# MAT104E PS

## Question 1.

Investigate the divergence or convergence of the following sequences

$i) x_n = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots + \frac{1}{(2n-1)(2n+1)}$

$ii) x_n = \sqrt[n]{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2}}$

$iii) x_n = \frac{1 - 2 + 3 - 4 + 5 - 6 + \cdots + (2n-1) - 2n}{\sqrt{n^2+1}}$.

## Answer.

i) Note that $\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$. So, we get $x_n = \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} \cdots + \frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) \implies \lim_{x\to\infty} x_n = \frac{1}{2}$. Hence $\{x_n\}$ is convergent.

ii) $\sqrt[n]{\frac{1}{n^2} + \frac{1}{n^2} + \frac{1}{n^2} + \cdots + \frac{1}{n^2}} \le x_n \le \sqrt[n]{1 + 1 \cdots + 1} \implies \sqrt[n]{\frac{1}{n}} \le x_n \le \sqrt[n]{n}$. As we know that $\lim_{x\to\infty} \sqrt[n]{n} = 1$. By Sandwich Theorem, we get $\lim_{x\to\infty} x_n = 1$ which implies that $\{x_n\}$ is convergent.

**Answer.**

iii) $1 - 2 + 3 - 4 + 5 - 6 + \cdots + (2n - 1) - 2n = -1 - 1 - 1 - \cdots - 1$. It has n-term. Then we get

$\lim_{x\to\infty} x_n = \lim_{x\to\infty} \dfrac{-n}{\sqrt{n^2 + 1}} = -1$.Thus $\{x_n\}$ is convergent

## Question 2.

Investigate the divergence or convergence of the following sequences $i) a_n = \dfrac{\sin^2 n}{2^n}$

$ii) a_n = \left(1 + \dfrac{1}{n+1}\right)^{5n}$

$iii) a_n = n \ln\left(\dfrac{n^2 - 1}{n^2}\right)$

$iv) a_n = 1 - (-1)^n \dfrac{n^2 + 2}{2n^2 + 1}$.

## Answer.

i) We know that $-1 \leq \sin n \leq 1$. It implies that $0 \leq \sin^2 n \leq 1 \implies 0 \leq \dfrac{\sin^2 n}{2^n} \leq \dfrac{1}{2^n}$.

Hence we obtain $0 \leq \lim\limits_{n\to\infty} \dfrac{\sin^2 n}{2^n} \leq \lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$. By Sandwich Theorem, we get $\lim a_n = 0$. Hence $\{a_n\}$ is convergent.

ii) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\left(1 + \dfrac{1}{n+1}\right)^{n+1} \left(1 + \dfrac{1}{n+1}\right)^{-1}\right)^5 = (e.1)^5 = e^5$.

Hence $\{a_n\}$ is convergent.

## Answer.

ii) **Second way :**

$\lim_{n\to\infty} a_n; 1^{\infty}$ . We can apply l'Hôpital's Rule if we first change the form to $0.\infty$ by taking the natural logarithm of $a_n$.

$\ln a_n = 5n \ln\left(1 + \dfrac{1}{n+1}\right)$. Then, we get

$$\begin{aligned}
\lim_{n\to\infty} \ln a_n &= \lim_{n\to\infty} 5n \ln\left(1 + \frac{1}{n+1}\right) \\
&= \lim_{n\to\infty} \frac{\ln\left(\frac{n+2}{n+1}\right)}{\frac{1}{5n}} \\
&= \lim_{n\to\infty} \frac{\frac{\frac{-1}{(n+1)^2}}{\frac{n+2}{n+1}}}{\frac{-1}{5n^2}} \\
&= \lim_{n\to\infty} \frac{5n^2}{(n+1)(n+2)} = 5.
\end{aligned}$$

Hence we have $\lim_{n\to\infty} a_n = e^5$ so that $\{a_n\}$ is convergent.

