Functions

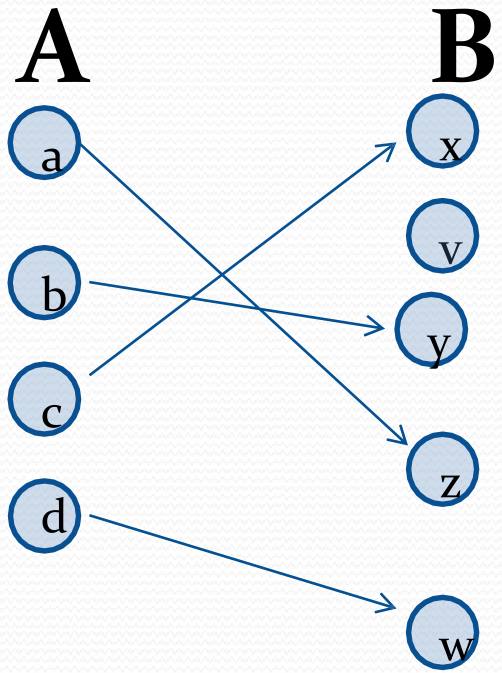
Section 2.3

Section Summary

- Definition of a Function.
  - Domain, Codomain
  - Image, Pre image
- Injection, Surjection, Bijection
- Inverse Function
- Function Composition
- Graphing Functions
- Floor, Ceiling, Factorial

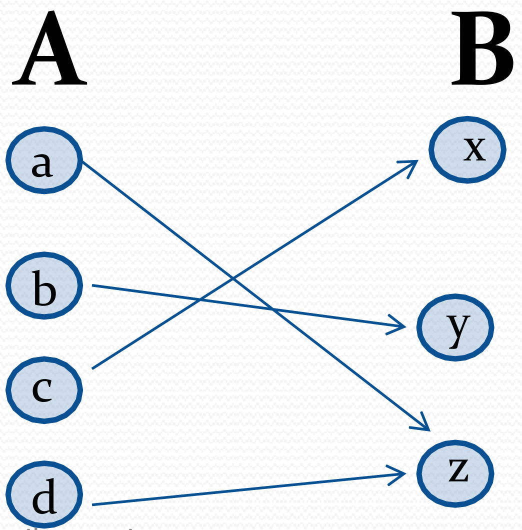
Injections

Definition: A function f is said to be *one-to-one* , or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.



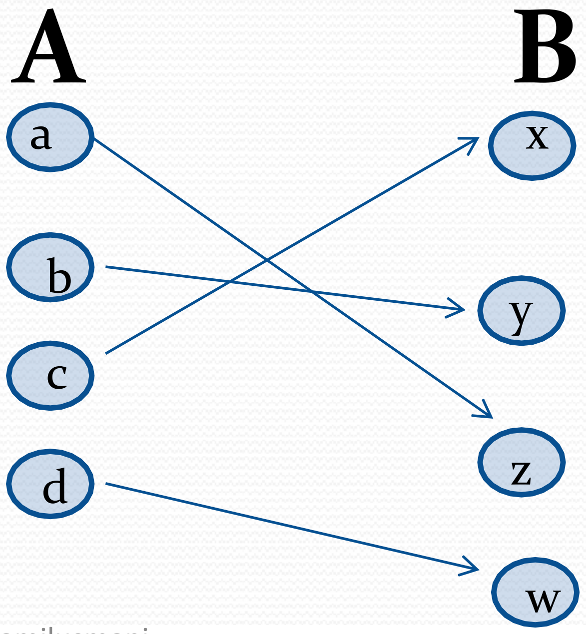
Surjections

Definition: A function *f* from *A* to *B* is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function *f* is called a *surjection* if it is onto.



Bijections

Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).



