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The Incenter/Excenter Lemma

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In this short note, we'll be considering the following very useful lemma.

Lemma. *Let ABC be a triangle with incenter I , A -excenter I_A , and denote by L the midpoint of arc BC . Show that L is the center of a circle through I, I_A, B, C .*

Proof. This is just angle chasing. Let $A = \angle BAC$, $B = \angle CBA$, $C = \angle ACB$, and note that A, I, L are collinear (as L is on the angle bisector). We are going to show that $LB = LI$, the other cases being similar.

First, notice that

$$\angle LBI = \angle LBC + \angle CBI = \angle LAC + \angle CBI = \angle IAC + \angle CBI = \frac{1}{2}A + \frac{1}{2}B.$$

However,

$$\angle BIL = \angle BAI + \angle ABI = \frac{1}{2}A + \frac{1}{2}B.$$

Hence, $\triangle BIL$ is isosceles. So $LB = LI$. The rest of the proof proceeds along these lines. \square

Now, let's see where this lemma has come up before...

1 Mild Embarrassments

Problem 1 (USAMO 1988). Triangle ABC has incenter I . Consider the triangle whose vertices are the circumcenters of $\triangle IAB$, $\triangle IBC$, $\triangle ICA$. Show that its circumcenter coincides with the circumcenter of $\triangle ABC$.

Problem 2 (CGMO 2012). The incircle of a triangle ABC is tangent to sides AB and AC at D and E respectively, and O is the circumcenter of triangle BCI . Prove that $\angle ODB = \angle OEC$.

Problem 3 (CHMMC Spring 2012). In triangle ABC , the angle bisector of $\angle A$ meets the perpendicular bisector of \overline{BC} at point D . The angle bisector of $\angle B$ meets the perpendicular bisector of \overline{AC} at point E . Let F be the intersection of the perpendicular bisectors of \overline{BC} and \overline{AC} . Find DF , given that $\angle ADF = 5^\circ$, $\angle BEF = 10^\circ$ and $AC = 3$.

Problem 4 (Nine-Point Circle). Let ABC be an acute triangle with orthocenter H . Let D, E, F be the feet of the altitudes from A, B, C to the opposite sides. Show that the midpoint of \overline{AH} lies on the circumcircle of $\triangle DEF$.

2 Some Short-Answer Problems

Problem 5 (HMMT 2011). Let $ABCD$ be a cyclic quadrilateral, and suppose that $BC = CD = 2$. Let I be the incenter of triangle ABD . If $AI = 2$ as well, find the minimum value of the length of diagonal BD .

Problem 6 (HMMT 2013). Let triangle ABC satisfy $2BC = AB + AC$ and have incenter I and circumcircle ω . Let D be the intersection of AI and ω (with A, D distinct). Prove that I is the midpoint of AD .

Problem 7 (Online Math Open 2014/F19). In triangle ABC , $AB = 3$, $AC = 5$, and $BC = 7$. Let E be the reflection of A over \overline{BC} , and let line BE meet the circumcircle of ABC again at D . Let I be the incenter of $\triangle ABD$. Compute $\cos \angle AEI$.

Problem 8 (NIMO 2012). Let $ABXC$ be a cyclic quadrilateral such that $\angle XAB = \angle XAC$. Let I be the incenter of triangle ABC and by D the foot of I on \overline{BC} . Given $AI = 25$, $ID = 7$, and $BC = 14$, find XI .

3 Intermediate Examples

Problem 9. Let ABC be an acute triangle such that $\angle A = 60^\circ$. Prove that $IH = IO$, where I, H, O are the incenter, orthocenter, and circumcenter.

Problem 10 (IMO 2006). Let ABC be a triangle with incenter I . A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

Problem 11 (APMO 2007). In triangle ABC , we have $AB > AC$ and $\angle A = 60^\circ$. Let I and H denote the incenter and orthocenter of the triangle. Show that $2\angle AHI = 3\angle B$.

Problem 12 (ELMO 2013, Evan Chen). Triangle ABC is inscribed in circle ω . A circle with chord BC intersects segments AB and AC again at S and R , respectively. Segments BR and CS meet at L , and rays LR and LS intersect ω at D and E , respectively. The internal angle bisector of $\angle BDE$ meets line ER at K . Prove that if $BE = BR$, then $\angle ELK = \frac{1}{2}\angle BCD$.

