AIME AMC Number Theory

Problem 1

Let a_n equal $6^n + 8^n$. Determine the remainder upon dividing a_{83} by 49.

Solution 1 First, we try to find a relationship between the numbers we're provided with and 49. We realize that $49 = 7^2$ and both 6 and 8 are greater or less than 7 by 1.

Expressing the numbers in terms of 7, we get $(7-1)^{83} + (7+1)^{83}$.

Applying the Binomial Theorem, half of our terms cancel out and we are left with $2(7^{83} + 3403 \cdot 7^{81} + \cdots + 83 \cdot 7)$. We realize that all of these terms are divisible by 49 except the final term.

After some quick division, our answer is $\boxed{035}$.

Solution 2

Since $\phi(49) = 42$ (the Euler's totient function), by Euler's Totient Theorem, $a^{42} \equiv 1 \pmod{49}$ where $\gcd(a, 49) = 1$. Thus $6^{83} + 8^{83} \equiv 6^{2(42)-1} + 8^{2(42)-1} \equiv 6^{-1} + 8^{-1} \equiv \frac{8+6}{48} \equiv \frac{14}{-1} \equiv \boxed{035} \pmod{49}$.

Alternatively, we could have noted that $a^b \equiv a^{b \pmod{\phi(n)}} \pmod{n}$. This way, we have $6^{83} \equiv 6^{83 \pmod{42}} \equiv 6^{-1} \pmod{49}$, and can finish the same way.

Problem 2 What is the largest 2-digit prime factor of the integer $\binom{200}{100}$?

Solution Expanding the binomial coefficient, we get $\binom{200}{100} = \dfrac{200!}{100!100!}$. Let the prime be p; then $10 \le p < 100$. If $p > 50$, then the factor of p appears twice in the denominator. Thus, we need p to appear as a factor three times in the numerator, or $3p < 200$. The largest such prime is $\boxed{061}$, which is our answer.