Quantifying the Shape of the Time Series using Topological Data Analysis

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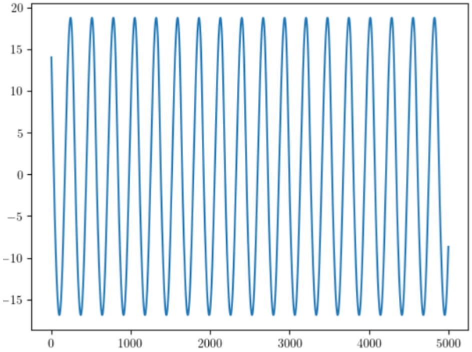
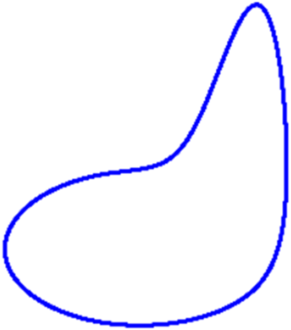
Time Series and Phase Space

Data is a shape, and shape is data.





Periodic time series and Phase space

Time Series and Phase Space

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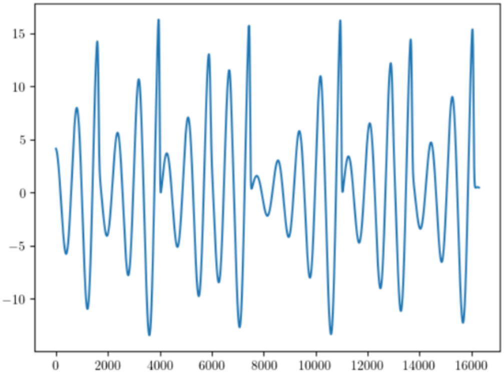

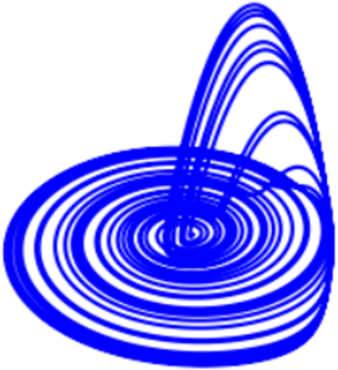

Chaotic time series and Phase space

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Rössler System

The Rössler system is described by the following system of three differential equations:

$$\dot{x} = -(wy + z), \tag{1}$$

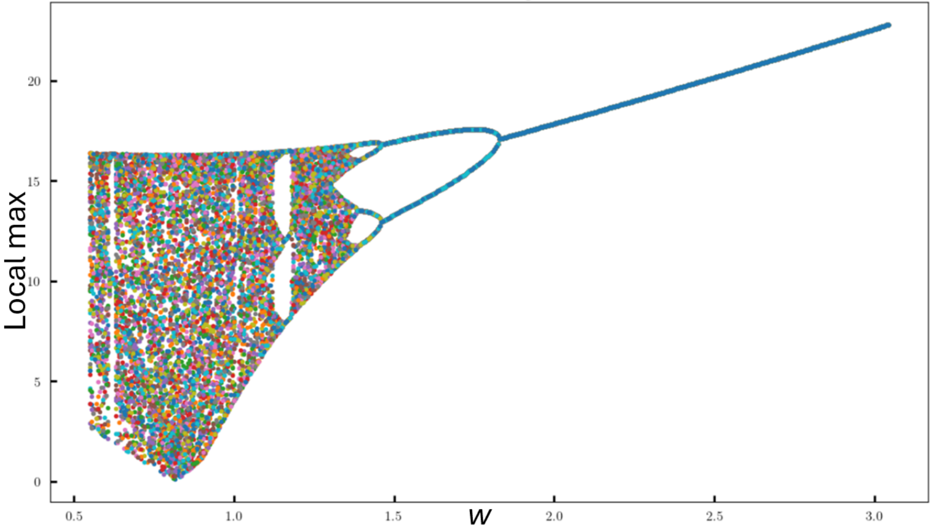
$$\dot{y} = wx + ay, \tag{2}$$

$$\dot{z} = b + z(x - c). \tag{3}$$

In this work, we explore the following fixed parameters: $a = 0.16$, $b = 0.1$, $c = 8.5$. We construct the bifurcation diagram in the range $0.58 \leq w \leq 3.04$ using w as a control parameter and it is a natural frequency.

