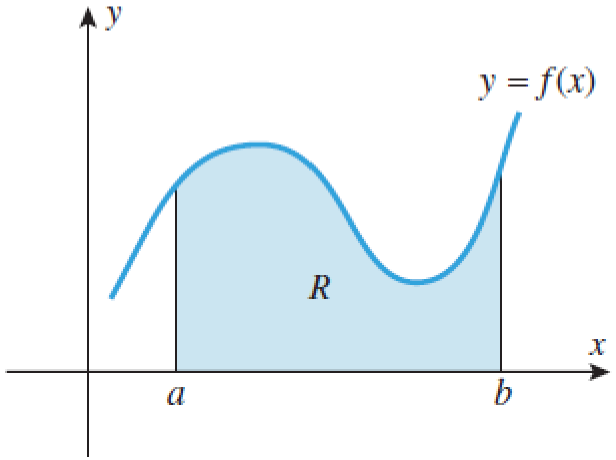
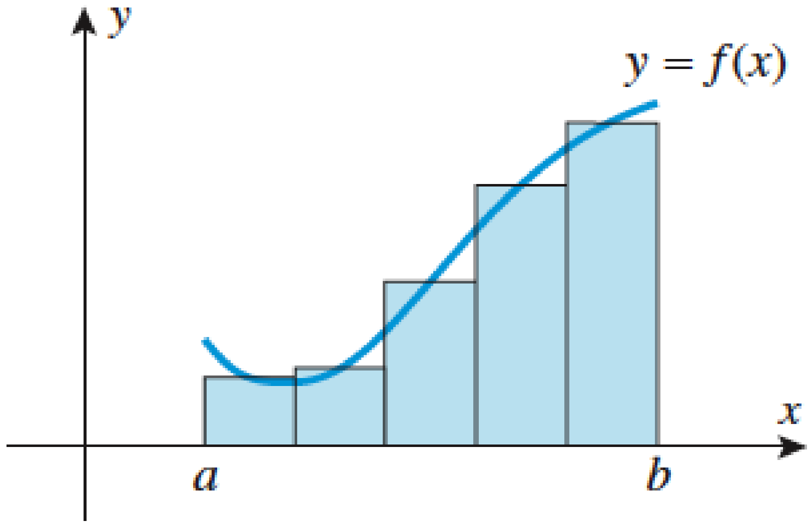
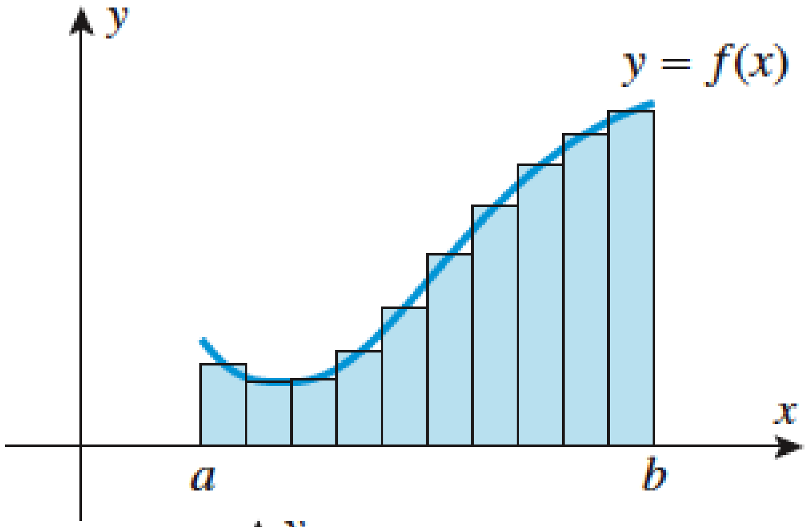
# *Riemann Sums + Definite Integral*

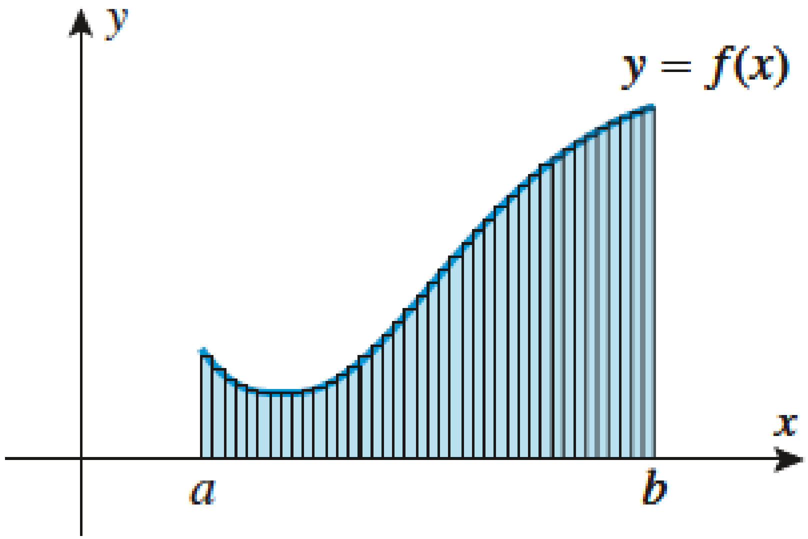
**5.1.1 THE AREA PROBLEM** Given a function $f$ that is continuous and nonnegative on an interval $[a, b]$, find the area between the graph of $f$ and the interval $[a, b]$ on the $x$-axis (Figure 5.1.2).

