Solution : Q1

Given that,

$X_1, X_2, \ldots, X_n \overset{iid}{\sim}$ Bernoulli(θ). So, the PDF of X is,

$$f(x) = \theta^x (1-\theta)^{1-x} \quad \text{where } x \in \{0,1\} \text{ and } 0<\theta<1$$

We know the mean of Bernoulli random variable X is,

$$E[X] = \theta \quad \text{and variance} \quad Var(\theta) = \theta(1-\theta)$$

a) The likelihood function can be written as,

$$L(\theta \mid x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i \mid \theta)$$

$$= \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i}$$

$$= \theta^{\sum_{i=1}^{n} x_i} \cdot (1-\theta)^{n-\sum_{i=1}^{n} x_i}$$

Now, by taking the natural log on both side, we will get the loglikelihood,

$$l(\theta) = \ln\left[\theta^{\sum_{i=1}^{n} x_i} \cdot (1-\theta)^{n-\sum_{i=1}^{n} x_i}\right]$$

$$= \sum_{i=1}^{n} x_i \ln(\theta) + \left(n - \sum_{i=1}^{n} x_i\right) \ln(1-\theta)$$

Now, the first derivative of $l(\theta)$ is,

$$l'(\theta) = \frac{\partial}{\partial \theta} l(\theta) = \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{n - \sum_{i=1}^{n} x_i}{1-\theta}$$

$$l'(\theta) = \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{n - \sum_{i=1}^{n} x_i}{1-\theta}$$

then, the 2nd derivative of $l(\theta)$ will be,

$$l''(\theta) = \frac{\partial}{\partial\theta}[l'(\theta)] = \frac{\partial}{\partial\theta}\left[\frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{n - \sum_{i=1}^{n} x_i}{1-\theta}\right]$$

$$= -\frac{\sum_{i=1}^{n} x_i}{\theta^2} - (-1)\frac{n - \sum_{i=1}^{n} x_i}{(1-\theta)^2}(-1) \quad \text{[Applying chain rule]}$$

$$= -\frac{\sum_{i=1}^{n} x_i}{\theta^2} - \frac{n - \sum_{i=1}^{n} x_i}{(1-\theta)^2}$$

Now,

Fisher information $I(\theta) = E[-l''(\theta)]$

$$\therefore I(\theta) = E\left[\frac{\sum_{i=1}^{n} x_i}{\theta^2} + \frac{n - \sum_{i=1}^{n} x_i}{(1-\theta)^2}\right]$$

$$= \frac{E[\sum_{i=1}^{n} x_i]}{\theta^2} + \frac{n - E[\sum_{i=1}^{n} x_i]}{(1-\theta)^2}$$

$$= \frac{n\theta}{\theta^2} + \frac{n - n\theta}{(1-\theta)^2} \quad \text{[since } E[\sum x_i] = \sum E[x_i] \text{ and } x_i\text{'s are iid so, } \sum E[x_i] = n\theta]$$

$$= \frac{n}{\theta} + \frac{n}{1-\theta}$$

Thus, $$I(\theta) = \frac{n - n\theta + n\theta}{\theta(1-\theta)} = \frac{n}{\theta(1-\theta)}.$$

b) Given that,

Prior on θ is Beta (α, β)

$$\therefore \quad f(\theta) = \frac{1}{Beta(\alpha,\beta)} \cdot \theta^{\alpha-1}(1-\theta)^{\beta-1} \quad ; \quad 0<\theta<1 \text{ and } \alpha, \beta > 0$$

Since, the normalizing part $\frac{1}{Beta(\alpha,\beta)}$ is parameter (θ) free,

we can write $f(\theta)$ as,

$$f(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

The likelihood we already get from part (a) which is,

$$L(\theta \mid x_1, \ldots, x_n) = \theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{n-\sum_{i=1}^{n} x_i}$$

We know that,

posterior $\propto$ likelihood $\times$ prior

$$\therefore \quad f(\theta \mid data) \propto L(\theta \mid Data) \cdot f(\theta)$$

$$\propto \theta^{\sum_{i=1}^{n} x_i} \cdot (1-\theta)^{n-\sum_{i=1}^{n} x_i} \cdot \theta^{\alpha-1} \cdot (1-\theta)^{\beta-1}$$

$$\propto \theta^{(\alpha+\sum_{i=1}^{n} x_i)-1} \cdot (1-\theta)^{(\beta+n-\sum_{i=1}^{n} x_i)-1}$$

As we can see, the above expression is the kernel of a Beta $\left(\alpha+\sum_{i=1}^{n} x_i, \; \beta+n-\sum_{i=1}^{n} x_i\right)$ PDF.

Thus, the posterior distribution of θ follows,

$$\theta | \text{data} \sim \text{Beta}\left(\alpha + \sum_{i=1}^{n} x_i ,\ \beta + n - \sum_{i=1}^{n} x_i\right).$$

Now, the posterior mean,

$$E[\theta | \text{Data}] = \frac{\alpha + \sum_{i=1}^{n} x_i}{\alpha + \sum_{i=1}^{n} x_i + \beta + n - \sum_{i=1}^{n} x_i} \quad \left[\text{if } X \sim \text{Beta}(a,b) \text{ then } E[X] = \frac{a}{a+b}\right]$$

$$= \frac{\alpha + \sum_{i=1}^{n} x_i}{\alpha + \beta + n}$$

Solution : Q2 :

(1) Given that,

$$f(x) = \begin{cases} cx^3 & \text{if } 0<x<2 \\ 0, & \text{Otherwise} \end{cases}$$

To be a valid density, the integral of $f(x)$ within the range $0<x<2$ must be equal to 1.

