National University of computer and emerging sciences

Foundation for Advancement of Science and Technology

Ex#9.5

Comparison, Ratio & Root Test

Comparison Test

In the comparison tests the idea is to compare a given series with a series that is known to be convergent or divergent. For instance, the series

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{2^n + 1}}$$

Comparison Test

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

reminds us of the series $\sum_{n=1}^{\infty} 1/2^n$, which is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$ and is therefore convergent. Because the series $\boxed{1}$ is so similar to a convergent series, we have the feeling that it too must be convergent. Indeed, it is. The inequality

$$\frac{1}{2^n + 1} < \frac{1}{2^n}$$

Comparison Test

The Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\sum a_n$ is also divergent.

Comparison Test

Step 1. Guess at whether the series $\sum u_k$ converges or diverges.

Step 2. Find a series that proves the guess to be correct. That is, if we guess that $\sum u_k$ diverges, we must find a divergent series whose terms are "smaller" than the corresponding terms of $\sum u_k$, and if we guess that $\sum u_k$ converges, we must find a convergent series whose terms are "bigger" than the corresponding terms of $\sum u_k$.

9.5.2 INFORMAL PRINCIPLE *Constant terms in the denominator of u_k can usually be deleted without affecting the convergence or divergence of the series.*

9.5.3 INFORMAL PRINCIPLE *If a polynomial in k appears as a factor in the numerator or denominator of u_k, all but the leading term in the polynomial can usually be discarded without affecting the convergence or divergence of the series.*

Determine whether the series $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

SOLUTION For large n the dominant term in the denominator is $2n^2$, so we compare the given series with the series $\sum 5/(2n^2)$. Observe that

$$\frac{5}{2n^2 + 4n + 3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator. (In the notation of the Comparison Test, a_n is the left side and b_n is the right side.) We know that

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it's a constant times a p-series with $p = 2 > 1$. Therefore

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

is convergent by part (i) of the Comparison Test.

▶ Example 1 Use the comparison test to determine whether the following series converge or diverge.

$$\text{(a)} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k} - \frac{1}{2}} \qquad \text{(b)} \sum_{k=1}^{\infty} \frac{1}{2k^2 + k}$$

Solution (a). According to Principle 9.5.2, we should be able to drop the constant in the denominator without affecting the convergence or divergence. Thus, the given series is likely to behave like

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \tag{1}$$

which is a divergent p-series $\left(p = \frac{1}{2}\right)$. Thus, we will guess that the given series diverges and try to prove this by finding a divergent series that is "smaller" than the given series. However, series (1) does the trick since

$$\frac{1}{\sqrt{k} - \frac{1}{2}} > \frac{1}{\sqrt{k}} \quad \text{for } k = 1, 2, \ldots$$

Thus, we have proved that the given series diverges.

Solution (b). According to Principle 9.5.3, we should be able to discard all but the leading term in the polynomial without affecting the convergence or divergence. Thus, the given series is likely to behave like

Solution (b). According to Principle 9.5.3, we should be able to discard all but the leading term in the polynomial without affecting the convergence or divergence. Thus, the given series is likely to behave like

$$\sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^2} \tag{2}$$

which converges since it is a constant times a convergent p-series ($p = 2$). Thus, we will guess that the given series converges and try to prove this by finding a convergent series that is "bigger" than the given series. However, series (2) does the trick since

$$\frac{1}{2k^2 + k} < \frac{1}{2k^2} \quad \text{for } k = 1, 2, \ldots$$

Thus, we have proved that the given series converges. ◀

1–2 Make a guess about the convergence or divergence of the series, and confirm your guess using the comparison test.

1. (a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5k^2 - k}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{3}{k - \frac{1}{4}}$

2. (a) $\displaystyle\sum_{k=2}^{\infty} \frac{k+1}{k^2 - k}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k^4 + k}$

1. (a) $\dfrac{1}{5k^2 - k} \le \dfrac{1}{5k^2 - k^2} = \dfrac{1}{4k^2}$, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{4k^2}$ converges, so the original series also converges.

(b) $\dfrac{3}{k - 1/4} > \dfrac{3}{k}$, $\displaystyle\sum_{k=1}^{\infty} \frac{3}{k}$ diverges, so the original series also diverges.

