Mathematical Modeling with Differential Equations

Exercise Set 8.1

1. $y' = 9x^2e^{x^3} = 3x^2y$ and $y(0) = 3$ by inspection.

3. **(a)** First order; $\dfrac{dy}{dx} = c$; $(1+x)\dfrac{dy}{dx} = (1+x)c = y$.

(b) Second order; $y' = c_1 \cos t - c_2 \sin t$, $y'' + y = -c_1 \sin t - c_2 \cos t + (c_1 \sin t + c_2 \cos t) = 0$.

5. False. It is a first-order equation, because it involves y and dy/dx, but not d^ny/dx^n for $n > 1$.

7. True. As mentioned in the marginal note after equation (2), the general solution of an n'th order differential equation usually involves n arbitrary constants.

9. **(a)** If $y = e^{-2x}$ then $y' = -2e^{-2x}$ and $y'' = 4e^{-2x}$, so $y'' + y' - 2y = 4e^{-2x} + (-2e^{-2x}) - 2e^{-2x} = 0$.
If $y = e^x$ then $y' = e^x$ and $y'' = e^x$, so $y'' + y' - 2y = e^x + e^x - 2e^x = 0$.

(b) If $y = c_1e^{-2x} + c_2e^x$ then $y' = -2c_1e^{-2x} + c_2e^x$ and $y'' = 4c_1e^{-2x} + c_2e^x$, so $y'' + y' - 2y = (4c_1e^{-2x} + c_2e^x) + (-2c_1e^{-2x} + c_2e^x) - 2(c_1e^{-2x} + c_2e^x) = 0$.

11. **(a)** If $y = e^{2x}$ then $y' = 2e^{2x}$ and $y'' = 4e^{2x}$, so $y'' - 4y' + 4y = 4e^{2x} - 4(2e^{2x}) + 4e^{2x} = 0$.
If $y = xe^{2x}$ then $y' = (2x+1)e^{2x}$ and $y'' = (4x+4)e^{2x}$, so $y'' - 4y' + 4y = (4x+4)e^{2x} - 4(2x+1)e^{2x} + 4xe^{2x} = 0$.

(b) If $y = c_1e^{2x} + c_2xe^{2x}$ then $y' = 2c_1e^{2x} + c_2(2x+1)e^{2x}$ and $y'' = 4c_1e^{2x} + c_2(4x+4)e^{2x}$, so $y'' - 4y' + 4y = (4c_1e^{2x} + c_2(4x+4)e^{2x}) - 4(2c_1e^{2x} + c_2(2x+1)e^{2x}) + 4(c_1e^{2x} + c_2xe^{2x}) = 0$.

13. **(a)** If $y = \sin 2x$ then $y' = 2\cos 2x$ and $y'' = -4\sin 2x$, so $y'' + 4y = -4\sin 2x + 4\sin 2x = 0$.
If $y = \cos 2x$ then $y' = -2\sin 2x$ and $y'' = -4\cos 2x$, so $y'' + 4y = -4\cos 2x + 4\cos 2x = 0$.

(b) If $y = c_1 \sin 2x + c_2 \cos 2x$ then $y' = 2c_1 \cos 2x - 2c_2 \sin 2x$ and $y'' = -4c_1 \sin 2x - 4c_2 \cos 2x$, so $y'' + 4y = (-4c_1 \sin 2x - 4c_2 \cos 2x) + 4(c_1 \sin 2x + c_2 \cos 2x) = 0$.

15. From Exercise 9, $y = c_1e^{-2x} + c_2e^x$ is a solution of the differential equation, with $y' = -2c_1e^{-2x} + c_2e^x$. Setting $y(0) = -1$ and $y'(0) = -4$ gives $c_1 + c_2 = -1$ and $-2c_1 + c_2 = -4$. So $c_1 = 1$, $c_2 = -2$, and $y = e^{-2x} - 2e^x$.

17. From Exercise 11, $y = c_1e^{2x} + c_2xe^{2x}$ is a solution of the differential equation, with $y' = 2c_1e^{2x} + c_2(2x+1)e^{2x}$. Setting $y(0) = 2$ and $y'(0) = 2$ gives $c_1 = 2$ and $2c_1 + c_2 = 2$, so $c_2 = -2$ and $y = 2e^{2x} - 2xe^{2x}$.

19. From Exercise 13, $y = c_1 \sin 2x + c_2 \cos 2x$ is a solution of the differential equation, with $y' = 2c_1 \cos 2x - 2c_2 \sin 2x$. Setting $y(0) = 1$ and $y'(0) = 2$ gives $c_2 = 1$ and $2c_1 = 2$, so $c_1 = 1$ and $y = \sin 2x + \cos 2x$.

21. $y' = 2 - 4x$, so $y = \int (2-4x)\,dx = -2x^2 + 2x + C$. Setting $y(0) = 3$ gives $C = 3$, so $y = -2x^2 + 2x + 3$.

23. If the solution has an inverse function $x(y)$ then, by equation (3) of Section 3.3, $\dfrac{dx}{dy} = \dfrac{1}{dy/dx} = y^{-2}$. So $x = \int y^{-2}\,dy = -y^{-1} + C$. When $x = 1$, $y = 2$, so $C = \dfrac{3}{2}$ and $x = \dfrac{3}{2} - y^{-1}$. Solving for y gives $y = \dfrac{2}{3-2x}$. The solution is valid for $x < \dfrac{3}{2}$.

25. By the product rule, $\dfrac{d}{dx}(x^2y) = x^2y' + 2xy = 0$, so $x^2y = C$ and $y = C/x^2$. Setting $y(1) = 2$ gives $C = 2$ so $y = 2/x^2$. The solution is valid for $x > 0$.

27. **(a)** $\dfrac{dy}{dt} = ky^2,\ y(0) = y_0,\ k > 0.$ **(b)** $\dfrac{dy}{dt} = -ky^2,\ y(0) = y_0,\ k > 0.$

29. **(a)** $\dfrac{ds}{dt} = \dfrac{1}{2}s.$ **(b)** $\dfrac{d^2s}{dt^2} = 2\dfrac{ds}{dt}.$

31. **(a)** Since $k > 0$ and $y > 0$, equation (3) gives $\dfrac{dy}{dt} = ky > 0$, so y is increasing.

(b) $\dfrac{d^2y}{dt^2} = \dfrac{d}{dt}(ky) = k\dfrac{dy}{dt} = k^2y > 0$, so y is concave upward.

