}{cosx}\, dx$
17. $\int \frac{cos^3\theta - 5}{cos^2\theta}\, d\theta$
18. $\int \frac{1}{1 + \sin x}\, dx$
19. Find the anti-derivative F(x) of f(x) = $\sqrt[3]{x^2}$ that satisfies F(1) = 2.
20. Find the general form of a function whose second derivative is $\sqrt{x}$

1. $\int x^8$ dx

Solution:

Integrate w.r.t 'x'

$\int x^8 \, dx = \frac{x^{8+1}}{8+1} + c = \frac{x^9}{9} + c$

2. $\int \frac{1}{x^6}$ dx

Solution:

Integrate w.r.t 'x'

$\int \frac{1}{x^6} \, dx = x^{-6}dx = \frac{x^{-6+1}}{-6+1} + c = \frac{x^{-5}}{-5} + c = \frac{-1}{5x^5} + c$

3. $\int x^{5/7}$ dx

Solution:

Integrate w.r.t 'x'

$\int x^{5/7}\, dx = \frac{x^{\frac{5}{7}+1}}{\frac{5}{7}+1} + c = \frac{x^{\frac{12}{7}}}{\frac{12}{7}} + c = \frac{7}{12} x^{\frac{12}{7}} + c$

4. $\int \sqrt[3]{x^2}$ dx

Solution:

$\int \sqrt[3]{x^2} \, dx = \int x^{\frac{2}{3}}\, dx = \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + c = \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + c = \frac{3}{5} \sqrt[3]{x^5} + c$

5. $\int \frac{4}{\sqrt{t}}$ dt

Solution:

$\int \frac{4}{\sqrt{t}} \, dt = 4 \int t^{\frac{-1}{2}}\, dt = 4 \cdot \frac{t^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + c = 4 \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c = 8\sqrt{t} + c$

6. $\int x^3 \sqrt{x}$ dx

Solution:

$\int x^3 \sqrt{x} \, dx = \int x^{3+\frac{1}{2}}\, dx = \frac{x^{\frac{7}{2}+1}}{\frac{7}{2}+1} + c = \frac{2}{9}\, x^{\frac{9}{2}} + c$

9. $\int (2+\ y^2)^2 dy$

Solution:

$\int (4+4y^2+y^4)dy$ = $4\int dy$ + $4\int y^2 dy$ + $\int y^4 dy$

= 4y + $\frac{4}{3}y^3$ +$\frac{y^5}{5}$ + c

10. $\int \frac{x^5+2x^2-1}{x^4}$ dx

Solution:

$\int \frac{x^5}{x^4}$ dx + $\int \frac{2x^2}{x^4}$ dx - $\int \frac{1}{x^4}$ dx

= $\int xdx$ + 2 $\int x^{-2}dx$ - $\int x^{-4}$dx

= $\frac{x^2}{2}$ - $\frac{2}{x}$ + $\frac{1}{3x^3}$ + c

15. $\int \frac{\sin x}{cos^2 x}$dx

Solution:

$\int \frac{\sin x}{cos^2 x}$dx

= $\int \frac{\sin x}{\cos x} . \frac{1}{\cos x}$ dx

= $\int \tan x \,.\, sec\ x\ dx$

= $\sec x$ + c

16. $\int \frac{\sin 2x}{cosx} dx$

Solution:

$\int \frac{\sin 2x}{cosx} dx$ = $\int \frac{2\sin x \cos x}{cosx} dx$ ($\sin 2x$ = $2 \sin x \cos x$)

(Double angle formula)

= 2 $\int \sin x\, dx$

= - 2 $\cos x$ + c

19. Find the anti-derivative F(x) of f(x) = $\sqrt[3]{x^2}$ that satisfies F(1) = 2.

Solution:

f(x) = $\sqrt[3]{x^2}$

Integrate w.r.t 'x' on both side

$\int$ f(x)dx = $\int \sqrt[3]{x^2}\, dx$

F(x) = $\int x^{\frac{2}{3}}\, dx = \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1}$ + c = $\frac{x^{\frac{5}{3}}}{\frac{5}{3}}$ + c = $\frac{3}{5}\sqrt[3]{x^5}$ + c

F(x) = $\frac{3}{5}\sqrt[3]{x^5}$ + c (1)

Now put F(1) = 2, in equation (1)

F(1) = $\frac{3}{5}\sqrt[3]{1^5}$ + c

2 = $\frac{3}{5}$ + c

c = $\frac{7}{5}$

Put the value of c in equation (1)

