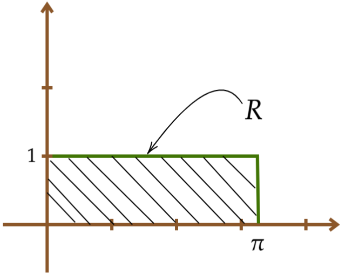
MAT104E PS

Question 1.

$\iint_R y\cos xy \, \mathrm{d}A = ?$ if $R = [0,\pi] \times [0,1]$.

Answer.



By using Fubini's Theorem,

$$\int_0^{\pi}\int_0^{1} y\cos(xy)\,\mathrm{d}y\mathrm{d}x = \int_0^{1}\int_0^{\pi} y\cos(xy)\,\mathrm{d}x\mathrm{d}y$$

$$= \int_0^{1}\Big[\sin(xy)\Big]_{x=0}^{x=\pi}\,\mathrm{d}y$$

$$= \int_0^{1}\sin(\pi y)\,\mathrm{d}y$$

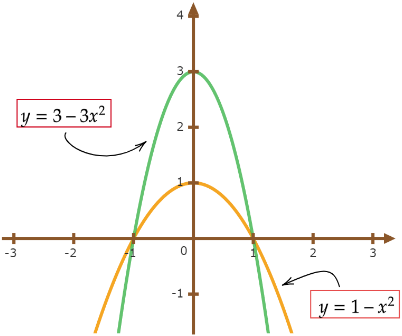
$$= -\frac{\cos(\pi y)}{\pi}\Bigg]_{y=0}^{y=1}$$

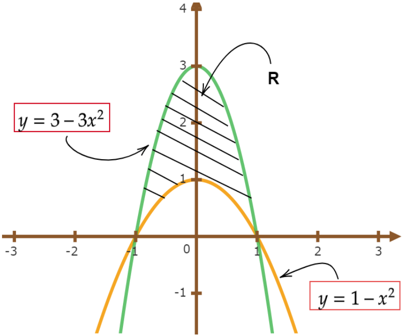
$$= 2/\pi$$

Question 2.

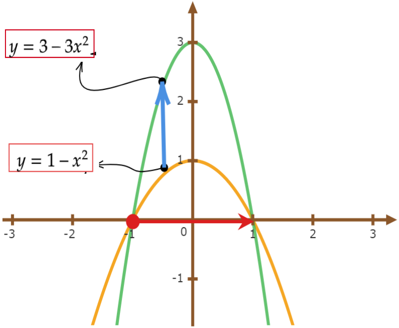
Evaluate $\iint_R (x+2y)\,\mathrm{d}A$ where R is the region enclosed by the curves $y = 1 - x^2,\ y = 3 - 3x^2$.

Answer.





Answer.



$$\iint_R (x+2y)\,\mathrm{d}A = \int_{-1}^{1}\int_{1-x^2}^{3-3x^2} (x+2y)\,\mathrm{d}y\mathrm{d}x$$

$$= \int_{-1}^{1} \left[xy + y^2\right]_{y=1-x^2}^{y=3-3x^2} \mathrm{d}x$$

$$= \int_{-1}^{1} \Big(x(3-3x^2) + (3-3x^2)^2 -$$

$$x(1-x^2) - (1-x^2)^2\Big)\,\mathrm{d}x$$

$$= \int_{-1}^{1} (8x^4 - 2x^3 - 16x^2 + 2x + 8)\,\mathrm{d}x$$

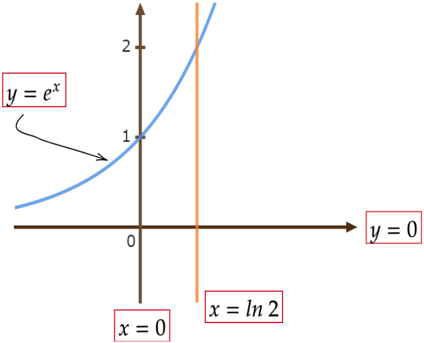
$$= \frac{8x^5}{5} - \frac{2x^4}{4} - \frac{16x^3}{3} + x^2 + 8x\Big]_{x=-1}^{x=1}$$

$$=128/5$$

Question 3.

