Chapter 9 .
Ex#9.1

Infinite Sequences & Series

Sequence :

A **sequence** *can* be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, \dots \dots \dots \dots, a_n$$

Where ***n*** is called the index of the sequence

The number a_1 is called the ***first term***, a_2 is the ***second term***, and in general a_n is the ***nth term*** .

9.1.1 DEFINITION A *sequence* is a function whose domain is a set of integers.

NOTATION The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}$$

Examples :

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ $\qquad a_n = \dfrac{n}{n+1}$ $\qquad \left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots, \dfrac{n}{n+1}, \ldots\right\}$

(b) $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}$ $\qquad a_n = \dfrac{(-1)^n(n+1)}{3^n}$ $\qquad \left\{-\dfrac{2}{3}, \dfrac{3}{9}, -\dfrac{4}{27}, \dfrac{5}{81}, \ldots, \dfrac{(-1)^n(n+1)}{3^n}, \ldots\right\}$

(c) $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$ $\qquad a_n = \sqrt{n-3},\ n \geqslant 3$ $\qquad \left\{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\right\}$

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Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{125}{125}, -\frac{6}{625}, \frac{7}{3125}, \dots\right\}$$

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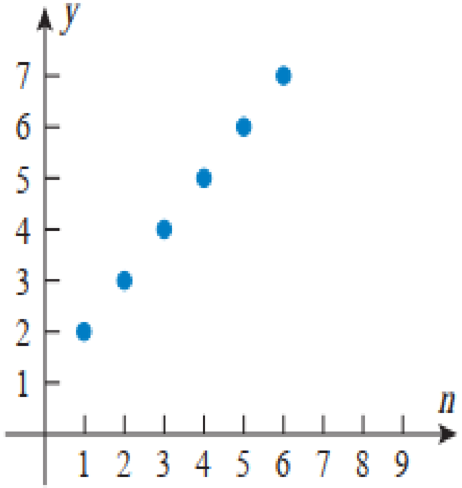
$$\boxed{a_n = (-1)^{n-1}\frac{n+2}{5^n}}$$

SEQUENCE	BRACE NOTATION
$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$	$\left\{\frac{n}{n+1}\right\}_{n=1}^{+\infty}$
$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$	$\left\{\frac{1}{2^n}\right\}_{n=1}^{+\infty}$
$\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots, (-1)^{n+1}\frac{n}{n+1}, \ldots$	$\left\{(-1)^{n+1}\frac{n}{n+1}\right\}_{n=1}^{+\infty}$
$1, 3, 5, 7, \ldots, 2n-1, \ldots$	$\{2n-1\}_{n=1}^{+\infty}$

