Topics in Differentiation

Exercise Set 3.1

1. (a) $1 + y + x\dfrac{dy}{dx} - 6x^2 = 0$, $\dfrac{dy}{dx} = \dfrac{6x^2 - y - 1}{x}$.

(b) $y = \dfrac{2 + 2x^3 - x}{x} = \dfrac{2}{x} + 2x^2 - 1$, $\dfrac{dy}{dx} = -\dfrac{2}{x^2} + 4x$.

(c) From part (a), $\dfrac{dy}{dx} = 6x - \dfrac{1}{x} - \dfrac{1}{x}y = 6x - \dfrac{1}{x} - \dfrac{1}{x}\left(\dfrac{2}{x} + 2x^2 - 1\right) = 4x - \dfrac{2}{x^2}$.

3. $2x + 2y\dfrac{dy}{dx} = 0$ so $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

5. $x^2\dfrac{dy}{dx} + 2xy + 3x(3y^2)\dfrac{dy}{dx} + 3y^3 - 1 = 0$, $(x^2 + 9xy^2)\dfrac{dy}{dx} = 1 - 2xy - 3y^3$, so $\dfrac{dy}{dx} = \dfrac{1 - 2xy - 3y^3}{x^2 + 9xy^2}$.

7. $-\dfrac{1}{2x^{3/2}} - \dfrac{\frac{dy}{dx}}{2y^{3/2}} = 0$, so $\dfrac{dy}{dx} = -\dfrac{y^{3/2}}{x^{3/2}}$.

9. $\cos(x^2y^2)\left[x^2(2y)\dfrac{dy}{dx} + 2xy^2\right] = 1$, so $\dfrac{dy}{dx} = \dfrac{1 - 2xy^2\cos(x^2y^2)}{2x^2y\cos(x^2y^2)}$.

11. $3\tan^2(xy^2 + y)\sec^2(xy^2 + y)\left(2xy\dfrac{dy}{dx} + y^2 + \dfrac{dy}{dx}\right) = 1$, so $\dfrac{dy}{dx} = \dfrac{1 - 3y^2\tan^2(xy^2 + y)\sec^2(xy^2 + y)}{3(2xy + 1)\tan^2(xy^2 + y)\sec^2(xy^2 + y)}$.

13. $4x - 6y\dfrac{dy}{dx} = 0$, $\dfrac{dy}{dx} = \dfrac{2x}{3y}$, $4 - 6\left(\dfrac{dy}{dx}\right)^2 - 6y\dfrac{d^2y}{dx^2} = 0$, so $\dfrac{d^2y}{dx^2} = -\dfrac{3\left(\frac{dy}{dx}\right)^2 - 2}{3y} = \dfrac{2(3y^2 - 2x^2)}{9y^3} = -\dfrac{8}{9y^3}$.

15. $\dfrac{dy}{dx} = -\dfrac{y}{x}$, $\dfrac{d^2y}{dx^2} = -\dfrac{x(dy/dx) - y(1)}{x^2} = -\dfrac{x(-y/x) - y}{x^2} = \dfrac{2y}{x^2}$.

17. $\dfrac{dy}{dx} = (1 + \cos y)^{-1}$, $\dfrac{d^2y}{dx^2} = -(1 + \cos y)^{-2}(-\sin y)\dfrac{dy}{dx} = \dfrac{\sin y}{(1 + \cos y)^3}$.

19. By implicit differentiation, $2x + 2y(dy/dx) = 0$, $\dfrac{dy}{dx} = -\dfrac{x}{y}$; at $(1/2, \sqrt{3}/2)$, $\dfrac{dy}{dx} = -\sqrt{3}/3$; at $(1/2, -\sqrt{3}/2)$, $\dfrac{dy}{dx} = +\sqrt{3}/3$. Directly, at the upper point $y = \sqrt{1 - x^2}$, $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{1 - x^2}} = -\dfrac{1/2}{\sqrt{3/4}} = -1/\sqrt{3}$ and at the lower point $y = -\sqrt{1 - x^2}$, $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{1 - x^2}} = +1/\sqrt{3}$.

21. False; $x = y^2$ defines two functions $y = \pm\sqrt{x}$. See Definition 3.1.1.

23. False; the equation is equivalent to $x^2 = y^2$ which is satisfied by $y = |x|$.

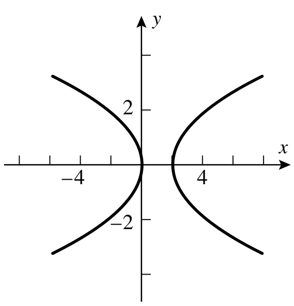
25. $4x^3 + 4y^3 \dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{x^3}{y^3} = -\dfrac{1}{15^{3/4}} \approx -0.1312$.

27. $4(x^2+y^2)\left(2x + 2y\dfrac{dy}{dx}\right) = 25\left(2x - 2y\dfrac{dy}{dx}\right)$, $\dfrac{dy}{dx} = \dfrac{x[25 - 4(x^2+y^2)]}{y[25 + 4(x^2+y^2)]}$; at $(3,1)$ $\dfrac{dy}{dx} = -9/13$.

29. $4a^3\dfrac{da}{dt} - 4t^3 = 6\left(a^2 + 2at\dfrac{da}{dt}\right)$, solve for $\dfrac{da}{dt}$ to get $\dfrac{da}{dt} = \dfrac{2t^3 + 3a^2}{2a^3 - 6at}$.

31. $2a^2\omega\dfrac{d\omega}{d\lambda} + 2b^2\lambda = 0$, so $\dfrac{d\omega}{d\lambda} = -\dfrac{b^2\lambda}{a^2\omega}$.

33. $2x + x\dfrac{dy}{dx} + y + 2y\dfrac{dy}{dx} = 0$. Substitute $y = -2x$ to obtain $-3x\dfrac{dy}{dx} = 0$. Since $x = \pm 1$ at the indicated points, $\dfrac{dy}{dx} = 0$ there.

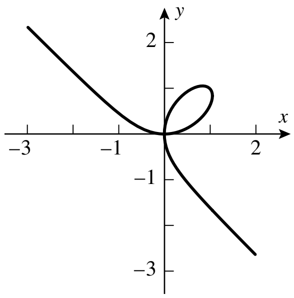
35. (a) 

(b) Implicit differentiation of the curve yields $(4y^3 + 2y)\dfrac{dy}{dx} = 2x - 1$, so $\dfrac{dy}{dx} = 0$ only if $x = 1/2$ but $y^4 + y^2 \geq 0$ so $x = 1/2$ is impossible.

(c) $x^2 - x - (y^4 + y^2) = 0$, so by the Quadratic Formula, $x = \dfrac{-1 \pm \sqrt{(2y^2+1)^2}}{2} = 1 + y^2$ or $-y^2$, and we have the two parabolas $x = -y^2, x = 1 + y^2$.

37. The point (1,1) is on the graph, so $1 + a = b$. The slope of the tangent line at (1,1) is $-4/3$; use implicit differentiation to get $\dfrac{dy}{dx} = -\dfrac{2xy}{x^2 + 2ay}$ so at (1,1), $-\dfrac{2}{1+2a} = -\dfrac{4}{3}$, $1 + 2a = 3/2$, $a = 1/4$ and hence $b = 1 + 1/4 = 5/4$.

39. We shall find when the curves intersect and check that the slopes are negative reciprocals. For the intersection solve the simultaneous equations $x^2 + (y-c)^2 = c^2$ and $(x-k)^2 + y^2 = k^2$ to obtain $cy = kx = \dfrac{1}{2}(x^2 + y^2)$. Thus $x^2 + y^2 = cy + kx$, or $y^2 - cy = -x^2 + kx$, and $\dfrac{y-c}{x} = -\dfrac{x-k}{y}$. Differentiating the two families yields (black) $\dfrac{dy}{dx} = -\dfrac{x}{y-c}$, and (gray) $\dfrac{dy}{dx} = -\dfrac{x-k}{y}$. But it was proven that these quantities are negative reciprocals of each other.



41. (a)

(b) $x \approx 0.84$

(c) Use implicit differentiation to get $dy/dx = (2y - 3x^2)/(3y^2 - 2x)$, so $dy/dx = 0$ if $y = (3/2)x^2$. Substitute this into $x^3 - 2xy + y^3 = 0$ to obtain $27x^6 - 16x^3 = 0, x^3 = 16/27$, $x = 2^{4/3}/3$ and hence $y = 2^{5/3}/3$.

43. By the chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx}$. Using implicit differentiation for $2y^3t + t^3y = 1$ we get $\dfrac{dy}{dt} = -\dfrac{2y^3 + 3t^2y}{6ty^2 + t^3}$, but $\dfrac{dt}{dx} = \dfrac{1}{\cos t}$, so $\dfrac{dy}{dx} = -\dfrac{2y^3 + 3t^2y}{(6ty^2 + t^3)\cos t}$.

Exercise Set 3.2

1. $\dfrac{1}{5x}(5) = \dfrac{1}{x}$.

3. $\dfrac{1}{1+x}$.

5. $\dfrac{1}{x^2-1}(2x) = \dfrac{2x}{x^2-1}$.

7. $\dfrac{d}{dx}\ln x - \dfrac{d}{dx}\ln(1+x^2) = \dfrac{1}{x} - \dfrac{2x}{1+x^2} = \dfrac{1-x^2}{x(1+x^2)}$.

9. $\dfrac{d}{dx}(2\ln x) = 2\dfrac{d}{dx}\ln x = \dfrac{2}{x}$.

11. $\dfrac{1}{2}(\ln x)^{-1/2}\left(\dfrac{1}{x}\right) = \dfrac{1}{2x\sqrt{\ln x}}$.

13. $\ln x + x\dfrac{1}{x} = 1 + \ln x$.

15. $2x\log_2(3-2x) + \dfrac{-2x^2}{(\ln 2)(3-2x)}$.

17. $\dfrac{2x(1+\log x) - x/(\ln 10)}{(1+\log x)^2}$.

19. $\dfrac{1}{\ln x}\left(\dfrac{1}{x}\right) = \dfrac{1}{x\ln x}$.

21. $\dfrac{1}{\tan x}(\sec^2 x) = \sec x \csc x$.

23. $-\sin(\ln x)\dfrac{1}{x}$.

25. $\dfrac{1}{\ln 10 \sin^2 x}(2\sin x\cos x) = 2\dfrac{\cot x}{\ln 10}$.

27. $\dfrac{d}{dx}\left[3\ln(x-1)+4\ln(x^2+1)\right] = \dfrac{3}{x-1}+\dfrac{8x}{x^2+1} = \dfrac{11x^2-8x+3}{(x-1)(x^2+1)}$.

29. $\dfrac{d}{dx}\left[\ln\cos x - \dfrac{1}{2}\ln(4-3x^2)\right] = -\tan x + \dfrac{3x}{4-3x^2}$

31. True, because $\dfrac{dy}{dx} = \dfrac{1}{x}$, so as $x = a \to 0^+$, the slope approaches infinity.

33. True; if $x > 0$ then $\dfrac{d}{dx}\ln|x| = 1/x$; if $x < 0$ then $\dfrac{d}{dx}\ln|x| = 1/x$.

35. $\ln|y| = \ln|x| + \dfrac{1}{3}\ln|1+x^2|$, so $\dfrac{dy}{dx} = x\sqrt[3]{1+x^2}\left[\dfrac{1}{x}+\dfrac{2x}{3(1+x^2)}\right]$.

