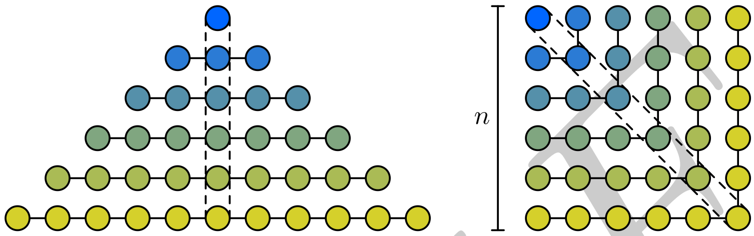
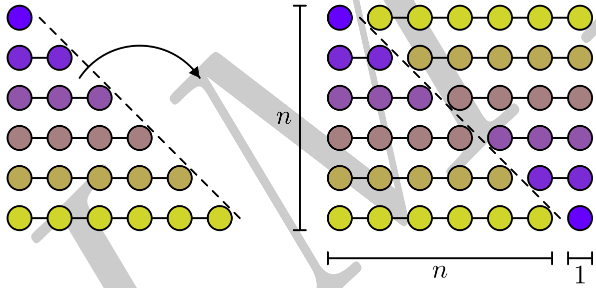
Proofs without words

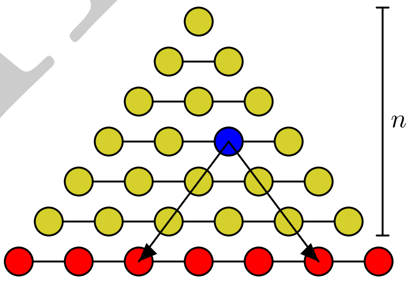
The following demonstrate proofs of various identities and theorems using pictures, inspired from this gallery.



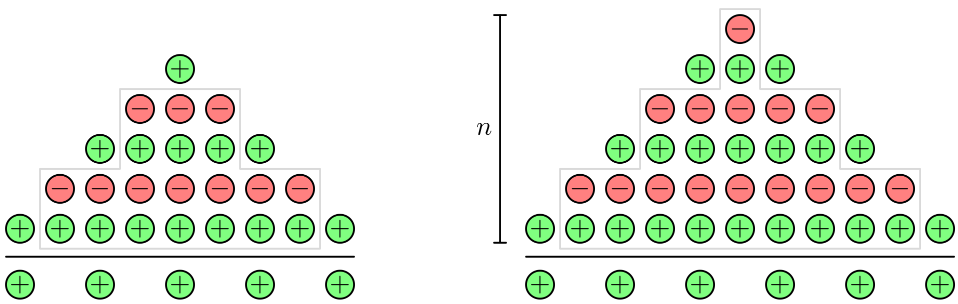
The sum of the first n odd natural numbers is n^2.



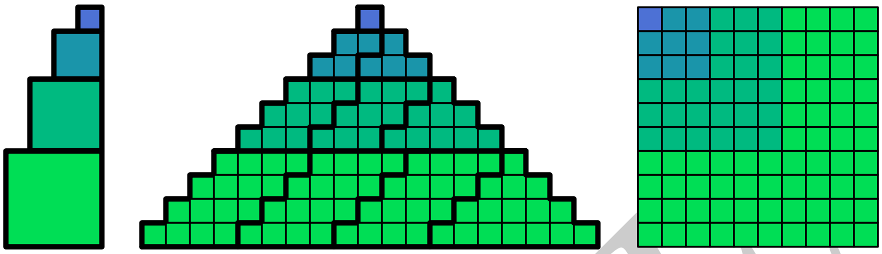
The sum of the first n positive integers is $\frac{n(n+1)}{2}$.



The sum of the first n positive integers is $\binom{n+1}{2}$.



The alternating sum of the first n odd natural numbers is $\sum_{k=1}^{n}(-1)^{n-k}(2k-1) = n$.



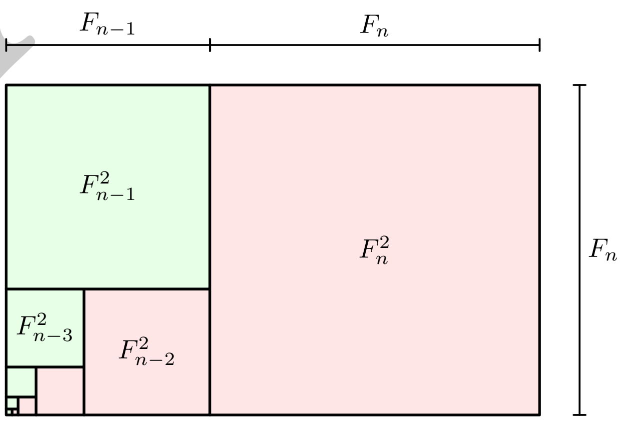
Nichomauss' Theorem: n^3 can be written as the sum of n consecutive integers, and consequently that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$.