33. **(a)** Both $y = 0$ and $y = L$ satisfy equation (6).

(b) The rate of growth is $\dfrac{dy}{dt} = ky(L-y)$; we wish to find the value of y which maximizes this. Since $\dfrac{d}{dy}[ky(L-y)] = k(L-2y)$, which is positive for $y < L/2$ and negative for $y > L/2$, the maximum growth rate occurs for $y = L/2$.

35. If $x = c_1\cos\left(\sqrt{\dfrac{k}{m}}\,t\right) + c_2\sin\left(\sqrt{\dfrac{k}{m}}\,t\right)$ then $\dfrac{dx}{dt} = c_2\sqrt{\dfrac{k}{m}}\cos\left(\sqrt{\dfrac{k}{m}}\,t\right) - c_1\sqrt{\dfrac{k}{m}}\sin\left(\sqrt{\dfrac{k}{m}}\,t\right)$ and $\dfrac{d^2x}{dt^2} = -c_1\dfrac{k}{m}\cos\left(\sqrt{\dfrac{k}{m}}\,t\right) - c_2\dfrac{k}{m}\sin\left(\sqrt{\dfrac{k}{m}}\,t\right) = -\dfrac{k}{m}x$. So $m\dfrac{d^2x}{dt^2} = -kx$; thus x satisfies the differential equation for the vibrating string.

Exercise Set 8.2

1. $\dfrac{1}{y}dy = \dfrac{1}{x}dx$, $\ln|y| = \ln|x| + C_1$, $\ln\left|\dfrac{y}{x}\right| = C_1$, $\dfrac{y}{x} = \pm e^{C_1} = C$, $y = Cx$, including $C = 0$ by inspection.

3. $\dfrac{dy}{1+y} = -\dfrac{x}{\sqrt{1+x^2}}dx$, $\ln|1+y| = -\sqrt{1+x^2} + C_1$, $1 + y = \pm e^{-\sqrt{1+x^2}}e^{C_1} = Ce^{-\sqrt{1+x^2}}$, $y = Ce^{-\sqrt{1+x^2}} - 1, C \neq 0$.

5. $\dfrac{2(1+y^2)}{y}\,dy = e^x dx,\ 2\ln|y| + y^2 = e^x + C$; by inspection, $y = 0$ is also a solution.

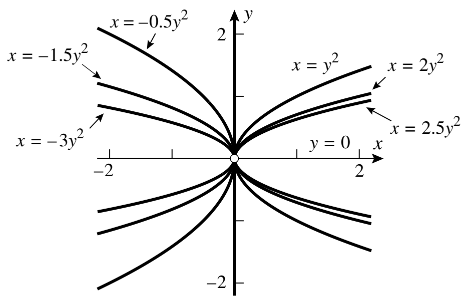
7. $e^y dy = \dfrac{\sin x}{\cos^2 x}dx = \sec x\tan x\,dx,\ e^y = \sec x + C,\ y = \ln(\sec x + C).$

9. $\dfrac{dy}{y^2 - y} = \dfrac{dx}{\sin x}$, $\int\left[-\dfrac{1}{y} + \dfrac{1}{y-1}\right]dy = \int \csc x\,dx,\ \ln\left|\dfrac{y-1}{y}\right| = \ln|\csc x - \cot x| + C_1$, $\dfrac{y-1}{y} = \pm e^{C_1}(\csc x - \cot x) = C(\csc x - \cot x),\ y = \dfrac{1}{1 - C(\csc x - \cot x)}, C \neq 0$; by inspection, $y = 0$ is also a solution, as is $y = 1$.

11. $(2y + \cos y)\,dy = 3x^2\,dx, y^2 + \sin y = x^3 + C, \pi^2 + \sin \pi = C, C = \pi^2, y^2 + \sin y = x^3 + \pi^2$.

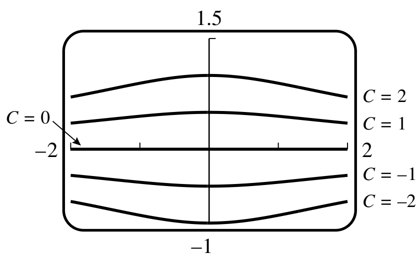
13. $2(y-1)\,dy = (2t+1)\,dt, y^2 - 2y = t^2 + t + C, 1 + 2 = C, C = 3, y^2 - 2y = t^2 + t + 3$.

15. **(a)** $\dfrac{dy}{y} = \dfrac{dx}{2x}$, $\ln|y| = \dfrac{1}{2}\ln|x| + C_1$, $|y| = C_2|x|^{1/2}$, $y^2 = Cx$; by inspection $y = 0$ is also a solution.

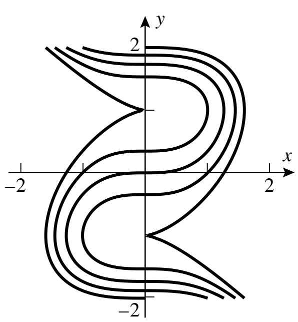


(b) $1^2 = C \cdot 2$, $C = 1/2$, $y^2 = x/2$.

17. $\dfrac{dy}{y} = -\dfrac{x\,dx}{x^2+4}$, $\ln|y| = -\dfrac{1}{2}\ln(x^2+4) + C_1$, $y = \dfrac{C}{\sqrt{x^2+4}}$.



19. $(1-y^2)\,dy = x^2\,dx$, $y - \dfrac{y^3}{3} = \dfrac{x^3}{3} + C_1$, $x^3 + y^3 - 3y = C$.



21. True. The equation can be rewritten as $\dfrac{1}{f(y)}\dfrac{dy}{dx} = 1$, which has the form (1).

23. True. After t minutes there will be $32 \cdot (1/2)^t$ grams left; when $t = 5$ there will be $32 \cdot (1/2)^5 = 1$ gram.

25. Of the solutions $y = \dfrac{1}{2x^2 - C}$, all pass through the point $\left(0, -\dfrac{1}{C}\right)$ and thus never through $(0,0)$. A solution of the initial value problem with $y(0) = 0$ is (by inspection) $y = 0$. The method of Example 1 fails in this case because it starts with a division by $y^2 = 0$.