F(x) = $\frac{3}{5}\sqrt[3]{x^5} + \frac{7}{5}$

20. Find the general form of a function whose second derivative is $\sqrt{x}$

Solution:

$f''(x) = \sqrt{x} = x^{\frac{1}{2}}$

Integrate w.r.t 'x' on both side

$\int f''(x)dx = \int x^{\frac{1}{2}}dx$

$f'(x) = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}$ + c

$f'(x) = \frac{2}{3}x^{\frac{3}{2}}$ + c

Again Integrate w.r.t 'x' on both side

$\int f'(x)dx = \int \frac{2}{3}\, x^{\frac{3}{2}}dx + c\int dx$

f(x) = $\frac{2}{3}\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}$ + c x + d

f(x) = $\frac{2}{3}\frac{x^{\frac{5}{2}}}{\frac{5}{2}}$ + c x + d

f(x) = $\frac{4}{15}x^{\frac{5}{2}}$+ c x + d (where c and d are constants of integration)

CONSTANTS, POWERS, EXPONENTIALS

1. $\int du = u + C$
2. $\int a\,du = a\int du = au + C$
3. $\int u^r\,du = \frac{u^{r+1}}{r+1} + C,\ r \neq -1$
4. $\int \frac{du}{u} = \ln|u| + C$
5. $\int e^u\,du = e^u + C$
6. $\int b^u\,du = \frac{b^u}{\ln b} + C,\ b > 0, b \neq 1$

TRIGONOMETRIC FUNCTIONS

7. $\int \sin u\,du = -\cos u + C$
8. $\int \cos u\,du = \sin u + C$
9. $\int \sec^2 u\,du = \tan u + C$
10. $\int \csc^2 u\,du = -\cot u + C$
11. $\int \sec u \tan u\,du = \sec u + C$
12. $\int \csc u \cot u\,du = -\csc u + C$
13. $\int \tan u\,du = -\ln|\cos u| + C$
14. $\int \cot u\,du = \ln|\sin u| + C$

HYPERBOLIC FUNCTIONS

15. $\int \sinh u\,du = \cosh u + C$
16. $\int \cosh u\,du = \sinh u + C$
17. $\int \operatorname{sech}^2 u\,du = \tanh u + C$
18. $\int \operatorname{csch}^2 u\,du = -\coth u + C$
19. $\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$
20. $\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$

ALGEBRAIC FUNCTIONS ($a > 0$)

21. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C \qquad (|u| < a)$
22. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$
23. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \qquad (0 < a < |u|)$

24. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \ln(u+\sqrt{u^2+a^2}) + C$

25. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C \qquad (0 < a < |u|)$

26. $\displaystyle\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C$

27. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C \qquad (0 < |u| < a)$

28. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

1–30 Evaluate the integrals by making appropriate u-substitutions and applying the formulas reviewed in this section.