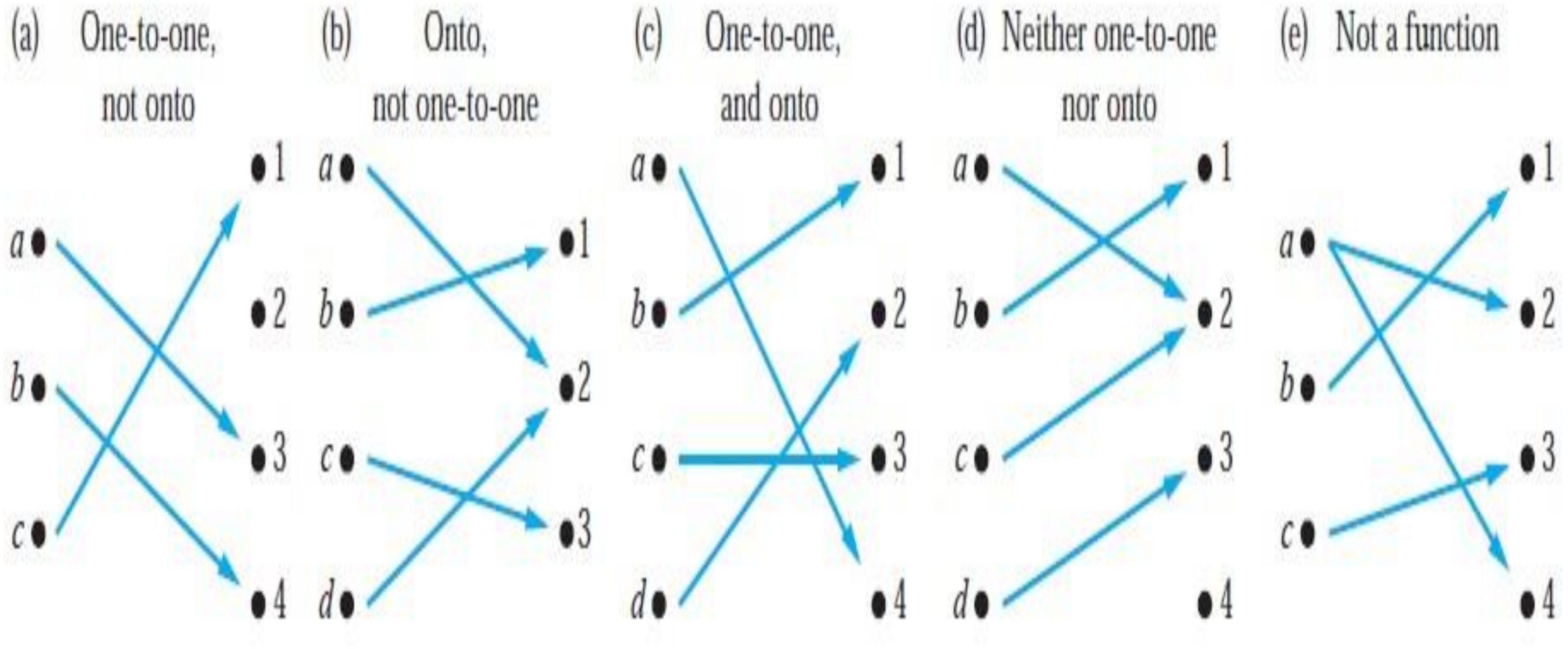
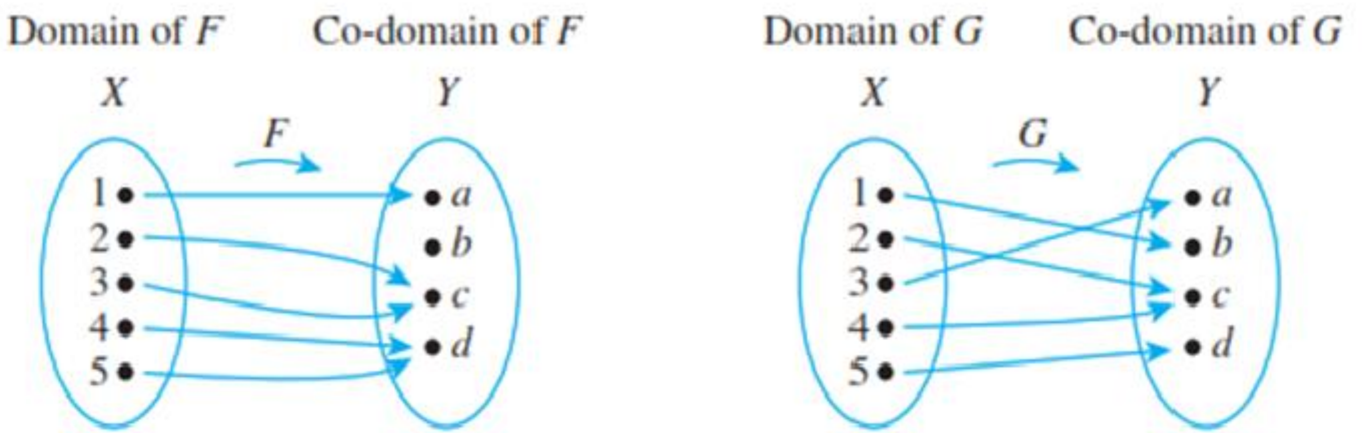


FIGURE 5 Examples of Different Types of Correspondences.

a. Do either of the arrow diagrams define onto functions?



b. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$. Define $H: X \to Y$ as follows: $H(1) = c$, $H(2) = a$, $H(3) = c$, $H(4) = b$. Define $K: X \to Y$ as follows: $K(1) = c$, $K(2) = b$, $K(3) = b$, and $K(4) = c$. Is either H or K onto?

