Question: Using the properties of an inner product, prove that

Using the properties of an inner product, prove that the Cauchy-Schwarz inequality holds for any "induced norm".

(a) Specifically, show that if $\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$, then for any $\boldsymbol{x}, \boldsymbol{y}$ we have $|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \leq \|\boldsymbol{x}\| \|\boldsymbol{y}\|$. [Hint: Consider writing $\boldsymbol{x}$ as a linear combination of $\boldsymbol{y}$ and the vector $\boldsymbol{z} = \boldsymbol{x} - \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{\|\boldsymbol{y}\|^2} \boldsymbol{y}$. What is $\langle \boldsymbol{z}, \boldsymbol{y} \rangle$? What can we infer from the previous problem?]

(b) *Optional: Show that equality holds if and only if $\boldsymbol{y} = a\boldsymbol{x}$ for some $a \in \mathbb{R}$.

Show transcribed image text

Expert Answer

Anonymous
answered this

(a) For $\alpha \in \mathbb{C}$,

$$0 \le \|x-\alpha y\|^2 = \langle x-\alpha y, x-\alpha y\rangle$$

$$= \langle x,x\rangle - \alpha\langle y,x\rangle - \bar{\alpha}\langle x,y\rangle + \alpha\bar{\alpha}\langle y,y\rangle$$

$$= \|x\|^2 - \alpha\langle y,x\rangle - \bar{\alpha}\langle x,y\rangle + |\alpha|^2\|y\|^2$$

For $y \neq 0$, let $\alpha = \dfrac{\langle x,y\rangle}{\|y\|^2}$

Then, $0 \le \|x\|^2 - \alpha\langle y,x\rangle - \dfrac{\overline{\langle x,y\rangle}}{\|y\|^2}\langle x,y\rangle + \dfrac{|\langle x,y\rangle|^2}{\|y\|^4}\cdot\|y\|^2$

$$= \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2} - \frac{|\langle x,y\rangle|^2}{\|y\|^2} + \frac{|\langle x,y\rangle|^2}{\|y\|^2}$$

$$= \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2}$$

i.e. $\|x\|^2\|y\|^2 \geq |\langle x,y\rangle|^2$

i.e. $|\langle x,y\rangle|^2 \le \|x\|\|y\|$

(b) If x and y are linearly dependent, then $y = kx$, for some $k \in \mathbb{C}$

L.H.S $= |\langle x,y\rangle|$ $= |\langle x,kx\rangle|$

$= |k||\langle x,x\rangle|$

$= |k|\,\|x\|^2$

R.H.S $= \|x\|\|y\|$ $= \|x\|\|kx\|$ $= |k|\,\|x\|^2$

conversely, if equality holds in cauchy schwarz inequality then the above computation becomes

$$\|y-ax\|^2 = 0$$

$$\Rightarrow \|y-ax\| = 0$$

$$\Rightarrow y-ax = 0$$

$$\Rightarrow y = ax \quad \text{for some } a \in \mathbb{R}.$$

[proved]

0 Comments

Was this answer helpful?

0

0

Questions viewed by other students

- I, we must also have Prove that if A and B are square N x N matrices, then if AB BA=I. Some hints to help you get started: = T. • An equivalent statement to BA=I is that, for any x ERN, BAx - • Think about what can you say about R(AB). • Think about what you can say about the relationship between R(AB) and R(B). • Recall that for any X E R(B), we can write x as a linear...

See answer

1) At the instant shown, the truck travels to the right at 3 m/s, while the pipe rolls counterclockwise at 6 rad/s without slipping at B. Determine the velocity of the pipe's center G. 15 m

See answer 80% (5 ratings)

Show more