Does shape matter?

Here we see the bifurcation diagram of a Rössler System.



Hypotheses

Visual inspection of time series signals only goes so far. We want to build a mathematical analysis of such data:

- Is it possible there are high-dimensional topological or geometric patterns in the time series of dynamical systems?
- How do we impose and construct geometry on this data?
- What do these patterns and features suggest?

We hypothesize that the right interpretation of time series signals could gain insight that is not apparent through standard methods, like Lyapunov exp, Poincaré, and Fast Fourier transform.

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What is TDA?

Topological Data Analysis (TDA) is a means to analyze high-dimensional data by looking at underlying topological and geometric patterns and structures. TDA is useful for analyzing data that:

- Lives in high-dimensional space.
- Lives in non-euclidean space or has non-Euclidean features that want to be analyzed.
- May not be easily reduced to a lower dimension without losing topological and geometric information.

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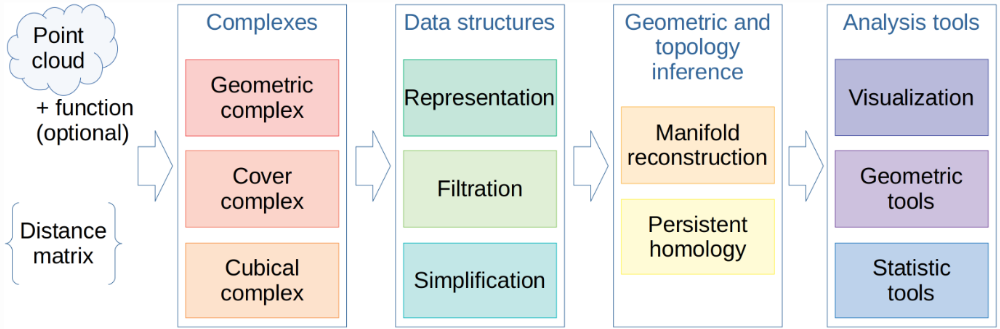
TDA-Pipeline



(Gudhi documentation: https://gudhi.inria.fr/introduction/)

Simplicial Complexes

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How do we construct and impose the geometry?

Simplex: A q-simplex is the convex hull of $(q+1)$ affinely independent points, called vertices.

Simplicial Complex

A simplicial complex Q is a finite set of non-empty simplices with two conditions:

- first, each face k of each simplex Q is a simplex Q.
- the intersection of the complexes must be a common face.