Increasing/ decreasing functions

- A function f is
  - increasing if $\forall x \forall y(x < y \rightarrow f(x) \leq f(y))$,
  - strictly increasing if $\forall x \forall y(x < y \rightarrow f(x) < f(y))$,
  - decreasing if $\forall x \forall y(x < y \rightarrow f(x) \geq f(y))$,
  - strictly decreasing if $\forall x \forall y(x < y \rightarrow f(x) > f(y))$,

where the universe of discourse is the domain of f.

Increasing/ decreasing functions

Example:

- Let g : **R** → **R**, where $g(x) = 2x - 1$. Is it increasing ?

Proof .

For $x>y$ holds $2x > 2y$ and subsequently $2x-1 > 2y-1$

Thus g is strictly increasing.

Relationship with one-to-one

- A function that is either strictly increasing or strictly decreasing must be one-to-one.

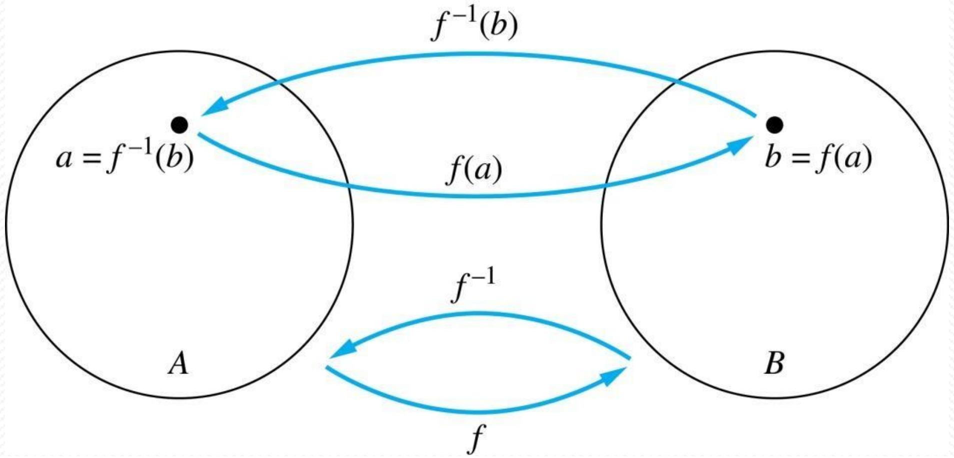
  Why?
- One-to-one function: A function is one-to-one if and only if $f(x) \neq f(y)$, whenever $x \neq y$.

- A function that is increasing, but not strictly increasing, or decreasing, but not strictly decreasing, is not one-to-one.

