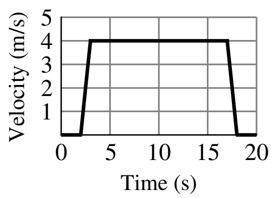
The Derivative

Exercise Set 2.1

1. **(a)** $m_{\tan} = (50 - 10)/(15 - 5) = 40/10 = 4$ m/s.

(b) 

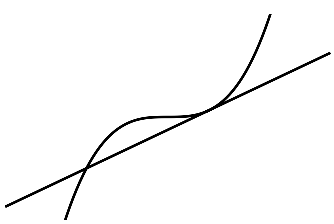
3. **(a)** $(10 - 10)/(3 - 0) = 0$ cm/s.

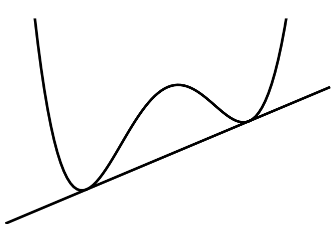
(b) $t = 0$, $t = 2$, $t = 4.2$, and $t = 8$ (horizontal tangent line).

(c) maximum: $t = 1$ (slope > 0), minimum: $t = 3$ (slope < 0).

(d) $(3 - 18)/(4 - 2) = -7.5$ cm/s (slope of estimated tangent line to curve at $t = 3$).

5. It is a straight line with slope equal to the velocity.

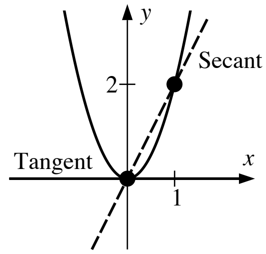
7. 

9. 

11. **(a)** $m_{\sec} = \dfrac{f(1) - f(0)}{1 - 0} = \dfrac{2}{1} = 2$

(b) $m_{\tan} = \lim\limits_{x_1 \to 0} \dfrac{f(x_1) - f(0)}{x_1 - 0} = \lim\limits_{x_1 \to 0} \dfrac{2x_1^2 - 0}{x_1 - 0} = \lim\limits_{x_1 \to 0} 2x_1 = 0$

(c) $m_{\tan} = \lim_{x_1 \to x_0} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \dfrac{2x_1^2 - 2x_0^2}{x_1 - x_0} = \lim_{x_1 \to x_0} (2x_1 + 2x_0) = 4x_0$



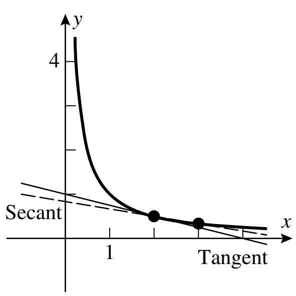
(d) The tangent line is the x-axis.

13. **(a)** $m_{\sec} = \dfrac{f(3) - f(2)}{3 - 2} = \dfrac{1/3 - 1/2}{1} = -\dfrac{1}{6}$

(b) $m_{\tan} = \lim_{x_1 \to 2} \dfrac{f(x_1) - f(2)}{x_1 - 2} = \lim_{x_1 \to 2} \dfrac{1/x_1 - 1/2}{x_1 - 2} = \lim_{x_1 \to 2} \dfrac{2 - x_1}{2x_1(x_1 - 2)} = \lim_{x_1 \to 2} \dfrac{-1}{2x_1} = -\dfrac{1}{4}$

(c) $m_{\tan} = \lim_{x_1 \to x_0} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \dfrac{1/x_1 - 1/x_0}{x_1 - x_0} = \lim_{x_1 \to x_0} \dfrac{x_0 - x_1}{x_0 x_1 (x_1 - x_0)} = \lim_{x_1 \to x_0} \dfrac{-1}{x_0 x_1} = -\dfrac{1}{x_0^2}$

(d)



15. **(a)** $m_{\tan} = \lim_{x_1 \to x_0} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \dfrac{(x_1^2 - 1) - (x_0^2 - 1)}{x_1 - x_0} = \lim_{x_1 \to x_0} \dfrac{(x_1^2 - x_0^2)}{x_1 - x_0} = \lim_{x_1 \to x_0} (x_1 + x_0) = 2x_0$

(b) $m_{\tan} = 2(-1) = -2$

17. **(a)** $m_{\tan} = \lim_{x_1 \to x_0} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \dfrac{(x_1 + \sqrt{x_1}\,) - (x_0 + \sqrt{x_0}\,)}{x_1 - x_0} = \lim_{x_1 \to x_0} \left(1 + \dfrac{1}{\sqrt{x_1} + \sqrt{x_0}}\right) = 1 + \dfrac{1}{2\sqrt{x_0}}$

(b) $m_{\tan} = 1 + \dfrac{1}{2\sqrt{1}} = \dfrac{3}{2}$

19. True. Let $x = 1 + h$.

21. False. Velocity represents the rate at which position changes.

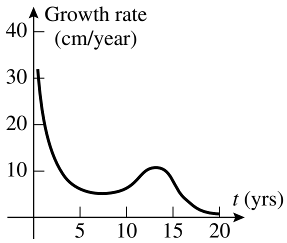
23. **(a)** 72°F at about 4:30 P.M. **(b)** About $(67 - 43)/6 = 4$°F/h.

(c) Decreasing most rapidly at about 9 P.M.; rate of change of temperature is about -7°F/h (slope of estimated tangent line to curve at 9 P.M.).

25. **(a)** During the first year after birth.

(b) About 6 cm/year (slope of estimated tangent line at age 5).

(c) The growth rate is greatest at about age 14; about 10 cm/year.

(d) 

27. **(a)** $0.3 \cdot 40^3 = 19{,}200$ ft **(b)** $v_{\text{ave}} = 19{,}200/40 = 480$ ft/s

(c) Solve $s = 0.3t^3 = 1000; t \approx 14.938$ so $v_{\text{ave}} \approx 1000/14.938 \approx 66.943$ ft/s.

(d) $v_{\text{inst}} = \lim_{h\to 0} \dfrac{0.3(40+h)^3 - 0.3 \cdot 40^3}{h} = \lim_{h\to 0} \dfrac{0.3(4800h + 120h^2 + h^3)}{h} = \lim_{h\to 0} 0.3(4800 + 120h + h^2) = 1440$ ft/s

29. **(a)** $v_{\text{ave}} = \dfrac{6(4)^4 - 6(2)^4}{4-2} = 720$ ft/min

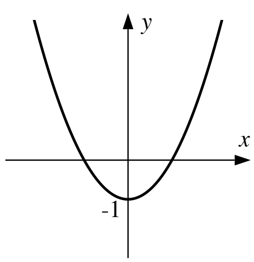
(b) $v_{\text{inst}} = \lim_{t_1\to 2} \dfrac{6t_1^4 - 6(2)^4}{t_1 - 2} = \lim_{t_1\to 2} \dfrac{6(t_1^4 - 16)}{t_1 - 2} = \lim_{t_1\to 2} \dfrac{6(t_1^2+4)(t_1^2-4)}{t_1 - 2} = \lim_{t_1\to 2} 6(t_1^2+4)(t_1+2) = 192$ ft/min

31. The instantaneous velocity at $t = 1$ equals the limit as $h \to 0$ of the average velocity during the interval between $t = 1$ and $t = 1 + h$.

Exercise Set 2.2

1. $f'(1) = 2.5,\ f'(3) = 0,\ f'(5) = -2.5,\ f'(6) = -1.$

3. **(a)** $f'(a)$ is the slope of the tangent line. **(b)** $f'(2) = m = 3$ **(c)** The same, $f'(2) = 3$.

5. 

7. $y - (-1) = 5(x - 3),\ y = 5x - 16$

9. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{2(x+h)^2 - 2x^2}{h} = \lim_{h\to 0} \dfrac{4xh + 2h^2}{h} = 4x$; $f'(1) = 4$ so the tangent line is given by $y - 2 = 4(x - 1)$, $y = 4x - 2$.

11. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{(x+h)^3 - x^3}{h} = \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2$; $f'(0) = 0$ so the tangent line is given by $y - 0 = 0(x - 0)$, $y = 0$.

13. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{x+1+h} - \sqrt{x+1}}{h} = \lim_{h\to 0} \dfrac{\sqrt{x+1+h} - \sqrt{x+1}}{h} \dfrac{\sqrt{x+1+h} + \sqrt{x+1}}{\sqrt{x+1+h} + \sqrt{x+1}} =$ $\lim_{h\to 0} \dfrac{h}{h(\sqrt{x+1+h} + \sqrt{x+1})} = \dfrac{1}{2\sqrt{x+1}}$; $f(8) = \sqrt{8+1} = 3$ and $f'(8) = \dfrac{1}{6}$ so the tangent line is given by

$y - 3 = \dfrac{1}{6}(x - 8)$, $y = \dfrac{1}{6}x + \dfrac{5}{3}$.

15. $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{\dfrac{1}{x + \Delta x} - \dfrac{1}{x}}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\dfrac{x - (x + \Delta x)}{x(x + \Delta x)}}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{-\Delta x}{x \Delta x (x + \Delta x)} = \lim\limits_{\Delta x \to 0} -\dfrac{1}{x(x + \Delta x)} = -\dfrac{1}{x^2}$.

17. $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{(x + \Delta x)^2 - (x + \Delta x) - (x^2 - x)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{2x\Delta x + (\Delta x)^2 - \Delta x}{\Delta x} = \lim\limits_{\Delta x \to 0} (2x - 1 + \Delta x) = 2x - 1$.

