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INTEGRATING RATIONAL FUNCTIONS
BY
PARTIAL FRACTIONS

In this section we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called *partial fractions*, that we already know how to integrate. To illustrate the method, observe that by taking the fractions $2/(x-1)$ and $1/(x+2)$ to a common denominator we obtain

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$$

this equation:

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right) dx$$
$$= 2\ln|x-1| - \ln|x+2| + C$$

To see how the method of partial fractions works in general, let's consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

NOTE:

- The integrand or *f(x)* should be ***proper fraction***.
- If *f(x) is* ***improper,***

  If f is *improper*, that is, $\deg(P) \geqslant \deg(Q)$, then we must take the preliminary step of dividing Q into P (by long division) until a remainder $R(x)$ is obtained such that $\deg(R) < \deg(Q)$. The division statement is

$$\boxed{1} \qquad f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

$$\text{Find} \int \frac{x^3 + x}{x - 1}\, dx.$$

Since the degree of the numerator is greater than the degree of the denominator, we first perform the long division.

$$\begin{array}{r}
x^2 + x + 2 \\
x - 1 \overline{\smash{)}\, x^3 \qquad + x} \\
\underline{x^3 - x^2 \qquad} \\
x^2 + x \\
\underline{x^2 - x} \\
2x \\
\underline{2x - 2} \\
2
\end{array}$$

$$\int \frac{x^3 + x}{x - 1}\, dx = \int \left(x^2 + x + 2 + \frac{2}{x - 1} \right) dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2 \ln|x - 1| + C$$

LINEAR FACTOR RULE For each factor of the form $(ax+b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$$

where $A_1, A_2, \ldots, A_m$ are constants to be determined. In the case where $m = 1$, only the first term in the sum appears.

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Evaluate $\int \frac{dx}{x^2 + x - 2}$.

The integrand is a proper rational function that can be written as

$$\frac{1}{x^2 + x - 2} = \frac{1}{(x-1)(x+2)}$$

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} \longrightarrow \text{Eq. 5}$$

where A and B are constants to be determined. Multiplying this expression through $(x-1)(x+2)$ yields

$$1 = A(x+2) + B(x-1) \longrightarrow \text{Eq. 6}$$

Setting $x = 1$ makes the second term in (6) drop out and yields $1 = 3A$ or $A = \frac{1}{3}$; and setting $x = -2$ makes the first term in (6) drop out and yields $1 = -3B$ or $B = -\frac{1}{3}$. Substituting these values in (5) yields the partial fraction decomposition

$$\frac{1}{(x-1)(x+2)} = \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}}{x+2}$$

The integration can now be completed as follows:

$$\int \frac{dx}{(x-1)(x+2)} = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{dx}{x+2}$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C$$

CASE I The denominator $Q(x)$ is a product of distinct linear factors.

Evaluate $\displaystyle\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx$.

$$2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x - 1)(x + 2)$$

Since the denominator has three distinct linear factors, the partial fraction decomposition of the integrand $\boxed{2}$ has the form

$$\boxed{3} \qquad \frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}$$

To determine the values of A, B, and C, we multiply both sides of this equation by the product of the denominators, $x(2x - 1)(x + 2)$, obtaining

$$\boxed{4} \qquad x^2 + 2x - 1 = A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1)$$

Expanding the right side of Equation 4 and writing it in the standard form for polynomials, we get

$$\boxed{5} \qquad x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A$$

$$
\begin{aligned}
2A + B + 2C &= 1 \\
3A + 2B - C &= 2 \\
-2A \qquad\qquad &= -1
\end{aligned}
$$

Solving, we get $A = \frac{1}{2}$, $B = \frac{1}{5}$, and $C = -\frac{1}{10}$, and so

$$
\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx = \int \left[\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} - \frac{1}{10}\frac{1}{x+2}\right] dx
$$

$$
= \tfrac{1}{2}\ln|x| + \tfrac{1}{10}\ln|2x-1| - \tfrac{1}{10}\ln|x+2| + K
$$

EXAMPLE 4 Find $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx$.

SOLUTION The first step is to divide. The result of long division is

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

The second step is to factor the denominator $Q(x) = x^3 - x^2 - x + 1$.

$$\begin{aligned} x^3 - x^2 - x + 1 &= (x - 1)(x^2 - 1) = (x - 1)(x - 1)(x + 1) \\ &= (x - 1)^2(x + 1) \end{aligned}$$

Since the linear factor $x - 1$ occurs twice, the partial fraction decomposition is

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Multiplying by the least common denominator, $(x-1)^2(x+1)$, we get

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

Solving, we obtain $A = 1$, $B = 2$, and $C = -1$, so

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx = \int \left[x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\right] dx$$

$$= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + K$$

$$= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + K \qquad \blacksquare$$

QUADRATIC FACTOR RULE For each factor of the form $(ax^2 + bx + c)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}$$

where $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$ are constants to be determined. In the case where $m = 1$, only the first term in the sum appears.

