תרגול אינפי:

תרגול דף תרגילים 2:

1. מצאו את האיבר הכללי של הסדרות הבאות:

(א)

$$\{2,4,6,8,16\} = \begin{cases} a_n = 2^n & n \in N \\ a_m = 2^{(m+1)} & m \in N_0 \end{cases}$$

(ב)

$$\left\{-\frac{4}{3}, \frac{5}{5}, -\frac{6}{7}, \frac{7}{9}\right\} = \left\{ a_n = \frac{(-1)^n \cdot (3+n)}{1+2n} \quad n \in N \right.$$

(ג)

$$\left\{\frac{1}{2}, \frac{1}{2}+\frac{3}{4}, \frac{1}{2}+\frac{3}{4}+\frac{5}{8}, \frac{1}{2}+\frac{3}{4}+\frac{5}{8}+\frac{7}{16}\right\} = \left\{ a_n = \sum_{i=1}^{n} \frac{1+2(i-1)}{2^i} = \sum_{i=1}^{n} \frac{2i-1}{2^i} \quad n \in N \right.$$

2. להוכיח את הגבול לפי ההגדרה:

(א)

$$lim_{n\to\infty}\left(2+\frac{1}{n}\right) = 2$$

יהי $\epsilon > 0$ כלשהו , נוכיח כי קיים N כך שלכל $n > N$ מתקיים $\left|\left(2+\frac{1}{n}\right)-2\right| < \epsilon$
נבצע חישוב עזר :

$$\left|2+\frac{1}{n}-2\right| = \left|\frac{1}{n}\right| \underset{n>0}{\leq} \frac{1}{n} < \epsilon \leftrightarrow \frac{1}{\epsilon} < n$$

לכן מצאנו $N = \frac{1}{\epsilon}$ כך ש $n > N$ שמקיים

$$\left|2+\frac{1}{n}-2\right| \leq \frac{1}{n} = \frac{1}{\frac{1}{\epsilon}} = \epsilon$$

(ב)

$$lim_{n\to\infty}\left(\frac{3n+4}{2n+1}\right)\neq 1$$

צריך למצוא שקיים $\epsilon>0$ כך שלכל $N>0$ קיים $N>n$ מתקיים $\left|\frac{3n+4}{2n+1}-1\right|\geq\epsilon$
נבצע חישוב עזר:

$$\left|\frac{3n+4}{2n+1}-1\right|=\left|\frac{3n+4-2n-1}{2n+1}\right|=\left|\frac{n-3}{2n+1}\right|\geq\left|\frac{n-3}{2n}\right|\underset{n>3}{\geq}\frac{n-3}{2n}>\epsilon\leftrightarrow\frac{n-3}{2\epsilon}>n=N$$

נקח $\frac{1}{8}=\frac{n-3}{2n}=\epsilon_4$ קיים $4=\frac{1}{\frac{2}{8}}=n$ כך משתקיים $\left|\frac{16}{9}-\frac{9}{9}\right|=\frac{7}{9}\geq\frac{1}{8}$

(ג)

$$lim_{n\to\infty}\frac{n}{n+1}=1$$

יהי $\epsilon>0$כלשהו, צריך להוכיח שקיים $N>0$, כך שלכל $N>n$מתקיים $\left|\frac{n}{n+1}-1\right|<\epsilon$
נבצע חישוב עזר:

$$\left|\frac{n}{n+1}-1\right|=\left|\frac{n-n-1}{n+1}\right|=\left|\frac{-1}{n+1}\right|=\underset{n>0>-1}{\frac{1}{n+1}}<\frac{1}{n}<\epsilon\leftrightarrow\frac{1}{\epsilon}<n=N$$

נחזור לתרגיל ,

$$\left|\frac{n}{n+1}-1\right|<\frac{1}{n}=\frac{1}{\frac{1}{\epsilon}}=\epsilon$$

