Geometric computer vision - part 1

Camera calibration, single-view metrology and epipolar geometry

Computer Vision – Fall 2023

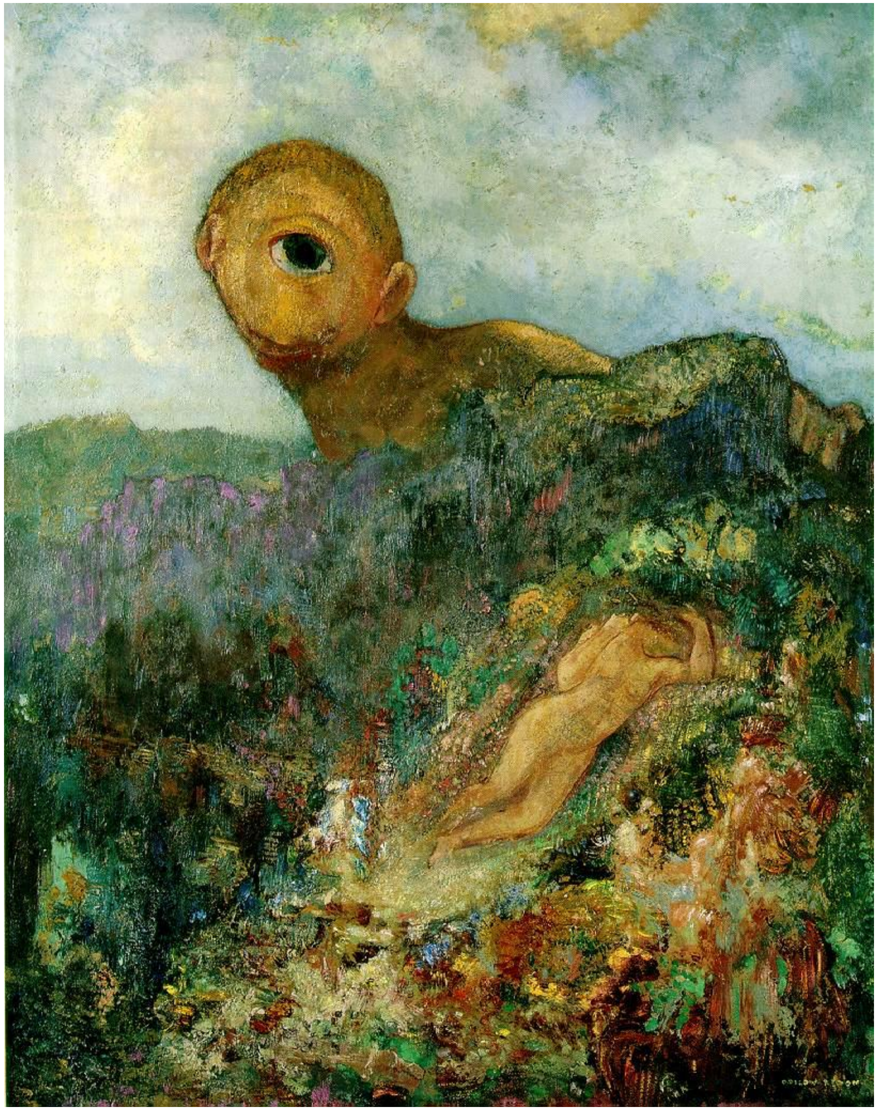
Instructor: Prof. Dr. Marius Leordeanu

For today

Reading:

Chapters 1 and 7.1 from Forsyth and Ponce

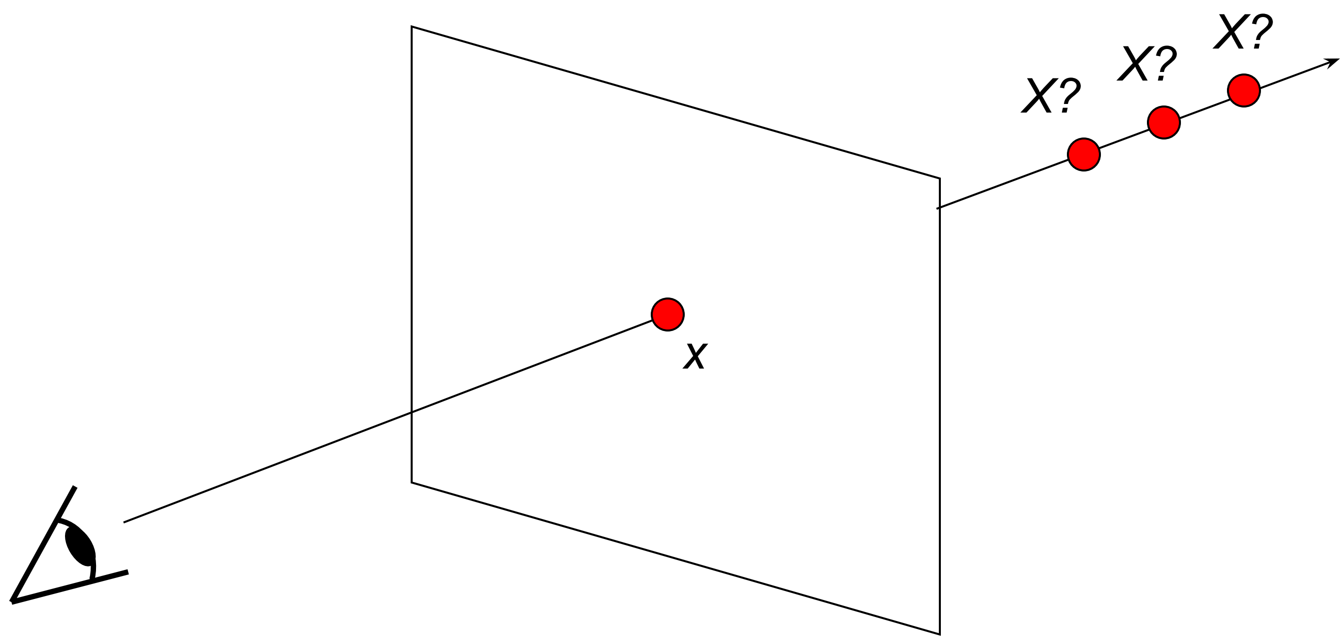
Part A: Calibrating a single camera



Odilon Redon, *Cyclops*, 1914

Our goal: Recovery of 3D structure

- Recovery of structure from one image is inherently ambiguous

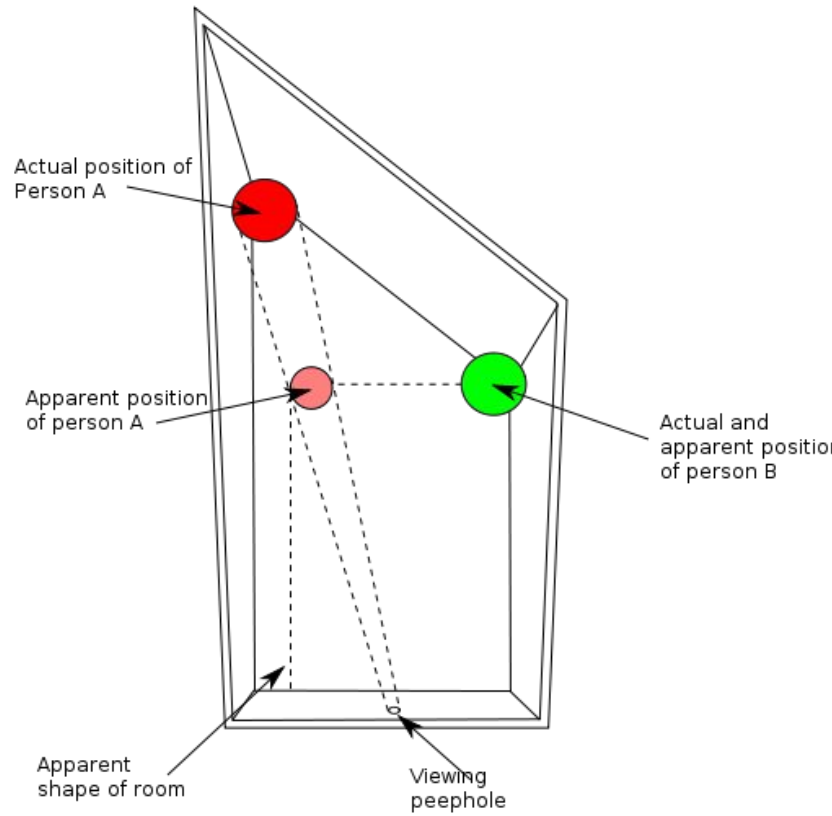


Single-view ambiguity

Single-view ambiguity

Rashad Alakbarov shadow sculptures

Single-view ambiguity

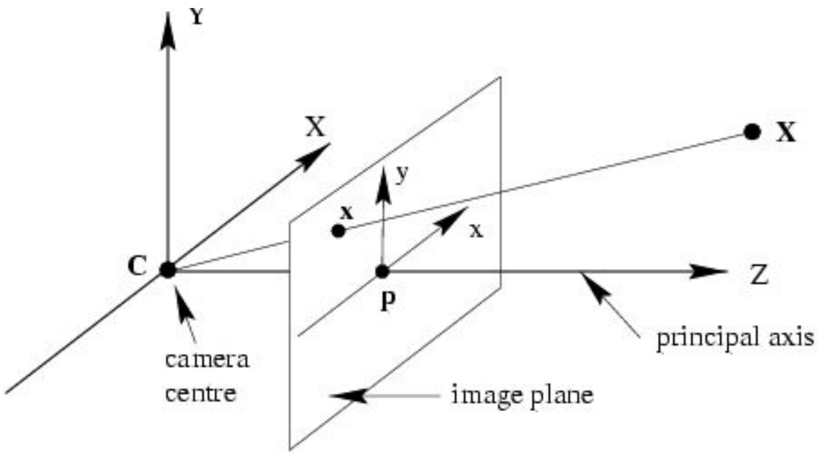


Ames room

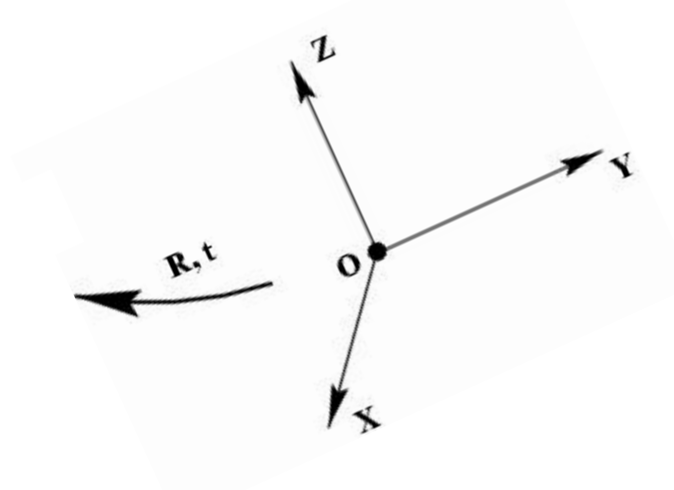
Our goal: Recovery of 3D structure

- We will need *multi-view geometry*

Review: Pinhole camera model



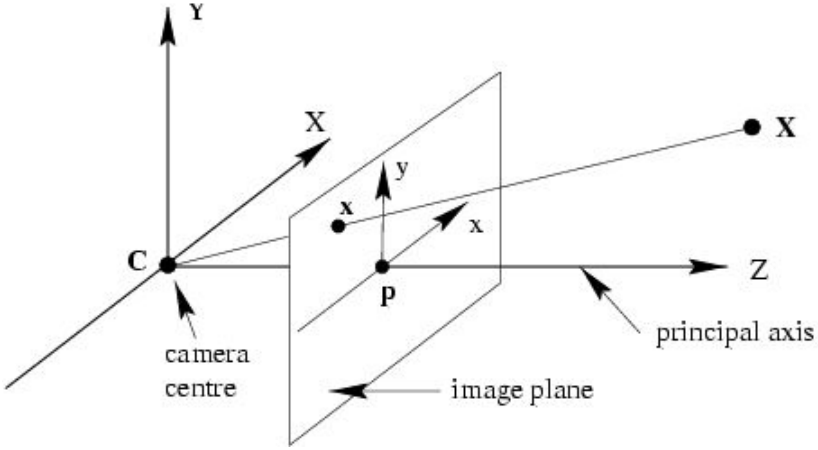
world coordinate system

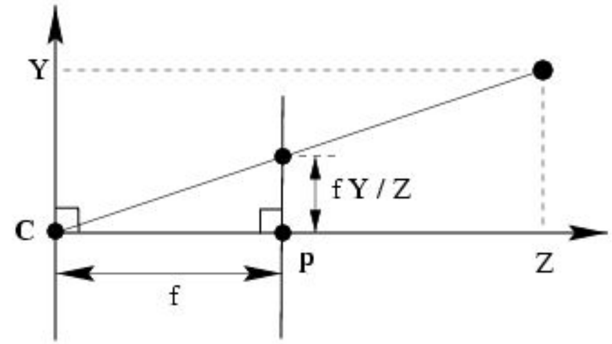


- **Normalized (camera) coordinate system:** camera center is at the origin, the *principal axis* is the z-axis, x and y axes of the image plane are parallel to x and y axes of the world

