Chapter 2

Basic Structures:

Sets and Functions

Sequence and Sums

Sets

Section 2.1

Section Summary

- Definition of sets
- Describing Sets
  - Roster Method
  - Set-Builder Notation
- Some Important Sets in Mathematics
- Empty Set and Universal Set
- Subsets and Set Equality
- Cardinality of Sets
- Tuples
- Cartesian Product

Introduction

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
  - Important for counting.
  - Programming languages have set operations.
- Set theory is an important branch of mathematics.
  - Many different systems of axioms have been used to develop set theory.
  - Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

Sets

- A set is an unordered collection of objects.
  - the students in this class
  - the chairs in this room
- The objects in a set are called the elements, or members of the set. A set is said to contain its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$

Describing a Set: Roster Method

- $S = \{a,b,c,d\}$
- Order not important

  $S = \{a,b,c,d\} = \{b,c,a,d\}$
- Each distinct object is either a member or not; listing more than once does not change the set.

  $S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$
- Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

  $S = \{a,b,c,d, \ldots\ldots,z\}$

Roster Method

- Set of all vowels in the English alphabet:

$$V = \{a,e,i,o,u\}$$

- Set of all odd positive integers less than 10:

$$O = \{1,3,5,7,9\}$$

- Set of all positive integers less than 100:

$$S = \{1,2,3,\ldots\ldots,99\}$$

- Set of all integers less than 0:

$$S = \{\ldots, -3,-2,-1\}$$

Some Important Sets

N = natural numbers $= \{0,1,2,3....\}$

Z = integers $= \{...,-3,-2,-1,0,1,2,3,...\}$

Z^+ = positive integers $= \{1,2,3,.....\}$

R = set of real numbers

R^+ = set of positive real numbers

C = set of complex numbers.

Q = set of rational numbers

Set-Builder Notation

- Specify the property or properties that all members must satisfy:

  $S = \{x \mid x \text{ is a positive integer less than } 100\}$

  $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  $O = \{x \in Z^+ \mid x \text{ is odd and } x < 10\}$
- A predicate may be used:

  $$S = \{x \mid P(x)\}$$
- Example: $S = \{x \mid \text{Prime}(x)\}$
- Positive rational numbers:

  $Q^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$

Universal Set and Empty Set

- The universal set U is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.
- The empty set is the set with no elements. Symbolized ∅, but {} also used.

Venn Diagram

John Venn (1834-1923)
Cambridge, UK

Some things to remember

- Sets can be elements of sets.

  $\{\{1,2,3\},a, \{b,c\}\}$

  $\{N,Z,Q,R\}$

- The empty set is different from a set containing the empty set.

  $\emptyset \neq \{ \emptyset \}$

Set Equality

Definition: Two sets are equal if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$.
- We write A = B if A and B are equal sets.

$$\{1,3,5\} = \{3, 5, 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Subsets

Definition: The set A is a subset of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \rightarrow x \in B)$ is true.
  1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$,for every set S.
  2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set S.

Showing a Set is or is not a Subset of Another Set

- **Showing that A is a Subset of B**: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.
- **Showing that A is not a Subset of B**: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Another look at Equality of Sets

- Recall that two sets A and B are equal, denoted by $A = B$, iff

$$\forall x(x \in A \leftrightarrow x \in B)$$

- Using logical equivalences we have that $A = B$ iff

$$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

- This is equivalent to

$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a proper subset of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

Venn Diagram

Set Cardinality

Definition: If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is finite. Otherwise it is infinite.

Definition: The cardinality of a finite set A, denoted by |A|, is the number of (distinct) elements of A.

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then |S| $= 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

Power Sets

Definition: The set of all subsets of a set A, denoted P(A), is called the power set of A.

Example: If A ={a,b} then

$$P(A) = \{\emptyset, \{a\},\{b\},\{a,b\}\}$$

- If a set has n elements, then the cardinality of the power set is 2^n. (In Chapters 5 and 6, we will discuss different ways to show this.)

Tuples

- The ordered n-tuple $(a_1,a_2,\ldots\ldots,a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
- Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called ordered pairs.
- The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product

René Descartes
(1596-1650)

Definition: The Cartesian Product of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) | a \in A \wedge b \in B\}$$

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

- **Definition**: A subset R of the Cartesian product $A \times B$ is called a relation from the set A to the set B. (Relations will be covered in depth in Chapter 9.)

