Complex Numbers

$z = x + iy \qquad Re\, z = x \ \& \ Im(z) = y$

$|z| = \sqrt{x^2 + y^2}$

$$\theta = \begin{cases} \tan^{-1} y/x & \text{if } x > 0 \\ \tan^{-1} y/x + \pi & \text{if } x < 0 \ \& \ y > 0 \\ \tan^{-1} y/x - \pi & \text{if } x < 0 \ \& \ y < 0 \end{cases}$$

$$\theta = \begin{cases} 0 & \text{if } x > 0 \ \& \ y = 0 \\ \pi & \text{if } x < 0 \ \& \ y = 0 \\ \pi/2 & \text{if } x = 0 \ \& \ y > 0 \\ -\pi/2 & \text{if } x = 0 \ \& \ y < 0 \end{cases}$$

Polar form of Complex Number

$z = r[\cos\theta + i \sin\theta]$

Example # 01 express the Complex Number $z = 1 + i$ in Polar form

Solution: Here $z = 1 + i \quad |z| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$

$\theta = \tan^{-1}(1/1) = \tan^{-1}(1) = \pi/4$

$r = \sqrt{2} \qquad \theta = \pi/4$

$z = \sqrt{2}\left[\cos \pi/4 + i \sin \pi/4\right]$

Example #02 Express $z = -1 + i$ in Polar form

Solution : Here $z = -1 + i \qquad r = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{-1}\right) + \pi \qquad$ As $x < 0, y > 0$

$\theta = -\tan^{-1}(1) + \pi$

$\theta = -\pi/4 + \pi = 3\pi/4$

Thus $z = \sqrt{2}\left[\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right]$

Exp #03 Express $z = -1 - i$ in Polar form

Sol : Here $z = -1 - i \qquad r = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{-1}{-1}\right) - \pi = \tan^{-1}(1) - \pi$

$\theta = \pi/4 - \pi = \frac{-3\pi}{4}$

$-1 - i = \sqrt{2}\left[\cos\left(\frac{-3\pi}{4}\right) + i \sin\left(\frac{-3\pi}{4}\right)\right]$

$$-1-i = \sqrt{2}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right]$$

$$-1-i = \sqrt{2}\ Cis\left(\frac{3\pi}{4}\right)$$

Exp#04 Express $z = +1-i$ in Polar form

Sol: Here $z = 1-i$ $\quad r = \sqrt{2}$

$$\theta = \tan^{-1}\left(\frac{-1}{1}\right) = -\tan^{-1}(1) = -\pi/4$$

$$z = r[\cos\theta + i\sin\theta]$$

$$1-i = \sqrt{2}\left[\cos(-\pi/4) + i\sin(-\pi/4)\right]$$

$$1-i = \sqrt{2}\left[\cos\pi/4 - i\sin\pi/4\right]$$

or

$$1-i = \sqrt{2}\ Cis(-\pi/4)$$

~~root~~ Principal argument θ if

$$-\pi < \theta \leq \pi$$

Root of a Complex Number

Q.1 Find roots of $\sqrt[3]{1-i}$

Sol: Here $z = 1-i \quad r = \sqrt{2} \quad \theta = \tan^{-1}\left(\frac{-1}{1}\right)$

$$\theta = \tan^{-1}(-1) = -\tan^{-1}(1) = -\pi/4$$

$$\theta = -\pi/4 + 2k\pi = \frac{8k\pi - \pi}{4} \quad k \in Z$$

$$\theta = (8k-1)\pi/4 \quad k \in Z$$

$$1-i = \sqrt{2}\left[\cos\frac{(8k-1)\pi}{4} + i\sin\frac{(8k-1)\pi}{4}\right]$$

$$(1-i)^{1/3} = (2)^{1/6}\left[\cos\frac{(8k-1)\pi}{4} + i\sin\frac{(8k-1)\pi}{4}\right]^{1/3}$$

$$(1-i)^{1/3} = (2)^{1/6}\left[\cos\frac{(8k-1)\pi}{12} + i\sin\frac{(8k-1)\pi}{12}\right]$$

Put $k=0$

$$2^{1/6}\left[\cos -\pi/12 + i\sin(-\pi/12)\right] = 2^{1/6}\operatorname{cis}\frac{-\pi}{12}$$

Put $k=1$ $\quad 2^{1/6}\left[\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right] = 2^{1/6}\operatorname{cis}\frac{7\pi}{12}$

Put $k=2$ $\quad 2^{1/6}\left[\cos\frac{15\pi}{12} + i\sin\frac{15\pi}{12}\right] = 2^{1/6}\operatorname{cis}\frac{5\pi}{4}$

Thus the roots of $\sqrt[3]{1-i}$ are

$$2^{1/6}\operatorname{Cis}(-\pi/12),\ 2^{1/6}\operatorname{Cis}\left(\frac{7\pi}{12}\right),\ \&\ 2^{1/6}\operatorname{Cis}\left(\frac{5\pi}{4}\right)$$

