Chapter 1

Permutations and Combinations

1.1 Two Basic Counting Principles

In our everyday lives, we often need to enumerate "events" such as, the arrangement of objects in a certain way, the partition of things under a certain condition, the distribution of items according to a certain specification, and so on. For instance, we may come across counting problems of the following types:

"*How many ways are there to arrange 5 boys and 3 girls in a row so that no two girls are adjacent?*"

"*How many ways are there to divide a group of 10 people into three groups consisting of 4, 3 and 2 people respectively, with 1 person rejected?* "

These are two very simple examples of counting problems related to what we call "permutations" and "combinations". Before we introduce in the next three sections what permutations and combinations are, we state in' this section two principles that are fundamental in all kinds of counting problems.

Theorem 1.1.1 (The Addition Principle (AP))

Assume that there are:

$$\begin{gathered} n_1 \text{ ways for the event } E_1 \text{ to occur,} \\ n_2 \text{ ways for the event } E_2 \text{ to occur,} \\ \vdots \\ n_k \text{ ways for the event } E_k \text{ to occur,} \end{gathered}$$

where $k \geq 1$ If these ways for the different events to occur are pairwise disjoint, then the number of ways for at least one of the events $E_1, E_2, \cdots$, or E_k to occur is $n_1+n_2+n_3+\cdots+n_k = \sum_{i=1}^{k} n_i$

Example 1.1.2

One can reach city Q from city P by sea, air and road. Suppose that there are 2 ways by sea, 3 ways by air and 2 ways by road (see Figure 1.1.1). Then by (AP), the total number of ways from P to Q by sea, air or road is $2 + 3 + 2 = 7$.

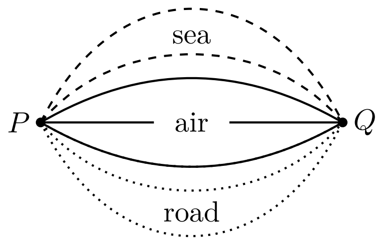


Figure 1.1:

An equivalent form of (AP), using set-theoretic terminology, is given below.

Lemma 1.1.3

Let $A_1, A_2, A_3, \cdots, A_k$ be any k finite sets, where $k \geq 1$. If the given sets are pairwise disjoint, i.e, $A_i \cap A_j = \phi$ for $i, j = 1, 2, \cdots, k, i \neq j$, then

$$\left|\bigcup_{i=0}^{k} A_i\right| = |A_1 \cup A_2 \cup \cdots \cup A_k| = \sum_{i=0}^{k} |A_i|$$

Example 1.1.4

Find the number of ordered pairs (x, y) of integers such that $x^2 + y^2 \leq 5$.

Solution. We may divide the problem into 6 disjoint cases: $x^2 + y^2 = 0, 1, \cdots, 5$.Thus for $i = 0, 1, \cdots, 5$, Let

$$S_i = \{(x, y) \,|\, x, y \in \mathbb{Z},\, x^2 + y^2 = i\}$$

It can be checked that

$$\begin{aligned}
S_0 &= \{(0,0)\}, \\
S_1 &= \{(1,0), (-1,0), (0,1), (0,-1)\}, \\
S_2 &= \{(1,1), (1,-1), (-1,1), (-1,-1)\}, \\
S_3 &= \phi, \\
S_4 &= \{(0,2), (0,-2), (2,0), (-2,0)\}, \mathit{and} \\
S_5 &= \{(1,2), (1,-2), (2,1), (2,-1), (-1,2), (-1,-2), (-2,1), (-2,-1)\}.
\end{aligned}$$

Thus by (AP), the desired number of ordered pairs is

$$\sum_{i=0}^{5} |S_i| = 1 + 4 + 4 + 0 + 4 + 8 = 21$$

□

Remark: 1) In the above example, one can find out the answer "21" simply by listing all the required ordered pairs (x, y). The above method, however, provides us with a systematical way to obtain the answer.
2) One may also divide the above problem into disjoint cases: $x^2 = 0, 1, \cdots, 5$, find out the number of required ordered pairs in each case, and obtain the desired answer by applying (AP).