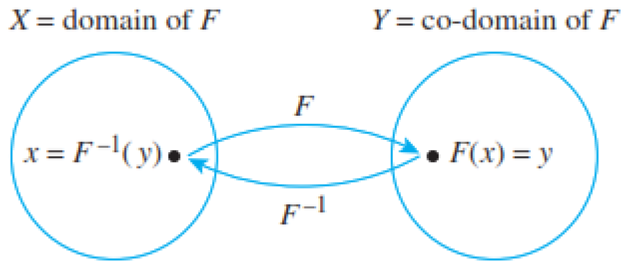
Inverse Functions

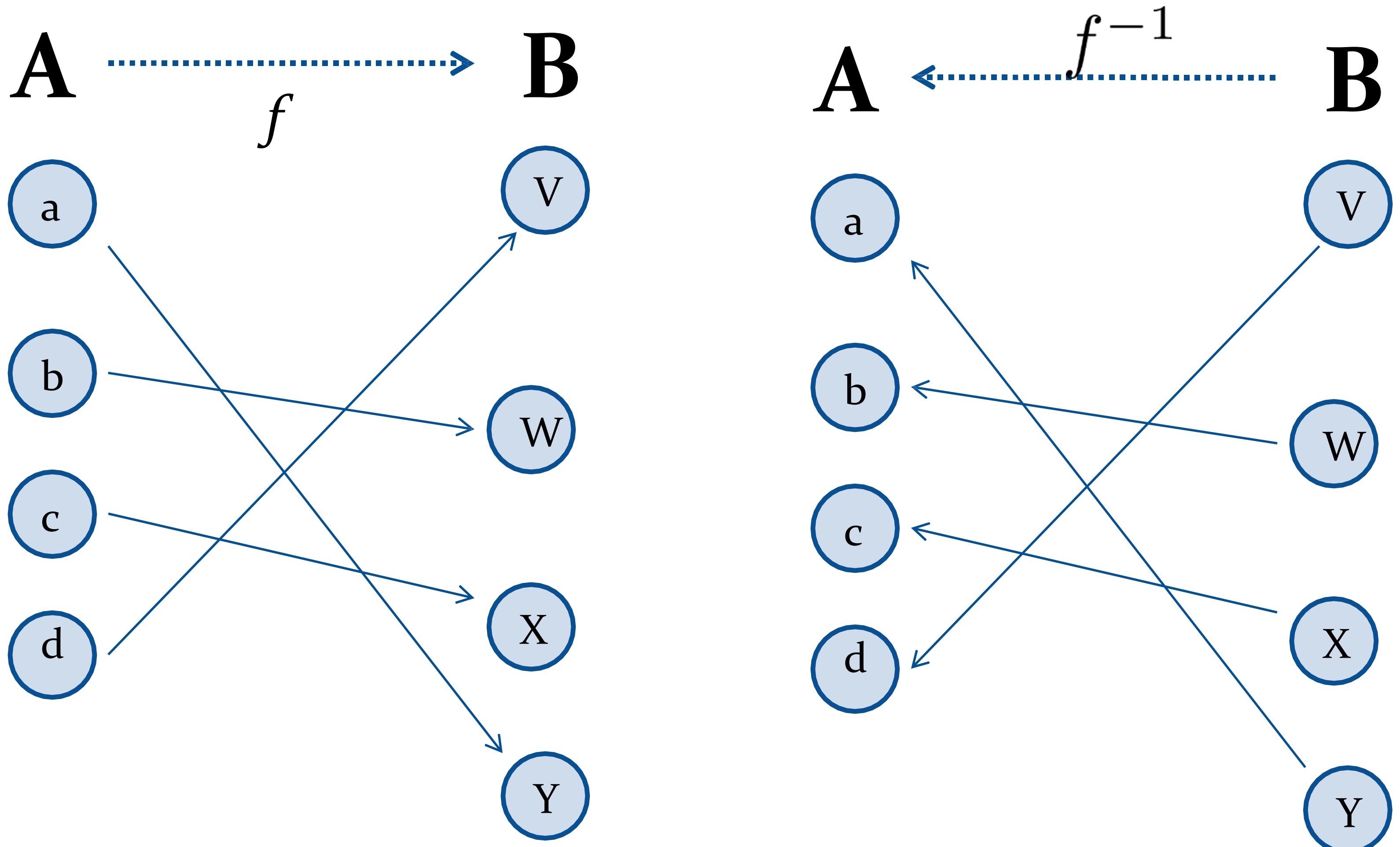
Definition: Let f be a bijection from A to B. Then the *inverse* of f, denoted f^{-1}, is the function from B to A defined as $$f^{-1}(y) = x \text{ iff } f(x) = y$$

No inverse exists unless f is a bijection. Why?



Inverse Functions





Questions

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence.
The inverse function f^{-1} reverses the correspondence given by f, so
$f^{-1}(1) = c$,
$f^{-1}(2) = a$, and
$f^{-1}(3) = b$.

Example 2: Let f: **Z** → **Z** be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

Example 3: Let f: **R** → **R** be such that $f(x) = x^2$

Is f invertible, and if so, what is its inverse?

Solution: The function f is not invertible because it is not one-to-one .

A Function from a Power Set to a Set of Strings

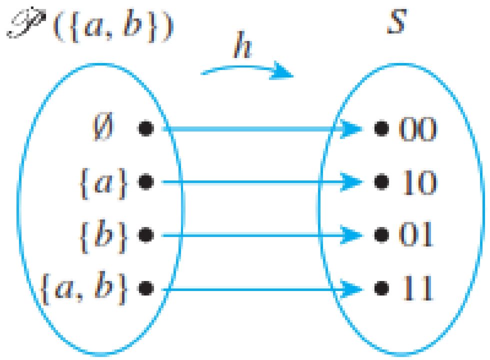
Example

Let $\mathscr{P}(\{a, b\})$ be the set of all subsets of $\{a, b\}$ and let S be the set of all strings of length 2 made up of 0's and 1's. Then $\mathscr{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $S = \{00, 01, 10, 11\}$.

Define a function h from $\mathscr{P}(\{a, b\})$ to S as follows:

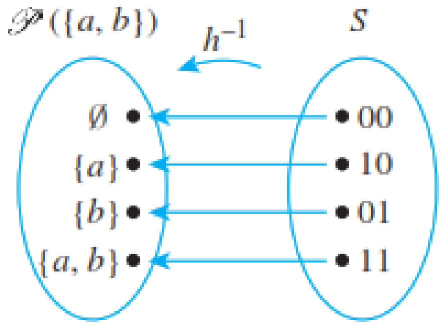
h

Subset of $\{a, b\}$	Status of a	Status of b	String in S
$\emptyset$	not in	not in	00
$\{a\}$	in	not in	10
$\{b\}$	not in	in	01
$\{a, b\}$	in	in	11



Is h a one-to-one correspondence?