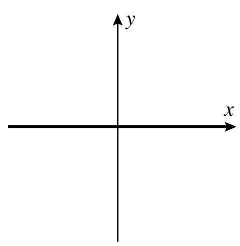
19. $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{\dfrac{1}{\sqrt{x + \Delta x}} - \dfrac{1}{\sqrt{x}}}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\sqrt{x} - \sqrt{x + \Delta x}}{\Delta x \sqrt{x}\sqrt{x + \Delta x}} = \lim\limits_{\Delta x \to 0} \dfrac{x - (x + \Delta x)}{\Delta x \sqrt{x}\sqrt{x + \Delta x}(\sqrt{x} + \sqrt{x + \Delta x})} =$

$= \lim\limits_{\Delta x \to 0} \dfrac{-1}{\sqrt{x}\sqrt{x + \Delta x}(\sqrt{x} + \sqrt{x + \Delta x})} = -\dfrac{1}{2x^{3/2}}$.

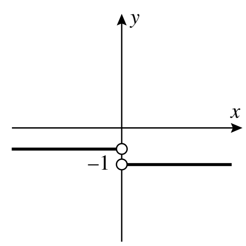
21. $f'(t) = \lim\limits_{h \to 0} \dfrac{f(t + h) - f(t)}{h} = \lim\limits_{h \to 0} \dfrac{[4(t + h)^2 + (t + h)] - [4t^2 + t]}{h} = \lim\limits_{h \to 0} \dfrac{4t^2 + 8th + 4h^2 + t + h - 4t^2 - t}{h} = \lim\limits_{h \to 0} \dfrac{8th + 4h^2 + h}{h} = \lim\limits_{h \to 0} (8t + 4h + 1) = 8t + 1$.

23. **(a)** D **(b)** F **(c)** B **(d)** C **(e)** A **(f)** E

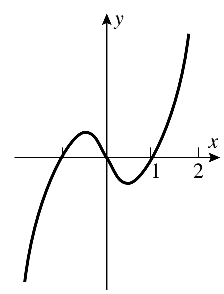
25. **(a)**



(b)



(c)



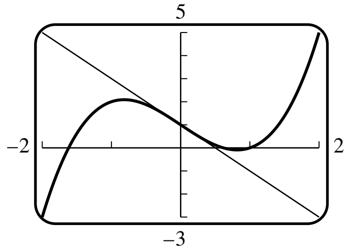
27. False. If the tangent line is horizontal then $f'(a) = 0$.

29. False. E.g. $|x|$ is continuous but not differentiable at $x = 0$.

31. **(a)** $f(x) = \sqrt{x}$ and $a = 1$ **(b)** $f(x) = x^2$ and $a = 3$

33. $\dfrac{dy}{dx} = \lim\limits_{h \to 0} \dfrac{(1 - (x + h)^2) - (1 - x^2)}{h} = \lim\limits_{h \to 0} \dfrac{-2xh - h^2}{h} = \lim\limits_{h \to 0} (-2x - h) = -2x$, and $\left.\dfrac{dy}{dx}\right|_{x=1} = -2$.

35. $y = -2x + 1$



37. **(b)**

w	1.5	1.1	1.01	1.001	1.0001	1.00001
$\dfrac{f(w) - f(1)}{w - 1}$	1.6569	1.4355	1.3911	1.3868	1.3863	1.3863

w	0.5	0.9	0.99	0.999	0.9999	0.99999
$\dfrac{f(w)-f(1)}{w-1}$	1.1716	1.3393	1.3815	1.3858	1.3863	1.3863

39. **(a)** $\dfrac{f(3)-f(1)}{3-1} = \dfrac{2.2-2.12}{2} = 0.04$; $\dfrac{f(2)-f(1)}{2-1} = \dfrac{2.34-2.12}{1} = 0.22$; $\dfrac{f(2)-f(0)}{2-0} = \dfrac{2.34-0.58}{2} = 0.88$.

(b) The tangent line at $x = 1$ appears to have slope about 0.8, so $\dfrac{f(2)-f(0)}{2-0}$ gives the best approximation and $\dfrac{f(3)-f(1)}{3-1}$ gives the worst.

41. **(a)** dollars/ft

(b) $f'(x)$ is roughly the price per additional foot.

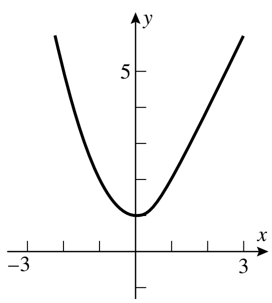
(c) If each additional foot costs extra money (this is to be expected) then $f'(x)$ remains positive.

(d) From the approximation $1000 = f'(300) \approx \dfrac{f(301)-f(300)}{301-300}$ we see that $f(301) \approx f(300) + 1000$, so the extra foot will cost around \$1000.

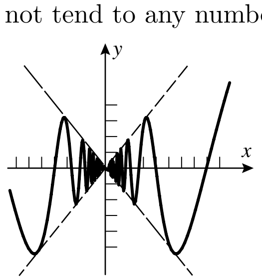
43. **(a)** $F \approx 200$ lb, $dF/d\theta \approx 50$ **(b)** $\mu = (dF/d\theta)/F \approx 50/200 = 0.25$

45. **(a)** $T \approx 115°$F, $dT/dt \approx -3.35°$F/min **(b)** $k = (dT/dt)/(T - T_0) \approx (-3.35)/(115-75) = -0.084$

47. $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = f(1)$, so f is continuous at $x = 1$. $\lim\limits_{h\to 0^-} \dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h\to 0^-} \dfrac{[(1+h)^2+1]-2}{h} = \lim\limits_{h\to 0^-}(2+h) = 2$; $\lim\limits_{h\to 0^+} \dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h\to 0^+} \dfrac{2(1+h)-2}{h} = \lim\limits_{h\to 0^+} 2 = 2$, so $f'(1) = 2$.



49. Since $-|x| \le x\sin(1/x) \le |x|$ it follows by the Squeezing Theorem (Theorem 1.6.4) that $\lim\limits_{x\to 0} x\sin(1/x) = 0$. The derivative cannot exist: consider $\dfrac{f(x)-f(0)}{x} = \sin(1/x)$. This function oscillates between -1 and $+1$ and does not tend to any number as x tends to zero.



51. Let $\epsilon = |f'(x_0)/2|$. Then there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$, then $\left|\dfrac{f(x)-f(x_0)}{x-x_0} - f'(x_0)\right| < \epsilon$. Since

$f'(x_0) > 0$ and $\epsilon = f'(x_0)/2$ it follows that $\dfrac{f(x)-f(x_0)}{x-x_0} > \epsilon > 0$. If $x = x_1 < x_0$ then $f(x_1) < f(x_0)$ and if $x = x_2 > x_0$ then $f(x_2) > f(x_0)$.

53. (a) Let $\epsilon = |m|/2$. Since $m \neq 0$, $\epsilon > 0$. Since $f(0) = f'(0) = 0$ we know there exists $\delta > 0$ such that $\left|\dfrac{f(0+h)-f(0)}{h}\right| < \epsilon$ whenever $0 < |h| < \delta$. It follows that $|f(h)| < \frac{1}{2}|hm|$ for $0 < |h| < \delta$. Replace h with x to get the result.

(b) For $0 < |x| < \delta, |f(x)| < \frac{1}{2}|mx|$. Moreover $|mx| = |mx - f(x) + f(x)| \leq |f(x) - mx| + |f(x)|$, which yields $|f(x) - mx| \geq |mx| - |f(x)| > \frac{1}{2}|mx| > |f(x)|$, i.e. $|f(x) - mx| > |f(x)|$.

(c) If any straight line $y = mx + b$ is to approximate the curve $y = f(x)$ for small values of x, then $b = 0$ since $f(0) = 0$. The inequality $|f(x) - mx| > |f(x)|$ can also be interpreted as $|f(x) - mx| > |f(x) - 0|$, i.e. the line $y = 0$ is a better approximation than is $y = mx$.

55. See discussion around Definition 2.2.2.

Exercise Set 2.3

1. $28x^6$, by Theorems 2.3.2 and 2.3.4.

3. $24x^7 + 2$, by Theorems 2.3.1, 2.3.2, 2.3.4, and 2.3.5.

5. 0, by Theorem 2.3.1.

7. $-\dfrac{1}{3}(7x^6 + 2)$, by Theorems 2.3.1, 2.3.2, 2.3.4, and 2.3.5.

9. $-3x^{-4} - 7x^{-8}$, by Theorems 2.3.3 and 2.3.5.

11. $24x^{-9} + 1/\sqrt{x}$, by Theorems 2.3.3, 2.3.4, and 2.3.5.

13. $f'(x) = ex^{e-1} - \sqrt{10}\,x^{-1-\sqrt{10}}$, by Theorems 2.3.3 and 2.3.5.

15. $(3x^2 + 1)^2 = 9x^4 + 6x^2 + 1$, so $f'(x) = 36x^3 + 12x$, by Theorems 2.3.1, 2.3.2, 2.3.4, and 2.3.5.

17. $y' = 10x - 3$, $y'(1) = 7$.

19. $2t - 1$, by Theorems 2.3.2 and 2.3.5.

21. $dy/dx = 1 + 2x + 3x^2 + 4x^3 + 5x^4$, $dy/dx|_{x=1} = 15$.

23. $y = (1 - x^2)(1 + x^2)(1 + x^4) = (1 - x^4)(1 + x^4) = 1 - x^8$, $\dfrac{dy}{dx} = -8x^7$, $dy/dx|_{x=1} = -8$.

25. $f'(1) \approx \dfrac{f(1.01) - f(1)}{0.01} = \dfrac{-0.999699 - (-1)}{0.01} = 0.0301$, and by differentiation, $f'(1) = 3(1)^2 - 3 = 0$.

27. The estimate will depend on your graphing utility and on how far you zoom in. Since $f'(x) = 1 - \dfrac{1}{x^2}$, the exact value is $f'(1) = 0$.

29. $32t$, by Theorems 2.3.2 and 2.3.4.

31. $3\pi r^2$, by Theorems 2.3.2 and 2.3.4.

33. True. By Theorems 2.3.4 and 2.3.5, $\dfrac{d}{dx}[f(x) - 8g(x)] = f'(x) - 8g'(x)$; substitute $x = 2$ to get the result.