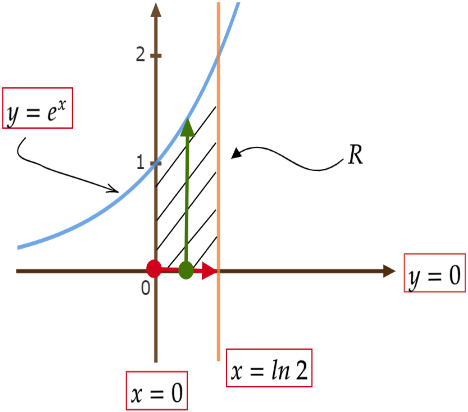
Calculate the area between $y = e^x,\ y = 0,\ x = 0,\ x = \ln 2$ using a double integral.

Answer.



$$\text{Area} = \iint_R \mathrm{d}A$$

Answer.



$$\text{Area} = \iint_R \mathrm{d}A$$

$$= \int_0^{\ln 2} \int_0^{e^x} \mathrm{d}y\mathrm{d}x$$

$$= \int_0^{\ln 2} e^x \,\mathrm{d}x = e^x\Big]_{x=0}^{x=\ln 2}$$

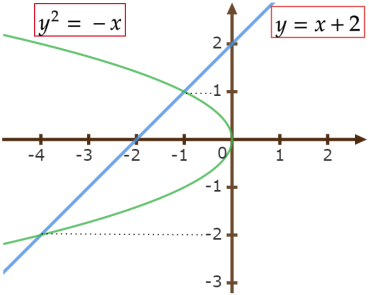
$$= e^{\ln 2} - e^0 = 1$$

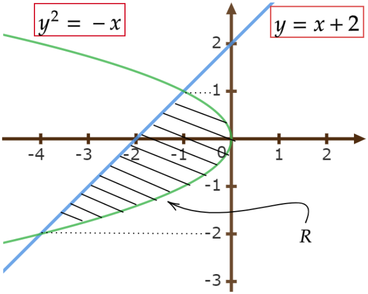
Question 4.

Find the area between $y^2 = -x$ and $y = x + 2$ with a double integral.

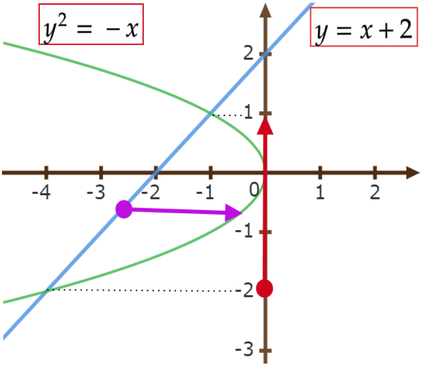
Answer.

$$\left.\begin{aligned} y &= x + 2 \\ y^2 &= -x \end{aligned}\right] \Longrightarrow y^2 = 2 - y \Rightarrow y^2 + y - 2 = 0 \Rightarrow y = -2 \text{ and } y = 1$$





Answer.



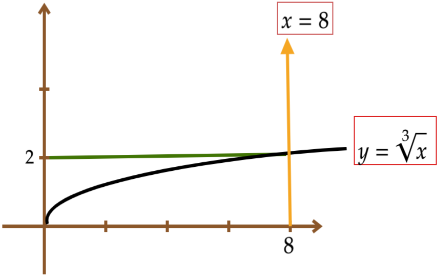
$$
\begin{aligned}
\text{Area} &= \iint_R \mathrm{d}A \\
&= \int_{-2}^{1} \int_{y-2}^{-y^2} \mathrm{d}x\mathrm{d}y \\
&= \int_{-2}^{1} (-y^2 - y + 2)\ \mathrm{d}y \\
&= -\frac{y^3}{3} - \frac{y^2}{2} + 2y \Big]_{y=-2}^{y=1} \\
&= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) \\
&= \frac{9}{2}
\end{aligned}
$$

Question 5.

Evaluate $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{1+y^4}\, dy dx$

Answer.