Cartesian Product

Definition: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n–tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $A \times B \times C = \{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,1,2)\}$

Truth Sets of Quantifiers

- Given a predicate P and a domain D, we define the truth set of P to be the set of elements in D for which P(x) is true. The truth set of P(x) is denoted by

$$\{x \in D | P(x)\}$$

- **Example**: The truth set of P(x) where the domain is the integers and P(x) is "$|x| = 1$" is the set {-1,1}

Set Operations

Section 2.2

Section Summary

- Set Operations
  - Union
  - Intersection
  - Complementation
  - Difference
- More on Set Cardinality
- Set Identities
- Proving Identities
- Membership Tables

Boolean Algebra

- Propositional calculus and set theory are both instances of an algebraic system called a Boolean Algebra. This is discussed in Chapter 12.
- The operators in set theory are analogous to the corresponding operator in propositional calculus.
- As always there must be a universal set U. All sets are assumed to be subsets of U.

Union

- **Definition**: Let A and B be sets. The union of the sets A and B, denoted by A ∪ B, is the set:

$$\{x | x \in A \vee x \in B\}$$

- **Example**: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

Solution: $\{1,2,3,4,5\}$

Venn Diagram for *A* ∪ B

Intersection

- **Definition**: The intersection of sets A and B, denoted by $A \cap B$, is

$$\{x | x \in A \wedge x \in B\}$$

- Note if the intersection is empty, then A and B are said to be disjoint.
- **Example**: What is? $\{1,2,3\} \cap \{3,4,5\}$?

  Solution: $\{3\}$
- **Example:**What is?

  $\{1,2,3\} \cap \{4,5,6\}$?

  Solution: $\emptyset$

Venn Diagram for $A \cap B$

Complement

Definition: If A is a set, then the complement of the A (with respect to U), denoted by $\bar{A}$ is the set U - A

$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c .)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

Venn Diagram for Complement

Difference

- **Definition**: Let A and B be sets. The difference of A and B, denoted by A – B, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$

Venn Diagram for $A - B$

The Cardinality of the Union of Two Sets

Inclusion-Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$

Venn Diagram for A, B, $A \cap B$, $A \cup B$

Example: Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

We will return to this principle in Chapter 6 and Chapter 8 where we will derive a formula for the cardinality of the union of n sets, where n is a positive integer.

Review Questions

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution: $\{1,2,3,4,5,6,7,8\}$
2. $A \cap B$

   Solution: $\{4,5\}$
3. $\bar{A}$

   Solution: $\{0,6,7,8,9,10\}$
4. $\bar{B}$

   Solution: $\{0,1,2,3,9,10\}$
5. $A - B$

   Solution: $\{1,2,3\}$
6. $B - A$

   Solution: $\{6,7,8\}$

Symmetric Difference (optional)

Definition: The symmetric difference of **A** and **B**, denoted by $A \oplus B$ is the set

$$(A - B) \cup (B - A)$$

Example:

U $= \{0,1,2,3,4,5,6,7,8,9,10\}$

A $= \{1,2,3,4,5\}$ B $=\{4,5,6,7,8\}$

What is:

- **Solution**: $\{1,2,3,6,7,8\}$

Venn Diagram

Set Identities

- Identity laws

$$A \cup \emptyset = A \qquad A \cap U = A$$

- Domination laws

$$A \cup U = U \qquad A \cap \emptyset = \emptyset$$

- Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

- Complementation law

$$\overline{(\overline{A})} = A$$

Continued on next slide →

Set Identities

- Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

- Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Continued on next slide →

Set Identities

- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

- Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \emptyset$$

Proving Set Identities

- Different ways to prove set identities:
  1. Prove that each set (side of the identity) is a subset of the other.
  2. Use set builder notation and propositional logic.
  3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Proving Set Identities

Example:

Let A, B, and C be sets. Show that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}.$$

Solution: We have

$$\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap (\overline{B \cap C}) && \text{by the first De Morgan law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.}
\end{aligned}$$

Proof of Second De Morgan Law

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Continued on next slide →

Proof of Second De Morgan Law

These steps show that:

$$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

Continued on next slide →

Proof of Second De Morgan Law

These steps show that:

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Set-Builder Notation: Second De Morgan Law

$$
\begin{array}{rcll}
\overline{A \cap B} & = & \{x | x \notin A \cap B\} & \text{by defn. of complement} \\
 & = & \{x | \neg(x \in (A \cap B))\} & \text{by defn. of does not belong symbol} \\
 & = & \{x | \neg(x \in A \wedge x \in B\} & \text{by defn. of intersection} \\
 & = & \{x | \neg(x \in A) \vee \neg(x \in B)\} & \text{by 1st De Morgan law} \\
 & & & \text{for Prop Logic} \\
 & = & \{x | x \notin A \vee x \notin B\} & \text{by defn. of not belong symbol} \\
 & = & \{x | x \in \overline{A} \vee x \in \overline{B}\} & \text{by defn. of complement} \\
 & = & \{x | x \in \overline{A} \cup \overline{B}\} & \text{by defn. of union} \\
 & = & \overline{A} \cup \overline{B} & \text{by meaning of notation}
\end{array}
$$

◀

Membership Table

Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Functions

Section 2.3

Section Summary

- Definition of a Function.
  - Domain, Codomain
  - Image, Pre image
- Injection, Surjection, Bijection
- Inverse Function
- Function Composition
- Graphing Functions
- Floor, Ceiling, Factorial

Functions

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted f: $A \to B$ is an assignment of each element of A to exactly one element of B. We write f(a) = b if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called *mappings* or *transformations*.