27. $\dfrac{dy}{dx} = xe^{-y}, e^y\,dy = x\,dx, e^y = \dfrac{x^2}{2} + C, x = 2$ when $y = 0$ so $1 = 2 + C, C = -1, e^y = x^2/2 - 1$, so $y = \ln(x^2/2 - 1)$.

29. **(a)** $\dfrac{dy}{dt} = 0.02y,\ y_0 = 10{,}000.$ **(b)** $y = 10{,}000e^{t/50}$.

(c) $T = \dfrac{1}{0.02}\ln 2 \approx 34.657$ h. **(d)** $45{,}000 = 10{,}000e^{t/50},\ t = 50\ln\dfrac{45{,}000}{10{,}000} \approx 75.20$ h.

31. **(a)** $\dfrac{dy}{dt} = -ky,\ y(0) = 5.0\times 10^7;\ 3.83 = T = \dfrac{1}{k}\ln 2$, so $k = \dfrac{\ln 2}{3.83} \approx 0.1810$.

(b) $y = 5.0\times 10^7 e^{-0.181t}$.

(c) $y(30) = 5.0\times 10^7 e^{-0.1810(30)} \approx 219{,}000$.

(d) $y(t) = (0.1)y_0 = y_0e^{-kt},\ -kt = \ln 0.1,\ t = -\dfrac{\ln 0.1}{0.1810} = 12.72$ days.

33. $100e^{0.02t} = 10{,}000,\ e^{0.02t} = 100,\ t = \dfrac{1}{0.02}\ln 100 \approx 230$ days.

35. $y(t) = y_0e^{-kt} = 10.0e^{-kt},\ 3.5 = 10.0e^{-k(5)},\ k = -\dfrac{1}{5}\ln\dfrac{3.5}{10.0} \approx 0.2100,\ T = \dfrac{1}{k}\ln 2 \approx 3.30$ days.

39. **(a)** $T = \dfrac{\ln 2}{k}$; and $\ln 2 \approx 0.6931$. If k is measured in percent, $k' = 100k$, then $T = \dfrac{\ln 2}{k} \approx \dfrac{69.31}{k'} \approx \dfrac{70}{k'}$.

(b) 70 yr **(c)** 20 yr **(d)** 7%

41. From (19), $y(t) = y_0e^{-0.000121t}$. If $0.27 = \dfrac{y(t)}{y_0} = e^{-0.000121t}$ then $t = -\dfrac{\ln 0.27}{0.000121} \approx 10{,}820$ yr, and if $0.30 = \dfrac{y(t)}{y_0}$ then $t = -\dfrac{\ln 0.30}{0.000121} \approx 9950$, or roughly between 9000 B.C. and 8000 B.C.

43. **(a)** Let $T_1 = 5730 - 40 = 5690,\ k_1 = \dfrac{\ln 2}{T_1} \approx 0.00012182;\ T_2 = 5730 + 40 = 5770, k_2 \approx 0.00012013$. With $y/y_0 = 0.92, 0.93,\ t_1 = -\dfrac{1}{k_1}\ln\dfrac{y}{y_0} = 684.5, 595.7;\ t_2 = -\dfrac{1}{k_2}\ln(y/y_0) = 694.1, 604.1$; in 1988 the shroud was at most 695 years old, which places its creation in or after the year 1293.

(b) Suppose T is the true half-life of carbon-14 and $T_1 = T(1 + r/100)$ is the false half-life. Then with $k = \dfrac{\ln 2}{T},\ k_1 = \dfrac{\ln 2}{T_1}$ we have the formulae $y(t) = y_0e^{-kt},\ y_1(t) = y_0e^{-k_1t}$. At a certain point in time a reading of the carbon-14 is taken resulting in a certain value y, which in the case of the true formula is given by $y = y(t)$ for some t, and in the case of the false formula is given by $y = y_1(t_1)$ for some t_1. If the true formula is used then the time t since the beginning is given by $t = -\dfrac{1}{k}\ln\dfrac{y}{y_0}$. If the false formula is used we get a false value $t_1 = -\dfrac{1}{k_1}\ln\dfrac{y}{y_0}$; note that in both cases the value y/y_0 is the same. Thus $t_1/t = k/k_1 = T_1/T = 1 + r/100$, so the percentage error in the time to be measured is the same as the percentage error in the half-life.

45. **(a)** If $y = y_0e^{kt}$, then $y_1 = y_0e^{kt_1},\ y_2 = y_0e^{kt_2}$, divide: $y_2/y_1 = e^{k(t_2-t_1)},\ k = \dfrac{1}{t_2 - t_1}\ln(y_2/y_1),\ T = \dfrac{\ln 2}{k} = \dfrac{(t_2 - t_1)\ln 2}{\ln(y_2/y_1)}$. If $y = y_0e^{-kt}$, then $y_1 = y_0e^{-kt_1},\ y_2 = y_0e^{-kt_2},\ y_2/y_1 = e^{-k(t_2-t_1)}, k = -\dfrac{1}{t_2 - t_1}\ln(y_2/y_1),\ T = \dfrac{\ln 2}{k} = -\dfrac{(t_2 - t_1)\ln 2}{\ln(y_2/y_1)}$. In either case, T is positive, so $T = \left|\dfrac{(t_2 - t_1)\ln 2}{\ln(y_2/y_1)}\right|$.

(b) In part (a) assume $t_2 = t_1 + 1$ and $y_2 = 1.25y_1$. Then $T = \dfrac{\ln 2}{\ln 1.25} \approx 3.1$ h.

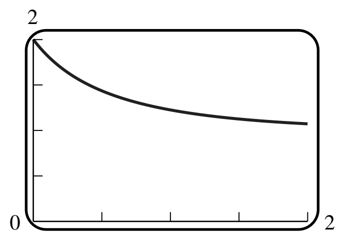
47. **(a)** $A = 1000e^{(0.08)(5)} = 1000e^{0.4} \approx \$1,491.82$.

(b) $Pe^{(0.08)(10)} = 10,000$, $Pe^{0.8} = 10,000$, $P = 10,000e^{-0.8} \approx \$4,493.29$.

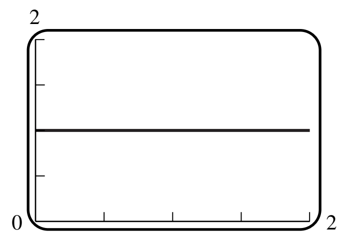
(c) From (11), with $k = r = 0.08$, $T = (\ln 2)/0.08 \approx 8.7$ years.