## Answer.

iii)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \ln\left(\frac{n^2-1}{n^2}\right)$$
$$= \lim_{n\to\infty} \frac{\ln\left(\frac{n^2-1}{n^2}\right)}{\frac{1}{n}}$$
$$= \lim_{n\to\infty} \frac{\frac{\frac{2n.n^2-(n^2-1)2n}{n^4}}{\frac{n^2-1}{n^2}}}{\frac{-1}{n^2}}$$
$$= \lim_{n\to\infty} \frac{-2n}{n^2-1} = 0$$

iv) For $n = 2k$, $a_{2k} = 1 - (-1)^{2k}\dfrac{(2k)^2+2}{2(2k)^2+1} = 1 - \dfrac{4k^2+2}{8k^2+1} \implies \lim\limits_{k\to\infty} a_{2k} = \dfrac{1}{2}$.

For $n = 2k+1$, $a_{2k+1} = 1 - (-1)^{2k+1}\dfrac{(2k+1)^2+2}{2(2k+1)^2+1} \implies \lim\limits_{k\to\infty} a_{2k+1} = \dfrac{3}{2}$.

Since $lim_{k\to\infty} a_{2k} \neq lim_{k\to\infty} a_{2k+1}$, the limit of $\{a_n\}$ does not exist.

Hence $\{a_n\}$ is divergent.

## Question 3.

Investigate the divergence or convergence of the following sequences $i) a_n = \sqrt[n]{n^2 + 2n}$
$ii) a_n = \sqrt[n]{10n}\frac{2^n}{n!}$
$iii) a_n = n - \sqrt{n^2 + 1}$.

## Answer.

i) $\ln a_n = \frac{\ln(n^2 + 2n)}{n}$. By using l'Hôpital's Rule, we have

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\ln(n^2 + 2n)}{n}$$
$$= \lim_{n\to\infty} \frac{\frac{2n+2}{n^2+2n}}{1} = 0.$$

$\lim_{n\to\infty} \ln a_n = 0 \implies \lim_{n\to\infty} a_n = e^0 = 1.$
Thus $\{x_n\}$ is convergent.
ii)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{10}\sqrt[n]{n}\frac{2^n}{n!}$$
$$= 1.1.0 = 0.$$

Thus $\{x_n\}$ is convergent.
iii)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n - \sqrt{n^2 + 1}$$
$$= \lim_{n\to\infty} (n - \sqrt{n^2 + 1}).\frac{n + \sqrt{n^2 + 1}}{n + \sqrt{n^2 + 1}}$$
$$= \lim_{n\to\infty} \frac{n^2 - (n^2 + 1)}{n + \sqrt{n^2 + 1}}$$
$$= \lim_{n\to\infty} \frac{-1}{n\left(1 + \sqrt{1 - \frac{1}{n^2}}\right)} = 0$$

Hence $\{x_n\}$ is convergent.

## Question 4.

Show that the sequence $a_n = \dfrac{1}{1+5} + \dfrac{1}{1+5^2} + \dfrac{1}{1+5^3} + \cdots + \dfrac{1}{1+5^n}$ converges by using the monotonic sequence theorem.

## Answer.

Note that

$$\begin{aligned} a_{n+1} - a_n &= \left(\frac{1}{1+5} + \frac{1}{1+5^2} + \cdots + \frac{1}{1+5^{n+1}}\right) - \left(\frac{1}{1+5} + \frac{1}{1+5^2} + \cdots + \frac{1}{1+5^n}\right) \\ &= \frac{1}{1+5^{n+1}} > 0 \end{aligned}$$

for all $n \geq 1$. Thus we get $a_{n+1} > a_n$ for all $n \geq 1$. The sequence is an increasing and it is bounded below by $a_1 = \dfrac{1}{6}$. On the other hand

$$\begin{aligned} a_n &= \frac{1}{1+5} + \frac{1}{1+5^2} + \frac{1}{1+5^3} + \cdots + \frac{1}{1+5^n} < \frac{1}{5} + \frac{1}{5^2} + \cdots + \frac{1}{5^n} \\ &= \frac{1}{5}\left(1 + \frac{1}{5} + \cdots + \frac{1}{5^{n-1}}\right) = \frac{1}{5}\left(\frac{1-(\frac{1}{5})^n}{1-\frac{1}{5}}\right) = \frac{1}{4}\left(1 - \left(\frac{1}{5}\right)^n\right) < \frac{1}{4} \end{aligned}$$

for all $n \geq 1$. So, $\{a_n\}$ is bounded above by $\dfrac{1}{4}$. Since $\{a_n\}$ is both bounded above and bounded below, it is a bounded. By Monotonic Sequence Theorem, $\{a_n\}$ convergences.