- Goal of camera calibration: go from *world* coordinate system to *image* coordinate system

Review: Pinhole camera model



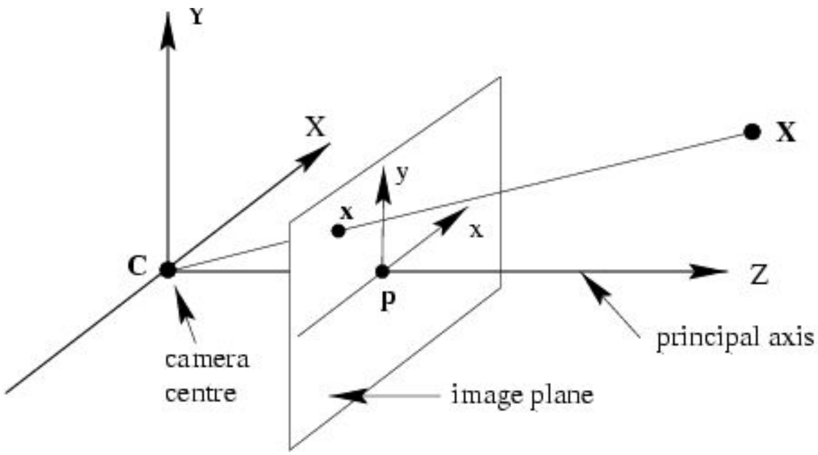


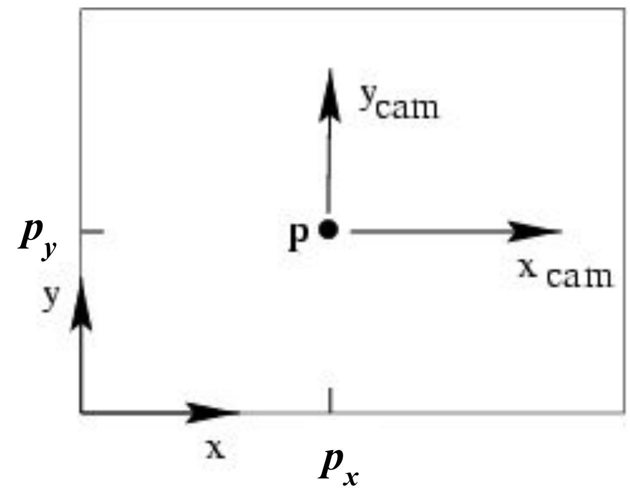
$$(X, Y, Z) \mapsto (f\,X/Z, f\,Y/Z)$$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} f\,X \\ f\,Y \\ Z \end{pmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \mathrm{P}\mathbf{X}$$

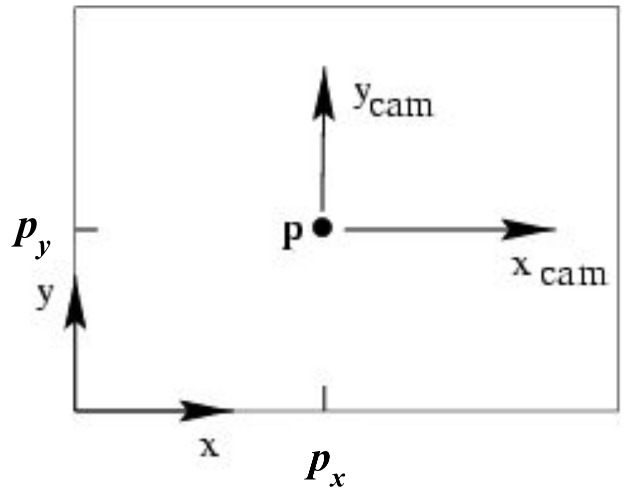
Principal point





- **Principal point (p):** point where principal axis intersects the image plane
- Normalized coordinate system: origin of the image is at the principal point
- Image coordinate system: origin is in the corner

Principal point offset

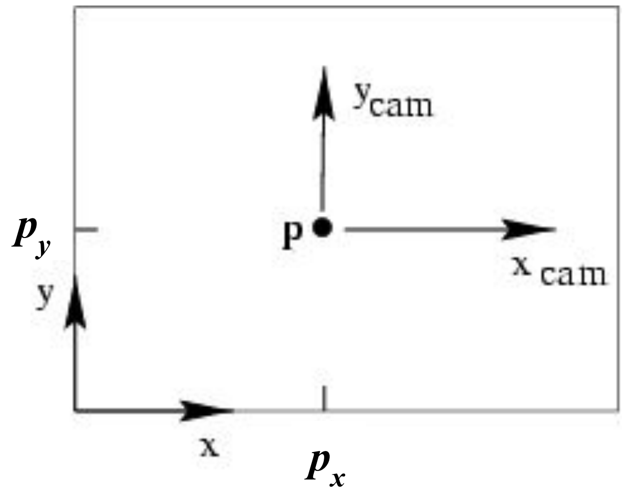


We want the principal point to map to (p_x, p_y) instead of (0,0)

$$(X, Y, Z) \mapsto (f\,X/Z + p_x, f\,Y/Z + p_y)$$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} f\,X + Z\,p_x \\ f\,Y + Z\,p_y \\ Z \end{pmatrix} = \begin{bmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Principal point offset