Theorem 1.1.5 (The Multiplication Principle (MP))

Assume that an event E can be decomposed into r ordered events $E_1, E_2, \cdots, E_r$ and that there are

$$\begin{gathered} n_1 \text{ ways for the event } E_1 \text{ to occur,} \\ n_2 \text{ ways for the event } E_2 \text{ to occur,} \\ \vdots \\ n_k \text{ ways for the event } E_r \text{ to occur,} \end{gathered}$$

Then the total number of ways for the event E to occur is given by:

$$n_1 \times n_2 \times n_3 \times \cdots \times n_r = \prod_{i=1}^{r} n_i$$

Example 1.1.6

To reach city D from city A, one has to pass through city B and then city C as shown in Figure 1.1.2. If there are 2 ways to travel from A to B, 5 ways from B to C, and 3 ways from C to D, then by (MP), the number of ways from A to D via B and C is given by $2 \times 5 \times 3 = 30$.

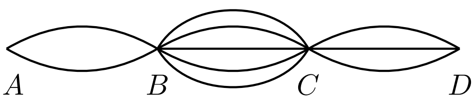


Figure 1.2

An equivalent form of (MP) using set-theoretic terminology, is stated below.

Lemma 1.1.7

Let,

$$\prod_{i=1}^{r} A_i = A_1 \times A_2 \times A_3 \cdots \times A_r = \{(a_1, a_2, a_3, \cdots, a_r) | a_i \in A_i,\ i = 1, 2, 3, \cdots, r\}$$

denote the cartesian product of the finite sets $A_1, A_2, A_i, \cdots, A_r$. Then

$$\left|\prod_{i=0}^{r} A_i\right| = |A_1| \times |A_2| \times \cdots \times |A_r| = \prod_{i=1}^{r} |A_i|$$

A sequence of numbers $a_1, a_2, a_3, \cdots, a_n$ is called a $k - ary\,sequence$, where $n, k \in \mathbb{N}$, if $a_i \in \{0, 1, \cdots, k-1\}$ for each $i = 1, 2, \cdots, n$. The length of the sequence $a_1, a_2, a_3, \cdots, a_n$ is defined to be n, which is the number of terms contained in the sequence. At times, such a sequence may be denoted by $(a_1, a_2, a_3, \cdots, a_n)$ A $k - ary\,sequence$ is also called a *binary*, *ternary*, or *quaternary* sequence when $k = 2, 3\,or\,4$, respectively. Thus, $\{000, 001, 010, 100, 011, 101, 110, 111\}$ is the set of all $8 (= 2^3)$ binary sequences of length 3. For given $k, n \in \mathbb{N}$, how many different $k - ary\,sequences$ of length n can we form? This will be discussed in the following example. You will find the result useful later on.

Example 1.1.8

To form a $k - ary\,sequence$ $a_1, a_2, a_3, \cdots, a_n$ of length n, we first select an a_1 from the set $B = \{0, 1, ..., k-1\}$; then an a_2 from the same set B; and so on until finally an a_n again from B. Since there are k choices in each step, the number of distinct $k - ary\,sequences$ of length n is, by (MP), $\underbrace{k \times k \times k \times \cdots \times k}_{n} = k^n$

Example 1.1.9

Find the number of positive divisors of 600, inclusive of 1 and 600 itself.

Solution. We first note that the number $'600'$ has a unique prime factorization, namely, $600 = 2^3 \times 3^1 \times 5^2$. It thus follows that a positive integer m is a divisor of 600 if and only if m is of the form $m = 2^a \times 3^b \times 5^c$, where $a, b, c \in \mathbb{Z}$ such that $0 < a < 3$, $0 < b < 1$ and $0 < c < 2$. Accordingly, the number of positive divisors of $'600'$ is the number of ways to form the triples (a, b, c) where $a \in \{0, 1, 2, 3\}, b \in \{0, 1\}$ and $c \in \{0, 1, 2\}$, which by (MP), is equal to $4 \times 2 \times 3 = 24$. □

Remark: By applying (MP) in a similar way, one obtains the following general result.
If a natural number n has as its prime factorization,

$$n = p_1^{k_1} \times p_2^{k_2} \times \cdots \times p_r^{k_r}$$

where the p_i's are distinct primes and the k_i 's are positive integers, then the number of positive divisors of n is given by $\prod_{i=1}^{r} (k_i + 1)$.