(ד)

$$lim_{n\to\infty}\frac{2n+1}{4n+3}=\frac{1}{2}$$

יהי $\epsilon>0$ כלשהו, צריך להוכיח שקיים $N>0$ כך שלכל $N>n$ מתקיים $\left|\frac{2n+1}{4n+3}-\frac{1}{2}\right|<\epsilon$
נבצע חישוב עזר:

$$\left|\frac{2n+1}{4n+3}-\frac{1}{2}\right|=\left|\frac{4n+2-4n-3}{2(4n+3)}\right|=\left|\frac{-1}{8n+6}\right|\underset{n>0}{=}\frac{1}{8n+6}<\frac{1}{8n}<\frac{1}{n}<\epsilon\leftrightarrow\frac{1}{\epsilon}<n$$

לכן מתקיים

$$\left|\frac{2n+1}{4n+3}-\frac{1}{2}\right|<\frac{1}{n}=\frac{1}{\frac{1}{\epsilon}}=\epsilon$$

(ה)

$$lim_{n\to\infty}\frac{n+8}{6n-7}\neq 1$$

קיים $\epsilon>0$, כך שלכל $N>0$ קיים $n>N$ מתקיים $\left|\frac{n+8}{6n-7}-1\right|>\epsilon$
נבצע חישוב עזר:

$$\left|\frac{n+8}{6n-7}-1\right|=\left|\frac{n+8-6n+7}{6n-7}\right|=\left|\frac{-5n+15}{6n-7}\right|\underset{n>3}{\geq}\frac{5n-15}{6n-7}>\frac{5(n-3)}{6n}>\frac{5(n-3)}{5n}=\frac{n-3}{n}$$

נקח $n>4$, $\epsilon=\frac{1}{4}$

$$\left|\frac{12}{17}-1\right|=\frac{5}{17}>\frac{1}{4}$$

3. חשב את הגבולות:

(א)

$$lim_{n\to\infty}\left(5+\frac{6}{n^2}\right)=lim_{n\to\infty}5+lim_{n\to\infty}\frac{6}{n^2}=5+0=5$$

(ב)

$$lim_{n\to\infty}\left(\frac{n^2}{n+1}-\frac{(n+1)^2}{n}\right)=lim_{n\to\infty}\left(\frac{n^3-(n+1)^3}{n\left(n+1\right)}\right)=lim_{n\to\infty}\left(\frac{n^3-\left(n^3-3n^2-3n+1\right)}{n(n+1)}\right)=lim_{n\to\infty}\left(\frac{3n^2+3n+1}{n^2+n}\right)$$

$$=lim_{n\to'\infty}\left(\frac{\cancel{n^2}\left(3+\frac{3}{n}+\frac{1}{n^2}\right)}{\cancel{n^2}\left(1+\frac{1}{n}\right)}\right)=\frac{3+0+0}{1+0}=3$$

(ג)

$$lim_{n\to\infty}\frac{2n^4-n^2+1}{n^2+9n+9n^4}\underset{t=n^2}{=}\frac{2t^2-t+1}{9t^2+t+\frac{9}{t}}=\frac{t^2\left(2-\frac{1}{t}+\frac{1}{t^2}\right)}{t^2\left(9+\frac{1}{t}+\frac{9}{t^3}\right)}=\frac{2}{9}$$

(ד)

$$lim_{n\to\infty}\left(\frac{n^4-n^3+n}{\left(n^2+2n+1\right)^2}\right)=lim_{n\to\infty}\left(\frac{n^4-n^3+n}{\left(n^2+2n+1\right)\left(n^2+2n+1\right)}\right)=lim_{n\to\infty}\left(\frac{n^4-n^3+n}{n^4+4n^3+6n^2+4n+1}\right)$$

$$lim_{n\to\infty}\left(\frac{n^4\left(1-\frac{1}{n}+\frac{1}{n^3}\right)}{n^4\left(1+\frac{4}{n}+\frac{6}{n^2}+\frac{4}{n^3}+\frac{1}{n^4}\right)}\right)=\frac{1}{1}=1$$

(ה)

$$lim_{n\to\infty}\left(\frac{n^4-n^2}{0.01n-n^2}\right)=lim_{n\to\infty}\left(\frac{n^4\left(1-\frac{1}{n^2}\right)}{n^2\left(\frac{0.01}{n}-1\right)}\right)=lim_{n\to\infty}\left(\frac{n^2\left(1-\frac{1}{n^2}\right)}{\frac{0.01}{n}-1}\right)=\infty$$

