National University of Computer and Emerging Sciences, Karachi.

FAST School of Computing

Assignment # 1-- Solution, Fall 2022

CS1005-Discrete Structures

<u>Instructions:</u> **Max. Points: 100**

1- This is hand written assignment. You have to submit the hard copy on 22nd September 2022 in your class as well as scan copy on Google classroom.
2- Just write the question number instead of writing the whole question.
3- You can only use A4 size paper for solving the assignment.
4- You have to write student ID and section on the top of each page.

---

1. Which of these sentences are propositions? What are the truth values of those that are propositions?

a) Boston is the capital of Massachusetts. Solution: Proposition. Hence True
b) Miami is the capital of Florida. Solution: Proposition. Hence False
c) 2 + 3 = 5. Solution: Proposition. Hence True
d) 5 + 7 = 10. Solution: Proposition. Hence False
e) x + 2 = 11. Solution: Not a Proposition.
f) Answer this question. Solution: Not a Proposition.

2. Suppose that Smartphone A has 256MB RAM and 32GB ROM, and the resolution of its camera is 8 MP; Smartphone B has 288 MB RAM and 64 GB ROM, and the resolution of its camera is 4 MP; and Smartphone C has 128 MB RAM and 32 GB ROM, and the resolution of its camera is 5 MP.
Express the given statements using Logical Connectives. Also determine the truth value of each of these propositions.
Solution:

	Smart Phone A	Smart Phone B	Smart Phone C
RAM	256 MB	288 MB	128 MB
ROM	32 GB	64 GB	32 GB
Camera resolution	8 MP	4 MP	5 MP

a) Smartphone B has the most RAM of these three smartphones.
Solution: True Statement: p
p: Smartphone B has the most RAM of these three smartphones.

b) Smartphone C has more ROM or a higher resolution camera than Smartphone B.
Solution: True Statement: $p \vee q$ (p OR q)
First identify propositions:
p: C has more ROM than B. FALSE q: C has a higher resolution camera than B TRUE

c) Smartphone B has more RAM, more ROM, and a higher resolution camera than Smartphone A.
Solution: False Statement: $p \wedge q \wedge r$ (p AND q AND r)
First identify propositions:
p: B has more RAM than A. TRUE q: B has more ROM than A. TRUE
r: B has a higher resolution camera than A. FALSE

d) If Smartphone B has more RAM and more ROM than Smartphone C, then it also has a higher resolution camera.
Solution: False **Statement: $(p \land q) \rightarrow r$ (IF (p AND q) THEN r)**
p: B has more RAM than C. TRUE **q: B has more ROM than C. TRUE**
r: B has a higher resolution camera than C. FALSE

e) Smartphone A has more RAM than Smartphone B if and only if Smartphone B has more RAM than Smartphone A.
Solution: False **Statement: $(p \leftrightarrow q)$ (p IF AND ONLY IF q)**
p: A has more RAM than B. FALSE **q: B has more RAM than A. TRUE**

3. Suppose that during the most recent fiscal year,
the annual revenue of Acme Computer was 138 billion dollars and its net profit was 8 billion dollars,
the annual revenue of Nadir Software was 87 billion dollars and its net profit was 5 billion dollars, and
the annual revenue of Quixote Media was 111 billion dollars and its net profit was 13 billion dollars.
Express the given statements using Logical Connectives. Also determine the truth value of each of these propositions for the most recent fiscal year.

a) Quixote Media had the largest annual revenue.
Solution: False (the largest annual revenue had Acme Computer, 138 billion dollars)
p: Quixote Media had the largest annual revenue.

b) Nadir Software had the lowest net profit and Acme Computer had the largest annual revenue.
Solution: True **Statement: $(p \land q)$**
p: Nadir Software had the lowest net profit. True **q: Acme Computer had the largest annual revenue. True**

c) Acme Computer had the largest net profit or Quixote Media had the largest net profit.
Solution: True **Statement: $(p \lor q)$**
p: Acme Computer had the largest net profit. False **q: Quixote Media had the largest net profit. True**

d) If Quixote Media had the smallest net profit, then Acme Computer had the largest annual revenue.
Solution: True **Statement: $(p \rightarrow q)$**
p: Quixote Media had the smallest net profit. False **q: Acme Computer had the largest annual revenue. True**

e) Nadir Software had the smallest net profit if and only if Acme Computer had the largest annual revenue.
Solution: True **Statement: $(p \leftrightarrow q)$**
p: Nadir Software had the smallest net profit. True **q: Acme Computer had the largest annual revenue. True**

