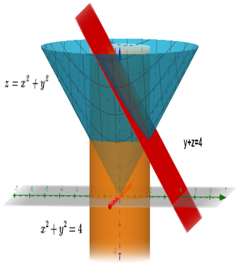
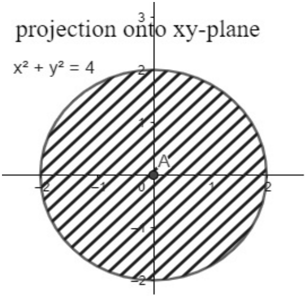
MAT104E PS

Question 1.

Find the volume of the space region enclosed below by the cone $z = \sqrt{x^2 + y^2}$ on the side by the cylinder $x^2 + y^2 = 4$ and on the top by the plane y+z=4

Answer.





Use the cylindirical coordinates:

$$\left.\begin{array}{ll} x & = r\cos\theta \\ y & = r\sin\theta \\ z & = z \end{array}\right] \implies z = \sqrt{x^2 + y^2} \Rightarrow z = r \text{ and } y + z = 4 \Rightarrow z = 4 - r\sin\theta$$

Answer.

$$\text{Volume}=\int_0^{2\pi}\int_0^{2}\int_r^{4-r\sin\theta} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} z\Big|_r^{4-r\sin\theta} r\,dr\,d\theta =$$

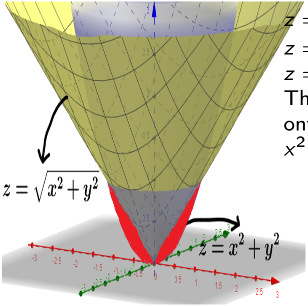
$$\int_0^{2\pi}\int_0^{2}(4-r\sin\theta-r)r\,dr\,d\theta = \int_0^{2\pi}\left[2r^2-\frac{r^3}{3}\sin\theta-\frac{r^3}{3}\right]_0^2 d\theta =$$

$$\int_0^{2\pi}(8-\frac{8}{3}\sin\theta-\frac{8}{3})d\theta = \left[\frac{16}{3}\theta+\frac{8}{3}\cos\theta\right]_0^{2\pi} = \frac{32\pi}{3}+\frac{8}{3}-\frac{8}{3}=\frac{32\pi}{3}$$

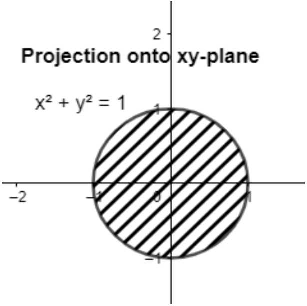
Question 2.

Let D be a space region bounded below by the surface $z = x^2 + y^2$ and above by the surface $z = \sqrt{x^2 + y^2}$ Calculate the volume of the solid.

Answer.



$z = x^2 + y^2$is a paraboloid
$z = \sqrt{x^2 + y^2}$ is half cone
$z = z^2 \Rightarrow z = 0$ or $z = 1$
The projection onto xy- plane is $x^2 + y^2 = 1$



Use the cylindirical coordinates:

$$\left.\begin{array}{ll} x & = r\cos\theta \\ y & = r\sin\theta \\ z & = z \end{array}\right] \Longrightarrow z = \sqrt{x^2 + y^2} \Rightarrow z = r \text{ and } z = x^2 + y^2 \Rightarrow z = r^2$$

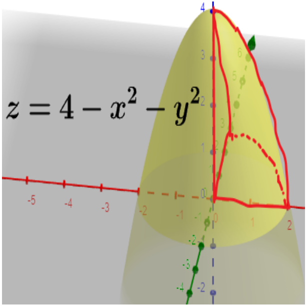
$$\text{Volume}=\int_0^{2\pi}\int_0^1\int_{r^2}^{r} z\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 rz\Big|_{r^2}^{r} dr\,d\theta = \int_0^{2\pi}\int_0^1 (r^2 - r^3)\,dr\,d\theta =$$

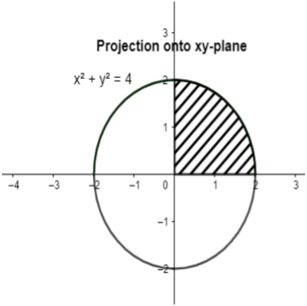
$$\int_0^{2\pi}\left[\frac{r3}{3} - \frac{r^4}{4}\right]_0^1 d\theta = \int_0^{2\pi}\frac{1}{12}\,d\theta = \frac{\theta}{12}\Big|_0^{2\pi} = \frac{\pi}{6}$$

Question 3.

Find the volume of the solid cut from the first octant by the surface $z = 4 - x^2 - y^2$.

Answer.





