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Lecture # 10 & 11

Differentiation

2.2.3 THEOREM *If a function f is differentiable at x_0, then f is continuous at x_0.*

2.3.1 THEOREM *The derivative of a constant function is 0; that is, if c is any real number, then*

$$\frac{d}{dx}[c] = 0 \tag{1}$$

2.3.2 THEOREM (*The Power Rule*) *If n is a positive integer, then*

$$\frac{d}{dx}[x^n] = nx^{n-1} \tag{5}$$

In words, *to differentiate a power function, decrease the constant exponent by one* and *multiply the resulting power function by the original exponent.*

2.3.3 THEOREM (*Extended Power Rule*) *If r is any real number, then*

$$\frac{d}{dx}[x^r] = rx^{r-1} \tag{7}$$

2.3.4 THEOREM (*Constant Multiple Rule*) *If f is differentiable at x and c is any real number, then cf is also differentiable at x and*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)] \tag{8}$$

In words, *a constant factor can be moved through a derivative sign.*

2.3.5 THEOREM (*Sum and Difference Rules*) *If f and g are differentiable at x, then so are $f+g$ and $f-g$ and*

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] \tag{9}$$

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)] \tag{10}$$

In words, *the derivative of a sum equals the sum of the derivatives*, and *the derivative of a difference equals the difference of the derivatives.*

EXERCISE SET 2.3 Graphing Utility

1–8 Find dy/dx.

1. $y = 4x^7$
2. $y = -3x^{12}$
3. $y = 3x^8 + 2x + 1$
4. $y = \frac{1}{2}(x^4 + 7)$
5. $y = \pi^3$
6. $y = \sqrt{2}x + (1/\sqrt{2})$
7. $y = -\frac{1}{3}(x^7 + 2x - 9)$
8. $y = \dfrac{x^2 + 1}{5}$

9–16 Find $f'(x)$.

9. $f(x) = x^{-3} + \dfrac{1}{x^7}$
10. $f(x) = \sqrt{x} + \dfrac{1}{x}$
11. $f(x) = -3x^{-8} + 2\sqrt{x}$
12. $f(x) = 7x^{-6} - 5\sqrt{x}$
13. $f(x) = x^e + \dfrac{1}{x^{\sqrt{10}}}$
14. $f(x) = \sqrt[3]{\dfrac{8}{x}}$
15. $f(x) = (3x^2 + 1)^2$
16. $f(x) = ax^3 + bx^2 + cx + d$ $(a, b, c, d$ constant$)$

21–24 Find $dy/dx|_{x=1}$.

21. $y = 1 + x + x^2 + x^3 + x^4 + x^5$
22. $y = \dfrac{1 + x + x^2 + x^3 + x^4 + x^5 + x^6}{x^3}$
23. $y = (1 - x)(1 + x)(1 + x^2)(1 + x^4)$

41–42 Find d^2y/dx^2.

41. (a) $y = 7x^3 - 5x^2 + x$ (b) $y = 12x^2 - 2x + 3$
(c) $y = \dfrac{x + 1}{x}$ (d) $y = (5x^2 - 3)(7x^3 + x)$

42. (a) $y = 4x^7 - 5x^3 + 2x$ (b) $y = 3x + 2$
(c) $y = \dfrac{3x - 2}{5x}$ (d) $y = (x^3 - 5)(2x + 3)$

43–44 Find y'''.

43. (a) $y = x^{-5} + x^5$ (b) $y = 1/x$
(c) $y = ax^3 + bx + c$ $(a, b, c$ constant$)$

44. (a) $y = 5x^2 - 4x + 7$ (b) $y = 3x^{-2} + 4x^{-1} + x$
(c) $y = ax^4 + bx^2 + c$ $(a, b, c$ constant$)$

45. Find
(a) $f'''(2)$, where $f(x) = 3x^2 - 2$
(b) $\left.\dfrac{d^2y}{dx^2}\right|_{x=1}$, where $y = 6x^5 - 4x^2$
(c) $\left.\dfrac{d^4}{dx^4}[x^{-3}]\right|_{x=1}$.

2.4.1 THEOREM (*The Product Rule*) *If f and g are differentiable at x, then so is the product $f \cdot g$, and*

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)] \tag{1}$$

In words, *the derivative of a product of two functions is the first function times the derivative of the second plus the second function times the derivative of the first.*

2.4.2 THEOREM (*The Quotient Rule*) *If f and g are both differentiable at x and if $g(x) \neq 0$, then f/g is differentiable at x and*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2} \tag{2}$$

In words, *the derivative of a quotient of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the denominator squared.*

Table 2.4.1

RULES FOR DIFFERENTIATION			
$\frac{d}{dx}[c] = 0$	$(f+g)' = f' + g'$	$(f \cdot g)' = f \cdot g' + g \cdot f'$	$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$
$(cf)' = cf'$	$(f-g)' = f' - g'$	$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$	$\frac{d}{dx}[x^r] = rx^{r-1}$

Find $\frac{dy}{dx}$ by using formulas.