Q#2 Solve the $z^2 - z + 1 + i = 0$

Solution: Here $a = 1 \quad b = -1 \quad c = 1+i$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1+i)}}{2(1)}$$

$$z = \frac{1 \pm \sqrt{1-4-4i}}{2} = \frac{1 \pm \sqrt{-3-4i}}{2}$$

Now $\sqrt{-3-4i} = ?$

$z = -3-4i \quad r = \sqrt{9+16} = 5$

$\theta = \tan^{-1}(-4/3) - \pi = \tan^{-1}\left(\frac{4}{3}\right) - \pi$

$\theta = \tan^{-1}(4/3) - \pi \quad \theta + 2k\pi \quad k \in \mathbb{Z}$

$$-3-4i = 5\left[\cos\left(\tan^{-1}(4/3)-\pi+2k\pi\right)\right.$$
$$\left.+\, i\sin\left(\tan^{-1} 4/3 - \pi + 2k\pi\right)\right]$$

$$-3+4i = 5\,\text{Cis}\left[\tan^{-1}\frac{4}{3} - \pi + 2k\pi\right]$$

$$(-3+4i)^{1/2} = \sqrt{5}\,\text{Cis}\left[\frac{\tan^{-1}\frac{4}{3}-\pi+2k\pi}{2}\right]$$

$k=0$

$$(-3+4i)^{1/2} = \sqrt{5}\,\text{Cis}\left(\tan^{-1}\frac{4}{3} - \pi\right]$$

$$= \sqrt{5}\left[-\cos\left(\tan^{-1}\frac{4}{3}\right) - \sin\tan^{-1} 4/3\right)$$

$$= -\sqrt{5}\,\text{Cis}\left(\tan^{-1} 4/3\right)$$

Thus

$$Z = \frac{1 \pm \left(-\sqrt{5}\,\text{Cis}\tan^{-1} 4/3\right)}{2}$$

$$Z = \frac{1}{2} \mp \sqrt{5}\,\text{Cis}\left(\tan^{-1} 4/3\right)$$

or

$$Z = \frac{1}{2} + \sqrt{5}\left(\cos\tan^{-1} 4/3 + i\sin\tan^{-1} 4/3\right)$$

$$,\ 1/2 - \sqrt{5}\left(\cos\tan^{-1} 4/3 + i\sin\tan^{-1} 4/3\right)$$

Show that

$$|z_1+z_2|^2+|z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$$

L.H.S $= |z_1+z_2|^2 + |z_1-z_2|^2 \qquad (|z_1|^2 = z_1\bar{z}_1)$

$$= (z_1+z_2)\overline{(z_1+z_2)} + (z_1-z_2)\overline{(z_1-z_2)}$$

$$= (z_1+z_2)(\bar{z}_1+\bar{z}_2) + (z_1-z_2)(\bar{z}_1-\bar{z}_2)$$

$$= z_1\bar{z}_1 + \cancel{z_1\bar{z}_2} + \cancel{z_2\bar{z}_1} + z_2\bar{z}_2$$
$$+ z_1\bar{z}_1 - \cancel{z_1\bar{z}_2} - \cancel{z_2\bar{z}_1} + z_2\bar{z}_2$$

$$= 2z_1\bar{z}_1 + 2z_2\bar{z}_2$$

$= 2[|z_1|^2+|z_2|^2]$ = R.H.S

Thus

$$|z_1+z_2|^2+|z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$$

Derivative of a Complex Number

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Q.1 Find $f'(z)$ if $f(z) = \bar{z}$

Sol: Here $f(z) = \bar{z}$ Then by definition

$$f'(z) = \lim_{\Delta z \to 0} \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{\bar{z}} + \overline{\Delta z} - \cancel{\bar{z}}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\overline{\Delta x + i\Delta y}}{\Delta x + i\Delta y}$$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

when $\Delta x \to 0$ $\Delta y = 0$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

when $\Delta x = 0$ & $\Delta y \to 0$

$$f'(z) = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = \lim_{\Delta y \to 0} (-1)$$

$$\cdot \; f'(z) = -1$$

Derivative of $f(z) = \bar{z}$ does not exists

check the continuity of $f(z)$ at $z=0$

if $f(z) = \dfrac{Im(z^2)}{|z|^2}$ $z \neq 0$

Solution: $f(z)$ is continuous at $z=0$ if

(i) $\lim\limits_{z \to 0} f(z)$ exists (ii) $f(0)$ is defined

(iii) $\lim\limits_{z \to 0} f(z) = f(0)$

Now $\lim\limits_{z \to 0} \dfrac{Im(z^2)}{|z|^2}$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{+2xy}{x^2+y^2}$

if $y = mx$ Then we have

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{2x(mx)}{x^2+(mx)^2}$$

$$= \lim_{x\to 0} \frac{2mx^2}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{2m}{1+m^2} = \frac{2m}{1+m^2}$$

Since limit depends on m so limit does not exists and the function is discontinuous at $z=0$

Cauchy Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

In Polar form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Q#1 if $u = -2xy$ then find v

Solution $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$$\frac{\partial v}{\partial y} = -2y$$

$$v = \int -2y\,\partial y + h(x)$$

$$v = -y^2 + h(x)$$

$$\frac{\partial v}{\partial x} = h'(x) \Rightarrow \quad h'(x) = -2x$$

$$h(x) = -x^2 + c$$

Thus $v = -y^2 - x^2 + c$

Hence $f(z) = u + iv = -2xy + i(c - x^2 - y^2)$

Q#02 Find "a" & "b" so that given function is hermonic and find hermonic conjug.

sol. $u = ax^3 + bxy$

Sol: $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \Rightarrow \frac{\partial v}{\partial y} = 3ax^2 + by$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$6ax = 0 \Rightarrow a = 0 \text{ \& } b = c$$

where c is any constant

Thus $u = cxy$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \Rightarrow \frac{\partial v}{\partial y} = cy$$

$$v = cy^2/2 + h(x)$$

No $\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y} \Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$$h'(x) = -cx \Rightarrow h(x) = -\frac{cx^2}{2} + d$$

Thus $v = -cx^2/2 + cy^2/2 + d$

$$v = \frac{c}{2}(y^2 - x^2) + d$$