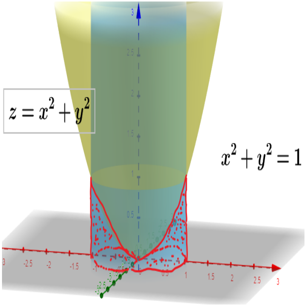
Use the cylindirical coordinates: $\left.\begin{array}{ll} x & = r\cos\theta \\ y & = r\sin\theta \\ z & = z \end{array}\right] \Longrightarrow z = 4 - x^2 - y^2 \Rightarrow z = 4 - r^2$

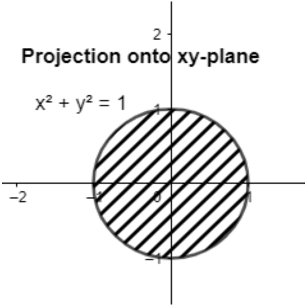
Volume$=\int_0^{\frac{\pi}{2}}\int_0^{2}\int_0^{4-r^2} r\,dz\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^{2} rz\Big|_0^{4-r^2} dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^{2}(4r-r^3)\,dr\,d\theta =$

$\int_0^{\frac{\pi}{2}}\left[2r^2-\frac{r^4}{4}\right]_0^2 d\theta = \int_0^{\frac{\pi}{2}} 4\,d\theta = 4\theta\Big|_0^{\frac{\pi}{2}} = 2\pi$

Question 4.

Find the volume of the space region bounded by the surfaces $z = x^2 + y^2$, $x^2 + y^2 = 1$ and the xy-plane.





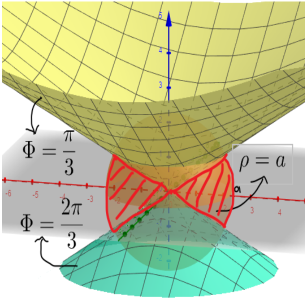
Use the cylindirical coordinates: $\left.\begin{array}{ll} x & = r\cos\theta \\ y & = r\sin\theta \\ z & = z \end{array}\right] \Longrightarrow z = x^2 + y^2 \Rightarrow z = r^2$

$$\text{Volume}=\int_0^{2\pi}\int_0^{1}\int_0^{r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{1} rz\Big|_0^{r^2} dr\,d\theta = \int_0^{2\pi}\int_0^{1} r^3\,dr\,d\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_0^{1} d\theta =$$

$$\int_0^{2\pi}\frac{1}{4}\,d\theta = \frac{\theta}{4}\Big|_0^{2\pi} = \frac{\pi}{2}$$

Question 5.

Find the volume of the portion of the solid $\rho \le a$ that lies between the cones $\phi = \dfrac{\pi}{3}$ and $\phi = \dfrac{2\pi}{3}$.



spherical

coordinates:

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ .y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \end{aligned}$$

$\Longrightarrow dzdydx \hookrightarrow \rho^2 \sin\phi\, d\rho d\phi d\theta$

$$\text{Volume}=\int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\int_0^{a} \rho^2 \sin\phi\, d\rho d\phi d\theta = \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin\phi \left.\frac{\rho^3}{3}\right|_0^a d\phi d\theta = \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{a^3}{3}\sin\phi\, d\phi d\theta =$$

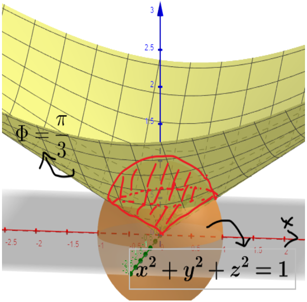
$$\int_0^{2\pi} -\frac{a^3}{3}\cos\phi\Big|_{\frac{\pi}{3}}^{\frac{2\pi}{3}} d\theta = \int_0^{2\pi}\frac{a^3}{3}d\theta = \frac{a^3}{3}\theta\Big|_0^{2\pi} = \frac{2\pi a^3}{3}$$

Question 6.

Set up triple integral for the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 1$ and below by the cone $\phi = \dfrac{\pi}{3}$

a) in rectangular coordinates
b) in spherical coordinates
Calculate the volume of the solid.

Answer.



$x^2 + y^2 + z^2 = 1 \Rightarrow \rho^2 = 1 \Rightarrow \rho = 1$

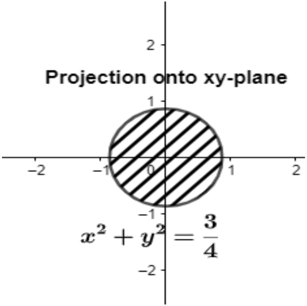
$\phi = \dfrac{\pi}{3} \Rightarrow \tan\phi = \sqrt{3}$

$\Rightarrow \dfrac{\rho\sin\phi}{\rho\cos\phi} = \sqrt{3} \Rightarrow z = \dfrac{\sqrt{x^2+y^2}}{\sqrt{3}}$

Since $x^2 + y^2 = 1 - z^2$ then

$z = \dfrac{\sqrt{1-z^2}}{\sqrt{3}} \Rightarrow z^2 = \dfrac{1}{4}$

So the intersection is $x^2 + y^2 = \dfrac{3}{4}$



Rectangular coordinates:

$$\text{Volume}=\int_{\frac{-\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}\int_{-\sqrt{\frac{3}{4}-x^2}}^{\sqrt{\frac{3}{4}-x^2}}\int_{\frac{\sqrt{x^2+y^2}}{\sqrt{3}}}^{\sqrt{1-x^2+y^2}} dzdydx$$

