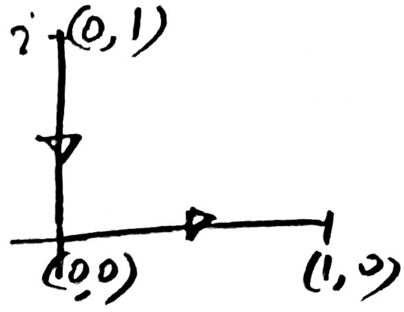
Question : $\int_C z e^{-z^2/2} dz$: c from i along the axes to 1 . .

Solution : Ist Method



$$\int_C z e^{-z^2/2} dz = \int_i^0 z e^{-z^2/2} dz + \int_0^1 z e^{-z^2/2} dz$$

$$= -\int_i^0 e^{-z^2/2} (-z\,dz) - \int_0^1 e^{-z^2/2} (-z) dz$$

$$= - e^{-z^2/2}\Big|_i^0 - e^{-z^2/2}\Big|_0^1$$

$$= -[e^0 - e^{1/2}] - [e^{-1/2} - 0]$$

$$= \cancel{-1} + e^{1/2} - e^{-1/2} \cancel{+1}$$

$$= 2\left(\frac{e^{1/2} - e^{-1/2}}{2}\right) = 2\sinh 1/2$$

2nd Method

$$\int_C z e^{-z^2/2} dz = \int_{C_1} z e^{-z^2/2} dz + \int_{C_2} z e^{-z^2/2} dz$$

C_1: i to 0 & C_2: 0 to 1

C_1: $(0,1)$ to $(0,0)$

$$\frac{x-0}{0-0} = \frac{y-1}{0-1} = t \Rightarrow x=0 \quad y=1-t$$

$z = x+iy$ when $y=1 \Rightarrow t=0$
when $y=0 \Rightarrow t=1$

$z = 0+i(1-t) \qquad dz = -i\,dt$

$$\int_{C_1} f(z)\,dz = \int_0^1 -i\, e^{-\frac{1}{2}(i(1-t))^2}\, i(1-t)\,dt$$

$$= \int_0^1 e^{\frac{1}{2}(1-t)^2}(1-t)\,dt$$

$$= -\int_0^1 e^{\frac{1}{2}(1-t)^2} - (1-t)\,dt$$

$$= -e^{\frac{1}{2}(1-t)^2}\Big|_0^1$$

$$= -(1-e^{1/2}) = -1+e^{1/2}$$

C_2 : $(0\ 0)$ to $(1,0)$

$$\frac{x-0}{1-0} = \frac{y-0}{0-0} = t \Rightarrow x=t \quad y=0$$

$z = t + 0i \qquad dz = dt$

when $x = 0 \Rightarrow t = 0$

when $x = 1 \Rightarrow t = 1$ so

$$\int_{C_2} f(z)\,dz = \int_0^1 e^{-\frac{1}{2}t^2}\, t\,dt = -\int_0^1 e^{-\frac{1}{2}t^2}(-t)\,dt$$

$$= -e^{-\frac{1}{2}t^2}\Big|_0^1 = -\left(e^{-\frac{1}{2}} - 1\right)$$

$$\int_{C_2} f(z)\,dz = -e^{-\frac{1}{2}} + 1$$

~~As~~ $\displaystyle\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$

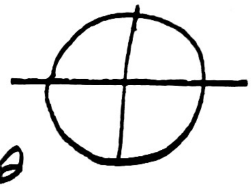
$$\int_C e^{-\frac{1}{2}z^2} z\,dz = \cancel{-1} + e^{\frac{1}{2}} - e^{-\frac{1}{2}} \cancel{+1}$$

$$= 2\left[\frac{e^{\frac{1}{2}} - e^{-\frac{1}{2}}}{2}\right] = 2\sinh\tfrac{1}{2}$$

Cauchy's Integral Theorem

If $f(z)$ is analytic in a simply connected domain D, Then for every simple closed Path C in D, $\oint_C f(z)\,dz = 0$

Question: Integrate $f(z)$ Counterclockwise around the unit circle indicating whether Cauchy's Integral Theorem applies if $f(z) = \operatorname{Re} z$