Functions

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted f: $A \to B$ is an assignment of each element of A to exactly one element of B. We write f(a) = b if b is the unique element of B assigned by the function f to the element a of A.

Functions

- A function f: $A \to B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.
- Specifically, a function f from A to B contains one, and only one ordered pair (a, b) for every element $a \in A$.

$$\forall x[x \in A \to \exists y[y \in B \wedge (x, y) \in f]]$$

and

$$\forall x, y_1, y_2[[(x, y_1) \in f \wedge (x, y_2)] \to y_1 = y_2]$$

Functions

Given a function $f: A \to B$:

- We say f *maps* A to B *or*
  f is a *mapping* from A to B.
- A is called the *domain* of f.
- B is called the *codomain* of f.
- If $f(a) = b$,
  - then b is called the *image* of a under f.
  - a is called the *preimage* of b.
- The range of f is the set of all images of points in **A** under f. We denote it by $f(A)$.
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.

Equal Functions

- Two functions are **equal** when they
  - have the same domain,
  - have the same codomain,
  - map each element of their common domain to the same element in their common codomain.
- If we change either the domain or the codomain of a function, then we obtain a different function.
- If we change the mapping of elements, then we also obtain a different function.

Representing Functions

- Functions may be specified in different ways:
  - An explicit statement of the assignment.

    Students and grades example.
  - A formula.

    $f(x) = x + 1$
  - A computer program.
    - A Java program that when given an integer n, produces the nth Fibonacci Number (covered in the next section and also inChapter 5).

Activity Time

What are the domain, codomain, and range of the following function

FIGURE 1 Assignment of Grades in a Discrete Mathematics Class.

Solution

- Let G be the function that assigns a grade to a student in our discrete mathematics class.
- Note that $G(\text{Adams}) = A$, for instance.
- The domain of G is the set {Adams, Chou, Goodfriend, Rodriguez, Stevens},
- The codomain is the set $\{A, B, C, D, F\}$.
- The range of G is the set $\{A, B, C, F\}$,

Questions

$f(\mathrm{a}) = ?$ z

The image of d is ? z

The domain of f is ? A

The codomain of f is ? B

The preimage of y is ? b

$f(A) = ?$

The preimage(s) of z is (are) ? {a,c,d}

Question on Functions and Sets

- If $f : A \to B$ and S is a subset of A, then

$$f(S) = \{f(s) | s \in S\}$$

f {a,b,c,} is ? {y,z}

f {c,d} is ? {z}

Injections

Definition: A function f is said to be *one-to-one* , or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.

Surjections

Definition: A function *f* from *A* to *B* is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function *f* is called a *surjection* if it is onto.

Bijections

Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).

FIGURE 5 Examples of Different Types of Correspondences.

real-valued

- A function is called **real-valued** if its codomain is the set of real numbers.

- Let f_1 and f_2 be functions from A to $\mathbf{R}$.
- Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbf{R}$ defined for all $x \in A$ by
  - $(f_1 + f_2)(x) = f_1(x) + f_2(x)$,
  - $(f_1 f_2)(x) = f_1(x) f_2(x)$.

real-valued Functions

Examples:

- **Assume**
  - $f1(x) = x - 1$
  - $f2(x) = x^3 + 1$

  then
  - $(f1 + f2)(x) = x^3 + x$
  - $(f1 * f2)(x) = x^4 - x^3 + x - 1.$

Showing that f is one-to-one or onto

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Showing that f is one-to-one or onto

Example 1: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution:

Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

Example 2:

Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.
Solution:

The function $f(x) = x^2$ is not one-to-one because, for instance, $f(1) = f(-1) = 1$, but $1 \neq -1$.

Increasing/ decreasing functions

- A function f whose domain and codomain are subsets of the set of real numbers is called *increasing* if $f(x) \leq f(y)$, and *strictly increasing* if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f.
- Similarly, f is called *decreasing* if $f(x) \geq f(y)$, and *strictly decreasing* if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f.