Spherical coordinates:

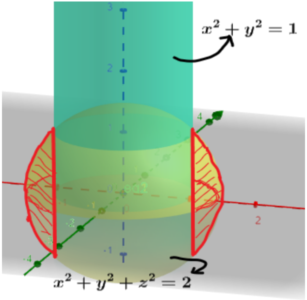
$$V=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^1 \rho^2\sin\phi d\rho d\phi d\theta=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\sin\phi\frac{\rho^3}{3}\Big|_0^1 d\phi d\theta=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\frac{1}{3}\sin\phi d\phi d\theta=$$

$$\int_0^{2\pi}-\frac{1}{3}\cos\phi\Big|_0^{\frac{\pi}{3}}d\theta=\int_0^{2\pi}\frac{1}{6}d\theta=\frac{\theta}{6}\Big|_0^{2\pi}=\frac{\pi}{3}$$

Question 7.

Find the volume of the space region inside the sphere $x^2 + y^2 + z^2 = 2$ outside the cylinder $x^2 + y^2 = 1$

Answer.



spherical coordinates:

$x \quad = \rho \sin\phi \cos\theta$

$.y \quad = \rho \sin\phi \sin\theta$

$z \quad = \rho \cos\phi$

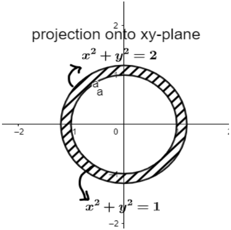
$x^2 + y^2 = 1 \Rightarrow \rho^2 \sin^2\phi = 1$

$\Rightarrow \rho \sin\phi = 1$ and

$x^2 + y^2 + z^2 = 2 \Rightarrow \rho = \sqrt{2}$ so the

intersection is $sin\phi = \dfrac{1}{\sqrt{2}} \Rightarrow \phi = \dfrac{\pi}{4}$ and

$\phi = \dfrac{3\pi}{4}$



$$V = \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{\frac{1}{\sin\phi}}^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$V=\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_{\frac{1}{\sin\phi}}^{\sqrt{2}} \rho^2 \sin\phi \, d\rho d\phi d\theta = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin\phi \frac{\rho^3}{3}\Bigg|_{\frac{1}{\sin\phi}}^{\sqrt{2}} d\phi d\theta =$$

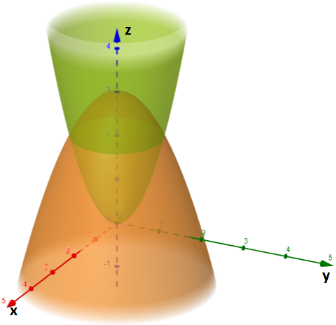
$$\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left[\frac{2\sqrt{2}}{3}\sin\phi - \frac{1}{3\sin^2\phi}\right] d\phi d\theta = \int_0^{2\pi} -\frac{2\sqrt{2}}{3}\cos\phi - \frac{1}{3}\tan\phi\Bigg|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta = \int_0^{2\pi} \frac{2}{3} d\theta =$$

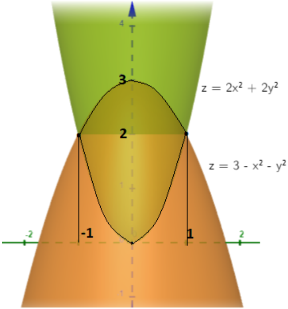
$$\frac{2\theta}{3}\Bigg|_0^{2\pi} = \frac{4\pi}{3}$$

Question 8.

Find the volume of the region bounded above by the paraboloid $z = 3 - x^2 - y^2$ and below by the paraboloid $z = 2x^2 + 2y^2$.

Answer.





Recall that Cartesian and cylindrical coordinates are related through the transformation equations $x = rcos\theta$, $y = rsin\theta$ and $z = z$ with $0 \leq \theta \leq 2\pi$.So, $r^2 = x^2 + y^2$.

Convert two paraboloids into cylindrical coordinates.

Answer.

$$z = 2x^2 + 2y^2 = 2r^2$$

$$z = 3 - x^2 - y^2 = 3 - r^2$$

So, z varies from $2r^2$ to $3 - r^2$.If $x = 0$ and $y = 0$, the first is a paraboloid starting at $z = 0$ and expanding upwards, and second is a paraboloid starting at $z = 3$ and expanding downwards.