4. Let p, q, and r be the propositions
p: You have the flu. **q: You miss the final examination.** **r: You pass the course.**
Express each of these propositions as an English sentence.
a) $p \rightarrow q$
Solution: If you have flu then you will miss the final examination.

b) $\neg q \leftrightarrow r$
Solution: You won't miss the final examination if and only if you pass the course.

c) $q \rightarrow \neg r$
Solution: If you miss the examination then you will be failing (not passing) the course.

d) $p \lor q \lor r$
Solution: You have the flu or you miss the final examination or you pass the course.
e) $(p \rightarrow \neg r) \lor (q \rightarrow \neg r)$
Solution: If you have the flu then you'll not pass the course or if you miss the final examination then you'll fail (not pass) the course.

f) $(p \land q) \lor (\neg q \land r)$
Solution: You have the flu and you miss the examination or you will not miss the final examination and you pass the course.

5. Let p, q, and r be the propositions
p: You get an A on the final exam. q: You do every exercise in this book. r: You get an A in this class.
Write these propositions using p, q, and r and logical connectives.

a) You get an A in this class, but you do not do every exercise in this book.
Solution: $r \land \neg q$

b) You get an A on the final, you do every exercise in this book, and you get an A in this class.
Solution: $p \land q \land r$

c) To get an A in this class, it is necessary for you to get an A on the final.
Solution: $r \rightarrow p$

d) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.
Solution: $p \land \neg q \land r$

e) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.
Solution: $(p \land q) \rightarrow r$

f) You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final.
Solution: $r \leftrightarrow (q \lor p)$

6. Write each of these statements in the form "if p, then q" in English. [Hint: Refer to the list of common ways to express conditional statements provided in this section.]

a) You send me an e-mail message only if I will remember to send you the address. "p only if q"
Solution: If you send me an e-mail message, then I will remember to send you the address.

b) To be a citizen of this country, it is sufficient that you were born in the United States.
"a sufficient condition for q is p"
Solution: If you were born in the United States, then you can be a citizen of the United States.

c) If you keep your textbook, it will be a useful reference in your future courses. "if p, q"
Solution: If you keep your textbook, then it will be a useful reference in your future courses.

d) The Red Wings will win the Stanley Cup if their goalie plays well. "q if p"
Solution: If the Red Wings's goalie plays well, they will win the Stanley Cup.

e) That you get the job implies that you had the best credentials. "p implies q"
Solution: If you get the job, then you had the best credentials.

f) The beach erodes whenever there is a storm. "q whenever p"
Solution: If there is a storm, then beach erodes.
g) It is necessary to have a valid password to log on to the server. "q is necessary for p"
Solution: If you can log on to the server, then you must have a valid password.

h) You will reach the summit unless you begin your climb too late. "q unless ¬p"
Solution: If you begin your climb too late, then you will not reach the summit.

7. "If it is sunny tomorrow, then I will go for a walk in the woods."

a) Describe at least five different ways to write the conditional statement $p \rightarrow q$ in English.
Solution: Following ways to express this conditional statement:

"if p, then q"	"p implies q"
"if p, q"	"p only if q"
"p is sufficient for q"	"a sufficient condition for q is p"
"q if p"	"q whenever p"
"q when p"	"q is necessary for p"
"a necessary condition for p is q"	"q follows from p"
"q unless ¬p"	"q provided that p"

b) State the converse, inverse and contrapositive of a conditional statement.
Solution:
Converse of $p \rightarrow q$: $q \rightarrow p$
Contrapositive of $p \rightarrow q$: $\neg q \rightarrow \neg p$
Inverse of $p \rightarrow q$: $\neg p \rightarrow \neg q$

c) Given a conditional statement $p \rightarrow q$, find the inverse of its inverse, the inverse of its converse, and the inverse of its contrapositive.
Solution:
Inverse: If it is not sunny tomorrow, then I will not go for a walk in the woods.
Inverse of inverse: If it is sunny tomorrow, then I will go for a walk in the woods.

Converse: If I go for a walk in the woods then it will be sunny tomorrow.
Inverse of converse: If I do not go for a walk in the woods then it will not be sunny tomorrow.