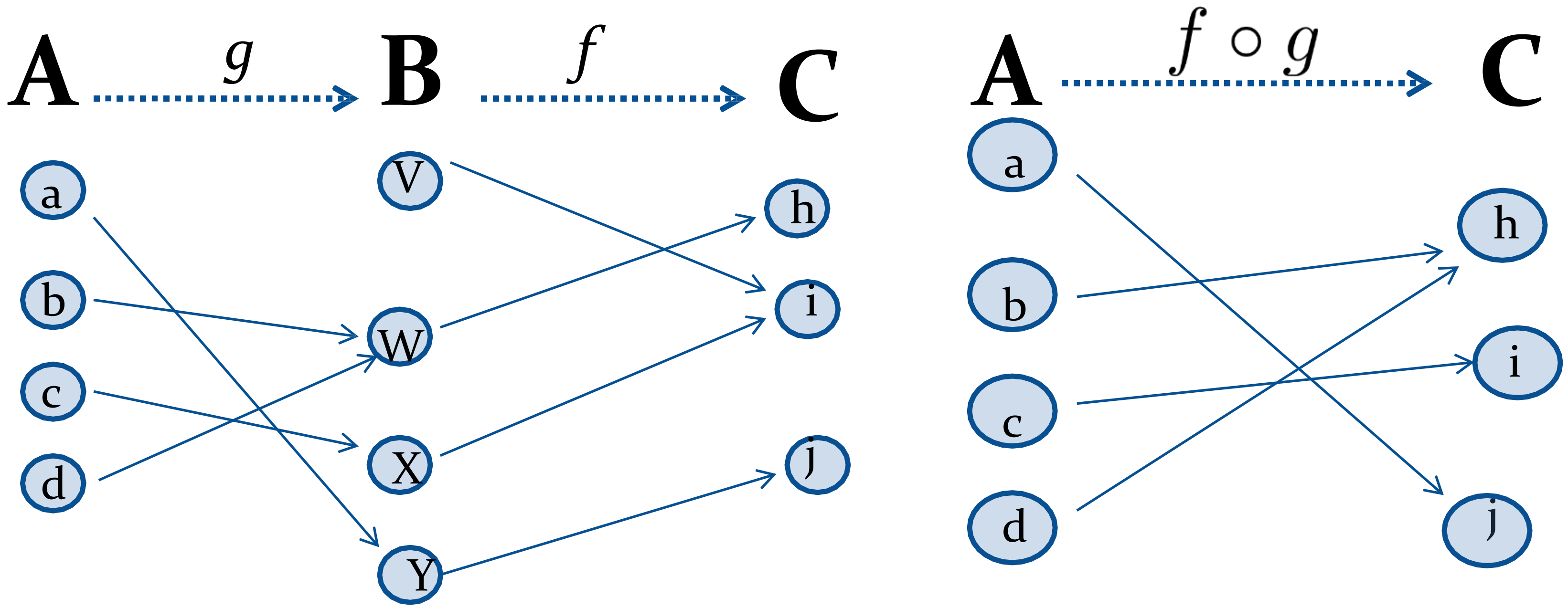
Define the inverse function for the one-to-one correspondence h



$h^{-1}(00) = \emptyset \quad h^{-1}(10) = \{a\}$

$h^{-1}(01) = \{b\} \quad h^{-1}(11) = \{a, b\}$

Composition



Composition

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$,
then

$$f(g(x)) = (2x + 1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Composition Questions

- **Example** 2: Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.
- What is the composition of f and g, and what is the composition of g and f.
- **Solution:** The composition $f \circ g$ is defined by
- $f \circ g\ (a) = f(g(a)) = f(b) = 2.$
- $f \circ g\ (b) = f(g(b)) = f(c) = 1.$
- $f \circ g\ (c) = f(g(c)) = f(a) = 3.$

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Composition Questions

Example 2: Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

What is the composition of f and g, and also the composition of g and f?

Solution:

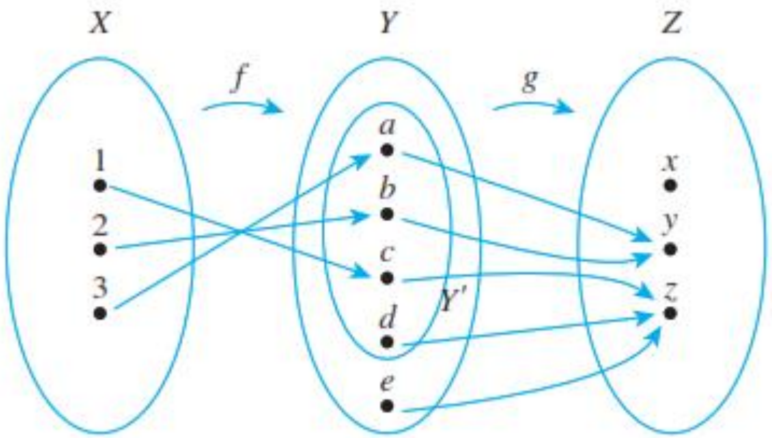
$f \circ g\ (x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