Find the intersection of the two paraboloids by setting their equations equal to each other.

$$3 - r^2 = 2r^2 \rightarrow r^2 = 1 \rightarrow r = 1$$

Set up an integral for the region:

$$V = \int_0^{2\pi} \int_0^1 \int_{2r^2}^{3-r^2} r \; dz \; dr \; d\theta$$

Answer.
Integrate with respect to z and simplify:

$$V = \int_0^{2\pi} \int_0^1 rz|_{z=2r^2}^{z=3-r^2} \ dr \ d\theta$$

$$= \int_0^{2\pi} \int_0^1 r(3 - r^2 - 2r^2) \ dr \ d\theta = \int_0^{2\pi} \int_0^1 (3r - 3r^3) \ dr \ d\theta$$

Integrate with respect to r and simplify:

$$V = \int_0^{2\pi} (\frac{3}{2}r^2 - \frac{3}{4}r^4)|_{r=0}^{r=1} \ d\theta = \int_0^{2\pi} \frac{3}{2} - \frac{3}{4} - 0 \ d\theta$$

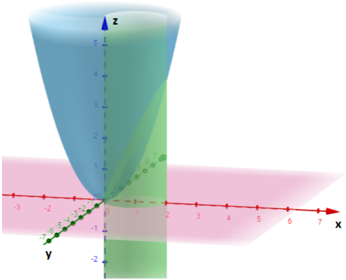
Integrate with respect to θ and simplify:

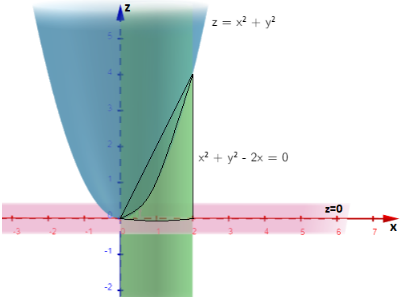
$$V = \int_0^{2\pi} \frac{3}{4} \ d\theta = (\frac{3}{4}\theta)|_{\theta=0}^{\theta=2\pi} \ d\theta = \frac{3}{2}\pi$$

Question 9.

Find the volume of the solid bounded by the surfaces $z = 0$, $z = x^2 + y^2, x^2 + y^2 - 2x = 0$.

Answer.





Answer.

Convert the surfaces into cylindrical coordinates.

$$z = 0$$
$$z = x^2 + y^2 = r^2$$
$$z = x^2 + y^2 - 2x \to r^2 - 2rcos\theta = 0$$

Answer.
z varies from 0 to r^2.Also,

$$r^2 - 2rcos\theta = 0 \rightarrow r(r - 2cos\theta) = 0 \rightarrow r = 0 \text{ or } r = 2cos\theta$$

r varies from 0 to $2cos\theta$. θ varies from $-\pi/2$ to $\pi/2$ by the geometry of solid.
Set up an integral for the region:

$$V = \int_{-\pi/2}^{\pi/2} \int_0^{2cos\theta} \int_0^{r^2} r \ dz \ dr \ d\theta$$

Integrate with respect to z and simplify:

$$V = \int_{-\pi/2}^{\pi/2} \int_0^{2cos\theta} rz|_{z=0}^{z=r^2} \ dr \ d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{2cos\theta} r(r^2 - 0) \ dr \ d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{2cos\theta} (r^3) \ dr \ d\theta$$

Answer.

Integrate with respect to r and simplify:

$$V = \int_{-\pi/2}^{\pi/2} (\frac{1}{4} r^4)|_{r=0}^{r=2cos\theta} \; d\theta = \int_{-\pi/2}^{\pi/2} (\frac{1}{4}(2cos\theta)^4 - 0) \; d\theta$$

Integrate with respect to θ and simplify:

$$V = 4\int_{-\pi/2}^{\pi/2} (cos^4\theta) \; d\theta$$

Remember,

$$cos(2\theta) = cos^2\theta - sin^2\theta = 2cos^2\theta - 1 \text{ since } cos^2\theta + sin^2\theta = 1$$

If we use double-angle identity for cosine, we get

$$cos^4\theta = (cos^2\theta)^2 = (\frac{cos(2\theta)+1}{2})^2 = \frac{cos(2\theta)^2 + 2cos(2\theta) + 1}{4}$$

$$= \frac{cos(4\theta)+1}{8} + \frac{2cos(2\theta)}{4} + \frac{1}{4}$$

Answer.

Now we can calculate the integral:

$$V = 4\int_{-\pi/2}^{\pi/2}(cos^4\theta)\ d\theta = \int_{-\pi/2}^{\pi/2}(\frac{cos(4\theta)}{2} + 2cos(2\theta) + \frac{3}{2})\ d\theta$$

$$= (\frac{sin(4\theta)}{8} + sin(2\theta) + \frac{3}{2}\theta)|_{\theta=-\pi/2}^{\theta=\pi/2} = \frac{3}{2}\pi$$

Question 10.