1. y = $4x^7$
2. y = -3 x^{12}
3. y = 3 x^8 + 2 x + 1
4. y = π^3
5. y = $\sqrt{2}$ x + $\frac{1}{\sqrt{2}}$

1. $y = 4x^7$

Solution:

Differentiate w.r.t. ‘x’

$\frac{dy}{dx} = \frac{d}{dx}(4x^7)$

$= 4\,\frac{d}{dx}(x^7)$

$= 4\,.\,7\,x^{7-1}\,\frac{d}{dx}(x)$

$= 28\,x^6\,(1)$

$\frac{dy}{dx} = 28\,x^6$

5. $y = \sqrt{2}\,x + \frac{1}{\sqrt{2}}$

Solution:

$\frac{dy}{dx} = \frac{d}{dx}(\sqrt{2}\,x + \frac{1}{\sqrt{2}})$

$= \frac{d}{dx}(\sqrt{2}\,x) + \frac{d}{dx}(\frac{1}{\sqrt{2}})$

$= \sqrt{2}\,\frac{d}{dx}(x) + 0$

$= \sqrt{2}\,(1)$

$\frac{dy}{dx} = \sqrt{2}$

Do the remaining parts by your own

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Product and Quotient Rule

1) $y = (3x^2 + 6)(2x - \frac{1}{4})$

Solution:

1) By using product rule of derivative

$$\frac{dy}{dx} = \frac{d}{dx}\left((3x^2 + 6)(2x - \frac{1}{4})\right)$$

$$= (3x^2 + 6)\frac{d}{dx}(2x - \frac{1}{4}) + (2x - \frac{1}{4})\frac{d}{dx}(3x^2 + 6)$$

$$= (3x^2 + 6)(2) + (2x - \frac{1}{4})(6x)$$

$$= 6x^2 + 12 + 12x^2 - \frac{3x}{2}$$

$$\frac{dy}{dx} = 18x^2 - \frac{3x}{2} + 12$$

2) $y = \frac{3}{\sqrt{x}+2}$

Solution:

2) $y = \frac{3}{\sqrt{x}+2} \cdot \frac{\sqrt{x}-2}{\sqrt{x}-2} = \frac{3\sqrt{x}-6}{x-4}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3\sqrt{x}-6}{x-4}\right)$$

By using quotient rule of derivatives

$$\frac{dy}{dx} = \frac{(x-4)\frac{d(3\sqrt{x}-6)}{dx} - (3\sqrt{x}-6)\frac{d(x-4)}{dx}}{(x-4)^2}$$

$$= \frac{(x-4)\frac{3}{2\sqrt{x}} - (3\sqrt{x}-6)(1)}{(x-4)^2}$$

$$= \frac{\frac{3x-12-6x+12\sqrt{x}}{2\sqrt{x}}}{(x-4)^2}$$

$$\frac{dy}{dx} = \frac{-3x-12+12\sqrt{x}}{2\sqrt{x}\,(x-4)^2}$$

EXERCISE SET 2.4 Graphing Utility

1–4 Compute the derivative of the given function $f(x)$ by (a) multiplying and then differentiating and (b) using the product rule. Verify that (a) and (b) yield the same result.

1. $f(x) = (x + 1)(2x - 1)$
2. $f(x) = (3x^2 - 1)(x^2 + 2)$
3. $f(x) = (x^2 + 1)(x^2 - 1)$
4. $f(x) = (x + 1)(x^2 - x + 1)$

21–24 Find $dy/dx|_{x=1}$.

21. $y = \dfrac{2x - 1}{x + 3}$
22. $y = \dfrac{4x + 1}{x^2 - 5}$
23. $y = \left(\dfrac{3x + 2}{x}\right)(x^{-5} + 1)$
24. $y = (2x^7 - x^2)\left(\dfrac{x - 1}{x + 1}\right)$

Do the following Questions

Ex # 2.3 (1-24,41-47)
Ex # 2.4 (1-24)

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Lecture # 12

*Differentiation (**Trigonometric Functions**)*

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Chain Rule

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x$$

1. f(x) = 2 $\cos x$ - 3 $\sin x$
2. f(x) = $\sin x \cos x$
3. f(x) = $\frac{\sin x}{x}$

Solution:

$\frac{dy}{dx} = 2\frac{dcosx}{dx} - 3\frac{dsinx}{dx}$

= 2 (- sin x)(1) – 3 (cos x)(1)

f'(x)= -2 sin x – 3 cos x

Do the remaining parts by your own

Ex # 2.5 (1-24)

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Derivatives of Logarithmic & Exponential functions

&

Chain Rule

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$$

$$\frac{d}{dx}[\ln |x|] = \frac{1}{x} \quad \text{if } x \neq 0$$

$$\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}, \quad x > 0$$

$$\frac{d}{dx}[e^x] = e^x$$

$$\frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[b^x] = b^x \ln b$$

$$\frac{d}{dx}[\log_b u] = \frac{1}{u \ln b} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[b^u] = b^u \ln b \cdot \frac{du}{dx}$$

Find $\frac{dy}{dx}$ $when$

1) Y = ln (x^3)

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}\ln(x^3)$$

$$= \frac{1}{x^3}\frac{d}{dx}(x^3)$$

$$= \frac{1}{x^3} \,.\, 3\,x^2 = \frac{3}{x}$$

2) Y = ln (sin x)

$$\frac{dy}{dx} = \frac{d}{dx}\ln(\sin x)$$

$$= \frac{1}{sinx}\frac{d}{dx}(\sin x)$$

$$= \frac{\cos x\,(1)}{sinx} = \cot x$$

Find $\frac{dy}{dx}$ $when$ Y = e^{-3x^2}

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}\; e^{-3x^2}$$

$$= e^{-3x^2}\,\frac{d}{dx}(-3x^2)$$

$$= e^{-3x^2}\,(-3\,\frac{d}{dx}x^2)$$

$$= e^{-3x^2}\,(-3 \,.\, 2\,x)$$

$$f'(x) = -6\,x\,e^{-3x^2}$$

2.6.1 THEOREM (*The Chain Rule*) *If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x. Moreover, if*

$$y = f(g(x)) \quad \textit{and} \quad u = g(x)$$

then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{1}$$

$$\frac{d}{dx}[f(g(x))] = (f \circ g)'(x) = f'(g(x))g'(x)$$

The derivative of $f(g(x))$ is the derivative of the outside function evaluated at the inside function times the derivative of the inside function.

Find derivative by using chain rule.

(i) y = $4\cos(x^3)$

(ii) y = $\tan(4x^3 + 1)$

(iii) y = $(x^2 - x + 1)^{23}$

(iv) y = $cos^2(\pi x)$

(v) y = $\sin\left(\sqrt{1 + \cos x}\right)$

(i) y = 4 $\cos(x^3)$

Solution:

Let u = x^3

y = 4 $\cos u$

By using chain rule

$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

$\frac{dy}{du}$ = - 4 $\sin u$ (1) = - 4 $\sin u$

$\frac{du}{dx}$ = 3 x^2

$\frac{dy}{dx}$ = - 4 $\sin u$. $3x^2$

$\frac{dy}{dx}$ = - 12 $x^2 \sin(x^3)$

v) y = $\sin\left(\sqrt{1+\cos x}\right)$

Solution:

Let u = $\sqrt{1+\cos x}$

y = $\sin u$

$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

$\frac{dy}{du}$ = $\cos u$

$\frac{du}{dx} = \frac{1}{2} (1+\cos x)^{\frac{1}{2}-1}$(0 – sin x (1))

$\frac{du}{dx} = \frac{-sinx}{2\sqrt{1+\cos x}}$

$\frac{dy}{dx}$ = $\cos u$. $\frac{-sinx}{2\sqrt{1+\cos x}}$

$\frac{dy}{dx}$ = - $\frac{sinx \cos(\sqrt{1+\cos x})}{2\sqrt{1+\cos x}}$

▶ **Example 2** Find dw/dt if $w = \tan x$ and $x = 4t^3 + t$.

Solution. In this case the chain rule computations take the form

$$\begin{aligned}
\frac{dw}{dt} &= \frac{dw}{dx} \cdot \frac{dx}{dt} \\
&= \frac{d}{dx}[\tan x] \cdot \frac{d}{dt}[4t^3 + t] \\
&= (\sec^2 x) \cdot (12t^2 + 1) \\
&= [\sec^2(4t^3 + t)] \cdot (12t^2 + 1) = (12t^2 + 1)\sec^2(4t^3 + t) \quad ◀
\end{aligned}$$

► **Example 3** (*Example 1 revisited*) Find $h'(x)$ if $h(x) = \cos(x^3)$.