Contrapositive: If I do not go for a walk in the woods then it will not be sunny tomorrow.
Inverse of its contrapositive: If I go for a walk in the woods then it will be sunny tomorrow.

8. Use De Morgan's laws to find the negation of each of the following statements.

a) Jan is rich and happy.	Solution: Jan is not rich or happy.
b) Carlos will bicycle or run tomorrow.	Solution: Carlos will not bicycle and not run tomorrow.
c) The fan is slow or it is very hot.	Solution: The fan is not slow and it is not very hot.
d) Akram is unfit and Saleem is injured.	Solution: Akram is not unfit or Saleem is not injured.

9. The following proposition uses the English connective "or". Determine from the context whether "or" is intended to be used in the inclusive or exclusive sense.

a) "Tonight, I will stay home or go out to a movie."
Solution:
Because the one alternative (staying home) precludes the other (going out), "or" is used in the exclusive sense.

b) "If you fail to make a payment on time or fail to pay the amount due, you will incur a penalty."
Solution:
You might both fail to make a payment on time and your late payment might be for an incorrect amount. Hence the inclusive "or" is used here.

c) "If I can't schedule the airline flight or if I can't get a hotel room, then I can't go on the trip."
Solution:
If both events happen, you won't go on the trip. Hence the inclusive "or" is used here.

d) "If you do not wear a shirt or do not wear shoes, then you will be denied service in the restaurant."
Solution:
It is implied that you won't be served if you fail to wear a shirt and also fail to wear shoes. Therefore, the inclusive "or" is used here.

10. Prove the following equivalences by using laws of logic:

a) $(p \wedge (\neg(\neg p \vee q))) \vee (p \wedge q) \equiv p$

Solution:

$\equiv (p \wedge (\neg(\neg p) \wedge q)) \vee (p \wedge q)$	De-Morgan Law
$\equiv (p \wedge (p \wedge \neg q)) \vee (p \wedge q)$	Double Negation
$\equiv ((p \wedge q) \vee (p \wedge \neg q)) \wedge ((p \wedge q) \vee (p))$	Distributive Law
$\equiv ((p \wedge q) \vee (p \wedge \neg q)) \wedge (p)$	Absorption Law
$\equiv (p \wedge (q \vee \neg q)) \wedge (p)$	Distributive Law
$\equiv ((p \wedge T) \wedge (p)$	Negation Law
$\equiv p \wedge p$	Idempotent Law
$\equiv p$	

b) $\neg(p \leftrightarrow q) \equiv (p \leftrightarrow \neg q)$

Solution:

$\neg(p \leftrightarrow q)$	
$\neg[(p \wedge q) \vee (\neg p \wedge \neg q)]$	by $(P \wedge Q) \vee (\neg P \wedge \neg Q)$
$\neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q)$	by De Morgan's Laws
$(\neg p \vee \neg q) \wedge (p \vee q)$	by De Morgan's Laws
$[\neg p \wedge (p \vee q)] \vee [\neg q \wedge (p \vee q)]$	by Distributive Laws
$[(\neg p \wedge p) \vee (\neg p \wedge q)] \vee [(\neg q \wedge p) \vee (\neg q \wedge q)]$	by Distributive Laws
$[F \vee (\neg p \wedge q)] \vee [(\neg q \wedge p) \vee F]$	by Negation Laws
$(\neg p \wedge q) \vee (\neg q \wedge p)$	by Domination Laws
$p \leftrightarrow \neg q$	by $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

OR

$$
\begin{aligned}
&(p \leftrightarrow \neg q) \\
&\equiv (p \rightarrow \neg q) \land (\neg q \rightarrow p) && \text{Equivalence Law on 0} \\
&\equiv (\neg p \lor \neg q) \land (q \lor p) && \text{Implication Law on} \\
&\equiv \neg(\neg((\neg p \lor \neg q) \land (q \lor p))) && \text{Double negation on 2} \\
&\equiv \neg(\neg(\neg p \lor \neg q) \lor \neg(q \lor p)) && \text{De Morgan's Law...} \\
&\equiv \neg((p \land q) \lor (\neg q \land \neg p)) && \text{De Morgan's Law} \\
&\equiv \neg((p \lor \neg q) \land (p \lor \neg p) \land (q \lor \neg q) \land (q \lor \neg p)) && \text{Distribution Law} \\
&\equiv \neg((p \lor \neg q) \land (q \lor \neg p)) && \text{Identity Law} \\
&\equiv \neg((q \rightarrow p) \land (p \rightarrow q)) && \text{Implication Law} \\
&\equiv \neg(p \leftrightarrow q) && \text{Equivalence Law}
\end{aligned}
$$