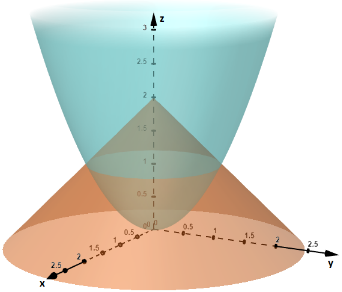
Let D be the solid bounded above by the cone $z = 2 - \sqrt{x^2 + y^2}$ and below by the paraboloid $z = x^2 + y^2$. Write the triple integral expressing the volume of D as an iterated integral in terms of

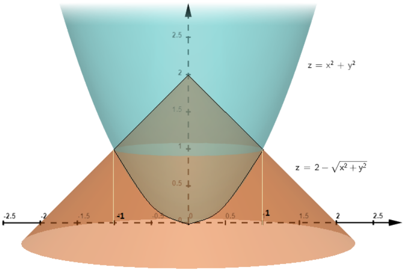
a. cartesian coordinates

b. cylindrical coordinates

Find the volume of D by calculating one of the triple integral in a or b.

Answer.





Answer.

a. Cartesian coordinates:

z changes from $x^2 + y^2$ to $2 - \sqrt{x^2 + y^2}$. Let find the intersections.

$$2 - \sqrt{x^2 + y^2} = x^2 + y^2 \to (\sqrt{x^2 + y^2})^2 + \sqrt{x^2 + y^2} - 2 = 0$$

Let say $a = \sqrt{x^2 + y^2}$, then the last equation becomes
$a^2 + a - 2 = 0 \to a_1 = -2$ and $a_2 = 1$.
So, $\sqrt{x^2 + y^2}$ should be 1.

$$\sqrt{x^2 + y^2} = 1 \to x^2 + y^2 = 1 \to y = \pm\sqrt{1 - x^2} \text{ where } x \in [-1, 1]$$

y changes from $-\sqrt{1 - x^2}$ to $\sqrt{1 - x^2}$ and x changes from -1 to 1.

$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-\sqrt{x^2+y^2}} dz \ dy \ dx$$

Answer.

b. Cylindrical coordinates:

Convert cone and paraboloid into cylindrical coordinates.

$$z = 2 - \sqrt{x^2 + y^2} = 2 - r$$

$$z = x^2 + y^2 = r^2$$

So, z varies from r^2 to $2 - r$. If $x = 0$ and $y = 0$, the first is a cone starting at $z = 2$ and expanding downwards, and second is a paraboloid starting at $z = 0$ and expanding upwards.

Find the intersection of cone and paraboloid by setting their equations equal to each other.

$$2 - r = r^2 \rightarrow r = 1.$$

Set up an integral for the region:

$$V = \int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r} r \; dz \; dr \; d\theta$$

Answer.
Integrate with respect to z and simplify:

$$V = \int_0^{2\pi} \int_0^1 rz|_{z=r^2}^{z=2-r} \ dr \ d\theta$$

$$= \int_0^{2\pi} \int_0^1 r(2 - r - r^2) \ dr \ d\theta = \int_0^{2\pi} \int_0^1 (2r - r^2 - r^3) \ dr \ d\theta$$

Integrate with respect to r and simplify:

$$V = \int_0^{2\pi} (r^2 - \frac{r^2}{3} - \frac{r^4}{4})|_{r=0}^{r=1} \ d\theta = \int_0^{2\pi} 1 - \frac{1}{3} - \frac{1}{4} - 0 \ d\theta$$

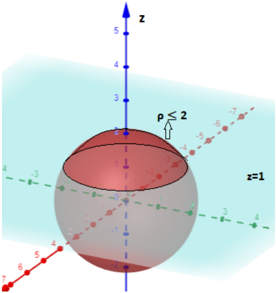
Integrate with respect to θ and simplify:

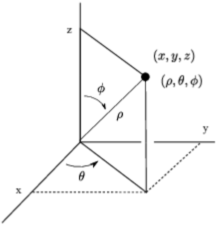
$$V = \int_0^{2\pi} \frac{5}{12} \ d\theta = \frac{5}{6}\pi$$

Question 11.

Find the volume of the smaller region cut from the solid sphere $\rho \leq 2$ by the plane $z = 1$.

Answer.





Recall the relationships between rectangular coordinates and spherical coordinates. From spherical coordinates to rectangular coordinates:

$$x = \rho sin\phi cos\theta$$

$$y = \rho sin\phi sin\theta$$

$$z = \rho cos\phi$$

$$\text{and } \rho^2 = x^2 + y^2 + z^2$$

So, $z = \rho cos\phi = 1 \rightarrow \rho = sec\phi$ (below) and $\rho = 2$(above).

Answer.

If $z = 1$ and ρ is 2, $z = \rho cos\phi = 1 \rightarrow 2cos\phi = 1 \rightarrow \phi = \frac{\pi}{3}$.

Let calculate the triple integral in spherical coordinates:

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_{sec\phi}^{2} \rho^2 sin\phi \ d\rho \ d\phi \ d\theta$$

Integrate with respect to ρ and simplify:

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} (\frac{1}{3}\rho^3 sin\phi)|_{sec\phi}^{2} \ d\phi \ d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \int_0^{\frac{\pi}{3}} (8sin\phi - sec^3\phi sin\phi) \ d\phi \ d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \int_0^{\frac{\pi}{3}} (8sin\phi - sec^2\phi tan\phi) \ d\phi \ d\theta$$

Answer.