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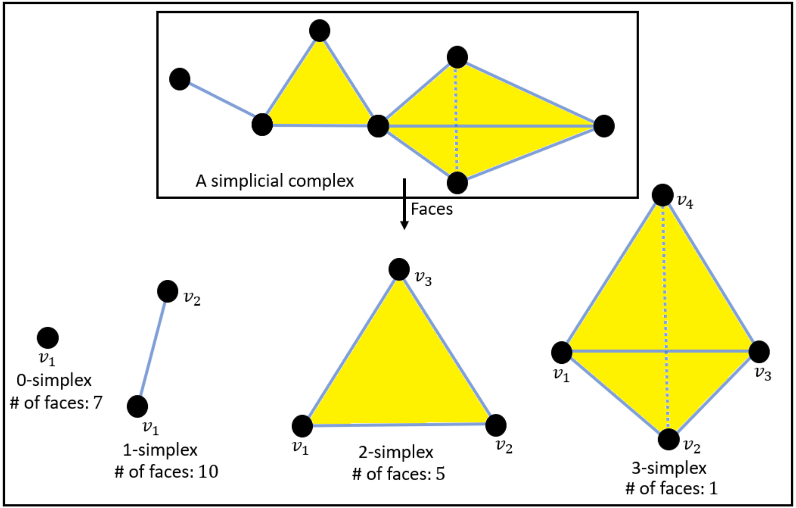
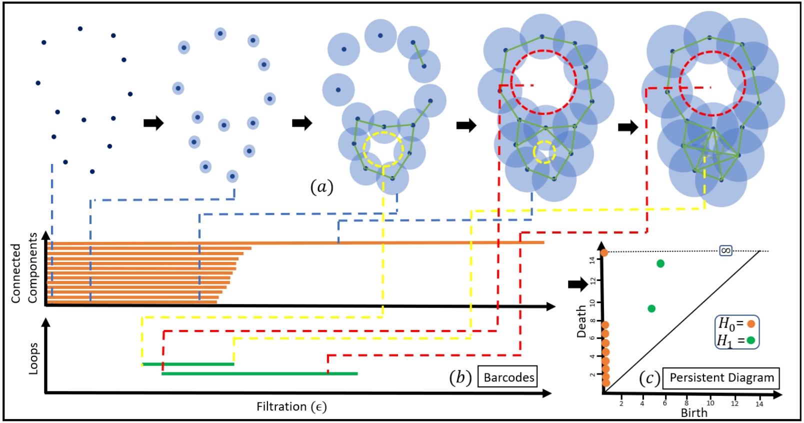
Simplicial Complexes

A simplicial complex with its building blocks.



Point cloud and Persistence Diagram



(a) Filtered subcomplexes (yellow and red circles) form around each data point. (b) A persistence barcode that depicts homological features like connected components and loops. c) A persistence diagram, visualizing the birth and death of topological features across different scales.

Betti Curve and Persistent Landscape

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Machine Learning feature:

Betti Curve: the Betti curve $\beta_D^{gm}(u)$ counts the number of features in the PD that are "alive" at u, i.e., the number of points (b_i, d_i) for which $b_i \leq u < d_i$. More precisely, it can be expressed as:

$$\beta_D^{gm} : \#\{(b_i, d_i)\, i \in D^{gm} \mid b_i \leq u < d_i\}.$$

Persistent Landscape: A commonly used functional summary of a PD is the persistence landscape. Intuitively, this is constructed by rotating the PD 45° clockwise.

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Machine Learning feature

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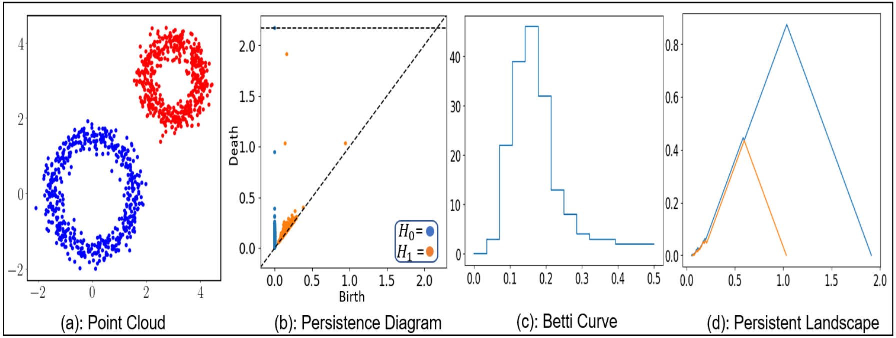

(a) Point cloud representation of two circles. (b) Persistence diagram showing the birth and death of topological features from the point cloud. (c) Betti curve illustrating the number of persistent features as a function of the parameter scale. (d) Persistence landscape depicting the evolution of prominent loops identified in the persistence diagram.