c) $\neg p \leftrightarrow q \equiv p \leftrightarrow \neg q$

Solution:

$$
\begin{aligned}
&\neg p \leftrightarrow q \\
&\Leftrightarrow (\neg p \rightarrow q) \land (q \rightarrow \neg p) && \text{Biconditional Equivalence} \\
&\Leftrightarrow (\neg\neg p \lor q) \land (\neg q \lor \neg p) && \text{Implication Equivalence (x2)} \\
&\Leftrightarrow (p \lor q) \land (\neg q \lor \neg p) && \text{Double Negation} \\
&\Leftrightarrow (q \lor p) \land (\neg p \lor \neg q) && \text{Commutative} \\
&\Leftrightarrow (\neg\neg q \lor p) \land (\neg p \lor \neg q) && \text{Double Negation} \\
&\Leftrightarrow (\neg q \rightarrow p) \land (p \rightarrow \neg q) && \text{Implication Equivalence (x2)} \\
&\Leftrightarrow p \leftrightarrow \neg q && \text{Biconditional Equivalence}
\end{aligned}
$$

d) $(p \land q) \rightarrow (p \rightarrow q) \equiv T$

Solution:

$$
\begin{aligned}
&(p \land q) \rightarrow (p \rightarrow q) \\
&\equiv \neg(p \land q) \lor (p \rightarrow q) && \text{Law of Implication} \\
&\equiv \neg(p \land q) \lor (\neg p \lor q) && \text{Law of Implication} \\
&\equiv (\neg p \lor \neg q) \lor (\neg p \lor q) && \text{De Morgan's Law} \\
&\equiv (\neg p) \lor (\neg q \lor (\neg p \lor q)) && \text{Associative Law} \\
&\equiv (\neg p) \lor ((\neg p \lor q) \lor \neg q) && \text{Commutative Law} \\
&\equiv (\neg p) \lor (\neg p \lor (q \lor \neg q)) && \text{Associative Law} \\
&\equiv (\neg p) \lor (\neg p \lor \mathbf{T}) && \text{Negation Law} \\
&\equiv (\neg p) \lor (\mathbf{T}) && \text{Domination Law} \\
&\equiv \mathbf{T} && \text{Domination Law}
\end{aligned}
$$

e) $\neg(p \lor \neg(p \land q)) \equiv F$

Solution:

$$
\begin{aligned}
&\neg(p \lor \neg(p \land q)) \\
&\Leftrightarrow \neg p \land \neg(\neg(p \land q)) && \textit{De Morgan's Law} \\
&\Leftrightarrow \neg p \land (p \land q) && \textit{Double Negation Law} \\
&\Leftrightarrow (\neg p \land p) \land q && \textit{Associative Law} \\
&\Leftrightarrow F \land q && \textit{Contradiction} \\
&\Leftrightarrow F && \textit{Domination Law and Commutative Law}
\end{aligned}
$$

11. Using Truth table, show that these compound propositions are logically equivalent or not.

a) $(p \rightarrow r) \land (q \rightarrow r)$ and $(p \lor q) \rightarrow r$

Solution: Equivalent

b) $(p \rightarrow q) \lor (p \rightarrow r)$ and $p \rightarrow (q \lor r)$

Solution: Equivalent
c) $(p \rightarrow q) \rightarrow (r \rightarrow s)$ and $(p \rightarrow r) \rightarrow (q \rightarrow s)$
Solution: Not equivalent

12. Let P(m, n) be the statement "m divides n," where the domain for both variables consists of all positive integers. (By "m divides n" we mean that n = km for some integer k.) Determine the truth values of each of these statements.

a) P(4, 5)	Solution: False
b) P(2, 4)	Solution: True
c) $\forall m \forall n P(m, n)$	Solution: False
d) $\exists m \forall n P(m, n)$	Solution: True
e) $\exists n \forall m P(m, n)$	Solution: False
f) $\forall n P(1, n)$	Solution: True

13. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.