In unit Circle $x = \cos\theta$ $y = \sin\theta$

$$z = x + iy \Rightarrow z = \cos\theta + i\sin\theta \quad 0 \le \theta \le 2\pi$$

$$dz = [-\sin\theta + i\cos\theta]\,d\theta \qquad \operatorname{Re} z = \cos\theta$$

$$\oint_C f(z)\,dz = \int_0^{2\pi} \cos\theta \cdot [-\sin\theta + i\cos\theta]\,d\theta$$

$$= \int_0^{2\pi} (-\cos\theta\sin\theta + i\cos^2\theta)\,d\theta$$

$$= \int_0^{2\pi} \cos\theta(-\sin\theta)\,d\theta + i\int_0^{2\pi} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$\oint_C f(z)\,dz = \frac{\cos^2\theta}{2}\Big|_0^{2\pi} + i\left[\frac{1}{2}\theta + \frac{1}{2}\frac{\sin 2\theta}{2}\right]\Big|_0^{2\pi}$$

$$= \frac{1}{2}(\cancel{1}-\cancel{1}) + i\left[\pi - 0 + 0\right]$$

$$= i\pi$$

Thus $\oint_C f(z)\,dz = i\pi$

Here Cauchy's integral theorem is not applicable because the given function is not analytic.

Question: $\oint_C e^{z^2/2}\,dz$ over the unit circle

Solution: Since $e^{z^2/2}$ is analytic in entire domain i.e., in unit circle ~~so~~ according to the Cauchy's integral theorem we can write $\oint_C f(z)\,dz = \oint_C e^{+z^2/2}\,dz = 0$

$$\oint (z-z_0)^m dz = \begin{cases} 2\pi i & m=-1 \\ 0 & m \neq -1 \end{cases}$$

Question : $\oint_C \frac{dz}{2z-i}$ $\quad C : |z| = 3$ (counter clockwise)

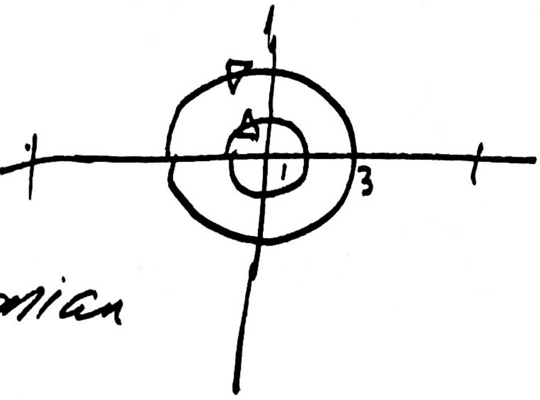
Solution : $\oint_C \frac{dz}{2z-i} = \frac{1}{2}\oint_C \frac{dz}{z-\frac{1}{2}i}$

Here $z_0 = \frac{1}{2}i$ and $m=-1$ so

$$\oint_C \frac{dz}{2z-i} = \frac{1}{2}(2\pi i) = \pi i$$

Question $\oint_C \frac{\cos z}{z} dz$, C consists of $|z| = 1$ (counter)

and $|z| = 3$ (Clockwise)



Solution:- Since $f(z)$ is analytic in the given domian

So $\oint_C f(z) dz = 0$

Cauchy's Integral formula

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

$$\oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{n}(z_0)$$

Question $\oint_C \frac{\cos z}{z}\,dz$ over the unit circle

Solution : Here ~~f~~ the given function is not analytic at $z=0$ which is inside unit circle (Domain) & $z_0 = 0$ $f(z) = \cos z$

Then according to Cauchy's Integral formula we have

$$\oint_C \frac{\cos z}{z}\,dz = 2\pi i \cos(0) = 2\pi i$$

Question: $\oint_c \frac{dz}{2z-i} dz \quad |z|=3$

Solution: $\oint_c \frac{dz}{2z-i} = \oint_c \frac{\frac{1}{2}dz}{z-\frac{1}{2}i} \quad |z|=3$

Here the given function is not analytic at $z_0 = \frac{1}{2}i$ so according to Cauchy's Integral formula $f(z) = \frac{1}{2} \quad z_0 = \frac{1}{2}i$