49. **(a)** Given $\dfrac{dy}{dt} = k\left(1 - \dfrac{y}{L}\right)y$, separation of variables yields $\left(\dfrac{1}{y} + \dfrac{1}{L-y}\right)dy = k\,dt$ so that $\ln\dfrac{y}{L-y} = \ln y - \ln(L-y) = kt + C$. The initial condition gives $C = \ln\dfrac{y_0}{L-y_0}$ so $\ln\dfrac{y}{L-y} = kt + \ln\dfrac{y_0}{L-y_0}$, $\dfrac{y}{L-y} = e^{kt}\dfrac{y_0}{L-y_0}$, and $y(t) = \dfrac{y_0 L}{y_0 + (L-y_0)e^{-kt}}$.

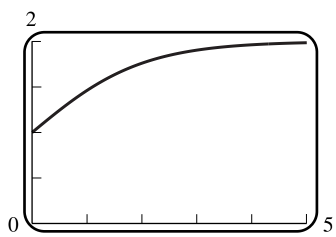
(b) If $y_0 > 0$ then $y_0 + (L-y_0)e^{-kt} = Le^{-kt} + y_0(1-e^{-kt}) > 0$ for all $t \ge 0$, so $y(t)$ exists for all such t. Since $\lim\limits_{t\to+\infty} e^{-kt} = 0$, $\lim\limits_{t\to+\infty} y(t) = \dfrac{y_0 L}{y_0 + (L-y_0)\cdot 0} = L$. (Note that for $y_0 < 0$ the solution "blows up" at $t = -\dfrac{1}{k}\ln\dfrac{-y_0}{L-y_0}$, so $\lim\limits_{t\to+\infty} y(t)$ is undefined.)



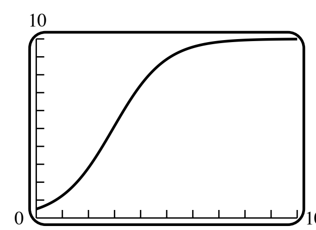
51. **(a)** $k = L = 1, y_0 = 2$.



(b) $k = L = y_0 = 1$.



(c) $k = y_0 = 1, L = 2$.



(d) $k = 1, y_0 = 0.5, L = 10$.

53. $y_0 \approx 2$, $L \approx 8$; since the curve $y = \dfrac{2\cdot 8}{2 + 6e^{-kt}}$ passes through the point $(2,4)$, $4 = \dfrac{16}{2+6e^{-2k}}$, $6e^{-2k} = 2$, $k = \dfrac{1}{2}\ln 3 \approx 0.5493$.

55. **(a)** $y_0 = 5$. **(b)** $L = 12$. **(c)** $k = 1$.

(d) $L/2 = 6 = \dfrac{60}{5 + 7e^{-t}}$, $5 + 7e^{-t} = 10$, $t = -\ln(5/7) \approx 0.3365$.

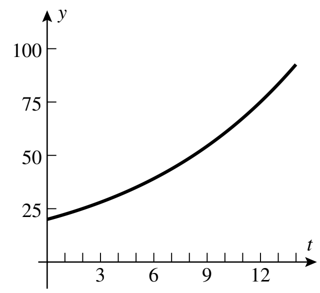
(e) $\dfrac{dy}{dt} = \dfrac{1}{12}y(12-y)$, $y(0) = 5$.

57. **(a)** Assume that $y(t)$ students have had the flu t days after the break. If the disease spreads as predicted by equation (6) of Section 8.1 and if nobody is immune, then Exercise 50 gives $y(t) = \dfrac{y_0 L}{y_0 + (L-y_0)e^{-kLt}}$, where $y_0 = 20$ and $L = 1000$. So $y(t) = \dfrac{20000}{20 + 980e^{-1000kt}} = \dfrac{1000}{1 + 49e^{-1000kt}}$. Using $y(5) = 35$ we find that $k = -\dfrac{\ln(193/343)}{5000}$. Hence $y = \dfrac{1000}{1 + 49(193/343)^{t/5}}$.

(b)

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$y(t)$	20	22	25	28	31	35	39	44	49	54	61	67	75	83	93

(c)



59. **(a)** From Exercise 58 with $T_0 = 95$ and $T_a = 21$, we have $T = 21 + 74e^{-kt}$ for some $k > 0$.

(b) $85 = T(1) = 21 + 74e^{-k}$, $k = -\ln\dfrac{64}{74} = -\ln\dfrac{32}{37}$, $T = 21 + 74e^{t\ln(32/37)} = 21 + 74\left(\dfrac{32}{37}\right)^t$, $T = 51$ when $\dfrac{30}{74} = \left(\dfrac{32}{37}\right)^t$, $t = \dfrac{\ln(30/74)}{\ln(32/37)} \approx 6.22$ min.

61. **(a)** $\dfrac{dv}{dt} = \dfrac{ck}{m_0 - kt} - g, v = -c\ln(m_0 - kt) - gt + C; v = 0$ when $t = 0$ so $0 = -c\ln m_0 + C$, $C = c\ln m_0, v = c\ln m_0 - c\ln(m_0 - kt) - gt = c\ln\dfrac{m_0}{m_0 - kt} - gt$.

(b) $m_0 - kt = 0.2m_0$ when $t = 100$, so $v = 2500\ln\dfrac{m_0}{0.2m_0} - 9.8(100) = 2500\ln 5 - 980 \approx 3044\,\mathrm{m/s}$.

63. **(a)** $A(h) = \pi(1)^2 = \pi, \pi\dfrac{dh}{dt} = -0.025\sqrt{h}, \dfrac{\pi}{\sqrt{h}}dh = -0.025dt, 2\pi\sqrt{h} = -0.025t + C; h = 4$ when $t = 0$, so $4\pi = C, 2\pi\sqrt{h} = -0.025t + 4\pi, \sqrt{h} = 2 - \dfrac{0.025}{2\pi}t, h \approx (2 - 0.003979\,t)^2$.

(b) $h = 0$ when $t \approx 2/0.003979 \approx 502.6$ s ≈ 8.4 min.