## Question 5.

Show that the sequence $a_n = \dfrac{1}{1+n} + \dfrac{1}{2+n} + \dfrac{1}{3+n} + \cdots + \dfrac{1}{n+n}$ converges by using the monotonic sequence theorem.

## Answer.

Note that, for all $n \geq 1$

$$\begin{aligned} a_{n+\mathbf{1}} - a_n &= \left( \frac{1}{2+n} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} \right) - \left( \frac{1}{1+n} + \frac{1}{2+n} + \cdots + \frac{1}{2n} \right) \\ &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{(2n+1)(2n+2)} > 0. \end{aligned}$$

Thus we get $a_{n+\mathbf{1}} > a_n$ for all $n \geq 1$. The sequence is increasing and it is bounded below by $a_{\mathbf{1}} = \dfrac{1}{2}$. On the other hand,

$a_n = \dfrac{1}{1+n} + \dfrac{1}{2+n} + \cdots + \dfrac{1}{n+n} < \dfrac{1}{n} + \dfrac{1}{n} + \cdots + \dfrac{1}{n} = n\dfrac{1}{n} = 1$ for all $n \geq 1$.

Therefore, $\{a_n\}$ is bounded above by 1.

Since $\{a_n\}$ is both bounded above and bounded below, it is a bounded.

By Monotonic Sequence Theorem, $\{a_n\}$ convergences.

## Question 6.

Let the sequence $\{a_n\}$ be defined recursively by $a_1 = 1$, $a_{n+1} = 1 + \dfrac{2}{3}a_n$, $n \geq 1$.
Find the $n$-th term of the sequence $\{a_n\}$
Find the limit off the sequence $\{a_n\}$.

## Answer.

Observe that

$$\begin{aligned}
a_{n+1} &= 1 + \frac{2}{3}a_n \\
&= 1 + \frac{2}{3}\left(1 + \frac{2}{3}a_{n-1}\right) = 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 a_{n-1} \\
&= 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 \left(1 + \frac{2}{3}a_{n-2}\right) = 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 a_{n-2} \\
&= \ldots = 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots + (\frac{2}{3})^{n-1} + \left(\frac{2}{3}\right)^n a_{n-(n-1)} \\
&= 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots + \left(\frac{2}{3}\right)^{n-1} + \left(\frac{2}{3}\right)^n a_1 = 3\left(1 - (\frac{2}{3})^{n+1}\right).
\end{aligned}$$

for all $n \geq 1$. Therefore, we get $a_n = 3\left(1 - \left(\dfrac{2}{3}\right)^n\right)$ for all $n \geq 1$.

Then, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 3\left(1 - \left(\dfrac{2}{3}\right)^n\right) = 3.$

## Question 7.

Show that the sequence $a_n = \dfrac{2n-1}{2n+3}$ convergences by using the monotonic sequence theorem.

## Answer.

Note that $a_n = \dfrac{2n-1}{2n+3} = 1 - \dfrac{4}{2n+3}$ for all $n \geq 1$. Then we get

$$\begin{aligned} a_{n+1} - a_n &= 1 - \frac{4}{2n+5} - \left(1 - \frac{4}{2n+3}\right) \\ &= \frac{8}{(2n+5)(2n+3)} > 0 \end{aligned}$$

for all $n \geq 1$. Thus, $\{a_n\}$ is increasing.
On the other hand, $a_n < 1$ for all $n \geq 1$.
By the monotonic sequence theorem, $\{a_n\}$ is convergent.

## Question 8.

Investigate the divergence or convergence of the sequence $a_n = \dfrac{(n!)^2}{(2n)!}$.

## Answer.

We can see that,

$$
\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(n+1)!^2}{(2n+2)!}\frac{(2n)!}{(n!)^2} \\
&= \frac{((n+1)n!)^2}{(2n+2)!}\frac{(2n)!}{(n!)^2} \\
&= \frac{(n+1)^2}{(2n+1)(2n+2)} \\
&= \frac{n+1}{4n+2} < 1.
\end{aligned}
$$

for all $n \geq 1$. $a_n > 0$ and $\dfrac{a_{n+1}}{a_n} < 1$ yield that $a_{n+1} < a_n$. It means that $\{a_n\}$ is a decreasing sequence. On the other hand, due to $0 < a_n < \dfrac{1}{2}$ for all $n \geq 1$, $\{a_n\}$ is bounded. Hence $\{a_n\}$ is convergent by monotone sequence theorem.