a) $\exists x(x^2 = 2)$	Solution: False
b) $\exists x(x^2 = -1)$	Solution: False
c) $\forall x(x^2 + 2 \geq 1)$	Solution: True
d) $\exists x(x^2 = x)$	Solution: True

14. Let F(x, y) be the statement "x can fool y," where the domain consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody can fool Bob.	Solution: $\forall x F(x, Bob)$
b) Alice can fool everybody.	Solution: $\forall y F(Alice, y)$
c) Everybody can fool somebody.	Solution: $\forall x \exists y F(x, y)$
d) There is no one who can fool everybody.	Solution: $\neg\exists x \forall y F(x, y)$
e) Everyone can be fooled by somebody.	Solution: $\forall y \exists x F(x, y)$

15. Let P(x) be the statement "x can speak Russian" and let Q(x) be the statement "x knows the computer language C++." Express each of these sentences in terms of P(x), Q(x), quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

a) There is a student at your school who can speak Russian and who knows C++.
Solution: $\exists x (P(x) \wedge Q(x))$
b) There is a student at your school who can speak Russian but who doesn't know C++.
Solution: $\exists x (P(x) \wedge \neg Q(x))$
c) Every student at your school either can speak Russian or knows C++.
Solution: $\forall x (P(x) \vee Q(x))$
d) No student at your school can speak Russian or knows C++.
Solution: $\neg\exists x (P(x) \vee Q(x))$

16. Let Q(x, y) be the statement "x has sent an e-mail message to y," where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.

a) $\exists x \exists y Q(x, y)$
Solution: There is some student in your class who has sent a message to some student in your class.
b) $\exists x \forall y Q(x, y)$
Solution: There is some student in your class who has sent a message to every student in your class.
c) $\forall x \exists y Q(x, y)$
Solution: Every student in your class has sent a message to at least one student in your class.

d) $\exists y \forall x\ Q(x, y)$
Solution: There is a student in your class who has been sent a message by every student in your class.
e) $\forall y \exists x\ Q(x, y)$
Solution: Every student in your class has been sent a message from at least one student in your class.
f) $\forall x \forall y\ Q(x, y)$
Solution: Every student in the class has sent a message to every student in the class.

17. Let P(x, y) be the statement "Student x has taken class y," where the domain for x consists of all students in your class and for y consists of all computer science courses at your school. Express each of these quantifications in English.
a) $\exists x \exists y\ P(x, y)$
Solution: At least one student has taken class y.
b) $\exists x \forall y\ P(x, y)$
Solution: At least one student has taken all classes.
c) $\forall x \exists y\ P(x, y)$
Solution: Every student has taken at least one class.
d) $\exists y \forall x\ P(x, y)$
Solution: At least one class has all students.
e) $\forall y \exists x\ P(x, y)$
Solution: Every class has at least one student.
f) $\forall x \forall y\ P(x,y)$
Solution: All students are taking all classes.

18. What rule of inference is used in each of these arguments?

a) Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.
Solution: Addition
b) Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.
Solution: Simplification
c) If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.
Solution: Modus ponens
d) If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.
Solution: Modus Tollens
e) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.
Solution: Hypothetical Syllogism

19. By using Laws of inference, show that the following statement is valid:

a) If today is Tuesday, I have a test in Mathematics or Economics. If my Economics professor is sick, I will not have a test in Economics. Today is Tuesday, and my Economics professor is sick. Therefore, I will have a test in Mathematics.
Solution:

Converting to logical notation:

t = today is Tuesday

m = I have a test in Mathematics

e = I have a test in Economics

s = My Economics Professor is sick.

So the argument is:

$t \longrightarrow (m \vee e)$

$s \longrightarrow \neg e$

$\underline{t \wedge s}$

$\therefore m$

1.	$t \wedge s$	premise
2.	t	from (1) by simplification Law
3.	$t \longrightarrow (m \vee e)$	premise
4.	$m \vee e$	from (2) and (3) by modus ponens
5.	s	from (1) by simplification Law
6.	$s \longrightarrow \neg e$	premise
7.	$\neg e$	from (5) and (6) by modus ponens
8.	$e \vee m$	from (4) by commutative law
9.	m	from (7) and (8) by elimination Law

b) If Ali is a lawyer, then he is ambitious. If Ali is an early riser then he does not like chocolates. If Ali is ambitious then he is an early riser. Therefore, if Ali is a lawyer then he does not like chocolates.