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Methodology

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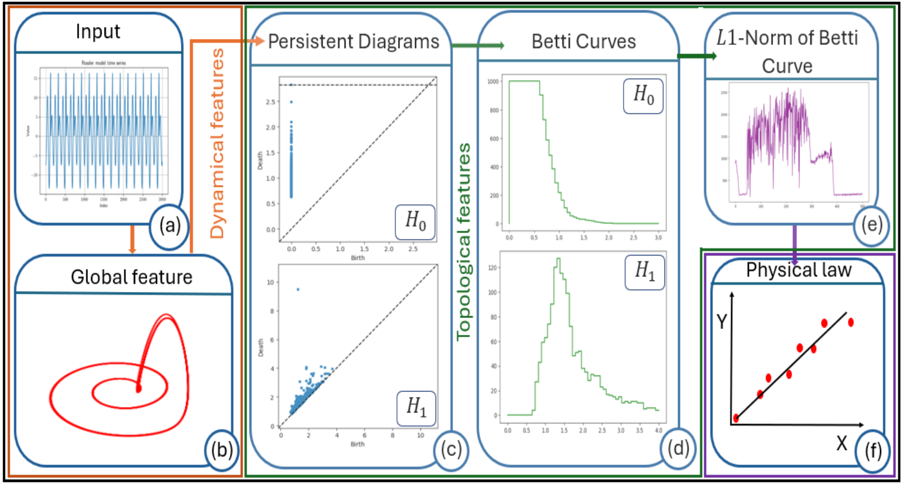

Results (CROCKER Plot)

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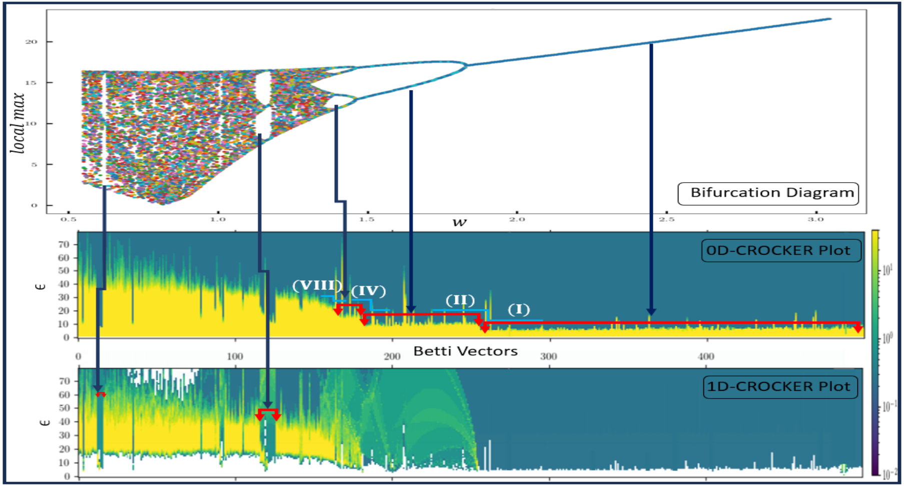
An interactive visualization technique of TDA that plot the homological features w.r.t change in the system state variable.



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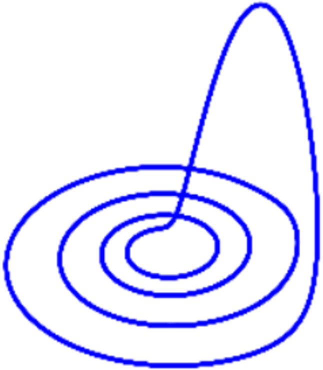
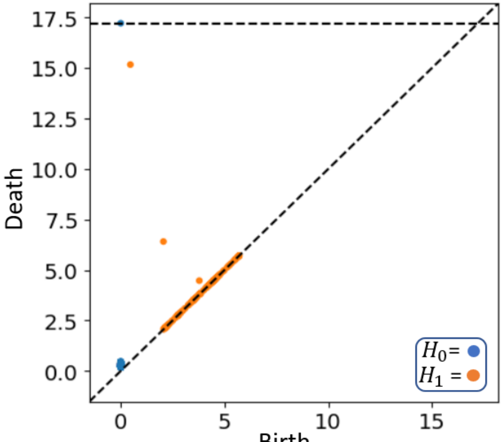
Results

Phase space trajectory and persistence diagram for $w = 1$, showing a periodic regime.