37. $\ln|y| = \dfrac{1}{3}\ln|x^2-8| + \dfrac{1}{2}\ln|x^3+1| - \ln|x^6-7x+5|$, so

$$\frac{dy}{dx} = \frac{(x^2-8)^{1/3}\sqrt{x^3+1}}{x^6-7x+5}\left[\frac{2x}{3(x^2-8)}+\frac{3x^2}{2(x^3+1)}-\frac{6x^5-7}{x^6-7x+5}\right].$$

39. **(a)** $\log_x e = \dfrac{\ln e}{\ln x} = \dfrac{1}{\ln x}$, so $\dfrac{d}{dx}[\log_x e] = -\dfrac{1}{x(\ln x)^2}$.

(b) $\log_x 2 = \dfrac{\ln 2}{\ln x}$, so $\dfrac{d}{dx}[\log_x 2] = -\dfrac{\ln 2}{x(\ln x)^2}$.

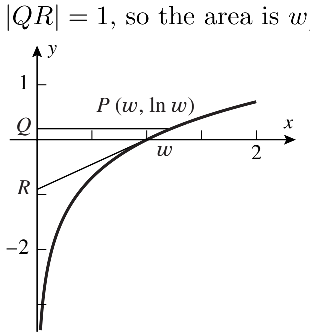
41. $f'(x_0) = \dfrac{1}{x_0} = e,\ y-(-1) = e(x-x_0) = ex-1,\ y = ex-2$.

43. $f(x_0) = f(-e) = 1,\ f'(x)|_{x=-e} = -\dfrac{1}{e},\ y-1 = -\dfrac{1}{e}(x+e),\ y = -\dfrac{1}{e}x$.

45. **(a)** Let the equation of the tangent line be $y = mx$ and suppose that it meets the curve at (x_0, y_0). Then $m = \left.\dfrac{1}{x}\right|_{x=x_0} = \dfrac{1}{x_0}$ and $y_0 = mx_0 + b = \ln x_0$. So $m = \dfrac{1}{x_0} = \dfrac{\ln x_0}{x_0}$ and $\ln x_0 = 1, x_0 = e, m = \dfrac{1}{e}$ and the equation of the tangent line is $y = \dfrac{1}{e}x$.

(b) Let $y = mx + b$ be a line tangent to the curve at (x_0, y_0). Then b is the y-intercept and the slope of the tangent line is $m = \dfrac{1}{x_0}$. Moreover, at the point of tangency, $mx_0 + b = \ln x_0$ or $\dfrac{1}{x_0}x_0 + b = \ln x_0$, $b = \ln x_0 - 1$, as required.

47. The area of the triangle PQR is given by the formula $|PQ||QR|/2$. $|PQ| = w$, and, by Exercise 45 part (b), $|QR| = 1$, so the area is $w/2$.



49. If $x = 0$ then $y = \ln e = 1$, and $\dfrac{dy}{dx} = \dfrac{1}{x+e}$. But $e^y = x + e$, so $\dfrac{dy}{dx} = \dfrac{1}{e^y} = e^{-y}$.

51. Let $y = \ln(x + a)$. Following Exercise 49 we get $\dfrac{dy}{dx} = \dfrac{1}{x+a} = e^{-y}$, and when $x = 0, y = \ln(a) = 0$ if $a = 1$, so let $a = 1$, then $y = \ln(x + 1)$.

53. **(a)** Set $f(x) = \ln(1 + 3x)$. Then $f'(x) = \dfrac{3}{1+3x}$, $f'(0) = 3$. But $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x} = \lim\limits_{x\to 0} \dfrac{\ln(1+3x)}{x}$.

(b) Set $f(x) = \ln(1 - 5x)$. Then $f'(x) = \dfrac{-5}{1-5x}$, $f'(0) = -5$. But $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x} = \lim\limits_{x\to 0} \dfrac{\ln(1-5x)}{x}$.

55. **(a)** Let $f(x) = \ln(\cos x)$, then $f(0) = \ln(\cos 0) = \ln 1 = 0$, so $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x} = \lim\limits_{x\to 0} \dfrac{\ln(\cos x)}{x}$, and $f'(0) = -\tan 0 = 0$.

(b) Let $f(x) = x^{\sqrt{2}}$, then $f(1) = 1$, so $f'(1) = \lim\limits_{h\to 0} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h\to 0} \dfrac{(1+h)^{\sqrt{2}} - 1}{h}$, and $f'(x) = \sqrt{2}x^{\sqrt{2}-1}, f'(1) = \sqrt{2}$.

57. Differentiating implicitly gives $0 = \dfrac{1}{p}\dfrac{dp}{dt} - \dfrac{1}{2.3 - 0.0046p}(-0.0046)\dfrac{dp}{dt} - 2.3$, from which $\dfrac{dp}{dt} = 0.0046p(500 - p)$ as claimed.

Exercise Set 3.3

1. **(a)** $f'(x) = 5x^4 + 3x^2 + 1 \geq 1$ so f is increasing and one-to-one on $-\infty < x < +\infty$.

(b) $f(1) = 3$ so $1 = f^{-1}(3)$; $\dfrac{d}{dx}f^{-1}(x) = \dfrac{1}{f'(f^{-1}(x))}$, $(f^{-1})'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{9}$.

3. $f^{-1}(x) = \dfrac{2}{x} - 3$, so directly $\dfrac{d}{dx}f^{-1}(x) = -\dfrac{2}{x^2}$. Using Formula (2), $f'(x) = \dfrac{-2}{(x+3)^2}$, so $\dfrac{1}{f'(f^{-1}(x))} = -(1/2)(f^{-1}(x) + 3)^2$, and $\dfrac{d}{dx}f^{-1}(x) = -(1/2)\left(\dfrac{2}{x}\right)^2 = -\dfrac{2}{x^2}$.

5. **(a)** $f'(x) = 2x + 8$; $f' < 0$ on $(-\infty, -4)$ and $f' > 0$ on $(-4, +\infty)$; not enough information. By inspection, $f(1) = 10 = f(-9)$, so not one-to-one.

(b) $f'(x) = 10x^4 + 3x^2 + 3 \geq 3 > 0$; $f'(x)$ is positive for all x, so f is one-to-one.

(c) $f'(x) = 2 + \cos x \geq 1 > 0$ for all x, so f is one-to-one.

(d) $f'(x) = -(\ln 2)\left(\frac{1}{2}\right)^x < 0$ because $\ln 2 > 0$, so f is one-to-one for all x.

7. $y = f^{-1}(x)$, $x = f(y) = 5y^3 + y - 7$, $\dfrac{dx}{dy} = 15y^2 + 1$, $\dfrac{dy}{dx} = \dfrac{1}{15y^2 + 1}$; check: $1 = 15y^2\dfrac{dy}{dx} + \dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \dfrac{1}{15y^2 + 1}$.

9. $y = f^{-1}(x)$, $x = f(y) = 2y^5 + y^3 + 1$, $\dfrac{dx}{dy} = 10y^4 + 3y^2$, $\dfrac{dy}{dx} = \dfrac{1}{10y^4 + 3y^2}$; check: $1 = 10y^4\dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \dfrac{1}{10y^4 + 3y^2}$.

11. Let $P(a, b)$ be given, not on the line $y = x$. Let Q_1 be its reflection across the line $y = x$, yet to be determined. Let Q have coordinates (b, a).

(a) Since P does not lie on $y = x$, we have $a \neq b$, i.e. $P \neq Q$ since they have different abscissas. The line $\vec{PQ}$ has slope $(b-a)/(a-b) = -1$ which is the negative reciprocal of $m = 1$ and so the two lines are perpendicular.

(b) Let (c, d) be the midpoint of the segment PQ. Then $c = (a+b)/2$ and $d = (b+a)/2$ so $c = d$ and the midpoint is on $y = x$.

(c) Let $Q(c, d)$ be the reflection of P through $y = x$. By definition this means P and Q lie on a line perpendicular to the line $y = x$ and the midpoint of P and Q lies on $y = x$.

(d) Since the line through P and Q is perpendicular to the line $y = x$ it is parallel to the line through P and Q_1; since both pass through P they are the same line. Finally, since the midpoints of P and Q_1 and of P and Q both lie on $y = x$, they are the same point, and consequently $Q = Q_1$.

13. If $x < y$ then $f(x) \le f(y)$ and $g(x) \le g(y)$; thus $f(x) + g(x) \le f(y) + g(y)$. Moreover, $g(x) \le g(y)$, so $f(g(x)) \le f(g(y))$. Note that $f(x)g(x)$ need not be increasing, e.g. $f(x) = g(x) = x$, both increasing for all x, yet $f(x)g(x) = x^2$, not an increasing function.

15. $\dfrac{dy}{dx} = 7e^{7x}$.

17. $\dfrac{dy}{dx} = x^3e^x + 3x^2e^x = x^2e^x(x+3)$.

19. $\dfrac{dy}{dx} = \dfrac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} = \dfrac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2} = 4/(e^x + e^{-x})^2$.

21. $\dfrac{dy}{dx} = (x\sec^2 x + \tan x)e^{x\tan x}$.

23. $\dfrac{dy}{dx} = (1 - 3e^{3x})e^{(x - e^{3x})}$.

25. $\dfrac{dy}{dx} = \dfrac{(x-1)e^{-x}}{1 - xe^{-x}} = \dfrac{x-1}{e^x - x}$.

27. $f'(x) = 2^x \ln 2$; $y = 2^x$, $\ln y = x\ln 2$, $\dfrac{1}{y}y' = \ln 2$, $y' = y\ln 2 = 2^x \ln 2$.

29. $f'(x) = \pi^{\sin x}(\ln \pi)\cos x$; $y = \pi^{\sin x}$, $\ln y = (\sin x)\ln \pi$, $\dfrac{1}{y}y' = (\ln \pi)\cos x$, $y' = \pi^{\sin x}(\ln \pi)\cos x$.

31. $\ln y = (\ln x)\ln(x^3 - 2x)$, $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{3x^2 - 2}{x^3 - 2x}\ln x + \dfrac{1}{x}\ln(x^3 - 2x)$, $\dfrac{dy}{dx} = (x^3 - 2x)^{\ln x}\left[\dfrac{3x^2 - 2}{x^3 - 2x}\ln x + \dfrac{1}{x}\ln(x^3 - 2x)\right]$.

33. $\ln y = (\tan x)\ln(\ln x)$, $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x\ln x}\tan x + (\sec^2 x)\ln(\ln x)$, $\dfrac{dy}{dx} = (\ln x)^{\tan x}\left[\dfrac{\tan x}{x\ln x} + (\sec^2 x)\ln(\ln x)\right]$.

35. $\ln y = (\ln x)(\ln(\ln x))$, $\dfrac{dy/dx}{y} = (1/x)(\ln(\ln x)) + (\ln x)\dfrac{1/x}{\ln x} = (1/x)(1 + \ln(\ln x))$, $dy/dx = \dfrac{1}{x}(\ln x)^{\ln x}(1 + \ln\ln x)$.

37. $\dfrac{dy}{dx} = (3x^2 - 4x)e^x + (x^3 - 2x^2 + 1)e^x = (x^3 + x^2 - 4x + 1)e^x$.

39. $\dfrac{dy}{dx} = (2x + \dfrac{1}{2\sqrt{x}})3^x + (x^2 + \sqrt{x})3^x \ln 3$.

41. $\dfrac{dy}{dx} = 4^{3\sin x - e^x} \ln 4 (3\cos x - e^x)$.

43. $\dfrac{dy}{dx} = \dfrac{3}{\sqrt{1-(3x)^2}} = \dfrac{3}{\sqrt{1-9x^2}}$.

45. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-1/x^2}}(-1/x^2) = -\dfrac{1}{|x|\sqrt{x^2-1}}$.