In the above examples, we have seen how (AP) and (MP) were separately used to solve some counting problems. Very often, solving a more complicated problem may require a 'joint' application of both (AP) and (MP). To illustrate this, we give the following example.

Example 1.1.10

Let $X = \{1, 2, 3 \cdots 100\}$ and let

$$S = \{(a, b, c)|a, b, c \in X, a < b \,\text{and}\, a < c\}$$

Find $|S|$.

Solution. The problem may be divided into disjoint cases by considering $a = 1, 2, \cdots, 99$. For $a = k \in \{1, 2, \cdots, 99\}$, the number of choices for b is $100 - k$ and that for c is also $100 - k$. Thus the number of required ordered triples (k, b, c) is $(100 - k)^2$, by (MP). Since k takes on the values $1, 2, \cdots, 99$ by applying (AP), we have

$$|S| = 99^2 + 98^2 + \cdots + 1^2.$$

Using the formula $\sum\limits_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$, we finally obtain

$$|S| = \frac{1}{6} \times 99 \times 100 \times 199 = 328350.$$

$\square$

As mathematical statements, both (AP) and (MP) are really 'trivial'. This could be a reason why they are very often neglected by students. Actually, they are very fundamental in solving counting problems. As we shall witness in this book, a given counting problem, no matter how complicated it is, can always be 'decomposed' into some simpler 'sub-problems' that in turn can be counted by using (AP) and/or (MP).

1.2 Permutations

At the beginning of Section 1.1, we mentioned the following problem: "How many ways are there to arrange 5 boys and 3 girls in a row so that no two girls are adjacent?" This is a typical example of a more general problem of arranging some distinct objects subject to certain additional conditions.

Let $A = \{a_1, a_2, \cdots, a_n\}$ be a given set of n distinct objects. For $0 \leq r \leq n$, an $r-$permutation of A is a way of arranging any r of the objects of A in a row. When $r = n$, an $n-$permutation of A is simply called a permutation of A.

Example 1.2.1

Let $A = \{a, b, c, d\}$. All the 3-permutations of A are shown below:

$$\begin{aligned}
&abc, acb, bac, bca, cab, cba\\
&abd, adb, bad, bda, dab, dba\\
&acd, adc, cad, cda, dac, dca\\
&bcd, bdc, cbd, cdb, dbc, dcb
\end{aligned}$$

There are altogether 24 in number. .

Let ${}^nP_r$ denote the number of $r - permutations$ of A. Thus ${}^3P_4 = 24$ as shown in Example 1.2.1. In what follows, we shall derive a formula for ${}^rP_n$ by applying (MP).

An $r - permutation$ of A can be formed in r steps, as described below: First, we choose an object from A and put it in the first position (see Figure 1.2.1). Next we choose an object from the remaining ones in A and put it in the second position. We proceed on until the $r - th$ step in which we choose an object from the remaining $(n - r + 1)$ elements in A and put it in the $r - th$ position.

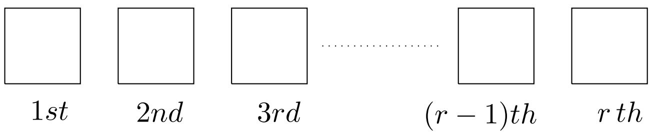


Figure 1.3:

There are n choices in step $1, (n - 1)$ choices in step $2, \cdots, n - (r - 1)$ choices in step r. Thus by (MP),

$${}^nP_r = n(n - 1)(n - 2) \cdots (n - r + 1).$$

If we use the factorial notation: $n! = n \times (n - 1) \times \cdots \times 2 \times 1$, then

$${}^nP_r = \frac{n!}{(n - r)!}$$

Remark. By convention, $O! = 1$. Note that ${}^nP_0 = 1$ and ${}^nP_n = n!$.

Example 1.2.2

Let $E = \{a, b, c, \cdots, x, y, z\}$ be the set of the 26 English alphabets. Find the number of 5-letter words that can be formed from E such that the first and last letters are distinct vowels and the remaining three are distinct consonants.

