BDMO Number Theory

01. Let $\underbrace{444\cdots\cdots\cdots4}_{2018}\underbrace{888\cdots\cdots\cdots8}_{2017}9$ The number has 2018 digits of 4, followed by 2017 digits of 8 and one digit of 9. Find the sum of the digits of square root of this number.

02. $f:\mathbb{R}\to\mathbb{R}$ an injective function such that $f(f(x))=f(2x+1)$. What is the value of $f(2016)$?

03. Given that $f(x,y)=f(xy,\frac{x}{y})$ where $y\neq 0$ if $f(x^4,y^4)+f(x^2,y^2)=16$ then $f(x,y)=$?

04. $f:\{\mathbb{R}-0\}\to\mathbb{R}$ is such a function that $f(xy)=\dfrac{f(x)}{y}$. If $f(2012)=1$ then $f(2013)=$?

05. $A=\{a_1,a_2,a_3,a_4,.........,a_{100}\}$, $B=\{b_1,b_2,b_3,b_4,.........,b_{50}\}$, and $f:A\to B$ is a function. If $f(a_1)\leq f(a_2)\leq f(a_3)\leq.........\leq f(a_{100})$ Then how many different function f possible?

06. For an injective function $f:\mathbb{R}\to\mathbb{R}$, $f(x+f(y))=2012+f(x+y)$ then $f(2013)=$?

07. $f:\mathbb{Z}\to\mathbb{Z}$, $f(n+1)=2f(n)-f(n-1)$ and $f(-4)=20, f(-6)=40$ for any $x\in\mathbb{Z}, f(x)+f(-x)=$?

08. Given that ,$[f(x^2,y)+f(x,y^2)]^2=4f(x^2,y^2).f(x,y)$.Find all the values of a for which $f(x^2,a).f(a,x^2)=f(x,a).f(a,x)$ will be true.

09. A function $f:\mathbb{R}\to\mathbb{R}$ is defined in such a way that $f(x).f(y)=f(x+y)$, for $a\in n$, $\sum\limits_{k=1}^{n} f(a+k)=16(2^n-1), f(1)=2$ then what is the value of a?

10. $f(y)=y$ repeats y times, for example $f(3)=333, f(5)=55555$. If $a=f(2001)+f(2002)+f(2003)+f(2004)+.........+f(2015)$. What is the remainder upon division of a by 3?

11. $f(n)=$ sum of the squares of digits of n. $f_2(n)=f(f(n))$, $f_2(n)=f(f(f(n)))$ *etc.* Then $f_{14}(3)=$?

12. For all positive integer x, $f(f(x))=4x+3$ and for one positive value of integer k, $f(5^k)=5^k\times 2^{k-2}+2^{k-3}$. $f(2015)=$?

13. For all positive integers x,y; $f(x)\geq 0$ and $f(xy)=f(x)+f(y)$ If the digit at the one's of x is 6, then $f(x)=0$ and If $f(1920)=420$ then $f(2015)=$?

14. For x,y are positive integers $f(x)=x^2+4$, $f(y)=x^2+23$ then $f(x+y)=$?

15. A function $f:\mathbb{R}\to\mathbb{R}$ is defined in such way that $f(x+2)=f(x)-\dfrac{1}{f(x+1)}$. $f(1)=2, f(2)=1007$; for $k\in R$ if $f(k)=0$ then what is the value of k?

16. If $xf(x)f(f(x^2))f(f(f(x^2)))=2013^2$, $\mid f(2013)\mid=?$

17. Consider a function $f:\mathbb{N}_0\to\mathbb{N}_0$ following the relations:
$(i) f(0)=0$;
$(ii) f(np)=f(n)$;
$(iii) f(n)=n+f(\lfloor\frac{n}{p}\rfloor)$. Where n is not divisible by p. Here $p>1$ is a positive integer, $\mathbb{N}_0$ is the set of all nonnegative integers and $\lfloor x\rfloor$ is the largest integer smaller or equal to x. Let, a_k be the maximum value of $f(n)$ for $0\le n\le p^k$. Find a_k.

18. $F:\mathbb{N}\to\mathbb{N}, F(1)=1, F(X)=F\left(\dfrac{X}{2016}\right)$ How many elements are there in the range of the relation?

19. $f(x)+f(-x)=x^2+(b^2-5b+6)x+1$. What is the largest possible value of b?