35. False. $\dfrac{d}{dx}[4f(x) + x^3]\Big|_{x=2} = (4f'(x) + 3x^2)\big|_{x=2} = 4f'(2) + 3 \cdot 2^2 = 32$

37. **(a)** $\dfrac{dV}{dr} = 4\pi r^2$ **(b)** $\dfrac{dV}{dr}\Big|_{r=5} = 4\pi(5)^2 = 100\pi$

39. $y - 2 = 5(x + 3)$, $y = 5x + 17$.

41. **(a)** $dy/dx = 21x^2 - 10x + 1$, $d^2y/dx^2 = 42x - 10$ **(b)** $dy/dx = 24x - 2$, $d^2y/dx^2 = 24$

(c) $dy/dx = -1/x^2$, $d^2y/dx^2 = 2/x^3$ **(d)** $dy/dx = 175x^4 - 48x^2 - 3$, $d^2y/dx^2 = 700x^3 - 96x$

43. **(a)** $y' = -5x^{-6} + 5x^4$, $y'' = 30x^{-7} + 20x^3$, $y''' = -210x^{-8} + 60x^2$

(b) $y = x^{-1}$, $y' = -x^{-2}$, $y'' = 2x^{-3}$, $y''' = -6x^{-4}$

(c) $y' = 3ax^2 + b$, $y'' = 6ax$, $y''' = 6a$

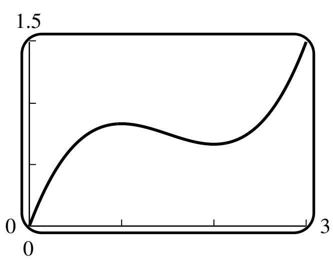
45. **(a)** $f'(x) = 6x$, $f''(x) = 6$, $f'''(x) = 0$, $f'''(2) = 0$

(b) $\dfrac{dy}{dx} = 30x^4 - 8x$, $\dfrac{d^2y}{dx^2} = 120x^3 - 8$, $\dfrac{d^2y}{dx^2}\Big|_{x=1} = 112$

(c) $\dfrac{d}{dx}\left[x^{-3}\right] = -3x^{-4}$, $\dfrac{d^2}{dx^2}\left[x^{-3}\right] = 12x^{-5}$, $\dfrac{d^3}{dx^3}\left[x^{-3}\right] = -60x^{-6}$, $\dfrac{d^4}{dx^4}\left[x^{-3}\right] = 360x^{-7}$, $\dfrac{d^4}{dx^4}\left[x^{-3}\right]\Big|_{x=1} = 360$

47. $y' = 3x^2 + 3$, $y'' = 6x$, and $y''' = 6$ so $y''' + xy'' - 2y' = 6 + x(6x) - 2(3x^2 + 3) = 6 + 6x^2 - 6x^2 - 6 = 0$.

49. The graph has a horizontal tangent at points where $\dfrac{dy}{dx} = 0$, but $\dfrac{dy}{dx} = x^2 - 3x + 2 = (x - 1)(x - 2) = 0$ if $x = 1, 2$. The corresponding values of y are $5/6$ and $2/3$ so the tangent line is horizontal at $(1, 5/6)$ and $(2, 2/3)$.



51. The y-intercept is -2 so the point $(0, -2)$ is on the graph; $-2 = a(0)^2 + b(0) + c$, $c = -2$. The x-intercept is 1 so the point (1,0) is on the graph; $0 = a + b - 2$. The slope is $dy/dx = 2ax + b$; at $x = 0$ the slope is b so $b = -1$, thus $a = 3$. The function is $y = 3x^2 - x - 2$.

53. The points $(-1, 1)$ and $(2, 4)$ are on the secant line so its slope is $(4 - 1)/(2 + 1) = 1$. The slope of the tangent line to $y = x^2$ is $y' = 2x$ so $2x = 1$, $x = 1/2$.

55. $y' = -2x$, so at any point (x_0, y_0) on $y = 1 - x^2$ the tangent line is $y - y_0 = -2x_0(x - x_0)$, or $y = -2x_0x + x_0^2 + 1$. The point $(2, 0)$ is to be on the line, so $0 = -4x_0 + x_0^2 + 1$, $x_0^2 - 4x_0 + 1 = 0$. Use the quadratic formula to get $x_0 = \dfrac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$. The points are $(2 + \sqrt{3}, -6 - 4\sqrt{3})$ and $(2 - \sqrt{3}, -6 + 4\sqrt{3})$.

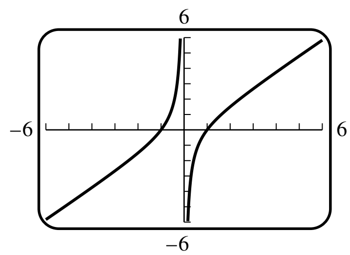
57. $y' = 3ax^2 + b$; the tangent line at $x = x_0$ is $y - y_0 = (3ax_0^2 + b)(x - x_0)$ where $y_0 = ax_0^3 + bx_0$. Solve with $y = ax^3 + bx$ to get

$$\begin{aligned}(ax^3 + bx) - (ax_0^3 + bx_0) &= (3ax_0^2 + b)(x - x_0)\\ ax^3 + bx - ax_0^3 - bx_0 &= 3ax_0^2x - 3ax_0^3 + bx - bx_0\\ x^3 - 3x_0^2x + 2x_0^3 &= 0\\ (x - x_0)(x^2 + xx_0 - 2x_0^2) &= 0\\ (x - x_0)^2(x + 2x_0) &= 0, \text{ so } x = -2x_0.\end{aligned}$$

59. $y' = -\dfrac{1}{x^2}$; the tangent line at $x = x_0$ is $y - y_0 = -\dfrac{1}{x_0^2}(x - x_0)$, or $y = -\dfrac{x}{x_0^2} + \dfrac{2}{x_0}$. The tangent line crosses the x-axis at $2x_0$, the y-axis at $2/x_0$, so that the area of the triangle is $\dfrac{1}{2}(2/x_0)(2x_0) = 2$.

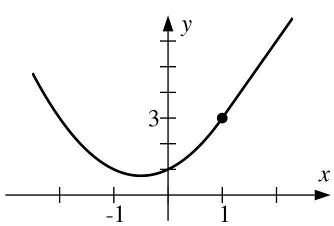
61. $F = GmMr^{-2}$, $\dfrac{dF}{dr} = -2GmMr^{-3} = -\dfrac{2GmM}{r^3}$

63. Since $dT/dx = (1/2)0.453x^{-1/2} = 0.2265/\sqrt{x}$, we have $dT/dx|_{x=9} = 0.2265/3 = 0.0755$ s/m.



65. $f'(x) = 1 + 1/x^2 > 0$ for all $x \neq 0$

67. f is continuous at 1 because $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = f(1)$; also $\lim_{x\to1^-} f'(x) = \lim_{x\to1^-} (2x+1) = 3$ and $\lim_{x\to1^+} f'(x) = \lim_{x\to1^+} 3 = 3$ so f is differentiable at 1, and the derivative equals 3.



69. f is continuous at 1 because $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = f(1)$. Also, $\lim_{x->1^-} \dfrac{f(x) - f(1)}{x - 1}$ equals the derivative of x^2 at $x = 1$, namely $2x|_{x=1} = 2$, while $\lim_{x->1^+} \dfrac{f(x) - f(1)}{x - 1}$ equals the derivative of $\sqrt{x}$ at $x = 1$, namely $\left.\dfrac{1}{2\sqrt{x}}\right|_{x=1} = \dfrac{1}{2}$. Since these are not equal, f is not differentiable at $x = 1$.

71. **(a)** $f(x) = 3x - 2$ if $x \geq 2/3$, $f(x) = -3x + 2$ if $x < 2/3$ so f is differentiable everywhere except perhaps at $2/3$. f is continuous at $2/3$, also $\lim_{x\to2/3^-} f'(x) = \lim_{x\to2/3^-} (-3) = -3$ and $\lim_{x\to2/3^+} f'(x) = \lim_{x\to2/3^+} (3) = 3$ so f is not differentiable at $x = 2/3$.

(b) $f(x) = x^2 - 4$ if $|x| \geq 2$, $f(x) = -x^2 + 4$ if $|x| < 2$ so f is differentiable everywhere except perhaps at ±2. f is continuous at -2 and 2, also $\lim_{x\to2^-} f'(x) = \lim_{x\to2^-} (-2x) = -4$ and $\lim_{x\to2^+} f'(x) = \lim_{x\to2^+} (2x) = 4$ so f is not differentiable at $x = 2$. Similarly, f is not differentiable at $x = -2$.

73. **(a)**

$$\frac{d^2}{dx^2}[cf(x)] = \frac{d}{dx}\left[\frac{d}{dx}[cf(x)]\right] = \frac{d}{dx}\left[c\frac{d}{dx}[f(x)]\right] = c\frac{d}{dx}\left[\frac{d}{dx}[f(x)]\right] = c\frac{d^2}{dx^2}[f(x)]$$

$$\frac{d^2}{dx^2}[f(x)+g(x)] = \frac{d}{dx}\left[\frac{d}{dx}[f(x)+g(x)]\right] = \frac{d}{dx}\left[\frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]\right] = \frac{d^2}{dx^2}[f(x)] + \frac{d^2}{dx^2}[g(x)]$$

(b) Yes, by repeated application of the procedure illustrated in part **(a)**.

75. **(a)** $f'(x) = nx^{n-1}$, $f''(x) = n(n-1)x^{n-2}$, $f'''(x) = n(n-1)(n-2)x^{n-3}, \ldots, f^{(n)}(x) = n(n-1)(n-2)\cdots 1$

(b) From part **(a)**, $f^{(k)}(x) = k(k-1)(k-2)\cdots 1$ so $f^{(k+1)}(x) = 0$ thus $f^{(n)}(x) = 0$ if $n > k$.

(c) From parts **(a)** and **(b)**, $f^{(n)}(x) = a_n n(n-1)(n-2)\cdots 1$.