2. (a) $\dfrac{k+1}{k^2 - k} > \dfrac{k}{k^2} = \dfrac{1}{k}$, $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k}$ diverges, so the original series also diverges.

(b) $\dfrac{2}{k^4 + k} < \dfrac{2}{k^4}$, $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k^4}$ converges, so the original series also converges.

THE LIMIT COMPARISON TEST

9.5.4 THEOREM (*The Limit Comparison Test*) *Let $\sum a_k$ and $\sum b_k$ be series with positive terms and suppose that*

$$\rho = \lim_{k \to +\infty} \frac{a_k}{b_k}$$

If ρ is finite and $\rho > 0$, then the series both converge or both diverge.

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

$$a_n = \frac{1}{2^n - 1} \qquad b_n = \frac{1}{2^n}$$

and obtain

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1/(2^n - 1)}{1/2^n} = \lim_{n \to \infty} \frac{2^n}{2^n - 1} = \lim_{n \to \infty} \frac{1}{1 - 1/2^n} = 1 > 0$$

Since this limit exists and $\sum 1/2^n$ is a convergent geometric series, the given series converges by the Limit Comparison Test.

▶ **Example 2** Use the limit comparison test to determine whether the following series converge or diverge.

$$\text{(a)} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1} \qquad \text{(b)} \sum_{k=1}^{\infty} \frac{1}{2k^2+k} \qquad \text{(c)} \sum_{k=1}^{\infty} \frac{3k^3-2k^2+4}{k^7-k^3+2}$$

Solution (a). As in Example 1, Principle 9.5.2 suggests that the series is likely to behave like the divergent p-series (1). To prove that the given series diverges, we will apply the limit comparison test with

$$a_k = \frac{1}{\sqrt{k}+1} \quad \text{and} \quad b_k = \frac{1}{\sqrt{k}}$$

We obtain

$$\rho = \lim_{k\to+\infty} \frac{a_k}{b_k} = \lim_{k\to+\infty} \frac{\sqrt{k}}{\sqrt{k}+1} = \lim_{k\to+\infty} \frac{1}{1+\dfrac{1}{\sqrt{k}}} = 1$$

Since ρ is finite and positive, it follows from Theorem 9.5.4 that the given series diverges.

Solution (b). As in Example 1, Principle 9.5.3 suggests that the series is likely to behave like the convergent series (2). To prove that the given series converges, we will apply the limit comparison test with

$$a_k = \frac{1}{2k^2 + k} \quad \text{and} \quad b_k = \frac{1}{2k^2}$$

We obtain

$$\rho = \lim_{k \to +\infty} \frac{a_k}{b_k} = \lim_{k \to +\infty} \frac{2k^2}{2k^2 + k} = \lim_{k \to +\infty} \frac{2}{2 + \dfrac{1}{k}} = 1$$

Since ρ is finite and positive, it follows from Theorem 9.5.4 that the given series converges, which agrees with the conclusion reached in Example 1 using the comparison test.

Solution (c). From Principle 9.5.3, the series is likely to behave like

$$\sum_{k=1}^{\infty} \frac{3k^3}{k^7} = \sum_{k=1}^{\infty} \frac{3}{k^4} \tag{3}$$

which converges since it is a constant times a convergent p-series. Thus, the given series is likely to converge. To prove this, we will apply the limit comparison test to series (3) and the given series. We obtain

$$\rho = \lim_{k \to +\infty} \frac{\dfrac{3k^3 - 2k^2 + 4}{k^7 - k^3 + 2}}{\dfrac{3}{k^4}} = \lim_{k \to +\infty} \frac{3k^7 - 2k^6 + 4k^4}{3k^7 - 3k^3 + 6} = 1$$

Since ρ is finite and nonzero, it follows from Theorem 9.5.4 that the given series converges, since (3) converges. ◀

5–10 Use the limit comparison test to determine whether the series converges.

5. $\sum_{k=1}^{\infty} \frac{4k^2 - 2k + 6}{8k^7 + k - 8}$

6. $\sum_{k=1}^{\infty} \frac{1}{9k + 6}$

7. $\sum_{k=1}^{\infty} \frac{5}{3^k + 1}$

8. $\sum_{k=1}^{\infty} \frac{k(k + 3)}{(k + 1)(k + 2)(k + 5)}$

9. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{8k^2 - 3k}}$

10. $\sum_{k=1}^{\infty} \frac{1}{(2k + 3)^{17}}$

5. Compare with the convergent series $\sum_{k=1}^{\infty} \frac{1}{k^5}$, $\rho = \lim_{k\to+\infty} \frac{4k^7 - 2k^6 + 6k^5}{8k^7 + k - 8} = 1/2$, which is finite and positive, therefore the original series converges.

