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BDMO functions

01. Let $(fog)(x) = f(g(x))$ and $f^n(x) = (fof^{n-1})(x)$ where $f^1(x) = f(x)$ consider $f(x) = \cos x$ and $g(x) = \sin^{-1} x$ Find $(fog)^{2012}(x)$

02. $f : \mathbb{R} \to \mathbb{R}$ an injective function such that $f(f(x)) = f(2x+1)$. What is the value of $f(2016)$?

03. Given that $f(x, y) = f(xy, \frac{x}{y})$ where $y \neq 0$ if $f(x^4, y^4) + f(x^2, y^2) = 16$ then $f(x, y) =$?

04. $f : \{\mathbb{R} - 0\} \to \mathbb{R}$ is such a function that $f(xy) = \dfrac{f(x)}{y}$. If $f(2012) = 1$ then $f(2013) =$?

05. $A = \{a_1, a_2, a_3, a_4, ..., a_{100}\}$, $B = \{b_1, b_2, b_3, b_4, ..., b_{50}\}$, and $f : A \to B$ is a function. If $f(a_1) \leq f(a_2) \leq f(a_3) \leq .... \leq f(a_{100})$ Then how many different function f possible?

06. For an injective function $f : \mathbb{R} \to \mathbb{R}$, $f(x + f(y)) = 2012 + f(x + y)$ then $f(2013) =$?

07. $f : \mathbb{Z} \to \mathbb{Z}$, $f(n + 1) = 2f(n) - f(n - 1)$ and $f(-4) = 20, f(-6) = 40$ for any $x \in \mathbb{Z}, f(x) + f(-x) =$?

08. Given that ,$[f(x^2, y) + f(x, y^2)]^2 = 4f(x^2, y^2).f(x, y)$.Find all the values of a for which $f(x^2, a).f(a,x^2) = f(x, a).f(a, x)$ will be true.

09. A function $f : \mathbb{R} \to \mathbb{R}$ is defined in such a way that $f(x).f(y) = f(x + y)$, for $a \in n, \sum_{k=1}^{n} f(a + k) = 16(2^n - 1), f(1) = 2$ then what is the value of a?

10. $f(y) = y$ repeats y times, for example $f(3) = 333, f(5) = 55555$. If $a = f(2001) + f(2002) + f(2003) + f(2004) + ... + f(2015)$. What is the remainder upon division of a by 3?

11. $f(n) =$ sum of the squares of digits of n. $f_2(n) = f(f(n))$, $f_2(n) = f(f(f(n)))$ *etc*. Then $f_{14}(3) =$?

12. For all positive integer x, $f(f(x)) = 4x + 3$ and for one positive value of integer k, $f(5^k) = 5^k \times 2^{k-2} + 2^{k-3}$. $f(2015) =$?

13. For all positive integers $x, y; f(x) \geq 0$ and $f(xy) = f(x) + f(y)$ If the digit at the ones of x is 6, then $f(x) = 0$ and If $f(1920) = 420$ then $f(2015) =$?

14. For x, y are positive integers $f(x) = x^2 + 4$, $f(y) = x^2 + 23$ then $f(x + y) =$?

15. A function $f : \mathbb{R} \to \mathbb{R}$ is defined in such way that $f(x+2) = f(x) - \frac{1}{f(x+1)}$. $f(1) = 2, f(2) = 1007$; for $k \in R$ if $f(k) = 0$ then what is the value of k?

16. If $xf(x)f(f(x^2))f(f(f(x^2))) = 2013^2$, $\mid f(2013) \mid =$?

17. Consider a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ following the relations:
$(i) f(0) = 0$;
$(ii) f(np) = f(n)$;
$(iii) f(n) = n + f(\lfloor \frac{n}{p} \rfloor)$. Where n is not divisible by p. Here $p > 1$ is a positive integer, $\mathbb{N}_0$ is the set of all nonnegative integers and $\lfloor x \rfloor$ is the largest integer smaller or equal to x. Let, a_k be the maximum value of $f(n)$ for $0 \le n \le p^k$. Find a_k.

18. $F : \mathbb{N} \to \mathbb{N}, F(1) = 1, F(X) = F\left(\frac{X}{2016}\right)$ How many elements are there in the range of the relation?

19. $f(x) + f(-x) = x^2 + (b^2 - 5b + 6)x + 1$. What is the largest possible value of b?