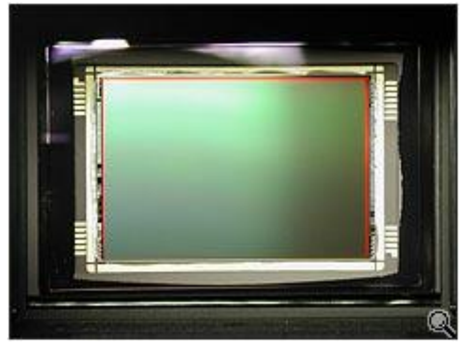
principal point: (p_x, p_y)

$$\begin{pmatrix} fX + Zp_x \\ fY + Zp_y \\ Z \end{pmatrix} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$K = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$ calibration matrix

$$\mathrm{P} = \mathrm{K}[\mathrm{I} \mid 0]$$

Pixel coordinates

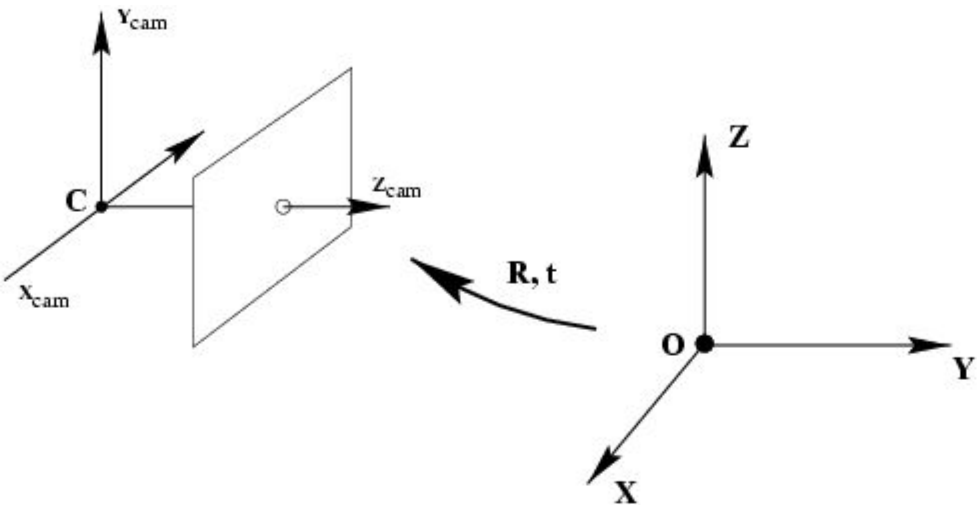


Pixel size: $\dfrac{1}{m_x} \times \dfrac{1}{m_y}$

m_x pixels per meter in horizontal direction,
m_y pixels per meter in vertical direction

$$K = \underbrace{\begin{bmatrix} m_x & & \\ & m_y & \\ & & 1 \end{bmatrix}}_{\text{pixels/m}} \underbrace{\begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}}_{\text{m}} = \underbrace{\begin{bmatrix} \alpha_x & & \beta_x \\ & \alpha_y & \beta_y \\ & & 1 \end{bmatrix}}_{\text{pixels}}$$

Camera rotation and translation



- In general, the *camera* coordinate frame will be related to the *world* coordinate frame by a rotation and a translation

- Conversion from world to camera coordinate system (in non-homogeneous coordinates):

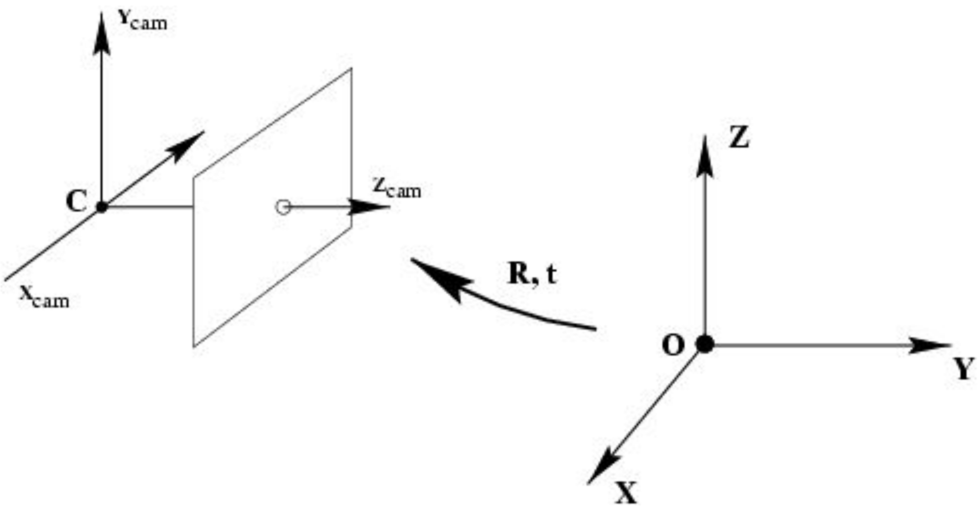
$$\widetilde{X}_{cam} = R\left(\widetilde{X} - \widetilde{C}\right)$$

coords. of point in camera frame

coords. of a point in world frame

coords. of camera center in world frame

Camera rotation and translation



$$\tilde{X}_{cam} = R\left(\tilde{X} - \tilde{C}\right)$$

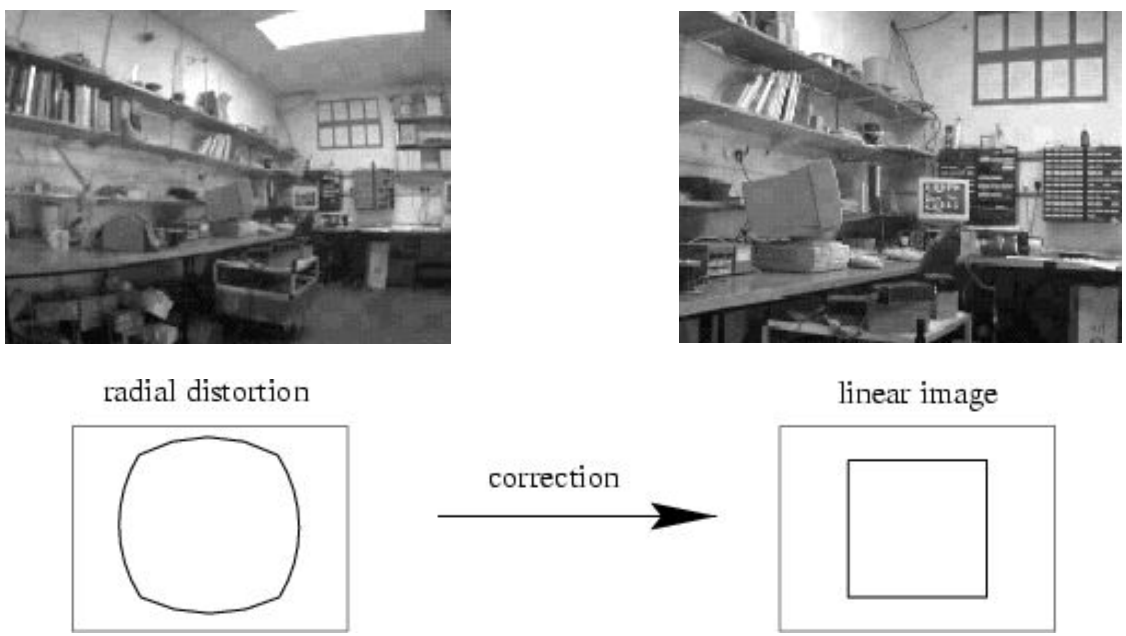
$$X_{cam} = \begin{pmatrix} \tilde{X}_{cam} \\ 1 \end{pmatrix} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \tilde{X} \\ 1 \end{pmatrix} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} X$$