CASE III $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

Evaluate $\displaystyle\int \frac{x^2+x-2}{3x^3-x^2+3x-1}\,dx.$

Solution. The denominator in the integrand can be factored by grouping:

$$3x^3 - x^2 + 3x - 1 = x^2(3x-1) + (3x-1) = (3x-1)(x^2+1)$$

By the linear factor rule, the factor $3x-1$ introduces one term, namely,

$$\frac{A}{3x-1}$$

and by the quadratic factor rule, the factor x^2+1 introduces one term, namely,

$$\frac{Bx+C}{x^2+1}$$

Thus, the partial fraction decomposition is

$$\frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1} \tag{10}$$

Multiplying by $(3x-1)(x^2+1)$ yields

$$x^2+x-2 = A(x^2+1) + (Bx+C)(3x-1) \tag{11}$$

$$x^2 + x - 2 = (A + 3B)x^2 + (-B + 3C)x + (A - C)$$

Equating corresponding coefficients gives

$$\begin{aligned} A + 3B \phantom{{}+3C} &= 1 \\ - B + 3C &= 1 \\ A \phantom{{}+3B} - C &= -2 \end{aligned}$$

To solve this system, subtract the third equation from the first to eliminate A. Then use the resulting equation together with the second equation to solve for B and C. Finally, determine A from the first or third equation. This yields (verify)

$$A = -\frac{7}{5}, \quad B = \frac{4}{5}, \quad C = \frac{3}{5}$$

Thus, (10) becomes

$$\frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{-\frac{7}{5}}{3x-1} + \frac{\frac{4}{5}x+\frac{3}{5}}{x^2+1}$$

and

$$\begin{aligned}\int \frac{x^2+x-2}{(3x-1)(x^2+1)}\,dx &= -\frac{7}{5}\int \frac{dx}{3x-1} + \frac{4}{5}\int \frac{x}{x^2+1}\,dx + \frac{3}{5}\int \frac{dx}{x^2+1} \\ &= -\frac{7}{15}\ln|3x-1| + \frac{2}{5}\ln(x^2+1) + \frac{3}{5}\tan^{-1}x + C\end{aligned}$$

CASE IV $Q(x)$ contains a repeated irreducible quadratic factor.

EXAMPLE 7 Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

SOLUTION

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx + D}{x^2+x+1} + \frac{Ex + F}{x^2+1} + \frac{Gx + H}{(x^2+1)^2} + \frac{Ix + J}{(x^2+1)^3}$$

9–34 Evaluate the integral.

9. $\displaystyle\int \frac{dx}{x^2 - 3x - 4}$

10. $\displaystyle\int \frac{dx}{x^2 - 6x - 7}$

11. $\displaystyle\int \frac{11x + 17}{2x^2 + 7x - 4}\, dx$

12. $\displaystyle\int \frac{5x - 5}{3x^2 - 8x - 3}\, dx$

13. $\displaystyle\int \frac{2x^2 - 9x - 9}{x^3 - 9x}\, dx$

14. $\displaystyle\int \frac{dx}{x(x^2 - 1)}$

15. $\displaystyle\int \frac{x^2 - 8}{x + 3}\, dx$

16. $\displaystyle\int \frac{x^2 + 1}{x - 1}\, dx$

17. $\displaystyle\int \frac{3x^2 - 10}{x^2 - 4x + 4}\,dx$

18. $\displaystyle\int \frac{x^2}{x^2 - 3x + 2}\,dx$

19. $\displaystyle\int \frac{2x - 3}{x^2 - 3x - 10}\,dx$

20. $\displaystyle\int \frac{3x + 1}{3x^2 + 2x - 1}\,dx$

21. $\displaystyle\int \frac{x^5 + x^2 + 2}{x^3 - x}\,dx$

22. $\displaystyle\int \frac{x^5 - 4x^3 + 1}{x^3 - 4x}\,dx$

23. $\displaystyle\int \frac{2x^2 + 3}{x(x - 1)^2}\,dx$

24. $\displaystyle\int \frac{3x^2 - x + 1}{x^3 - x^2}\,dx$

25. $\displaystyle\int \frac{2x^2 - 10x + 4}{(x + 1)(x - 3)^2}\,dx$

26. $\displaystyle\int \frac{2x^2 - 2x - 1}{x^3 - x^2}\,dx$

27. $\displaystyle\int \frac{x^2}{(x + 1)^3}\,dx$

28. $\displaystyle\int \frac{2x^2 + 3x + 3}{(x + 1)^3}\,dx$

29. $\displaystyle\int \frac{2x^2 - 1}{(4x-1)(x^2+1)}\,dx$

30. $\displaystyle\int \frac{dx}{x^3 + 2x}$

31. $\displaystyle\int \frac{x^3 + 3x^2 + x + 9}{(x^2+1)(x^2+3)}\,dx$

32. $\displaystyle\int \frac{x^3 + x^2 + x + 2}{(x^2+1)(x^2+2)}\,dx$

33. $\displaystyle\int \frac{x^3 - 2x^2 + 2x - 2}{x^2+1}\,dx$

34. $\displaystyle\int \frac{x^4 + 6x^3 + 10x^2 + x}{x^2 + 6x + 10}\,dx$

Do Questions (9-30) from Ex # 7.5

7.6

u = tan(x/2) substitution

Functions that consist of finitely many sums, differences, quotients, and products of $\sin x$ and $\cos x$ are called ***rational functions of sin x and cos x***. Some examples are

$$\frac{\sin x + 3\cos^2 x}{\cos x + 4\sin x}, \quad \frac{\sin x}{1 + \cos x - \cos^2 x}, \quad \frac{3\sin^5 x}{1 + 4\sin x}$$