1. $\int (4-2x)^3\,dx$

2. $\int 3\sqrt{4+2x}\,dx$

3. $\int x\sec^2(x^2)\,dx$

4. $\int 4x\tan(x^2)\,dx$

5. $\int \frac{\sin 3x}{2+\cos 3x}\,dx$

6. $\int \frac{1}{9+4x^2}\,dx$

7. $\int e^x \sinh(e^x)\,dx$

8. $\int \frac{\sec(\ln x)\tan(\ln x)}{x}\,dx$

9. $\int e^{\tan x}\sec^2 x\,dx$

10. $\int \frac{x}{\sqrt{1-x^4}}\,dx$

11. $\int \cos^5 5x \sin 5x\,dx$

12. $\int \frac{\cos x}{\sin x\sqrt{\sin^2 x+1}}\,dx$

13. $\displaystyle\int \frac{e^x}{\sqrt{4+e^{2x}}}\,dx$

14. $\displaystyle\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx$

15. $\displaystyle\int \frac{e^{\sqrt{x-1}}}{\sqrt{x-1}}\,dx$

16. $\displaystyle\int (x+1)\cot(x^2+2x)\,dx$

17. $\displaystyle\int \frac{\cosh\sqrt{x}}{\sqrt{x}}\,dx$

18. $\displaystyle\int \frac{dx}{x(\ln x)^2}$

19. $\displaystyle\int \frac{dx}{\sqrt{x}\,3^{\sqrt{x}}}$

20. $\displaystyle\int \sec(\sin\theta)\tan(\sin\theta)\cos\theta\,d\theta$

21. $\displaystyle\int \frac{\operatorname{csch}^2(2/x)}{x^2}\,dx$

22. $\displaystyle\int \frac{dx}{\sqrt{x^2-4}}$

23. $\displaystyle\int \frac{e^{-x}}{4-e^{-2x}}\,dx$

24. $\displaystyle\int \frac{\cos(\ln x)}{x}\,dx$

25. $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

26. $\displaystyle\int \frac{\sinh(x^{-1/2})}{x^{3/2}}\,dx$

27. $\displaystyle\int \frac{x}{\csc(x^2)}\,dx$

28. $\displaystyle\int \frac{e^x}{\sqrt{4-e^{2x}}}\,dx$

29. $\displaystyle\int x4^{-x^2}\,dx$

30. $\displaystyle\int 2^{\pi x}\,dx$

Do Questions (1-30) from Ex # 7.1

Ex # 7.2 integration by parts and reduction formulas

By parts formula :

$$\int u.vdx = u.\int vdx - \int\left(u'.\int vdx\right)dx$$

LIATE

There is another useful strategy for choosing u and dv that can be applied when the integrand is a product of two functions from *different* categories in the list

Logarithmic, **I**nverse trigonometric, **A**lgebraic, **T**rigonometric, **E**xponential

In this case you will often be successful if you take u to be the function whose category occurs earlier in the list and take dv to be the rest of the integrand. The acronym LIATE will help you to remember the order. The method does not work all the time, but it works often enough to be useful.

Tabular Integration by Parts

Step 1. Differentiate $p(x)$ repeatedly until you obtain 0, and list the results in the first column.

Step 2. Integrate $f(x)$ repeatedly and list the results in the second column.

Step 3. Draw an arrow from each entry in the first column to the entry that is one row down in the second column.

Step 4. Label the arrows with alternating $+$ and $-$ signs, starting with a $+$.

Step 5. For each arrow, form the product of the expressions at its tip and tail and then multiply that product by $+1$ or -1 in accordance with the sign on the arrow. Add the results to obtain the value of the integral.

This process is illustrated in Figure 7.2.1 for the integral $\int (x^2 - x)\cos x\, dx$.

REPEATED DIFFERENTIATION		REPEATED INTEGRATION
$x^2 - x$	$+$	$\cos x$
$2x - 1$	$-$	$\sin x$
2	$+$	$-\cos x$
0		$-\sin x$

$$\boxed{\begin{aligned}\int (x^2 - x)\cos x\, dx &= (x^2 - x)\sin x + (2x - 1)\cos x - 2\sin x + C \\ &= (x^2 - x - 2)\sin x + (2x - 1)\cos x + C\end{aligned}}$$

Figure 7.2.1

REDUCTION FORMULAE

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

Example 8 Evaluate $\int \cos^4 x\,dx$.

Solution. From (10) with $n = 4$

$$\int \cos^4 x\,dx = \tfrac{1}{4}\cos^3 x \sin x + \tfrac{3}{4}\int \cos^2 x\,dx$$

Now apply (10) with $n = 2$.

$$= \tfrac{1}{4}\cos^3 x \sin x + \tfrac{3}{4}\left(\tfrac{1}{2}\cos x \sin x + \tfrac{1}{2}\int dx\right)$$

$$= \tfrac{1}{4}\cos^3 x \sin x + \tfrac{3}{8}\cos x \sin x + \tfrac{3}{8}x + C$$

INTEGRATING PRODUCTS OF SINES AND COSINES

If m and n are positive integers, then the integral

$$\int \sin^m x \cos^n x \, dx$$

► **Example 2** Evaluate

$$\text{(a)} \int \sin^4 x \cos^5 x \, dx \qquad \text{(b)} \int \sin^4 x \cos^4 x \, dx$$

Table 7.3.1

INTEGRATING PRODUCTS OF SINES AND COSINES

$\int \sin^m x \cos^n x \, dx$	PROCEDURE	RELEVANT IDENTITIES
n odd	• Split off a factor of $\cos x$. • Apply the relevant identity. • Make the substitution $u = \sin x$.	$\cos^2 x = 1 - \sin^2 x$
m odd	• Split off a factor of $\sin x$. • Apply the relevant identity. • Make the substitution $u = \cos x$.	$\sin^2 x = 1 - \cos^2 x$
$\begin{cases} m \text{ even} \\ n \text{ even} \end{cases}$	• Use the relevant identities to reduce the powers on $\sin x$ and $\cos x$.	$\begin{cases} \sin^2 x = \frac{1}{2}(1 - \cos 2x) \\ \cos^2 x = \frac{1}{2}(1 + \cos 2x) \end{cases}$