47. $\dfrac{dy}{dx} = \dfrac{3x^2}{1+(x^3)^2} = \dfrac{3x^2}{1+x^6}$.

49. $y = 1/\tan x = \cot x$, $dy/dx = -\csc^2 x$.

51. $\dfrac{dy}{dx} = \dfrac{e^x}{|x|\sqrt{x^2-1}} + e^x \sec^{-1} x$.

53. $\dfrac{dy}{dx} = 0$.

55. $\dfrac{dy}{dx} = 0$.

57. $\dfrac{dy}{dx} = -\dfrac{1}{1+x}\left(\dfrac{1}{2}x^{-1/2}\right) = -\dfrac{1}{2(1+x)\sqrt{x}}$.

59. False; $y = Ae^x$ also satisfies $\dfrac{dy}{dx} = y$.

61. True; examine the cases $x > 0$ and $x < 0$ separately.

63. **(a)** Let $x = f(y) = \cot y,\ 0 < y < \pi,\ -\infty < x < +\infty$. Then f is differentiable and one-to-one and $f'(f^{-1}(x)) = -\csc^2(\cot^{-1} x) = -x^2 - 1 \neq 0$, and $\dfrac{d}{dx}[\cot^{-1} x]\Big|_{x=0} = \lim\limits_{x\to 0} \dfrac{1}{f'(f^{-1}(x))} = -\lim\limits_{x\to 0} \dfrac{1}{x^2+1} = -1$.

(b) If $x \neq 0$ then, from Exercise 48(a) of Section 0.4, $\dfrac{d}{dx}\cot^{-1} x = \dfrac{d}{dx}\tan^{-1}\dfrac{1}{x} = -\dfrac{1}{x^2}\dfrac{1}{1+(1/x)^2} = -\dfrac{1}{x^2+1}$. For $x = 0$, part (a) shows the same; thus for $-\infty < x < +\infty$, $\dfrac{d}{dx}[\cot^{-1} x] = -\dfrac{1}{x^2+1}$.

(c) For $-\infty < u < +\infty$, by the chain rule it follows that $\dfrac{d}{dx}[\cot^{-1} u] = -\dfrac{1}{u^2+1}\dfrac{du}{dx}$.

65. $x^3 + x\tan^{-1} y = e^y$, $3x^2 + \dfrac{x}{1+y^2}y' + \tan^{-1} y = e^y y'$, $y' = \dfrac{(3x^2+\tan^{-1} y)(1+y^2)}{(1+y^2)e^y - x}$.

67. **(a)** $f(x) = x^3 - 3x^2 + 2x = x(x-1)(x-2)$ so $f(0) = f(1) = f(2) = 0$ thus f is not one-to-one.

(b) $f'(x) = 3x^2 - 6x + 2$, $f'(x) = 0$ when $x = \dfrac{6 \pm \sqrt{36-24}}{6} = 1 \pm \sqrt{3}/3$. $f'(x) > 0$ (f is increasing) if $x < 1 - \sqrt{3}/3$, $f'(x) < 0$ (f is decreasing) if $1 - \sqrt{3}/3 < x < 1 + \sqrt{3}/3$, so $f(x)$ takes on values less than $f(1-\sqrt{3}/3)$ on both sides of $1 - \sqrt{3}/3$ thus $1 - \sqrt{3}/3$ is the largest value of k.

69. **(a)** $f'(x) = 4x^3 + 3x^2 = (4x+3)x^2 = 0$ only at $x = 0$. But on $[0, 2]$, f' has no sign change, so f is one-to-one.

(b) $F'(x) = 2f'(2g(x))g'(x)$ so $F'(3) = 2f'(2g(3))g'(3)$. By inspection $f(1) = 3$, so $g(3) = f^{-1}(3) = 1$ and $g'(3) = (f^{-1})'(3) = 1/f'(f^{-1}(3)) = 1/f'(1) = 1/7$ because $f'(x) = 4x^3 + 3x^2$. Thus $F'(3) = 2f'(2)(1/7) =$

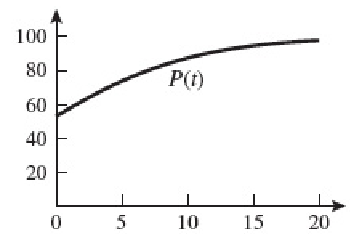
$2(44)(1/7) = 88/7$. $F(3) = f(2g(3)) = f(2\cdot 1) = f(2) = 25$, so the line tangent to $F(x)$ at $(3, 25)$ has the equation $y - 25 = (88/7)(x - 3)$, $y = (88/7)x - 89/7$.

71. $y = Ae^{kt}$, $dy/dt = kAe^{kt} = k(Ae^{kt}) = ky$.

73. **(a)** $y' = -xe^{-x} + e^{-x} = e^{-x}(1 - x)$, $xy' = xe^{-x}(1 - x) = y(1 - x)$.

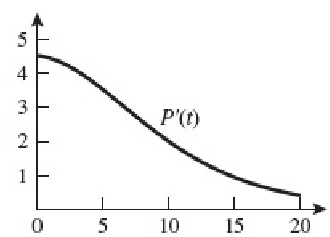
(b) $y' = -x^2e^{-x^2/2} + e^{-x^2/2} = e^{-x^2/2}(1 - x^2)$, $xy' = xe^{-x^2/2}(1 - x^2) = y(1 - x^2)$.

75. **(a)**



(b) The percentage converges to 100%, full coverage of broadband internet access. The limit of the expression in the denominator is clearly 53 as $t \to \infty$.

(c) The rate converges to 0 according to the graph.



77. $f(x) = e^{3x}$, $f'(0) = \lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = 3e^{3x}\big|_{x=0} = 3$.

79. $\lim_{h\to 0} \dfrac{10^h - 1}{h} = \dfrac{d}{dx}10^x\bigg|_{x=0} = \dfrac{d}{dx}e^{x\ln 10}\bigg|_{x=0} = \ln 10$.

81. $\lim_{\Delta x\to 0} \dfrac{9[\sin^{-1}(\frac{\sqrt{3}}{2} + \Delta x)]^2 - \pi^2}{\Delta x} = \dfrac{d}{dx}(3\sin^{-1} x)^2\bigg|_{x=\frac{\sqrt{3}}{2}} = 2(3\sin^{-1} x)\dfrac{3}{\sqrt{1 - x^2}}\bigg|_{x=\frac{\sqrt{3}}{2}} = 2(3\dfrac{\pi}{3})\dfrac{3}{\sqrt{1 - (3/4)}} = 12\pi$.

83. $\lim_{k\to 0^+} 9.8\,\dfrac{1 - e^{-kt}}{k} = 9.8 \lim_{k\to 0^+} \dfrac{1 - e^{-kt}}{k} = 9.8\,\dfrac{d}{dk}(-e^{-kt})\big|_{k=0} = 9.8\,t$, so if the fluid offers no resistance, then the speed will increase at a constant rate of 9.8 m/s^2.

Exercise Set 3.4

1. $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$

(a) $\dfrac{dy}{dt} = 3(2) = 6$. **(b)** $-1 = 3\dfrac{dx}{dt}$, $\dfrac{dx}{dt} = -\dfrac{1}{3}$.

3. $8x\dfrac{dx}{dt} + 18y\dfrac{dy}{dt} = 0$

(a) $8\dfrac{1}{2\sqrt{2}}\cdot 3 + 18\dfrac{1}{3\sqrt{2}}\dfrac{dy}{dt} = 0$, $\dfrac{dy}{dt} = -2$. **(b)** $8\left(\dfrac{1}{3}\right)\dfrac{dx}{dt} - 18\dfrac{\sqrt{5}}{9}\cdot 8 = 0$, $\dfrac{dx}{dt} = 6\sqrt{5}$.

5. **(b)** $A = x^2$.

(c) $\dfrac{dA}{dt} = 2x\dfrac{dx}{dt}$.

(d) Find $\left.\dfrac{dA}{dt}\right|_{x=3}$ given that $\left.\dfrac{dx}{dt}\right|_{x=3} = 2$. From part (c), $\left.\dfrac{dA}{dt}\right|_{x=3} = 2(3)(2) = 12\ \text{ft}^2/\text{min}$.

7. **(a)** $V = \pi r^2 h$, so $\dfrac{dV}{dt} = \pi\left(r^2\dfrac{dh}{dt} + 2rh\dfrac{dr}{dt}\right)$.

(b) Find $\left.\dfrac{dV}{dt}\right|_{\substack{h=6,\\ r=10}}$ given that $\left.\dfrac{dh}{dt}\right|_{\substack{h=6,\\ r=10}} = 1$ and $\left.\dfrac{dr}{dt}\right|_{\substack{h=6,\\ r=10}} = -1$. From part (a), $\left.\dfrac{dV}{dt}\right|_{\substack{h=6,\\ r=10}} = \pi[10^2(1)+2(10)(6)(-1)] = -20\pi\ \text{in}^3/\text{s}$; the volume is decreasing.

9. **(a)** $\tan\theta = \dfrac{y}{x}$, so $\sec^2\theta\dfrac{d\theta}{dt} = \dfrac{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}}{x^2}$, $\dfrac{d\theta}{dt} = \dfrac{\cos^2\theta}{x^2}\left(x\dfrac{dy}{dt} - y\dfrac{dx}{dt}\right)$.

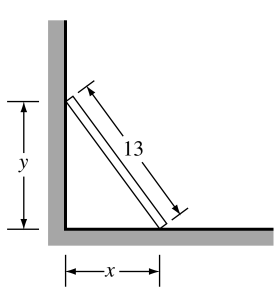
(b) Find $\left.\dfrac{d\theta}{dt}\right|_{\substack{x=2,\\ y=2}}$ given that $\left.\dfrac{dx}{dt}\right|_{\substack{x=2,\\ y=2}} = 1$ and $\left.\dfrac{dy}{dt}\right|_{\substack{x=2,\\ y=2}} = -\dfrac{1}{4}$. When $x = 2$ and $y = 2$, $\tan\theta = 2/2 = 1$ so $\theta = \dfrac{\pi}{4}$ and $\cos\theta = \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$. Thus from part (a), $\left.\dfrac{d\theta}{dt}\right|_{\substack{x=2,\\ y=2}} = \dfrac{(1/\sqrt{2})^2}{2^2}\left[2\left(-\dfrac{1}{4}\right) - 2(1)\right] = -\dfrac{5}{16}$ rad/s; θ is decreasing.

11. Let A be the area swept out, and θ the angle through which the minute hand has rotated. Find $\dfrac{dA}{dt}$ given that $\dfrac{d\theta}{dt} = \dfrac{\pi}{30}$ rad/min; $A = \dfrac{1}{2}r^2\theta = 8\theta$, so $\dfrac{dA}{dt} = 8\dfrac{d\theta}{dt} = \dfrac{4\pi}{15}\ \text{in}^2/\text{min}$.

13. Find $\left.\dfrac{dr}{dt}\right|_{A=9}$ given that $\dfrac{dA}{dt} = 6$. From $A = \pi r^2$ we get $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$ so $\dfrac{dr}{dt} = \dfrac{1}{2\pi r}\dfrac{dA}{dt}$. If $A = 9$ then $\pi r^2 = 9$, $r = 3/\sqrt{\pi}$ so $\left.\dfrac{dr}{dt}\right|_{A=9} = \dfrac{1}{2\pi(3/\sqrt{\pi})}(6) = 1/\sqrt{\pi}$ mi/h.