V	C	C	C	V

Figure 1.4:

Solution. There are 5 vowels and 21 consonants in E. A required 5-letter word can be formed in the following way.
Step 1. Choose a 2-permutation of $\{a, e, i, o, u\}$ and then put the first vowel in the 1^{st} position and the second vowel in the 5^{th} position (see Figure 1.4).
Step 2. Choose a 3-permutation of $E \setminus \{a, e, i, o, u\}$ and put the $1st$, $2nd$ and $3rd$ consonants of the permutation in the $2nd$, $3rd$ and $4th$ positions respectively (see Figure 1.4).

There are ${}^5P_2$ choices in Step 1 and ${}^{21}P_3$ choices in Step 2. Thus by (MP), the number of such 5-1etter words is given by

$$ {}^5P_2 \times {}^{21}P_3 = (5 \times 4) \times (21 \times 20 \times 19) = 159600. $$

$\square$

Example 1.2.3

There are 7 boys and 3 girls in a gathering. In how many ways can they be arranged in a row so that
(i) the 3 girls form a single block (i.e. there is no boy between any two of the girls)?
(ii) the two end-positions are occupied by boys and no girls are adjacent?

Solution. (i) Since the 3 girls must be together, we can treat them as a single entity. The number of ways to arrange 7 boys together with this entity is $(7 + 1)!$. As the girls can permute among themselves within the entity in 3! ways, the desired number of ways is, by (MP),

$$ 8! \times 3! $$

(ii) We first consider the arrangements of boys and then those of girls. There are 7! ways to arrange the boys. Fix an arbitrary one of the arrangements. Since the end-positions are occupied by boys, there are only 6 spaces available for the 3 girls G_1, G_2 and G_3 .

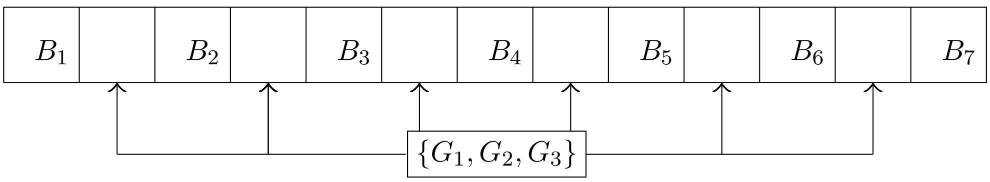


Figure 1.5:

G_1 has 6 choices. Since no two girls are adjacent, G_2 has 5 choices and G_3 has 4. Thus by (MP), the number of such arrangements is

$$7! \times 6 \times 5 \times 4.$$

□

Remark Example 1.2.3 can also be solved by considering the arrangements for the girls first. This will be discussed in Example 1.7.2.

Example 1.2.4

Between 20000 and 70000, find the number of even integers in which no digit is repeated.

Solution. Let *abcde* be a required even integer. As shown in the following diagram, the 1st digit a can be chosen from $\{2, 3, 4, 5, 6\}$ and the 5th digit e can be chosen from $\{0, 2, 4, 6, 8\}$. Since $\{2, 3, 4, 5, 6\} \cap \{0, 2, 4, 6, 8\} = \{2, 4, 6\}$, we divide the problem into 2 disjoint cases:

Case 1. $a \in \{2, 4, 6\}$. In this case, a has 3 choices, e then has $4(= 5 - 1)$ choices, and *bcd* has ${}^{(10-2)}P_3 = {}^8P_3$ choices. By (MP), there are

$$3 \times 4 \times {}^8P_3 = 4032$$

such even numbers.

Case 2. $a \in \{3, 5\}$. In this case, a has 2 choices, e has 5 choices and again *bcd* has ${}^8P_3$ choices. By (MP), there are

$$2 \times 5 \times {}^8P_3 = 3360$$

such even numbers
. Now, by (AP), the total number of required even numbers is $4032 + 3360 = 7392$. □

Example 1.2.5

Let S be the set of natural numbers whose digits are chosen from $\{1, 3, 5, 7\}$ such that no digits are repeated. Find

$$(i)\ |S| \quad (ii)\ \sum_{n \in S} n$$

Solution. (i) We divide S into 4 disjoint subsets consisting of:
(1) 1− digit numbers: $1, 3, 5, 7$;
(2) 2− digit numbers: $13, 15, \cdots$;
(3) 3− digit numbers: $135, 137, \cdots$;
(4) 4− digit numbers: $1357, 1375, \cdots$;

and find $|S|$ by applying (AP). Thus for $i = 1, 2, 3, 4$, let S_i denote the set of $i-$digit natural numbers formed by $1, 3, 5, 7$ with no repetition. Then $S = S_1 \cup S_2 \cup S_3 \cup S_4$ and by (AP),

$$\begin{aligned}|S| = \sum_{i=1}^{4} |S_i| &= {}^4P_1 + {}^4P_2 + {}^4P_3 + {}^4P_4 \\ &= 4 + 12 + 24 + 24\end{aligned}$$

