Exponential function

Exercise # 13.5

Q# 05 If $z = 1-3\pi i$ Then find e^z in the form of $u+iv$ and $|e^z|$

Solution: Given that $z = 1-3\pi i$ then

$$e^z = e^{1-3\pi i} = e^1 . e^{-3\pi i}$$

$$= e^1[\cos(-3\pi) + i\sin(-3\pi)]$$

$$= e^1[\cos 3\pi - i\sin 3\pi]$$

$$= e^1[-1-0i] = -e^1$$

$$\Rightarrow e^z = -e$$

$$|e^z| = \sqrt{(-e)^2} = e$$

Q# 15 Find Re and Im of $\exp(-z^2)$

Solution: $\exp(-z^2) = e^{-z^2} = e^{-[x+iy]^2}$

$$e^{-z^2} = e^{-[x^2-y^2+2ixy]}$$

$$= e^{y^2-x^2} \cdot e^{-2xyi}$$

$$e^{-z^2} = e^{y^2-x^2}\left[\cos(-2xy) + i\sin(-2xy)\right]$$

$$= e^{y^2-x^2}\left(\cos 2xy - i\sin 2xy\right)$$

$$= e^{y^2-x^2}\cos 2xy - i\, e^{y^2-x^2}\sin 2xy$$

$$\Rightarrow \operatorname{Re}\left[e^{-z^2}\right] = e^{y^2-x^2}\cos 2xy$$

$$\operatorname{Im}\left[e^{-z^2}\right] = -e^{y^2-x^2}\sin 2xy$$

Q#19 Solve $e^z = 1$

Solution $e^z = 1 \Rightarrow e^{x+iy} = 1+0i$

$$\Rightarrow e^x\left[\cos y + i\sin y\right] = 1+0i$$

$$\Rightarrow e^x\cos y = 1 \;\&\; e^x\sin y = 0$$

$$\Rightarrow e^x \neq 0 \Rightarrow \sin y = 0 \Rightarrow y = n\pi \quad n\in\mathbb{Z}$$

$$\Rightarrow e^x\cos n\pi = 1 \quad n \text{ will be even}$$

$$\Rightarrow e^x = 1 \Rightarrow x = 0$$

$$z = 0 + i\,2k\pi \quad k\in\mathbb{Z}$$

Q#9 Express $\cosh(-2+i)$ in $u+iv$ form

Sol: Given that $\cosh(-2+i)$

As we now $\cosh z = \cosh x \cos y + i \sinh x \sin y$

$\Rightarrow \cosh z = \cosh(-2)\cos 1 + i \sinh(-2)\sin(1)$

$= \cosh 2 \cos(1) - i \sinh 2 \sin(1)$

$\Rightarrow u = \cosh 2 \cos 1 \quad v = -\sinh 2 \sin(1)$

Q# 17 Solve $\cosh 2z = 0$

Sol: $\dfrac{e^{2z} + e^{-2z}}{2} = 0$ As $\cosh 2z = \dfrac{e^{2z} + e^{-2z}}{2}$

$\Rightarrow e^{2z} + e^{-2z} = 0$

$e^{4z} + 1 = 0$

$\Rightarrow e^{4z} = -1$

$4z = \ln(-1)$

$4z = \ln(1) + i(\pi + 2k\pi)$

$z = \frac{i}{4}(\pi + 2k\pi) \quad k \in \mathbb{Z}$ $\qquad \frac{4}{=}$

$z = i(1+2k)\pi/4 \quad k \in \mathbb{Z}$

Alternative method

$\cosh 2z = 0$

$\Rightarrow \cosh z = \cosh(2x + i2y) = 0$

$\cosh 2x \cos 2y + i \sinh 2x \sin 2y = 0$

$\Rightarrow \cosh 2x \cos 2y = 0 \quad \Rightarrow \quad \cos 2y = 0$

$\sinh 2x \sin 2y = 0 \qquad 2y = (2k+1)\pi/2$

$y = (2k+1)\pi/4 \qquad \sinh 2x = 0 \Rightarrow x = 0$

$z = x + iy = 0 + i(2k+1)\pi/4$

$z = i(2k+1)\pi/4$

Logarithm (EXERCISE # 13.7)