## Question 9.

Show that the recursively defined sequence $a_1 = \frac{1}{4}$, $a_{n+1} = \frac{4a_n}{2a_n + 1}$, $n \geq 1$ convergences.

## Answer.

Observe that, for all $n \geq 1$

$$\begin{aligned} a_{n+1} - a_n &= \frac{4a_n}{2a_n + 1} - \frac{4a_{n-1}}{2a_{n-1} + 1} \\ &= 4\frac{2a_n a_{n-1} + a_n - 2a_n a_{n-1} - a_{n-1}}{(2a_{n+1} + 1)(2a_n + 1)} \\ &= 4\frac{a_n - a_{n-1}}{(2a_{n+1} + 1)(2a_n + 1)}. \end{aligned}$$

Since $a_n > 0$ for all n, we have $2a_{n+1} + 1 > 0$ and $2a_n + 1 > 0$. Then we can easily see that $a_{n+1} - a_n$ and $a_n - a_{n-1}$ have the same sign for all $n \geq 1$.

$a_2 - a_1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12} > 0 \implies a_{n+1} - a_n > 0$. Therefore $\{a_n\}$ is an increasing sequence.For all $n \geq 1$,

$a_{n+1} - a_n = \frac{4a_n}{2a_n + 1} - a_n = \frac{a_n(3 - 2a_n)}{2a_n + 1} \implies 3 - 2a_n > 0 \implies a_n < \frac{3}{2}$ which implies that $\{a_n\}$ is bounded above.

By monotonic sequence theorem, the sequence convergences.

## Question 10.

Show that the recursively defined sequence $a_1 = 2$, $a_{n+1} = \dfrac{3+2a_n}{2+a_n}$, $n \geq 1$ convergences. Find the limit of the sequence.

## Answer.

Observe that

$$\begin{aligned} a_{n+1} - a_n &= \frac{3+2a_n}{2+a_n} - \frac{3+2a_{n-1}}{2+a_{n-1}} \\ &= \frac{a_n - a_{n-1}}{(2+a_n)(2+a_{n-1})} \end{aligned}$$

for all $n \geq 1$. Since $a_n > 0$ for all n, we obtain $2 + a_n > 0$ and $2 + a_{n-1} > 0$. Then we can easily see that $a_{n+1} - a_n$ and $a_n - a_{n-1}$ have the same sign for all $n \geq 1$.
$a_2 - a_1 = \dfrac{7}{4} - 2 = \dfrac{-1}{4} < 0 \implies a_{n+1} - a_n < 0$ for all $n \geq 1$. Therefore $\{a_n\}$ is a decreasing sequence and bounded above by 2.
We have $0 < a_n < 2$ for all $n \geq 1$. By monotonic sequence theorem, the sequence convergences.
Let $\lim\limits_{n\to\infty} a_n = L$. Then $\lim\limits_{n\to\infty} a_{n+1} = L$. So,
$L = \dfrac{3+2L}{2+L} \implies L^2 - 3 = 0 \implies L = -\sqrt{3}$ or $L = \sqrt{3}$.
Since $0 < a_n$ for all $n$, $L = \sqrt{3}$.

## Question 11.

Investigate the convergence of the series $\sum_{n=1}^{\infty} n^2 \tan^2(\frac{1}{n})$

## Answer.

Let $a_n = n^2 \tan^2(\frac{1}{n})$

$$
\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} n^2 \tan^2(\frac{1}{n})(0.\infty) \\
&\implies \lim_{n\to\infty} (\frac{\tan(\frac{1}{n})}{\frac{1}{n}})^2 \\
&\implies \lim_{n\to\infty} (\frac{\sin(\frac{1}{n})}{\cos(\frac{1}{n})\frac{1}{n}})^2 \\
(\lim_{n\to\infty} \cos(\frac{1}{n}) = 1) \qquad &\implies \lim_{n\to\infty} (\frac{\sin(\frac{1}{n})}{\frac{1}{n}})^2 \\
u = (\frac{1}{n}) \qquad &\implies \lim_{u\to 0} (\frac{\sin u}{u})^2 = 1.
\end{aligned}
$$

Since, $\lim_{n\to\infty} a_n \neq 0$ the series diverges from the $n^{th}$ term test.