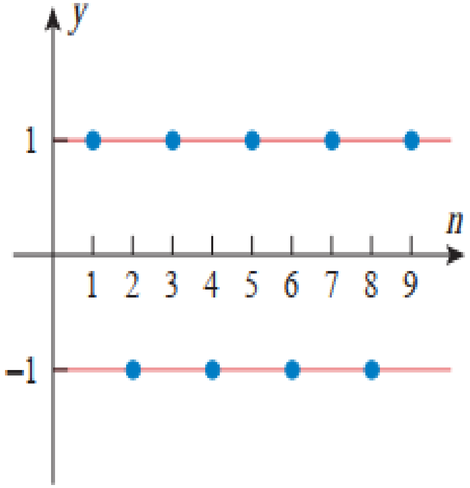
LIMIT OF A SEQUENCE

$\{n+1\}$



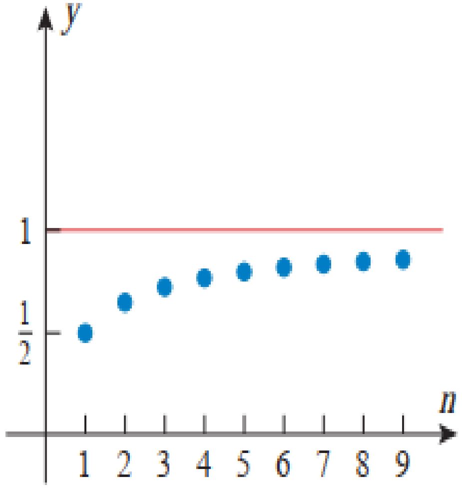
$\left\{n+1\right\}_{n=1}^{+\infty}$

$\{(-1)^{n+1}\}$



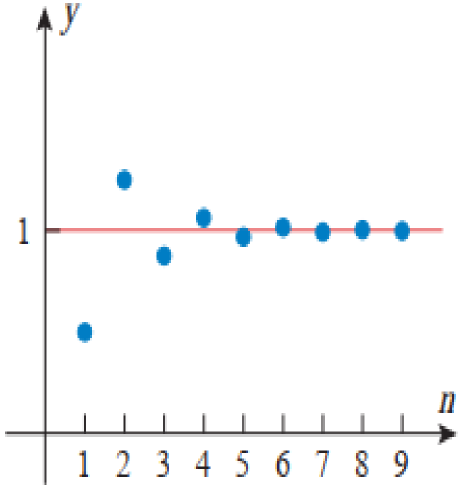
$\left\{(-1)^{n+1}\right\}_{n=1}^{+\infty}$

$\{n/(n+1)\}$



$\left\{\frac{n}{n+1}\right\}_{n=1}^{+\infty}$

$\left\{1+\left(-\frac{1}{2}\right)^n\right\}$



$\left\{1+\left(-\frac{1}{2}\right)^n\right\}_{n=1}^{+\infty}$

1 **Definition** A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \to \infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n \to \infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

9.1.3 THEOREM *Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ converge to limits L_1 and L_2, respectively, and c is a constant. Then:*

(*a*) $\displaystyle\lim_{n\to+\infty} c = c$

(*b*) $\displaystyle\lim_{n\to+\infty} ca_n = c\lim_{n\to+\infty} a_n = cL_1$

(*c*) $\displaystyle\lim_{n\to+\infty} (a_n + b_n) = \lim_{n\to+\infty} a_n + \lim_{n\to+\infty} b_n = L_1 + L_2$

(*d*) $\displaystyle\lim_{n\to+\infty} (a_n - b_n) = \lim_{n\to+\infty} a_n - \lim_{n\to+\infty} b_n = L_1 - L_2$

(*e*) $\displaystyle\lim_{n\to+\infty} (a_n b_n) = \lim_{n\to+\infty} a_n \cdot \lim_{n\to+\infty} b_n = L_1 L_2$

(*f*) $\displaystyle\lim_{n\to+\infty} \left(\frac{a_n}{b_n}\right) = \frac{\displaystyle\lim_{n\to+\infty} a_n}{\displaystyle\lim_{n\to+\infty} b_n} = \frac{L_1}{L_2} \quad (\text{if } L_2 \neq 0)$

▸ **Example 3** In each part, determine whether the sequence converges or diverges by examining the limit as $n \to +\infty$.

(a) $\left\{\dfrac{n}{2n+1}\right\}_{n=1}^{+\infty}$ (b) $\left\{(-1)^{n+1}\dfrac{n}{2n+1}\right\}_{n=1}^{+\infty}$

(c) $\left\{(-1)^{n+1}\dfrac{1}{n}\right\}_{n=1}^{+\infty}$ (d) $\{8-2n\}_{n=1}^{+\infty}$

Solution (a). Dividing numerator and denominator by n and using Theorem 9.1.3 yields

$$\lim_{n\to+\infty}\frac{n}{2n+1}=\lim_{n\to+\infty}\frac{1}{2+1/n}=\frac{\lim\limits_{n\to+\infty}1}{\lim\limits_{n\to+\infty}(2+1/n)}=\frac{\lim\limits_{n\to+\infty}1}{\lim\limits_{n\to+\infty}2+\lim\limits_{n\to+\infty}1/n}$$
$$=\frac{1}{2+0}=\frac{1}{2}$$

Thus, the sequence converges to $\frac{1}{2}$.