(ii) Let $\alpha = \sum_{n \in S} n$. It is tedious to determine α by summing up all the 64 numbers in S. Instead, we use the following method.

Let α_1 denote the sum of unit-digits of the numbers in S; α_2 that of ten-digits of the numbers in S; α_3 that of hundred-digits of the numbers in S; and α_4 that of thousand-digits of the numbers in S. Then

$$\alpha = \alpha_1 + 10\alpha_2 + 100\alpha_3 + 1000\alpha_4$$

We first count α_1. Clearly, the sum of unit-digits of the numbers in S_1 is

$$1 + 3 + 5 + 7 = 16.$$

In S_2, there are ${}^3P_1$ numbers whose unit-digits are, respectively, 1, 3, 5 and 7. Thus the sum of the unit-digits of the number is S_2 is

$${}^3P_1 \times (1 + 3 + 5 + 7) = 3 \times 16 = 48.$$

In $S3$, there are ${}^3P_2$ numbers whose unit-digits are, respectively, 1, 3, 5 and 7. Thus the sum of unit-digits of the numbers in S_3 is

$${}^3P_2 \times (1 + 3 + 5 + 7) = 6 \times 16 = 96.$$

In S_4, there are ${}^3P_3$ numbers whose unit-digits are, respectively, 1, 3, 5 and 7. Thus the sum of unit-digits of the numbers in S_4 is

$${}^3P_3 \times (1 + 3 + 5 + 7) = 6 \times 16 = 96.$$

Hence by (AP),

$$\alpha_1 = 16 + 48 + 96 + 96 = 256.$$

Similarly, we have:

$$\begin{aligned} &\alpha_2 = {}^3P_1 \times (1 + 3 + 5 + 7) + {}^3P_2 \times (1 + 3 + 5 + 7) + {}^3P_3 \times (1 + 3 + 5 + 7) = 240; \\ &\alpha_3 = ({}^3P_2 + {}^3P_3) \times (1 + 3 + 5 + 7) = 192; \\ and\quad &\alpha_4 = {}^3P_3 \times (1 + 3 + 5 + 7) = 96. \end{aligned}$$

Thus,

$$\begin{aligned}\alpha &= \alpha_1 + 10\alpha_2 + 100\alpha_3 + 1000\alpha_4 \\ &= 256 + 2400 + 19200 + 96000 \\ &= 117856.\end{aligned}$$

□

Remark: There is a shortcut to compute the sum $\alpha = \sum(n|n \in S)$ in part (ii). Observe that the 4 numbers in S_1 can be paired off as $\{1,7\}$ and $\{3,5\}$ so that the sum of the two numbers in each pair is equal to 8 and the 12 numbers in S_2 can be paired off as $\{13,75\}$, $\{15,73\}$, $\{17,71\}$, $\{35,53\}, \cdots$ so that the sum of the two numbers in each pair is 88. Likewise, the 24 numbers in S_3 and the 24 numbers in S_4 can be paired off so that the sum of the two numbers in each pair is equal to 888 and 8888 respectively. Thus

$$\begin{aligned}\alpha &= 8 \times \frac{4}{2} + 88 \times \frac{12}{2} + 888 \times \frac{24}{2} + 8888 \times \frac{24}{2} \\ &= 117856.\end{aligned}$$

1.3 Circular Permutation

The permutations discussed in Section 1.2 involved arrangements of objects in a row. There are permutations which require arranging objects in a circular closed curve. These are called circular permutations. Consider the problem of arranging 3 distinct objects a, b, c in 3 positions around a circle. Suppose the 3 positions are numbered (1), (2) and (3) as shown in Figure 1.6 Then the three arrangements of a, b, c shown in the figure can be viewed as the permutations:

$$abc, \; cab, \; bca$$

respectively.