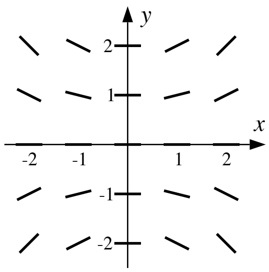
65. $\dfrac{dv}{dt} = -\dfrac{1}{32}v^2, \dfrac{1}{v^2}dv = -\dfrac{1}{32}dt, -\dfrac{1}{v} = -\dfrac{1}{32}t + C; v = 128$ when $t = 0$ so $-\dfrac{1}{128} = C$, $-\dfrac{1}{v} = -\dfrac{1}{32}t - \dfrac{1}{128}, v = \dfrac{128}{4t+1}$ cm/s. But $v = \dfrac{dx}{dt}$ so $\dfrac{dx}{dt} = \dfrac{128}{4t+1}, x = 32\ln(4t+1) + C_1$; $x = 0$ when $t = 0$ so $C_1 = 0, x = 32\ln(4t+1)$ cm.

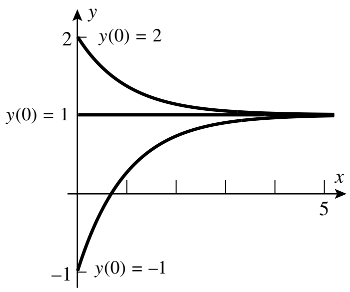
67. Suppose that $H(y) = G(x) + C$. Then $\dfrac{dH}{dy}\dfrac{dy}{dx} = G'(x)$. But $\dfrac{dH}{dy} = h(y)$ and $\dfrac{dG}{dx} = g(x)$, hence $y(x)$ is a solution of (1).

69. If $h(y) = 0$ then (1) implies that $g(x) = 0$, so $h(y)\,dy = 0 = g(x)\,dx$. Otherwise the slope of L is $\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)}$. Since (x_1, y_1) and (x_2, y_2) lie on L, we have $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{g(x)}{h(y)}$. So $h(y)(y_2 - y_1) = g(x)(x_2 - x_1)$; i.e. $h(y)\,dy = g(x)\,dx$.

71. Suppose that $y = f(x)$ satisfies $h(y)\dfrac{dy}{dx} = g(x)$. Integrating both sides of this with respect to x gives $\displaystyle\int h(y)\frac{dy}{dx}\,dx = \int g(x)\,dx$, so $\displaystyle\int h(f(x))f'(x)\,dx = \int g(x)\,dx$. By equation (2) of Section 5.3 with f replaced by h, g replaced by f, and F replaced by $\displaystyle\int h(y)\,dy$, the left side equals $F(f(x)) = F(y)$. Thus $\displaystyle\int h(y)\,dy = \int g(x)\,dx$.

Exercise Set 8.3

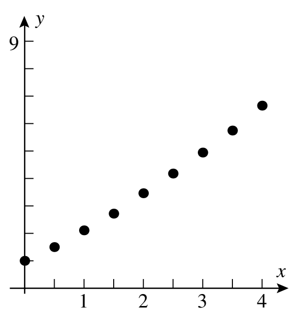
1. 

3. 

5. $\lim_{x\to+\infty} y = 1$.

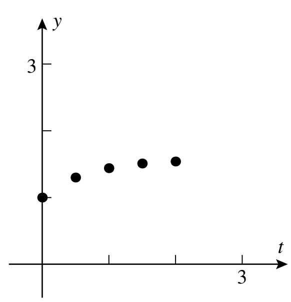
7. $y_0 = 1,\ y_{n+1} = y_n + \dfrac{1}{2}y_n^{1/3}$.

n	0	1	2	3	4	5	6	7	8
x_n	0	0.5	1	1.5	2	2.5	3	3.5	4
y_n	1	1.5	2.07	2.71	3.41	4.16	4.96	5.81	6.71



9. $y_0 = 1,\ y_{n+1} = y_n + \dfrac{1}{2}\cos y_n$.

n	0	1	2	3	4
t_n	0	0.5	1	1.5	2
y_n	1	1.27	1.42	1.49	1.53



11. $h = 1/5,\ y_0 = 1,\ y_{n+1} = y_n + \dfrac{1}{5}\sin(\pi n/5)$.

n	0	1	2	3	4	5
t_n	0	0.2	0.4	0.6	0.8	1.0
y_n	0.00	0.00	0.12	0.31	0.50	0.62

13. True. $\dfrac{dy}{dx} = e^{xy} > 0$ for all x and y. So, for any integral curve, y is an increasing function of x.

15. True. Every cubic polynomial has at least one real root. If $p(y_0) = 0$ then $y = y_0$ is an integral curve that is a horizontal line.

17. **(b)** $y\,dy = -x\,dx,\ y^2/2 = -x^2/2 + C_1,\ x^2 + y^2 = C$; if $y(0) = 1$ then $C = 1$ so $y(1/2) = \sqrt{3}/2$.

19. **(b)** The equation $y' = 1 - y$ is separable: $\dfrac{dy}{1-y} = dx$, so $\displaystyle\int \frac{dy}{1-y} = \int dx$, $-\ln|1-y| = x + C$. Substituting $x = 0$ and $y = -1$ gives $C = -\ln 2$, so $x = \ln 2 - \ln|1-y| = \ln\left|\dfrac{2}{1-y}\right|$. Since the integral curve stays below the line $y = 1$, we can drop the absolute value signs: $x = \ln\dfrac{2}{1-y}$ and $y = 1 - 2e^{-x}$. Solving $y = 0$ shows that the x-intercept is $\ln 2 \approx 0.693$.

21. **(a)** The slope field does not vary with x, hence along a given parallel line all values are equal since they only depend on the height y.

(b) As in part (a), the slope field does not vary with x; it is independent of x.

(c) From $G(y) - x = C$ we obtain $\dfrac{d}{dx}(G(y) - x) = \dfrac{1}{f(y)}\dfrac{dy}{dx} - 1 = \dfrac{d}{dx}C = 0$, i.e. $\dfrac{dy}{dx} = f(y)$.

23. **(a)** By implicit differentiation, $y^3 + 3xy^2\dfrac{dy}{dx} - 2xy - x^2\dfrac{dy}{dx} = 0,\ \dfrac{dy}{dx} = \dfrac{2xy - y^3}{3xy^2 - x^2}$.

(b) If $y(x)$ is an integral curve of the slope field in part (a), then $\dfrac{d}{dx}\{x[y(x)]^3 - x^2y(x)\} = [y(x)]^3 + 3xy(x)^2y'(x) - 2xy(x) - x^2y'(x) = 0$, so the integral curve must be of the form $x[y(x)]^3 - x^2y(x) = C$.