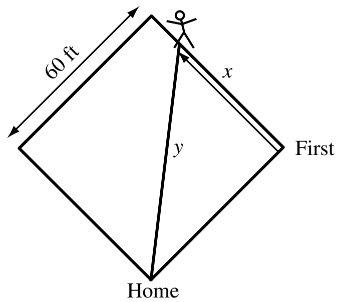
15. Find $\left.\dfrac{dV}{dt}\right|_{r=9}$ given that $\dfrac{dr}{dt} = -15$. From $V = \dfrac{4}{3}\pi r^3$ we get $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$ so $\left.\dfrac{dV}{dt}\right|_{r=9} = 4\pi(9)^2(-15) = -4860\pi$. Air must be removed at the rate of $4860\pi\ \text{cm}^3/\text{min}$.

17. Find $\left.\dfrac{dx}{dt}\right|_{y=5}$ given that $\dfrac{dy}{dt} = -2$. From $x^2+y^2 = 13^2$ we get $2x\dfrac{dx}{dt}+2y\dfrac{dy}{dt} = 0$ so $\dfrac{dx}{dt} = -\dfrac{y}{x}\dfrac{dy}{dt}$. Use $x^2+y^2 = 169$ to find that $x = 12$ when $y = 5$ so $\left.\dfrac{dx}{dt}\right|_{y=5} = -\dfrac{5}{12}(-2) = \dfrac{5}{6}$ ft/s.

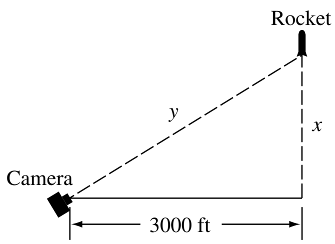


19. Let x denote the distance from first base and y the distance from home plate. Then $x^2+60^2 = y^2$ and $2x\dfrac{dx}{dt} = 2y\dfrac{dy}{dt}$.

When $x = 50$ then $y = 10\sqrt{61}$ so $\dfrac{dy}{dt} = \dfrac{x}{y}\dfrac{dx}{dt} = \dfrac{50}{10\sqrt{61}}(25) = \dfrac{125}{\sqrt{61}}$ ft/s.



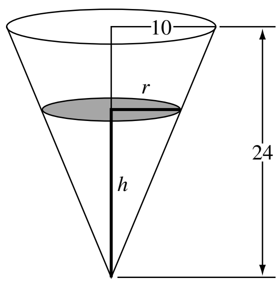
21. Find $\left.\dfrac{dy}{dt}\right|_{x=4000}$ given that $\left.\dfrac{dx}{dt}\right|_{x=4000} = 880$. From $y^2 = x^2 + 3000^2$ we get $2y\dfrac{dy}{dt} = 2x\dfrac{dx}{dt}$ so $\dfrac{dy}{dt} = \dfrac{x}{y}\dfrac{dx}{dt}$. If $x = 4000$, then $y = 5000$ so $\left.\dfrac{dy}{dt}\right|_{x=4000} = \dfrac{4000}{5000}(880) = 704$ ft/s.



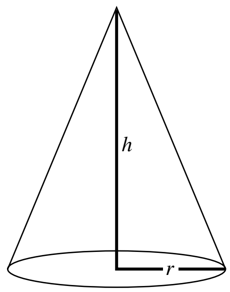
23. **(a)** If x denotes the altitude, then $r - x = 3960$, the radius of the Earth. $\theta = 0$ at perigee, so $r = 4995/1.12 \approx 4460$; the altitude is $x = 4460 - 3960 = 500$ miles. $\theta = \pi$ at apogee, so $r = 4995/0.88 \approx 5676$; the altitude is $x = 5676 - 3960 = 1716$ miles.

(b) If $\theta = 120^\circ$, then $r = 4995/0.94 \approx 5314$; the altitude is $5314 - 3960 = 1354$ miles. The rate of change of the altitude is given by $\dfrac{dx}{dt} = \dfrac{dr}{dt} = \dfrac{dr}{d\theta}\dfrac{d\theta}{dt} = \dfrac{4995(0.12\sin\theta)}{(1 + 0.12\cos\theta)^2}\dfrac{d\theta}{dt}$. Use $\theta = 120^\circ$ and $d\theta/dt = 2.7^\circ/\text{min} = (2.7)(\pi/180)$ rad/min to get $dr/dt \approx 27.7$ mi/min.

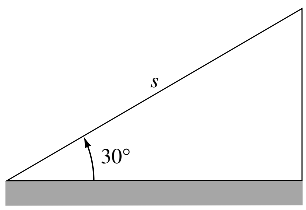
25. Find $\left.\dfrac{dh}{dt}\right|_{h=16}$ given that $\dfrac{dV}{dt} = 20$. The volume of water in the tank at a depth h is $V = \dfrac{1}{3}\pi r^2 h$. Use similar triangles (see figure) to get $\dfrac{r}{h} = \dfrac{10}{24}$ so $r = \dfrac{5}{12}h$ thus $V = \dfrac{1}{3}\pi\left(\dfrac{5}{12}h\right)^2 h = \dfrac{25}{432}\pi h^3$, $\dfrac{dV}{dt} = \dfrac{25}{144}\pi h^2\dfrac{dh}{dt}$; $\dfrac{dh}{dt} = \dfrac{144}{25\pi h^2}\dfrac{dV}{dt}$, $\left.\dfrac{dh}{dt}\right|_{h=16} = \dfrac{144}{25\pi(16)^2}(20) = \dfrac{9}{20\pi}$ ft/min.



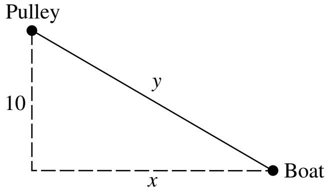
27. Find $\left.\dfrac{dV}{dt}\right|_{h=10}$ given that $\dfrac{dh}{dt} = 5$. $V = \dfrac{1}{3}\pi r^2 h$, but $r = \dfrac{1}{2}h$ so $V = \dfrac{1}{3}\pi\left(\dfrac{h}{2}\right)^2 h = \dfrac{1}{12}\pi h^3$, $\dfrac{dV}{dt} = \dfrac{1}{4}\pi h^2\dfrac{dh}{dt}$, $\left.\dfrac{dV}{dt}\right|_{h=10} = \dfrac{1}{4}\pi(10)^2(5) = 125\pi$ ft^3/min.



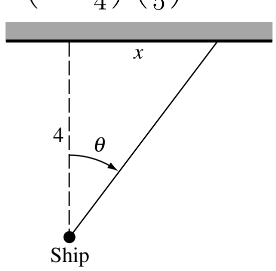
29. With s and h as shown in the figure, we want to find $\dfrac{dh}{dt}$ given that $\dfrac{ds}{dt} = 500$. From the figure, $h = s\sin 30^\circ = \dfrac{1}{2}s$ so $\dfrac{dh}{dt} = \dfrac{1}{2}\dfrac{ds}{dt} = \dfrac{1}{2}(500) = 250$ mi/h.



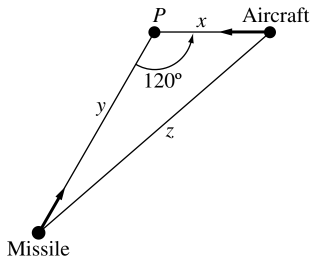
31. Find $\dfrac{dy}{dt}$ given that $\left.\dfrac{dx}{dt}\right|_{y=125} = -12$. From $x^2 + 10^2 = y^2$ we get $2x\dfrac{dx}{dt} = 2y\dfrac{dy}{dt}$ so $\dfrac{dy}{dt} = \dfrac{x}{y}\dfrac{dx}{dt}$. Use $x^2 + 100 = y^2$ to find that $x = \sqrt{15,525} = 15\sqrt{69}$ when $y = 125$ so $\dfrac{dy}{dt} = \dfrac{15\sqrt{69}}{125}(-12) = -\dfrac{36\sqrt{69}}{25}$. The rope must be pulled at the rate of $\dfrac{36\sqrt{69}}{25}$ ft/min.



33. Find $\left.\dfrac{dx}{dt}\right|_{\theta=\pi/4}$ given that $\dfrac{d\theta}{dt} = \dfrac{2\pi}{10} = \dfrac{\pi}{5}$ rad/s. Then $x = 4\tan\theta$ (see figure) so $\dfrac{dx}{dt} = 4\sec^2\theta\dfrac{d\theta}{dt}$, $\left.\dfrac{dx}{dt}\right|_{\theta=\pi/4} = 4\left(\sec^2\dfrac{\pi}{4}\right)\left(\dfrac{\pi}{5}\right) = 8\pi/5$ km/s.



35. We wish to find $\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\ y=4}}$ given $\dfrac{dx}{dt} = -600$ and $\left.\dfrac{dy}{dt}\right|_{\substack{x=2,\\ y=4}} = -1200$ (see figure). From the law of cosines, $z^2 = x^2 + y^2 - 2xy\cos 120^\circ = x^2 + y^2 - 2xy(-1/2) = x^2 + y^2 + xy$, so $2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} + x\dfrac{dy}{dt} + y\dfrac{dx}{dt}$, $\dfrac{dz}{dt} = \dfrac{1}{2z}\left[(2x+y)\dfrac{dx}{dt} + (2y+x)\dfrac{dy}{dt}\right]$. When $x = 2$ and $y = 4$, $z^2 = 2^2 + 4^2 + (2)(4) = 28$, so $z = \sqrt{28} = 2\sqrt{7}$, thus $\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\ y=4}} = \dfrac{1}{2(2\sqrt{7})}[(2(2)+4)(-600) + (2(4)+2)(-1200)] = -\dfrac{4200}{\sqrt{7}} = -600\sqrt{7}$ mi/h; the distance between missile and aircraft is decreasing at the rate of $600\sqrt{7}$ mi/h.



37. **(a)** We want $\left.\dfrac{dy}{dt}\right|_{\substack{x=1,\\y=2}}$ given that $\left.\dfrac{dx}{dt}\right|_{\substack{x=1,\\y=2}} = 6$. For convenience, first rewrite the equation as $xy^3 = \dfrac{8}{5} + \dfrac{8}{5}y^2$ then $3xy^2\dfrac{dy}{dt} + y^3\dfrac{dx}{dt} = \dfrac{16}{5}y\dfrac{dy}{dt}$, $\dfrac{dy}{dt} = \dfrac{y^3}{\dfrac{16}{5}y - 3xy^2}\dfrac{dx}{dt}$, so $\left.\dfrac{dy}{dt}\right|_{\substack{x=1,\\y=2}} = \dfrac{2^3}{\dfrac{16}{5}(2) - 3(1)2^2}(6) = -60/7$ units/s.

(b) Falling, because $\dfrac{dy}{dt} < 0$.

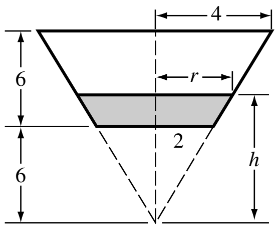
39. The coordinates of P are $(x, 2x)$, so the distance between P and the point $(3, 0)$ is $D = \sqrt{(x-3)^2 + (2x-0)^2} = \sqrt{5x^2 - 6x + 9}$. Find $\left.\dfrac{dD}{dt}\right|_{x=3}$ given that $\left.\dfrac{dx}{dt}\right|_{x=3} = -2$. $\dfrac{dD}{dt} = \dfrac{5x-3}{\sqrt{5x^2-6x+9}}\dfrac{dx}{dt}$, so $\left.\dfrac{dD}{dt}\right|_{x=3} = \dfrac{12}{\sqrt{36}}(-2) = -4$ units/s.