$\ln z = \ln r + i\theta$ & $\operatorname{Ln} z = \ln r + i\theta$

Q# 8 if $z = 8 + 8i$ then $\operatorname{Ln} z = ?$

Sol: Given that $z = 8 + 8i$

then $r = \sqrt{(8)^2 + (8)^2} = \sqrt{64 + 64} = \sqrt{128}$
$= 8\sqrt{2}$

$\theta = \tan^{-1} \frac{8}{8} = \tan^{-1} 1 = \pi/4$

$\operatorname{Ln}(8 + 8i) = \ln \sqrt{128} + i\,\pi/4$

$\operatorname{Ln}(8 + 8i) = \frac{1}{2} \ln(128) + i\,\pi/4$

Q# 15 Find $\ln(e^i)$

Sol $z = e^i = \cos(1) + i\sin(1)$

$r = \sqrt{\cos^2(1) + \sin^2(1)} = \sqrt{1} = 1$

$\theta = \tan^{-1}\left[\frac{\sin 1}{\cos(1)}\right] = \tan^{-1}\tan(1) = 1 + 2k\pi$

$\ln(e^i) = \ln(1) + i(1 + 2k\pi)$

$\ln e^{i} = 0 + i(1 + 2k\pi)$

Q 19 Solve for z if $\ln z = 4 - 3i$

Sol given that $\ln z = 4 - 3i$

$\Rightarrow \ln r + i\theta = 4 - 3i$

$\Rightarrow \theta = -3$ & $\ln r = 4$

$\Rightarrow r = e^{4}$ & $\tan^{-1} y/x = -3$

$x^2 + y^2 = e^{8}$ & $y/x = -\tan 3$

$y = -x \tan 3$

$x^2 + x^2 \tan^2 3 = e^{8}$

$x^2 [1 + \tan^2 3] = e^{8}$

$x^2 = \dfrac{e^{8}}{\sec^2 3} = e^{8} \cos^2 3$

$x = \pm e^{4} \cos 3$

$y = \mp e^{4} \sin 3$

$z = \pm e^{4} (\cos 3 - i \sin 3)$

Alternative method

Q.19 $\ln z = 4-3i \Rightarrow z = e^{4-3i}$

$$z = e^{4}[\cos(-3) + i\sin(-3)]$$

$$z = e^{4}(\cos 3 - i\sin 3)$$

Q# 23 Find $\operatorname{Ln}(1+i)^{1-i}$

Sol $\operatorname{Ln}(1+i)^{1-i} = (1-i)\operatorname{Ln}(1+i)$

$$= (1-i)(\ln\sqrt{2} + i\,\pi/4)$$

$$= (\ln\sqrt{2} + \pi/4) + i(\pi/4 - \ln\sqrt{2})$$

$$= (\tfrac{1}{2}\ln 2 + \pi/4) + i(\pi/4 - \tfrac{1}{2}\ln 2)$$

$$(1+i)^{1-i} = e^{(\frac{1}{2}\ln 2 + \pi/4) + i(\pi/4 - \frac{1}{2}\ln 2)}$$

Complex Integration

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,\dot{z}(t)\,dt \quad a \le t \le b$$

Q#21 Integrate $\int_C \operatorname{Re} z\,dz$

C : the shortest Path from $1+i$ to $5+5i$

Solution $(1,1),(5,5)$

$$\frac{x-1}{5-1} = \frac{y-1}{5-1} \Rightarrow \frac{x-1}{4} = \frac{y-1}{4} \Rightarrow x = y$$

$$x = y = t \Rightarrow x = t \quad y = t$$

$$z = x+iy \Rightarrow z = t+it$$

$$\operatorname{Re}(z) = \operatorname{Re}(t+it) = t$$

$$\dot{z}(t) = 1+i$$

when $z = 1+i \Rightarrow t+it = 1+i \quad t=1$

when $z = 5+5i \Rightarrow t+it = 5+5i \quad t=5$

$$\int_C \operatorname{Re}(z)\,dz = \int_1^5 t(1+i)\,dt = (1+i)\cdot\frac{t^2}{2}\Big|_1^5 = \frac{(1+i)}{2}[24]$$

$$= 12+12i$$