13/03/21 Ps-12

13.9

Lagrange Multipliers

↳ To find relative extrema of '3' variables under some constraints

$f(x,y) = \ldots\ldots$ $\qquad x^2+y^2=9$ (constraint)

$\nabla f = f_x\hat{i} + f_y\hat{j}$ $\qquad x^2+y^2-9 = g(x,y)$

$\nabla f = \lambda \nabla g$

$f_x\hat{i} + f_y\hat{j} = \lambda(g_x\hat{i} + g_y\hat{j})$

$f_x = \lambda g_x$

$f_y = \lambda g_y$

Ex $f(x,y) = xy$; $4x^2+8y^2=16$

$\nabla f = \lambda \nabla g$

$(y\hat{i} + x\hat{j}) = \lambda(8x\hat{i} + 16y\hat{j})$

(i) $\Rightarrow y = 8x\lambda$

$\lambda = \frac{y}{8x}$ —(A)

(ii) $\Rightarrow x = 16y\lambda$

$\lambda = \frac{x}{16y}$ —(B)

by (A) and (B)

$\frac{y}{8x} = \frac{x}{16y}$

$2y^2 = x^2$

$x = \pm\sqrt{2}\,y$

$4(\pm\sqrt{2}y)^2 + 8y^2 = 16$

$4(2y^2) + 8y^2 = 16$

$\boxed{y = \pm 1}$

when $y = 1$; When $y = -1$

$x = \pm\sqrt{2}$

$x = \sqrt{2}\,(-1) = -\sqrt{2}$

$= (-\sqrt{2})(-1) = \sqrt{2}$

$(\sqrt{2}, 1), (-\sqrt{2}, 1)$

$(-\sqrt{2}, -1), (\sqrt{2}, -1)$

$(\sqrt{2}, 1) = \sqrt{2} = 1.41 \rightarrow$ maximum

$(-\sqrt{2}, 1) = -1.41 \rightarrow$ minimum

$(-\sqrt{2}, -1) = 1.41 \rightarrow$ maximum

$(\sqrt{2}, -1) = -1.41 \rightarrow$ minimum

Max. at $(\sqrt{2}, 1), (-\sqrt{2}, -1)$ and value is 1.41

Min. at $(-\sqrt{2}, 1), (\sqrt{2}, -1)$ and value is -1.41

7. $f(x,y) = 4x^3 + y^2$; $2x^2 + y^2 = 1$

$\nabla f = \lambda \nabla g$

$(12x^2\hat{i} + 2y\hat{j}) = \lambda\,(4x\hat{i} + 2y\hat{j})$

(i) $\Rightarrow 12x^2 = 4x\lambda$

$12x^2 - 4x\lambda = 0$

$4x\,(3x - \lambda) = 0$

$x = 0$, $x = \frac{\lambda}{3}$

(ii) $\Rightarrow 2y = 2y\lambda$

$2y - 2y\lambda = 0$

$2y\,(1 - \lambda) = 0$

$y = 0$, $\lambda = 1$

when $y=0$

$$2x^2+y^2=1$$

$$2x^2+0=1$$

$$x=\pm\frac{1}{\sqrt{2}} \qquad \left(\pm\frac{1}{\sqrt{2}},0\right)$$

when $x=0$

$$y^2=1 \Rightarrow y=\pm 1$$

$$(0,\pm 1)$$

when $x=\frac{1}{3}$

$$\frac{2}{9}+y^2=1 \Rightarrow y=\pm\frac{\sqrt{7}}{3}$$

$$\left(\frac{1}{3},\frac{\pm\sqrt{7}}{3}\right)$$

$(1/\sqrt{2},0)$	1.41	$\rightarrow$ maximum
$(-1/\sqrt{2},0)$	-1.41	$\rightarrow$ minimum
$(0,1)$	1	
$(0,-1)$	1	
$(1/3,\sqrt{7}/3)$	0.925	
$(1/3,-\sqrt{7}/3)$	0.925	

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10. $f(x,y,z) = 3x + 6y + 2z \; ; \; 2x^2 + 4y^2 + z^2 = 70$

$\nabla f = \lambda \nabla g$

$(3\hat{i} + 6\hat{j} + 2\hat{k}) = \lambda (4x\hat{i} + 8y\hat{j} + 2z\hat{k})$

(i) $\Rightarrow 3 = 4x\lambda$

$\lambda = \frac{3}{4x}$

(ii) $\Rightarrow 6 = 8y\lambda$

$\lambda = \frac{3}{4y}$

(iii) $\Rightarrow 2 = 2z\lambda$

$\lambda = \frac{1}{z}$

Solving (i) and (ii)

$\frac{3}{4x} = \frac{3}{4y}$

$x = y$ — (A)

Solving (ii) and (iii)

$\frac{3}{4y} = \frac{1}{z}$

$z = \frac{4y}{3}$ — (B)

$\because 2x^2 + 4y^2 + z^2 = 70$

By (A) and (B)