$$x = K\left[I \mid 0\right] X_{cam} = K\left[R \mid -R\tilde{C}\right] X \qquad P = K\left[R \mid t\right], \qquad t = -R\tilde{C}$$

Camera parameters $\mathbf{P} = \mathbf{K}[\mathbf{R} \quad \mathbf{t}]$

- Intrinsic parameters
  - Principal point coordinates
  - Focal length
  - Pixel magnification factors
  - *Skew (non-rectangular pixels)*
  - *Radial distortion*

$$\mathbf{K} = \begin{bmatrix} m_x & & \\ & m_y & \\ & & 1 \end{bmatrix} \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} = \begin{bmatrix} \alpha_x & & \beta_x \\ & \alpha_y & \beta_y \\ & & 1 \end{bmatrix}$$



Camera parameters $\mathbf{P} = \mathbf{K}[\mathbf{R} \quad \mathbf{t}]$

- Intrinsic parameters
  - Principal point coordinates
  - Focal length
  - Pixel magnification factors
  - *Skew (non-rectangular pixels)*
  - *Radial distortion*
- Extrinsic parameters
  - Rotation and translation relative to world coordinate system

$$\mathbf{P} = \mathbf{K}[\mathbf{R} \quad -\mathbf{R}\tilde{\mathbf{C}}]$$

coords. of camera center in world frame

  - What is the projection of the camera center?

$$\mathbf{P}\mathbf{C} = \mathbf{K}[\mathbf{R} \quad -\mathbf{R}\tilde{\mathbf{C}}]\begin{bmatrix}\tilde{\mathbf{C}} \\ 1\end{bmatrix} = 0$$

The camera center is the *null space* of the projection matrix!

Camera calibration

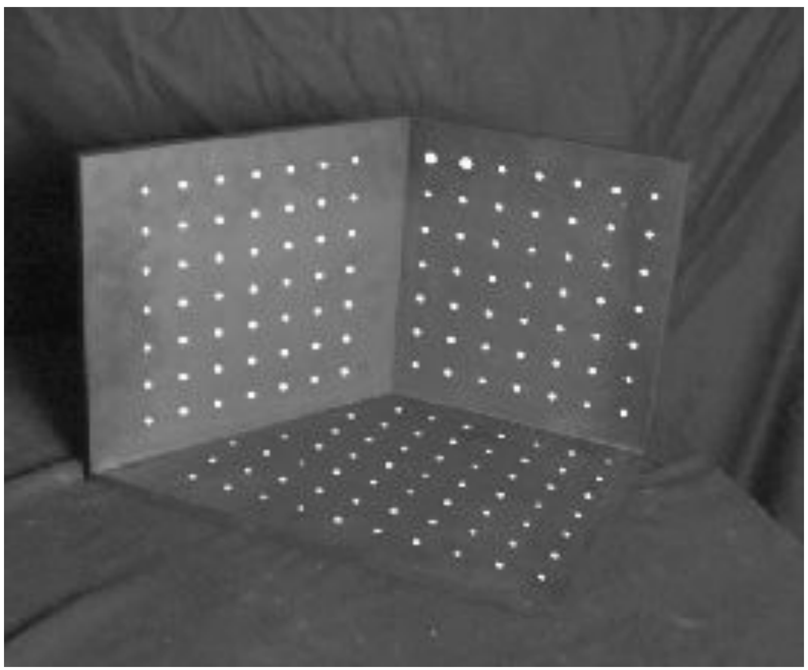
$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

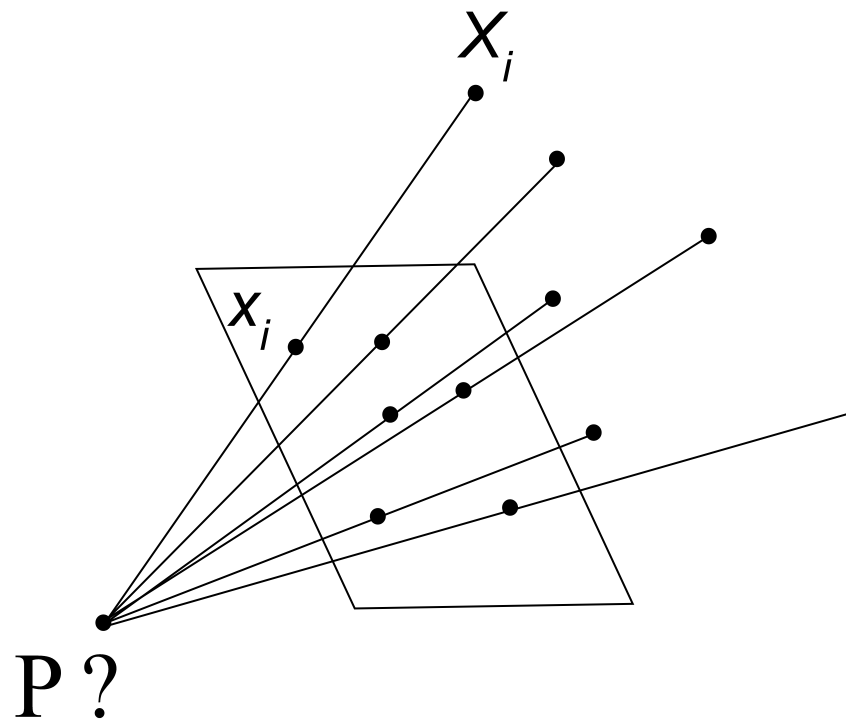
$$\begin{bmatrix}\lambda x \\ \lambda y \\ \lambda\end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \\ 1\end{bmatrix}$$