## Question 12.

Show that

$$1 + \frac{1}{x} + \frac{1}{x^2} + ... = \frac{x}{x-1}$$

assuming $x > 1$, then express the repeating decimal 0.131313... as a rational number.

## Answer.

Note that if $x > 1$, then $\frac{1}{x} < 1$. In this case, the left side is the sum of an infinite geometric progression. Using the formula $S = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$, we can write the left side as

$$1 + \frac{1}{x} + \frac{1}{x^2} + ... = \sum_{n=0}^{\infty} (\frac{1}{x})^n = \frac{1}{1 - \frac{1}{x}} = \frac{x}{x-1}$$

so,

$$0.131313... = \frac{13}{100} + \frac{13}{10000} + \frac{13}{1000000} + ... =$$

$$(\frac{13}{100})(1 + \frac{1}{100} + \frac{13}{10000} + ...) = (\frac{13}{100})\frac{100}{100-1} = \frac{13}{99} \ (r = \frac{1}{100}, x = 100)$$

## Question 13.

$\lim\limits_{n\to\infty} a_n = a$ ise $\sum\limits_{n=1}^{\infty}(a_n - a_{n+1}) = a_1 - a$ olduğunu gösterin.

Bu özelliği kullanarak $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\dfrac{2n+1}{n(n+1)}$ serisinin toplamını hesaplayın.

## Answer.

$$S_N = \sum_{n=1}^{N}(a_n - a_{n+1}) = a_1 - a_2 + a_2 - a_3 + \ldots + a_N - a_{N+1} = a_1 - a_{N+1}$$

$$\begin{aligned} \sum_{N=1}^{\infty}(a_n - a_{n+1}) &= \lim_{N\to\infty} S_N \\ &= \lim_{N\to\infty} a_1 - a_{N+1} \\ (\lim_{N\to\infty} a_{N+1} = \lim_{N\to\infty} a_N = a) \quad &= a_1 - a. \end{aligned}$$

Bu özellik yardımıyla,

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n+1}{n(n+1)} = \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n} + \frac{1}{n+1}\right)$$

$$
\begin{aligned}
=&\sum_{n=1}^{\infty}((-1)^{n+1}\frac{1}{n}+(-1)^{n+1}\frac{1}{n+1})\\
=&\sum_{n=1}^{\infty}((-1)^{n}\frac{-1}{n}+(-1)^{n+1}\frac{1}{n+1})\\
=&-\sum_{n=1}^{\infty}((-1)^{n}\frac{1}{n}-(-1)^{n+1}\frac{1}{n+1})\\
=&-(-1-0)\\
=&1\ (a_n=\frac{(-1)^n}{n}, a_1=1, a=\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{n}=0).
\end{aligned}
$$

## Question 14.

Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n+1 \ln(n+1)}$ converges or diverges.

## Answer.

The funcion $f(x) = \frac{1}{(x+1)\ln(x+1)}$ is continuous and positive.

$f'(x) = -\frac{1+\ln(1+x)}{(1+x)\ln(x+1))^2} < 0$ on $[1, \infty)$ (decreasing)

Thus, we can apply Integral Test.

$$\begin{aligned}
\int_1^{\infty} \frac{dx}{(x+1)\ln(x+1)} &= \lim_{a\to\infty} \int_1^{a} \frac{dx}{(x+1)\ln(x+1)} \\
&= \lim_{a\to\infty} \ln\ln(x+1)\Big|_1^a \\
&= \lim_{a\to\infty} \ln\ln(a+1) - \ln\ln(2) \\
&= \infty
\end{aligned}$$

The Integral approaches infinity. Therefore, the series diverges.

## Question 15.

Prove that if $\sum_{n=1}^{\infty} a_n$ is convergent series of non-negative terms, then $\sum_{n=1}^{\infty} {a_n}^2$ converges.