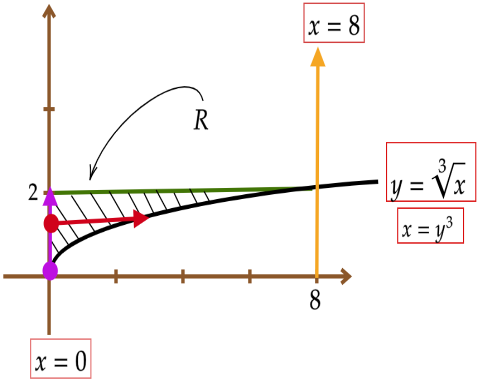
R : the region bounded by the curves $y = \sqrt[3]{x}$ and $y = 2$ between $x = 0$ and $x = 8$.



Answer.

By using Fubini's theorem,

$$\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{1+y^4}\, dydx = \int_0^2 \int_0^{y^3} \frac{1}{1+y^4}\, dxdy$$



$$= \int_0^2 \left[\frac{x}{1+y^4}\right]_{x=0}^{x=y^3} dy$$

$$= \int_0^2 \frac{y^3}{1+y^4}\, dy$$

$$= \frac{1}{4} \ln|1+y^4| \Big]_{y=0}^{y=2}$$

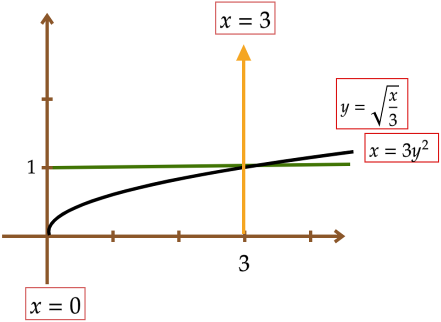
$$= \frac{ln 17}{4}$$

Question 6.

Evaluate $\int_{0}^{3}\int_{\sqrt{\frac{x}{3}}}^{1} e^{y^3}\, dydx$

Answer.

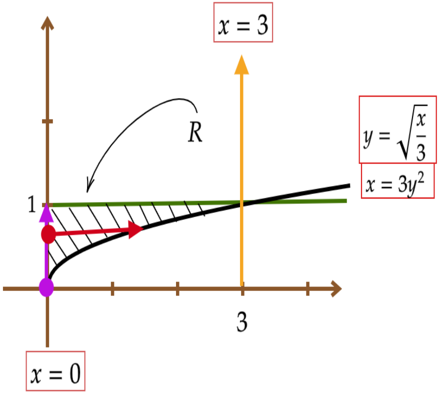
R : the region bounded by the curves $y = \sqrt{\frac{x}{3}}$ and $y = 1$ between $x = 0$ and $x = 3$.



Answer.

By using Fubini's theorem,

$$\int_0^3 \int_{\sqrt{\frac{x}{3}}}^1 e^{y^3}\, dydx = \int_0^1 \int_0^{3y^2} e^{y^3}\, dxdy$$



$$= \int_0^1 \left[xe^{y^3} \right]_{x=0}^{x=3y^2} dy$$

$$= \int_0^1 3y^2 e^{y^3}\, dy$$

$$= e^{y^3} \Big]_{y=0}^{y=1}$$

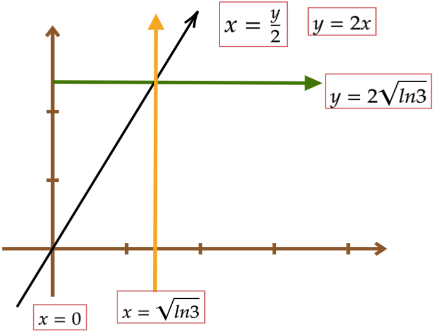
$$= e - 1$$

Question 7.

Evaluate $\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2} \, dxdy$

Answer.

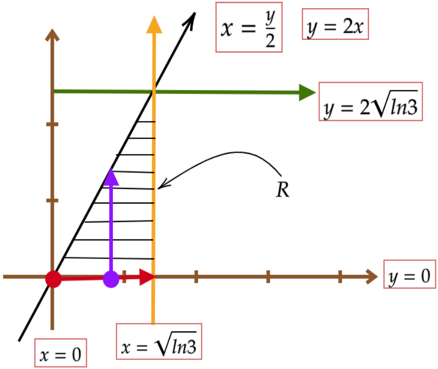
R : the region bounded by the curves $x = \dfrac{y}{2}$ and $x = \sqrt{ln3}$ between $y = 0$ and $y = 2\sqrt{\ln 3}$.