Source: D. Hoiem

Camera calibration

- Given *n* points with known 3D coordinates $\boldsymbol{X}_i$ and known image projections $\boldsymbol{x}_i$, estimate the camera parameters





Camera calibration: Linear method

$$\lambda \mathbf{x}_i = \mathbf{P}\mathbf{X}_i \qquad \mathbf{x}_i \times \mathbf{P}\mathbf{X}_i = 0 \qquad \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} \times \begin{bmatrix} \mathbf{P}_1^T \mathbf{X}_i \\ \mathbf{P}_2^T \mathbf{X}_i \\ \mathbf{P}_3^T \mathbf{X}_i \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & -\mathbf{X}_i^T & y_i \mathbf{X}_i^T \\ \mathbf{X}_i^T & 0 & -x_i \mathbf{X}_i^T \\ -y_i \mathbf{X}_i^T & x_i \mathbf{X}_i^T & 0 \end{bmatrix} \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix} = 0$$

Two linearly independent equations

Camera calibration: Linear method

$$\begin{bmatrix} 0^T & \mathbf{X}_1^T & -y_1\mathbf{X}_1^T \\ \mathbf{X}_1^T & 0^T & -x_1\mathbf{X}_1^T \\ \vdots & \vdots & \vdots \\ 0^T & \mathbf{X}_n^T & -y_n\mathbf{X}_n^T \\ \mathbf{X}_n^T & 0^T & -x_n\mathbf{X}_n^T \end{bmatrix} \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix} = 0 \qquad \mathbf{Ap} = \mathbf{0}$$

- P has 11 degrees of freedom
- One 2D/3D correspondence gives us two linearly independent equations
  - 6 correspondences needed for a minimal solution
- Homogeneous least squares: find $\mathbf{p}$ minimizing $\|\mathbf{Ap}\|^2$
  - Solution given by eigenvector of $\mathbf{A}^\mathbf{T}\mathbf{A}$ with smallest eigenvalue

Camera calibration: Linear method

$$\begin{bmatrix} 0^T & \mathbf{X}_1^T & -y_1\mathbf{X}_1^T \\ \mathbf{X}_1^T & 0^T & -x_1\mathbf{X}_1^T \\ \vdots & \vdots & \vdots \\ 0^T & \mathbf{X}_n^T & -y_n\mathbf{X}_n^T \\ \mathbf{X}_n^T & 0^T & -x_n\mathbf{X}_n^T \end{bmatrix} \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix} = 0 \qquad \mathbf{Ap} = \mathbf{0}$$

- Note: for coplanar points that satisfy $\mathbf{\Pi}^T\mathbf{X}=0$, we will get degenerate solutions $(\mathbf{\Pi},\mathbf{0},\mathbf{0})$, $(\mathbf{0},\mathbf{\Pi},\mathbf{0})$, or $(\mathbf{0},\mathbf{0},\mathbf{\Pi})$

Camera calibration: Linear method

- The linear method only estimates the entries of the projection matrix:

$$\begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

- What we ultimately want is a decomposition of this matrix into the intrinsic and extrinsic parameters:

$$\mathbf{x} = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}\mathbf{X}$$

- State-of-the-art methods use nonlinear optimization to solve for the parameter values directly

Camera calibration: Linear method

- Advantages: easy to formulate and solve
- Disadvantages
  - Doesn’t directly tell you camera parameters
  - Doesn’t model radial distortion
  - Can’t impose constraints, such as known focal length and orthogonality

- Non-linear methods are preferred
  - Define error as sum of squared distances between measured 2D points and estimated projections of 3D points
  - Minimize error using Newton’s method or other non-linear optimization

Source: D. Hoiem

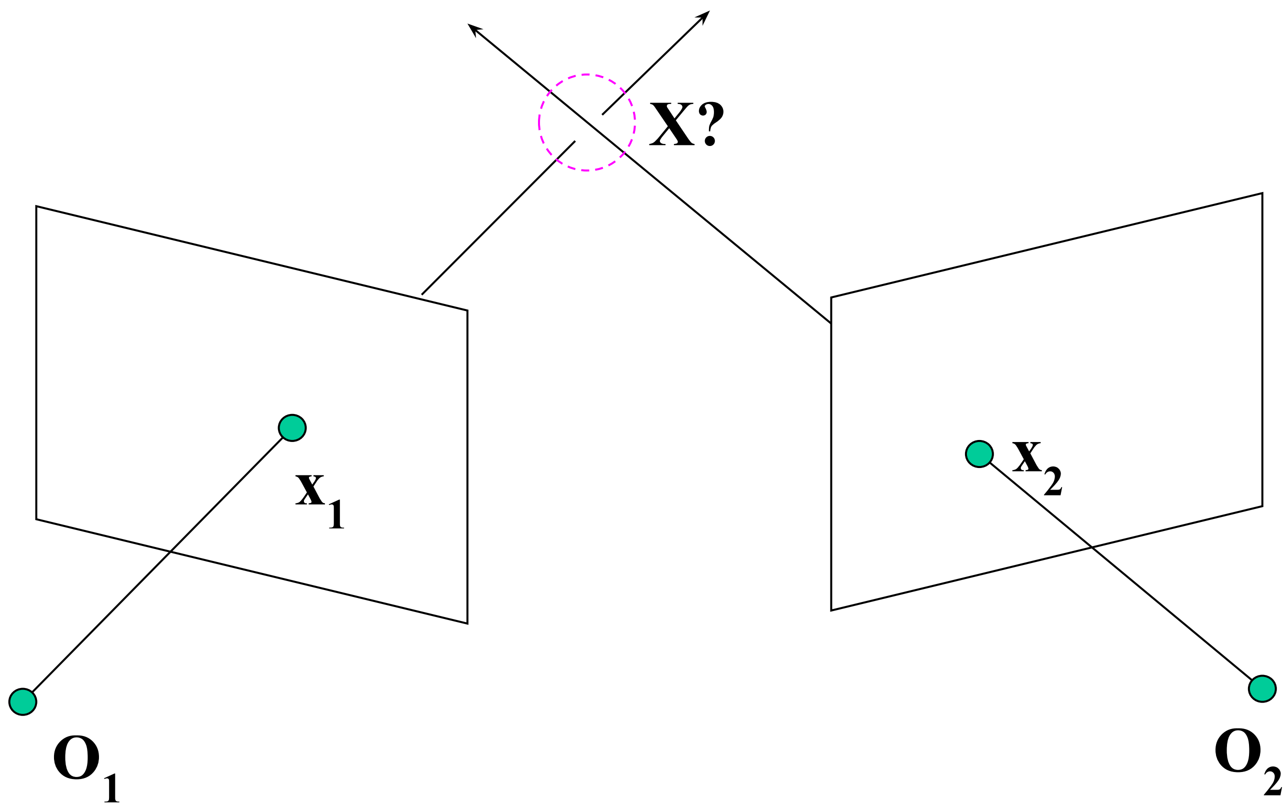
A taste of multi-view geometry: Triangulation