6. Compare with the divergent series $\sum_{k=1}^{\infty} \frac{1}{k}$, $\rho = \lim_{k\to+\infty} \frac{k}{9k+6} = 1/9$, which is finite and positive, therefore the original series diverges.

7. Compare with the convergent series $\sum_{k=1}^{\infty} \frac{5}{3^k}$, $\rho = \lim_{k\to+\infty} \frac{3^k}{3^k+1} = 1$, which is finite and positive, therefore the original series converges.

8. Compare with the divergent series $\sum_{k=1}^{\infty} \frac{1}{k}$, $\rho = \lim_{k\to+\infty} \frac{k^2(k+3)}{(k+1)(k+2)(k+5)} = 1$, which is finite and positive, therefore the original series diverges.

9. Compare with the divergent series $\sum_{k=1}^{\infty} \frac{1}{k^{2/3}}$, $\rho = \lim_{k\to+\infty} \frac{k^{2/3}}{(8k^2-3k)^{1/3}} = \lim_{k\to+\infty} \frac{1}{(8-3/k)^{1/3}} = 1/2$, which is finite and positive, therefore the original series diverges.

10. Compare with the convergent series $\sum_{k=1}^{\infty} \frac{1}{k^{17}}$, $\rho = \lim_{k\to+\infty} \frac{k^{17}}{(2k+3)^{17}} = \lim_{k\to+\infty} \frac{1}{(2+3/k)^{17}} = 1/2^{17}$, which is finite and positive, therefore the original series converges.

THE RATIO TEST

9.5.5 THEOREM (*The Ratio Test*) *Let* $\sum u_k$ *be a series with positive terms and suppose that*

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k}$$

(a) *If* $\rho < 1$, *the series converges.*

(b) *If* $\rho > 1$ *or* $\rho = +\infty$, *the series diverges.*

(c) *If* $\rho = 1$, *the series may converge or diverge, so that another test must be tried.*

▶ **Example 3** Each of the following series has positive terms, so the ratio test applies. In each part, use the ratio test to determine whether the following series converge or diverge.

$$\text{(a)} \sum_{k=1}^{\infty} \frac{1}{k!} \qquad \text{(b)} \sum_{k=1}^{\infty} \frac{k}{2^k} \qquad \text{(c)} \sum_{k=1}^{\infty} \frac{k^k}{k!} \qquad \text{(d)} \sum_{k=3}^{\infty} \frac{(2k)!}{4^k} \qquad \text{(e)} \sum_{k=1}^{\infty} \frac{1}{2k-1}$$

Solution (a). The series converges, since

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \to +\infty} \frac{1/(k+1)!}{1/k!} = \lim_{k \to +\infty} \frac{k!}{(k+1)!} = \lim_{k \to +\infty} \frac{1}{k+1} = 0 < 1$$

Solution (b). The series converges, since

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \to +\infty} \frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k} = \frac{1}{2} \lim_{k \to +\infty} \frac{k+1}{k} = \frac{1}{2} < 1$$

Solution (c). The series diverges, since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k} = \lim_{k\to+\infty} \frac{(k+1)^k}{k^k} = \lim_{k\to+\infty} \left(1+\frac{1}{k}\right)^k = e > 1$$

See Formula (4) of Section 6.1

Solution (d). The series diverges, since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{[2(k+1)]!}{4^{k+1}} \cdot \frac{4^k}{(2k)!} = \lim_{k\to+\infty} \left(\frac{(2k+2)!}{(2k)!} \cdot \frac{1}{4}\right)$$

$$= \lim_{k\to+\infty} \left(\frac{(2k+2)(2k+1)(2k)!}{(2k)!} \cdot \frac{1}{4}\right) = \frac{1}{4} \lim_{k\to+\infty} (2k+2)(2k+1) = +\infty$$

Solution (e). The ratio test is of no help since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{1}{2(k+1)-1} \cdot \frac{2k-1}{1} = \lim_{k\to+\infty} \frac{2k-1}{2k+1} = 1$$