Problem 13 (Online Math Open 2012/F27). Let ABC be a triangle with circumcircle ω . Let the bisector of $\angle ABC$ meet segment AC at D and circle ω at $M \neq B$. The circumcircle of $\triangle BDC$ meets line AB at $E \neq B$, and CE meets ω at $P \neq C$. The bisector of $\angle PMC$ meets segment AC at $Q \neq C$. Given that $PQ = MC$, determine the degree measure of $\angle ABC$.

4 Harder Tasks

Problem 14 (Iran 2001). Let ABC be a triangle with incenter I and A -excenter I_A . Let M be the midpoint of arc BC not containing A , and let N denote the midpoint of arc MBA . Lines NI and NI_A intersect the circumcircle of ABC at S and T . Prove that the lines ST , BC and AI are concurrent.

Problem 15 (Online Math Open 2014/F26). Let ABC be a triangle with $AB = 26$, $AC = 28$, $BC = 30$. Let X, Y, Z be the midpoints of arcs BC, CA, AB (not containing the opposite vertices) respectively on the circumcircle of ABC . Let P be the midpoint of arc BC containing point A . Suppose lines BP and XZ meet at M , while lines CP and XY meet at N . Find the square of the distance from X to MN .

Problem 16 (Euler). Let ABC be a triangle with incenter I and circumcenter O . Show that $IO^2 = R(R - 2r)$, where R and r are the circumradius and inradius of $\triangle ABC$, respectively.

Problem 17 (IMO 2010). Let I be the incenter of a triangle ABC and let Γ be its circumcircle. Let the line AI intersect Γ again at D . Let E be a point on the arc BDC and F a point on the side BC such that

$$\angle BAF = \angle CAE < \frac{1}{2}\angle BAC.$$

Finally, let G be the midpoint of \overline{IF} . Prove that \overline{DG} and \overline{EI} intersect on Γ .

5 Bonus Problems

Problem 18 (Russia 2014). Let ABC be a triangle with $AB > BC$ and circumcircle Ω . Points M, N lie on the sides AB, BC respectively, such that $AM = CN$. Lines MN and AC meet at K . Let P be the incenter of the triangle AMK , and let Q be the K -excenter of the triangle CNK . If R is midpoint of arc ABC of Ω then prove that $RP = RQ$.

Problem 19. Let ABC be a triangle with circumcircle Ω , and let D be any point on \overline{BC} . We draw a *curvilinear incircle* tangent to \overline{AD} at L , to \overline{BC} at K and internally tangent to Ω . Show that the incenter of triangle ABC lies on \overline{KL} .

6 Hints to the Problems

1. Tautological.
2. Who is O ?
3. Point F is the circumcenter of $\triangle ABC$. Who are D and E ?
4. What is the incenter of $\triangle DEF$? What is the D -excenter?
5. Show that $AC = 4$.
6. Apply Ptolemy's Theorem.
7. Who is C ? Erase E .
8. Apply Ptolemy's Theorem.
9. Since $\angle BHC = \angle BIC = \angle BOC = 120^\circ$, points H and O now lie on the magic circle too. So $IH = IO$ is just an equality of certain arcs.
10. Use the angle condition to show that P also lies on the magic circle.
11. The point H lies on the magic circle. So $\angle IHC = \angle 180^\circ - \angle IBC$.
12. You need to do quite a bit of angle chasing. Show that R is the incenter of $\triangle CDE$. Who is B ?
13. Both M and P are arc midpoints. (Why?)
14. First show that S, T, I, I_A are concyclic, say by $NI \cdot NS = NM^2 = NI_A \cdot NT$.
15. Add the incenter I . Line MN is a tangent.
16. Add in point L , the midpoint of arc BC . By Power of a Point, it's equivalent to prove $AI \cdot IL = 2Rr$, which can be done with similar triangles.
17. Take a homothety with ratio 2 at I . This sends G to F and D to the A -excenter.
18. Construct arc midpoints on the circumcircles of both $\triangle AMJ$ and $\triangle CNK$. Use spiral similarity at R .
19. Let the tangency point to Ω be T , let M be the midpoint of arc BC , and let lines KL and AM meet at I . Show that M, K, T are collinear. Show that $ALIT$ is cyclic. Prove that $MI^2 = MK \cdot MT = MC^2 = MI^2$.