Solution (a). Since $n = 5$ is odd, we will follow the first procedure in Table 7.3.1:

$$\begin{aligned}
\int \sin^4 x \cos^5 x \, dx &= \int \sin^4 x \cos^4 x \cos x \, dx \\
&= \int \sin^4 x (1 - \sin^2 x)^2 \cos x \, dx \\
&= \int u^4 (1 - u^2)^2 \, du \\
&= \int (u^4 - 2u^6 + u^8) \, du \\
&= \tfrac{1}{5}u^5 - \tfrac{2}{7}u^7 + \tfrac{1}{9}u^9 + C \\
&= \tfrac{1}{5}\sin^5 x - \tfrac{2}{7}\sin^7 x + \tfrac{1}{9}\sin^9 x + C
\end{aligned}$$

$\int \sin^m x \cos^n x \, dx$	PROCEDURE	RELEVANT IDENTITIES
n odd	• Split off a factor of $\cos x$. • Apply the relevant identity. • Make the substitution $u = \sin x$.	$\cos^2 x = 1 - \sin^2 x$

Solution (b). Since $m = n = 4$, both exponents are even, so we will follow the third procedure in Table 7.3.1:

$$\int \sin^4 x \cos^4 x \, dx = \int (\sin^2 x)^2 (\cos^2 x)^2 \, dx$$

$$= \int \left(\tfrac{1}{2}[1 - \cos 2x]\right)^2 \left(\tfrac{1}{2}[1 + \cos 2x]\right)^2 \, dx$$

$$= \tfrac{1}{16} \int (1 - \cos^2 2x)^2 \, dx$$

$$= \tfrac{1}{16} \int \sin^4 2x \, dx$$

Note that this can be obtained more directly from the original integral using the identity $\sin x \cos x = \frac{1}{2} \sin 2x$.

$$= \tfrac{1}{32} \int \sin^4 u \, du$$

$u = 2x$
$du = 2\,dx$ or $dx = \frac{1}{2}\,du$

$$= \tfrac{1}{32} \left(\tfrac{3}{8}u - \tfrac{1}{4} \sin 2u + \tfrac{1}{32} \sin 4u\right) + C$$

Formula (13)

$$= \tfrac{3}{128}x - \tfrac{1}{128} \sin 4x + \tfrac{1}{1024} \sin 8x + C$$

$\begin{cases} m \text{ even} \\ n \text{ even} \end{cases}$	• Use the relevant identities to reduce the powers on $\sin x$ and $\cos x$.	$\begin{cases} \sin^2 x = \frac{1}{2}(1 - \cos 2x) \\ \cos^2 x = \frac{1}{2}(1 + \cos 2x) \end{cases}$

Integrals of the form

$$\int \sin mx \cos nx \, dx, \quad \int \sin mx \sin nx \, dx, \quad \int \cos mx \cos nx \, dx$$

can be found by using the trigonometric identities

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

to express the integrand as a sum or difference of sines and cosines.

► **Example 3** Evaluate $\int \sin 7x \cos 3x \, dx$.

Solution. Using (16) yields

$$\int \sin 7x \cos 3x \, dx = \tfrac{1}{2}\int (\sin 4x + \sin 10x)\, dx = -\tfrac{1}{8}\cos 4x - \tfrac{1}{20}\cos 10x + C$$

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$$

$$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

INTEGRATING PRODUCTS OF TANGENTS AND SECANTS

If m and n are positive integers, then the integral

$$\int \tan^m x \sec^n x \, dx$$

Table 7.3.2

INTEGRATING PRODUCTS OF TANGENTS AND SECANTS

$\int \tan^m x \sec^n x \, dx$	PROCEDURE	RELEVANT IDENTITIES
n even	• Split off a factor of $\sec^2 x$. • Apply the relevant identity. • Make the substitution $u = \tan x$.	$\sec^2 x = \tan^2 x + 1$
m odd	• Split off a factor of $\sec x \tan x$. • Apply the relevant identity. • Make the substitution $u = \sec x$.	$\tan^2 x = \sec^2 x - 1$
$\begin{cases} m \text{ even} \\ n \text{ odd} \end{cases}$	• Use the relevant identities to reduce the integrand to powers of $\sec x$ alone. • Then use the reduction formula for powers of $\sec x$.	$\tan^2 x = \sec^2 x - 1$

▶ **Example 4** Evaluate

(a) $\int \tan^2 x \sec^4 x \, dx$ (b) $\int \tan^3 x \sec^3 x \, dx$ (c) $\int \tan^2 x \sec x \, dx$