Answer.

By using Fubini's theorem,

$$\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{ln3}} e^{x^2}\, dxdy = \int_0^{\sqrt{ln3}} \int_0^{2x} e^{x^2}\, dydx$$



$$= \int_0^{\sqrt{ln3}} \left[ye^{x^2}\right]_{y=0}^{y=2x} \mathrm{d}x$$

$$= \int_0^{\sqrt{ln3}} 2xe^{x^2}\, \mathrm{d}x$$

$$= e^{x^2}\Big]_{x=0}^{x=\sqrt{ln3}}$$

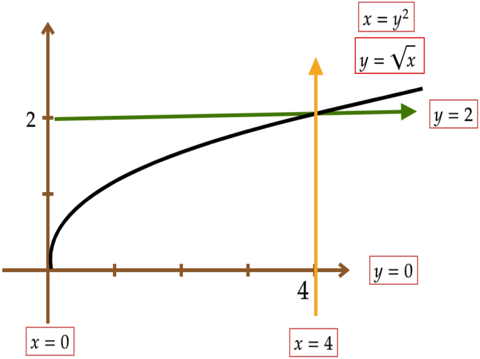
$$=3 - 1 = 2$$

Question 8.

Evaluate $\int_0^2 \int_{y^2}^4 \frac{3}{2} e^{(y/\sqrt{x})} \, dx dy$

Answer.

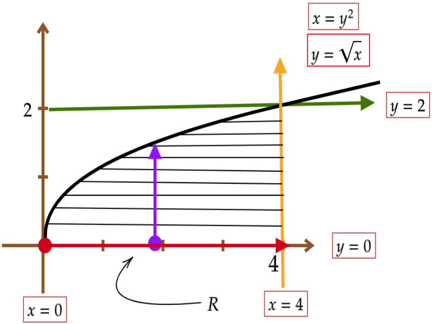
R : the region bounded by the curves $x = y^2$ and $x = 4$ between $y = 0$ and $y = 2$.



Answer.

By using Fubini's theorem,

$$\int_0^2 \int_{y^2}^4 \frac{3}{2} e^{y/\sqrt{x}}\, dx dy = \int_0^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\, dy dx$$



$$= \int_0^4 \left[\frac{3}{2} \sqrt{x} e^{y/\sqrt{x}} \right]_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^4 \left(\frac{3}{2} e \sqrt{x} - \frac{3}{2} \sqrt{x} \right) dx$$

$$= x^{3/2}(e-1) \Big]_{x=0}^{x=4}$$

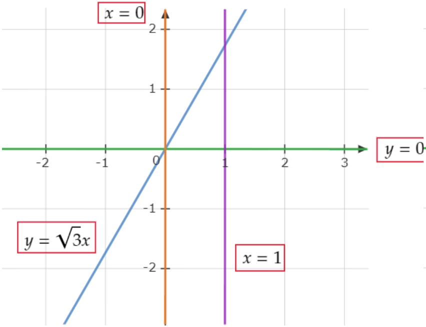
$$= 8(e-1)$$

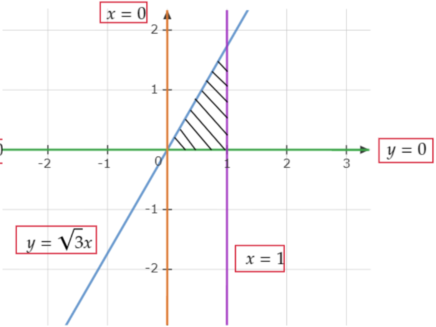
Question 9.

Evaluate the integral $\int_0^1 \int_0^{\sqrt{3}x} \frac{x}{y^2+x^2}\, dydx$ by transforming it to polar coordinates.

Answer.