Here, we use the same re-arrangement as the first proof on this page (the sum of first odd integers is a square). Here's another re-arrangement to see this:

This also suggests the following alternative proof:

An animated version of this proof can be found in this gallery.

The nth pentagonal number is the sum of n and three times the $n-1$th triangular number. If P_n denotes the nth pentagonal number, then $P_n = 3T_{n-1} + n$.



The identity $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n \cdot F_{n+1}$, where F_i is the ith Fibonacci number.

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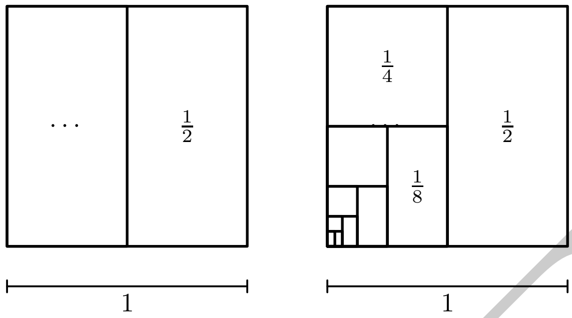
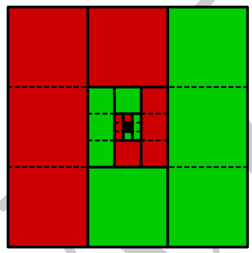
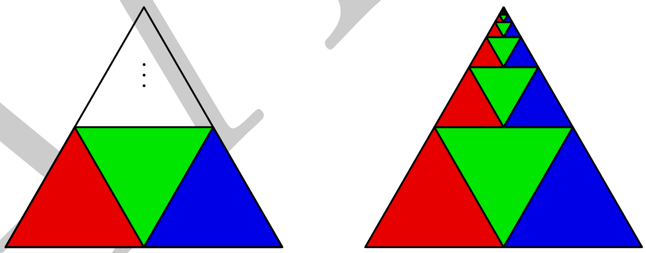


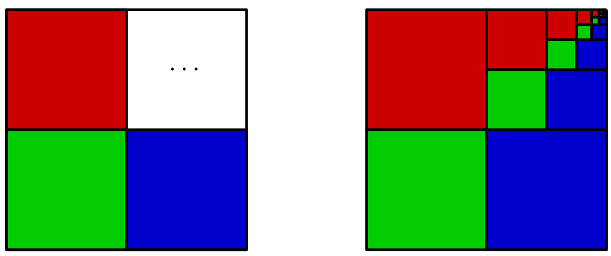
The infinite geometric series $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 1$.



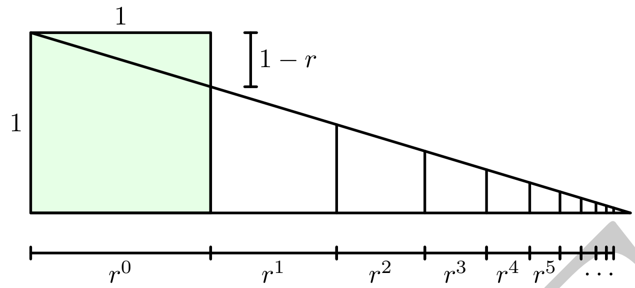
The infinite geometric series $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots = \frac{1}{2}$.



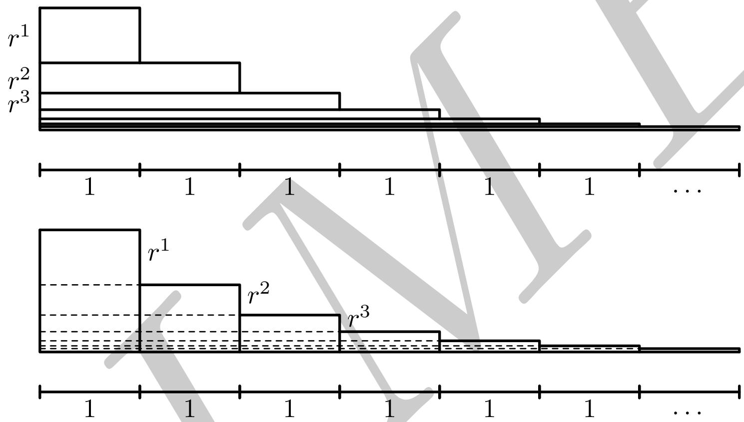
The infinite geometric series $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{1}{3}$.