Integrate with respect to ϕ and simplify:

$$\int_0^{\frac{\pi}{3}} (8sin\phi - sec^2\phi tan\phi)\ \ d\phi = 8\int_0^{\frac{\pi}{3}} sin\phi\ \ d\phi - \int_0^{\frac{\pi}{3}} sec^2\phi tan\phi\ \ d\phi$$

Let say $du = sec^2\phi d\phi$ where $u = tan\phi$.

$$\int u\ \ du = \frac{u^2}{2} = \frac{tan^2\phi}{2} + C$$

$$8\int_0^{\frac{\pi}{3}} sin\phi\ \ d\phi - \int_0^{\frac{\pi}{3}} sec^2\phi tan\phi\ \ d\phi = -8cos\phi|_0^{\frac{\pi}{3}} - \frac{tan^2\phi}{2}|_0^{\frac{\pi}{3}}$$

$$= (-4 + 8 - \frac{3}{2} + 0) = \frac{5}{2}$$

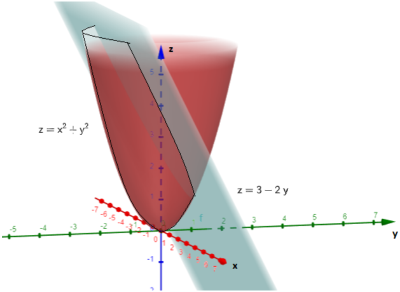
Integrate with respect to θ and simplify:

$$V = \frac{1}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{3}} (8sin\phi - sec^2\phi tan\phi)\ \ d\phi\ \ d\theta = \frac{1}{3}\int_0^{2\pi} \frac{5}{2}\ \ d\theta = \frac{5}{6}\theta|_{\theta=0}^{\theta=2\pi} = \frac{5}{3}\pi$$

Question 12.

Find the volume of the region D lying above $z = 3 - 2y$ and below $z = x^2 + y^2$ by using triple integral.

Answer.



Calculate the triple integral in cylindrical coordinates.

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = 3 - 2y \rightarrow z = 3 - 2r\sin\theta$$
$$z = x^2 + y^2 \rightarrow z = r^2$$

z varies from r^2 to $3 - 2r\sin\theta$.
r varies from 0 to 2 and θ varies from 0 to 2π.

Answer.

Set up an integral for the region:

$$V = \int_0^{2\pi} \int_0^2 \int_{r^2}^{3-2rsin\theta} r \ \ dz \ \ dr \ \ d\theta$$

Integrate with respect to z and simplify:

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^2 z|_{r^2}^{3-2rsin\theta} \ \ rdr \ \ d\theta \\ &= \int_0^{2\pi} \int_0^2 (3 - 2rsin\theta - r^2) \ \ rdr \ \ d\theta \end{aligned}$$

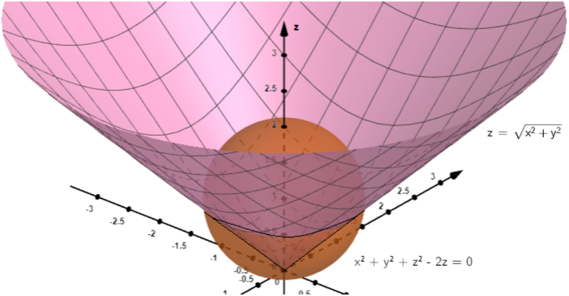
Integrate with respect to r and θ:

$$\begin{aligned} V &= \int_0^{2\pi} (\frac{3r^2}{2} - \frac{2r^3}{3} sin\theta - \frac{r^4}{4})|_{r=0}^{r=2} \ \ d\theta \\ &= \int_0^{2\pi} (6 - \frac{16}{3} sin\theta - 4) \ \ d\theta = 4\pi \end{aligned}$$

Question 13.

Evaluate $\iiint\limits_D (x^2 + y^2 + z^2)\ dV$ where D is the solid bounded below by the surface $z = \sqrt{x^2 + y^2}$ and above $x^2 + y^2 + z^2 - 2z = 0$.

Answer.



For the sphere:

$$x^2 + y^2 + z^2 - 2z = 0 \to \rho^2 = 2\rho cos\phi \to \rho = 2cos\phi$$

Answer.

For the cone:

$$z = \sqrt{x^2 + y^2} \rightarrow \rho cos\phi = \sqrt{(\rho sin\phi cos\theta)^2 + (\rho sin\phi sin\theta)^2}$$

$$\rightarrow \rho cos\phi = \sqrt{\rho^2 sin^2\phi(cos^2\theta + sin^2\theta)} \rightarrow cos\phi = sin\phi \rightarrow \phi = \frac{\pi}{4}$$

Hence the integral becomes

$$\iiint\limits_D (x^2 + y^2 + z^2)\ dV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{2cos\phi} \rho^2 \rho^2 sin\phi\ d\rho\ d\phi\ d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} (\frac{\rho^5}{5})|_0^{2cos\phi} sin\phi\ d\phi\ d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \frac{32}{5} cos^5\phi sin\phi\ d\phi\ d\theta$$

Let say $du = -sin\phi d\phi$ where $u = cos\phi$.