Solution (a). Since $n = 4$ is even, we will follow the first procedure in Table 7.3.2:

$$\begin{aligned}
\int \tan^2 x \sec^4 x \, dx &= \int \tan^2 x \sec^2 x \sec^2 x \, dx \\
&= \int \tan^2 x (\tan^2 x + 1) \sec^2 x \, dx \\
&= \int u^2 (u^2 + 1) \, du \\
&= \tfrac{1}{5} u^5 + \tfrac{1}{3} u^3 + C = \tfrac{1}{5} \tan^5 x + \tfrac{1}{3} \tan^3 x + C
\end{aligned}$$

n even	• Split off a factor of $\sec^2 x$. • Apply the relevant identity. • Make the substitution $u = \tan x$.	$\sec^2 x = \tan^2 x + 1$

Solution (b). Since $m = 3$ is odd, we will follow the second procedure in Table 7.3.2:

$$\begin{aligned}\int \tan^3 x \sec^3 x\,dx &= \int \tan^2 x \sec^2 x (\sec x \tan x)\,dx \\ &= \int (\sec^2 x - 1)\sec^2 x (\sec x \tan x)\,dx \\ &= \int (u^2 - 1)u^2\,du \\ &= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}\sec^5 x - \tfrac{1}{3}\sec^3 x + C\end{aligned}$$

m odd	• Split off a factor of $\sec x \tan x$. • Apply the relevant identity. • Make the substitution $u = \sec x$.	$\tan^2 x = \sec^2 x - 1$

Solution (c). Since $m = 2$ is even and $n = 1$ is odd, we will follow the third procedure in Table 7.3.2:

$$\begin{aligned}\int \tan^2 x \sec x\,dx &= \int (\sec^2 x - 1)\sec x\,dx \\ &= \int \sec^3 x\,dx - \int \sec x\,dx \qquad \boxed{\text{See (26) and (22)}} \\ &= \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln|\sec x + \tan x| - \ln|\sec x + \tan x| + C \\ &= \tfrac{1}{2}\sec x \tan x - \tfrac{1}{2}\ln|\sec x + \tan x| + C \blacktriangleleft\end{aligned}$$

$\begin{cases} m \text{ even} \\ n \text{ odd}\end{cases}$	• Use the relevant identities to reduce the integrand to powers of $\sec x$ alone. • Then use the reduction formula for powers of $\sec x$.	$\tan^2 x = \sec^2 x - 1$

1–38 Evaluate the integral.

1. $\int xe^{-2x}\,dx$

2. $\int xe^{3x}\,dx$

3. $\int x^2e^x\,dx$

4. $\int x^2e^{-2x}\,dx$

5. $\int x\sin 3x\,dx$

6. $\int x\cos 2x\,dx$

7. $\int x^2\cos x\,dx$

8. $\int x^2\sin x\,dx$

9. $\int x\ln x\,dx$

10. $\int \sqrt{x}\ln x\,dx$

11. $\int (\ln x)^2\,dx$

12. $\int \frac{\ln x}{\sqrt{x}}\,dx$

13. $\int \ln(3x-2)\,dx$

14. $\int \ln(x^2+4)\,dx$

15. $\int \sin^{-1}x\,dx$

16. $\int \cos^{-1}(2x)\,dx$

17. $\int \tan^{-1}(3x)\,dx$

18. $\int x\tan^{-1}x\,dx$

19. $\int e^x\sin x\,dx$

20. $\int e^{3x}\cos 2x\,dx$

21. $\int \sin(\ln x)\,dx$

22. $\int \cos(\ln x)\,dx$

23. $\int x\sec^2 x\,dx$

24. $\int x\tan^2 x\,dx$

25. $\int x^3e^{x^2}\,dx$

26. $\int \frac{xe^x}{(x+1)^2}\,dx$

27. $\int_0^2 xe^{2x}\,dx$

28. $\int_0^1 xe^{-5x}\,dx$

29. $\int_1^e x^2\ln x\,dx$

30. $\int_{\sqrt{e}}^e \frac{\ln x}{x^2}\,dx$

31. $\int_{-1}^1 \ln(x+2)\,dx$

32. $\int_0^{\sqrt{3}/2} \sin^{-1}x\,dx$

33. $\int_2^4 \sec^{-1}\sqrt{\theta}\,d\theta$

34. $\int_1^2 x\sec^{-1}x\,dx$

35. $\int_0^\pi x\sin 2x\,dx$

36. $\int_0^\pi (x+x\cos x)\,dx$

37. $\int_1^3 \sqrt{x}\tan^{-1}\sqrt{x}\,dx$

38. $\int_0^2 \ln(x^2+1)\,dx$

colony.