$g \circ f\ (x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

Example: Class Activity

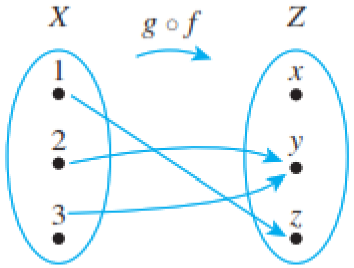
Composition of Functions Defined on Finite Sets

Let $X = \{1, 2, 3\}$, $Y' = \{a, b, c, d\}$, $Y = \{a, b, c, d, e\}$, and $Z = \{x, y, z\}$. Define functions $f: X \to Y'$ and $g: Y \to Z$ by the arrow diagrams below.



Draw the arrow diagram for $g \circ f$. What is the range of $g \circ f$?

Solution:



$$(g \circ f)(1) = g(f(1)) = g(c) = z$$
$$(g \circ f)(2) = g(f(2)) = g(b) = y$$
$$(g \circ f)(3) = g(f(3)) = g(a) = y$$

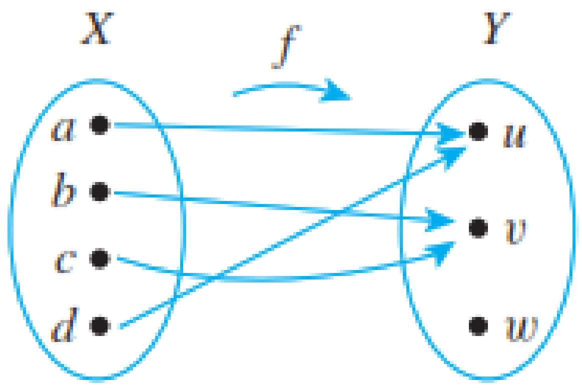
The range of $g \circ f$ is $\{y, z\}$.

Example:

Class Activity

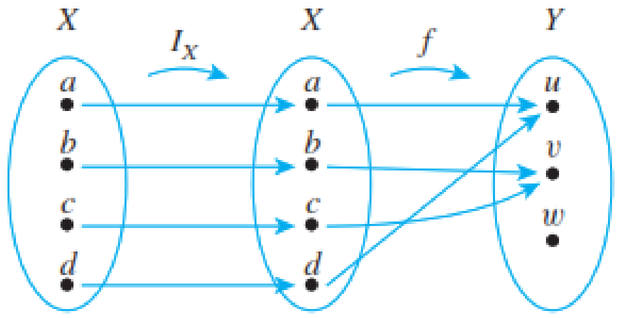
Composition with the Identity Function

Let $X = \{a, b, c, d\}$ and $Y = \{u, v, w\}$, and suppose $f: X \to Y$ is given by the arrow



Find $f \circ I_X$ and $I_Y \circ f$.

Solution:



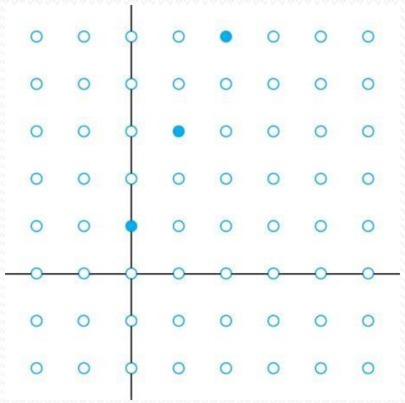
$$\begin{aligned}
(f \circ I_X)(a) &= f(I_X(a)) = f(a) = u \\
(f \circ I_X)(b) &= f(I_X(b)) = f(b) = v \\
(f \circ I_X)(c) &= f(I_X(c)) = f(c) = v \\
(f \circ I_X)(d) &= f(I_X(d)) = f(d) = u
\end{aligned}$$

Note that for all elements x in X,

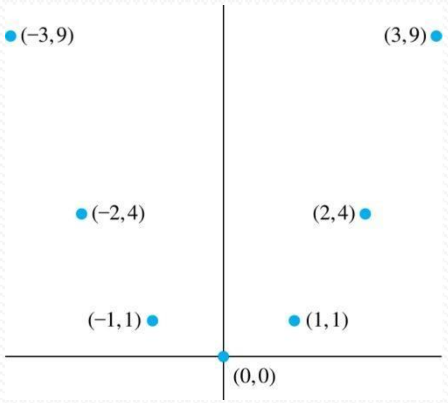
$$(f \circ I_X)(x) = f(x).$$

Graphs of Functions

- Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$ from Z to Z



Graph of $f(x) = x^2$ from Z to Z

Some Important Functions

- The *floor* function, denoted

$$f(x) = \lfloor x \rfloor$$

is the largest integer less than or equal to *x*.