(b) $\left\{(-1)^{n+1}\dfrac{n}{2n+1}\right\}_{n=1}^{+\infty}$

Solution (b). This sequence is the same as that in part (a), except for the factor of $(-1)^{n+1}$, which oscillates between $+1$ and -1. Thus, the terms in this sequence oscillate between positive and negative values, with the odd-numbered terms being identical to those in part (a) and the even-numbered terms being the negatives of those in part (a). Since the sequence in part (a) has a limit of $\frac{1}{2}$, it follows that the odd-numbered terms in this sequence approach $\frac{1}{2}$, and the even-numbered terms approach $-\frac{1}{2}$. Therefore, this sequence has no limit—it diverges.

(c) $\left\{(-1)^{n+1}\dfrac{1}{n}\right\}_{n=1}^{+\infty}$

Solution (c). Since $1/n \to 0$, the product $(-1)^{n+1}(1/n)$ oscillates between positive and negative values, with the odd-numbered terms approaching 0 through positive values and the even-numbered terms approaching 0 through negative values. Thus,

$$\lim_{n\to+\infty} (-1)^{n+1}\frac{1}{n} = 0$$

so the sequence converges to 0.

(d) $\{8-2n\}_{n=1}^{+\infty}$

Solution (d). $\lim_{n\to+\infty} (8-2n) = -\infty$, so the sequence $\{8-2n\}_{n=1}^{+\infty}$ diverges. ◀

Example 4 In each part, determine whether the sequence converges, and if so, find its limit.

(a) $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots$ (b) $1, 2, 2^2, 2^3, \ldots, 2^n, \ldots$

Solution. Replacing n by x in the first sequence produces the power function $(1/2)^x$, and replacing n by x in the second sequence produces the power function 2^x. Now recall that if $0 < b < 1$, then $b^x \to 0$ as $x \to +\infty$, and if $b > 1$, then $b^x \to +\infty$ as $x \to +\infty$ (Figure 1.8.1). Thus,

$$\lim_{n \to +\infty} \frac{1}{2^n} = 0 \quad \text{and} \quad \lim_{n \to +\infty} 2^n = +\infty$$

So, the sequence $\{1/2^n\}$ converges to 0, but the sequence $\{2^n\}$ diverges. ◀

Example 5 Find the limit of the sequence $\left\{\dfrac{n}{e^n}\right\}_{n=1}^{+\infty}$.

Solution. The expression

$$\lim_{n \to +\infty} \frac{n}{e^n}$$

is an indeterminate form of type ∞/∞, so L'Hôpital's rule is indicated. However, we cannot apply this rule directly to n/e^n because the functions n and e^n have been defined here only at the positive integers, and hence are not differentiable functions. To circumvent this problem we extend the domains of these functions to all real numbers, here implied by replacing n by x, and apply L'Hôpital's rule to the limit of the quotient x/e^x. This yields

$$\lim_{x \to +\infty} \frac{x}{e^x} = \lim_{x \to +\infty} \frac{1}{e^x} = 0$$

from which we can conclude that

$$\lim_{n \to +\infty} \frac{n}{e^n} = 0 \blacktriangleleft$$

▶ **Example 6** Show that $\lim_{n \to +\infty} \sqrt[n]{n} = 1$.

Solution.

$$\lim_{n \to +\infty} \sqrt[n]{n} = \lim_{n \to +\infty} n^{1/n} = \lim_{n \to +\infty} e^{(1/n)\ln n} = e^0 = 1$$

By L'Hôpital's rule applied to $(1/x)\ln x$ ◀

Sometimes the even-numbered and odd-numbered terms of a sequence behave sufficiently differently that it is desirable to investigate their convergence separately. The following theorem, whose proof is omitted, is helpful for that purpose.