Increasing/ decreasing functions

- A function f is
  - increasing if $\forall x \forall y (x < y \rightarrow f(x) \leq f(y))$,
  - strictly increasing if $\forall x \forall y (x < y \rightarrow f(x) < f(y))$,
  - decreasing if $\forall x \forall y (x < y \rightarrow f(x) \geq f(y))$,
  - strictly decreasing if $\forall x \forall y (x < y \rightarrow f(x) > f(y))$,

where the universe of discourse is the domain of f.

Increasing/ decreasing functions

Example:

- Let $g : \mathbf{R} \to \mathbf{R}$, where $g(x) = 2x - 1$. Is it increasing ?

Proof .

For $x>y$ holds $2x > 2y$ and subsequently $2x-1 > 2y-1$

Thus g is strictly increasing.

Relationship with one-to-one

- A function that is either strictly increasing or strictly decreasing must be one-to-one.

  Why?
- One-to-one function: A function is one-to-one if and only if $f(x) \neq f(y)$, whenever $x \neq y$.

- A function that is increasing, but not strictly increasing, or decreasing, but not strictly decreasing, is not one-to-one.

The *identity function*

- Let A be a set. The *identity function* on A is the function $iA : A \to A$, where

$$iA(x) = x$$

- for all $x \in A$.

- The function iA is one-to-one and onto, so it is a bijection.

The *identity function*

Example:

- *Let* $A = \{1, 2, 3\}$

Solution:

$i\,A(1) = 1$

$i\,A(2) = 2$

$i\,A(3) = 3$

Inverse Functions

Definition: Let f be a bijection from A to B. Then the *inverse* of f, denoted f^{-1}, is the function from B to A defined as $f^{-1}(y) = x$ iff $f(x) = y$

No inverse exists unless f is a bijection. Why?

Inverse Functions

Questions

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Questions

Example 2: Let f: **Z** → **Z** be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

Questions

Example 3: Let f: **R** → **R** be such that $f(x) = x^2$. Is f invertible, and if so, what is its inverse?

Solution: The function f is not invertible because it is not one-to-one .

Composition

- **Definition**: Let f: $B \to C$, g: $A \to B$. The *composition of f with g*, denoted $f \circ g$ is the function from A to C defined by

$$f \circ g(x) = f(g(x))$$

Composition

Composition

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$, then

$$f(g(x)) = (2x+1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Composition Questions

- **Example** 2: Let g be the function from the set $\{a,b,c\}$ to itself that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.
- What is the composition of f and g, and what is the composition and f.
- **Solution:** The composition $f \circ g$ is defined by
- $f \circ g\ (a) = f(g(a)) = f(b) = 2.$
- $f \circ g\ (b) = f(g(b)) = f(c) = 1.$
- $f \circ g\ (c) = f(g(c)) = f(a) = 3.$

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Composition Questions

Example 2: Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

What is the composition of f and g, and also the composition of g and f?

Solution:

$f \circ g\ (x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

$g \circ f\ (x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

Graphs of Functions

- Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.

Graph of $f(n) = 2n + 1$ from Z to Z

Graph of $f(x) = x^2$ from Z to Z

Some Important Functions

- The *floor* function, denoted

$$f(x) = \lfloor x \rfloor$$

is the largest integer less than or equal to *x*.

- The *ceiling* function, denoted

$$f(x) = \lceil x \rceil$$

is the smallest integer greater than or equal to *x*

Example: $\lceil 3.5 \rceil = 4$ $\lfloor 3.5 \rfloor = 3$

$\lceil -1.5 \rceil = -1$ $\lfloor -1.5 \rfloor = -2$

Floor and Ceiling Functions

Graph of (a) Floor and (b) Ceiling Functions

Floor Functions

- Determine whether the function from R to Z is Injective OR Surjective.

$$f(\mathrm{a}) = \lfloor a/2 \rfloor$$

Solution:

- It is Surjective (onto function). This can be shown by an example; $f\ (0) = 0$, and $f\ (1) = 0$.

Ceiling Functions

- Determine whether the function from R to Z is Injective OR Surjective.

$$f(a) = \lceil a/2 \rceil$$

Solution:

- It is Surjective (onto function). This can be shown by an example; $f(1) = 1$, and $f(2) = 1$.

Floor and Ceiling Functions

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, $\boldsymbol{x}$ is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$
(1d)	$\lceil x \rceil = n$ if and only if $x \leq n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Proving Properties of Functions

Example: Prove that x is a real number, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

Solution: Let $x = n + \varepsilon$, where n is an integer and $0 \le \varepsilon < 1$.

Case 1: $\varepsilon < ½$

- $2x = 2n + 2\varepsilon$ and $\lfloor 2x \rfloor = 2n$, since $0 \le 2\varepsilon < 1$.
- $\lfloor x + 1/2 \rfloor = n$, since $x + ½ = n + (1/2 + \varepsilon)$ and $0 \le ½ + \varepsilon < 1$.
- Hence, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n$.