69. Use reduction formula (9) to evaluate

(a) $\int \sin^4 x\,dx$ (b) $\int_0^{\pi/2} \sin^5 x\,dx$.

70. Use reduction formula (10) to evaluate

(a) $\int \cos^5 x\,dx$ (b) $\int_0^{\pi/2} \cos^6 x\,dx$.

71. Derive reduction formula (9).

72. In each part, use integration by parts or other methods to derive the reduction formula.

(a) $\displaystyle\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$

(b) $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$

(c) $\displaystyle\int x^n e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx$

Lecture # 25

TRIGONOMETRIC SUBSTITUTION

CONSTANTS, POWERS, EXPONENTIALS

1. $\int du = u + C$
2. $\int a\,du = a\int du = au + C$
3. $\int u^r\,du = \dfrac{u^{r+1}}{r+1} + C,\ r \neq -1$
4. $\int \dfrac{du}{u} = \ln|u| + C$
5. $\int e^u\,du = e^u + C$
6. $\int b^u\,du = \dfrac{b^u}{\ln b} + C,\ b > 0, b \neq 1$

TRIGONOMETRIC FUNCTIONS

7. $\int \sin u\,du = -\cos u + C$
8. $\int \cos u\,du = \sin u + C$
9. $\int \sec^2 u\,du = \tan u + C$
10. $\int \csc^2 u\,du = -\cot u + C$
11. $\int \sec u \tan u\,du = \sec u + C$
12. $\int \csc u \cot u\,du = -\csc u + C$
13. $\int \tan u\,du = -\ln|\cos u| + C$
14. $\int \cot u\,du = \ln|\sin u| + C$

HYPERBOLIC FUNCTIONS

15. $\int \sinh u\,du = \cosh u + C$
16. $\int \cosh u\,du = \sinh u + C$
17. $\int \operatorname{sech}^2 u\,du = \tanh u + C$
18. $\int \operatorname{csch}^2 u\,du = -\coth u + C$
19. $\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$
20. $\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$

ALGEBRAIC FUNCTIONS ($a > 0$)

21. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\dfrac{u}{a} + C \qquad (|u| < a)$
22. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\dfrac{u}{a} + C$
23. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{u}{a}\right| + C \qquad (0 < a < |u|)$

THE METHOD OF TRIGONOMETRIC SUBSTITUTION

To start, we will be concerned with integrals that contain expressions of the form

$$\sqrt{a^2 - x^2}, \quad \sqrt{x^2 + a^2}, \quad \sqrt{x^2 - a^2}$$

in which a is a positive constant. The basic idea for evaluating such integrals is to make a substitution for x that will eliminate the radical. For example, to eliminate the radical in the expression $\sqrt{a^2 - x^2}$, we can make the substitution

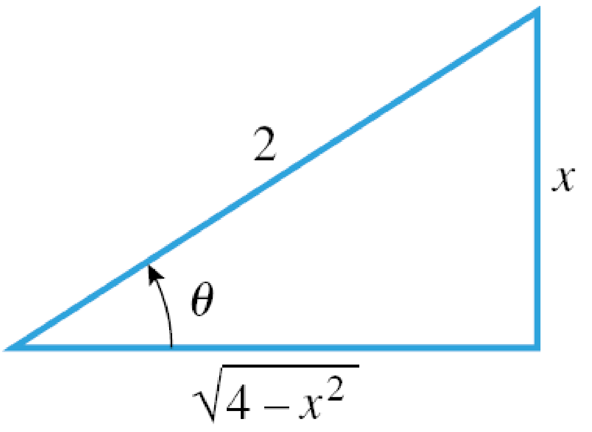
$$x = a \sin\theta, \quad -\pi/2 \le \theta \le \pi/2 \tag{1}$$

which yields

$$\begin{aligned} \sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} \\ &= a\sqrt{\cos^2\theta} = a|\cos\theta| = a\cos\theta \end{aligned}$$

$\boxed{\cos\theta \ge 0 \text{ since } -\pi/2 \le \theta \le \pi/2}$

▶ Example 1 Evaluate $\int \frac{dx}{x^2\sqrt{4 - x^2}}$.