9.1.4 THEOREM *A sequence converges to a limit L if and only if the sequences of even-numbered terms and odd-numbered terms both converge to L.*

Example 7 The sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{2^3}, \frac{1}{3^3}, \ldots$$

converges to 0, since the even-numbered terms and the odd-numbered terms both converge to 0, and the sequence

$$1, \tfrac{1}{2}, 1, \tfrac{1}{3}, 1, \tfrac{1}{4}, \ldots$$

diverges, since the odd-numbered terms converge to 1 and the even-numbered terms converge to 0. ◀

9.1.5 THEOREM (*The Squeezing Theorem for Sequences*) *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences such that*

$$a_n \le b_n \le c_n \quad (\text{for all values of } n \text{ beyond some index } N)$$

If the sequences $\{a_n\}$ and $\{c_n\}$ have a common limit L as $n \to +\infty$, then $\{b_n\}$ also has the limit L as $n \to +\infty$.

6 Theorem If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

1. In each part, find a formula for the general term of the sequence, starting with $n = 1$.

(a) $1, \dfrac{1}{3}, \dfrac{1}{9}, \dfrac{1}{27}, \ldots$ (b) $1, -\dfrac{1}{3}, \dfrac{1}{9}, -\dfrac{1}{27}, \ldots$

(c) $\dfrac{1}{2}, \dfrac{3}{4}, \dfrac{5}{6}, \dfrac{7}{8}, \ldots$ (d) $\dfrac{1}{\sqrt{\pi}}, \dfrac{4}{\sqrt[3]{\pi}}, \dfrac{9}{\sqrt[4]{\pi}}, \dfrac{16}{\sqrt[5]{\pi}}, \ldots$

Exercise Set 9.1

1. (a) $\dfrac{1}{3^{n-1}}$ (b) $\dfrac{(-1)^{n-1}}{3^{n-1}}$ (c) $\dfrac{2n-1}{2n}$ (d) $\dfrac{n^2}{\pi^{1/(n+1)}}$

7–22 Write out the first five terms of the sequence, determine whether the sequence converges, and if so find its limit.

7. $\left\{\dfrac{n}{n+2}\right\}_{n=1}^{+\infty}$

8. $\left\{\dfrac{n^2}{2n+1}\right\}_{n=1}^{+\infty}$

9. $\{2\}_{n=1}^{+\infty}$

10. $\left\{\ln\left(\dfrac{1}{n}\right)\right\}_{n=1}^{+\infty}$

11. $\left\{\dfrac{\ln n}{n}\right\}_{n=1}^{+\infty}$

12. $\left\{n\sin\dfrac{\pi}{n}\right\}_{n=1}^{+\infty}$

13. $\{1+(-1)^n\}_{n=1}^{+\infty}$

14. $\left\{\dfrac{(-1)^{n+1}}{n^2}\right\}_{n=1}^{+\infty}$

15. $\left\{(-1)^n\dfrac{2n^3}{n^3+1}\right\}_{n=1}^{+\infty}$

16. $\left\{\dfrac{n}{2^n}\right\}_{n=1}^{+\infty}$

17. $\left\{\dfrac{(n+1)(n+2)}{2n^2}\right\}_{n=1}^{+\infty}$

18. $\left\{\dfrac{\pi^n}{4^n}\right\}_{n=1}^{+\infty}$

19. $\{n^2e^{-n}\}_{n=1}^{+\infty}$

20. $\{\sqrt{n^2+3n}-n\}_{n=1}^{+\infty}$

21. $\left\{\left(\dfrac{n+3}{n+1}\right)^n\right\}_{n=1}^{+\infty}$

22. $\left\{\left(1-\dfrac{2}{n}\right)^n\right\}_{n=1}^{+\infty}$

7. $1/3,\ 2/4,\ 3/5,\ 4/6,\ 5/7, \ldots$; $\lim\limits_{n\to+\infty} \dfrac{n}{n+2} = 1$, converges.

8. $1/3,\ 4/5,\ 9/7,\ 16/9,\ 25/11, \ldots$; $\lim\limits_{n\to+\infty} \dfrac{n^2}{2n+1} = +\infty$, diverges.