(c) $x[y(x)]^3 - x^2y(x) = 2$.

25. **(a)** For any n, y_n is the value of the discrete approximation at the right endpoint, that, is an approximation of $y(1)$. By increasing the number of subdivisions of the interval $[0, 1]$ one might expect more accuracy, and hence in the limit $y(1)$.

(b) For a fixed value of n we have, for $k = 1, 2, \ldots, n$, $y_k = y_{k-1} + y_{k-1}\dfrac{1}{n} = \dfrac{n+1}{n}y_{k-1}$. In particular $y_n = \dfrac{n+1}{n}y_{n-1} = \left(\dfrac{n+1}{n}\right)^2 y_{n-2} = \ldots = \left(\dfrac{n+1}{n}\right)^n y_0 = \left(\dfrac{n+1}{n}\right)^n$. Consequently, $\displaystyle\lim_{n\to+\infty} y_n = \lim_{n\to+\infty}\left(\frac{n+1}{n}\right)^n = e$, which is the (correct) value $y = e^x\Big|_{x=1}$.

27. Visual inspection of the slope field may show where the integral curves are increasing, decreasing, concave up, or concave down. It may also help to identify asymptotes for the integral curves. For example, in Exercise 3 we see that $y = 1$ is an integral curve that is an asymptote of all other integral curves. Those curves with $y < 1$ are increasing and concave down; those with $y > 1$ are decreasing and concave up.

Exercise Set 8.4

1. $\mu = e^{\int 4\,dx} = e^{4x}$, $e^{4x}y = \displaystyle\int e^x\,dx = e^x + C$, $y = e^{-3x} + Ce^{-4x}$.

3. $\mu = e^{\int dx} = e^x$, $e^xy = \displaystyle\int e^x\cos(e^x)dx = \sin(e^x) + C$, $y = e^{-x}\sin(e^x) + Ce^{-x}$.

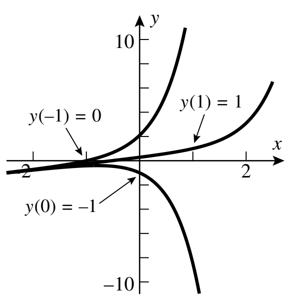
5. $\dfrac{dy}{dx} + \dfrac{x}{x^2+1}y = 0, \mu = e^{\int (x/(x^2+1))dx} = e^{\frac{1}{2}\ln(x^2+1)} = \sqrt{x^2+1}$, $\dfrac{d}{dx}\left[y\sqrt{x^2+1}\right] = 0$, $y\sqrt{x^2+1} = C$, $y = \dfrac{C}{\sqrt{x^2+1}}$.

7. $\dfrac{dy}{dx}+\dfrac{1}{x}y=1,\ \mu=e^{\int(1/x)dx}=e^{\ln x}=x,\ \dfrac{d}{dx}[xy]=x,\ xy=\dfrac{1}{2}x^2+C,\ y=\dfrac{x}{2}+\dfrac{C}{x},\ 2=y(1)=\dfrac{1}{2}+C, C=\dfrac{3}{2}, y=\dfrac{x}{2}+\dfrac{3}{2x}.$

9. $\mu=e^{-2\int x\,dx}=e^{-x^2},\ e^{-x^2}y=\displaystyle\int 2xe^{-x^2}dx=-e^{-x^2}+C,\ y=-1+Ce^{x^2},\ 3=-1+C,\ C=4,\ y=-1+4e^{x^2}.$

11. False. If y_1 and y_2 both satisfy $\dfrac{dy}{dx}+p(x)y=q(x)$ then $\dfrac{d}{dx}(y_1+y_2)+p(x)(y_1+y_2)=2q(x)$. Unless $q(x)=0$ for all x, y_1+y_2 is not a solution of the original differential equation.

13. True. The concentration in the tank will approach the concentration in the solution flowing into the tank.

15. 

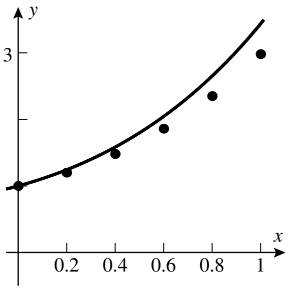
17. It appears that $\displaystyle\lim_{x\to+\infty} y=\begin{cases}+\infty, & \text{if } y_0\ge 1/4;\\ -\infty, & \text{if } y_0<1/4.\end{cases}$ To confirm this, we solve the equation using the method of integrating factors: $\dfrac{dy}{dx}-2y=-x,\ \mu=e^{-2\int dx}=e^{-2x},\ \dfrac{d}{dx}\left[ye^{-2x}\right]=-xe^{-2x},\ ye^{-2x}=\dfrac{1}{4}(2x+1)e^{-2x}+C,$ $y=\dfrac{1}{4}(2x+1)+Ce^{2x}$. Setting $y(0)=y_0$ gives $C=y_0-\dfrac{1}{4}$, so $y=\dfrac{1}{4}(2x+1)+\left(y_0-\dfrac{1}{4}\right)e^{2x}$. If $y_0=1/4$, then $y=\dfrac{1}{4}(2x+1)\to+\infty$ as $x\to+\infty$. Otherwise, we rewrite the solution as $y=e^{2x}\left(y_0-\dfrac{1}{4}+\dfrac{2x+1}{4e^{2x}}\right)$; since $\displaystyle\lim_{x\to+\infty}\frac{2x+1}{4e^{2x}}=0$, we obtain the conjectured limit.

19. (a) $y_0=1,\ y_{n+1}=y_n+(x_n+y_n)(0.2)=(x_n+6y_n)/5.$

n	0	1	2	3	4	5
x_n	0	0.2	0.4	0.6	0.8	1.0
y_n	1	1.20	1.48	1.86	2.35	2.98

(b) $y'-y=x,\ \mu=e^{-x},\ \dfrac{d}{dx}\left[ye^{-x}\right]=xe^{-x},\ ye^{-x}=-(x+1)e^{-x}+C,\ 1=-1+C,\ C=2,\ y=-(x+1)+2e^x.$

x_n	0	0.2	0.4	0.6	0.8	1.0
$y(x_n)$	1	1.24	1.58	2.04	2.65	3.44
abs. error	0	0.04	0.10	0.19	0.30	0.46
perc. error	0	3	6	9	11	13



(c)

21. $\frac{dy}{dt}$ = rate in − rate out, where y is the amount of salt at time t, $\frac{dy}{dt} = (4)(2) - \left(\frac{y}{50}\right)(2) = 8 - \frac{1}{25}y$, so $\frac{dy}{dt} + \frac{1}{25}y = 8$ and $y(0) = 25$. $\mu = e^{\int(1/25)dt} = e^{t/25}$, $e^{t/25}y = \int 8e^{t/25}dt = 200e^{t/25} + C$, $y = 200 + Ce^{-t/25}$, $25 = 200 + C$, $C = -175$,

(a) $y = 200 - 175e^{-t/25}$ oz. **(b)** when $t = 25$, $y = 200 - 175e^{-1} \approx 136$ oz.