Another proof of the identity $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{1}{3}$.



The infinite geometric series $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.



The arithmetic-geometric series $\sum_{n=1}^{\infty} nr^n = \sum_{n=1}^{\infty} \sum_{i=n}^{\infty} r^i = \sum_{n=1}^{\infty} \frac{r^n}{1-r} = \frac{r}{(1-r)^2}$, also known as Gabriel's staircase.[2]

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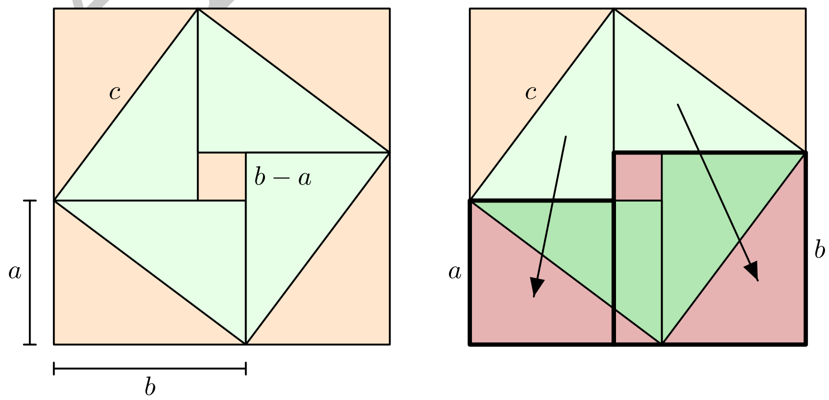
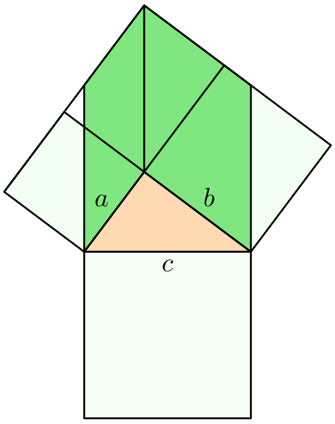
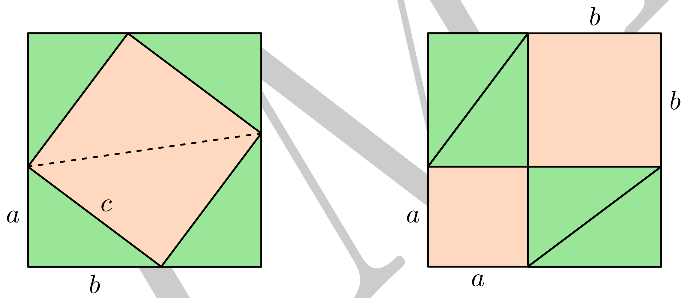


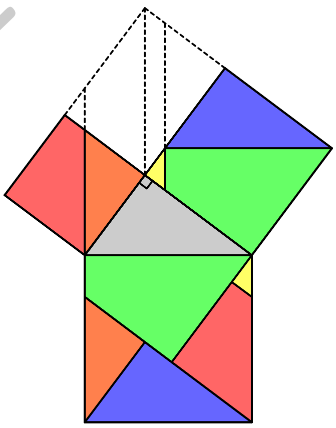
The Pythagorean Theorem (first of many proofs): the left diagram shows that $c^2 = 4 \cdot \frac{ab}{2} + (b-a)^2 = a^2 + b^2$, and the right diagram shows a second proof by re-arranging the first diagram (the area of the shaded part is equal to $a^2 + b^2$, but it is also the re-arranged version of the oblique square, which has area c^2).[3]



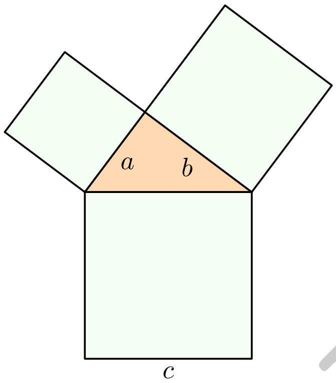
Another proof of the Pythagorean Theorem (animated version).