## Answer.

If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$
According to Limit Comparison Test (L.C.T),

$$\lim_{n\to\infty} \frac{{a_n}^2}{a_n} = \lim_{n\to\infty} a_n = 0$$

So, $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} {a_n}^2$ converges too.

## Question 16.

Determine whether the series $\sum_{n=1}^{\infty} \frac{2n^3+7}{n^4 \sin^2 n}$ converges or diverges.

## Answer.

$2n^3 + 7 > 2n^3 \implies \frac{2n^3+7}{n^4 \sin^2 n} > \frac{2n^3}{n^4 \sin^2 n} > \frac{2}{n \sin^2 n} (*)$
and we also know , $0 \leqslant \sin^2 n \leqslant 1$ so,
$n \sin^2 n \leqslant n \implies \frac{1}{n} \leqslant \frac{1}{n \sin^2 n} (**)$
With the help of $(*)$ and $(**)$,

$$\frac{2n^3+7}{n^4 \sin^2 n} > \frac{2}{n \sin^2 n} \geq \frac{2}{n}$$

$$\implies \sum_{n=1}^{\infty} \frac{2n^3+7}{n^4 \sin^2 n} > \sum_{n=1}^{\infty} \frac{2}{n} \; (divergent, p = 1).$$

According to Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{2n^3+7}{n^4 \sin^2 n}$ is divergent.

## Question 17.

Determine whether the series $\sum_{n=1}^{\infty} \frac{n!}{5^n}$ converges or diverges.

## Answer.

$$
\begin{aligned}
a_n = \frac{n!}{5^n}, a_(n+1) = \frac{(n+1)!}{5^{n+1}} = \frac{(n+1)n!}{5^n 5} = \frac{n+1}{5} a_n \\
\implies \lim_{n\to\infty} \left| \frac{a_n + 1}{a_n} \right| \\
\implies \lim_{n\to\infty} \left| \frac{\frac{n+1}{5} a_n}{a_n} \right| \\
\implies \infty.
\end{aligned}
$$

According to Ratio Test the series is divergent.

## Question 18.

Determine whether the series $\sum_{n=1}^{\infty} \frac{n^n}{2^{1+n}}$ converges or diverges.

## Answer.

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{2^{1+n}}} \\
&= \lim_{n\to\infty} \frac{n}{2^{\frac{1}{n}}2} \\
&= \infty.
\end{aligned}$$

According to root test, this series is divergent.

## Question 19.

Determine whether the series $\sum_{n=2}^{\infty} \frac{\cos n\pi}{\sqrt{n}}$ converges (absolute, conditionally) or diverges.

## Answer.

$$\sum_{n=2}^{\infty} \frac{\cos n\pi}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} (u_n = \frac{1}{\sqrt{n}})$$

$$\implies \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

$$\implies \mathrm{u}_n = \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} = u_{(n+1)}$$

The two conditions of the test met and so by the Alterne Series Test the series is convergent.

$$\sum_{n=2}^{\infty} \left| \frac{\cos n\pi}{\sqrt{n}} \right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \quad (p = \frac{1}{2} < 1, divergent)$$

Therefore, this series is not absolutely convergent. It is however conditionally convergent since the series itself does converges.

## Question 20.

Find the interval of absolute convergence,
interval of convergence,
radius of convergence, of the power series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n+\sqrt{n}}$

Determine the $x$ values, if any, for which the power series converges conditionally.

## Answer.

Let's use the Ratio Test to find the interval of absolute convergence:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 &\implies \lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{n+1+\sqrt{n+1}} . \frac{n+\sqrt{n}}{(x+2)^n} \right| < 1 \\
&\implies |x+2| . \lim_{n\to\infty} \left( \frac{n+\sqrt{n}}{n+1+\sqrt{n+1}} \right) < 1 \\
&\implies |x+2| < 1 \\
&\implies -3 < x < -1.
\end{aligned}$$

Now, let's investigate the convergence at endpoints:

- When $x=-3$, we have the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n+\sqrt{n}}$ which is convergent by the Alternating Series Test, but $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n+\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{n+\sqrt{n}}$ is divergent by the Limit Comparison Test comparing with the divergent $p$-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$. So, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n+\sqrt{n}}$ converges conditionally.
- When $x=-1$, we have the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+\sqrt{n}}$ which is divergent by the Limit Comparison Test.