20. $f(3m)=\dfrac{3f(m)}{3}$, $f(3m+2)=\dfrac{(m+2)f(m+2)}{3}$, $f(3m+1)=\dfrac{(m+1)f(m+1)}{3}$, $f(2016)=?$

21. A function $f:\mathbb{N}\to\mathbb{N}$ is defined such that $f(x)$ is equal to the number of divisors of x. For example, $f(6)=4$. The least value of x, which satisfies the equation $f(x)=2016$ can be written as $a\times b^2$, where a, has no square divisors. Find the value of b.

Other Function Problems

01. A certain function f has the properties that $f(3x)=3f(x)$ for all positive real values of x, and that $f(x)=1-|x-2|$ for $1\le x\le 3$. Find the smallest x for which $f(x)=f(77)$.

02. How many functions $f:\{1,2,3,4,5\}\to\{1,2,3,4,5\}$ satisfy $f(f(x))=f(x)\quad\forall x\in\{1,2,3,4,5\}$?

03. A function $f(x)$ has the property that, for all positive x, $3f(x)+7f(\frac{2016}{x})=2x$.What is the value of $f(8)$?

04. If $f(x)$ is a function taking real numbers to real numbers such that for all real $x\neq 0,1$,

$$f(x)+f(\frac{1}{1-x})=(2x-1)^2+f(1-\frac{1}{x})$$

Find $f(3)=?$

Solution to the Other Function's Problems

02. Note that if $f(x) = x$ then $f(f(x)) = f(x)$. We will casework on the number of $x \in \{1, 2, 3, 4, 5\}$ such that $f(x) = x$. If there are k numbers such that $f(x) = x$ (where $0 \le k \le 5$), then each of those k numbers satisfies $f(f(x)) = f(x)$. For the other $5-k$ numbers, if we choose $f(x) = c$ where c is not one of the k numbers, then $f(f(x)) = f(c) \neq c$ because only those k numbers have the property that $f(x) = x$. So $f(f(x)) \neq c$, and since $f(x) = c$ we have that $f(f(x)) \neq f(x)$. So c has to be one of the k numbers, in which case everything works because $f(f(x)) = f(c) = c = f(x)$.

So now to compute the answer: For each of $0 \le k \le 5$, first choose k numbers to have $f(x) = x$, and then for each of the other $5-k$ numbers, there are k choices for their output. So the answer is $\sum\limits_{k=0}^{5} \binom{5}{k} k^{5-k}$.

03. $3f(x) + 7f(\frac{2016}{x}) = 2x$
$x = 8 \Rightarrow 3f(8) + 7f(\frac{2016}{8}) = 2 \cdot 8 = 16$
$\Rightarrow 3f(8) + 7f(252) = 16$
$x = 252 \Rightarrow 3f(252) + 7f(3) = 2 \cdot 252 = 504$

$$\begin{cases} 3\overbrace{f(8)}^{a}+7\overbrace{f(252)}^{b}=16 \\ 3f(252)+7f(3)=504 \end{cases} \Rightarrow \begin{cases} 3a+7b=16 \\ 7a+3b=504 \end{cases} \Rightarrow a = \frac{\begin{vmatrix} 16 & 7 \\ 504 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 7 \\ 7 & 3 \end{vmatrix}} = \tfrac{16\cdot3-7\cdot504}{3\cdot3-7\cdot7}$$

or Let $P(x)$ be the assertion $3f(x) + 7f(\dfrac{2016}{x}) = 2x$ $P(\dfrac{2016}{x}) : 3f(\dfrac{2016}{x}) + 7f(x) = \dfrac{4032}{x}$ Solving the system occurs $f(x) = \dfrac{28224 - 6x^2}{40x}$ $\therefore f(8) = 87$

04. $x = 1 - \dfrac{1}{x} : f\left(1 - \dfrac{1}{x}\right) + f(x) = \left(1 - \dfrac{2}{x}\right)^2 + f\left(\dfrac{1}{1-x}\right)$

$\longrightarrow f(x) - f\left(\dfrac{1}{1-x}\right) = \left(1 - \dfrac{2}{x}\right)^2 - f\left(1 - \dfrac{1}{x}\right)$

$f(x) + f\left(\dfrac{1}{1-x}\right) = (2x-1)^2 + f\left(1 - \dfrac{1}{x}\right)$

$$f(x) = \frac{(2x-1)^2 + \left(1 - \dfrac{2}{x}\right)^2}{2} \longrightarrow f(3) = \frac{113}{9}$$