$\boxed{x = 2\sin\theta}$

Solution. To eliminate the radical we make the substitution

$$x = 2\sin\theta, \quad dx = 2\cos\theta\, d\theta$$

This yields

$$\int \frac{dx}{x^2\sqrt{4 - x^2}} = \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4 - 4\sin^2\theta}}$$

$$= \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2(2\cos\theta)} = \frac{1}{4}\int \frac{d\theta}{\sin^2\theta}$$

$$= \frac{1}{4}\int \csc^2\theta\, d\theta = -\frac{1}{4}\cot\theta + C \qquad (2)$$

From that figure we obtain

$$\cot\theta = \frac{\sqrt{4-x^2}}{x}$$

Substituting this in (2) yields

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C \blacktriangleleft$$

Evaluate $\displaystyle\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}$.

Method 1.

Using the result from Example 1 with the x-limits of integration yields

$$\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\left[\frac{\sqrt{4-x^2}}{x}\right]_1^{\sqrt{2}} = -\frac{1}{4}\left[1-\sqrt{3}\right] = \frac{\sqrt{3}-1}{4}$$

Method 2.

The substitution $x = 2\sin\theta$ can be expressed as $x/2 = \sin\theta$ or $\theta = \sin^{-1}(x/2)$, so the θ-limits that correspond to $x = 1$ and $x = \sqrt{2}$ are

$$x = 1: \qquad \theta = \sin^{-1}(1/2) = \pi/6$$

$$x = \sqrt{2}: \quad \theta = \sin^{-1}(\sqrt{2}/2) = \pi/4$$

Thus, from (2) in Example 1 we obtain

$$\int_{1}^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = \frac{1}{4}\int_{\pi/6}^{\pi/4} \csc^2\theta\, d\theta$$

Convert x-limits to θ-limits.

$$= -\frac{1}{4}\left[\cot\theta\right]_{\pi/6}^{\pi/4} = -\frac{1}{4}\left[1-\sqrt{3}\right] = \frac{\sqrt{3}-1}{4}$$

Table 7.4.1

TRIGONOMETRIC SUBSTITUTIONS

EXPRESSION IN THE INTEGRAND	SUBSTITUTION	RESTRICTION ON θ	SIMPLIFICATION
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$	$-\pi/2 \le \theta \le \pi/2$	$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$	$-\pi/2 < \theta < \pi/2$	$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$	$\begin{cases} 0 \le \theta < \pi/2 & (\text{if } x \ge a) \\ \pi/2 < \theta \le \pi & (\text{if } x \le -a) \end{cases}$	$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

V EXAMPLE 2 Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

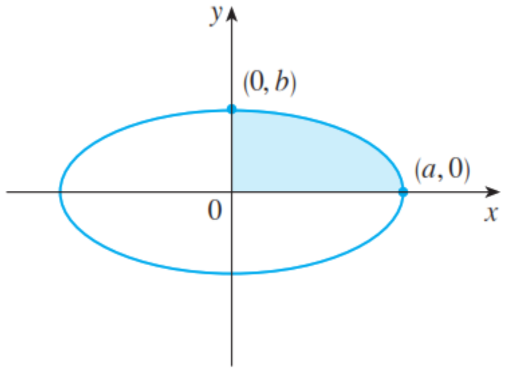


FIGURE 2
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

SOLUTION Solving the equation of the ellipse for y, we get

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2} \qquad \text{or} \qquad y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

Because the ellipse is symmetric with respect to both axes, the total area A is four times the area in the first quadrant (see Figure 2). The part of the ellipse in the first quadrant is given by the function

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad 0 \leqslant x \leqslant a$$

and so

$$\tfrac{1}{4}A = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

To evaluate this integral we substitute $x = a \sin \theta$. Then $dx = a \cos \theta \, d\theta$. To change the limits of integration we note that when $x = 0$, $\sin \theta = 0$, so $\theta = 0$; when $x = a$, $\sin \theta = 1$, so $\theta = \pi/2$. Also

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = \sqrt{a^2 \cos^2\theta} = a\,|\cos \theta| = a \cos \theta$$

since $0 \leqslant \theta \leqslant \pi/2$. Therefore

$$\begin{aligned} A &= 4\,\frac{b}{a} \int_0^a \sqrt{a^2 - x^2}\, dx = 4\,\frac{b}{a} \int_0^{\pi/2} a \cos \theta \cdot a \cos \theta \, d\theta \\ &= 4ab \int_0^{\pi/2} \cos^2\theta \, d\theta = 4ab \int_0^{\pi/2} \tfrac{1}{2}(1 + \cos 2\theta)\, d\theta \\ &= 2ab\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{\pi/2} = 2ab\left(\frac{\pi}{2} + 0 - 0\right) = \pi ab \end{aligned}$$

We have shown that the area of an ellipse with semiaxes a and b is πab. In particular, taking $a = b = r$, we have proved the famous formula that the area of a circle with radius r is πr^2.