Answer.

$$\int cos^5\phi sin\phi \ \ d\phi = -\int u^5 du = -\frac{u^6}{6} + C = -\frac{cos^6\phi}{6} + C$$

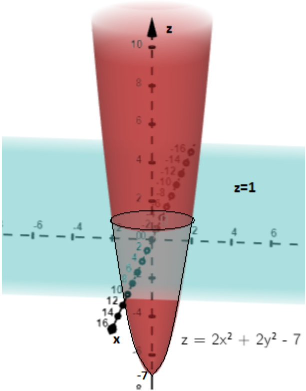
$$\iiint\limits_D (x^2 + y^2 + z^2) \ \ dV = \int_0^{2\pi} -\frac{32}{5}\frac{cos^6\phi}{6}\Big|_{\phi=0}^{\phi=\frac{\pi}{4}} \ \ d\theta = \int_0^{2\pi} -\frac{32}{5}(\frac{1}{48} - \frac{1}{6}) \ \ d\theta$$

$$= \int_0^{2\pi} \frac{14}{15} \ \ d\theta = \frac{14}{15}\theta|_0^{2\pi} = \frac{28}{15}\pi$$

Question 14.

Evaluate $\iiint\limits_D 4xy \;\, dV$ where D is the region bounded by the $z = 2x^2 + 2y^2 - 7$ and $z = 1$.

Answer.



From the sketch, we can say that z varies from $2x^2 + 2y^2 - 7$ to 1.
We can determine the equation of the disk by setting the two equations from the problem statement equal.

$$2x^2 + 2y^2 - 7 = 1 \rightarrow x^2 + y^2 = 4$$

So, $x^2 + y^2 \leq 4$ and we will use cylindrical coordinates for this integral.(Don't forget to convert the $z = 2x^2 + 2y^2 - 7$ into cylindrical coordinates.)

Answer.

$$\iiint\limits_D 4xy \ \ dV = \int_0^{2\pi}\int_0^2\int_{2r^2-7}^{1}(4rcos\theta rsin\theta)r \ \ dz \ \ dr \ \ d\theta$$

$$= \int_0^{2\pi}\int_0^2 (4r^3cos\theta sin\theta)z|_{2r^2-7}^{1} \ \ dr \ \ d\theta$$

$$= \int_0^{2\pi}\int_0^2 (4r^3cos\theta sin\theta) - (8r^5cos\theta sin\theta) + (28r^3cos\theta sin\theta) \ \ dr \ \ d\theta$$

$$= \int_0^{2\pi}(cos\theta sin\theta)(r^4 - \frac{8r^6}{6} + 7r^4)|_{r=0}^{r=2} \ \ d\theta$$

$$= \int_0^{2\pi}(cos\theta sin\theta)(8r^4 - \frac{4r^6}{3})|_{r=0}^{r=2} \ \ d\theta$$

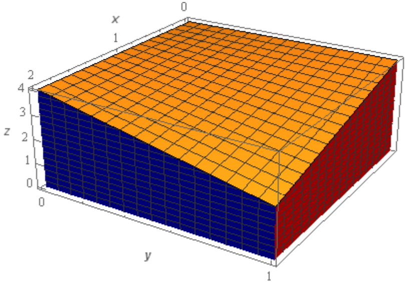
$$= \int_0^{2\pi}\frac{128}{3}(cos\theta sin\theta) \ \ d\theta = \int_0^{2\pi}\frac{64}{3}sin(2\theta) \ \ d\theta$$

$$= -\frac{32}{3}cos(2\theta)|_0^{2\pi} = 0$$

Question 15.

Use a triple integral to determine the volume of the region below $z = 4 - xy$ and above the region in the xy-plane defined by $0 \leq x \leq 2, 0 \leq y \leq 1$.

Answer.



z varies from 0 to $4 - xy$. The limits for x and y are given in the problem.
The volume of this solid is,

$$V = \iiint\limits_D dV = \int_0^2 \int_0^1 \int_0^{4-xy} dzdydx$$

Now we can evaluate the integral.

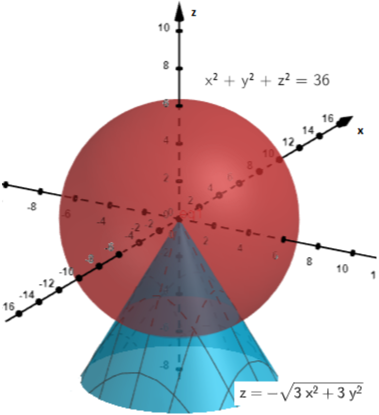
Answer.

$$
\begin{aligned}
V &= \int_0^2 \int_0^1 z|_0^{4-xy} dydx = \int_0^2 \int_0^1 (4 - xy)dydx \\
&= \int_0^2 (4y - \frac{xy^2}{2})|_{y=0}^{y=1} \ dx = \int_0^2 (4 - \frac{x}{2})dx \\
&= (4x - \frac{x^2}{4})|_0^2 = 7
\end{aligned}
$$

Question 16.