$\frac{70}{9} y^2 = 70$

$\boxed{y = \pm 3}$

6, 8, 9

When $y = 3$

$(3, 3, 4)$

When $y = -3$

$(-3, -3, -4)$

$f(x, y, z) = 3x + 6y + 2z$

$f(3, 3, 4) = 35 \rightarrow$ maximum

$f(-3, -3, -4) = -35 \rightarrow$ minimum

9. $f(x, y, z) = 2x + y - \lambda z$; $x^2 + y^2 + z^2 = 4$

$\nabla f = \lambda \nabla g$

$(2\hat{i} + \hat{j} - 2\hat{k}) = \lambda(2x\hat{i} + 2y\hat{j} + 2z\hat{k})$

(i) $\Rightarrow \cancel{2} = \cancel{2}x\lambda$

$\boxed{\lambda = \frac{1}{x}}$

(ii) $\Rightarrow 1 = 2\lambda y$

$\boxed{\lambda = \frac{1}{2y}}$

(iii) $\Rightarrow -2 = 2z\lambda$

$\boxed{\lambda = \frac{-1}{z}}$

Solving (i) and (ii)

$\frac{1}{x} = \frac{1}{2y} \Rightarrow \boxed{x = 2y}$ —(A)

solving (ii) and (iii)

$$\frac{1}{2y} = \frac{-1}{2}$$

$$\boxed{z = -2y} \quad -(B)$$

$\because x^2+y^2+z^2 = 4$

$(2y)^2+y^2+(2y)^2 = 4$

$4y^2+y^2+4y^2 = 4$

$9y^2 = 4$

$y = \pm\frac{2}{3}$

When $y = \frac{2}{3}$

$\left(\frac{4}{3}, \frac{2}{3}, \frac{-4}{3}\right)$

When $y = -\frac{2}{3}$

$\left(\frac{-4}{3}, \frac{-2}{3}, \frac{+4}{3}\right)$

$\because f(x,y,z) = 2x+y-2z$

$f\left(\frac{4}{3}, \frac{2}{3}, \frac{-4}{3}\right) = \frac{8}{3} + \frac{2}{3} + \frac{8}{3}$

$= \frac{18}{3} = 6 \rightarrow$ maximum

$f\left(\frac{-4}{3}, \frac{-2}{3}, \frac{+4}{3}\right) = -\frac{8}{3} - \frac{2}{3} - \frac{8}{3}$

$= \frac{-18}{3} = -6 \rightarrow$ minimum

$F(x,y) = x - 3y - 1 \; ; \; x^2 + 3y^2 = 16$

$(\hat{i} - 3\hat{j}) = \lambda(2x\hat{i} + 6y\hat{j})$

(i) $\Rightarrow$ $1 = \lambda 2x$

$$\boxed{\lambda = \frac{1}{2x}}$$

(ii) $\Rightarrow$ $-3 = 6y\lambda$

$-1 = 2y\lambda$

$$\boxed{\lambda = \frac{-1}{2y}}$$

Solving (i) and (ii)

$$\frac{1}{2x} = \frac{-1}{2y}$$

$$\boxed{x = -y}$$

$\because$ $x^2 + 3y^2 = 16$

$(-y)^2 + 3y^2 = 16$

$4y^2 = 16$

$$\boxed{y = \pm 2}$$

when $y = 2$; when $y = -2$

$x = -2$, $x = +2$

$(-2, 2)$ $(2, -2)$

$\because$ $f(x,y) = x - 3y - 1$

$f(-2,2) = -2 - 6 - 1 = -9$ $\rightarrow$ min

$f(2,-2) = 2 + 6 - 1 = 7$ $\rightarrow$ max

20/03/24

$f(x,y)$

$a \le x \le b$, $c \le y \le d$

Fubini's Theorem:-

$$\int_c^d \left(\int_a^b f(x,y)\,dx \right) dy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

iff a, b, c, d are constants

Ex : 14.1 (1-16)

① $\int_0^1 \int_0^2 (x+3)\,dy\,dx$

$= \int_0^1 (x+3)y \Big|_0^2 \cdot dx$

$= \int_0^1 2(x+3) \cdot dx$

$= 2\left(\frac{x^2}{2} + 3x\right)\Big|_0^1$

$= x^2 + 6x \Big|_0^1$

$= \boxed{7}$

9a $\int_0^1 \int_0^1 \frac{x}{(xy+1)^2} \cdot dy\,dx$

$= \int_0^1 \int_0^1 x(xy+1)^{-2} \cdot dy\,dx$

$= \int_0^1 \frac{(xy+1)^{-1}}{-1} \Big|_0^1 \cdot dx$

$= \int_0^1 \frac{-1}{(xy+1)} \Big|_0^1 dx$

$= \int_0^1 \frac{-1}{x+1} \cdot dx$

$= \left(-\ln(x+1) + x\right) \Big|_0^1$

$= [-\ln(2) + 1] - (0+0)$

$= 1 - \ln(2)$

10 $\int_{\pi/2}^{\pi} \int_1^2 x\cos xy \cdot dy\,dx$

$\int_{\pi/2}^{\pi} \frac{\sin xy}{\cancel{x}} \Big|_1^2 dx$

$\int_{\pi/2}^{\pi} \sin xy \Big|_1^2 \cdot dx$

$\int_{\pi/2}^{\pi} \sin 2x - \sin x \cdot dx$

$\frac{-\cos 2x}{2} + \cos x \Big|_{\pi/2}^{\pi}$

$$= \left(\frac{-1}{2} - 1\right) - \left(\frac{1}{2}\right)$$

$$\boxed{= -2}$$

$$2 - \int_1^3 \int_{-1}^1 (2x - 4y)\, dy\, dx$$

$$= \int_1^3 2xy - \frac{4y^2}{2} \Big|_{-1}^1 \, dx$$

$$\cancel{\int_1^3 \left(2y - \frac{4y^2}{2}\right) - \left(-2y - \frac{4y^2}{2}\right) dx}$$