Case 2: $\varepsilon \ge ½$

- $2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$ and $\lfloor 2x \rfloor = 2n + 1$, since $0 \le 2\varepsilon - 1 < 1$.
- $\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \varepsilon) \rfloor = \lfloor n + 1 + (\varepsilon - 1/2) \rfloor = n + 1$ since $0 \le \varepsilon - 1/2 < 1$.
- Hence, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1$. ◀

Factorial Function

Definition: f: **N** → **Z**$^+$, denoted by $f(n) = n!$ is the product of the first n positive integers when n is a nonnegative integer.

$f(n) = 1 \cdot 2 \cdots (n-1) \cdot n, \qquad f(0) = 0! = 1$

Examples:

$f(1) = 1! = 1$

$f(2) = 2! = 1 \cdot 2 = 2$

$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$

$f(20) = 2{,}432{,}902{,}008{,}176{,}640{,}000.$

Stirling's Formula:

$$n! \sim \sqrt{2\pi n}(n/e)^n$$

$$f(n) \sim g(n) \doteq lim_{n\to\infty} f(n)/g(n) = 1$$

Sequences and Series(Sums)

Section Summary

- Sequences.
  - Examples: Geometric Progression, Arithmetic Progression
- Recurrence Relations
  - Example: Fibonacci Sequence
- Summations

Introduction

- Sequences are ordered lists of elements.
- EXAMPLES:
  - 1, 2, 3, 5, 8
  - 1, 3, 9, 27, 81, …….
  - 1, 2, 3, 4, 5, …
  - 4, 8, 12, 16, 20,…
  - 2, 4, 8, 16, 32, …
  - 1, 1/2, 1/3, 1/4, 1/5, …
  - 1, 4, 9, 16, 25, …
  - 1, -1, 1, -1, 1, -1, …
- Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.
- We will introduce the terminology to represent sequences and sums of the terms in the sequences.

SEQUENCES IN COMPUTER PROGRAMMING:

- An important data type in computer programming consists of finite sequences known as one-dimensional arrays; a single variable in which a sequence of variables may be stored.

EXAMPLE:

- The names of k students in a class may be represented by an array of k elements “name” as:

name [0], name[1], name[2], …, name[k-1]

Sequences

Definition: A *sequence* is a function from a subset of the integers (usually either the set $\{0, 1, 2, 3, 4, \ldots..\}$ or $\{1, 2, 3, 4, \ldots.\}$) to a set *S*.

- The notation a_n is used to denote the image of the integer *n*. We can think of a_n as the equivalent of *f(n)* where *f* is a function from $\{0,1,2,\ldots..\}$ to *S*. We call a_n a *term* of the sequence.

OR

A sequence is just a list of elements usually written in a row.

Sequences

Example: Consider the sequence $\{a_n\}$ where

$$a_n = \frac{1}{n} \qquad \{a_n\} = \{a_1, a_2, a_3, \ldots\}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$$

Sequences

EXAMPLE:

Write the first four terms of the sequence defined by the formula: $b_j = 1 + 2^j$, for all integers $j \geq 0$

SOLUTION:

- $b_0 = 1 + 2^0 = 1 + 1 = 2$
- $b_1 = 1 + 2^1 = 1 + 2 = 3$
- $b_2 = 1 + 2^2 = 1 + 4 = 5$
- $b_3 = 1 + 2^3 = 1 + 8 = 9$

REMARK:

The formula $b_j = 1 + 2^j$, for all integers $j \geq 0$ defines an infinite sequence having infinite number of values.

Sequences

- **EXERCISE:**

Compute the first six terms of the sequence defined by the formula

- Cn = 1+ $(-1)^n$ for all integers n ≥ 0

SOLUTION :

- C0 = 1 + $(-1)^0$ = 1 + 1 = 2
- C1 = 1 + $(-1)^1$ = 1 + (-1) = 0
- C2 = 1 + $(-1)^2$ = 1 + 1 = 2
- C3 = 1 + $(-1)^3$ = 1 + (-1) = 0
- C4 = 1 + $(-1)^4$ = 1 + 1 = 2
- C5 = 1 + $(-1)^5$ = 1 + (-1) = 0

REMARK:

1) If n is even, then Cn = 2 and if n is odd, then Cn = 0. Hence, the sequence oscillates endlessly between 2 and 0.

2) An infinite sequence may have only a finite number of values.