- The *ceiling* function, denoted

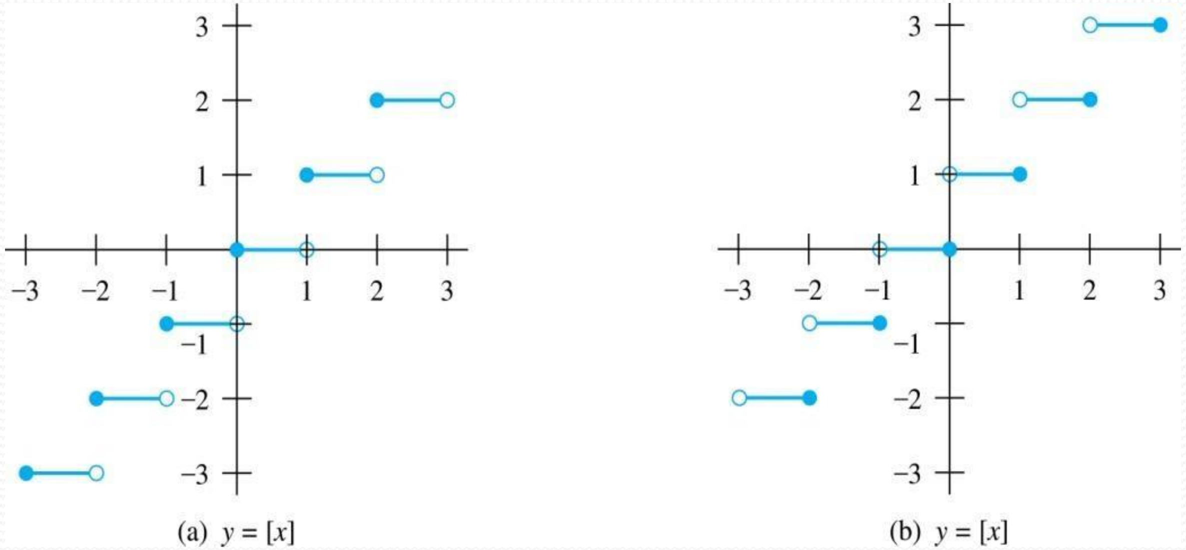
$$f(x) = \lceil x \rceil$$

is the smallest integer greater than or equal to *x*

Example: $\lceil 3.5 \rceil = 4$ $\lfloor 3.5 \rfloor = 3$

$\lceil -1.5 \rceil = -1$ $\lfloor -1.5 \rfloor = -2$

Floor and Ceiling Functions



Graph of (a) Floor and (b) Ceiling Functions

Floor Functions

- Determine whether the function from R to Z is Injective OR Surjective.

  $$f(\mathrm{a}) = \lfloor a/2 \rfloor$$

Solution:

- It is Surjective (onto function). This can be shown by an example; f (0) = 0, and f (1) = 0.

Ceiling Functions

- Determine whether the function from R to Z is Injective OR Surjective.

  $f(a) = \lceil a/2 \rceil$

Solution:

- It is Surjective (onto function). This can be shown by an example; $f(1) = 1$, and $f(2) = 1$.

Floor and Ceiling Functions

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Activity Time

What are the domain, codomain, and range of the following function

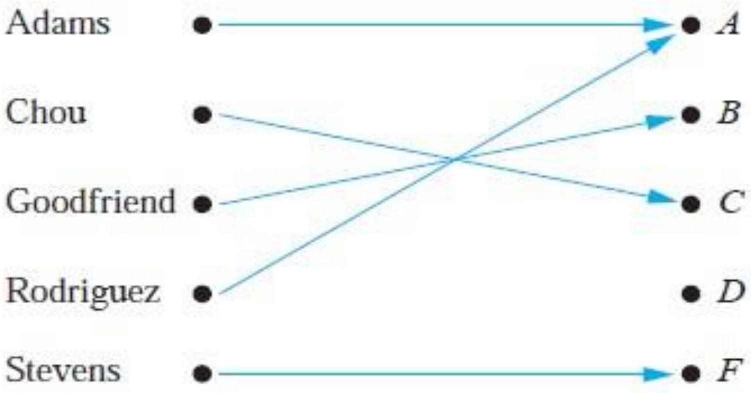


FIGURE 1 Assignment of Grades in a Discrete Mathematics Class.