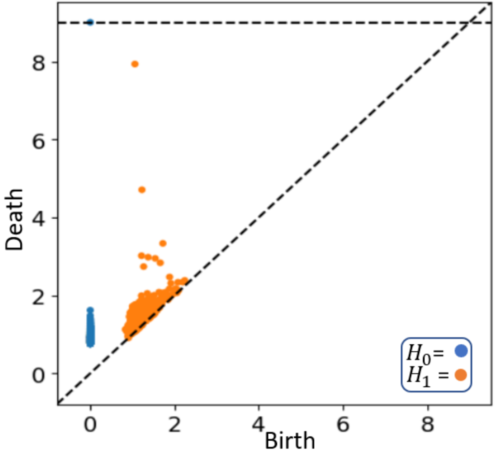




Results

Phase space trajectory and persistence diagram for $w = 1.03$, illustrating a chaotic regime with numerous loopy structures.

Phase Space

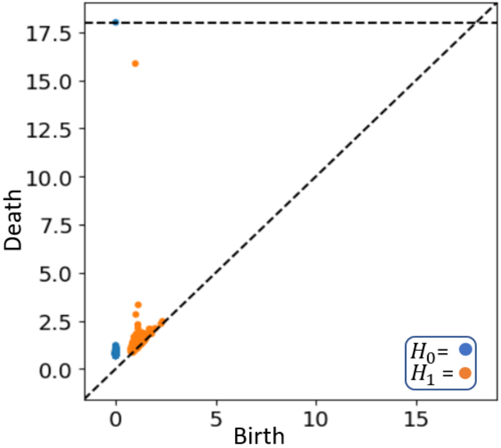


Persistence Diagram

Results

Phase space trajectory and persistence diagram for $w = 1.28$, showing a chaotic regime.

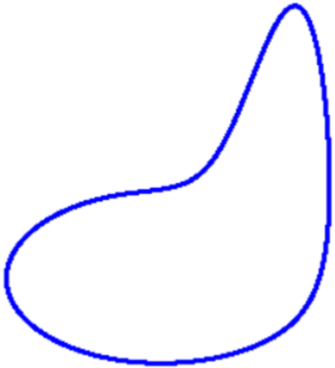
Phase Space



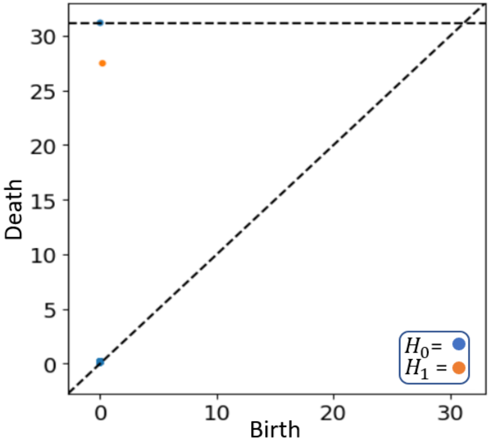
Persistence Diagram

Results

Phase space trajectory and persistence diagram for $w = 1.89$, depicting a period-one regime.

