Divisbility

1. Let $m \geq 4$ be an integer that is not prime. Prove that there exists an integer n with $2 \leq n \leq m-1$, so that m divides $n^n - n$.

2. Prove that there exist infinitely many integers n which satisfy $2017^2 | 1^n + 2^n + ... + 2017^n$.

3. Find all triples (a, b, c) of positive integers such that if n is not divisible by any prime less than 2014, then $n + c$ divides $a^n + b^n + n$.

4. Let k be a positive integer. Define n_k to be the number with decimal representation 70...01 where there are exactly k zeroes. Prove the following assertions:

   a) None of the numbers n_k is divisible by 13.
   b) Infinitely many of the numbers n_k are divisible by 17.

5. Let α be a positive real number that is not an integer and let $n = \left\lfloor \dfrac{1}{\alpha - \lfloor \alpha \rfloor} \right\rfloor$ Prove that $\lfloor (n+1)\,\alpha \rfloor - 1$ is divisible by $n + 1$

6. Determine all pairs of positive integers (a, b) such that $\dfrac{a^2}{2ab^2 - b^3 + 1}$ is a positive integer.

7. Let p- a prime number. Prove: $p^2 | \dbinom{2p}{p} - 2$. and $p^2 | \dbinom{np}{p} - n$, where n is a natural number, $n > 1$.