Thus,

- the interval of absolute convergence is $(-3,-1)$,
- the interval of convergence is $[-3,-1)$,
- the radius of convergence is 1,
- the series converges conditionally at $x=-3$.

## Question 21.

Let $\sum_{n=0}^{\infty} n^3(2x-3)^n$

Find the interval of absolute convergence,
interval of convergence,
radius of convergence, of the power series.

Determine the $x$ values, if any, for which the power series converges conditionally.

## Answer.

Let's use the Ratio Test to find the interval of absolute convergence:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 &\implies \lim_{n\to\infty}\left|\frac{(n+1)^3(2x-3)^{n+1}}{n^3(2x-3)^n}\right| < 1 \\
&\implies |2x-3|\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3 < 1 \\
&\implies |2x-3| < 1 \\
&\implies 1 < x < 2.
\end{aligned}$$

Now, let's investigate the convergence at endpoints:

- When $x = 1$, we have the series $\sum_{n=1}^{\infty} (-1)^n n^3$ which is divergent by the nth-term Test.
- When $x = 2$, we have the series $\sum_{n=1}^{\infty} n^3$ which is divergent by the nth-term Test.

Thus,

- the interval of absolute convergence is $(1, 2)$,
- the interval of convergence is $(1, 2)$,
- the radius of convergence is $\frac{1}{2}$,
- there are no values for which the series converges conditionally.

## Question 22.

Using Maclaurin series of $\sin x$ find the Maclaurin series of $f(x) = x^3 \sin(2x)$ and find the interval of convergence of the obtained series.

## Answer.

Recall that the Maclaurin series of $\sin x$ is

$$\sin x = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Then,

$$\sin 2x = \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}.$$

Thus, the Maclaurin series of $f(x)$ is

$$f(x) = x^3 \sin 2x = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+4}}{(2n+1)!}.$$

Let's use the Ratio Test to find the interval of convergence:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 &\implies \lim_{n\to\infty}\left|\frac{(-1)^{n+1}2^{2n+3}x^{2n+6}}{(2n+3)!}\cdot\frac{(2n+1)!}{(-1)^n 2^{2n+1}x^{2n+4}}\right| < 1 \\
&\implies (2x)^2.\lim_{n\to\infty}\left(\frac{1}{(2n+3)(2n+2)}\right) < 1 \\
&\implies 0 < 1.
\end{aligned}$$

Thus, the series converges for all x, i.e., the interval of convergence is $(-\infty, \infty)$.

## Question 23.

Find the Maclaurin series of $f(x) = \ln(x+4)$.

## Answer.

We can express the fraction $\dfrac{1}{4+x}$ as

$$\frac{1}{4+x} = \frac{1}{4}.\frac{1}{1+\left(\frac{x}{4}\right)} = \frac{1}{4}.\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{4}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{4^{n+1}}.$$

Then, after term-by-term integration, we have that

$$\int \frac{1}{4+x}\,dx = C + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n.4^n}.$$

For $x = 0$,we get $C = \ln(4)$. Thus,

$$\ln(x+4) = \ln(4) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n.4^n}.$$

## Question 24.

Let $f(x) = \int_0^x \cos t^2 \, dt$. Find the Maclaurin series of $f(x)$.

## Answer.

The power series of $\cos t$ is

$$\cos t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!}.$$

Then, using the substitution $t \to t^2$, we get

$$\cos t^2 = \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n}}{(2n)!}.$$

Then, after term-by-term integration from 0 to $x$, we have that

$$\begin{aligned} \int_0^x \cos t^2 \, dt &= \left( t - \frac{t^5}{5.2!} + \frac{t^9}{9.4!} - \cdots \right) \Bigg|_0^x \\ &= x - \frac{x^5}{5.2!} + \frac{x^9}{9.4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(4n+1)(2n)!}. \end{aligned}$$

## Question 25.

Express $\int e^x \, dx$ as a power series.

## Answer.

We have the fact that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Then, after term-by-term integration, we have

$$\int e^x \, dx = C + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = C + \sum_{n=1}^{\infty} \frac{x^n}{n!}.$$

## Question 26.

Let $f(x) = \cos(x-1)$

i) Find $f^{(n)}(x)$

ii) Find the Taylor series $f(x)$ at $x = 1$

iii)Using *ii*) find the sum of the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{(2\pi)^{2n}}{(2n)!}$.