77. Let $g(x) = x^n, f(x) = (mx+b)^n$. Use Exercise 52 in Section 2.2, but with f and g permuted. If $x_0 = mx_1 + b$ then Exercise 52 says that f is differentiable at x_1 and $f'(x_1) = mg'(x_0)$. Since $g'(x_0) = nx_0^{n-1}$, the result follows.

79. $f(x) = 27x^3 - 27x^2 + 9x - 1$ so $f'(x) = 81x^2 - 54x + 9 = 3 \cdot 3(3x-1)^2$, as predicted by Exercise 75.

81. $f(x) = 3(2x+1)^{-2}$ so $f'(x) = 3(-2)2(2x+1)^{-3} = -12/(2x+1)^3$.

83. $f(x) = \dfrac{2x^2+4x+2+1}{(x+1)^2} = 2 + (x+1)^{-2}$, so $f'(x) = -2(x+1)^{-3} = -2/(x+1)^3$.

Exercise Set 2.4

1. **(a)** $f(x) = 2x^2 + x - 1$, $f'(x) = 4x + 1$ **(b)** $f'(x) = (x+1)\cdot(2) + (2x-1)\cdot(1) = 4x + 1$

3. **(a)** $f(x) = x^4 - 1$, $f'(x) = 4x^3$ **(b)** $f'(x) = (x^2+1)\cdot(2x) + (x^2-1)\cdot(2x) = 4x^3$

5. $f'(x) = (3x^2+6)\dfrac{d}{dx}\left(2x - \dfrac{1}{4}\right) + \left(2x - \dfrac{1}{4}\right)\dfrac{d}{dx}(3x^2+6) = (3x^2+6)(2) + \left(2x - \dfrac{1}{4}\right)(6x) = 18x^2 - \dfrac{3}{2}x + 12$

7. $f'(x) = (x^3+7x^2-8)\dfrac{d}{dx}(2x^{-3}+x^{-4}) + (2x^{-3}+x^{-4})\dfrac{d}{dx}(x^3+7x^2-8) = (x^3+7x^2-8)(-6x^{-4}-4x^{-5}) +$
$+(2x^{-3}+x^{-4})(3x^2+14x) = -15x^{-2} - 14x^{-3} + 48x^{-4} + 32x^{-5}$

9. $f'(x) = 1\cdot(x^2+2x+4) + (x-2)\cdot(2x+2) = 3x^2$

11. $f'(x) = \dfrac{(x^2+1)\frac{d}{dx}(3x+4) - (3x+4)\frac{d}{dx}(x^2+1)}{(x^2+1)^2} = \dfrac{(x^2+1)\cdot 3 - (3x+4)\cdot 2x}{(x^2+1)^2} = \dfrac{-3x^2-8x+3}{(x^2+1)^2}$

13. $f'(x) = \dfrac{(3x-4)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(3x-4)}{(3x-4)^2} = \dfrac{(3x-4)\cdot 2x - x^2\cdot 3}{(3x-4)^2} = \dfrac{3x^2-8x}{(3x-4)^2}$

15. $f(x) = \dfrac{2x^{3/2} + x - 2x^{1/2} - 1}{x+3}$, so

$$f'(x) = \frac{(x+3)\frac{d}{dx}(2x^{3/2}+x-2x^{1/2}-1) - (2x^{3/2}+x-2x^{1/2}-1)\frac{d}{dx}(x+3)}{(x+3)^2} =$$

$$= \frac{(x+3)\cdot(3x^{1/2}+1-x^{-1/2}) - (2x^{3/2}+x-2x^{1/2}-1)\cdot 1}{(x+3)^2} = \frac{x^{3/2}+10x^{1/2}+4-3x^{-1/2}}{(x+3)^2}$$

17. This could be computed by two applications of the product rule, but it's simpler to expand $f(x)$: $f(x) = 14x + 21 + 7x^{-1} + 2x^{-2} + 3x^{-3} + x^{-4}$, so $f'(x) = 14 - 7x^{-2} - 4x^{-3} - 9x^{-4} - 4x^{-5}$.

19. In general, $\dfrac{d}{dx}\left[g(x)^2\right] = 2g(x)g'(x)$ and $\dfrac{d}{dx}\left[g(x)^3\right] = \dfrac{d}{dx}\left[g(x)^2 g(x)\right] = g(x)^2 g'(x) + g(x)\dfrac{d}{dx}\left[g(x)^2\right] = g(x)^2 g'(x) + g(x)\cdot 2g(x)g'(x) = 3g(x)^2 g'(x)$.
Letting $g(x) = x^7 + 2x - 3$, we have $f'(x) = 3(x^7 + 2x - 3)^2(7x^6 + 2)$.

21. $\dfrac{dy}{dx} = \dfrac{(x+3)\cdot 2 - (2x-1)\cdot 1}{(x+3)^2} = \dfrac{7}{(x+3)^2}$, so $\left.\dfrac{dy}{dx}\right|_{x=1} = \dfrac{7}{16}$.

23. $\dfrac{dy}{dx} = \left(\dfrac{3x+2}{x}\right)\dfrac{d}{dx}\left(x^{-5}+1\right) + \left(x^{-5}+1\right)\dfrac{d}{dx}\left(\dfrac{3x+2}{x}\right) = \left(\dfrac{3x+2}{x}\right)\left(-5x^{-6}\right) + \left(x^{-5}+1\right)\left[\dfrac{x(3)-(3x+2)(1)}{x^2}\right] = \left(\dfrac{3x+2}{x}\right)\left(-5x^{-6}\right) + \left(x^{-5}+1\right)\left(-\dfrac{2}{x^2}\right)$; so $\left.\dfrac{dy}{dx}\right|_{x=1} = 5(-5) + 2(-2) = -29$.

25. $f'(x) = \dfrac{(x^2+1)\cdot 1 - x\cdot 2x}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}$, so $f'(1) = 0$.

27. (a) $g'(x) = \sqrt{x}f'(x) + \dfrac{1}{2\sqrt{x}}f(x)$, $g'(4) = (2)(-5) + \dfrac{1}{4}(3) = -37/4$.

(b) $g'(x) = \dfrac{xf'(x) - f(x)}{x^2}$, $g'(4) = \dfrac{(4)(-5) - 3}{16} = -23/16$.

29. (a) $F'(x) = 5f'(x) + 2g'(x), F'(2) = 5(4) + 2(-5) = 10$.

(b) $F'(x) = f'(x) - 3g'(x), F'(2) = 4 - 3(-5) = 19$.

(c) $F'(x) = f(x)g'(x) + g(x)f'(x), F'(2) = (-1)(-5) + (1)(4) = 9$.

(d) $F'(x) = [g(x)f'(x) - f(x)g'(x)]/g^2(x), F'(2) = [(1)(4) - (-1)(-5)]/(1)^2 = -1$.

31. $\dfrac{dy}{dx} = \dfrac{2x(x+2) - (x^2-1)}{(x+2)^2}$, $\dfrac{dy}{dx} = 0$ if $x^2 + 4x + 1 = 0$. By the quadratic formula, $x = \dfrac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$. The tangent line is horizontal at $x = -2 \pm \sqrt{3}$.

33. The tangent line is parallel to the line $y = x$ when it has slope 1. $\dfrac{dy}{dx} = \dfrac{2x(x+1) - (x^2+1)}{(x+1)^2} = \dfrac{x^2+2x-1}{(x+1)^2} = 1$ if $x^2 + 2x - 1 = (x+1)^2$, which reduces to $-1 = +1$, impossible. Thus the tangent line is never parallel to the line $y = x$.

35. Fix x_0. The slope of the tangent line to the curve $y = \dfrac{1}{x+4}$ at the point $(x_0, 1/(x_0+4))$ is given by $\dfrac{dy}{dx} = \left.\dfrac{-1}{(x+4)^2}\right|_{x=x_0} = \dfrac{-1}{(x_0+4)^2}$. The tangent line to the curve at (x_0, y_0) thus has the equation $y - y_0 = \dfrac{-(x-x_0)}{(x_0+4)^2}$, and this line passes through the origin if its constant term $y_0 - x_0\dfrac{-1}{(x_0+4)^2}$ is zero. Then $\dfrac{1}{x_0+4} = \dfrac{-x_0}{(x_0+4)^2}$, so $x_0 + 4 = -x_0$, $x_0 = -2$.

37. (a) Their tangent lines at the intersection point must be perpendicular.

(b) They intersect when $\dfrac{1}{x} = \dfrac{1}{2-x}$, $x = 2 - x$, $x = 1$, $y = 1$. The first curve has derivative $y = -\dfrac{1}{x^2}$, so the slope when $x = 1$ is -1. Second curve has derivative $y = \dfrac{1}{(2-x)^2}$ so the slope when $x = 1$ is 1. Since the two slopes are negative reciprocals of each other, the tangent lines are perpendicular at the point $(1, 1)$.

39. $F'(x) = xf'(x) + f(x)$, $F''(x) = xf''(x) + f'(x) + f'(x) = xf''(x) + 2f'(x)$.

41. $R'(p) = p \cdot f'(p) + f(p) \cdot 1 = f(p) + pf'(p)$, so $R'(120) = 9000 + 120 \cdot (-60) = 1800$. Increasing the price by a small amount Δp dollars would increase the revenue by about $1800\Delta p$ dollars.

43. $f(x) = \dfrac{1}{x^n}$ so $f'(x) = \dfrac{x^n \cdot (0) - 1 \cdot (nx^{n-1})}{x^{2n}} = -\dfrac{n}{x^{n+1}} = -nx^{-n-1}$.