Evaluate $\iiint\limits_D x^2 \; dV$ where D is the region above $x^2+y^2+z^2=36$ and inside $z=-\sqrt{3x^2+3y^2}$.

Answer.



For the sphere:

$$x^2+y^2+z^2=36 \rightarrow \rho^2=36$$
$$\rightarrow \rho=6$$

For the cone:

$$z=-\sqrt{3x^2+3y^2}$$
$$\rho cos\phi=-\sqrt{3}\rho sin\phi$$
$$tan\phi=-\frac{1}{\sqrt{3}} \rightarrow \phi=\frac{5\pi}{6}$$

Answer.

Hence the integral becomes

$$\iiint\limits_D (x^2)\ dV = \int_0^{2\pi}\int_{\frac{5\pi}{6}}^{\pi}\int_0^6 (\rho sin\phi cos\theta)^2\rho^2 sin\phi\ \ d\rho\ \ d\phi\ \ d\theta$$

$$= \int_0^{2\pi}\int_{\frac{5\pi}{6}0}^{\pi}\int_0^6 \rho^4 sin^3\phi cos^2\theta\ \ d\rho\ \ d\phi\ \ d\theta$$

$$= \int_0^{2\pi}\int_{\frac{5\pi}{6}}^{\pi}(sin^3\phi cos^2\theta)\frac{\rho^5}{5}|_0^6\ \ d\phi\ \ d\theta$$

$$= \int_0^{2\pi}\int_{\frac{5\pi}{6}}^{\pi}(sin^3\phi cos^2\theta)\frac{7776}{5}\ \ d\phi\ \ d\theta$$

$$= \int_0^{2\pi}\int_{\frac{5\pi}{6}}^{\pi}\frac{7776}{5}(1-cos^2\phi)sin\phi cos^2\theta\ \ d\phi\ \ d\theta$$

$$= \int_0^{2\pi} -\frac{7776}{5}cos^2\theta(cos\phi - \frac{cos^3\phi}{3})|_{\phi=\frac{5\pi}{6}}^{\phi=\pi}\ \ d\theta$$

Answer.

$$= \int_0^{2\pi} \frac{7776}{5}(\frac{2}{3} - \frac{3\sqrt{3}}{8})cos^2\theta \;\; d\theta$$

$$= \int_0^{2\pi} \frac{7776}{5}(\frac{2}{3} - \frac{3\sqrt{3}}{8})(\frac{cos(2\theta) + 1}{2}) \;\; d\theta$$

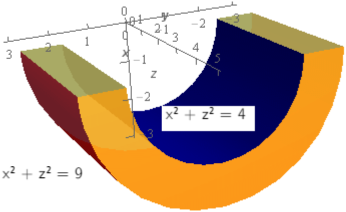
$$= \frac{7776}{5}(\frac{2}{3} - \frac{3\sqrt{3}}{8})(\frac{sin(2\theta)}{4} + \frac{\theta}{2})|_0^{2\pi}$$

$$= \frac{7776}{5}(\frac{2}{3} - \frac{3\sqrt{3}}{8})\pi$$

Question 17.

Evaluate $\iiint\limits_D e^{-x^2-z^2}\ dV$ where D is the region between the two cylinders $x^2+z^2=4$ and $x^2+z^2=9$ with $1 \le y \le 5$ and $z \le 0$.

Answer.



We know that $x^2+z^2=4$ and $x^2+z^2=9$ are cylinders of radius 2 and 3 respectively that are centered on the y-axis. $z \le 0$ tells us that we will only have the lower half of each of the cylinders.
$x = rcos\theta$, $z = rsin\theta$ and $y = y$ with $0 \le \theta \le 2\pi$.So, r varies from 2 to 3, y varies from 1 to 5 and θ varies from π to 2π.

Answer.

Here are the cylindrical coordinates for this problem.

$$\iiint\limits_D e^{-x^2-z^2}\ dV = \int_\pi^{2\pi}\int_2^3\int_1^5 e^{-r^2} r\ dy\ dr\ d\theta$$

$$= \int_\pi^{2\pi}\int_2^3 (e^{-r^2} r) y|_1^5\ dr\ d\theta = \int_\pi^{2\pi}\int_2^3 4(e^{-r^2} r)\ dr\ d\theta$$

$$= \int_\pi^{2\pi} -2(e^{-r^2})|_2^3\ d\theta = \int_\pi^{2\pi} 2(e^{-4}-e^{-9})\ d\theta$$

$$= 2(e^{-4}-e^{-9})\theta|_\pi^{2\pi} = 2(e^{-4}-e^{-9})\pi$$