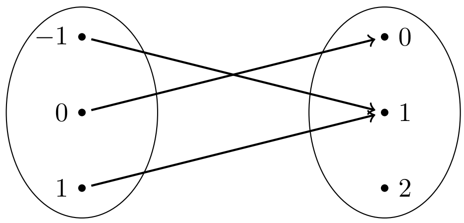
1 Definition

Function is a special case of relation, from a non empty set A to a non empty set B, that associates each member of A to a unique member of B. Symbolically, we write $f : A \to B$. We read it as "f is a function from A to B". Set A is called domain of f and set B is called co-domain of f. For example, let $A = \{1, 0, 1\}$ and $B = \{0, 1, 2\}$. Then $A \times B = \{(1, 0), (1, 1), (1, 2), (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$ Now, "$f : A \to B$ defined by $f(x) = x^2$" is the function such that $f = \{(1, 1), (0, 0), (1, 1)\}$. f can also be show diagramatically by following picture:



Every function say $f : A \to B$ satisfies the following conditions:
(i) $f \subseteq A \times B$, (ii) $\forall a \in A \Rightarrow (a, f(a)) \in f$ and (iii) $(a, b) \in f \,\&\, (a, c) \in f \Rightarrow b = c$

Example 1.1

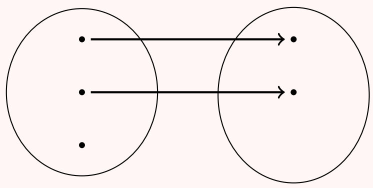
Which of the following correspondences can be called a function ?

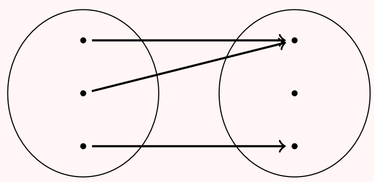
$(a) f(x) = x^3;$	$\{-1, 0, 1\} \to \{0, 1, 2, 3\}$
$(b) f(x) = \pm\sqrt{x};$	$\{0, 1, 4\} \to \{-2, -1, 0, 1, 2\}$
$(c) f(x) = \sqrt{x};$	$\{0, 1, 4\} \to \{-2, -1, 0, 1, 2\}$
$(d) f(x) = -\sqrt{x};$	$\{0, 1, 4\} \to \{-2, -1, 0, 1, 2\}$

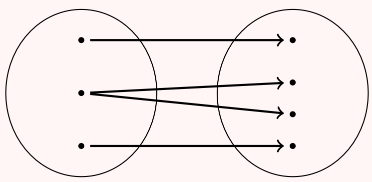
Solution. $f(x)$ in (c) & (d) are functions as definition of function is satisfied. while in case of (a) the given relation is not a function, as $f(-1) \notin$ codomain. Hence definition of function is not satisfied. While in case of (b), the given relation is not a function, as $f(1) = \pm 1$ and $f(4) = \pm 2$, *i.e.* element 1 as well as 4 in domain is related with two elements of codomain. Hence definition of function is not satisfied ☐

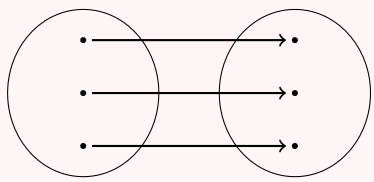
Example 1.2

Which of the following pictorial diagrams represent the function?

(a) 

(b) 

(c) 

(d) 

Solution. (b) & (d). In (a) one element of domain has no image, while in (c) one element of domain has two images in codomain □

Example 1.3

Let $g(x)$ be a function defined on $[-1, 1]$. If the area of the equilateral triangle with two of its vertices at $(0, 0)$ & $(x, g(x))$ is $\frac{3}{4}$ sq.units, then the function $g(x)$ may be-

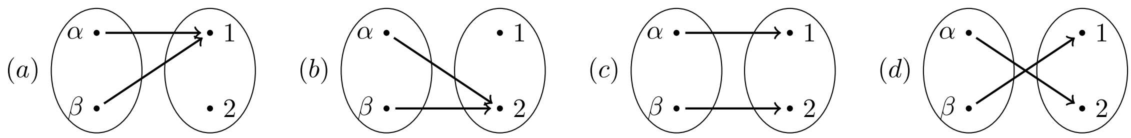
$(a)\, g(x) = \pm\sqrt{1-x^2}$ $\quad$ $(b)\, g(x) = \sqrt{1-x^2}$ $\quad$ $(c)\, g(x) = -\sqrt{1-x^2}$ $\quad$ $(d)\, g(x) = \sqrt{1+x^2}$

Solution. Answer is (b). □

Example 1.4

Represent all possible functions defined from $\{\alpha, \beta\}$ to $\{1, 2\}$.

Solution.



□

2 Domain, Co-domain & Range of a Function :

Let $f : A \to B$, then the set A is known as the domain of f & the set B is known as co-domain of f. If a member $'a'$ of A is associated to the member $'b'$ of B, then $'b'$ is called the $f-$**image** of $'a'$ and we write $b = f(a)$. Further $'a'$ is called a pre-image of $'b'$. The set $\{f(a) : \forall a \in A\}$ is called the range of f and is denoted by $f(A)$. Clearly $f(A) \subseteq B$.
Sometimes if only definition of $f(x)$ is given (domain and codomain are not mentioned), then domain is set of those values of $'x'$ for which $f(x)$ is defined, while codomain is considered to be $(-\infty, \infty)$ A function whose domain and range both are sets of real numbers is called a **real function**. Conventionally the word "*FUNCTION*" is used only as the meaning of real function.