Sequences

EXAMPLE:

Write the first four terms of the sequence defined by

$$C_n = \frac{(-1)^n n}{n+1} \quad \text{for all integers } n \geq 1$$

SOLUTION:

$$C_1 = \frac{(-1)^1(1)}{1+1} = \frac{-1}{2}, C_2 = \frac{(-1)^2(2)}{2+1} = \frac{2}{3}, C_3 = \frac{(-1)^3(3)}{3+1} = \frac{-3}{4}$$

$$And \quad fourth \ \ term \ is C_4 = \frac{(-1)^4(4)}{4+1} = \frac{4}{5}$$

REMARK: A sequence whose terms alternate in sign is called an alternating sequence.

Sequences

Find explicit formulas for sequences with the initial terms given:

1) 0, 1, -2, 3, -4, 5, …

SOLUTION:

$a_n = (-1)^{n+1} n$ for all integers $n \geq 0$

2) $1 - \frac{1}{2}, \frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \frac{1}{4} - \frac{1}{5}, \ldots$

SOLUTION:

$b_k = \frac{1}{k} - \frac{1}{k+1}$ for all integers $n \geq 1$

Sequences

3) 2, 6, 12, 20, 30, 42, 56, …

SOLUTION:

$$C_n = n\,(n+1) \text{ for all integers } n \geq 1$$

4) 1/4, 2/9, 3/16, 4/25, 5/36, 6/49, …

SOLUTION:

$$d_i = \frac{i}{(i+1)^2} \quad \text{for all integers} \quad i \geq 1$$

OR

$$d_j = \frac{j+1}{(j+2)^2} \quad \text{for all integers} \quad j \geq 0$$

Arithmetic Progression OR Sequences

- A sequence in which every term after the first is obtained from the preceding term by adding a constant number is called an arithmetic sequence or arithmetic progression (A.P.)
- The constant number, being the difference of any two consecutive terms is called the common difference of A.P., commonly denoted by “d”.

EXAMPLES:

1. 5, 9, 13, 17, … (common difference = 4)
2. 0, -5, -10, -15, … (common difference = -5)
3. x + a, x + 3a, x + 5a, … (common difference = 2a)

Arithmetic Sequences

GENERAL TERM OF AN ARITHMETIC SEQUENCE:

Let **a** be the first term and **d** be the common difference of an arithmetic sequence. Then the sequence is:

$$a, a+d, a+2d, a+3d, \ldots$$

If a_i, for $i \geq 1$, represents the terms of the sequence then

a_1 = first term = $a = a + (1-1)\,d$

a_2 = second term = $a + d = a + (2-1)\,d$

a_3 = third term = $a + 2d = a + (3-1)\,d$

By symmetry

a_n = nth term = $a + (n - 1)d$ for all integers $n \geq 1$.

Arithmetic Sequences

Examples:

1. Let $a = -1$ and $d = 4$:

$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}$$

2. Let $a = 7$ and $d = -3$:

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

3. Let $a = 1$ and $d = 2$:

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Arithmetic Sequences

EXAMPLE:

Find the 20th term of the arithmetic sequence

3, 9, 15, 21, …

SOLUTION:

- Here a = first term = 3
- d = common difference = 9 - 3 = 6
- n = term number = 20
- a_{20} = value of 20th term = ?
- Since $a_n = a + (n - 1)\,d;\ n \geq 1$

$\therefore\ a_{20} = 3 + (20 - 1)\,6$

$= 3 + 114$

$= 117$

Arithmetic Sequences

EXERCISE:

Find the 36th term of the arithmetic sequence whose 3rd term is 7 and 8th term is 17.

SOLUTION:

Let **a** be the first term and **d** be the common difference of the arithmetic sequence.

Then $a_n = a + (n - 1)d \quad n \geq 1$

$\Rightarrow a_3 = a + (3 - 1)\,d$ and $a_8 = a + (8 - 1)\,d$

Given that $a_3 = 7$ and $a_8 = 17$. Therefore

$7 = a + 2d$……(1) and $17 = a + 7d$……..(2)

Subtracting (1) from (2), we get,

$10 = 5d \qquad \Rightarrow d = 2$

Substituting d = 2 in (1) we have

$7 = a + 2(2)$ which gives $a = 3$

Arithmetic Sequences

Thus, $a_n = a + (n - 1)\, d$

$a_n = 3 + (n - 1)\, 2$ (using values of a and d)

Hence the value of 36th term is

$$a_{36} = 3 + (36 - 1)\, 2$$

$$= 3 + 70$$

$$a_{36} = 73$$

Geometric Progression OR Sequence

- A sequence in which every term after the first is obtained from the preceding term by multiplying it with a constant number is called a geometric sequence or geometric progression (G.P.)
- The constant number, being the ratio of any two consecutive terms is called the common ratio of the G.P. commonly denoted by “r”.
- **EXAMPLE:**

1. 1, 2, 4, 8, 16, … (common ratio = 2)