Solution:

$P \rightarrow q$

$\underline{q \rightarrow r}$

$p \rightarrow r$

(Hypothetical syllogism)

$P \rightarrow r$

$\underline{r \rightarrow \neg s}$

$P \rightarrow \neg s$

(Hypothetical syllogism)

Conclusion: $p \rightarrow \neg s$: if Ali is a lawyer then he does not like chocolates.

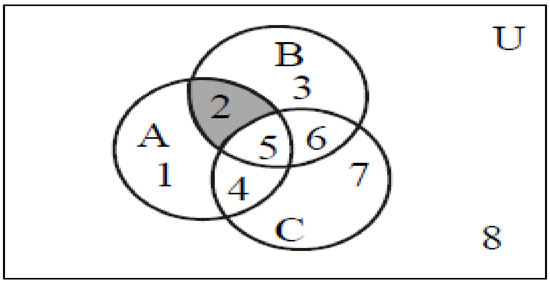
20. Let U= {1, 2, 3, 4, 5, 6, 7,8}, A = {1, 2, 4, 5}, B = {2, 3, 5, 6}, and C = {4, 5, 6, 7}. Find:

a) $(A \cap B) \cap \overline{C}$ b) $\overline{A} \cup (B \cup C)$ c) $(A - B) \cap C$ d) $(A \cap \overline{B}) \cup \overline{C}$

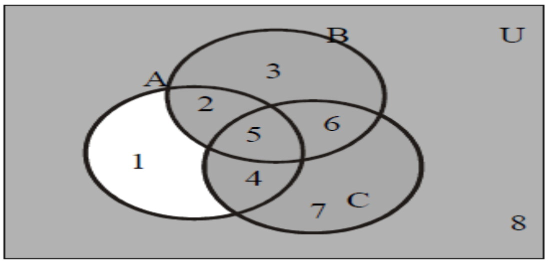
Draw the Venn diagrams for each of these combinations of the sets *A*, *B* and *C*.

Solution:

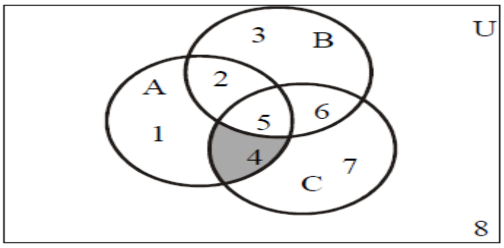
a) $(A \cap B) \cap \overline{C}$ = {2}



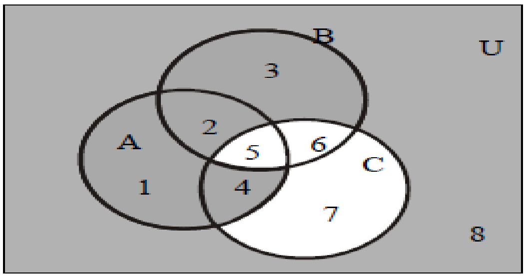
b) $\overline{A} \cup (B \cup C)$ = {2, 3, 4, 5, 6, 7}



c) (A − B) ∩ C = {4}



d) (A ∩$\overline{B}$) ∪ $\overline{C}$ = {1, 2, 3, 4}



21. Prove or disprove the following expression by using the set identities:

a) (A − (A⋂ B)) ⋂ (B − (A⋂ B)) = Φ

Solution:

= (A ⋂ $\overline{(A \cap B)}$) ⋂ (B ⋂ $\overline{(A \cap B)}$)	(A − B) = A ⋂ $\overline{B}$
= (A ⋂ B) ⋂ ($\overline{(A \cap B)}$ ⋂ $\overline{(A \cap B)}$)	Associative Law
= (A ⋂ B) ⋂ ($\overline{(A \cap B)}$)	(A⋂ A) = A
= Φ	(A⋂ $\overline{A}$) = Φ

b) (A − B) ∪ (A⋂ B) = A

Solution:

LHS	$= (A - B) \cup (A \cap B)$	
	$= (A \cap B^c) \cup (A \cap B)$	(Alternative representation for set difference)
	$= A \cap (B^c \cup B)$	Distributive Law
	$= A \cap U$	Complement Law
	$= A$	Identity Law
	$=$ RHS	(proved)

c) (A − B) − C = (A − C) − B

Solution:

$$
\begin{aligned}
\text{LHS} &= A - (A - B) \\
&= A - (A \cap B^c) && \text{Alternative representation for set difference} \\
&= A \cap (A \cap B^c)^c && \text{Alternative representation for set difference} \\
&= A \cap (A^c \cup (B^c)^c) && \text{DeMorgan's Law} \\
&= A \cap (A^c \cup B) && \text{Double Complement Law} \\
&= (A \cap A^c) \cup (A \cap B) && \text{Distributive Law} \\
&= \varnothing \cup (A \cap B) && \text{Complement Law} \\
&= A \cap B && \text{Identity Law} \\
&= \text{RHS} && \text{(proved)}
\end{aligned}
$$

d) $\overline{(\overline{B} \cup (\overline{B} - A))} = B$

Solution:

$(B^c \cup (B^c - A))^c = (B^c \cup (B^c \cap A^c))^c$

Alternative representation for set difference

$$
\begin{aligned}
&= (B^c)^c \cap (B^c \cap A^c)^c && \text{DeMorgan's Law} \\
&= B \cap ((B^c)^c \cup (A^c)^c) && \text{DeMorgan's Law} \\
&= B \cap (B \cup A) && \text{Double Complement Law} \\
&= B && \text{Absorption Law}
\end{aligned}
$$

22. a) Suppose that in a bushel of 100 apples there are 20 that have worms in them and 15 that have bruises. Only those apples with neither worms nor bruises can be sold. If there are 10 bruised apples that have worms in them, how many of the 100 apples can be sold?

Solution:

Total number of apples, n(A) = 100
Number of apples with worm, n(W) = 20
Number of apples with bruises, n(B) = 15
Number of apples with both worms and bruises, n(W∩B) = 10
Number of apples with worms or bruises,
n(W∪B) = n(W) + n(B) – n(W∩B) = 20 + 15 – 10 = 25
Number of apples can be sold = n(A) – n(W∪B) = 100 – 25 = 75

b) In a University of 1000 students, 350 like Computer Science and 450 like Software Engineering. 100 students like both CS & SE. How many like either of them and how many like neither?

Solution:

Total number of students, n(μ) = 1000
Number of Computer Science students, n(CS) = 350
Number of Software Engineering students, n(SE) = 450
Number of students who like both, n(CS ∩ SE) = 100
Number of students who like either of them,
n(CS ∪ SE) = n(CS) + n(SE) – n(CS ∩ SE)= 450 + 350 – 100 = 700
Number of students who like neither = n(μ) – n(CS ∪ SE) = 1000 – 700 = 300

c) In a survey on the gelato preferences of college students, the following data was obtained:
78 like mixed berry, 32 like Irish cream, 57 like tiramisu, 13 like both mixed berry and Irish cream, 21 like both Irish cream and tiramisu, 16 like both tiramisu and mixed berry, 5 like all three flavors, and 14 like none of these three flavors. How many students were surveyed?

Solution:

Let the set of students who like mixed berry be M, those who like tiramisu be T and those who like Irish cream be I. Then, by the inclusion-exclusion principle, the number of students who like at least one of the flavors is

$|M \cup T \cup I| = |M| + |T| + |I| - |M \cap T| - |T \cap I| - |M \cap I| + |M \cap T \cap I|$
$= 78 + 32 + 57 - 16 - 21 - 13 + 5 = 122$
Now there are an additional 14 who like none of these three flavors, so the total number of students surveyed was 122 + 14 = 136.

d) Use set-builder notation and logical equivalences to prove the following. A X (B ∩ C) = (A X B) ∩ (A X C)
Solution: L.H.S

$$
\begin{aligned}
&= \{(x,y) : (x \in A) \wedge (y \in B \cap C)\} && \text{(def. of } \times\text{)} \\
&= \{(x,y) : (x \in A) \wedge (y \in B) \wedge (y \in C)\} && \text{(def. of } \cap\text{)} \\
&= \{(x,y) : (x \in A) \wedge (x \in A) \wedge (y \in B) \wedge (y \in C)\} && (P = P \wedge P) \\
&= \{(x,y) : ((x \in A) \wedge (y \in B)) \wedge ((x \in A) \wedge (y \in C))\} && \text{(rearrange)} \\
&= \{(x,y) : (x \in A) \wedge (y \in B)\} \cap \{(x,y) : (x \in A) \wedge (y \in C)\} && \text{(def. of } \cap\text{)} \\
&= (A \times B) \cap (A \times C) && \text{(def. of } \times\text{)}
\end{aligned}
$$