23. **(a)** $\frac{dy}{dt}$ = rate in − rate out $= 145 - 0.09y$, $y(0) = y_0 = 300$.

(b) $\frac{dy}{dt} + 0.09y = 145$; $\mu = e^{\int 0.09\,dt} = e^{0.09t}$, thus $e^{0.09t}y = \int 145e^{0.09t}\,dt = \frac{14500}{9}e^{0.09t} + C$, i.e. $y = \frac{14500}{9} + Ce^{-0.09t}$. Also, $300 = y(0) = \frac{14500}{9} + C$, thus $C = -\frac{11800}{9}$. We obtain $y = \frac{1}{9}(14500 - 11800e^{-0.09t})$.

(c) $\lim_{t\to\infty} y(t) = \frac{14500}{9} \approx 1611$ (μg), thus the man should modify his diet.

25. **(a)** $\frac{dv}{dt} + \frac{c}{m}v = -g, \mu = e^{(c/m)\int dt} = e^{ct/m}, \frac{d}{dt}\left[ve^{ct/m}\right] = -ge^{ct/m}, ve^{ct/m} = -\frac{gm}{c}e^{ct/m} + C, v = -\frac{gm}{c} + Ce^{-ct/m}$, but $v_0 = v(0) = -\frac{gm}{c} + C, C = v_0 + \frac{gm}{c}, v = -\frac{gm}{c} + \left(v_0 + \frac{gm}{c}\right)e^{-ct/m}$.

(b) Replace $\frac{mg}{c}$ with v_τ and $-ct/m$ with $-gt/v_\tau$ in (16).

(c) From part (b), $s(t) = C - v_\tau t - (v_0 + v_\tau)\frac{v_\tau}{g}e^{-gt/v_\tau}$; $s_0 = s(0) = C - (v_0 + v_\tau)\frac{v_\tau}{g}$, $C = s_0 + (v_0 + v_\tau)\frac{v_\tau}{g}$, $s(t) = s_0 - v_\tau t + \frac{v_\tau}{g}(v_0 + v_\tau)\left(1 - e^{-gt/v_\tau}\right)$.

27. $\frac{dI}{dt} + \frac{R}{L}I = \frac{V(t)}{L}, \mu = e^{(R/L)\int dt} = e^{Rt/L}, \frac{d}{dt}(e^{Rt/L}I) = \frac{V(t)}{L}e^{Rt/L}, Ie^{Rt/L} = I(0) + \frac{1}{L}\int_0^t V(u)e^{Ru/L}du$, so $I(t) = I(0)e^{-Rt/L} + \frac{1}{L}e^{-Rt/L}\int_0^t V(u)e^{Ru/L}du$.

(a) $I(t) = \frac{1}{5}e^{-2t}\int_0^t 20e^{2u}du = 2e^{-2t}e^{2u}\Big]_0^t = 2\left(1 - e^{-2t}\right)$ A. **(b)** $\lim_{t\to+\infty} I(t) = 2$ A.

29. **(a)** Let $y = \frac{1}{\mu}[H(x) + C]$ where $\mu = e^{P(x)}$, $\frac{dP}{dx} = p(x)$, $\frac{d}{dx}H(x) = \mu q$, and C is an arbitrary constant. Then $\frac{dy}{dx} + p(x)y = \frac{1}{\mu}H'(x) - \frac{\mu'}{\mu^2}[H(x) + C] + p(x)y = q - \frac{p}{\mu}[H(x) + C] + p(x)y = q$.

(b) Given the initial value problem, let $C = \mu(x_0)y_0 - H(x_0)$. Then $y = \frac{1}{\mu}[H(x) + C]$ is a solution of the initial value problem with $y(x_0) = y_0$. This shows that the initial value problem has a solution. To show uniqueness, suppose $u(x)$ also satisfies (3) together with $u(x_0) = y_0$. Following the arguments in the text we arrive at

$u(x) = \dfrac{1}{\mu}[H(x) + C]$ for some constant C. The initial condition requires $C = \mu(x_0)y_0 - H(x_0)$, and thus $u(x)$ is identical with $y(x)$.

Chapter 8 Review Exercises

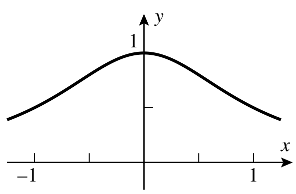
1. **(a)** Linear. **(b)** Both. **(c)** Separable. **(d)** Neither.

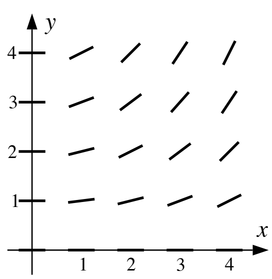
3. $\dfrac{dy}{1+y^2} = x^2\,dx, \tan^{-1} y = \dfrac{1}{3}x^3 + C, y = \tan\left(\dfrac{1}{3}x^3 + C\right)$.

5. $\left(\dfrac{1}{y} + y\right)\,dy = e^x dx,\ \ln|y| + y^2/2 = e^x + C$; by inspection, $y = 0$ is also a solution.

7. $\left(\dfrac{1}{y^5} + \dfrac{1}{y}\right)\,dy = \dfrac{dx}{x},\ -\dfrac{1}{4}y^{-4} + \ln|y| = \ln|x| + C;\ -\dfrac{1}{4} = C,\ y^{-4} + 4\ln(x/y) = 1$.

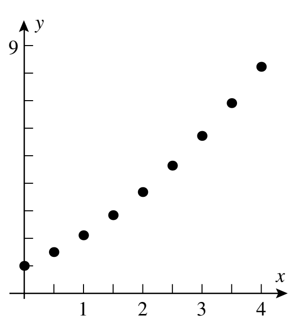
9. $\dfrac{dy}{y^2} = -2x\,dx,\ -\dfrac{1}{y} = -x^2 + C,\ -1 = C,\ y = 1/(x^2+1)$.