41. Solve $\dfrac{dx}{dt} = 3\dfrac{dy}{dt}$ given $y = x/(x^2+1)$. Then $y(x^2+1) = x$. Differentiating with respect to x, $(x^2+1)\dfrac{dy}{dx} + y(2x) = 1$. But $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1}{3}$ so $(x^2+1)\dfrac{1}{3} + 2xy = 1$, $x^2+1+6xy = 3$, $x^2+1+6x^2/(x^2+1) = 3$, $(x^2+1)^2 + 6x^2 - 3x^2 - 3 = 0$, $x^4 + 5x^2 - 2 = 0$. By the quadratic formula applied to x^2 we obtain $x^2 = (-5 \pm \sqrt{25+8})/2$. The minus sign is spurious since x^2 cannot be negative, so $x^2 = (-5+\sqrt{33})/2$, and $x = \pm\sqrt{(-5+\sqrt{33})/2}$.

43. Find $\left.\dfrac{dS}{dt}\right|_{s=10}$ given that $\left.\dfrac{ds}{dt}\right|_{s=10} = -2$. From $\dfrac{1}{s} + \dfrac{1}{S} = \dfrac{1}{6}$ we get $-\dfrac{1}{s^2}\dfrac{ds}{dt} - \dfrac{1}{S^2}\dfrac{dS}{dt} = 0$, so $\dfrac{dS}{dt} = -\dfrac{S^2}{s^2}\dfrac{ds}{dt}$. If $s = 10$, then $\dfrac{1}{10} + \dfrac{1}{S} = \dfrac{1}{6}$ which gives $S = 15$. So $\left.\dfrac{dS}{dt}\right|_{s=10} = -\dfrac{225}{100}(-2) = 4.5$ cm/s. The image is moving away from the lens.

45. Let r be the radius, V the volume, and A the surface area of a sphere. Show that $\dfrac{dr}{dt}$ is a constant given that $\dfrac{dV}{dt} = -kA$, where k is a positive constant. Because $V = \dfrac{4}{3}\pi r^3$, $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$. But it is given that $\dfrac{dV}{dt} = -kA$ or, because $A = 4\pi r^2$, $\dfrac{dV}{dt} = -4\pi r^2 k$ which when substituted into the previous equation for $\dfrac{dV}{dt}$ gives $-4\pi r^2 k = 4\pi r^2\dfrac{dr}{dt}$, and $\dfrac{dr}{dt} = -k$.

47. Extend sides of cup to complete the cone and let V_0 be the volume of the portion added, then (see figure) $V = \dfrac{1}{3}\pi r^2 h - V_0$ where $\dfrac{r}{h} = \dfrac{4}{12} = \dfrac{1}{3}$ so $r = \dfrac{1}{3}h$ and $V = \dfrac{1}{3}\pi\left(\dfrac{h}{3}\right)^2 h - V_0 = \dfrac{1}{27}\pi h^3 - V_0$, $\dfrac{dV}{dt} = \dfrac{1}{9}\pi h^2\dfrac{dh}{dt}$, $\dfrac{dh}{dt} = \dfrac{9}{\pi h^2}\dfrac{dV}{dt}$, $\left.\dfrac{dh}{dt}\right|_{h=9} = \dfrac{9}{\pi(9)^2}(20) = \dfrac{20}{9\pi}$ cm/s.



Exercise Set 3.5

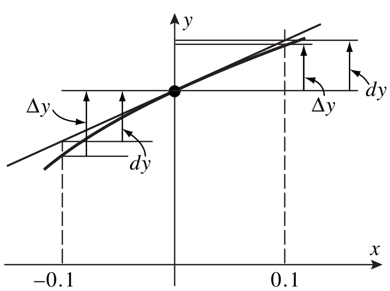
1. (a) $f(x) \approx f(1) + f'(1)(x-1) = 1 + 3(x-1)$.

(b) $f(1+\Delta x) \approx f(1) + f'(1)\Delta x = 1 + 3\Delta x$.

(c) From part (a), $(1.02)^3 \approx 1 + 3(0.02) = 1.06$. From part (b), $(1.02)^3 \approx 1 + 3(0.02) = 1.06$.

3. (a) $f(x) \approx f(x_0) + f'(x_0)(x-x_0) = 1 + (1/(2\sqrt{1})(x-0) = 1 + (1/2)x$, so with $x_0 = 0$ and $x = -0.1$, we have $\sqrt{0.9} = f(-0.1) \approx 1 + (1/2)(-0.1) = 1 - 0.05 = 0.95$. With $x = 0.1$ we have $\sqrt{1.1} = f(0.1) \approx 1 + (1/2)(0.1) = 1.05$.

(b)



5. $f(x) = (1+x)^{15}$ and $x_0 = 0$. Thus $(1+x)^{15} \approx f(x_0) + f'(x_0)(x-x_0) = 1 + 15(1)^{14}(x-0) = 1 + 15x$.

7. $\tan x \approx \tan(0) + \sec^2(0)(x-0) = x$.

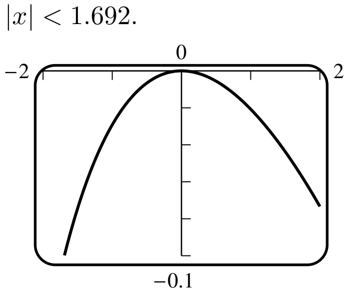
9. $x_0 = 0, f(x) = e^x, f'(x) = e^x, f'(x_0) = 1$, hence $e^x \approx 1 + 1 \cdot x = 1 + x$.

11. $x^4 \approx (1)^4 + 4(1)^3(x-1)$. Set $\Delta x = x - 1$; then $x = \Delta x + 1$ and $(1+\Delta x)^4 = 1 + 4\Delta x$.

13. $\dfrac{1}{2+x} \approx \dfrac{1}{2+1} - \dfrac{1}{(2+1)^2}(x-1)$, and $2 + x = 3 + \Delta x$, so $\dfrac{1}{3+\Delta x} \approx \dfrac{1}{3} - \dfrac{1}{9}\Delta x$.

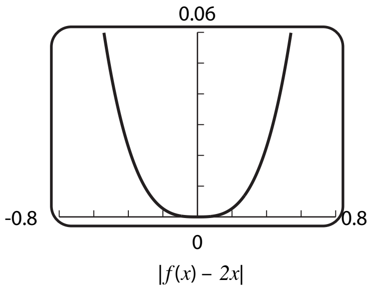
15. Let $f(x) = \tan^{-1} x, f(1) = \pi/4, f'(1) = 1/2, \tan^{-1}(1+\Delta x) \approx \dfrac{\pi}{4} + \dfrac{1}{2}\Delta x$.

17. $f(x) = \sqrt{x+3}$ and $x_0 = 0$, so $\sqrt{x+3} \approx \sqrt{3} + \dfrac{1}{2\sqrt{3}}(x-0) = \sqrt{3} + \dfrac{1}{2\sqrt{3}}x$, and $\left|f(x) - \left(\sqrt{3} + \dfrac{1}{2\sqrt{3}}x\right)\right| < 0.1$ if $|x| < 1.692$.



$\left|f(x) - \left(\sqrt{3} + \frac{1}{2\sqrt{3}}x\right)\right|$

19. $\tan 2x \approx \tan 0 + (\sec^2 0)(2x - 0) = 2x$, and $|\tan 2x - 2x| < 0.1$ if $|x| < 0.3158$.



21. (a) The local linear approximation $\sin x \approx x$ gives $\sin 1^\circ = \sin(\pi/180) \approx \pi/180 = 0.0174533$ and a calculator gives $\sin 1^\circ = 0.0174524$. The relative error $|\sin(\pi/180) - (\pi/180)|/(\sin \pi/180) = 0.000051$ is very small, so for such a small value of x the approximation is very good.

(b) Use $x_0 = 45^\circ$ (this assumes you know, or can approximate, $\sqrt{2}/2$).

(c) $44^\circ = \dfrac{44\pi}{180}$ radians, and $45^\circ = \dfrac{45\pi}{180} = \dfrac{\pi}{4}$ radians. With $x = \dfrac{44\pi}{180}$ and $x_0 = \dfrac{\pi}{4}$ we obtain $\sin 44^\circ = \sin \dfrac{44\pi}{180} \approx \sin \dfrac{\pi}{4} + \left(\cos \dfrac{\pi}{4}\right)\left(\dfrac{44\pi}{180} - \dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\left(\dfrac{-\pi}{180}\right) = 0.694765$. With a calculator, $\sin 44^\circ = 0.694658$.

23. $f(x) = x^4$, $f'(x) = 4x^3$, $x_0 = 3$, $\Delta x = 0.02$; $(3.02)^4 \approx 3^4 + (108)(0.02) = 81 + 2.16 = 83.16$.

25. $f(x) = \sqrt{x}$, $f'(x) = \dfrac{1}{2\sqrt{x}}$, $x_0 = 64$, $\Delta x = 1$; $\sqrt{65} \approx \sqrt{64} + \dfrac{1}{16}(1) = 8 + \dfrac{1}{16} = 8.0625$.

27. $f(x) = \sqrt{x}$, $f'(x) = \dfrac{1}{2\sqrt{x}}$, $x_0 = 81$, $\Delta x = -0.1$; $\sqrt{80.9} \approx \sqrt{81} + \dfrac{1}{18}(-0.1) \approx 8.9944$.

29. $f(x) = \sin x$, $f'(x) = \cos x$, $x_0 = 0$, $\Delta x = 0.1$; $\sin 0.1 \approx \sin 0 + (\cos 0)(0.1) = 0.1$.

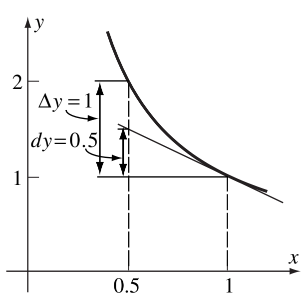
31. $f(x) = \cos x$, $f'(x) = -\sin x$, $x_0 = \pi/6$, $\Delta x = \pi/180$; $\cos 31^\circ \approx \cos 30^\circ + \left(-\dfrac{1}{2}\right)\left(\dfrac{\pi}{180}\right) = \dfrac{\sqrt{3}}{2} - \dfrac{\pi}{360} \approx 0.8573$.

33. $\tan^{-1}(1 + \Delta x) \approx \dfrac{\pi}{4} + \dfrac{1}{2}\Delta x, \Delta x = -0.01, \tan^{-1} 0.99 \approx \dfrac{\pi}{4} - 0.005 \approx 0.780398$.

35. $\sqrt[3]{8.24} = 8^{1/3}\sqrt[3]{1.03} \approx 2(1 + \frac{1}{3}0.03) \approx 2.02$, and $4.08^{3/2} = 4^{3/2}1.02^{3/2} = 8(1 + 0.02(3/2)) = 8.24$.

37. (a) $dy = (-1/x^2)dx = (-1)(-0.5) = 0.5$ and $\Delta y = 1/(x + \Delta x) - 1/x = 1/(1 - 0.5) - 1/1 = 2 - 1 = 1$.

(b)



39. $dy = 3x^2 dx$; $\Delta y = (x + \Delta x)^3 - x^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3 = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$.

41. $dy = (2x-2)dx$; $\Delta y = [(x+\Delta x)^2 - 2(x+\Delta x) + 1] - [x^2 - 2x + 1] = x^2 + 2x\,\Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - x^2 + 2x - 1 = 2x\,\Delta x + (\Delta x)^2 - 2\Delta x$.

43. **(a)** $dy = (12x^2 - 14x)dx$.

(b) $dy = x\,d(\cos x) + \cos x\,dx = x(-\sin x)dx + \cos x dx = (-x\sin x + \cos x)dx$.