## Answer.

i)

$$
\begin{aligned}
f(x) = \cos(x-1) &\implies f'(x) = -\sin(x-1) \\
&\implies f''(x) = -\cos(x-1) \\
&\implies f'''(x) = \sin(x-1) \\
&\implies f^{(4)}(x) = \cos(x-1) \\
&\qquad \vdots \\
&\implies f^{(n)}(x) = \begin{cases} \cos(x-1) & , \ n = 4k \\ -\sin(x-1) & , \ n = 4k+1 \\ -\cos(x-1) & , \ n = 4k+2 \\ \sin(x-1) & , \ n = 4k+3 \end{cases}.
\end{aligned}
$$

Moreover, $f^{(n)}(1) = \begin{cases} 1 & , \ n = 4k \\ 0 & , \ n = 4k+1 \\ -1 & , \ n = 4k+2 \\ 0 & , \ n = 4k+3 \end{cases}.$

ii) The Taylor series of $f(x)$ at $x = 1$ is defined as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{(n)!}(x-1)^n.$$

So,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{n}(1)}{(n)!}(x-1)^n \\ &= 1 + 0 - \frac{(x-1)^2}{2!} + 0 + \frac{(x-1)^4}{4!} + 0 - \frac{(x-1)^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{2n}}{(2n)!} \end{aligned}$$

iii) Using ii),we can write the following

$$\cos(x-1)=\sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^{2n}}{(2n)!}=1+\sum_{n=1}^{\infty}(-1)^n\frac{(x-1)^{2n}}{(2n)!}$$

.

Thus, taking $x=2\pi+1$, we have that

$$\cos(2\pi)=1+\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi)^{2n}}{(2n)!}$$

$$\Longrightarrow\ \sum_{n=1}^{\infty}(-1)^n\frac{(2\pi)^{2n}}{(2n)!}=0.$$

## Question 27.

Find the power series representation of $f(x) = \dfrac{1}{2+x}$ in powers of $(x-1)$.

Find the interval of convergence of the obtained series.

## Answer.

We can express the fraction $\dfrac{1}{2+x}$ as

$$\begin{aligned}
\frac{1}{2+x} &= \frac{1}{3+(x-1)} \\
&= \frac{1}{3(1+(\frac{x-1}{3}))} \\
&= \frac{1}{3}.\frac{1}{1+(\frac{x-1}{3})} \\
&= \frac{1}{3}.\sum_{n=0}^{\infty}(-1)^n\left(\frac{x-1}{3}\right)^n \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^n}{3^{n+1}}.
\end{aligned}$$

Now, let's use the Ratio Test to find the interval of absolute convergence:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1 &\implies \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-1)^{n+1}}{3^{n+1}}.\frac{3^{n+1}}{(-1)^n(x-1)^n}\right|<1\\
&\implies |x-1|.\lim_{n\to\infty}\left|-\frac{1}{3}\right|<1\\
&\implies \frac{|x-1|}{3}<1\\
&\implies -2<x<4.
\end{aligned}$$

Let's investigate the convergence at endpoints:

- When $x=-2$, we have the series $\displaystyle\sum_{n=0}^{\infty}\frac{1}{3}$ which is divergent by the nth-term Test.
- When $x=4$, we have the series $\displaystyle\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)$ which is divergent by the nth-term Test.

## Question 28.

Express $\int e^{-x^3} dx$ as a power series

Taking the first two terms in the series find an approximate value for the integral $\int_0^{1/2} e^{-x^3} dx$.

## Answer.

We have the fact that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.That is,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

Substituting $-x^3$ instead of x in above equation, we get

$$e^{-x^3} = 1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \cdots .$$

Term-by-integrating both sides of previous equation, we have

$$\int e^{-x^3} \ dx = C + x - \frac{x^4}{4.1!} + \frac{x^7}{7.2!} - \frac{x^{10}}{10.3!} + \cdots = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{(3n+1)n!}.$$

So, taking the first two term in the series, we have

$$\int_0^{1/2} e^{-x^3}\, dx \approx \left( x - \frac{x^4}{4} \right)\Big|_0^{1/2} = \frac{31}{64}.$$