Exercise Set 2.5

1. $f'(x) = -4\sin x + 2\cos x$

3. $f'(x) = 4x^2 \sin x - 8x\cos x$

5. $f'(x) = \dfrac{\sin x(5 + \sin x) - \cos x(5 - \cos x)}{(5 + \sin x)^2} = \dfrac{1 + 5(\sin x - \cos x)}{(5 + \sin x)^2}$

7. $f'(x) = \sec x \tan x - \sqrt{2}\sec^2 x$

9. $f'(x) = -4\csc x \cot x + \csc^2 x$

11. $f'(x) = \sec x(\sec^2 x) + (\tan x)(\sec x \tan x) = \sec^3 x + \sec x \tan^2 x$

13. $f'(x) = \dfrac{(1 + \csc x)(-\csc^2 x) - \cot x(0 - \csc x \cot x)}{(1 + \csc x)^2} = \dfrac{\csc x(-\csc x - \csc^2 x + \cot^2 x)}{(1 + \csc x)^2}$, but $1 + \cot^2 x = \csc^2 x$ (identity), thus $\cot^2 x - \csc^2 x = -1$, so $f'(x) = \dfrac{\csc x(-\csc x - 1)}{(1 + \csc x)^2} = -\dfrac{\csc x}{1 + \csc x}$.

15. $f(x) = \sin^2 x + \cos^2 x = 1$ (identity), so $f'(x) = 0$.

17. $f(x) = \dfrac{\tan x}{1 + x\tan x}$ (because $\sin x \sec x = (\sin x)(1/\cos x) = \tan x$), so $f'(x) = \dfrac{(1 + x\tan x)(\sec^2 x) - \tan x[x(\sec^2 x) + (\tan x)(1)]}{(1 + x\tan x)^2} = \dfrac{\sec^2 x - \tan^2 x}{(1 + x\tan x)^2} = \dfrac{1}{(1 + x\tan x)^2}$ (because $\sec^2 x - \tan^2 x = 1$).

19. $dy/dx = -x\sin x + \cos x$, $d^2y/dx^2 = -x\cos x - \sin x - \sin x = -x\cos x - 2\sin x$

21. $dy/dx = x(\cos x) + (\sin x)(1) - 3(-\sin x) = x\cos x + 4\sin x$,
$d^2y/dx^2 = x(-\sin x) + (\cos x)(1) + 4\cos x = -x\sin x + 5\cos x$

23. $dy/dx = (\sin x)(-\sin x) + (\cos x)(\cos x) = \cos^2 x - \sin^2 x$,
$d^2y/dx^2 = (\cos x)(-\sin x) + (\cos x)(-\sin x) - [(\sin x)(\cos x) + (\sin x)(\cos x)] = -4\sin x\cos x$

25. Let $f(x) = \tan x$, then $f'(x) = \sec^2 x$.

(a) $f(0) = 0$ and $f'(0) = 1$, so $y - 0 = (1)(x - 0)$, $y = x$.

(b) $f\left(\dfrac{\pi}{4}\right) = 1$ and $f'\left(\dfrac{\pi}{4}\right) = 2$, so $y - 1 = 2\left(x - \dfrac{\pi}{4}\right)$, $y = 2x - \dfrac{\pi}{2} + 1$.

(c) $f\left(-\dfrac{\pi}{4}\right) = -1$ and $f'\left(-\dfrac{\pi}{4}\right) = 2$, so $y + 1 = 2\left(x + \dfrac{\pi}{4}\right)$, $y = 2x + \dfrac{\pi}{2} - 1$.

27. **(a)** If $y = x\sin x$ then $y' = \sin x + x\cos x$ and $y'' = 2\cos x - x\sin x$ so $y'' + y = 2\cos x$.

(b) Differentiate the result of part **(a)** twice more to get $y^{(4)} + y'' = -2\cos x$.

29. **(a)** $f'(x) = \cos x = 0$ at $x = \pm\pi/2, \pm 3\pi/2$.

(b) $f'(x) = 1 - \sin x = 0$ at $x = -3\pi/2, \pi/2$.

(c) $f'(x) = \sec^2 x \geq 1$ always, so no horizontal tangent line.

(d) $f'(x) = \sec x \tan x = 0$ when $\sin x = 0$, $x = \pm 2\pi, \pm\pi, 0$.

31. $x = 10\sin\theta$, $dx/d\theta = 10\cos\theta$; if $\theta = 60°$, then $dx/d\theta = 10(1/2) = 5$ ft/rad $= \pi/36$ ft/deg ≈ 0.087 ft/deg.

33. $D = 50\tan\theta, dD/d\theta = 50\sec^2\theta$; if $\theta = 45°$, then $dD/d\theta = 50(\sqrt{2})^2 = 100$ m/rad $= 5\pi/9$ m/deg ≈ 1.75 m/deg.

35. False. $g'(x) = f(x)\cos x + f'(x)\sin x$

37. True. $f(x) = \dfrac{\sin x}{\cos x} = \tan x$, so $f'(x) = \sec^2 x$.

39. $\dfrac{d^4}{dx^4}\sin x = \sin x$, so $\dfrac{d^{4k}}{dx^{4k}}\sin x = \sin x$; $\dfrac{d^{87}}{dx^{87}}\sin x = \dfrac{d^3}{dx^3}\dfrac{d^{4\cdot 21}}{dx^{4\cdot 21}}\sin x = \dfrac{d^3}{dx^3}\sin x = -\cos x$.

41. $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, and $f^{(4)}(x) = \cos x$ with higher order derivatives repeating this pattern, so $f^{(n)}(x) = \sin x$ for $n = 3, 7, 11, \ldots$

43. **(a)** all x **(b)** all x **(c)** $x \neq \pi/2 + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

(d) $x \neq n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ **(e)** $x \neq \pi/2 + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ **(f)** $x \neq n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

(g) $x \neq (2n+1)\pi$, $n = 0, \pm 1, \pm 2, \ldots$ **(h)** $x \neq n\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$ **(i)** all x

45. $\dfrac{d}{dx}\sin x = \lim\limits_{w\to x}\dfrac{\sin w - \sin x}{w - x} = \lim\limits_{w\to x}\dfrac{2\sin\frac{w-x}{2}\cos\frac{w+x}{2}}{w-x} = \lim\limits_{w\to x}\dfrac{\sin\frac{w-x}{2}}{\frac{w-x}{2}}\cos\dfrac{w+x}{2} = 1\cdot\cos x = \cos x$.

47. **(a)** $\lim\limits_{h\to 0}\dfrac{\tan h}{h} = \lim\limits_{h\to 0}\dfrac{\left(\dfrac{\sin h}{\cos h}\right)}{h} = \lim\limits_{h\to 0}\dfrac{\left(\dfrac{\sin h}{h}\right)}{\cos h} = \dfrac{1}{1} = 1$.

(b) $\dfrac{d}{dx}[\tan x] = \lim\limits_{h\to 0}\dfrac{\tan(x+h) - \tan x}{h} = \lim\limits_{h\to 0}\dfrac{\dfrac{\tan x + \tan h}{1 - \tan x\tan h} - \tan x}{h} = \lim\limits_{h\to 0}\dfrac{\tan x + \tan h - \tan x + \tan^2 x\tan h}{h(1 - \tan x\tan h)} =$
$\lim\limits_{h\to 0}\dfrac{\tan h(1+\tan^2 x)}{h(1 - \tan x\tan h)} = \lim\limits_{h\to 0}\dfrac{\tan h\sec^2 x}{h(1-\tan x\tan h)} = \sec^2 x\lim\limits_{h\to 0}\dfrac{\dfrac{\tan h}{h}}{1 - \tan x\tan h} = \sec^2 x\dfrac{\lim\limits_{h\to 0}\dfrac{\tan h}{h}}{\lim\limits_{h\to 0}(1 - \tan x\tan h)} = \sec^2 x$.

49. By Exercises 49 and 50 of Section 1.6, we have $\lim\limits_{h\to 0}\dfrac{\sin h}{h} = \dfrac{\pi}{180}$ and $\lim\limits_{h\to 0}\dfrac{\cos h - 1}{h} = 0$. Therefore:

(a) $\dfrac{d}{dx}[\sin x] = \lim\limits_{h\to 0}\dfrac{\sin(x+h) - \sin x}{h} = \sin x\lim\limits_{h\to 0}\dfrac{\cos h - 1}{h} + \cos x\lim\limits_{h\to 0}\dfrac{\sin h}{h} = (\sin x)(0) + (\cos x)(\pi/180) =$
$\dfrac{\pi}{180}\cos x$.

(b) $\dfrac{d}{dx}[\cos x] = \lim\limits_{h\to 0}\dfrac{\cos(x+h) - \cos x}{h} = \lim\limits_{h\to 0}\dfrac{\cos x\cos h - \sin x\sin h - \cos x}{h} = \cos x\lim\limits_{h\to 0}\dfrac{\cos h - 1}{h} - \sin x\lim\limits_{h\to 0}\dfrac{\sin h}{h} =$
$0\cdot\cos x - \dfrac{\pi}{180}\cdot\sin x = -\dfrac{\pi}{180}\sin x$.

Exercise Set 2.6

1. $(f\circ g)'(x) = f'(g(x))g'(x)$, so $(f\circ g)'(0) = f'(g(0))g'(0) = f'(0)(3) = (2)(3) = 6$.

3. **(a)** $(f\circ g)(x) = f(g(x)) = (2x-3)^5$ and $(f\circ g)'(x) = f'(g(x))g'(x) = 5(2x-3)^4(2) = 10(2x-3)^4$.

(b) $(g\circ f)(x) = g(f(x)) = 2x^5 - 3$ and $(g\circ f)'(x) = g'(f(x))f'(x) = 2(5x^4) = 10x^4$.

5. **(a)** $F'(x) = f'(g(x))g'(x)$, $F'(3) = f'(g(3))g'(3) = -1(7) = -7$.

(b) $G'(x) = g'(f(x))f'(x)$, $G'(3) = g'(f(3))f'(3) = 4(-2) = -8$.

7. $f'(x) = 37(x^3+2x)^{36}\dfrac{d}{dx}(x^3+2x) = 37(x^3+2x)^{36}(3x^2+2)$.