45. **(a)** $dy = \left(\sqrt{1-x} - \dfrac{x}{2\sqrt{1-x}}\right)dx = \dfrac{2-3x}{2\sqrt{1-x}}dx$.

(b) $dy = -17(1+x)^{-18}dx$.

47. False; $dy = (dy/dx)dx$.

49. False; they are equal whenever the function is linear.

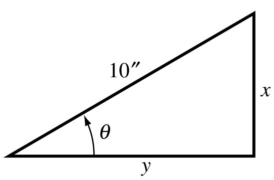
51. $dy = \dfrac{3}{2\sqrt{3x-2}}dx$, $x = 2$, $dx = 0.03$; $\Delta y \approx dy = \dfrac{3}{4}(0.03) = 0.0225$.

53. $dy = \dfrac{1-x^2}{(x^2+1)^2}dx$, $x = 2$, $dx = -0.04$; $\Delta y \approx dy = \left(-\dfrac{3}{25}\right)(-0.04) = 0.0048$.

55. **(a)** $A = x^2$ where x is the length of a side; $dA = 2x\,dx = 2(10)(\pm 0.1) = \pm 2$ ft^2.

(b) Relative error in x is within $\dfrac{dx}{x} = \dfrac{\pm 0.1}{10} = \pm 0.01$ so percentage error in x is $\pm 1\%$; relative error in A is within $\dfrac{dA}{A} = \dfrac{2x\,dx}{x^2} = 2\dfrac{dx}{x} = 2(\pm 0.01) = \pm 0.02$ so percentage error in A is $\pm 2\%$.

57. **(a)** $x = 10\sin\theta$, $y = 10\cos\theta$ (see figure), $dx = 10\cos\theta d\theta = 10\left(\cos\dfrac{\pi}{6}\right)\left(\pm\dfrac{\pi}{180}\right) = 10\left(\dfrac{\sqrt{3}}{2}\right)\left(\pm\dfrac{\pi}{180}\right) \approx \pm 0.151$ in, $dy = -10(\sin\theta)d\theta = -10\left(\sin\dfrac{\pi}{6}\right)\left(\pm\dfrac{\pi}{180}\right) = -10\left(\dfrac{1}{2}\right)\left(\pm\dfrac{\pi}{180}\right) \approx \pm 0.087$ in.



(b) Relative error in x is within $\dfrac{dx}{x} = (\cot\theta)d\theta = \left(\cot\dfrac{\pi}{6}\right)\left(\pm\dfrac{\pi}{180}\right) = \sqrt{3}\left(\pm\dfrac{\pi}{180}\right) \approx \pm 0.030$, so percentage error in x is $\approx \pm 3.0\%$; relative error in y is within $\dfrac{dy}{y} = -\tan\theta d\theta = -\left(\tan\dfrac{\pi}{6}\right)\left(\pm\dfrac{\pi}{180}\right) = -\dfrac{1}{\sqrt{3}}\left(\pm\dfrac{\pi}{180}\right) \approx \pm 0.010$, so percentage error in y is $\approx \pm 1.0\%$.

59. $\dfrac{dR}{R} = \dfrac{(-2k/r^3)dr}{(k/r^2)} = -2\dfrac{dr}{r}$, but $\dfrac{dr}{r} = \pm 0.05$ so $\dfrac{dR}{R} = -2(\pm 0.05) = \pm 0.10$; percentage error in R is $\pm 10\%$.

61. $A = \dfrac{1}{4}(4)^2\sin 2\theta = 4\sin 2\theta$ thus $dA = 8\cos 2\theta d\theta$ so, with $\theta = 30° = \pi/6$ radians and $d\theta = \pm 15' = \pm 1/4° = \pm\pi/720$ radians, $dA = 8\cos(\pi/3)(\pm\pi/720) = \pm\pi/180 \approx \pm 0.017$ cm^2.

63. $V = x^3$ where x is the length of a side; $\dfrac{dV}{V} = \dfrac{3x^2dx}{x^3} = 3\dfrac{dx}{x}$, but $\dfrac{dx}{x} = \pm 0.02$, so $\dfrac{dV}{V} = 3(\pm 0.02) = \pm 0.06$; percentage error in V is $\pm 6\%$.

65. $A = \dfrac{1}{4}\pi D^2$ where D is the diameter of the circle; $\dfrac{dA}{A} = \dfrac{(\pi D/2)dD}{\pi D^2/4} = 2\dfrac{dD}{D}$, but $\dfrac{dA}{A} = \pm 0.01$ so $2\dfrac{dD}{D} = \pm 0.01$, $\dfrac{dD}{D} = \pm 0.005$; maximum permissible percentage error in D is $\pm 0.5\%$.

67. V = volume of cylindrical rod $= \pi r^2 h = \pi r^2(15) = 15\pi r^2$; approximate ΔV by dV if $r = 2.5$ and $dr = \Delta r = 0.1$. $dV = 30\pi r\,dr = 30\pi(2.5)(0.1) \approx 23.5619$ cm^3.

69. Differentiating $R = \log_{10}(A/A_0)$, we obtain $\dfrac{dR}{dA} = \dfrac{1}{A\ln 10}$. Thus $dR = \dfrac{1}{\ln 10}\dfrac{dA}{A}$, and $\Delta R \approx dR \approx 0.4343\dfrac{dA}{A}$.

Exercise Set 3.6

1. **(a)** $\lim\limits_{x\to 2} \dfrac{x^2-4}{x^2+2x-8} = \lim\limits_{x\to 2} \dfrac{(x-2)(x+2)}{(x+4)(x-2)} = \lim\limits_{x\to 2} \dfrac{x+2}{x+4} = \dfrac{2}{3}$ or, using L'Hôpital's rule,

$\lim\limits_{x\to 2} \dfrac{x^2-4}{x^2+2x-8} = \lim\limits_{x\to 2} \dfrac{2x}{2x+2} = \dfrac{2}{3}$.

(b) $\lim\limits_{x\to+\infty} \dfrac{2x-5}{3x+7} = \dfrac{2 - \lim\limits_{x\to+\infty} \dfrac{5}{x}}{3 + \lim\limits_{x\to+\infty} \dfrac{7}{x}} = \dfrac{2}{3}$ or, using L'Hôpital's rule, $\lim\limits_{x\to+\infty} \dfrac{2x-5}{3x+7} = \lim\limits_{x\to+\infty} \dfrac{2}{3} = \dfrac{2}{3}$.

3. True; $\ln x$ is not defined for negative x.

5. False; apply L'Hôpital's rule n times.

7. $\lim\limits_{x\to 0} \dfrac{e^x}{\cos x} = 1$.

9. $\lim\limits_{\theta\to 0} \dfrac{\sec^2\theta}{1} = 1$.

11. $\lim\limits_{x\to\pi^+} \dfrac{\cos x}{1} = -1$.

13. $\lim\limits_{x\to+\infty} \dfrac{1/x}{1} = 0$.

15. $\lim\limits_{x\to 0^+} \dfrac{-\csc^2 x}{1/x} = \lim\limits_{x\to 0^+} \dfrac{-x}{\sin^2 x} = \lim\limits_{x\to 0^+} \dfrac{-1}{2\sin x\cos x} = -\infty$.

17. $\lim\limits_{x\to+\infty} \dfrac{100x^{99}}{e^x} = \lim\limits_{x\to+\infty} \dfrac{(100)(99)x^{98}}{e^x} = \cdots = \lim\limits_{x\to+\infty} \dfrac{(100)(99)(98)\cdots(1)}{e^x} = 0$.

19. $\lim\limits_{x\to 0} \dfrac{2/\sqrt{1-4x^2}}{1} = 2$.

21. $\lim\limits_{x\to+\infty} xe^{-x} = \lim\limits_{x\to+\infty} \dfrac{x}{e^x} = \lim\limits_{x\to+\infty} \dfrac{1}{e^x} = 0$.

23. $\lim\limits_{x\to+\infty} x\sin(\pi/x) = \lim\limits_{x\to+\infty} \dfrac{\sin(\pi/x)}{1/x} = \lim\limits_{x\to+\infty} \dfrac{(-\pi/x^2)\cos(\pi/x)}{-1/x^2} = \lim\limits_{x\to+\infty} \pi\cos(\pi/x) = \pi$.

25. $\lim\limits_{x\to(\pi/2)^-} \sec 3x\cos 5x = \lim\limits_{x\to(\pi/2)^-} \dfrac{\cos 5x}{\cos 3x} = \lim\limits_{x\to(\pi/2)^-} \dfrac{-5\sin 5x}{-3\sin 3x} = \dfrac{-5(+1)}{(-3)(-1)} = -\dfrac{5}{3}$.

27. $y = (1-3/x)^x$, $\lim\limits_{x\to+\infty} \ln y = \lim\limits_{x\to+\infty} \dfrac{\ln(1-3/x)}{1/x} = \lim\limits_{x\to+\infty} \dfrac{-3}{1-3/x} = -3$, $\lim\limits_{x\to+\infty} y = e^{-3}$.

29. $y = (e^x+x)^{1/x}$, $\lim\limits_{x\to 0} \ln y = \lim\limits_{x\to 0} \dfrac{\ln(e^x+x)}{x} = \lim\limits_{x\to 0} \dfrac{e^x+1}{e^x+x} = 2$, $\lim\limits_{x\to 0} y = e^2$.

31. $y = (2-x)^{\tan(\pi x/2)}$, $\lim\limits_{x\to 1} \ln y = \lim\limits_{x\to 1} \dfrac{\ln(2-x)}{\cot(\pi x/2)} = \lim\limits_{x\to 1} \dfrac{2\sin^2(\pi x/2)}{\pi(2-x)} = 2/\pi$, $\lim\limits_{x\to 1} y = e^{2/\pi}$.

33. $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right) = \lim\limits_{x\to 0} \dfrac{x - \sin x}{x\sin x} = \lim\limits_{x\to 0} \dfrac{1-\cos x}{x\cos x + \sin x} = \lim\limits_{x\to 0} \dfrac{\sin x}{2\cos x - x\sin x} = 0.$

35. $\lim\limits_{x\to+\infty} \dfrac{(x^2+x)-x^2}{\sqrt{x^2+x}+x} = \lim\limits_{x\to+\infty} \dfrac{x}{\sqrt{x^2+x}+x} = \lim\limits_{x\to+\infty} \dfrac{1}{\sqrt{1+1/x}+1} = 1/2.$

37. $\lim\limits_{x\to+\infty} [x - \ln(x^2+1)] = \lim\limits_{x\to+\infty} [\ln e^x - \ln(x^2+1)] = \lim\limits_{x\to+\infty} \ln \dfrac{e^x}{x^2+1}$, $\lim\limits_{x\to+\infty} \dfrac{e^x}{x^2+1} = \lim\limits_{x\to+\infty} \dfrac{e^x}{2x} = \lim\limits_{x\to+\infty} \dfrac{e^x}{2} = +\infty$, so $\lim\limits_{x\to+\infty} [x - \ln(x^2+1)] = +\infty$

39. $y = x^{\sin x}$, $\ln y = \sin x \ln x$, $\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} \dfrac{\ln x}{\csc x} = \lim\limits_{x\to 0^+} \dfrac{1/x}{-\csc x \cot x} = \lim\limits_{x\to 0^+} \left(\dfrac{\sin x}{x}\right)(-\tan x) = 1(-0) = 0$, so $\lim\limits_{x\to 0^+} x^{\sin x} = \lim\limits_{x\to 0^+} y = e^0 = 1.$