(ו)

$$lim_{n\to\infty}\left(n-\sqrt{n+a}\sqrt{n+b}\right)=lim_{n\to\infty}\left(\frac{n-\sqrt{n+a}\sqrt{n+b}}{1}\cdot\frac{n+\sqrt{n+a}\sqrt{n+b}}{n+\sqrt{n+a}\sqrt{n+b}}\right)=lim_{n\to\infty}\left(\frac{n^2-(n+a)(n+b)}{n+\sqrt{n+a}\sqrt{n+b}}\right)=$$

$$=lim_{n\to\infty}\left(\frac{n^2-\left(n^2+nb+na+ab\right)}{n+\sqrt{n\left(1+\frac{a}{n}\right)}\sqrt{n\left(1+\frac{b}{n}\right)}}\right)=lim_{n\to\infty}\frac{-n(a+b+\frac{ab}{n})}{n(1+\sqrt{1+\frac{a}{n}}\sqrt{1+\frac{b}{n}})}=-\frac{a+b}{2}$$

(ז)

$$lim_{n\to\infty}\frac{\sqrt{n+4}-\sqrt{n+1}}{\sqrt{n}}=lim_{n\to\infty}\left(\frac{\sqrt{n+4}-\sqrt{n+1}}{\sqrt{n}}\cdot\frac{\sqrt{n+4}+\sqrt{n+1}}{\sqrt{n+4}+\sqrt{n+1}}\right)=lim_{n\to\infty}\left(\frac{n+4-(n+1)}{\sqrt{n}\left(\sqrt{n+4}+\sqrt{n+1}\right)}\right)=$$

$$=lim_{n\to\infty}\left(\frac{3}{\sqrt{n}\left(\sqrt{n}\sqrt{1+\frac{4}{n}}+\sqrt{n}\sqrt{1+\frac{1}{n}}\right)}\right)=lim_{n\to\infty}\frac{3}{n\left(\sqrt{1+\frac{4}{n}}+\sqrt{1+\frac{1}{n}}\right)}=0$$

(ח)

$$lim_{n\to\infty}n\left[\left(27+\frac{1}{n}\right)^{\frac{1}{3}}-3\right]=lim_{n\to\infty}n\left[3\left(1+\frac{1}{27n}\right)^{\frac{1}{3}}-3\right]=lim_{n\to\infty}3n\left[1+\frac{1}{3}\cdot\frac{1}{27n}+O\left(\frac{1}{n^2}\right)-1\right]=$$

$$==lim_{n\to\infty}3n\left(\frac{1}{3}\cdot\frac{1}{27n}+O\left(\frac{1}{n^2}\right)\right)=\frac{1}{27}$$

4. מציאת הגבולות הבאים:

(א)

$$lim_{n\to\infty}\frac{1+2+2^2+....+2^n}{1+5+5^2+....+5^n}=lim_{n\to\infty}\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n=lim_{n\to\infty}\frac{\frac{2}{5}^n-1}{\frac{2}{5}-1}=lim_{n\to\infty}\frac{\frac{2}{5}^n-1}{-\frac{3}{5}}=\frac{5}{3}\left(1-\frac{2}{5}^n\right)=\frac{5}{3}$$

(ב)

$$lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+....+\frac{n-1}{n^2}\right)=lim_{n\to\infty}\frac{1}{n^2}\left(\sum_{i=1}^{n-1}i\right)=lim_{n\to\infty}\frac{1}{n^2}\left(\frac{n-1(n)}{2}\right)=lim_{n\to\infty}\frac{n^2-n}{2n^2}=\frac{1}{2}$$

(ג)

$$lim_{n\to\infty}\left(\frac{1+3+5....+(2n+1)}{n+1}-\frac{2n+1}{2}\right)=lim_{n\to\infty}\left(\frac{1}{n+1}\left(\sum_{i=1}^{n}(2n+1)\right)-\frac{2n+1}{2}\right)=$$

$$=lim_{n\to\infty}\frac{1}{n+1}\left(\frac{n\left(2+(n-1)2\right)}{2}\right)-\frac{2n+1}{2}=lim_{n\to\infty}\frac{n^2}{n+1}-\frac{2n+1}{2}=$$

$$=lim_{n\to\infty}\frac{2n^2-(2n+1)(n+1)}{2(n+1)}=lim_{n\to\infty}\frac{-3n-1}{2n+2}=-\frac{3}{2}$$