9. $f'(x) = -2\left(x^3 - \dfrac{7}{x}\right)^{-3}\dfrac{d}{dx}\left(x^3 - \dfrac{7}{x}\right) = -2\left(x^3 - \dfrac{7}{x}\right)^{-3}\left(3x^2 + \dfrac{7}{x^2}\right)$.

11. $f(x) = 4(3x^2-2x+1)^{-3}$, $f'(x) = -12(3x^2-2x+1)^{-4}\dfrac{d}{dx}(3x^2-2x+1) = -12(3x^2-2x+1)^{-4}(6x-2) = \dfrac{24(1-3x)}{(3x^2-2x+1)^4}$.

13. $f'(x) = \dfrac{1}{2\sqrt{4+\sqrt{3x}}}\,\dfrac{d}{dx}(4+\sqrt{3x}) = \dfrac{\sqrt{3}}{4\sqrt{x}\sqrt{4+\sqrt{3x}}}$.

15. $f'(x) = \cos(1/x^2)\dfrac{d}{dx}(1/x^2) = -\dfrac{2}{x^3}\cos(1/x^2)$.

17. $f'(x) = 20\cos^4 x\dfrac{d}{dx}(\cos x) = 20\cos^4 x(-\sin x) = -20\cos^4 x\sin x$.

19. $f'(x) = 2\cos(3\sqrt{x})\dfrac{d}{dx}[\cos(3\sqrt{x})] = -2\cos(3\sqrt{x})\sin(3\sqrt{x})\dfrac{d}{dx}(3\sqrt{x}) = -\dfrac{3\cos(3\sqrt{x})\sin(3\sqrt{x})}{\sqrt{x}}$.

21. $f'(x) = 4\sec(x^7)\dfrac{d}{dx}[\sec(x^7)] = 4\sec(x^7)\sec(x^7)\tan(x^7)\dfrac{d}{dx}(x^7) = 28x^6\sec^2(x^7)\tan(x^7)$.

23. $f'(x) = \dfrac{1}{2\sqrt{\cos(5x)}}\dfrac{d}{dx}[\cos(5x)] = -\dfrac{5\sin(5x)}{2\sqrt{\cos(5x)}}$.

25. $f'(x) = -3\left[x + \csc(x^3+3)\right]^{-4}\dfrac{d}{dx}\left[x+\csc(x^3+3)\right] =$

$= -3\left[x+\csc(x^3+3)\right]^{-4}\left[1 - \csc(x^3+3)\cot(x^3+3)\dfrac{d}{dx}(x^3+3)\right] =$

$= -3\left[x+\csc(x^3+3)\right]^{-4}\left[1 - 3x^2\csc(x^3+3)\cot(x^3+3)\right]$.

27. $\dfrac{dy}{dx} = x^3(2\sin 5x)\dfrac{d}{dx}(\sin 5x) + 3x^2\sin^2 5x = 10x^3\sin 5x\cos 5x + 3x^2\sin^2 5x$.

29. $\dfrac{dy}{dx} = x^5\sec\left(\dfrac{1}{x}\right)\tan\left(\dfrac{1}{x}\right)\dfrac{d}{dx}\left(\dfrac{1}{x}\right) + \sec\left(\dfrac{1}{x}\right)(5x^4) = x^5\sec\left(\dfrac{1}{x}\right)\tan\left(\dfrac{1}{x}\right)\left(-\dfrac{1}{x^2}\right) + 5x^4\sec\left(\dfrac{1}{x}\right) =$

$= -x^3\sec\left(\dfrac{1}{x}\right)\tan\left(\dfrac{1}{x}\right) + 5x^4\sec\left(\dfrac{1}{x}\right)$.

31. $\dfrac{dy}{dx} = -\sin(\cos x)\dfrac{d}{dx}(\cos x) = -\sin(\cos x)(-\sin x) = \sin(\cos x)\sin x.$

33. $\dfrac{dy}{dx} = 3\cos^2(\sin 2x)\dfrac{d}{dx}[\cos(\sin 2x)] = 3\cos^2(\sin 2x)[-\sin(\sin 2x)]\dfrac{d}{dx}(\sin 2x) = -6\cos^2(\sin 2x)\sin(\sin 2x)\cos 2x.$

35. $\dfrac{dy}{dx} = (5x+8)^7\dfrac{d}{dx}(1-\sqrt{x})^6 + (1-\sqrt{x})^6\dfrac{d}{dx}(5x+8)^7 = 6(5x+8)^7(1-\sqrt{x})^5\dfrac{-1}{2\sqrt{x}} + 7\cdot 5(1-\sqrt{x})^6(5x+8)^6 =$
$\dfrac{-3}{\sqrt{x}}(5x+8)^7(1-\sqrt{x})^5 + 35(1-\sqrt{x})^6(5x+8)^6.$

37. $\dfrac{dy}{dx} = 3\left[\dfrac{x-5}{2x+1}\right]^2 \dfrac{d}{dx}\left[\dfrac{x-5}{2x+1}\right] = 3\left[\dfrac{x-5}{2x+1}\right]^2\cdot\dfrac{11}{(2x+1)^2} = \dfrac{33(x-5)^2}{(2x+1)^4}.$

39. $\dfrac{dy}{dx} = \dfrac{(4x^2-1)^8(3)(2x+3)^2(2)-(2x+3)^3(8)(4x^2-1)^7(8x)}{(4x^2-1)^{16}} = \dfrac{2(2x+3)^2(4x^2-1)^7[3(4x^2-1)-32x(2x+3)]}{(4x^2-1)^{16}} =$
$-\dfrac{2(2x+3)^2(52x^2+96x+3)}{(4x^2-1)^9}.$

41. $\dfrac{dy}{dx} = 5\left[x\sin 2x + \tan^4(x^7)\right]^4 \dfrac{d}{dx}\left[x\sin 2x\tan^4(x^7)\right] =$
$= 5\left[x\sin 2x + \tan^4(x^7)\right]^4\left[x\cos 2x\dfrac{d}{dx}(2x) + \sin 2x + 4\tan^3(x^7)\dfrac{d}{dx}\tan(x^7)\right] =$
$= 5\left[x\sin 2x + \tan^4(x^7)\right]^4\left[2x\cos 2x + \sin 2x + 28x^6\tan^3(x^7)\sec^2(x^7)\right].$

43. $\dfrac{dy}{dx} = \cos 3x - 3x\sin 3x$; if $x=\pi$ then $\dfrac{dy}{dx} = -1$ and $y=-\pi$, so the equation of the tangent line is $y+\pi = -(x-\pi)$, or $y=-x$.

45. $\dfrac{dy}{dx} = -3\sec^3(\pi/2-x)\tan(\pi/2-x)$; if $x=-\pi/2$ then $\dfrac{dy}{dx} = 0, y=-1$, so the equation of the tangent line is $y+1=0$, or $y=-1$

47. $\dfrac{dy}{dx} = \sec^2(4x^2)\dfrac{d}{dx}(4x^2) = 8x\sec^2(4x^2)$, $\left.\dfrac{dy}{dx}\right|_{x=\sqrt{\pi}} = 8\sqrt{\pi}\sec^2(4\pi) = 8\sqrt{\pi}$. When $x=\sqrt{\pi}$, $y=\tan(4\pi)=0$, so the equation of the tangent line is $y = 8\sqrt{\pi}(x-\sqrt{\pi}) = 8\sqrt{\pi}x - 8\pi$.

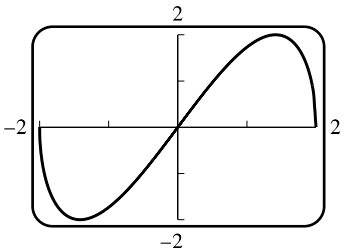
49. $\dfrac{dy}{dx} = 2x\sqrt{5-x^2} + \dfrac{x^2}{2\sqrt{5-x^2}}(-2x)$, $\left.\dfrac{dy}{dx}\right|_{x=1} = 4-1/2 = 7/2$. When $x=1, y=2$, so the equation of the tangent line is $y-2 = (7/2)(x-1)$, or $y = \dfrac{7}{2}x - \dfrac{3}{2}$.

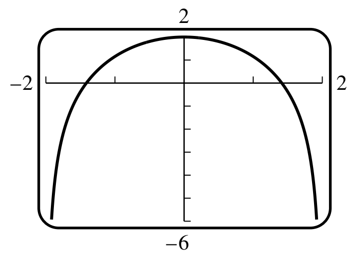
51. $\dfrac{dy}{dx} = x(-\sin(5x))\dfrac{d}{dx}(5x) + \cos(5x) - 2\sin x\dfrac{d}{dx}(\sin x) = -5x\sin(5x) + \cos(5x) - 2\sin x\cos x =$
$= -5x\sin(5x) + \cos(5x) - \sin(2x),$
$\dfrac{d^2y}{dx^2} = -5x\cos(5x)\dfrac{d}{dx}(5x) - 5\sin(5x) - \sin(5x)\dfrac{d}{dx}(5x) - \cos(2x)\dfrac{d}{dx}(2x) = -25x\cos(5x) - 10\sin(5x) - 2\cos(2x).$

53. $\dfrac{dy}{dx} = \dfrac{(1-x)+(1+x)}{(1-x)^2} = \dfrac{2}{(1-x)^2} = 2(1-x)^{-2}$ and $\dfrac{d^2y}{dx^2} = -2(2)(-1)(1-x)^{-3} = 4(1-x)^{-3}.$

55. $y = \cot^3(\pi-\theta) = -\cot^3\theta$ so $dy/dx = 3\cot^2\theta\csc^2\theta.$

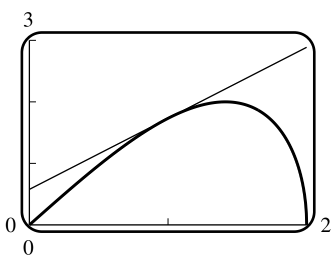
57. $\dfrac{d}{d\omega}[a\cos^2\pi\omega + b\sin^2\pi\omega] = -2\pi a\cos\pi\omega\sin\pi\omega + 2\pi b\sin\pi\omega\cos\pi\omega = \pi(b-a)(2\sin\pi\omega\cos\pi\omega) = \pi(b-a)\sin 2\pi\omega.$

59. **(a)** 



(c) $f'(x) = x\dfrac{-x}{\sqrt{4-x^2}} + \sqrt{4-x^2} = \dfrac{4-2x^2}{\sqrt{4-x^2}}$.

