# Residue Integration Method



I $\oint_C f(z)\,dz = 2\pi i\, b_1$ where $b_1 = \operatorname*{Res}_{z=z_0} f(z)$

$$b_1 = \operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z-z_0) f(z)$$

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)} \quad \text{if } f(z) = \frac{p(z)}{q(z)}$$

$$b_1 = \lim_{z\to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

$$\oint_C f(z) = 2\pi i \sum_{n=1}^{k} \operatorname*{Res}_{z=z_n} f(z)$$

**Q1** Find all singular points of $f(z) = \dfrac{z^2}{z^4-1}$

Solution: For singular point we have

$$z^4-1=0 \Rightarrow (z^2-1)(z^2+1)=0$$

$$z^2+1=0 \quad \& \quad z^2-1=0$$

$$z=\pm i \quad \& \quad z=\pm 1$$

Thus all singular point are $\{\pm 1, \pm i\}$

Q#2 evaluate the integral $f(z) = \tan \pi z$
when $C: |z| = 1$

Solution: $f(z) = \tan \pi z = \dfrac{\sin \pi z}{\cos \pi z}$

For Singular Points $\cos \pi z = 0$

$\pi z = (2n+1)\pi/2 \quad = z = (2n+1)\frac{1}{2} \quad n \in Z$

$z = 1/2 \,, -1/2$

for $z = 1/2 \qquad b_1 = \dfrac{\sin \pi/2}{-\pi \sin \pi/2} = -1/\pi$

$$b_1 = \frac{\sin(-\frac{1}{2}\pi)}{-\pi \sin(-\frac{1}{2}\pi)} = \frac{-1}{\pi} = -1/\pi$$

$$\oint_c f(z)dz = 2\pi i\left(-\frac{1}{\pi} - \frac{1}{\pi}\right) = +2i(-2)$$
$$= -4i$$

$$\oint_c \tan \pi z \, dz = -4i$$

Q# 3 $\oint_c \frac{\sin \pi z}{z^4} dz \quad c: |z-i| = 2$ 

Solution: Here $f(z) = \frac{\sin \pi z}{z^4} \quad |z-i| = 2$

$z^4 = 0 \quad z = 0$ for order $= 4$

$m = 4 \quad z_0 = 0$

$$b_1 = \lim_{z \to 0} \frac{1}{4-1} \frac{d^3}{dz^3}\left[\cancel{(z-0)^4} \frac{\sin \pi z}{\cancel{z^4}}\right]$$

$$= \frac{1}{3} \lim_{z \to 0} \frac{d^3}{dz^3} \sin \pi z$$

$$= \frac{1}{3} \lim_{z \to 0} \left[-\pi^3 \cos \pi z\right]$$

$$= -\pi^3/3$$

$$\oint_c \frac{\sin \pi z}{z^4} dz = 2\pi i \left(-\pi^3/3\right) = -\frac{2}{3}\pi^4 i$$

# Residue Integration of Real integrals

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$d\theta = \frac{dz}{iz}$$

Cauchy Principal value

$$\text{Pr.V} \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z) + \pi i \sum \operatorname{Res} f(z)$$

Q4 Evaluate the integral $\displaystyle\int_0^{2\pi} \frac{d\theta}{7+6\cos\theta}$

Solution: $$\int_0^{2\pi} \frac{d\theta}{7+6\cos\theta} = \oint_C \frac{1}{7+\cancel{6}^{3}\,\frac{1}{\cancel{2}}\left(z+\frac{1}{z}\right)} \cdot \frac{dz}{iz}$$

$$= -i\oint_C \frac{\cancel{z}}{7z+3z^2+3}\,\frac{dz}{i\cancel{z}} = -i\oint \frac{dz}{3z^2+7z+3}$$

$$3z^2+7z+3=0$$

$a=3 \quad b=7 \quad c=3$

$$z=\frac{-7\pm\sqrt{49-36}}{6}=\frac{-7\pm\sqrt{13}}{6}$$

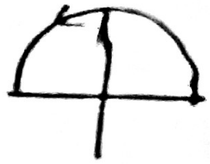
if $z=\dfrac{-7+\sqrt{13}}{6}$ lies in $|z|=1$

$z=\dfrac{-7-\sqrt{13}}{6}$ out side $|z|=1$

$$b_1=\lim_{z\to\frac{-7+\sqrt{13}}{6}}\left[\frac{1}{3z^2+7z+3}\right]=\lim_{z\to z_0}\frac{1}{6z+7}$$

$$b_1=\frac{1}{\cancel{6}\left(\frac{-7+\sqrt{13}}{\cancel{6}}\right)+7}=\frac{1}{\sqrt{13}}$$

$$\oint_C\frac{dz}{3z^2+7z+3}=-i\left(2\pi i\,\frac{1}{\sqrt{13}}\right)=\frac{2\pi}{\sqrt{13}}$$