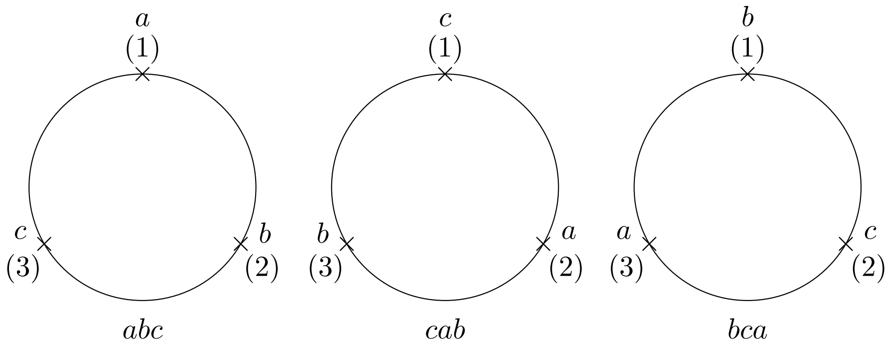


Figure 1.6:

In this case, such "*circular permutations*" are identical with the usual permutations, and thus there is nothing new worth discussing. To get something interesting, let us now neglect the numbering of the positions (and thus only "relative positions" of objects are concerned). As shown in Figure 1.3.2, any of the 3 arrangements is a rotation of every other; i.e., the relative positions of the objects are invariant under rotation. In this case, we shall agree to say that the 3 arrangements of Figure 1.3.2 are identical. In general, two circular permutations of the same objects are identical if anyone of them can be obtained by a rotation of the other.

Let A be a set of n distinct objects. For $0 < r < n$, an $r-$ *circular permutation* of A is a circular permutation of any r distinct objects taken from A. Let Q_n^r denote the number of

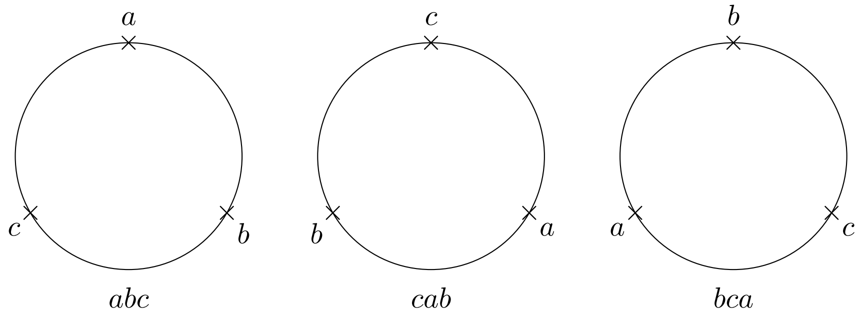


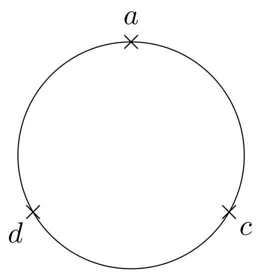
Figure 1.7:

$r - circular\,permutations$ of A. We shall derive a formula for Q_n^r.

Let $A = \{a, b, c, d\}$. There are altogether ${}^4P_3(= 24)$ 3-permutations of A and they are shown in Example 1.2.1. These 24 3-permutations are re-grouped into 8 subsets as shown below:

abc	cab	bca	acb	bac	cba
abd	dab	bda	adb	bad	dba
acd	dac	cda	adc	cad	dca
bed	dbc	cdb	bdc	cbd	deb

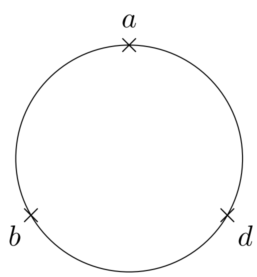
It is noted that every 3-circular permutation of A gives rise to a unique such subset. For instance,



$$\Longrightarrow \quad \{acd,\ dac,\ cda\}$$

Conversely, every such subset corresponds to a unique 3-circular permutation of A. For instance,