(ד)

$$lim_{n\to\infty}\left[\frac{7}{10}+\frac{29}{100}+...+\frac{5^n+2^n}{10^n}\right]=lim_{n\to\infty}\left[\left(\frac{1}{2}+\frac{1}{5}\right)+\left(\frac{1}{2^2}+\frac{1}{5^2}\right)+....+\left(\frac{1}{2^n}+\frac{1}{5^n}\right)\right]=$$

$$=lim_{n\to\infty}\left[\left(\frac{1}{2}+...+\frac{1}{2^n}\right)+\left(\frac{1}{5}+....+\frac{1}{5^n}\right)\right]=lim_{n\to\infty}\left(\frac{\frac{1}{2}\left(\frac{1}{2}^n-1\right)}{\frac{1}{2}-1}\right)+\left(\frac{\frac{1}{5}\left(\frac{1}{5}^n-1\right)}{\frac{1}{5}-1}\right)=$$

$$=lim_{n\to\infty}\left(1-\frac{1}{2}^n\right)+\left(\frac{1}{\cancel{5}}\cdot\frac{\cancel{5}}{4}\left(1-\frac{1}{5}^n\right)\right)=\frac{5}{4}$$

(ה)

$$lim_{n\to\infty}\frac{n^2+n!+(n+1)\,(n+1)!}{(n+2)!}=lim_{n\to\infty}\frac{n^2+n!+(n+1)^2n!}{n!(n+1)(n+2)}=lim_{n\to\infty}\frac{\cancel{n!}\left(2n^2+2n+1\right)}{\cancel{n!}\,(n^2+3n+2)}=2$$

(ו)

$$lim_{n\to\infty}\frac{2^{n+1}+3^n}{2^n+3^{n+1}}=lim_{n\to\infty}\frac{\cancel{3^n}\left(2\cdot\left(\frac{2}{3}\right)^n+1\right)}{\cancel{3^n}\left(\left(\frac{2}{3}\right)^n+3\right)}=\frac{1}{3}$$

(ז)

$$lim_{n\to\infty}\left(\frac{2^n+4^n}{2\cdot 9^n-8^n}\right)=lim_{n\to\infty}\frac{9^n\left(\frac{2}{9}^n+\frac{4}{9}^n\right)}{9^n\left(2-\frac{8}{9}^n\right)}=\frac{0}{2}=0$$

5. חשב את הגבולות

(א)

$$0=lim_{n\to\infty}\frac{-n}{n^2+1}\le lim_{n\to\infty}\frac{nsin\,(n)}{n^2+1}\le lim_{n\to\infty}\frac{n}{n^2+1}=\frac{n(1)}{n(n+\frac{1}{n})}=0$$

(ב

$$0 = lim_{n\to\infty}\frac{-1}{2^n} \leq lim_{n\to\infty}\frac{cosn}{2^n} \leq lim_{n\to\infty}\frac{1}{2^n} = 0$$

(ג

$$4 = lim_{n\to\infty}\sqrt[n]{4^n} \leq lim_{n\to\infty}\sqrt[n]{3^n+4^n} \leq lim_{n\to\infty}\sqrt[n]{2\cdot 4^n} = 4$$

(ד

$$0 = lim_{n\to\infty}\frac{-n^{\frac{2}{3}}}{n+1} \leq lim_{n\to\infty}\frac{\sqrt[3]{n^2}cos(n!)}{n+1} \leq lim_{n\to\infty}\frac{n^{\frac{2}{3}}}{n+1} = lim_{n\to\infty}\frac{n^{\frac{2}{3}}\cdot 1}{n^{\frac{2}{3}}\left(n^{\frac{1}{3}}+\frac{1}{n^{\frac{2}{3}}}\right)} = 0$$

(ה

$$lim_{n\to\infty}\sqrt[n]{2^n+(-1)^n+7^n} = lim_{n\to\infty}\sqrt[n]{7^n\left(\frac{2}{7}^n-\frac{1}{7}^n+1\right)} = 7$$