9. $2, 2, 2, 2, 2, \ldots$; $\lim\limits_{n\to+\infty} 2 = 2$, converges.

10. $\ln 1,\ \ln\frac{1}{2},\ \ln\frac{1}{3},\ \ln\frac{1}{4},\ \ln\frac{1}{5}, \ldots$; $\lim\limits_{n\to+\infty} \ln(1/n) = -\infty$, diverges.

11. $\dfrac{\ln 1}{1}, \dfrac{\ln 2}{2}, \dfrac{\ln 3}{3}, \dfrac{\ln 4}{4}, \dfrac{\ln 5}{5}, \ldots$; $\lim\limits_{n\to+\infty} \dfrac{\ln n}{n} = \lim\limits_{n\to+\infty} \dfrac{1}{n} = 0$ $\left(\text{apply L'Hôpital's Rule to } \dfrac{\ln x}{x}\right)$, converges.

12. $\sin\pi,\ 2\sin(\pi/2),\ 3\sin(\pi/3),\ 4\sin(\pi/4),\ 5\sin(\pi/5), \ldots$; $\lim\limits_{n\to+\infty} n\sin(\pi/n) = \lim\limits_{n\to+\infty} \dfrac{\sin(\pi/n)}{1/n}$; but using L'Hospital's rule, $\lim\limits_{x\to+\infty} \dfrac{\sin(\pi/x)}{1/x} = \lim\limits_{x\to+\infty} \dfrac{(-\pi/x^2)\cos(\pi/x)}{-1/x^2} = \pi$, so the sequence also converges to π.

13. $0, 2, 0, 2, 0, \ldots$; diverges.

14. $1,\ -1/4,\ 1/9,\ -1/16,\ 1/25, \ldots$; $\lim\limits_{n\to+\infty} \dfrac{(-1)^{n+1}}{n^2} = 0$, converges.

15. $-1,\ 16/9,\ -54/28,\ 128/65,\ -250/126, \ldots$; diverges because odd-numbered terms approach -2, even-numbered terms approach 2.

16. $1/2, 2/4, 3/8, 4/16, 5/32, \ldots$; using L'Hospital's rule, $\lim\limits_{x\to+\infty} \dfrac{x}{2^x} = \lim\limits_{x\to+\infty} \dfrac{1}{2^x \ln 2} = 0$, so the sequence also converges to 0.

17. $6/2, 12/8, 20/18, 30/32, 42/50, \ldots$; $\lim\limits_{n\to+\infty} \dfrac{1}{2}(1+1/n)(1+2/n) = 1/2$, converges.

18. $\pi/4, \pi^2/4^2, \pi^3/4^3, \pi^4/4^4, \pi^5/4^5, \ldots$; $\lim\limits_{n\to+\infty} (\pi/4)^n = 0$, converges.

19. $e^{-1}, 4e^{-2}, 9e^{-3}, 16e^{-4}, 25e^{-5}, \ldots$; using L'Hospital's rule, $\lim\limits_{x\to+\infty} x^2 e^{-x} = \lim\limits_{x\to+\infty} \dfrac{x^2}{e^x} = \lim\limits_{x\to+\infty} \dfrac{2x}{e^x} = \lim\limits_{x\to+\infty} \dfrac{2}{e^x} = 0$, so $\lim\limits_{n\to+\infty} n^2 e^{-n} = 0$, converges.

20. $1, \sqrt{10}-2, \sqrt{18}-3, \sqrt{28}-4, \sqrt{40}-5, \ldots$; $\lim\limits_{n\to+\infty} (\sqrt{n^2+3n}-n) = \lim\limits_{n\to+\infty} \dfrac{3n}{\sqrt{n^2+3n}+n} = \lim\limits_{n\to+\infty} \dfrac{3}{\sqrt{1+3/n}+1} = \dfrac{3}{2}$, converges.