However, the integral test proves that the series diverges since

$$\int_1^{+\infty} \frac{dx}{2x-1} = \lim_{b\to+\infty} \int_1^b \frac{dx}{2x-1} = \lim_{b\to+\infty} \frac{1}{2}\ln(2x-1)\Big]_1^b = +\infty$$

Both the comparison test and the limit comparison test would also have worked here (verify). ◀

Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$

Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$

We use the Ratio Test with $a_n = (-1)^n n^3/3^n$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\dfrac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$

Converging

Practice Examples on Page # 634 & 635

The Root Test

9.5.6 THEOREM (*The Root Test*) *Let $\sum u_k$ be a series with positive terms and suppose that*

$$\rho = \lim_{k \to +\infty} \sqrt[k]{u_k} = \lim_{k \to +\infty} (u_k)^{1/k}$$

(*a*) *If $\rho < 1$, the series converges.*

(*b*) *If $\rho > 1$ or $\rho = +\infty$, the series diverges.*

(*c*) *If $\rho = 1$, the series may converge or diverge, so that another test must be tried.*

Test the convergence of the series $\sum_{n=1}^{\infty} \left(\dfrac{2n+3}{3n+2} \right)^n$.

$$a_n = \left(\frac{2n+3}{3n+2} \right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2 + \dfrac{3}{n}}{3 + \dfrac{2}{n}} \to \frac{2}{3} < 1$$

Thus the given series converges by the Root Test.

▶ **Example 4** Use the root test to determine whether the following series converge or diverge.

$$\text{(a)}\ \sum_{k=2}^{\infty}\left(\frac{4k-5}{2k+1}\right)^k \qquad \text{(b)}\ \sum_{k=1}^{\infty}\frac{1}{(\ln(k+1))^k}$$

Solution (a). The series diverges, since

$$\rho = \lim_{k\to+\infty}(u_k)^{1/k} = \lim_{k\to+\infty}\frac{4k-5}{2k+1} = 2 > 1$$

Solution (b). The series converges, since

$$\rho = \lim_{k\to+\infty}(u_k)^{1/k} = \lim_{k\to+\infty}\frac{1}{\ln(k+1)} = 0 < 1 \blacktriangleleft$$

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

$$\text{(i)} \quad b_{n+1} \leqslant b_n \qquad \text{for all } n$$

$$\text{(ii)} \quad \lim_{n \to \infty} b_n = 0$$

then the series is convergent.

1 **Definition** A series $\Sigma\, a_n$ is called **absolutely convergent** if the series of absolute values $\Sigma\, |a_n|$ is convergent.

2 **Definition** A series $\Sigma\, a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

d) Is the given series convergent or divergent? If it is convergent, its it absolutely convergent or conditionally convergent?