Evaluate $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\,dx$, assuming that $x \geq 5$.

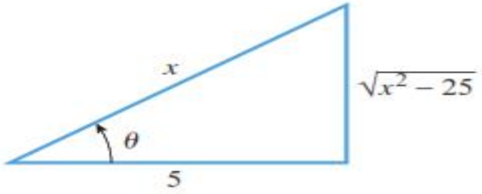
Solution. The integrand involves a radical of the form $\sqrt{x^2-a^2}$ with $a=5$, so from Table 7.4.1 we make the substitution

$$x = 5\sec\theta, \quad 0 \leq \theta < \pi/2$$

$$\frac{dx}{d\theta} = 5\sec\theta\tan\theta \quad \text{or} \quad dx = 5\sec\theta\tan\theta\,d\theta$$

Thus,

$$\begin{aligned}
\int \frac{\sqrt{x^2-25}}{x}\,dx &= \int \frac{\sqrt{25\sec^2\theta-25}}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta \\
&= \int \frac{5|\tan\theta|}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta \\
&= 5\int \tan^2\theta\,d\theta \qquad \boxed{\tan\theta \geq 0 \text{ since } 0 \leq \theta < \pi/2} \\
&= 5\int (\sec^2\theta - 1)\,d\theta = 5\tan\theta - 5\theta + C
\end{aligned}$$



$x = 5\sec\theta$

Figure 7.4.5

To express the solution in terms of x, we will represent the substitution $x = 5\sec\theta$ geometrically by the triangle in Figure 7.4.5, from which we obtain

$$\tan\theta = \frac{\sqrt{x^2-25}}{5}$$

From this and the fact that the substitution can be expressed as $\theta = \sec^{-1}(x/5)$, we obtain

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right) + C$$

$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\begin{cases} 0 \leq \theta < \pi/2 & (\text{if } x \geq a) \\ \pi/2 < \theta \leq \pi & (\text{if } x \leq -a) \end{cases}$	$x^2-a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

INTEGRALS INVOLVING $ax^2 + bx + c$

HINT: First completing the square, then making an appropriate substitution.

Evaluate $\int \frac{x}{x^2 - 4x + 8}\, dx.$

Solution. Completing the square yields

$$x^2 - 4x + 8 = (x^2 - 4x + 4) + 8 - 4 = (x - 2)^2 + 4$$

Thus, the substitution $u = x - 2, \quad du = dx$

yields

$$\int \frac{x}{x^2 - 4x + 8}\, dx = \int \frac{x}{(x-2)^2 + 4}\, dx = \int \frac{u+2}{u^2 + 4}\, du$$

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$$= \int \frac{u}{u^2+4}\,du + 2\int \frac{du}{u^2+4}$$

$$= \frac{1}{2}\int \frac{2u}{u^2+4}\,du + 2\int \frac{du}{u^2+4}$$

$$= \frac{1}{2}\ln(u^2+4) + 2\left(\frac{1}{2}\right)\tan^{-1}\frac{u}{2} + C$$

$$= \frac{1}{2}\ln[(x-2)^2+4] + \tan^{-1}\left(\frac{x-2}{2}\right) + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

16. $\displaystyle\int \frac{dx}{1+2x^2+x^4}$

$1+2x^2+x^4=(1+x^2)^2$, $x=\tan\theta$, $dx=\sec^2\theta\,d\theta$, $\displaystyle\int\frac{1}{\sec^2\theta}d\theta=\int\cos^2\theta\,d\theta=\frac{1}{2}\int(1+\cos 2\theta)d\theta=\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta+C=\frac{1}{2}\theta+\frac{1}{2}\sin\theta\cos\theta+C=\frac{1}{2}\tan^{-1}x+\frac{x}{2(1+x^2)}+C.$

22. $\displaystyle\int_0^{1/2}\frac{dx}{(1-x^2)^2}$

$x=\sin\theta$, $dx=\cos\theta\,d\theta$, $\displaystyle\int_0^{\pi/6}\sec^3\theta\,d\theta=\left[\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\ln|\sec\theta+\tan\theta|\right]_0^{\pi/6}=\left(\frac{1}{2}\frac{2}{\sqrt{3}}\frac{1}{\sqrt{3}}+\frac{1}{2}\ln(\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)=\frac{1}{3}+\frac{1}{4}\ln 3.$

Do Questions (1-25,37-48) from Ex # 7.4