Another proof of the Pythagorean Theorem; the left-hand diagram suggests the identity $c^2 = (a+b)^2 - 4 \cdot \frac{ab}{2} = a^2 + b^2$, and the right-hand diagram offers another re-arrangement proof.



A dissection proof of the Pythagorean Theorem. (Cut-the-knot)

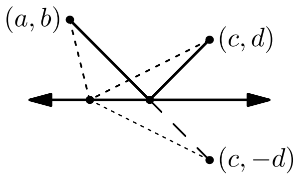


COMING: The last proof of the Pythagorean Theorem we shall present on this page, this one by dissection.

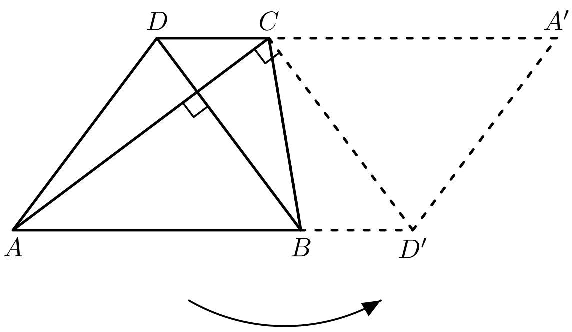
The area of a triangle is given by $A = \frac{1}{2} \cdot r \cdot (a+b+c) = rs$, where r is the inradius and s is the semiperimeter.[10] (Comment: we do not need to re-arrange the triangles to a trapezoid to see this, but this re-arrangement works due to alternate interior angles/angle bisector properties of the incenter.)

The area of a parallelogram with adjacent side vectors $\overrightarrow{(a,b)}, \overrightarrow{(c,d)}$ is given by $\overrightarrow{(a,b)} \times \overrightarrow{(c,d)} = ad - bc$.

The area of a dodecagon is $3R^2$, where R is the circumradius.



The smallest distance necessary to travel between (a,b), the x-axis, and then (c,d) for $b, d > 0$ is given by $\sqrt{(a-c)^2 + (b+d)^2}$.[4]



In trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$, then $\overline{AC} \perp \overline{BD} \iff AC^2 + BD^2 = (AB + CD)^2$.

Varignon's theorem: the area of the outer parallelogram is twice the area of the quadrilateral and four times the area of the midpoint parallelogram, so the midpoint parallelogram of a (convex) quadrilateral has area 1/2 of the quadrilateral.

Proof for Volume of a Cone: http://www.mathematische-basteleien.de/wuerfel16.gif

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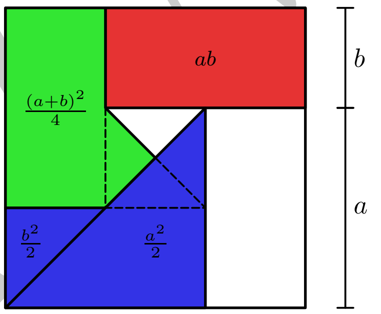
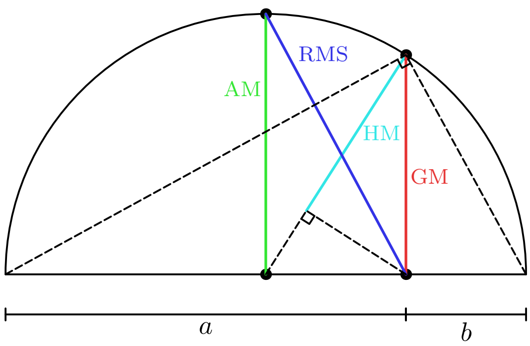
$\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4}$ from $\begin{cases} \sin^2 x + \cos^2 x = 1 \\ \sin x = \cos(\pi/2 - x) \end{cases}$. (Source)

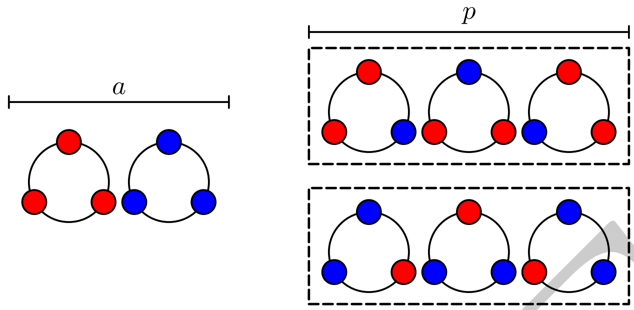
$\alpha > 0 \implies \int_0^1 \left(x^\alpha + x^{1/\alpha} \right) dx = 1$. (Source)



The Root-Mean Square-Arithmetic Mean-Geometric Mean inequality, $ab \leq \frac{(a+b)^2}{4} \leq \frac{a^2+b^2}{2}$.