20. $f(3m) = \frac{3f(m)}{3}$, $f(3m+2) = \frac{(m+2)f(m+2)}{3}$, $f(3m+1) = \frac{(m+1)f(m+1)}{3}$, $f(2016) =$?

21. A function $f : \mathbb{N} \to \mathbb{N}$ is defined such that $f(x)$ is equal to the number of divisors of x. For example, $f(6) = 4$. The least value of x, which satisfies the equation $f(x) = 2016$ can be written as $a \times b^2$, where a, has no square divisors. Find the value of b.

Other Function Problems

01. A certain function f has the properties that $f(3x) = 3f(x)$ for all positive real values of x, and that $f(x) = 1 - |x - 2|$ for $1 \le x \le 3$. Find the smallest x for which $f(x) = f(77)$.

02. How many functions $f : \{1,2,3,4,5\} \to \{1,2,3,4,5\}$ satisfy $f(f(x)) = f(x) \quad \forall x \in \{1,2,3,4,5\}$?

03. A function $f(x)$ has the property that, for all positive x, $3f(x) + 7f(\frac{2016}{x}) = 2x$.What is the value of $f(8)$?

04. If $f(x)$ is a function taking real numbers to real numbers such that for all real $x \neq 0, 1$,

$$f(x) + f(\frac{1}{1-x}) = (2x-1)^2 + f(1 - \frac{1}{x})$$

Find $f(3) =$?

Solution to the Other Function's Problems

02. Note that if $f(x) = x$ then $f(f(x)) = f(x)$. We will casework on the number of $x \in \{1, 2, 3, 4, 5\}$ such that $f(x) = x$. If there are k numbers such that $f(x) = x$ (where $0 \leq k \leq 5$), then each of those k numbers satisfies $f(f(x)) = f(x)$. For the other $5 - k$ numbers, if we choose $f(x) = c$ where c is not one of the k numbers, then $f(f(x)) = f(c) \neq c$ because only those k numbers have the property that $f(x) = x$. So $f(f(x)) \neq c$, and since $f(x) = c$ we have that $f(f(x)) \neq f(x)$. So c has to be one of the k numbers, in which case everything works because $f(f(x)) = f(c) = c = f(x)$.

So now to compute the answer: For each of $0 \leq k \leq 5$, first choose k numbers to have $f(x) = x$, and then for each of the other $5 - k$ numbers, there are k choices for their output. So the answer is $\sum\limits_{k=0}^{5} \binom{5}{k} k^{5-k}$.

03. $3f(x) + 7f(\frac{2016}{x}) = 2x$
$x = 8 \Rightarrow 3f(8) + 7f(\frac{2016}{8}) = 2 \cdot 8 = 16$
$\Rightarrow 3f(8) + 7f(252) = 16$
$x = 252 \Rightarrow 3f(252) + 7f(3) = 2 \cdot 252 = 504$

$$\begin{cases} 3\overbrace{f(8)}^{a} + 7\overbrace{f(252)}^{b} = 16 \\ 3f(252) + 7f(3) = 504 \end{cases} \Rightarrow \begin{cases} 3a + 7b = 16 \\ 7a + 3b = 504 \end{cases} \Rightarrow a = \frac{\begin{vmatrix} 16 & 7 \\ 504 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 7 \\ 7 & 3 \end{vmatrix}} = \frac{16 \cdot 3 - 7 \cdot 504}{3 \cdot 3 - 7 \cdot 7}$$

or Let $P(x)$ be the assertion $3f(x) + 7f(\dfrac{2016}{x}) = 2x$ $P(\dfrac{2016}{x}) : 3f(\dfrac{2016}{x}) + 7f(x) = \dfrac{4032}{x}$ Solving the system occurs $f(x) = \dfrac{28224 - 6x^2}{40x}$ $\therefore f(8) = 87$

04. $x = 1 - \dfrac{1}{x} : f\left(1 - \dfrac{1}{x}\right) + f(x) = \left(1 - \dfrac{2}{x}\right)^2 + f\left(\dfrac{1}{1-x}\right)$

$\longrightarrow f(x) - f\left(\dfrac{1}{1-x}\right) = \left(1 - \dfrac{2}{x}\right)^2 - f\left(1 - \dfrac{1}{x}\right)$

$f(x) + f\left(\dfrac{1}{1-x}\right) = (2x-1)^2 + f\left(1 - \dfrac{1}{x}\right)$

$$f(x) = \frac{(2x-1)^2 + \left(1 - \frac{2}{x}\right)^2}{2} \longrightarrow f(3) = \frac{113}{9}$$