$\Rightarrow \quad f(z_0) = \frac{1}{2}$

Thus $\oint_c f(z)dz = 2\pi i(\frac{1}{2}) = \pi i$

Q Integrate $\oint_c \frac{\sin z}{(z-i\frac{\pi}{2})^4} dz \quad |z|=2$

Solution: Here $F(z) = \frac{\sin z}{(z-i\frac{\pi}{2})^2}$ is not analytic in $|z|=2$ so $z_0 = i\frac{\pi}{2} \quad n=3$

$f(z) = \sin z \Rightarrow f'(z) = \cos z, \; f''(z) = -\sin z$

$f'''(z_0) = -\cos i\frac{\pi}{2} = -\cosh\frac{\pi}{2}$ =

Thus $\oint_c F(z)dz = \oint_c \frac{\sin z}{(z-i\pi/2)^4} dz = \frac{2\pi i}{3!} f'''(z_0)$

$$\Rightarrow \oint_c \frac{\sin z}{(z-i\pi/2)^4} dz = \frac{2\pi i}{3!}(-\cosh \pi/2)$$

$$= -\frac{2\pi i}{3!} \cosh \pi/2$$

Question $\frac{\tan \pi z}{z^2}$ c: the ellipse $16x^2+y^2=1$ counterclockwise

Solution: Here $f(z) = \frac{\tan \pi z}{z^2}$ is not analytic in the ellipse $16x^2+y^2=1$ so $z_0=0$ with $n=1$ and $f(z) = \tan \pi z \Rightarrow f'(z) = \pi \sec^2 \pi z$

$f'(0) = \pi \sec^2(0) = \pi$

Thus $\oint_c F(z)dz = \oint_c \frac{\tan \pi z}{z^2} dz = \frac{2\pi i}{1!} f'(z_0)$

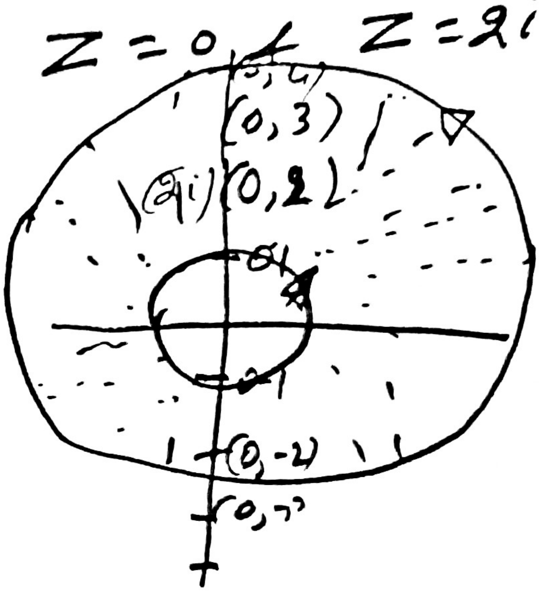
$$= 2\pi i(\pi)$$

$$= i2\pi^2$$

Question $\dfrac{e^{2z}}{z(z-2i)^2}$ c: Consists of $|z-i|=3$ (counterclockwise) and $|z|=1$ (clockwise)

Solution: $f(z)=\dfrac{e^{2z}}{z(z-2i)^2}$

Singularities of $f(z)$ are $z=0$ & $z=2i$



$z_0=2i$ lies in the domain with $n=1$

$f(z)=e^{2z}/z$

Then $f'(z)=\dfrac{2z\,e^{2z}-e^{2z}}{z^2}$

$f'(2i)=\dfrac{4i\,e^{4i}-e^{4i}}{-4}$

Thus $\oint_c f(z)\,dz=\dfrac{2\pi i}{1!}f'(z_0)$

$$\oint_c \frac{e^{2z}}{z(z-2i)^2}\,dz=2\pi i\cdot\left|\frac{e^{4i}-4i\,e^{4i}}{4}\right|$$

$$\oint_C \frac{e^{2z}}{z(z-2i)^2}\,dz = i\frac{\pi}{2}e^{4i} + 2e^{4i}$$

$$= (4+i\pi)\,e^{4i}/2$$