21. $2, (5/3)^2, (6/4)^3, (7/5)^4, (8/6)^5, \ldots$; let $y = \left[\dfrac{x+3}{x+1}\right]^x$, converges because $\lim\limits_{x\to+\infty} \ln y = \lim\limits_{x\to+\infty} \dfrac{\ln \dfrac{x+3}{x+1}}{1/x} = \lim\limits_{x\to+\infty} \dfrac{2x^2}{(x+1)(x+3)} = 2$, so $\lim\limits_{n\to+\infty} \left[\dfrac{n+3}{n+1}\right]^n = e^2$.

22. $-1, 0, (1/3)^3, (2/4)^4, (3/5)^5, \ldots$; let $y = (1-2/x)^x$, converges because $\lim\limits_{x\to+\infty} \ln y = \lim\limits_{x\to+\infty} \dfrac{\ln(1-2/x)}{1/x} = \lim\limits_{x\to+\infty} \dfrac{-2}{1-2/x} = -2$, $\lim\limits_{n\to+\infty} (1-2/n)^n = \lim\limits_{x\to+\infty} y = e^{-2}$.

23–30 Find the general term of the sequence, starting with $n = 1$, determine whether the sequence converges, and if so find its limit.

23. $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$

24. $0, \frac{1}{2^2}, \frac{2}{3^2}, \frac{3}{4^2}, \ldots$

25. $\frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, -\frac{1}{81}, \ldots$

26. $-1, 2, -3, 4, -5, \ldots$

27. $\left(1 - \frac{1}{2}\right), \left(\frac{1}{3} - \frac{1}{2}\right), \left(\frac{1}{3} - \frac{1}{4}\right), \left(\frac{1}{5} - \frac{1}{4}\right), \ldots$

28. $3, \frac{3}{2}, \frac{3}{2^2}, \frac{3}{2^3}, \ldots$

29. $(\sqrt{2} - \sqrt{3}), (\sqrt{3} - \sqrt{4}), (\sqrt{4} - \sqrt{5}), \ldots$

30. $\frac{1}{3^5}, -\frac{1}{3^6}, \frac{1}{3^7}, -\frac{1}{3^8}, \ldots$

23. $\left\{\dfrac{2n-1}{2n}\right\}_{n=1}^{+\infty}$; $\lim\limits_{n\to+\infty}\dfrac{2n-1}{2n}=1$, converges.

24. $\left\{\dfrac{n-1}{n^2}\right\}_{n=1}^{+\infty}$; $\lim\limits_{n\to+\infty}\dfrac{n-1}{n^2}=0$, converges.

25. $\left\{(-1)^{n-1}\dfrac{1}{3^n}\right\}_{n=1}^{+\infty}$; $\lim\limits_{n\to+\infty}\dfrac{(-1)^{n-1}}{3^n}=0$, converges.

26. $\{(-1)^n n\}_{n=1}^{+\infty}$; diverges because odd-numbered terms tend toward $-\infty$, even-numbered terms tend toward $+\infty$.

27. $\left\{(-1)^{n+1}\left(\dfrac{1}{n}-\dfrac{1}{n+1}\right)\right\}_{n=1}^{+\infty}$; the sequence converges to 0.

28. $\{3/2^{n-1}\}_{n=1}^{+\infty}$; $\lim\limits_{n\to+\infty}3/2^{n-1}=0$, converges.

29. $\{\sqrt{n+1}-\sqrt{n+2}\}_{n=1}^{+\infty}$; converges because $\lim\limits_{n\to+\infty}(\sqrt{n+1}-\sqrt{n+2})=\lim\limits_{n\to+\infty}\dfrac{(n+1)-(n+2)}{\sqrt{n+1}+\sqrt{n+2}}=$

$=\lim\limits_{n\to+\infty}\dfrac{-1}{\sqrt{n+1}+\sqrt{n+2}}=0.$

30. $\{(-1)^{n+1}/3^{n+4}\}_{n=1}^{+\infty}$; $\lim\limits_{n\to+\infty}(-1)^{n+1}/3^{n+4}=0$, converges.

Do Questions (1, 7-30) from Ex # 9.1