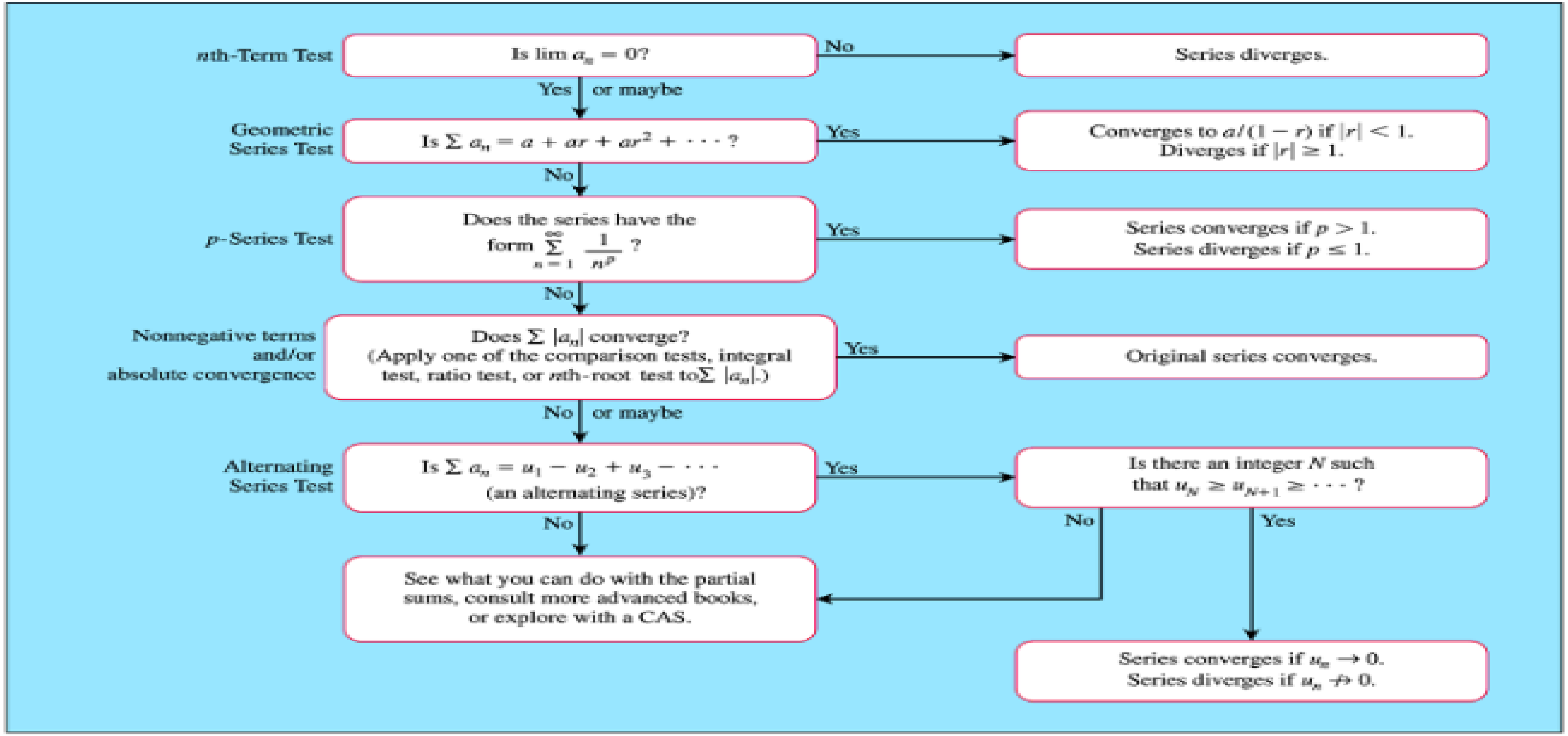
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = -\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}}$$

Converges by the alternating series test.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}}$$

Diverges since it is a p-series with $p < 1$. The Given series is conditionally convergent.

Procedure for determining Convergence



Do Q. 25 - 49 from Ex # 9.5

11–16 Use the ratio test to determine whether the series converges. If the test is inconclusive, then say so.

11. $\sum_{k=1}^{\infty} \frac{3^k}{k!}$ **12.** $\sum_{k=1}^{\infty} \frac{4^k}{k^2}$ **13.** $\sum_{k=1}^{\infty} \frac{1}{5k}$

14. $\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k$ **15.** $\sum_{k=1}^{\infty} \frac{k!}{k^3}$ **16.** $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$

17–20 Use the root test to determine whether the series converges. If the test is inconclusive, then say so.

17. $\sum_{k=1}^{\infty} \left(\frac{3k+2}{2k-1}\right)^k$ **18.** $\sum_{k=1}^{\infty} \left(\frac{k}{100}\right)^k$

19. $\sum_{k=1}^{\infty} \frac{k}{5^k}$ **20.** $\sum_{k=1}^{\infty} (1-e^{-k})^k$

11. $\rho = \lim_{k\to+\infty} \frac{3^{k+1}/(k+1)!}{3^k/k!} = \lim_{k\to+\infty} \frac{3}{k+1} = 0 < 1$, the series converges.

12. $\rho = \lim_{k\to+\infty} \frac{4^{k+1}/(k+1)^2}{4^k/k^2} = \lim_{k\to+\infty} \frac{4k^2}{(k+1)^2} = 4 > 1$, the series diverges.

13. $\rho = \lim_{k\to+\infty} \frac{k}{k+1} = 1$, the result is inconclusive.

14. $\rho = \lim_{k\to+\infty} \frac{(k+1)(1/2)^{k+1}}{k(1/2)^k} = \lim_{k\to+\infty} \frac{k+1}{2k} = 1/2 < 1$, the series converges.