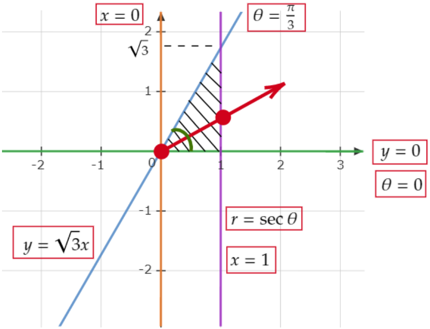
The region is bounded by the lines $y = 0$ and $y = \sqrt{3}x$ between $x = 0$ and $x = 1$.





Answer.

Let $x = r\cos\theta$, $y = r\sin\theta$. Replace $dydx$ by $rdrd\theta$.



$$y = \sqrt{3}x \Longrightarrow \tan\theta = \sqrt{3} \Longrightarrow \theta = \frac{\pi}{3}$$

$$x = 1 \Longrightarrow r\cos\theta = 1 \Longrightarrow r = \sec\theta$$

$$\int_0^1 \int_0^{\sqrt{3}x} \frac{x}{y^2 + x^2}\, dydx = \int_0^{\frac{\pi}{3}} \int_0^{\sec\theta} \frac{r\cos\theta}{r^2} rdrd\theta = \int_0^{\frac{\pi}{3}} \int_0^{\sec\theta} \cos\theta drd\theta$$

$$= \int_0^{\frac{\pi}{3}} \left(r\Big|_0^{\sec\theta} \right) \cos\theta d\theta = \int_0^{\frac{\pi}{3}} \sec\theta\cos\theta d\theta = \int_0^{\frac{\pi}{3}} d\theta$$

$$= \theta\Big|_0^{\frac{\pi}{3}} = \frac{\pi}{3}$$

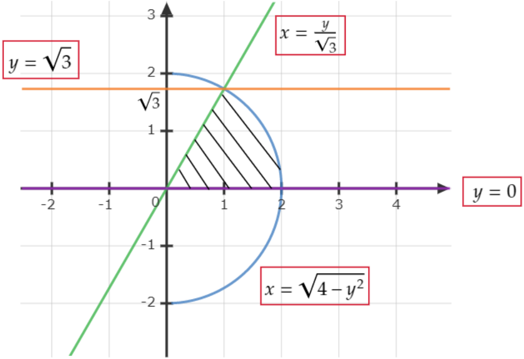
Question 10.

$$\int_0^{\sqrt{3}} \int_{\frac{y}{\sqrt{3}}}^{\sqrt{4-y^2}} \sqrt{(4-x^2-y^2)^3}\, dxdy$$

Evaluate the given integral by transforming it to polar coordinates.

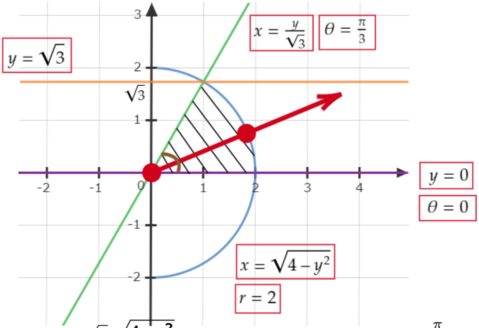
Answer.

The region is bounded by the line $x = \dfrac{y}{\sqrt{3}}$ and the curve $x = \sqrt{4-y^2}$ between $y = 0$ and $y = \sqrt{3}$.



Answer.

Let $x = r\cos\theta$, $y = r\sin\theta$. Replace $dxdy$ by $rdrd\theta$.



$$x = \sqrt{4-y^2} \implies x^2+y^2=4$$
$$\implies r=2$$
$$x=\frac{y}{\sqrt{3}} \implies \tan\theta=\sqrt{3}$$
$$\implies \theta=\frac{\pi}{3}$$

$$I=\int_0^{\sqrt{3}}\int_{\frac{y}{\sqrt{3}}}^{\sqrt{4-y^2}}\sqrt{(4-x^2-y^2)^3}\,dxdy=\int_0^{\frac{\pi}{3}}\int_0^2\sqrt{(4-r^2)^3}\,rdrd\theta$$