## 5.1.1 The Area Problem




y
y = f(x)
x
a
b


y
y = f(x)
x
a
b

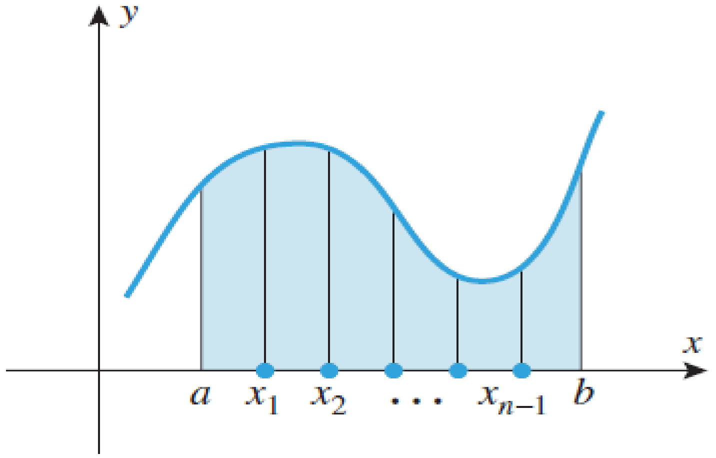

y
y = f(x)
x
a
b

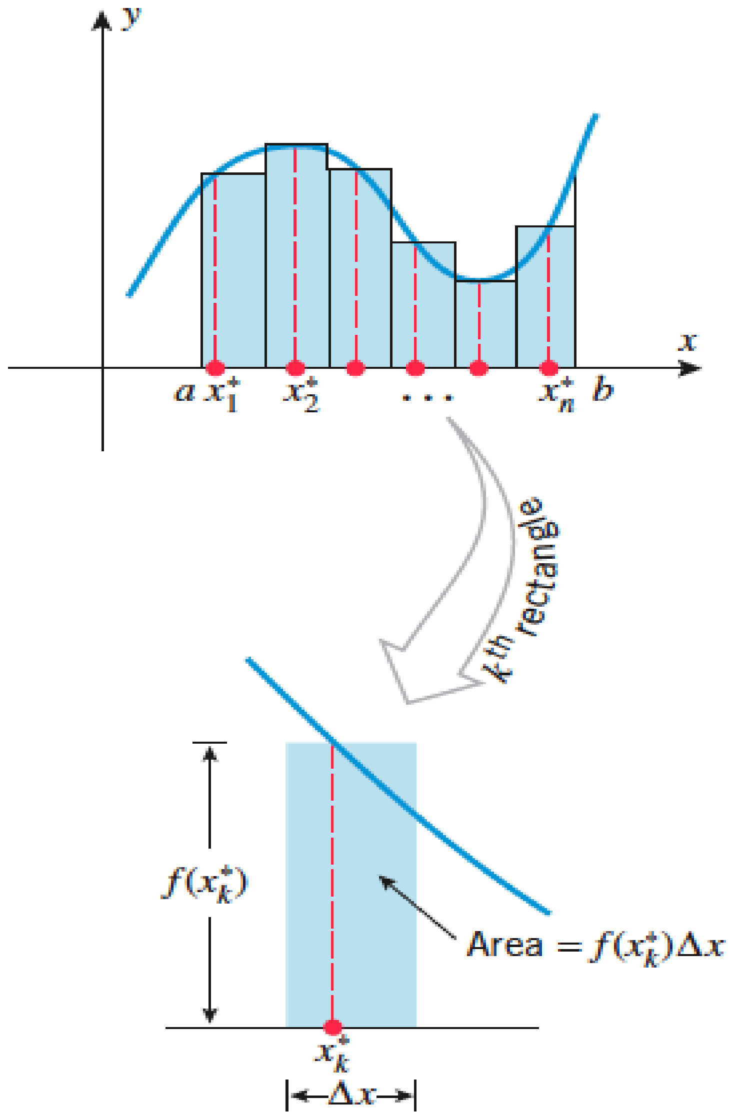
**5.4.2 THEOREM**

(a) $$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

(b) $$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(c) $$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$


y
x
a
$x_1$
$x_2$
. . .
$x_{n-1}$
b


y
x
a
$x_1^*$
$x_2^*$
. . .
$x_n^*$
b
$k^{th}$ rectangle
$f(x_k^*)$
Area $= f(x_k^*)\Delta x$
$x_k^*$
$\Delta x$