2. Determine whether f is a function from **Z** to **R** if

a) $f(n) = \pm n$.

b) $f(n) = \sqrt{n^2 + 1}$.

c) $f(n) = 1/(n^2 - 4)$.

12. Determine whether each of these functions from **Z** to **Z** is one-to-one.

a) $f(n) = n - 1$

b) $f(n) = n^2 + 1$

c) $f(n) = n^3$

d) $f(n) = \lceil n/2 \rceil$

22. What is the cardinality of each of these sets?

a) $\varnothing$

b) $\{\varnothing\}$

c) $\{\varnothing, \{\varnothing\}\}$

d) $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$

34. Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find

a) $A \times B \times C$.

b) $C \times B \times A$.

c) $C \times A \times B$.

d) $B \times B \times B$.

35. Find A^2 if

a) $A = \{0, 1, 3\}$.

b) $A = \{1, 2, a, b\}$.

36. Find A^3 if

a) $A = \{a\}$.

b) $A = \{0, a\}$.

8. Find these values.

a) $\lfloor 1.1 \rfloor$

b) $\lceil 1.1 \rceil$

c) $\lfloor -0.1 \rfloor$

d) $\lceil -0.1 \rceil$

e) $\lceil 2.99 \rceil$

f) $\lceil -2.99 \rceil$

g) $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$

h) $\lceil \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil + \frac{1}{2} \rceil$

9. Find these values.

a) $\lceil \frac{3}{4} \rceil$

b) $\lfloor \frac{7}{8} \rfloor$

c) $\lceil -\frac{3}{4} \rceil$

d) $\lfloor -\frac{7}{8} \rfloor$

e) $\lceil 3 \rceil$

f) $\lfloor -1 \rfloor$

g) $\lfloor \frac{1}{2} + \lceil \frac{3}{2} \rceil \rfloor$

h) $\lfloor \frac{1}{2} \cdot \lfloor \frac{5}{2} \rfloor \rfloor$

Solution:

8. We simply round up or down in each case.

a) 1

b) 2

c) -1

d) 0

e) 3

f) -2

g) $\lfloor \frac{1}{2} + 1 \rfloor = \lfloor \frac{3}{2} \rfloor = 1$

h) $\lceil 0 + 1 + \frac{1}{2} \rceil = \lceil \frac{3}{2} \rceil = 2$

10. Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one.

a) $f(a) = b, f(b) = a, f(c) = c, f(d) = d$

b) $f(a) = b, f(b) = b, f(c) = d, f(d) = c$

c) $f(a) = d, f(b) = b, f(c) = c, f(d) = d$

Solution:

a) This is one-to-one.

b) This is not one-to-one, since b is the image of both a and b.

c) This is not one-to-one, since d is the image of both a and d.

13. Let f and g be functions from $\{1, 2, 3, 4\}$ to $\{a, b, c, d\}$ and from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$, respectively, with $f(1) = d, f(2) = c, f(3) = a$, and $f(4) = b$, and $g(a) = 2$, $g(b) = 1, g(c) = 3$, and $g(d) = 2$.

a) Is f one-to-one? Is g one-to-one?

b) Is f onto? Is g onto?

c) Does either f or g have an inverse? If so, find this inverse.

Home activity

22. Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = -3x + 4$

b) $f(x) = -3x^2 + 7$

c) $f(x) = (x + 1)/(x + 2)$

d) $f(x) = x^5 + 1$

Home Activity

30. Let $S = \{-1, 0, 2, 4, 7\}$. Find $f(S)$ if

a) $f(x) = 1$.

b) $f(x) = 2x + 1$.

c) $f(x) = \lceil x/5 \rceil$.

d) $f(x) = \lfloor (x^2 + 1)/3 \rfloor$.

Solution:

we simply need to compute the values $f(-1)$, $f(0)$, $f(2)$, $f(4)$, and $f(7)$ and collect the values into a set.

a) $\{1\}$ (all five values are the same)

b) $\{-1, 1, 5, 8, 15\}$

c) $\{0, 1, 2\}$

d) $\{0, 1, 5, 16\}$

38. Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from **R** to **R**.

40. Let $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, and d are constants. Determine necessary and sufficient conditions on the constants a, b, c, and d so that $f \circ g = g \circ f$.

44. Let f be the function from **R** to **R** defined by $f(x) = x^2$. Find

a) $f^{-1}(\{1\})$.

b) $f^{-1}(\{x \mid 0 < x < 1\})$.

c) $f^{-1}(\{x \mid x > 4\})$.

45. Let $g(x) = \lfloor x \rfloor$. Find

a) $g^{-1}(\{0\})$.

b) $g^{-1}(\{-1, 0, 1\})$.

c) $g^{-1}(\{x \mid 0 < x < 1\})$.