23. Let A= {a, b, c, d} and B = {a, b, c, d}. Consider the following functions:

a) *f(a) = b, f(b) = a, f(c) = c, f(d) = d* **b) *f(a) = b, f(b) = b, f(c) = d, f(d) = c***
c) *f(a) = d, f(b) = b, f(c) = c, f(d) = d* **d) *f(a) = c, f(b) = a, f(c) =b, f(d) = d***

(i) Determine the Domain, Co-domain and Range of the functions.

Solution: (a)	**Domain: {a, b, c, d}**	**Co-domain: {a, b, c, d}**	**Range: {a, b, c, d}**
(b)	**Domain: {a, b, c, d}**	**Co-domain: {a, b, c, d}**	**Range: {b, c, d}**
(c)	**Domain: {a, b, c, d}**	**Co-domain: {a, b, c, d}**	**Range: {b, c, d}**
(d)	**Domain: {a, b, c, d}**	**Co-domain: {a, b, c, d}**	**Range: {a, b, c, d}**

(ii) Determine whether the functions are Injective, Surjective and Bijective or not?
(a) Injective and Bijective **(b) Neither** **(c) Neither** **(d) Injective and Bijective**

(iii) Determine the inverse of function if exists.
(a) $f^{-1}(b) = a$, $f^{-1}(a) = b$, $f^{-1}(c) = c$, $f^{-1}(d) = d$ ***(b) Not exists***
(c) $f^{-1}(c) = a$, $f^{-1}(a) = b$, $f^{-1}(b) = c$, $f^{-1}(d) = d$ ***(d) Not exists***

24. **(a) *Let* $f(x) = \lfloor \frac{x^2}{3} \rfloor$, Find f (S) if :**
(i) S = {-2,-1,0,1,2,3} **Solution: f (S) = {1,0,0,0,1,3}**
(ii) S = {0,1,2,3,4,5} **Solution: f (S) = {0,0,1,3,5,8}**
(iii) S = {1,5,7,11} **Solution: f (S) = {0,8,16,40}**
(iv) S = {2,6,10,14} **Solution: f (S) = {1,12,33,65}**

(b) (i) $\lceil \frac{3}{4} \rceil$ Solution: 1 **(ii) $\lfloor \frac{7}{8} \rfloor$ Solution: 0** **iii) $\lceil -\frac{3}{4} \rceil$ Solution: 0**

(iv) $\lfloor -\frac{7}{8} \rfloor$ Solution: -1 **(v) $\lceil 3 \rceil$ Solution: 3** **(vi) $\lfloor -1 \rfloor$ Solution: -1**
(vii) $\lfloor \frac{1}{2} + \lceil \frac{3}{2} \rceil \rfloor$ Solution: 2 **(viii) $\lfloor \frac{1}{2} \cdot \lfloor \frac{5}{2} \rfloor \rfloor$ Solution: 1**

(c) Prove or disproof that if x is a real number, then $\lfloor -x \rfloor = -\lceil x \rceil$ and $\lceil -x \rceil = -\lfloor x \rfloor$.

Solution: $\left\lfloor -\frac{3}{4} \right\rfloor = -\left\lceil \frac{3}{4} \right\rceil$ -----> $-\mathbf{1} = -\mathbf{1}$

$\left\lceil -\frac{3}{4} \right\rceil = -\left\lfloor \frac{3}{4} \right\rfloor$ -----> 0 = 0

Hence, It's a disproof.

25. Let f and g be the functions from the set of integers to the set of integers defined by $f(a) = 2a + 3$ and $g(a) = 3a + 2$.
(a) What is the composition of f and g? What is the composition of g and f?
Solution: $(f \circ g)(a) = f(g(a)) = f(3a + 2) = 2(3a + 2) + 3 = 6a + 7$.
$(g \circ f)(a) = g(f(a)) = g(2a + 3) = 3(2a + 3) + 2 = 6a + 11$.

(b) Which type of function f and g are?
Solution: f and g are not injective, surjective and bijective.

(c) Are f and g invertible?
Solution: Hence f and g are not invertible.