2. 3, - 3/2, 3/4, - 3/8, … (common ratio = - 1/2)

3. 0.1, 0.01, 0.001, 0.0001, … (common ratio = 0.1 = 1/10)

GENERAL TERM OF A GEOMETRIC SEQUENCE:

Let **a** be the first tem and **r** be the common ratio of a geometric sequence. Then the sequence is $a, ar, ar^2, ar^3, \ldots$

If a_i, for $i \geq 1$ represent the terms of the sequence, then

a_1 = first term = $a = ar^{1-1}$

a_2 = second term = $ar = ar^{2-1}$

a_3 = third term = $ar^2 = ar^{3-1}$

……

……

a_n = nth term = ar^{n-1}; for all integers $n \geq 1$

Geometric Progression

Examples:

1. Let $a = 1$ and $r = -1$. Then:

$$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$$

2. Let $a = 2$ and $r = 5$. Then:

$$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$$

3. Let $a = 6$ and $r = 1/3$. Then:

$$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\}$$

Geometric Sequence

EXAMPLE:

Find the 8th term of the following geometric sequence

$$4, 12, 36, 108, \ldots$$

SOLUTION:

Here a = first term = 4

r = common ratio = $\dfrac{12}{4} = 3$

n = term number = 8

a_8 = value of 8th term = ?

Since $a_n = ar^{n-1}; \quad n \geq 1$

$\Rightarrow \quad a_8 = (4)(3)^{8-1}$

$= 4\,(2187)$

$= 8748$

Geometric Sequence

EXERCISE:

Write the geometric sequence with positive terms whose second term is 9 and fourth term is 1.

SOLUTION:

Let **a** be the first term and **r** be the common ratio of the geometric sequence. Then

$$a_n = ar^{n-1} \qquad n \geq 1$$

Now $\quad a_2 = ar^{2-1}$

$\Rightarrow \quad 9 = ar \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

Also $\quad a_4 = ar^{4-1}$

$1 = ar^3 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

Dividing (2) by (1), we get,

$$\frac{1}{9} = \frac{ar^3}{ar}$$

$$\Rightarrow \quad \frac{1}{9} = r^2$$

$$\Rightarrow \quad r = \frac{1}{3} \qquad \left(\text{rejecting } r = -\frac{1}{3}\right)$$

Substituting r = 1/3 in (1), we get

$$9 = a\left(\frac{1}{3}\right)$$

$$\Rightarrow \quad a = 9 \times 3 = 27$$

Hence the geometric sequence is

27, 9, 3, 1, 1/3, 1/9, …

Useful Sequences

TABLE 1 **Some Useful Sequences.**

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, . . .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, . . .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, . . .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, . . .
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

SERIES

The sum of the terms of a sequence forms a series. If $a_1, a_2, a_3, \ldots$ represent a sequence of numbers, then the corresponding series is

$$a_1 + a_2 + a_3 + \ldots = \sum_{k=1}^{\infty} a_k$$

SUMMATIONS

SUMMATION NOTATION:

The capital Greek letter sigma $\sum$ is used to write a sum in a short hand notation.

where k varies from 1 to n represents the sum given in expanded form by

$= a_1 + a_2 + a_3 + \ldots + a_n$

More generally if m and n are integers and $m \leq n$, then the summation from k equal m to n of a_k is

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

Here **k** is called the index of the summation; **m** the lower limit of the summation and **n** the upper limit of the summation.

SUMMATIONS

COMPUTING SUMMATIONS:

Let $a_0 = 2$, $a_1 = 3$, $a_2 = -2$, $a_3 = 1$ and $a_4 = 0$.Compute each of the summations:

$$(a) \quad \sum_{i=0}^{4} a_i \qquad (b) \quad \sum_{j=0}^{2} a_{2j} \qquad (c) \quad \sum_{k=1}^{1} a_k$$

SOLUTION:

(a) $$\sum_{i=0}^{4} a_i = a_0 + a_1 + a_2 + a_3 + a_4$$
$$= 2 + 3 + (-2) + 1 + 0 = 4$$

(b) $$\sum_{j=0}^{2} a_{2j} = a_0 + a_2 + a_4$$
$$= 2 + (-2) + 0 = 0$$

(c) $$\sum_{k=1}^{1} a_k = a_1$$
$$= 3$$

ARITHMETIC SERIES:

The sum of the terms of an arithmetic sequence forms an arithmetic series (A.S). For example

$$1 + 3 + 5 + 7 + \ldots$$

is an arithmetic series of positive odd integers.