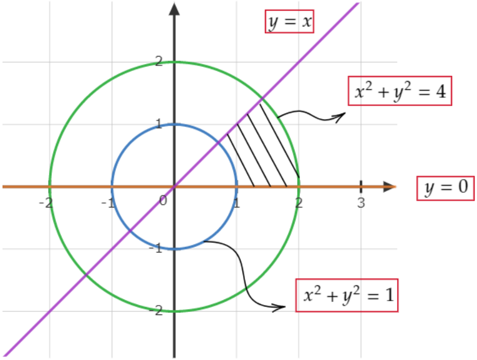
Let $u=4-r^2$. Then $du=-2rdr$. When $r=0, u=4$. When $r=2, u=0$.

$$I=\int_0^{\frac{\pi}{3}}\int_4^0 -\frac{u^{\frac{3}{2}}}{2}\,dud\theta=\int_0^{\frac{\pi}{3}}\left(-\frac{1}{5}u^{\frac{5}{2}}\Big|_4^0\right)d\theta=\int_0^{\frac{\pi}{3}}\frac{32}{5}d\theta=\frac{32}{5}\theta\Big|_0^{\frac{\pi}{3}}=\frac{32\pi}{15}$$

Question 11.

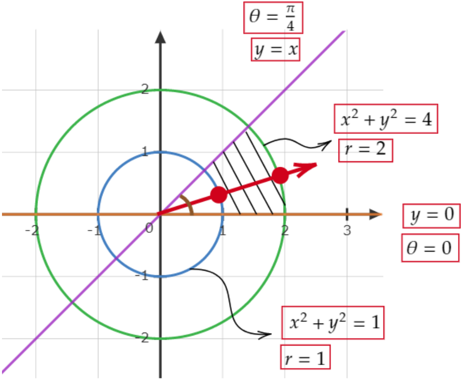
Evaluate $\iint\limits_R \tan^{-1}\left(\frac{y}{x}\right) dA$ where R is the region defined by $1 \leq x^2 + y^2 \leq 4$, $0 \leq y \leq x$.

Answer.



Answer.

Let $x = r\cos\theta$, $y = r\sin\theta$. Replace dA by $rdrd\theta$.



$$x^2 + y^2 = 1 \implies r = 1$$
$$x^2 + y^2 = 4 \implies r = 2$$
$$y = x \implies \tan\theta = 1 \implies \theta = \frac{\pi}{4}$$

$$\iint_R \tan^{-1}\left(\frac{y}{x}\right) dA = \int_0^{\frac{\pi}{4}} \int_1^2 \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta}\right) rdrd\theta = \int_0^{\frac{\pi}{4}} \int_1^2 \tan^{-1}(\tan\theta)\, rdrd\theta$$
$$= \int_0^{\frac{\pi}{4}} \int_1^2 \theta rdrd\theta = \int_0^{\frac{\pi}{4}} \left(\frac{r^2}{2}\Big|_1^2\right) \theta d\theta = \int_0^{\frac{\pi}{4}} \frac{3}{2}\theta d\theta = \frac{3\theta^2}{4}\Big|_0^{\frac{\pi}{4}} = \frac{3\pi^2}{48}$$

Question 12.

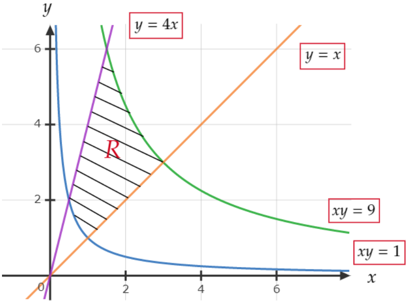
Use the transformation $x = \frac{u}{v}$, $y = uv$ with $u > 0$ and $v > 0$ to evaluate the integral $\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy$ where R is the region bounded by the curves $xy = 1$, $xy = 9$, and the lines $y = x$, $y = 4x$ in the first quadrant.

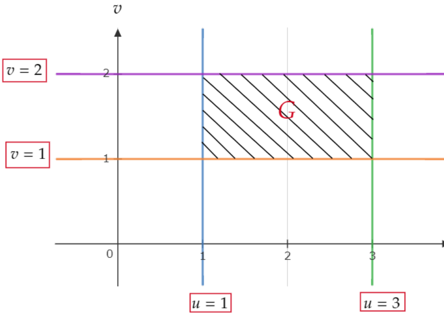
Answer.