$$\{abd,\ dab,\ bda\} \quad \Longrightarrow$$



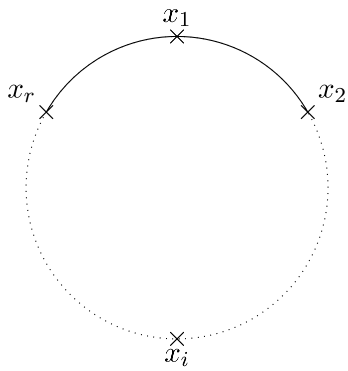
Thus we see that

$$Q_3^4 = \frac{24}{3} = 8$$

Example 1.3.1

tells us that $Q_3^4 = \frac{1}{3}\,{}^4P_3$. What is the relation between Q_r^n and ${}^nP_r$ in general?

Solution. A circular permutation of r distinct objects $x_1, x_2, \cdots, x_r$ shown below:



gives rise to a unique subset of $r\; r-permutations$:

$$x_1x_2\cdots x_r,\; x_rx_1x_2\cdots x_{r-1}, \cdots, x_2x_3\cdots x_rx_1$$

obtained through a rotation of the circular permutation. Conversely, every such subset of $r\; r-permutations$ of A corresponds to a unique $r-circular$ permutation of A. Since all the $r-permutations$ of A can be equally divided into such subsets, we have

$$Q_r^n = \frac{{}^nP_r}{r}$$

In particular

$$Q_n^n = \frac{{}^nP_n}{n} = (n-1)!$$

□

Example 1.3.2

In how many ways can 5 boys and 3 girls be seated around a table if
(i) there is no restriction?
(ii) boy B_1 and girl G_1 are not adjacent?
(iii) no girls are adjacent?

Solution. (i) The number of ways is $Q_8^8 = 7!$.
(ii) The 5 boys and 2 girls not including G_1 can be seated in $(7-1)!$ ways. Given such an arrangement as shown in Figure 1.3.3, G_1 has $5(= 7-2)$ choices for a seat not adjacent to B_1. Thus the desired number of ways is

$$6! \times 5 = 3600.$$

We may obtain another solution by using what we call the Principle of Complementation as given below:

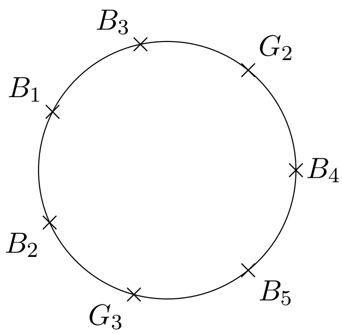


Figure 1.8:

Theorem 1.3.3 (Principle of Complementation (CP))
If A is a subset of a finite universal set U, then

$$|U \setminus A| = |U| - |A|.$$

Now, the number of ways to arrange the 5 boys and 3 girls around a table so that boy B_1 and girl G_1 are adjacent (treating $\{B_1, G_1\}$ as an entity) is

$$(7-1)! \times 2 = 1440.$$

Thus the desired number of ways is by (CP),

$$7! - 1440 = 3600.$$

(iii) We first seat the 5 boys around the table in $(5-1)! = 4!$ ways. Given such an arrangement as shown in Figure 1.3.4, there are 5 ways to seat girl G_1 . As no girls are adjacent, G_2 and G_3 have 4 and 3 choices respectively. Thus the desired number of ways is

$$4! \times 5 \times 4 \times 3 = 1440.$$

□

Example 1.3.4
Find the number of ways to seat n married couples around a table in each of the following cases:
(i) Men and women alternate;
(ii) Every woman is next to her husband.

Solution. (i) The n men can first be seated in $(n-1)!$ ways. The n women can then be seated in the n spaces between two men in $n!$ ways. Thus the number of such arrangements is $(n-1)! \times n!$.
(ii) Each couple is first treated as an entity. The number of ways to arrange the n entities around

the table is $(n-1)!$. Since the two people in each entity can be permuted in 2! ways, the desired number of ways is

$$(n-1)! \times 2^n$$

□

Remark:A famous and much more difficult problem related to the above problem is the following: How many ways are there to seat n married couples ($n \geq 3$) around a table such that men and women alternate and each woman is not adjacent to her husband? This problem, known as the problem of menages, was first introduced by the French mathematician Francis Edward Anatole Lucas (1842 - 1891). A solution to this problem will be given in Chapter 4.