- Given projections of a 3D point in two or more images (with known camera matrices), find the coordinates of the point



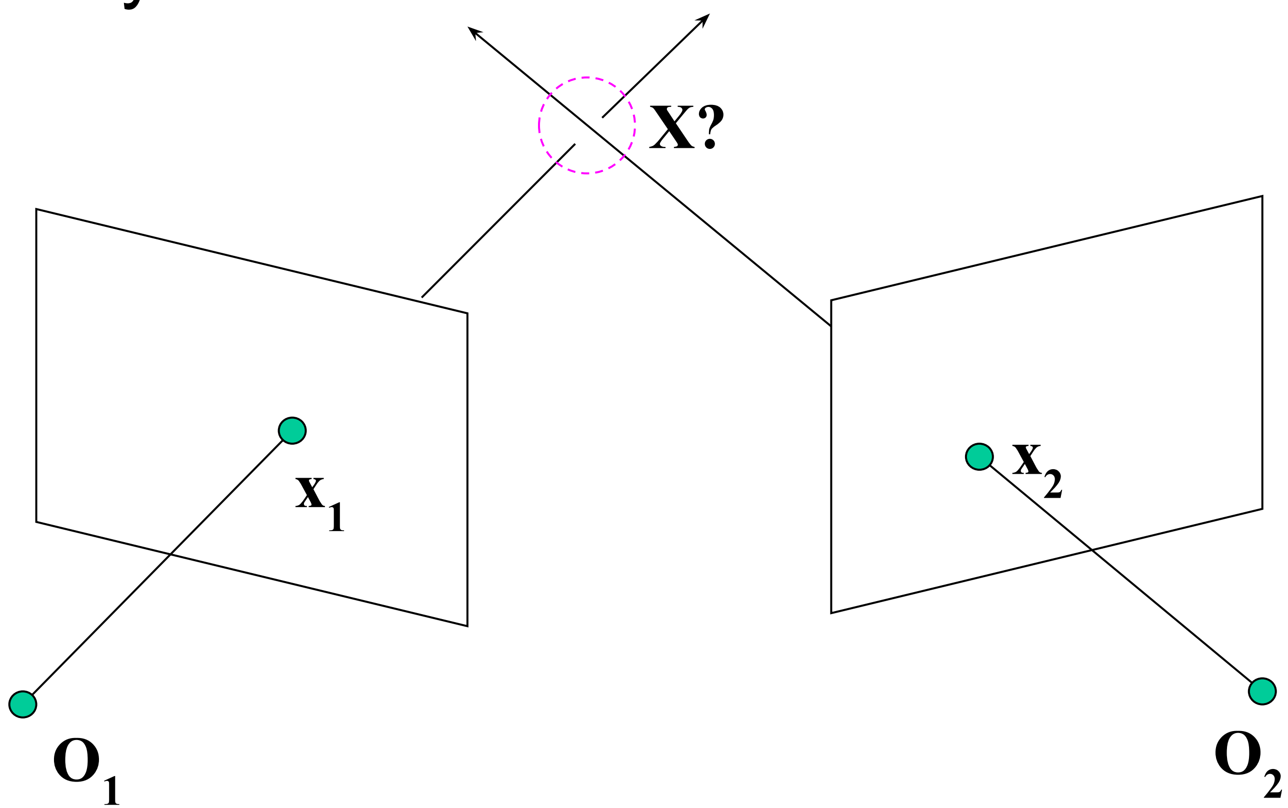
Triangulation

- Given projections of a 3D point in two or more images (with known camera matrices), find the coordinates of the point