$$= \int_1^3 2xy - 2y^2 \Big|_{-1}^1 \, dx$$

$$= \int_1^3 (2x - 2) - (-2x - 2)\, dx$$

$$= \int_1^3 (2x - \cancel{2} + 2x + \cancel{2})\, dx$$

$$= \int_1^3 4x \, dx$$

$$= \frac{4x^2}{2} \Big|_1^3$$

$$= 2x^2 \Big|_1^3$$

$$= 2(3)^2 - 2(1)^2$$

$$= 2(9) - 2$$

$$= 18 - 2$$

$$= 16$$

5. $\int_0^{\ln 3}\int_0^{\ln 2} e^{x+y}\, dy\, dx$

$= \int_0^{\ln 3} \frac{e^{x+y}}{1}\Big|_0^{\ln 2} dx$

$= \int_0^{\ln 3} \left(e^{x+\ln 2}\right) - \left(e^{x+0}\right) dx$

$= \int_0^{\ln 3} \left(e^{x+\ln 2} - e^{x}\right) dx$

$= \int_0^{\ln 3} e^{x}\left(e^{\ln 2} - 1\right) dx$

$= \left(e^{\ln 2} - 1\right) e^{x}\Big|_0^{\ln 3}$

$= \left(e^{\ln 2} - 1\right)\left(e^{\ln 3} - e^{0}\right)$

$= \left(e^{\ln 2} - 1\right)\left(e^{\ln 3} - 1\right)$

$= (2-1)(3-1)$

$= 1(2)$

$= 2$

$\underline{\underline{\text{Ill}}}$ $\int_0^{\ln 2}\int_0^{1} xy\, e^{y^2 x}\, dy\, dx$

$= \int_0^{\ln 2} \frac{1}{2}\int_0^{1} \left(2xy\, e^{y^2 x}\right) dy\, dx$

$= \int_0^{\ln 2} \frac{e^{y^2 x}}{2}\Big|_0^{1} dx$

$= \int_0^{\ln 2} \left(\frac{e^{x}}{2} - \frac{e^{0}}{2}\right) dx \quad = \int_0^{\ln 2} \left(\frac{e^{x}}{2} - \frac{1}{2}\right) dx$

$= \frac{1}{2}\int_0^{\ln 2} \left(e^{x} - 1\right) dx$

$= \frac{1}{2}\left[\left(e^{\ln 2} - \ln 2\right) - \left(e^{0} - 1\right)\right] \quad = \frac{1}{2}\left[\left[2(1) - \ln(2)\right] - 0\right]$

$= 1 - \frac{\ln 2}{2}$

$\because dA = dx\,dy$ or $dA = dy\,dx$

(as per our wish)

$$\text{Q14} \quad \int_0^1 \int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\, dx\, dy$$

$$= \int_0^1 y \int_0^1 x\,(x^2+y^2+1)^{-1/2}\, dx\, dy$$

$$= \int_0^1 y \, \frac{1}{2} \int_0^1 2x\,(x^2+y^2+1)^{-1/2}\, dx\, dy$$

$$= \int_0^1 \frac{y}{2} \left. \frac{(x^2+y^2+1)^{1/2}}{1/2} \right|_0^1 dy$$

$$= \int_0^1 y\,(x^2+y^2+1)^{1/2} \Big|_0^1 \cdot dy$$

$$= \int_0^1 \left[y\,(y^2+2)^{1/2} - y\,(y^2+1)^{1/2} \right] dy$$

$$= \frac{1}{2} \int_0^1 \left[2y\,(y^2+2)^{1/2} - 2y\,(y^2+1)^{1/2} \right] dy$$

$$= \frac{1}{2} \left. \left(\frac{(y^2+2)^{3/2}}{3/2} - \frac{(y^2+1)^{3/2}}{3/2} \right) \right|_0^1 \text{ ans}$$

$$= \frac{(y^2+2)^{3/2}}{3} - \frac{(y^2+1)^{3/2}}{3}$$

$$= \left(\frac{(3)^{3/2}}{3} - \frac{(2)^{3/2}}{3} \right) - \left(\frac{(2)^{3/2}}{3} - \frac{(1)^{3/2}}{3} \right)$$

$$= \frac{(3)^{3/2}}{3} - \frac{(2)^{3/2}}{3} - \frac{(2)^{3/2}}{3} + \frac{1}{3}$$

$$= \frac{(3)^{3/2}}{3} \text{ ans}$$