11. 

13. $y_0 = 1,\ y_{n+1} = y_n + \sqrt{y_n}/2$.

n	0	1	2	3	4	5	6	7	8
x_n	0	0.5	1	1.5	2	2.5	3	3.5	4
y_n	1	1.50	2.11	2.84	3.68	4.64	5.72	6.91	8.23



15. $h = 1/5,\ y_0 = 1,\ y_{n+1} = y_n + \dfrac{1}{5}\cos(2\pi n/5)$.

n	0	1	2	3	4	5
t_n	0	0.2	0.4	0.6	0.8	1.0
y_n	1.00	1.20	1.26	1.10	0.94	1.00

17. **(a)** $k = \dfrac{\ln 2}{5} \approx 0.1386$; $y \approx 2e^{0.1386t}$. **(b)** $y(t) = 5e^{0.015t}$.

(c) $y = y_0 e^{kt}$, $1 = y_0 e^k$, $100 = y_0 e^{10k}$. We obtain that $100 = e^{9k}$, $k = \dfrac{1}{9}\ln 100 \approx 0.5117$, $y \approx y_0 e^{0.5117t}$; also $y(1) = 1$, so $y_0 = e^{-0.5117} \approx 0.5995$, $y \approx 0.5995e^{0.5117t}$.

(d) $\dfrac{\ln 2}{T} \approx 0.1386$, $1 = y(1) \approx y_0 e^{0.1386}$, $y_0 \approx e^{-0.1386} \approx 0.8706$, $y \approx 0.8706e^{0.1386t}$.

19. $\mu = e^{\int 3\,dx} = e^{3x}$, $e^{3x}y = \displaystyle\int e^x\,dx = e^x + C$, $y = e^{-2x} + Ce^{-3x}$.

21. $\mu = e^{-\int x\,dx} = e^{-x^2/2}$, $e^{-x^2/2}y = \displaystyle\int xe^{-x^2/2}dx = -e^{-x^2/2} + C$, $y = -1 + Ce^{x^2/2}$, $3 = -1 + C$, $C = 4$, $y = -1 + 4e^{x^2/2}$.

23. By inspection, the left side of the equation is $\dfrac{d}{dx}(y\cosh x)$, so $\dfrac{d}{dx}(y\cosh x) = \cosh^2 x = \dfrac{1}{2}(1 + \cosh 2x)$ and $y\cosh x = \dfrac{1}{2}x + \dfrac{1}{4}\sinh 2x + C = \dfrac{1}{2}(x + \sinh x\cosh x) + C$. When $x = 0$, $y = 2$ so $2 = C$, and $y = 2\,\mathrm{sech}\,x + \dfrac{1}{2}(x\,\mathrm{sech}\,x + \sinh x)$.

25. Assume the tank contains $y(t)$ oz of salt at time t. Then $y_0 = 0$ and for $0 < t < 15$, $\dfrac{dy}{dt} = 5\cdot 10 - \dfrac{y}{1000}10 = (50 - y/100)$ oz/min, with solution $y = 5000 + Ce^{-t/100}$. But $y(0) = 0$ so $C = -5000$, $y = 5000(1 - e^{-t/100})$ for $0 \le t \le 15$, and $y(15) = 5000(1 - e^{-0.15})$. For $15 < t < 30$, $\dfrac{dy}{dt} = 0 - \dfrac{y}{1000}5$, $y = C_1 e^{-t/200}$, $C_1 e^{-0.075} = y(15) = 5000(1 - e^{-0.15})$, $C_1 = 5000(e^{0.075} - e^{-0.075})$, $y = 5000(e^{0.075} - e^{-0.075})e^{-t/200}$, $y(30) = 5000(e^{0.075} - e^{-0.075})e^{-0.15} \approx 646.14$ oz.

Chapter 8 Making Connections

1. **(a)** $u(x) = q - p\,y(x)$ so $\dfrac{du}{dx} = -p\dfrac{dy}{dx} = -p(q - py(x)) = (-p)u(x)$. If $p < 0$ then $-p > 0$ so $u(x)$ grows exponentially. If $p > 0$ then $-p < 0$ so $u(x)$ decays exponentially.

(b) From (a), $u(x) = 4 - 2y(x)$ satisfies $\dfrac{du}{dx} = -2u(x)$, so equation (14) of Section 8.2 gives $u(x) = u_0 e^{-2x}$ for some constant u_0. Since $u(0) = 4 - 2y(0) = 6$, we have $u(x) = 6e^{-2x}$; hence $y(x) = 2 - 3e^{-2x}$.

3. **(a)** $\dfrac{du}{dx} = \dfrac{d}{dx}\left(\dfrac{y}{x}\right) = \dfrac{x\dfrac{dy}{dx} - y}{x^2} = \dfrac{x\,f\left(\dfrac{y}{x}\right) - y}{x^2}$. Since $y = ux$, $\dfrac{du}{dx} = \dfrac{x\,f(u) - ux}{x^2} = \dfrac{f(u) - u}{x}$ and $\dfrac{1}{f(u) - u}\dfrac{du}{dx} = \dfrac{1}{x}$.

(b) $\dfrac{dy}{dx} = \dfrac{x - y}{x + y} = \dfrac{1 - y/x}{1 + y/x}$ has the form given in (a), with $f(t) = \dfrac{1 - t}{1 + t}$. So $\dfrac{1}{\dfrac{1 - u}{1 + u} - u}\dfrac{du}{dx} = \dfrac{1}{x}$, $\dfrac{1 + u}{1 - 2u - u^2}du = \dfrac{dx}{x}$, $\displaystyle\int \dfrac{1 + u}{1 - 2u - u^2}du = \int \dfrac{dx}{x}$, $-\dfrac{1}{2}\ln|1 - 2u - u^2| = \ln|x| + C_1$, and $|1 - 2u - u^2| = e^{-2C_1}x^{-2}$. Hence $1 - 2u - u^2 =$

Cx^{-2} where C is either e^{-2C_1} or $-e^{-2C_1}$. Substituting $u = \dfrac{y}{x}$ gives $1 - \dfrac{2y}{x} - \dfrac{y^2}{x^2} = Cx^{-2}$, and $x^2 - 2xy - y^2 = C$.