# Area Under a Curve

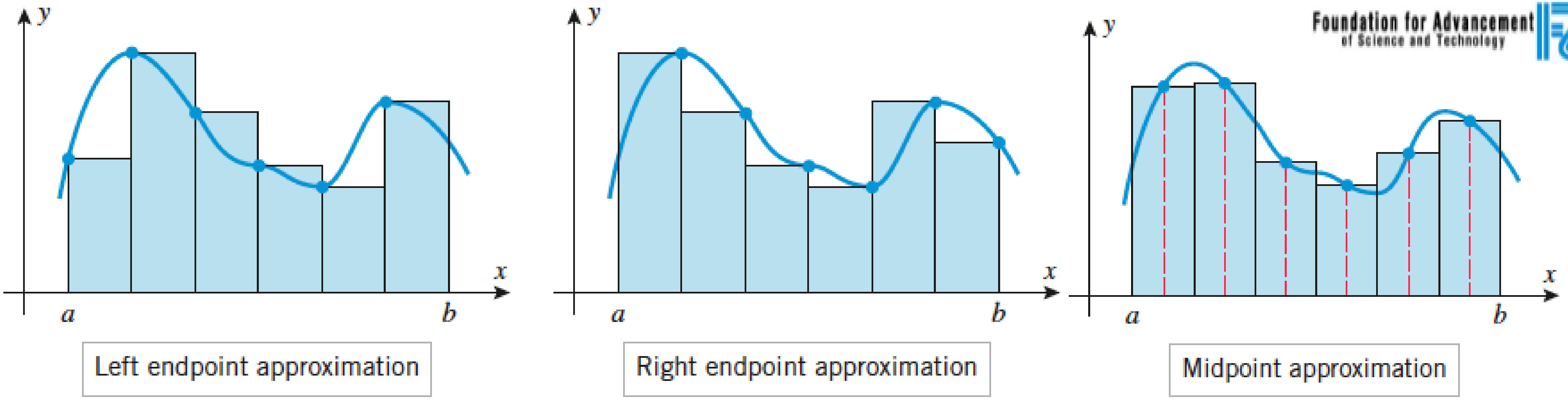
**5.4.3 DEFINITION (*Area Under a Curve*)** If the function $f$ is continuous on $[a, b]$ and if $f(x) \geq 0$ for all $x$ in $[a, b]$, then the ***area*** $A$ under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \tag{2}$$

**5.4.4 THEOREM**

(a) $\displaystyle \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} 1 = 1$  (b) $\displaystyle \lim_{n \to +\infty} \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{1}{2}$

(c) $\displaystyle \lim_{n \to +\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{1}{3}$  (d) $\displaystyle \lim_{n \to +\infty} \frac{1}{n^4} \sum_{k=1}^{n} k^3 = \frac{1}{4}$



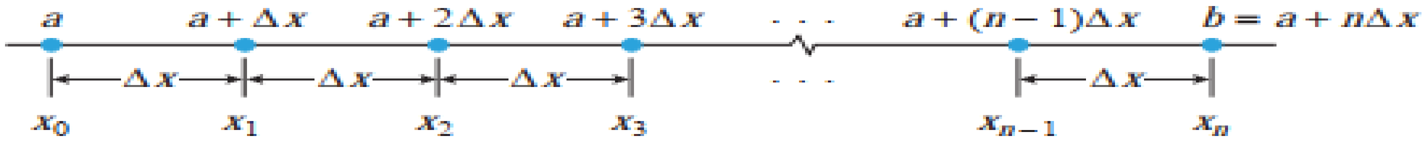

(a) (b) (c)

Thus, the left endpoint, right endpoint, and midpoint choices for $x_1^*, x_2^*, \ldots, x_n^*$ are given by

$$x_k^* = x_{k-1} = a + (k-1)\Delta x \qquad \boxed{\text{Left endpoint}} \qquad (3)$$

$$x_k^* = x_k = a + k\Delta x \qquad \boxed{\text{Right endpoint}} \qquad (4)$$

$$x_k^* = \tfrac{1}{2}(x_{k-1} + x_k) = a + \left(k - \tfrac{1}{2}\right)\Delta x \qquad \boxed{\text{Midpoint}} \qquad (5)$$




► **Figure 5.4.6**

► **Example 4** Use Definition 5.4.3 with $x_k^*$ as the right endpoint of each subinterval to find the area between the graph of $f(x) = x^2$ and the interval $[0, 1]$.