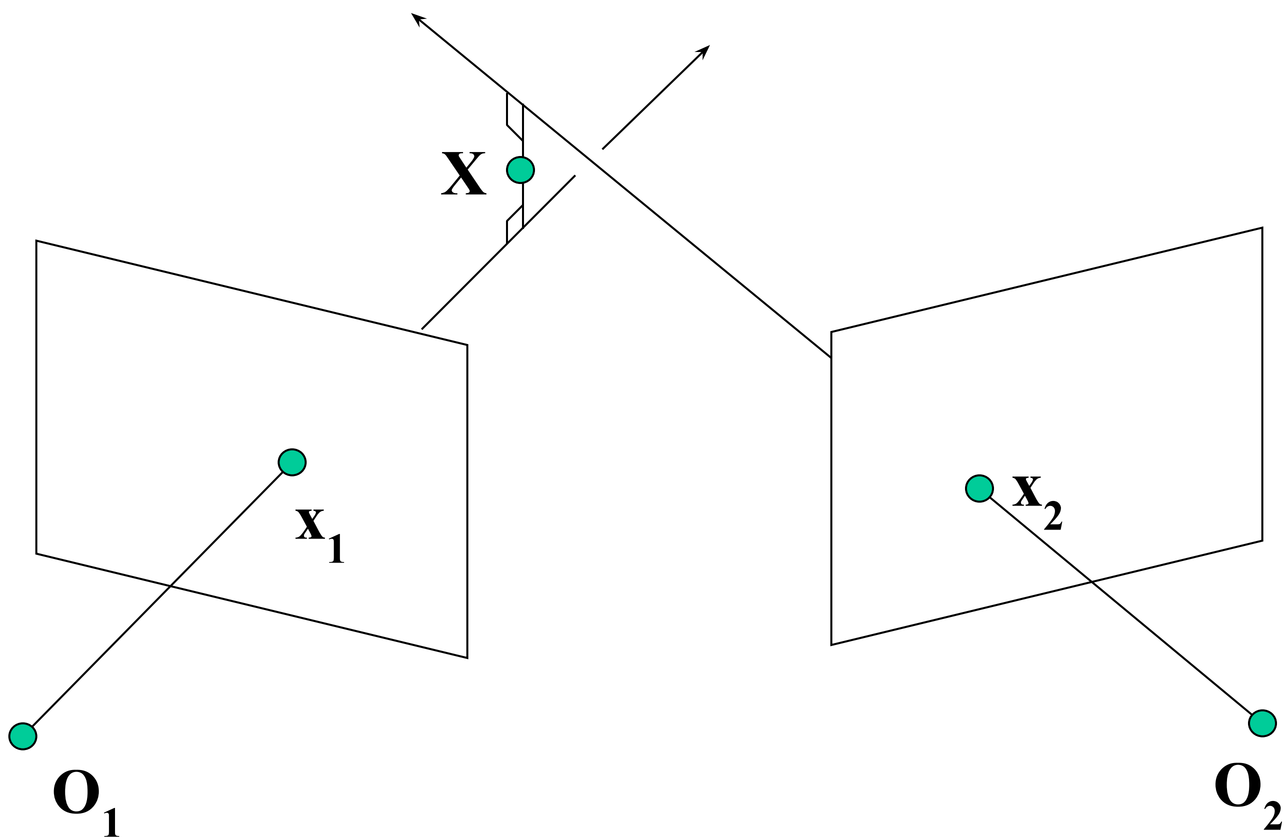
Triangulation

- We want to intersect the two visual rays corresponding to $\mathbf{x}_1$ and $\mathbf{x}_2$, but because of noise and numerical errors, they don't meet exactly



Triangulation: Geometric approach

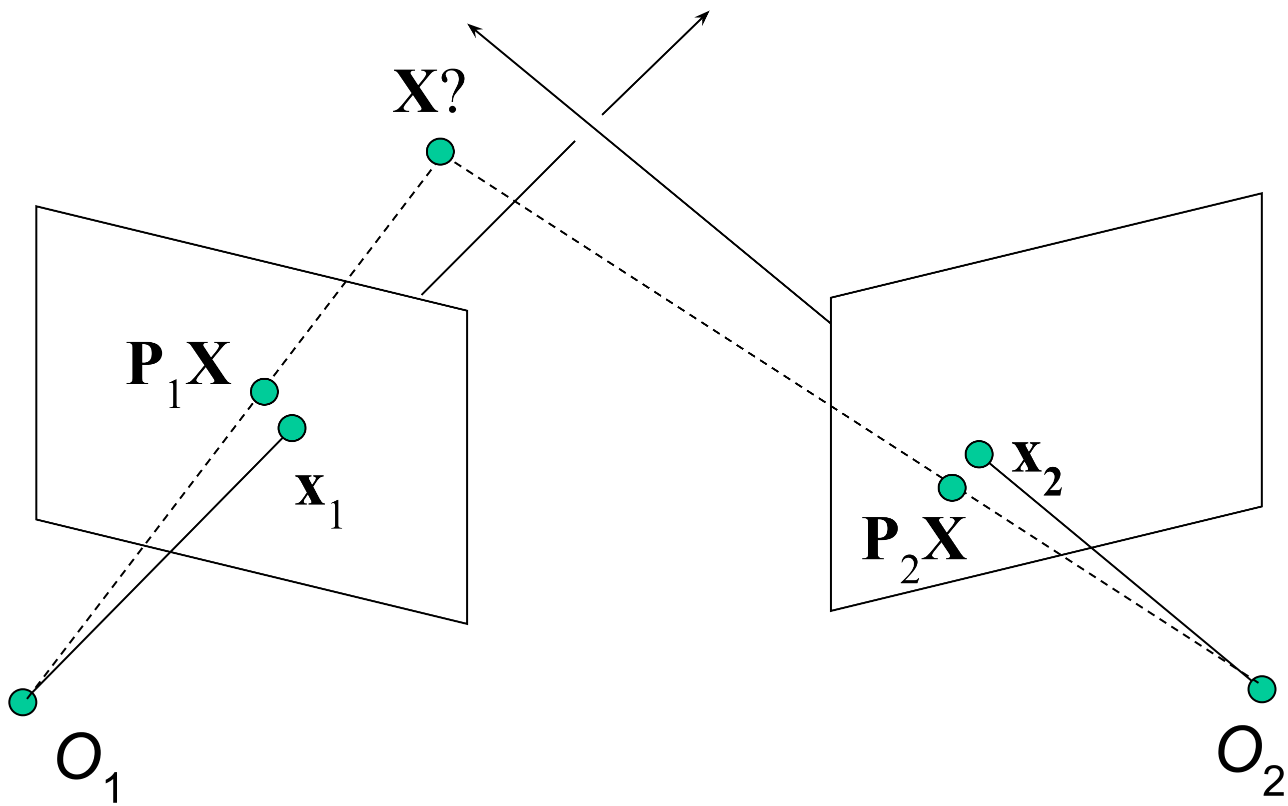
- Find shortest segment connecting the two viewing rays and let **X** be the midpoint of that segment



Triangulation: Nonlinear approach

Find X that minimizes

$$d^2(\mathbf{x_1}, \mathbf{P_1X}) + d^2(\mathbf{x_2}, \mathbf{P_2X})$$



Triangulation: Linear approach

$$\lambda_1 \mathbf{x_1} = \mathbf{P_1 X} \qquad \mathbf{x_1} \times \mathbf{P_1 X} = \mathbf{0} \qquad [\mathbf{x_{1\times}}]\mathbf{P_1 X} = \mathbf{0}$$

$$\lambda_2 \mathbf{x_2} = \mathbf{P_2 X} \qquad \mathbf{x_2} \times \mathbf{P_2 X} = \mathbf{0} \qquad [\mathbf{x_{2\times}}]\mathbf{P_2 X} = \mathbf{0}$$

Cross product as matrix multiplication:

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times]\mathbf{b}$$

Triangulation: Linear approach

$$\lambda_1 \mathbf{x_1} = \mathbf{P_1 X} \qquad \mathbf{x_1} \times \mathbf{P_1 X} = \mathbf{0} \qquad [\mathbf{x_{1\times}}]\mathbf{P_1 X} = \mathbf{0}$$

$$\lambda_2 \mathbf{x_2} = \mathbf{P_2 X} \qquad \mathbf{x_2} \times \mathbf{P_2 X} = \mathbf{0} \qquad [\mathbf{x_{2\times}}]\mathbf{P_2 X} = \mathbf{0}$$

Two independent equations each in terms of three unknown entries of **X**