(d) $f(1) = \sqrt{3}$ and $f'(1) = \dfrac{2}{\sqrt{3}}$ so the tangent line has the equation $y - \sqrt{3} = \dfrac{2}{\sqrt{3}}(x-1)$.



61. False. $\dfrac{d}{dx}[\sqrt{y}\,] = \dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx} = \dfrac{f'(x)}{2\sqrt{f(x)}}$.

63. False. $dy/dx = -\sin[g(x)]\,g'(x)$.

65. **(a)** $dy/dt = -A\omega \sin \omega t, d^2y/dt^2 = -A\omega^2 \cos \omega t = -\omega^2 y$

(b) One complete oscillation occurs when ωt increases over an interval of length 2π, or if t increases over an interval of length $2\pi/\omega$.

(c) $f = 1/T$

(d) Amplitude = 0.6 cm, $T = 2\pi/15$ s/oscillation, $f = 15/(2\pi)$ oscillations/s.

67. By the chain rule, $\dfrac{d}{dx}\left[\sqrt{x+f(x)}\right] = \dfrac{1+f'(x)}{2\sqrt{x+f(x)}}$. From the graph, $f(x) = \dfrac{4}{3}x + 5$ for $x < 0$, so $f(-1) = \dfrac{11}{3}$, $f'(-1) = \dfrac{4}{3}$, and $\dfrac{d}{dx}\left[\sqrt{x+f(x)}\right]\Big|_{x=-1} = \dfrac{7/3}{2\sqrt{8/3}} = \dfrac{7\sqrt{6}}{24}$.

69. **(a)** $p \approx 10$ lb/in^2, $dp/dh \approx -2$ lb/in^2/mi. **(b)** $\dfrac{dp}{dt} = \dfrac{dp}{dh}\dfrac{dh}{dt} \approx (-2)(0.3) = -0.6$ lb/in^2/s.

71. With $u = \sin x$, $\dfrac{d}{dx}(|\sin x|) = \dfrac{d}{dx}(|u|) = \dfrac{d}{du}(|u|)\dfrac{du}{dx} = \dfrac{d}{du}(|u|)\cos x = \begin{cases} \cos x, & u > 0 \\ -\cos x, & u < 0 \end{cases} = \begin{cases} \cos x, & \sin x > 0 \\ -\cos x, & \sin x < 0 \end{cases}$
$= \begin{cases} \cos x, & 0 < x < \pi \\ -\cos x, & -\pi < x < 0 \end{cases}$

73. **(a)** For $x \neq 0, |f(x)| \leq |x|$, and $\lim_{x\to 0} |x| = 0$, so by the Squeezing Theorem, $\lim_{x\to 0} f(x) = 0$.

(b) If $f'(0)$ were to exist, then the limit (as x approaches 0) $\dfrac{f(x)-f(0)}{x-0} = \sin(1/x)$ would have to exist, but it doesn't.

(c) For $x \neq 0$, $f'(x) = x\left(\cos\dfrac{1}{x}\right)\left(-\dfrac{1}{x^2}\right) + \sin\dfrac{1}{x} = -\dfrac{1}{x}\cos\dfrac{1}{x} + \sin\dfrac{1}{x}$.

(d) If $x = \dfrac{1}{2\pi n}$ for an integer $n \neq 0$, then $f'(x) = -2\pi n\cos(2\pi n) + \sin(2\pi n) = -2\pi n$. This approaches $+\infty$ as $n \to -\infty$, so there are points x arbitrarily close to 0 where $f'(x)$ becomes arbitrarily large. Hence $\lim_{x\to 0} f'(x)$ does not exist.

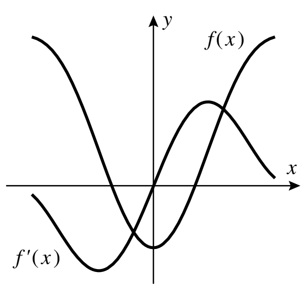
75. **(a)** $g'(x) = 3[f(x)]^2 f'(x)$, $g'(2) = 3[f(2)]^2 f'(2) = 3(1)^2(7) = 21$.

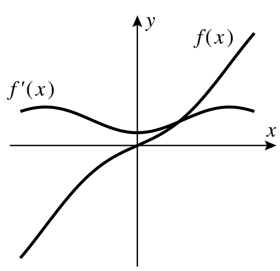
(b) $h'(x) = f'(x^3)(3x^2)$, $h'(2) = f'(8)(12) = (-3)(12) = -36$.

77. $F'(x) = f'(g(x))g'(x) = f'(\sqrt{3x-1})\dfrac{3}{2\sqrt{3x-1}} = \dfrac{\sqrt{3x-1}}{(3x-1)+1}\,\dfrac{3}{2\sqrt{3x-1}} = \dfrac{1}{2x}$.

79. $\dfrac{d}{dx}[f(3x)] = f'(3x)\dfrac{d}{dx}(3x) = 3f'(3x) = 6x$, so $f'(3x) = 2x$. Let $u = 3x$ to get $f'(u) = \dfrac{2}{3}u$; $\dfrac{d}{dx}[f(x)] = f'(x) = \dfrac{2}{3}x$.

81. For an even function, the graph is symmetric about the y-axis; the slope of the tangent line at $(a, f(a))$ is the negative of the slope of the tangent line at $(-a, f(-a))$. For an odd function, the graph is symmetric about the origin; the slope of the tangent line at $(a, f(a))$ is the same as the slope of the tangent line at $(-a, f(-a))$.





83. $\dfrac{d}{dx}[f(g(h(x)))] = \dfrac{d}{dx}[f(g(u))],\ u = h(x),\ \dfrac{d}{du}[f(g(u))]\dfrac{du}{dx} = f'(g(u))g'(u)\dfrac{du}{dx} = f'(g(h(x)))g'(h(x))h'(x)$.

Chapter 2 Review Exercises

3. **(a)** $m_{\tan} = \lim_{w\to x}\dfrac{f(w)-f(x)}{w-x} = \lim_{w\to x}\dfrac{(w^2+1)-(x^2+1)}{w-x} = \lim_{w\to x}\dfrac{w^2-x^2}{w-x} = \lim_{w\to x}(w+x) = 2x$.

(b) $m_{\tan} = 2(2) = 4$.

5. $v_{\text{inst}} = \lim_{h\to 0}\dfrac{3(h+1)^{2.5}+580h-3}{10h} = 58 + \dfrac{1}{10}\,\dfrac{d}{dx}3x^{2.5}\Big|_{x=1} = 58 + \dfrac{1}{10}(2.5)(3)(1)^{1.5} = 58.75$ ft/s.

7. **(a)** $v_{\text{ave}} = \dfrac{[3(3)^2+3]-[3(1)^2+1]}{3-1} = 13$ mi/h.

(b) $v_{\text{inst}} = \lim_{t_1\to 1}\dfrac{(3t_1^2+t_1)-4}{t_1-1} = \lim_{t_1\to 1}\dfrac{(3t_1+4)(t_1-1)}{t_1-1} = \lim_{t_1\to 1}(3t_1+4) = 7$ mi/h.

9. **(a)** $\dfrac{dy}{dx} = \lim_{h\to 0} \dfrac{\sqrt{9-4(x+h)} - \sqrt{9-4x}}{h} = \lim_{h\to 0} \dfrac{9-4(x+h)-(9-4x)}{h(\sqrt{9-4(x+h)}+\sqrt{9-4x})} =$

$= \lim_{h\to 0} \dfrac{-4h}{h(\sqrt{9-4(x+h)}+\sqrt{9-4x})} = \dfrac{-4}{2\sqrt{9-4x}} = \dfrac{-2}{\sqrt{9-4x}}.$

(b) $\dfrac{dy}{dx} = \lim_{h\to 0} \dfrac{\dfrac{x+h}{x+h+1} - \dfrac{x}{x+1}}{h} = \lim_{h\to 0} \dfrac{(x+h)(x+1)-x(x+h+1)}{h(x+h+1)(x+1)} = \lim_{h\to 0} \dfrac{h}{h(x+h+1)(x+1)} = \dfrac{1}{(x+1)^2}.$

11. **(a)** $x = -2, -1, 1, 3$ **(b)** $(-\infty, -2)$, $(-1, 1)$, $(3, +\infty)$ **(c)** $(-2, -1)$, $(1, 3)$

(d) $g''(x) = f''(x)\sin x + 2f'(x)\cos x - f(x)\sin x$; $g''(0) = 2f'(0)\cos 0 = 2(2)(1) = 4$

13. **(a)** The slope of the tangent line $\approx \dfrac{10-2.2}{2050-1950} = 0.078$ billion, so in 2000 the world population was increasing at the rate of about 78 million per year.

(b) $\dfrac{dN/dt}{N} \approx \dfrac{0.078}{6} = 0.013 = 1.3$ %/year

15. **(a)** $f'(x) = 2x\sin x + x^2\cos x$ **(c)** $f''(x) = 4x\cos x + (2-x^2)\sin x$

17. **(a)** $f'(x) = \dfrac{6x^2+8x-17}{(3x+2)^2}$ **(c)** $f''(x) = \dfrac{118}{(3x+2)^3}$

19. **(a)** $\dfrac{dW}{dt} = 200(t-15)$; at $t=5$, $\dfrac{dW}{dt} = -2000$; the water is running out at the rate of 2000 gal/min.