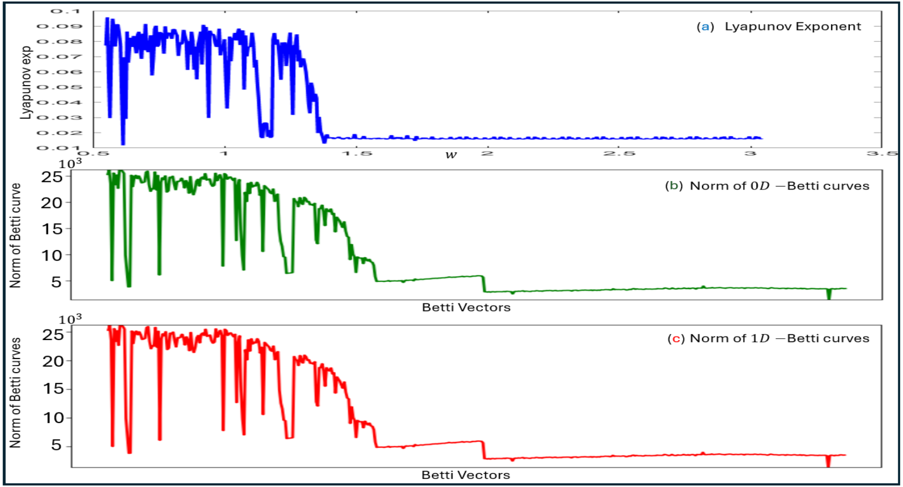


Phase Space



Persistence Diagram

Results



(a) Leading Lyapunov exponent versus w. (b) L1-norm of 0-dimensional Betti curves. (c) L1-norm of 1-dimensional Betti curves.

Physical law

Three different regression models have been applied; the goal is to construct a curve that captures the overall trend of the data:

Reciprocal Model

$$y = \frac{1}{ax + b}$$

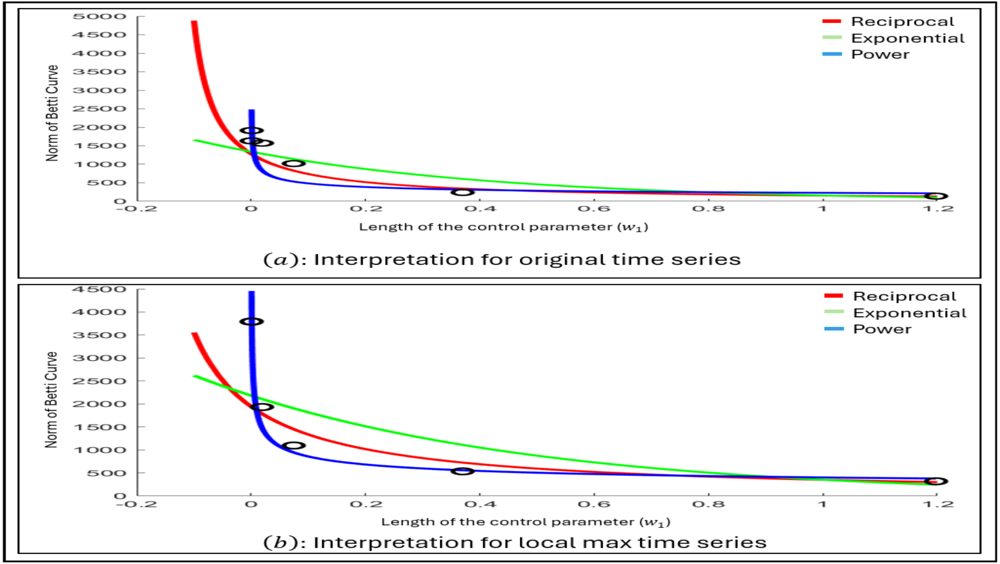
Exponential Model

$$y = ae^{bx}$$

Power law Model

$$y = ax^b$$

Physical law



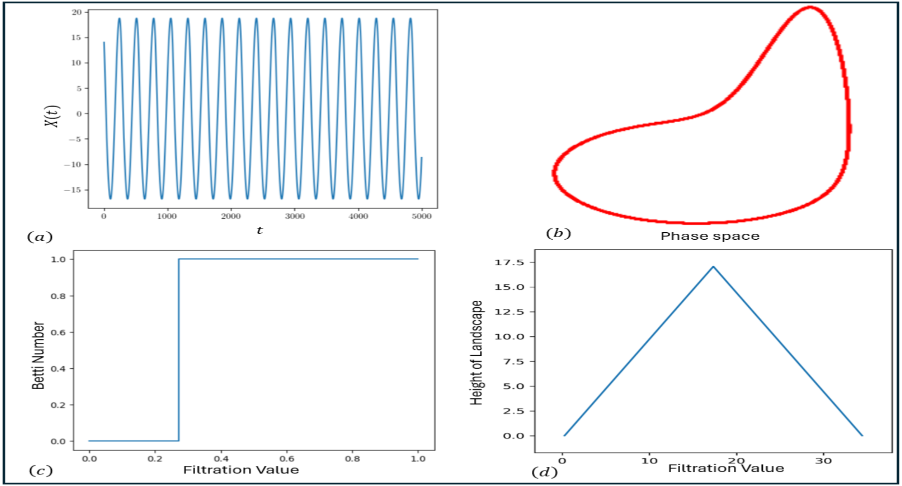
Results

Table: Comparison of r^2 values for different models.

Model	r^2**-Original Time Series**	r^2**-Localmax Time Series**
Reciprocal model	0.94	0.93
Exponential model	0.81	0.70
Power law model	0.43	0.94

Results

(a) Time series and (b) phase space trajectory of period 1. (c) Betti curve, and (d) persistence landscape, indicating a single loop-like structure in the phase space



Results

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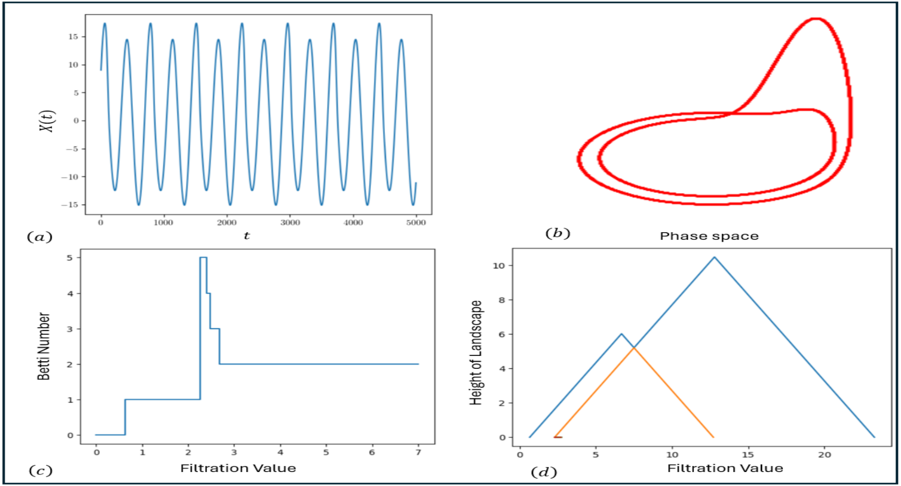
(a) Time series and (b) phase space trajectory of period 2. (c) Betti curve, and (d) persistence landscape, confirm the presence of 2 loops in the phase space