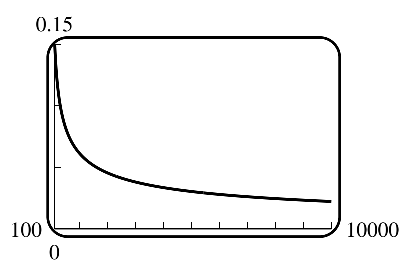
41. $y = \left[-\dfrac{1}{\ln x}\right]^x$, $\ln y = x \ln\left[-\frac{1}{\ln x}\right]$, $\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} \dfrac{\ln\left[-\frac{1}{\ln x}\right]}{1/x} = \lim\limits_{x\to 0^+} \left(-\dfrac{1}{x\ln x}\right)(-x^2) = -\lim\limits_{x\to 0^+} \dfrac{x}{\ln x} = 0$, so $\lim\limits_{x\to 0^+} y = e^0 = 1.$

43. $y = (\ln x)^{1/x}$, $\ln y = (1/x)\ln\ln x$, $\lim\limits_{x\to+\infty} \ln y = \lim\limits_{x\to+\infty} \dfrac{\ln\ln x}{x} = \lim\limits_{x\to+\infty} \dfrac{1/(x\ln x)}{1} = 0$, so $\lim\limits_{x\to+\infty} y = 1.$

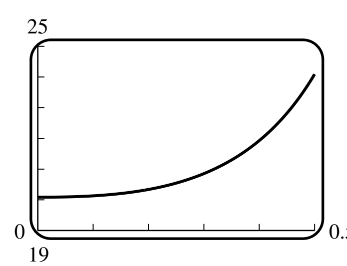
45. $y = (\tan x)^{\pi/2-x}$, $\ln y = (\pi/2 - x)\ln\tan x$, $\lim\limits_{x\to(\pi/2)^-} \ln y = \lim\limits_{x\to(\pi/2)^-} \dfrac{\ln\tan x}{1/(\pi/2-x)} = \lim\limits_{x\to(\pi/2)^-} \dfrac{(\sec^2 x/\tan x)}{1/(\pi/2-x)^2} = \lim\limits_{x\to(\pi/2)^-} \dfrac{(\pi/2-x)}{\cos x}\dfrac{(\pi/2-x)}{\sin x} = \lim\limits_{x\to(\pi/2)^-} \dfrac{(\pi/2-x)}{\cos x} \lim\limits_{x\to(\pi/2)^-} \dfrac{(\pi/2-x)}{\sin x} = 1\cdot 0 = 0$, so $\lim\limits_{x\to(\pi/2)^-} y = 1.$

47. (a) L'Hôpital's rule does not apply to the problem $\lim\limits_{x\to 1} \dfrac{3x^2-2x+1}{3x^2-2x}$ because it is not an indeterminate form.

(b) $\lim\limits_{x\to 1} \dfrac{3x^2-2x+1}{3x^2-2x} = 2.$

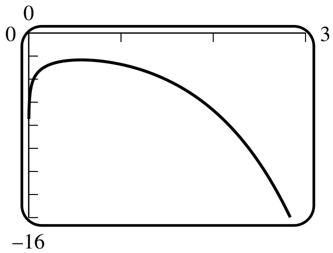


49. $\lim\limits_{x\to+\infty} \dfrac{1/(x\ln x)}{1/(2\sqrt{x})} = \lim\limits_{x\to+\infty} \dfrac{2}{\sqrt{x}\ln x} = 0.$

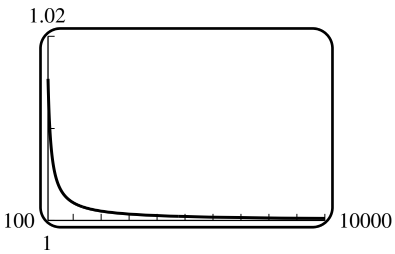


51. $y = (\sin x)^{3/\ln x}$, $\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} \dfrac{3\ln\sin x}{\ln x} = \lim\limits_{x\to 0^+} (3\cos x)\dfrac{x}{\sin x} = 3$, $\lim\limits_{x\to 0^+} y = e^3.$

53. $\ln x - e^x = \ln x - \dfrac{1}{e^{-x}} = \dfrac{e^{-x}\ln x - 1}{e^{-x}}$; $\lim_{x\to+\infty} e^{-x}\ln x = \lim_{x\to+\infty}\dfrac{\ln x}{e^x} = \lim_{x\to+\infty}\dfrac{1/x}{e^x} = 0$ by L'Hôpital's rule, so $\lim_{x\to+\infty}[\ln x - e^x] = \lim_{x\to+\infty}\dfrac{e^{-x}\ln x - 1}{e^{-x}} = -\infty$; no horizontal asymptote.



55. $y = (\ln x)^{1/x}$, $\lim_{x\to+\infty}\ln y = \lim_{x\to+\infty}\dfrac{\ln(\ln x)}{x} = \lim_{x\to+\infty}\dfrac{1}{x\ln x} = 0$; $\lim_{x\to+\infty} y = 1$, $y = 1$ is the horizontal asymptote.



57. **(a)** 0 **(b)** $+\infty$ **(c)** 0 **(d)** $-\infty$ **(e)** $+\infty$ **(f)** $-\infty$

59. $\lim_{x\to+\infty}\dfrac{1+2\cos 2x}{1}$ does not exist, nor is it $\pm\infty$; $\lim_{x\to+\infty}\dfrac{x+\sin 2x}{x} = \lim_{x\to+\infty}\left(1+\dfrac{\sin 2x}{x}\right) = 1$.

61. $\lim_{x\to+\infty}(2 + x\cos 2x + \sin 2x)$ does not exist, nor is it $\pm\infty$; $\lim_{x\to+\infty}\dfrac{x(2+\sin 2x)}{x+1} = \lim_{x\to+\infty}\dfrac{2+\sin 2x}{1+1/x}$, which does not exist because $\sin 2x$ oscillates between -1 and 1 as $x\to+\infty$.

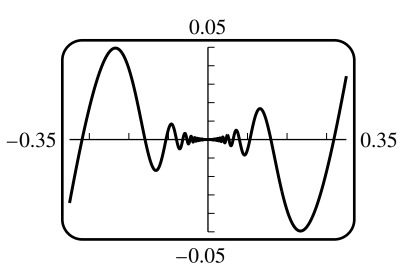
63. $\lim_{R\to 0^+}\dfrac{\frac{Vt}{L}e^{-Rt/L}}{1} = \dfrac{Vt}{L}$.

65. **(b)** $\lim_{x\to+\infty} x(k^{1/x}-1) = \lim_{t\to 0^+}\dfrac{k^t-1}{t} = \lim_{t\to 0^+}\dfrac{(\ln k)k^t}{1} = \ln k$.

(c) $\ln 0.3 = -1.20397$, $1024\left(\sqrt[1024]{0.3}-1\right) = -1.20327$; $\ln 2 = 0.69315$, $1024\left(\sqrt[1024]{2}-1\right) = 0.69338$.

67. **(a)** No; $\sin(1/x)$ oscillates as $x\to 0$.

(b)



(c) For the limit as $x\to 0^+$ use the Squeezing Theorem together with the inequalities $-x^2 \le x^2\sin(1/x) \le x^2$. For $x\to 0^-$ do the same; thus $\lim_{x\to 0} f(x) = 0$.

69. $\lim_{x\to 0^+} \dfrac{\sin(1/x)}{(\sin x)/x}$, $\lim_{x\to 0^+} \dfrac{\sin x}{x} = 1$ but $\lim_{x\to 0^+} \sin(1/x)$ does not exist because $\sin(1/x)$ oscillates between -1 and 1 as $x \to +\infty$, so $\lim_{x\to 0^+} \dfrac{x\sin(1/x)}{\sin x}$ does not exist.

Chapter 3 Review Exercises

1. **(a)** $3x^2 + x\dfrac{dy}{dx} + y - 2 = 0$, $\dfrac{dy}{dx} = \dfrac{2 - y - 3x^2}{x}$.

(b) $y = (1 + 2x - x^3)/x = 1/x + 2 - x^2, dy/dx = -1/x^2 - 2x$.

(c) $\dfrac{dy}{dx} = \dfrac{2 - (1/x + 2 - x^2) - 3x^2}{x} = -1/x^2 - 2x$.

3. $-\dfrac{1}{y^2}\dfrac{dy}{dx} - \dfrac{1}{x^2} = 0$ so $\dfrac{dy}{dx} = -\dfrac{y^2}{x^2}$.

5. $\left(x\dfrac{dy}{dx} + y\right)\sec(xy)\tan(xy) = \dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \dfrac{y\sec(xy)\tan(xy)}{1 - x\sec(xy)\tan(xy)}$.

7. $\dfrac{dy}{dx} = \dfrac{3x}{4y}$, $\dfrac{d^2y}{dx^2} = \dfrac{(4y)(3) - (3x)(4dy/dx)}{16y^2} = \dfrac{12y - 12x(3x/(4y))}{16y^2} = \dfrac{12y^2 - 9x^2}{16y^3} = \dfrac{-3(3x^2 - 4y^2)}{16y^3}$, but $3x^2 - 4y^2 = 7$ so $\dfrac{d^2y}{dx^2} = \dfrac{-3(7)}{16y^3} = -\dfrac{21}{16y^3}$.

9. $\dfrac{dy}{dx} = \tan(\pi y/2) + x(\pi/2)\dfrac{dy}{dx}\sec^2(\pi y/2)$, $\left.\dfrac{dy}{dx}\right|_{y=1/2} = 1 + (\pi/4)\left.\dfrac{dy}{dx}\right|_{y=1/2}(2)$, $\left.\dfrac{dy}{dx}\right|_{y=1/2} = \dfrac{2}{2-\pi}$.

11. Substitute $y = mx$ into $x^2 + xy + y^2 = 4$ to get $x^2 + mx^2 + m^2x^2 = 4$, which has distinct solutions $x = \pm 2/\sqrt{m^2 + m + 1}$. They are distinct because $m^2 + m + 1 = (m + 1/2)^2 + 3/4 \geq 3/4$, so $m^2 + m + 1$ is never zero. Note that the points of intersection occur in pairs (x_0, y_0) and $(-x_0, -y_0)$. By implicit differentiation, the slope of the tangent line to the ellipse is given by $dy/dx = -(2x + y)/(x + 2y)$. Since the slope is unchanged if we replace (x, y) with $(-x, -y)$, it follows that the slopes are equal at the two point of intersection. Finally we must examine the special case $x = 0$ which cannot be written in the form $y = mx$. If $x = 0$ then $y = \pm 2$, and the formula for dy/dx gives $dy/dx = -1/2$, so the slopes are equal.

13. By implicit differentiation, $3x^2 - y - xy' + 3y^2y' = 0$, so $y' = (3x^2 - y)/(x - 3y^2)$. This derivative exists except when $x = 3y^2$. Substituting this into the original equation $x^3 - xy + y^3 = 0$, one has $27y^6 - 3y^3 + y^3 = 0$, $y^3(27y^3 - 2) = 0$. The unique solution in the first quadrant is $y = 2^{1/3}/3, x = 3y^2 = 2^{2/3}/3$

15. $y = \ln(x+1) + 2\ln(x+2) - 3\ln(x+3) - 4\ln(x+4)$, $dy/dx = \dfrac{1}{x+1} + \dfrac{2}{x+2} - \dfrac{3}{x+3} - \dfrac{4}{x+4}$.

17. $\dfrac{dy}{dx} = \dfrac{1}{2x}(2) = 1/x$.

19. $\dfrac{dy}{dx} = \dfrac{1}{3x(\ln x + 1)^{2/3}}$.

21. $\dfrac{dy}{dx} = \log_{10}\ln x = \dfrac{\ln\ln x}{\ln 10}$, $y' = \dfrac{1}{(\ln 10)(x\ln x)}$.