***Solution.*** The length of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

so it follows from (4) that

$$x_k^* = a + k\Delta x = \frac{k}{n}$$

Thus,

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n} (x_k^*)^2 \Delta x = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} k^2$$

$$= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \qquad \boxed{\text{Part } (b) \text{ of Theorem 5.4.2}}$$

$$= \frac{1}{6}\left(\frac{n}{n} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n}\right) = \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

from which it follows that

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n \to +\infty} \left[\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = \frac{1}{3}$$

▶ **Example 5** Use Definition 4.4.3 with $x_k^*$ as the midpoint of each subinterval to find the area under the parabola $y = f(x) = 9 - x^2$ and over the interval $[0, 3]$.

***Solution.*** Each subinterval has length

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

so it follows from (5) that

$$x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x = \left(k - \frac{1}{2}\right)\left(\frac{3}{n}\right)$$

Thus,

$$\begin{aligned} f(x_k^*)\Delta x = [9 - (x_k^*)^2]\Delta x &= \left[9 - \left(k - \frac{1}{2}\right)^2\left(\frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right) \\ &= \left[9 - \left(k^2 - k + \frac{1}{4}\right)\left(\frac{9}{n^2}\right)\right]\left(\frac{3}{n}\right) \\ &= \frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3} \end{aligned}$$

from which it follows that

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$= \lim_{n \to +\infty} \sum_{k=1}^{n} \left( \frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3} \right)$$

$$= 27\left[1 - \frac{1}{3} + 0 \cdot \frac{1}{2} - 0 \cdot 1\right] = 18$$ Theorem 4.4.4 ◀

**35–40** Use Definition 4.4.3 with $x_k^*$ as the *right* endpoint of each subinterval to find the area under the curve $y = f(x)$ over the specified interval. ■

**35.** $f(x) = x/2;\ [1, 4]$

**36.** $f(x) = 5 - x;\ [0, 5]$

**37.** $f(x) = 9 - x^2;\ [0, 3]$

**38.** $f(x) = 4 - \frac{1}{4}x^2;\ [0, 3]$

**39.** $f(x) = x^3;\ [2, 6]$

**40.** $f(x) = 1 - x^3;\ [-3, -1]$

**41–44** Use Definition 4.4.3 with $x_k^*$ as the *left* endpoint of each subinterval to find the area under the curve $y = f(x)$ over the specified interval. ■

**41.** $f(x) = x/2;\ [1, 4]$

**42.** $f(x) = 5 - x;\ [0, 5]$

**43.** $f(x) = 9 - x^2;\ [0, 3]$

**44.** $f(x) = 4 - \frac{1}{4}x^2;\ [0, 3]$

**45–48** Use Definition 4.4.3 with $x_k^*$ as the *midpoint* of each subinterval to find the area under the curve $y = f(x)$ over the specified interval. ■

**45.** $f(x) = 2x;\ [0, 4]$

**46.** $f(x) = 6 - x;\ [1, 5]$

**47.** $f(x) = x^2;\ [0, 1]$

**48.** $f(x) = x^2;\ [-1, 1]$

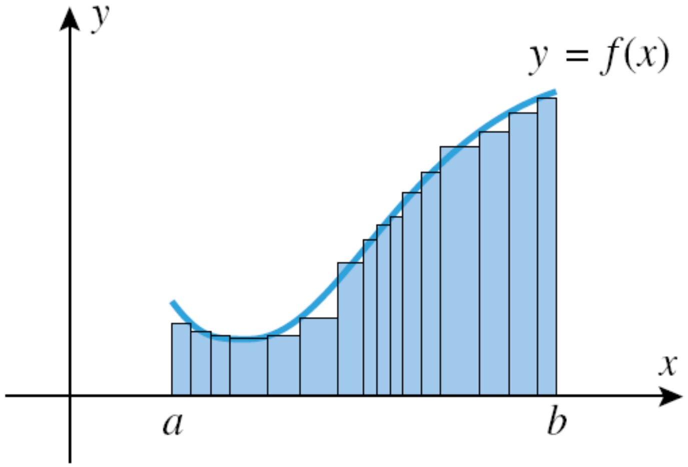
**5.5.1 DEFINITION** A function $f$ is said to be ***integrable*** on a finite closed interval $[a, b]$ if the limit

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

exists and does not depend on the choice of partitions or on the choice of the points $x_k^*$ in the subintervals. When this is the case we denote the limit by the symbol

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

which is called the ***definite integral*** of $f$ from $a$ to $b$. The numbers $a$ and $b$ are called the ***lower limit of integration*** and the ***upper limit of integration***, respectively, and $f(x)$ is called the ***integrand***.



# Do Questions (35-48) from Ex # 4.4

**5.5.1 DEFINITION** A function $f$ is said to be ***integrable*** on a finite closed interval $[a, b]$ if the limit

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