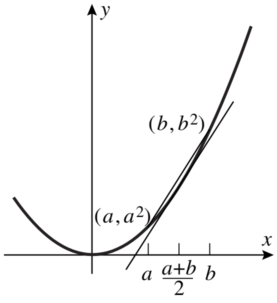
(b) $\dfrac{W(5)-W(0)}{5-0} = \dfrac{10000-22500}{5} = -2500$; the average rate of flow out is 2500 gal/min.

21. **(a)** $f'(x) = 2x, f'(1.8) = 3.6$ **(b)** $f'(x) = (x^2-4x)/(x-2)^2, f'(3.5) = -7/9 \approx -0.777778$

23. f is continuous at $x=1$ because it is differentiable there, thus $\lim_{h\to 0} f(1+h) = f(1)$ and so $f(1) = 0$ because $\lim_{h\to 0} \dfrac{f(1+h)}{h}$ exists; $f'(1) = \lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \dfrac{f(1+h)}{h} = 5$.

25. The equation of such a line has the form $y = mx$. The points (x_0, y_0) which lie on both the line and the parabola and for which the slopes of both curves are equal satisfy $y_0 = mx_0 = x_0^3 - 9x_0^2 - 16x_0$, so that $m = x_0^2 - 9x_0 - 16$. By differentiating, the slope is also given by $m = 3x_0^2 - 18x_0 - 16$. Equating, we have $x_0^2 - 9x_0 - 16 = 3x_0^2 - 18x_0 - 16$, or $2x_0^2 - 9x_0 = 0$. The root $x_0 = 0$ corresponds to $m = -16, y_0 = 0$ and the root $x_0 = 9/2$ corresponds to $m = -145/4, y_0 = -1305/8$. So the line $y = -16x$ is tangent to the curve at the point $(0,0)$, and the line $y = -145x/4$ is tangent to the curve at the point $(9/2, -1305/8)$.

27. The slope of the tangent line is the derivative $y' = 2x\Big|_{x=\frac{1}{2}(a+b)} = a+b$. The slope of the secant is $\dfrac{a^2-b^2}{a-b} = a+b$, so they are equal.



29. **(a)** $8x^7 - \dfrac{3}{2\sqrt{x}} - 15x^{-4}$ **(b)** $2 \cdot 101(2x+1)^{100}(5x^2-7) + 10x(2x+1)^{101} = (2x+1)^{100}(1030x^2+10x-1414)$

31. **(a)** $2(x-1)\sqrt{3x+1} + \dfrac{3}{2\sqrt{3x+1}}(x-1)^2 = \dfrac{(x-1)(15x+1)}{2\sqrt{3x+1}}$

(b) $3\left(\dfrac{3x+1}{x^2}\right)^2 \dfrac{x^2(3)-(3x+1)(2x)}{x^4} = -\dfrac{3(3x+1)^2(3x+2)}{x^7}$

33. Set $f'(x) = 0$: $f'(x) = 6(2)(2x+7)^5(x-2)^5 + 5(2x+7)^6(x-2)^4 = 0$, so $2x+7=0$ or $x-2=0$ or, factoring out $(2x+7)^5(x-2)^4$, $12(x-2)+5(2x+7)=0$. This reduces to $x=-7/2$, $x=2$, or $22x+11=0$, so the tangent line is horizontal at $x = -7/2, 2, -1/2$.

35. Suppose the line is tangent to $y = x^2+1$ at (x_0, y_0) and tangent to $y = -x^2-1$ at (x_1, y_1). Since it's tangent to $y = x^2+1$, its slope is $2x_0$; since it's tangent to $y=-x^2-1$, its slope is $-2x_1$. Hence $x_1 = -x_0$ and $y_1 = -y_0$. Since the line passes through both points, its slope is $\dfrac{y_1-y_0}{x_1-x_0} = \dfrac{-2y_0}{-2x_0} = \dfrac{y_0}{x_0} = \dfrac{x_0^2+1}{x_0}$. Thus $2x_0 = \dfrac{x_0^2+1}{x_0}$, so $2x_0^2 = x_0^2+1$, $x_0^2=1$, and $x_0=\pm 1$. So there are two lines which are tangent to both graphs, namely $y=2x$ and $y=-2x$.

37. The line $y-x=2$ has slope $m_1 = 1$ so we set $m_2 = \dfrac{d}{dx}(3x-\tan x) = 3-\sec^2 x = 1$, or $\sec^2 x = 2$, $\sec x = \pm\sqrt{2}$ so $x = n\pi \pm \pi/4$ where $n = 0, \pm 1, \pm 2, \ldots$.

39. $3 = f(\pi/4) = (M+N)\sqrt{2}/2$ and $1 = f'(\pi/4) = (M-N)\sqrt{2}/2$. Add these two equations to get $4 = \sqrt{2}M$, $M = 2^{3/2}$. Subtract to obtain $2 = \sqrt{2}N$, $N = \sqrt{2}$. Thus $f(x) = 2\sqrt{2}\sin x + \sqrt{2}\cos x$. $f'\left(\dfrac{3\pi}{4}\right) = -3$, so the tangent line is $y - 1 = -3\left(x - \dfrac{3\pi}{4}\right)$.

41. $f'(x) = 2xf(x), f(2) = 5$

(a) $g(x) = f(\sec x), g'(x) = f'(\sec x)\sec x \tan x = 2\cdot 2f(2)\cdot 2\cdot\sqrt{3} = 40\sqrt{3}$.

(b) $h'(x) = 4\left[\dfrac{f(x)}{x-1}\right]^3 \dfrac{(x-1)f'(x)-f(x)}{(x-1)^2}, h'(2) = 4\dfrac{5^3}{1}\dfrac{f'(2)-f(2)}{1} = 4\cdot 5^3 \dfrac{2\cdot 2f(2)-f(2)}{1} = 4\cdot 5^3\cdot 3\cdot 5 = 7500$

Chapter 2 Making Connections

1. **(a)** By property (ii), $f(0) = f(0+0) = f(0)f(0)$, so $f(0) = 0$ or 1. By property (iii), $f(0) \neq 0$, so $f(0) = 1$.

(b) By property (ii), $f(x) = f\left(\dfrac{x}{2}+\dfrac{x}{2}\right) = f\left(\dfrac{x}{2}\right)^2 \geq 0$. If $f(x) = 0$, then $1 = f(0) = f(x+(-x)) = f(x)f(-x) = 0\cdot f(-x) = 0$, a contradiction. Hence $f(x) > 0$.

(c) $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\dfrac{f(x)f(h)-f(x)}{h} = \lim\limits_{h\to 0} f(x)\dfrac{f(h)-1}{h} = f(x)\lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{h} = f(x)f'(0) = f(x)$

3. **(a)** For brevity, we omit the "(x)" throughout.

$(f\cdot g\cdot h)' = \dfrac{d}{dx}[(f\cdot g)\cdot h] = (f\cdot g)\cdot\dfrac{dh}{dx} + h\cdot\dfrac{d}{dx}(f\cdot g) = f\cdot g\cdot h' + h\cdot\left(f\cdot\dfrac{dg}{dx} + g\cdot\dfrac{df}{dx}\right)$
$= f'\cdot g\cdot h + f\cdot g'\cdot h + f\cdot g\cdot h'$

(b) $(f\cdot g\cdot h\cdot k)' = \dfrac{d}{dx}[(f\cdot g\cdot h)\cdot k] = (f\cdot g\cdot h)\cdot\dfrac{dk}{dx} + k\cdot\dfrac{d}{dx}(f\cdot g\cdot h)$

$= f \cdot g \cdot h \cdot k' + k \cdot (f' \cdot g \cdot h + f \cdot g' \cdot h + f \cdot g \cdot h') = f' \cdot g \cdot h \cdot k + f \cdot g' \cdot h \cdot k + f \cdot g \cdot h' \cdot k + f \cdot g \cdot h \cdot k'$

(c) Theorem: If $n \geq 1$ and $f_1, \cdots, f_n$ are differentiable functions of x, then

$(f_1 \cdot f_2 \cdot \cdots \cdot f_n)' = \sum_{i=1}^{n} f_1 \cdot \cdots \cdot f_{i-1} \cdot f_i' \cdot f_{i+1} \cdot \cdots \cdot f_n.$

Proof: For $n = 1$ the statement is obviously true: $f_1' = f_1'$. If the statement is true for $n - 1$, then

$(f_1 \cdot f_2 \cdot \cdots \cdot f_n)' = \frac{d}{dx}[(f_1 \cdot f_2 \cdot \cdots \cdot f_{n-1}) \cdot f_n] = (f_1 \cdot f_2 \cdot \cdots \cdot f_{n-1}) \cdot f_n' + f_n \cdot (f_1 \cdot f_2 \cdot \cdots \cdot f_{n-1})'$

$= f_1 \cdot f_2 \cdot \cdots \cdot f_{n-1} \cdot f_n' + f_n \cdot \sum_{i=1}^{n-1} f_1 \cdot \cdots \cdot f_{i-1} \cdot f_i' \cdot f_{i+1} \cdot \cdots \cdot f_{n-1} = \sum_{i=1}^{n} f_1 \cdot \cdots \cdot f_{i-1} \cdot f_i' \cdot f_{i+1} \cdot \cdots \cdot f_n$

so the statement is true for n. By induction, it's true for all n.

5. (a) By the chain rule, $\frac{d}{dx}\left([g(x)]^{-1}\right) = -[g(x)]^{-2}g'(x) = -\frac{g'(x)}{[g(x)]^2}$. By the product rule,

$h'(x) = f(x).\frac{d}{dx}\left([g(x)]^{-1}\right) + [g(x)]^{-1}.\frac{d}{dx}[f(x)] = -\frac{f(x)g'(x)}{[g(x)]^2} + \frac{f'(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$

(b) By the product rule, $f'(x) = \frac{d}{dx}[h(x)g(x)] = h(x)g'(x) + g(x)h'(x)$. So

$h'(x) = \frac{1}{g(x)}[f'(x) - h(x)g'(x)] = \frac{1}{g(x)}\left[f'(x) - \frac{f(x)}{g(x)}g'(x)\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$