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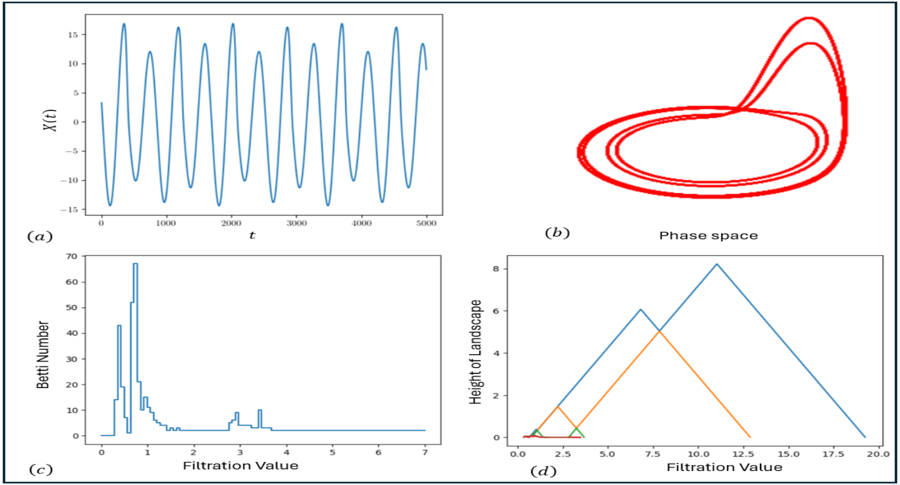
Results

(a) Time series and (b) phase space trajectory of period 4. (c) Betti curve, and (d) persistence landscape, confirm the presence of 4 loops in the phase space



Results

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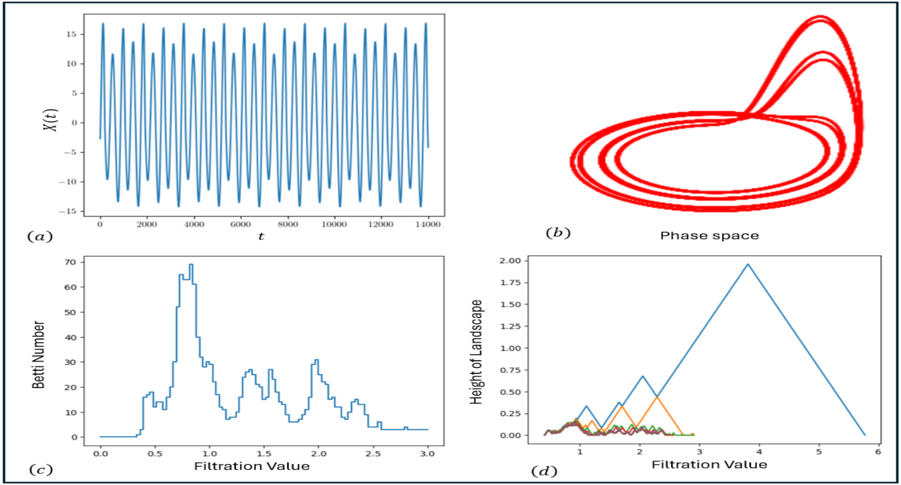
(a) Time series and (b) phase space trajectory of period 8. (c) Betti curve, and (d) persistence landscape, indicating a 8 loop-like structure in the phase space



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Conclusion

- *Persistent Homology:* Unveils topological insights in time series of dynamical systems.
- *Topology-Dynamics Link:* Betti vectors show a clear connection with system dynamics and the Lyapunov exponent.
- *Chaos Model:* A new model explains the transition to chaos using Betti vector L1-norms.
- *Future Work:* Apply to coupled oscillators and complex systems and integrate with machine learning.

References

- K. Mittal, S. Gupta, Topological characterization and early detection of bifurcations and chaos in complex systems using persistent homology, Chaos 27 (5) (2017).
- A. Zomorodian, G. Carlsson, Computing persistent homology, Discrete and Computational Geometry 33 (2) (2005) 249–274.
- A. N. Pisarchik, R. Jaimes-Reategui, Deterministic coherence resonance in coupled chaotic oscillators with frequency mismatch, Phys. Rev. E 92 (5) (2015) 050901.
- V. Tymchyshyn, M. Tsizh, F. Vazza, M. Baldi, On the evolution of Betti curves in the Cosmic web, arXiv preprint arXiv:2311.10029 (2023). . Bauer, Ripser: efficient computation of vietoris-rips persistence barcodes, arXiv preprint arXiv:1908.02518 (2019).

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