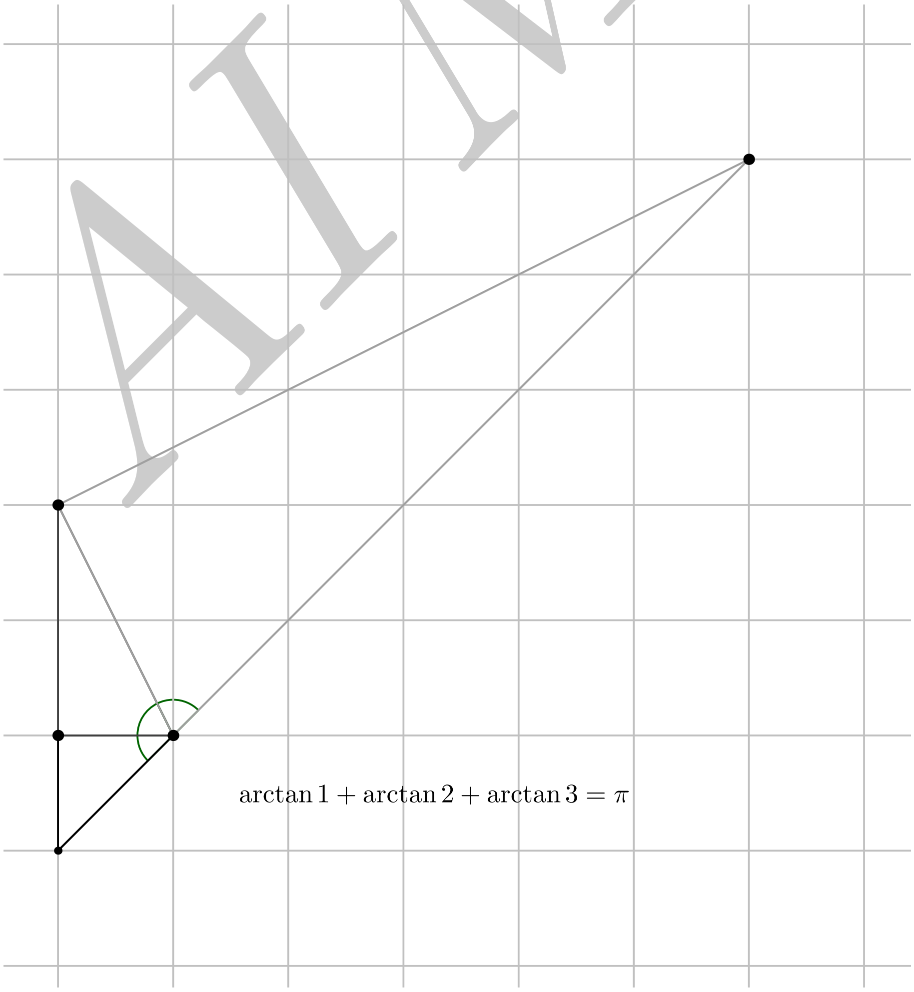
The Root-Mean Square-Arithmetic Mean-Geometric Mean-Harmonic mean Inequality.[5]



Fermat's Little Theorem: $a^p \equiv a \pmod{p}$ for $\gcd(a, p) = 1$ (above $a = 2, p = 3$).

There exists a homeomorphism, the stereographic projection, between the punctured hypersphere $S^n \setminus \{(1, \underbrace{0, \ldots, 0}_{n-1 \text{ zeroes}})\}$ and $\mathbb{R}^n$ for $n = 1, 2$.

Sum of arctangents formula:



Back to Top References MathOverflow Wolfram MathWorld Attributed to the Chinese text Zhou Bi Suan Jing. This is more of a proof without words of the AM-GM inequality $\frac{a+b}{2} \geq \sqrt{ab}$; though the lengths of the segments labeled RMS and HM can easily be verified to have values of $\sqrt{\frac{a^2+b^2}{2}}$, $\frac{2}{\frac{1}{a}+\frac{1}{b}}$, respectively, it might not be obvious from the diagram. It still serves as a useful graphical demonstration of the inequality.