Example 2.1

Find the domain of following functions :(i) $\quad f(x) = \sqrt{x^2 - 5}$ $\qquad (ii) \quad f(x) = \sin^{-1}(2x - 1)$

Solution. $(i) \quad f(x) = \sqrt{x^2 - 5}$ is real iff $x^2 - 5 \geq 0 \Rightarrow |x| \geq \sqrt{5} \Rightarrow x \leq \sqrt{5}$ *or* $x \geq \sqrt{5}$
$\therefore$ The domain of f is $(-\infty, -\sqrt{5}] \cup [\sqrt{5}, \infty)$ $\quad (ii) \quad -1 \leq 2x - 1 \leq 1 \quad \therefore$ domain is $x \in [0, 1]$ □

2.1 Algebraic Operations on Functions :

If f & g are real valued functions of x with domain set A and B respectively, then both f & g are defined in $A \cap B$. Now we define $(f + g)$, $(f - g)$, $(f \cdot g)$ & $\left(\frac{f}{g}\right)$ as follows:

$$\left.\begin{array}{l}(i)\ (f \pm g)(x) = f(x) \pm g(x) \\ (ii)\ (f \cdot g) = f(x) \cdot g(x)\end{array}\right\} \textit{domain in each case is } A \cap B$$

$$(iii)\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}\quad domain\ is\ \{x|x\in A\cap B such\ that\ \ g(x)\neq 0\}$$

★ For domain of $\phi(x)=\{f(x)\}^{g(x)}$, conventionally, the conditions are $f(x)>0$ and $g(x)$ must be defined.
★ For domain of $\phi(x)={}^{f(x)}C_{g(x)}$ or $\phi(x)={}^{f(x)}P_{g(x)}$ conditions of domain are $f(x)\geq g(x)$ and $f(x)\in\mathbb{N}$ and $g(x)\in\mathbb{N}$

Example 2.2

Find the domain of the following functions:
$(i)\ f(x)=\sqrt{\sin x}-\sqrt{16-x^2}\quad (ii)\ f(x)=\dfrac{3}{\sqrt{4-x^2}}\log(x^3-x)\quad (iii)\ f(x)=x^{\cos^{-1}x}$

Solution. (i) $\sqrt{\sin x}$ is real iff $\sin x\geq 0\iff x\in[2n\pi,(2n+1)\pi],\ n\in\mathbb{Z}$
$\sqrt{16-x^2}$ is real iff $16-x^2\geq 0\iff -4\leq x\leq 4$
Thus the domain of the given function is $\{x:x\in[2n\pi,(2n+1)\pi],\ n\in\mathbb{Z}\}\cap[-4,4]=[-4,-\pi]\cup[0,\pi]$
(ii) Domain of of $\sqrt{4-x^2}$ is $[-2,2]$ but here $\sqrt{4-x^2}\neq 0\ \therefore\ x\neq\pm 2\Rightarrow x\in(-2,2)$ again $\log(x^3-x)$ is defined for $x^3-x>0\ i.e\ x(x+1)(x-1)>0\quad\therefore$ The domain of $\log(x^3-x)$ is $(-1,0)\cup(1,\infty)$
Hence the domain of given function is $\{(-1,0)\cup(1,\infty)\}\cap(-2,2)=(-1,0)\cup(1,2)$.
$(iii)\ x>0\ and\ -1\leq x\leq 1\qquad\therefore$ domain is $(0,1]$ □

Assignment: Find the domain of the following functions:
$(i)\ f(x)=\dfrac{1}{\log(2-x)}+\sqrt{x+1}\qquad (ii)\ f(x)=\sqrt{1-x}-\sin^{-1}\dfrac{2x-1}{3}$
Answers: (i) $[-1,1)\cup(1,2)$ (ii) $[-1,1]$

2.2 Method of determining range:

Representing x in terms of y

Definition of the function is usually represented as y ($i.e.f(x)$ which is dependent variable) in terms of an expression of x (which is independent variable). To find range rewrite given definition so as to represent x in terms of an expression of y and thus obtain range (possible values of y). If $y=f(x)\iff x=g(y)$, then domain of $g(y)$ represents possible values of y, $i.e.$ range of $f(x)$.

Example 2.3

Find the range of $f(x)=\dfrac{x^2+x+1}{x^2+x-1}$

Solution. $f(x)=\dfrac{x^2+x+1}{x^2+x-1}$ where x^2+x+1 and x^2+x-1 have no common factor.
Let, $y=\dfrac{x^2+x+1}{x^2+x-1}\Rightarrow yx^2+yx-y=x^2+x+1\Rightarrow(y-1)x^2+(y-1)x-y-1=0$ if $y=1$ then the equation reduces to $-2=0$ which is not true. Further if $y\neq 1$ then $(y-1)x^2+(y-1)x-y-1=0$ is quadratic and has real roots if $(y-1)^2-4(y-1)(-y-1)\geq 0\quad\therefore$ if $y\leq -\dfrac{3}{5}$ *or* $y\geq 1$ but $y\neq 1$.
Thus the range of the given function is $(-\infty,-\dfrac{3}{5}]\cup(1,\infty)$ □

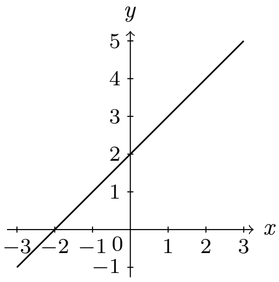
Graphical Method

Values covered on $y - axis$ by the graph of function is range

Example 2.4

Find the range of $f(x) = \dfrac{x^2-4}{x-2}$

Solution. $f(x) = \dfrac{x^2-4}{x-2} = x+2;\ x \neq 2$

$\therefore$ The graph of $f(x)$ would be: 

$\square$