Q5 Find Cauchy Principal value 

$$\int_{-\infty}^{\infty} \frac{x+5}{x^3-x}\, dz = \oint \frac{z+5}{z^3-z}\, dz$$

$$z^3 - z = 0 \qquad z(z^2-1) = 0 \qquad z = 0 \quad z = \pm 1$$

$$b_1 = z = z_0 = 0, \quad \lim_{z\to 0} \frac{z+5}{z^3 z} = \frac{5}{-1} = -5$$

$$b_2 = z = z_0 = 1 \quad \lim_{z\to 1} \frac{z+5}{z^3-z} = \frac{z+5}{3(1)^2-1}$$

$$b_2 = \operatorname*{Res}_{z=1} f(z) = \frac{6}{2} = 3$$

$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z\to -1} \frac{z+5}{z^3 z} = \frac{z+5}{3z^2-1}$$

$$\operatorname*{Res}_{z=-1} f(z) = \frac{-1+5}{3(-1)^2-1} = \frac{4}{2} = 2$$

$$\oint_c f(z)\,dz = 2\pi i \sum \operatorname{Res} f(z)$$

$$= \pi i(-5+3+2) = 0$$

Q6 Find fourier Series of

$f(x) = x^2 \quad -1 \le x \le 1 \quad p = 2 \quad l = 1$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{1}{2(1)}\int_{-1}^{1} x^2\,dx$$

$$a_0 = \frac{1}{2}\,\frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{3}\right) = \frac{1}{3}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi}{l}x\,dx$$

$$= \frac{1}{1}\int_{-1}^{1} x^2 \cos n\pi x\,dx$$

$$\begin{array}{ccc}
x^2 & & \cos n\pi x \\
 & + & \\
2x & & \cancel{n\pi}\,\dfrac{\sin n\pi x}{n\pi} \\
 & - & \\
2 & & -\dfrac{\cos n\pi x}{(n\pi)^2} \\
 & + & \\
0 & & -\dfrac{\sin n\pi x}{(n\pi)^3}
\end{array}$$

$$a_n = \left[x^2 \frac{\sin n\pi x}{n\pi} + 2x \frac{\cos n\pi x}{(n\pi)^2} - 2\frac{\sin n\pi x}{(n\pi)^3}\right]_{-1}^{1}$$

$$a_n = \left[0 + 2(1)\frac{\cos n\pi}{(n\pi)^2} - \frac{(-2)\cos n\pi}{(n\pi)^2}\right]$$

$$a_n = \frac{4\cos n\pi}{(n\pi)^2} = \frac{4(-1)^n}{n^2\pi^2}$$

$b_n = 0$ [due to even function]

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x$$

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} \cos n\pi x.$$

$$= \frac{1}{3} - \frac{4}{\pi^2}\left[\cos \pi x - \frac{1}{4}\cos 2\pi x + \frac{1}{9}\cos 3\pi x \cdots\right]$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, d\omega$$

$$F[f^{n}(x)] = (i\omega)^{n} F[f(x)]$$

Q7 Find fourier Transform of

$$f(x) = \begin{cases} K & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution: Here $f(x) = \begin{cases} K & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^{-1} 0\, e^{-i\omega x}\, dx + \int_{-1}^{1} K\, e^{-i\omega x}\, dx + \int_{1}^{\infty} 0\, e^{-i\omega x}\, dx \right]$$

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1} k\, e^{-i\omega x}\,dx = \frac{k\,e^{-i\omega x}}{-i\omega}\Big|_{-1}^{1}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{+2k}{\omega}\left(e^{-i\omega}-e^{i\omega}\right)\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[-\frac{2k}{\omega}\left(e^{i\omega}-e^{-i\omega}\right)\right]=\frac{1}{\sqrt{2\pi}}\left[-\frac{2ik}{\omega}\left(\frac{e^{i\omega}-e^{-i\omega}}{2i}\right)2i\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[+\frac{2k}{\omega}\sin\omega\right]$$

Thus $F(\omega)=\frac{1}{\sqrt{2\pi}}\frac{2k}{\omega}\sin\omega = \sqrt{\frac{2}{\pi}}\,\frac{k\sin\omega}{\omega}$

$$F(\omega)=\sqrt{\frac{2}{\pi}}\,\frac{k\sin\omega}{\omega}$$