15. $\rho = \lim_{k\to+\infty} \frac{(k+1)!/(k+1)^3}{k!/k^3} = \lim_{k\to+\infty} \frac{k^3}{(k+1)^2} = +\infty$, the series diverges.

16. $\rho = \lim_{k\to+\infty} \frac{(k+1)/[(k+1)^2+1]}{k/(k^2+1)} = \lim_{k\to+\infty} \frac{(k+1)(k^2+1)}{k(k^2+2k+2)} = 1$, the result is inconclusive.

17. $\rho = \lim_{k\to+\infty} \frac{3k+2}{2k-1} = 3/2 > 1$, the series diverges.

18. $\rho = \lim_{k\to+\infty} k/100 = +\infty$, the series diverges.

19. $\rho = \lim_{k\to+\infty} \frac{k^{1/k}}{5} = 1/5 < 1$, the series converges.

25–49 Use any method to determine whether the series converges.

25. $\sum_{k=0}^{\infty} \frac{7^k}{k!}$

26. $\sum_{k=1}^{\infty} \frac{1}{2k+1}$

27. $\sum_{k=1}^{\infty} \frac{k^2}{5^k}$

28. $\sum_{k=1}^{\infty} \frac{k!\,10^k}{3^k}$

29. $\sum_{k=1}^{\infty} k^{50} e^{-k}$

30. $\sum_{k=1}^{\infty} \frac{k^2}{k^3+1}$

31. $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^3+1}$

32. $\sum_{k=1}^{\infty} \frac{4}{2+3^k k}$

33. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k+1)}}$

34. $\sum_{k=1}^{\infty} \frac{2+(-1)^k}{5^k}$

35. $\sum_{k=1}^{\infty} \frac{2+\sqrt{k}}{(k+1)^3-1}$

36. $\sum_{k=1}^{\infty} \frac{4+|\cos x|}{k^3}$

37. $\sum_{k=1}^{\infty} \frac{1}{1+\sqrt{k}}$

38. $\sum_{k=1}^{\infty} \frac{k!}{k^k}$

39. $\sum_{k=1}^{\infty} \frac{\ln k}{e^k}$

40. $\sum_{k=1}^{\infty} \frac{k!}{e^{k^2}}$

41. $\sum_{k=0}^{\infty} \frac{(k+4)!}{4!\,k!\,4^k}$

42. $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}$

43. $\sum_{k=1}^{\infty} \frac{1}{4+2^{-k}}$

44. $\sum_{k=1}^{\infty} \frac{\sqrt{k}\ln k}{k^3+1}$

45. $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{k^2}$

46. $\sum_{k=1}^{\infty} \frac{5^k+k}{k!+3}$

47. $\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}$

48. $\sum_{k=1}^{\infty} \frac{[\pi(k+1)]^k}{k^{k+1}}$

49. $\sum_{k=1}^{\infty} \frac{\ln k}{3^k}$

25. Ratio Test, $\rho = \lim_{k\to+\infty} 7/(k+1) = 0$, converges.

26. Limit Comparison Test, compare with the divergent series $\sum_{k=1}^{\infty} 1/k$, $\rho = \lim_{k\to+\infty} \dfrac{k}{2k+1} = 1/2$, which is finite and positive, therefore the original series diverges.

27. Ratio Test, $\rho = \lim_{k\to+\infty} \dfrac{(k+1)^2}{5k^2} = 1/5 < 1$, converges.

28. Ratio Test, $\rho = \lim_{k\to+\infty} (10/3)(k+1) = +\infty$, diverges.

29. Ratio Test, $\rho = \lim_{k\to+\infty} e^{-1}(k+1)^{50}/k^{50} = e^{-1} < 1$, converges.

30. Limit Comparison Test, compare with the divergent series $\sum_{k=1}^{\infty} 1/k$.

31. Limit Comparison Test, compare with the convergent series $\sum_{k=1}^{\infty} 1/k^{5/2}$, $\rho = \lim_{k\to+\infty} \dfrac{k^3}{k^3+1} = 1$, converges.

32. $\dfrac{4}{2+3^k k} < \dfrac{4}{3^k k}$, $\sum_{k=1}^{\infty} \dfrac{4}{3^k k}$ converges (Ratio Test) so $\sum_{k=1}^{\infty} \dfrac{4}{2+k3^k}$ converges by the Comparison Test.