In general, if a is the first term and d the common difference of an arithmetic series, then the series is given as: $a + (a+d) + (a+2d) + \ldots$

SUM OF n TERMS OF AN ARITHMETIC SERIES:

Let a be the first term and d be the common difference of an arithmetic series. Then its nth term is:

$$a_n = a + (n - 1)d; \quad n \geq 1$$

If S_n denotes the sum of first n terms of the A.S, then

$$S_n = a + (a + d) + (a + 2d) + \ldots + [a + (n-1)\, d]$$
$$= a + (a+d) + (a + 2d) + \ldots + a_n$$
$$= a + (a+d) + (a + 2d) + \ldots + (a_n - d) + a_n \ldots\ldots\ldots(1)$$

where $a_n = a + (n - 1)\, d$

Rewriting the terms in the series in reverse order,

$$Sn = a_n + (a_n - d) + (a_n - 2d) + \ldots + (a + d) + a \ldots\ldots\ldots(2)$$

Adding (1) and (2) term by term, gives

$$2\, S_n = (a + a_n) + (a + a_n) + (a + a_n) + \ldots + (a + a_n) \qquad \text{(n terms)}$$
$$2\, S_n = n\,(a + a_n)$$
$$\Rightarrow \quad S_n = n(a + a_n)/2$$
$$S_n = n(a + l)/2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Where $l = an = a + (n - 1)d$

Therefore

$$S_n = n/2\, [a + a + (n - 1)\, d]$$
$$S_n\ n/2\, [2\, a + (n - 1)\, d] \ldots\ldots\ldots(4)$$

ARITHMETIC SERIES:

EXERCISE:

Find the sum of first n natural numbers.

SOLUTION:

Let $S_n = 1 + 2 + 3 + \ldots + n$

Clearly the right hand side forms an arithmetic series with

$a = 1, \quad d = 2 - 1 = 1 \qquad \text{and} \quad n = n$

$$\begin{aligned} \therefore \qquad S_n &= \frac{n}{2}\left[2a + (n-1)d\right] \\ &= \frac{n}{2}\left[2(1) + (n-1)(1)\right] \\ &= \frac{n}{2}\left[2 + n - 1\right] \\ &= \frac{n(n+1)}{2} \end{aligned}$$

GEOMETRIC SERIES:

The sum of the terms of a geometric sequence forms a geometric series (G.S.). For example

$$1 + 2 + 4 + 8 + 16 + \ldots$$

is geometric series.

In general, if **a** is the first term and **r** the common ratio of a geometric series, then the series is given as: $a + ar + ar^2 + ar^3 + \ldots$

Strings

Definition: A *string* is a finite sequence of characters from a finite set (an alphabet).

- Sequences of characters or bits are important in computer science.
- The *empty string* is represented by λ.
- The string *abcde* has *length* 5.

Recurrence Relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Questions about Recurrence Relations

Example 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,2,3,4,\ldots.$ and suppose that $a_0 = 2$. What are a_1, a_2 and a_3?

[Here $a_0 = 2$ is the initial condition.]

Solution: We see from the recurrence relation that

$$a_1 = a_0 + 3 = 2 + 3 = 5$$
$$a_2 = 5 + 3 = 8$$
$$a_3 = 8 + 3 = 11$$

Questions about Recurrence Relations

Example 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2,3,4,\ldots$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

[Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.]

Solution: We see from the recurrence relation that

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

Solving Recurrence Relations

- Finding a formula for the *n*th term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.
- Such a formula is called a *closed formula*.
- Various methods for solving recurrence relations will be covered in Chapter 8 where recurrence relations will be studied in greater depth.
- Here we illustrate by example the method of iteration in which we need to guess the formula. The guess can be proved correct by the method of induction (Chapter 5).

Fibonacci Sequence

Definition: Define the *Fibonacci sequence*, $f_0, f_1, f_2, \ldots,$ by:

- Initial Conditions: $f_0 = 0$, $f_1 = 1$
- Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Example: Find f_2, f_3, f_4, f_5 and f_6 .

Answer:

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$

Iterative Solution Example

Method 1: Working upward, forward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n\text{-}1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$$a_2 = 2 + 3$$
$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$
$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$

.

.

.

$$a_n = a_{n\text{-}1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$$

Iterative Solution Example

Method 2: Working downward, backward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n\text{-}1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$$
\begin{aligned}
a_n &= a_{n\text{-}1} + 3 \\
&= (a_{n\text{-}2} + 3) + 3 = a_{n\text{-}2} + 3 \cdot 2 \\
&= (a_{n\text{-}3} + 3\,) + 3 \cdot 2 = a_{n\text{-}3} + 3 \cdot 3 \\
&\quad . \\
&\quad . \\
&\quad . \\
&= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1)
\end{aligned}
$$

Some Useful Summation Formulae

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Geometric Series: We just proved this.

Later we will prove some of these by induction.

Proof in text (requires calculus)