Solution. We can think of h as a composition $f(g(x))$ in which $g(x) = x^3$ is the inside function and $f(x) = \cos x$ is the outside function. Thus, Formula (2) yields

$$h'(x) = \underbrace{f'(g(x))}_{\text{Derivative of the outside function evaluated at the inside function}} \cdot \underbrace{g'(x)}_{\text{Derivative of the inside function}}$$

$$= f'(x^3) \cdot 3x^2$$

$$= -\sin(x^3) \cdot 3x^2 = -3x^2 \sin(x^3)$$

which agrees with the result obtained in Example 1. ◄

Do Questions 7 - 40 from Ex # 2.6

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Lecture # 13

Chain Rule

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Implicit Differentiation

7–26 Find $f'(x)$.

7. $f(x) = (x^3 + 2x)^{37}$

8. $f(x) = (3x^2 + 2x - 1$

9. $f(x) = \left(x^3 - \dfrac{7}{x}\right)^{-2}$

10. $f(x) = \dfrac{1}{(x^5 - x + 1)^9}$

11. $f(x) = \dfrac{4}{(3x^2 - 2x + 1)^3}$

12. $f(x) = \sqrt{x^3 - 2x + 5}$

13. $f(x) = \sqrt{4 + \sqrt{3x}}$

14. $f(x) = \sqrt[3]{12 + \sqrt{x}}$

15. $f(x) = \sin\left(\dfrac{1}{x^2}\right)$

16. $f(x) = \tan\sqrt{x}$

17. $f(x) = 4\cos^5 x$

18. $f(x) = 4x + 5\sin^4 x$

19. $f(x) = \cos^2(3\sqrt{x})$

20. $f(x) = \tan^4(x^3)$

21. $f(x) = 2\sec^2(x^7)$

22. $f(x) = \cos^3\left(\dfrac{x}{x+1}\right)$

23. $f(x) = \sqrt{\cos(5x)}$

24. $f(x) = \sqrt{3x - \sin^2(4x)}$

25. $f(x) = [x + \csc(x^3 + 3)]^{-3}$

26. $f(x) = [x^4 - \sec(4x^2 - 2)]^{-4}$

27–40 Find dy/dx.

27. $y = x^3 \sin^2(5x)$

28. $y = \sqrt{x}\tan^3(\sqrt{x})$

29. $y = x^5 \sec(1/x)$

30. $y = \dfrac{\sin x}{\sec(3x+1)}$

31. $y = \cos(\cos x)$

32. $y = \sin(\tan 3x)$

33. $y = \cos^3(\sin 2x)$

34. $y = \dfrac{1 + \csc(x^2)}{1 - \cot(x^2)}$

35. $y = (5x + 8)^7 (1 - \sqrt{x})^6$

36. $y = (x^2 + x)^5 \sin^8 x$

Implicit Differentiation

Fortunately, we don't need to solve an equation for y in terms of x in order to find the derivative of y. Instead we can use the method of **implicit differentiation**. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'. In the examples and exercises of this section it is always assumed that the given equation determines y implicitly as a differentiable function of x so that the method of implicit differentiation can be applied.

Find derivatives by using implicit differentiation.

(i) $5y^2 + \sin y = x^2$

(ii) $7y^4 + x^3y + x = 4$ at (4,0)

(iii) y^2 - x + 1 = 0 at (2, -1)

$(i)\quad 5y^2 + \sin y = x^2$

Solution:

Differentiate w.r.t 'x' on both side

$\frac{d}{dx}(5y^2 + \sin y) = \frac{d}{dx}x^2$

5. 2y $\frac{dy}{dx}$ + $\cos y \frac{dy}{dx}$ = $2x^{2-1}$ (1)

10y $\frac{dy}{dx}$+ $\cos y \frac{dy}{dx}$ = 2x

$\frac{dy}{dx}$(10y + $\cos y$) = 2x

$\frac{dy}{dx} = \frac{2\,x}{10\,y + \cos y}$

Do remaining by your own

13–18 Find d^2y/dx^2 by implicit differentiation.

13. $2x^2 - 3y^2 = 4$

14. $x^3 + y^3 = 1$

15. $x^3y^3 - 4 = 0$

16. $xy + y^2 = 2$

17. $y + \sin y = x$

18. $x \cos y = y$

25–28 Use implicit differentiation to find the slope of the tangent line to the curve at the specified point, and check that your answer is consistent with the accompanying graph on the next page. ■

25. $x^4 + y^4 = 16$; $(1, \sqrt[4]{15})$ [*Lamé's special quartic*]

26. $y^3 + yx^2 + x^2 - 3y^2 = 0$; $(0, 3)$ [*trisectrix*]

27. $2(x^2 + y^2)^2 = 25(x^2 - y^2)$; $(3, 1)$ [*lemniscate*]

28. $x^{2/3} + y^{2/3} = 4$; $(-1, 3\sqrt{3})$ [*four-cusped hypocycloid*]

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1 + x^2}$$