Many rational functions of $\sin x$ and $\cos x$ can be evaluated by an ingenious method that was discovered by the mathematician Karl Weierstrass (see p. 102 for biography). The idea is to make the substitution

$$u = \tan(x/2), \quad -\pi/2 < x/2 < \pi/2$$

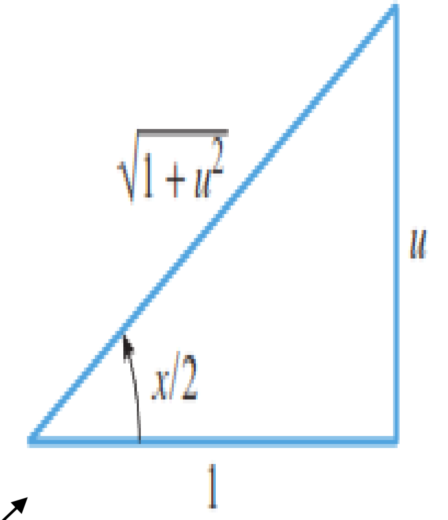
from which it follows that

$$x = 2\tan^{-1} u, \quad dx = \frac{2}{1+u^2}\,du$$

To implement this substitution we need to express $\sin x$ and $\cos x$ in terms of u. For this purpose we will use the identities

$$\sin x = 2\sin(x/2)\cos(x/2) \tag{3}$$

$$\cos x = \cos^2(x/2) - \sin^2(x/2) \tag{4}$$



$$\sin x = 2\left(\frac{u}{\sqrt{1+u^2}}\right)\left(\frac{1}{\sqrt{1+u^2}}\right) = \frac{2u}{1+u^2}$$

$$\cos x = \left(\frac{1}{\sqrt{1+u^2}}\right)^2 - \left(\frac{u}{\sqrt{1+u^2}}\right)^2 = \frac{1-u^2}{1+u^2}$$

$$\sin(x/2) = \frac{u}{\sqrt{1+u^2}} \quad \text{and} \quad \cos(x/2) = \frac{1}{\sqrt{1+u^2}}$$

$$\sin x = \frac{2u}{1+u^2}, \quad \cos x = \frac{1-u^2}{1+u^2}, \quad dx = \frac{2}{1+u^2}\,du$$

Evaluate $\displaystyle\int \frac{dx}{1-\sin x+\cos x}$.

Evaluate $\displaystyle\int \frac{dx}{1 - \sin x + \cos x}$.

$$\int \frac{dx}{1 - \sin x + \cos x} = \int \frac{\dfrac{2\,du}{1+u^2}}{1 - \left(\dfrac{2u}{1+u^2}\right) + \left(\dfrac{1-u^2}{1+u^2}\right)}$$

$$= \int \frac{2\,du}{(1+u^2) - 2u + (1-u^2)}$$

$$= \int \frac{du}{1-u} = -\ln|1-u| + C = -\ln|1 - \tan(x/2)| + C$$

65–70 (a) Make u-substitution (5) to convert the integrand to a rational function of u, and then evaluate the integral. (b) If you have a CAS, use it to evaluate the integral (no substitution), and then confirm that the result is equivalent to that in part (a).

65. $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

66. $\displaystyle\int \frac{dx}{2+\sin x}$

67. $\displaystyle\int \frac{d\theta}{1-\cos\theta}$

68. $\displaystyle\int \frac{dx}{4\sin x-3\cos x}$

69. $\displaystyle\int \frac{dx}{\sin x+\tan x}$

70. $\displaystyle\int \frac{\sin x}{\sin x+\tan x}\,dx$

Do Questions (65-70) from Ex # 7.6

69. $u=\tan(x/2)$, $\displaystyle\frac{1}{2}\int\frac{1-u^2}{u}du=\frac{1}{2}\int(1/u-u)du=\frac{1}{2}\ln|\tan(x/2)|-\frac{1}{4}\tan^2(x/2)+C.$

70. $u=\tan(x/2)$, $\displaystyle\int\frac{1-u^2}{1+u^2}\,du=-u+2\tan^{-1}u+C=x-\tan(x/2)+C.$

69. $u = \tan(x/2)$, $\frac{1}{2}\int \frac{1-u^2}{u}du = \frac{1}{2}\int (1/u - u)du = \frac{1}{2}\ln|\tan(x/2)| - \frac{1}{4}\tan^2(x/2) + C.$

70. $u = \tan(x/2)$, $\int \frac{1-u^2}{1+u^2}du = -u + 2\tan^{-1}u + C = x - \tan(x/2) + C.$

Do Questions (65-70) from Ex # 7.6