Boundaries:

$$
\begin{aligned}
xy &= 1 && \Longrightarrow \frac{u}{v}uv = 1 \;\; (u, v > 0) && \Longrightarrow u = 1 \\
xy &= 9 && \Longrightarrow \frac{u}{v}uv = 9 \;\; (u, v > 0) && \Longrightarrow u = 3 \\
y &= x && \Longrightarrow uv = \frac{u}{v} \;\; (u, v > 0) && \Longrightarrow v = 1 \\
y &= 4x && \Longrightarrow uv = 4\frac{u}{v} \;\; (u, v > 0) && \Longrightarrow v = 2
\end{aligned}
$$

Answer.





The Jakobian of the transformation is $J(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \dfrac{1}{v} & \dfrac{-u}{v^2} \\ v & u \end{vmatrix} = \dfrac{2u}{v}$

$$\iint\limits_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy} \right) dxdy = \iint\limits_G \left(\sqrt{\frac{uv}{\frac{u}{v}}} + \sqrt{\frac{u}{v} uv} \right) |J(u,v)|\, dudv = \int\limits_1^2 \int\limits_1^3 (v+u) \frac{2u}{v} dudv$$

$$= \int\limits_1^2 \left(u^2 + \frac{2u^3}{3v} \Bigg|_1^3 \right) dv = \int\limits_1^2 \left(8 + \frac{52}{3v} \right) dv = 8v + \frac{52}{3} \ln|v| \Bigg|_1^2 = 8 + \frac{52}{3} \ln 2$$

Question 13.

Evaluate $\displaystyle\iint_R 2(y-x)dxdy$

by applying the transformation $u = y - x, \;\; v = y + x$ and integrating over an appropriate region in the uv-plane. R is the region in xy-plane bounded by the lines $y = -x, \;\; y = -x + 1, \;\; y = x, \;\; y = x + 1.$

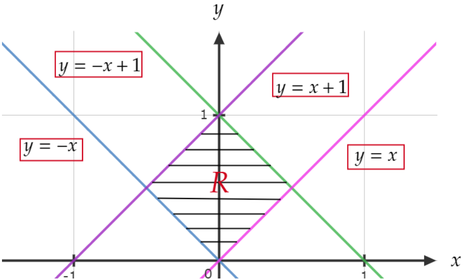
Answer.

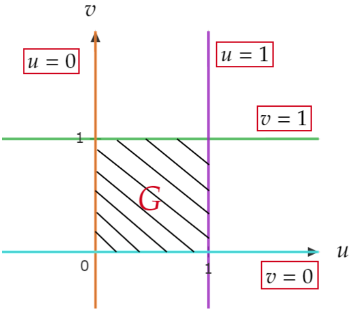
$$\left.\begin{array}{ll} u & = y - x \\ v & = y + x \end{array}\right] \implies x = \frac{v-u}{2} \text{ and } y = \frac{u+v}{2}$$

Boundaries:

$$\begin{array}{lll} y = -x & \implies \dfrac{u+v}{2} = -\dfrac{v-u}{2} & \implies v = 0 \\ y = -x + 1 & \implies \dfrac{u+v}{2} = -\dfrac{v-u}{2} + 1 & \implies v = 1 \\ y = x & \implies \dfrac{u+v}{2} = \dfrac{v-u}{2} & \implies u = 0 \\ y = x + 1 & \implies \dfrac{u+v}{2} = \dfrac{v-u}{2} + 1 & \implies u = 1 \end{array}$$

Answer.





The Jakobian of the transformation is $J(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} -\dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{1}{2} \end{vmatrix} = -\dfrac{1}{2}$

$$\iint\limits_R 2(y-x)dxdy = \iint\limits_G 2\left(\frac{u+v}{2} - \frac{v-u}{2}\right)|J(u,v)|\,dudv = \int\limits_0^1\int\limits_0^1 ududv$$

$$= \int\limits_0^1 \left(\frac{u^2}{2}\Big|_0^1\right) dv = \int\limits_0^1 \frac{1}{2}dv = \frac{v}{2}\Big|_0^1 = \frac{1}{2}$$