23. $y = \dfrac{3}{2}\ln x + \dfrac{1}{2}\ln(1 + x^4)$, $y' = \dfrac{3}{2x} + \dfrac{2x^3}{(1 + x^4)}$.

25. $y = x^2 + 1$ so $y' = 2x$.

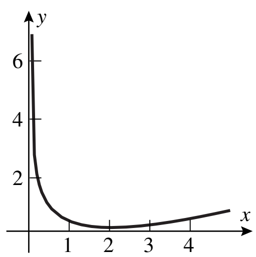
27. $y' = 2e^{\sqrt{x}} + 2xe^{\sqrt{x}}\dfrac{d}{dx}\sqrt{x} = 2e^{\sqrt{x}} + \sqrt{x}e^{\sqrt{x}}$.

29. $y' = \dfrac{2}{\pi(1+4x^2)}$.

31. $\ln y = e^x \ln x$, $\dfrac{y'}{y} = e^x\left(\dfrac{1}{x} + \ln x\right)$, $\dfrac{dy}{dx} = x^{e^x} e^x \left(\dfrac{1}{x} + \ln x\right) = e^x\left[x^{e^x - 1} + x^{e^x}\ln x\right]$.

33. $y' = \dfrac{2}{|2x+1|\sqrt{(2x+1)^2 - 1}}$.

35. $\ln y = 3\ln x - \dfrac{1}{2}\ln(x^2+1)$, $y'/y = \dfrac{3}{x} - \dfrac{x}{x^2+1}$, $y' = \dfrac{3x^2}{\sqrt{x^2+1}} - \dfrac{x^4}{(x^2+1)^{3/2}}$.

37. **(b)** 

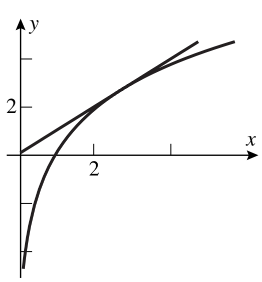
(c) $\dfrac{dy}{dx} = \dfrac{1}{2} - \dfrac{1}{x}$, so $\dfrac{dy}{dx} < 0$ at $x = 1$ and $\dfrac{dy}{dx} > 0$ at $x = e$.

(d) The slope is a continuous function which goes from a negative value to a positive value; therefore it must take the value zero between, by the Intermediate Value Theorem.

(e) $\dfrac{dy}{dx} = 0$ when $x = 2$.

39. Solve $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$ given $y = x\ln x$. Then $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = (1+\ln x)\dfrac{dx}{dt}$, so $1 + \ln x = 3$, $\ln x = 2$, $x = e^2$.

41. Set $y = \log_b x$ and solve $y' = 1$: $y' = \dfrac{1}{x\ln b} = 1$ so $x = \dfrac{1}{\ln b}$. The curves intersect when (x, x) lies on the graph of $y = \log_b x$, so $x = \log_b x$. From Formula (8), Section 1.6, $\log_b x = \dfrac{\ln x}{\ln b}$ from which $\ln x = 1$, $x = e$, $\ln b = 1/e$, $b = e^{1/e} \approx 1.4447$.



43. As long as $f' \neq 0$, g must be differentiable; this can be inferred from the graphs. Note that if $f' = 0$ at a point then g' cannot exist (infinite slope). (For example, $f(x) = x^3$ at $x = 0$).

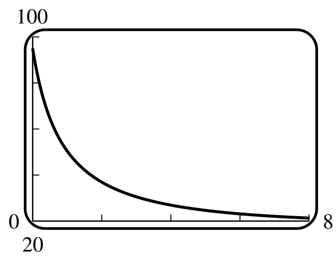
45. Let $P(x_0, y_0)$ be a point on $y = e^{3x}$ then $y_0 = e^{3x_0}$. $dy/dx = 3e^{3x}$ so $m_{\tan} = 3e^{3x_0}$ at P and an equation of the tangent line at P is $y - y_0 = 3e^{3x_0}(x - x_0)$, $y - e^{3x_0} = 3e^{3x_0}(x - x_0)$. If the line passes through the origin then $(0, 0)$ must satisfy the equation so $-e^{3x_0} = -3x_0 e^{3x_0}$ which gives $x_0 = 1/3$ and thus $y_0 = e$. The point is $(1/3, e)$.

47. $\ln y = 2x \ln 3 + 7x \ln 5$; $\dfrac{dy/dx}{y} = 2 \ln 3 + 7 \ln 5$, or $\dfrac{dy}{dx} = (2 \ln 3 + 7 \ln 5)y$.

49. $y' = ae^{ax} \sin bx + be^{ax} \cos bx$, and $y'' = (a^2 - b^2)e^{ax} \sin bx + 2abe^{ax} \cos bx$, so

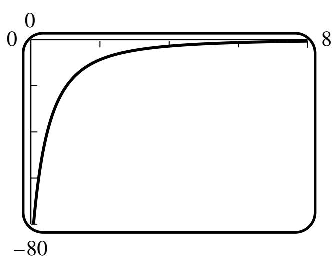
$$y'' - 2ay' + (a^2 + b^2)y = (a^2 - b^2)e^{ax} \sin bx + 2abe^{ax} \cos bx - 2a(ae^{ax} \sin bx + be^{ax} \cos bx) + (a^2 + b^2)e^{ax} \sin bx = 0.$$

51. (a)



(b) As t tends to $+\infty$, the population tends to 19: $\displaystyle\lim_{t\to+\infty} P(t) = \lim_{t\to+\infty} \frac{95}{5 - 4e^{-t/4}} = \frac{95}{5 - 4 \displaystyle\lim_{t\to+\infty} e^{-t/4}} = \frac{95}{5} = 19.$

(c) The rate of population growth tends to zero.



53. In the case $+\infty - (-\infty)$ the limit is $+\infty$; in the case $-\infty - (+\infty)$ the limit is $-\infty$, because large positive (negative) quantities are added to large positive (negative) quantities. The cases $+\infty - (+\infty)$ and $-\infty - (-\infty)$ are indeterminate; large numbers of opposite sign are subtracted, and more information about the sizes is needed.

55. $\displaystyle\lim_{x\to+\infty} (e^x - x^2) = \lim_{x\to+\infty} x^2(e^x/x^2 - 1)$, but $\displaystyle\lim_{x\to+\infty} \frac{e^x}{x^2} = \lim_{x\to+\infty} \frac{e^x}{2x} = \lim_{x\to+\infty} \frac{e^x}{2} = +\infty$, so $\displaystyle\lim_{x\to+\infty} (e^x/x^2 - 1) = +\infty$ and thus $\displaystyle\lim_{x\to+\infty} x^2(e^x/x^2 - 1) = +\infty$.

57. $\displaystyle\lim_{x\to0} \frac{x^2 e^x}{\sin^2 3x} = \left[\lim_{x\to0} \frac{3x}{\sin 3x}\right]^2 \left[\lim_{x\to0} \frac{e^x}{9}\right] = \frac{1}{9}.$

59. The boom is pulled in at the rate of 5 m/min, so the circumference $C = 2r\pi$ is changing at this rate, which means that $\dfrac{dr}{dt} = \dfrac{dC}{dt} \cdot \dfrac{1}{2\pi} = -5/(2\pi)$. $A = \pi r^2$ and $\dfrac{dr}{dt} = -5/(2\pi)$, so $\dfrac{dA}{dt} = \dfrac{dA}{dr}\dfrac{dr}{dt} = 2\pi r(-5/2\pi) = -250$, so the area is shrinking at a rate of 250 m^2/min.

61. (a) $\Delta x = 1.5 - 2 = -0.5$; $dy = \dfrac{-1}{(x-1)^2}\Delta x = \dfrac{-1}{(2-1)^2}(-0.5) = 0.5$; and $\Delta y = \dfrac{1}{(1.5-1)} - \dfrac{1}{(2-1)} = 2 - 1 = 1$.

(b) $\Delta x = 0 - (-\pi/4) = \pi/4$; $dy = \left(\sec^2(-\pi/4)\right)(\pi/4) = \pi/2$; and $\Delta y = \tan 0 - \tan(-\pi/4) = 1$.

(c) $\Delta x = 3 - 0 = 3$; $dy = \dfrac{-x}{\sqrt{25 - x^2}} = \dfrac{-0}{\sqrt{25 - (0)^2}}(3) = 0$; and $\Delta y = \sqrt{25 - 3^2} - \sqrt{25 - 0^2} = 4 - 5 = -1$.

63. (a) $h = 115 \tan\phi$, $dh = 115 \sec^2\phi\, d\phi$; with $\phi = 51° = \dfrac{51}{180}\pi$ radians and $d\phi = \pm 0.5° = \pm 0.5\left(\dfrac{\pi}{180}\right)$ radians, $h \pm dh = 115(1.2349) \pm 2.5340 = 142.0135 \pm 2.5340$, so the height lies between 139.48 m and 144.55 m.

(b) If $|dh| \le 5$ then $|d\phi| \le \dfrac{5}{115}\cos^2\dfrac{51}{180}\pi \approx 0.017$ radian, or $|d\phi| \le 0.98°$.

Chapter 3 Making Connections

1. (a) If $t > 0$ then $A(-t)$ is the amount K there was t time-units ago in order that there be 1 unit now, i.e. $K \cdot A(t) = 1$, so $K = \dfrac{1}{A(t)}$. But, as said above, $K = A(-t)$. So $A(-t) = \dfrac{1}{A(t)}$.

(b) If s and t are positive, then the amount 1 becomes $A(s)$ after s seconds, and that in turn is $A(s)A(t)$ after another t seconds, i.e. 1 becomes $A(s)A(t)$ after $s + t$ seconds. But this amount is also $A(s + t)$, so $A(s)A(t) = A(s+t)$. Now if $0 \le -s \le t$ then $A(-s)A(s+t) = A(t)$. From the first case, we get $A(s+t) = A(s)A(t)$. If $0 \le t \le -s$ then $A(s + t) = \dfrac{1}{A(-s-t)} = \dfrac{1}{A(-s)A(-t)} = A(s)A(t)$ by the previous cases. If s and t are both negative then by the first case, $A(s + t) = \dfrac{1}{A(-s-t)} = \dfrac{1}{A(-s)A(-t)} = A(s)A(t)$.

(c) If $n > 0$ then $A\left(\dfrac{1}{n}\right)A\left(\dfrac{1}{n}\right)\ldots A\left(\dfrac{1}{n}\right) = A\left(n\dfrac{1}{n}\right) = A(1)$, so $A\left(\dfrac{1}{n}\right) = A(1)^{1/n} = b^{1/n}$ from part (b). If $n < 0$ then by part (a), $A\left(\dfrac{1}{n}\right) = \dfrac{1}{A\left(-\frac{1}{n}\right)} = \dfrac{1}{A(1)^{-1/n}} = A(1)^{1/n} = b^{1/n}$.

(d) Let m, n be integers. Assume $n \ne 0$ and $m > 0$. Then $A\left(\dfrac{m}{n}\right) = A\left(\dfrac{1}{n}\right)^m = A(1)^{m/n} = b^{m/n}$.

(e) If f, g are continuous functions of t and f and g are equal on the rational numbers $\left\{\dfrac{m}{n} : n \ne 0\right\}$, then $f(t) = g(t)$ for all t. Because if x is irrational, then let t_n be a sequence of rational numbers which converges to x. Then for all $n > 0, f(t_n) = g(t_n)$ and thus $f(x) = \lim\limits_{n\to+\infty} f(t_n) = \lim\limits_{n\to+\infty} g(t_n) = g(x)$.