33. Limit Comparison Test, compare with the divergent series $\sum_{k=1}^{\infty} 1/k$, $\rho = \lim_{k\to+\infty} \dfrac{k}{\sqrt{k^2+k}} = 1$, diverges.

34. $\dfrac{2+(-1)^k}{5^k} \le \dfrac{3}{5^k}$, $\sum_{k=1}^{\infty} 3/5^k$ converges so $\sum_{k=1}^{\infty} \dfrac{2+(-1)^k}{5^k}$ converges by the Comparison Test.

35. Limit Comparison Test, compare with the convergent series $\sum_{k=1}^{\infty} \dfrac{1}{k^{5/2}}$, $\rho = \lim_{k\to+\infty} \dfrac{k^3+2k^{5/2}}{k^3+3k^2+3k} = 1$, converges.

36. $\dfrac{4+|\cos x|}{k^3} < \dfrac{5}{k^3}$, $\sum_{k=1}^{\infty} 5/k^3$ converges so $\sum_{k=1}^{\infty} \dfrac{4+|\cos x|}{k^3}$ converges.

37. Limit Comparison Test, compare with the divergent series $\sum_{k=1}^{\infty} 1/\sqrt{k}$.

38. Ratio Test, $\rho = \lim_{k\to+\infty} (1+1/k)^{-k} = 1/e < 1$, converges.

39. Ratio Test, $\rho = \lim_{k\to+\infty} \frac{\ln(k+1)}{e\ln k} = \lim_{k\to+\infty} \frac{k}{e(k+1)} = 1/e < 1$, converges.

40. Ratio Test, $\rho = \lim_{k\to+\infty} \frac{k+1}{e^{2k+1}} = \lim_{k\to+\infty} \frac{1}{2e^{2k+1}} = 0$, converges.

41. Ratio Test, $\rho = \lim_{k\to+\infty} \frac{k+5}{4(k+1)} = 1/4$, converges.

42. Root Test, $\rho = \lim_{k\to+\infty} \left(\frac{k}{k+1}\right)^k = \lim_{k\to+\infty} \frac{1}{(1+1/k)^k} = 1/e$, converges.

43. Diverges by the Divergence Test, because $\lim_{k\to+\infty} \frac{1}{4+2^{-k}} = 1/4 \neq 0$.

44. $\sum_{k=1}^{\infty} \frac{\sqrt{k}\ln k}{k^3+1} = \sum_{k=2}^{\infty} \frac{\sqrt{k}\ln k}{k^3+1}$ because $\ln 1 = 0$, $\frac{\sqrt{k}\ln k}{k^3+1} < \frac{k\ln k}{k^3} = \frac{\ln k}{k^2}$, $\int_2^{+\infty} \frac{\ln x}{x^2}dx = \lim_{\ell\to+\infty} \left(-\frac{\ln x}{x} - \frac{1}{x}\right)\Big]_2^{\ell} = \frac{1}{2}(\ln 2 + 1)$, so $\sum_{k=2}^{\infty} \frac{\ln k}{k^2}$ converges and so does $\sum_{k=1}^{\infty} \frac{\sqrt{k}\ln k}{k^3+1}$.

45. $\frac{\tan^{-1} k}{k^2} < \frac{\pi/2}{k^2}$, $\sum_{k=1}^{\infty} \frac{\pi/2}{k^2}$ converges so $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{k^2}$ converges.

46. $\frac{5^k + k}{k!+3} < \frac{5^k + 5^k}{k!} = \frac{2\,(5^k)}{k!}$, $\sum_{k=1}^{\infty} 2\left(\frac{5^k}{k!}\right)$ converges (Ratio Test), so $\sum_{k=1}^{\infty} \frac{5^k + k}{k!+3}$ converges.

47. Ratio Test, $\rho = \lim_{k\to+\infty} \frac{(k+1)^2}{(2k+2)(2k+1)} = 1/4$, converges.

48. Root Test: $\rho = \lim_{k\to+\infty} \frac{\pi(k+1)}{k^{1+1/k}} = \lim_{k\to+\infty} \pi\frac{k+1}{k} = \pi$, diverges.

49. $a_k = \frac{\ln k}{3^k}$, $\frac{a_{k+1}}{a_k} = \frac{\ln(k+1)}{\ln k}\frac{3^k}{3^{k+1}} \to \frac{1}{3}$, converges.