Now,

$$\int_x f(x)\, dx = 1$$

$$\Rightarrow \int_0^2 cx^3\, dx = 1$$

$$\Rightarrow c\left[\frac{x^4}{4}\right]_{x=0}^{x=2} = 1$$

$$\Rightarrow c[4-0] = 1$$

$$\Rightarrow 4c = 1$$

$$\therefore c = \frac{1}{4}$$

So,

$$f(x) = \begin{cases} \frac{1}{4}x^3 & \text{if } 0<x<2 \\ 0, & \text{otherwise.} \end{cases}$$

solution : Q2 (2) :

To perform inverse transform sampling, the following steps needs to be followed :

i) first, we need to calculate the cumulative distribution function (CDF) $F(x)$ for this given $f(x)$.

ii) then, find the inverse CDF $F^{-1}(u)$, where u is a random sample from Uniform $(0,1)$

iii) Generate 10,000 random samples u from Uniform$(0,1)$

iv) Finally, apply the inverse CDF to each u to get the corresponding values of x where x has PDF $f(x)$.

In other words, $F^{-1}(u_1), \ldots, F^{-1}(u_{10,000}) \sim f(x)$.

NOW, lets get the CDF first.

$$F(x) = \int_0^x f(t)\,dt = \cancel{\int_0^x \frac{1}{4} x^3}\,dt$$

$$= \int_0^x \frac{1}{4} t^3\,dt = \frac{1}{4} \cdot \left[\frac{t^4}{4}\right]_{t=0}^{t=x}$$

$$= \frac{1}{16} x^4$$

solution Q3:

Given that,

$$f(x,y) = \begin{cases} c\,(y\sqrt{x} + x\sqrt{y}) & \text{if } 0<x<2,\ 0<y<2 \\ 0, & \text{otherwise.} \end{cases}$$

(1) To be a valid density, $f(x,y)$ must integrate to 1 within the range of $0<x<2$ and $0<y<2$.

Now,

$$\int_y \int_x f(x,y)\,dx.dy = 1$$

$$\Rightarrow \int_0^2 \int_0^2 c\,(y\sqrt{x} + x\sqrt{y})\,dx.dy = 1$$

$$\Rightarrow c\left[\int_0^2 \left[\frac{2}{3}\cdot y\sqrt{x^3} + \frac{1}{2}x^2\sqrt{y}\right]_{x=0}^{x=2} dy\right] = 1$$

$$\Rightarrow c\left[\int_0^2 \left(\frac{2\sqrt{8}}{3}y + 2\sqrt{y}\right) dy\right] = 1$$

$$\Rightarrow c\left[\frac{2\sqrt{8}}{3}\cdot\frac{1}{2}y^2 + 2\cdot\frac{2}{3}\sqrt{y^3}\right]_{y=0}^{y=2} = 1$$

$$\Rightarrow c\left[\frac{4\sqrt{8}}{3} + \frac{4\sqrt{8}}{3}\right] = 1$$

$$\Rightarrow \frac{16\sqrt{2}}{3}c = 1$$

Thus, $c = \dfrac{3}{16\sqrt{2}}$

$$\therefore \quad f(x,y) = \begin{cases} \frac{3}{16\sqrt{2}}\left(y\sqrt{x} + x\sqrt{y}\right) & \text{if} \quad 0<x<2,\ 0<y<2 \\ 0, & \text{otherwise} \end{cases}$$

(2)

The conditional density $f(x|y) = \dfrac{f(x,y)}{f(y)}$

Now,

$$f(y) = \int_x f(x,y)\,dx$$

$$= \int_0^2 \frac{3}{16\sqrt{2}}\left(y\sqrt{x} + x\sqrt{y}\right)dx$$

$$= \frac{3}{16\sqrt{2}} \cdot \left[\frac{2}{3}y\sqrt{x^3} + \frac{1}{2}x^2\sqrt{y}\right]_{x=0}^{x=2}$$

$$= \frac{3}{16\sqrt{2}}\left[\frac{2\sqrt{8}}{3}y + 2\sqrt{y}\right]$$

$$= \frac{1}{4}y + \frac{3}{8\sqrt{2}}\sqrt{y}$$

$$\therefore \quad f(y) = \frac{1}{4}\left[y + \frac{3}{2\sqrt{2}}\sqrt{y}\right], \quad 0<y<2$$

Now,

$$f(x|y) = \frac{f(x,y)}{f(y)}$$

$$= \frac{3}{4\sqrt{2}} \frac{\cancel{\frac{3}{16\sqrt{2}}}\left(y\sqrt{x} + x\sqrt{y}\right)}{\cancel{\frac{1}{4}}\left(y + \frac{3}{2\sqrt{2}}\sqrt{y}\right)} \qquad 0<x<2,\ 0<y<2$$

Now, $f(y|x)$ can be written directly using the symmetry or can be calculated as follows. Both results in same expression.

$$f(y|x) = \frac{f(x,y)}{f(x)}$$

$$\therefore \quad f(x) = \int_0^2 f(x,y)\,dy$$

$$= \int_0^2 \frac{3}{16\sqrt{2}} \left(y\sqrt{x} + x\sqrt{y}\right) dy$$

$$= \frac{3}{16\sqrt{2}} \left[\frac{1}{2} y^2\sqrt{x} + \frac{2}{3} x\sqrt{y^3} \right]_{y=0}^{y=2}$$

$$= \frac{3}{16\sqrt{2}} \left[2\sqrt{x} + \frac{2\sqrt{8}}{3} x \right]$$

$$= \frac{3}{8\sqrt{2}} \sqrt{x} + \frac{1}{4} x$$

$$= \frac{1}{4} \left[x + \frac{3}{2\sqrt{2}} \sqrt{x} \right] \quad ; \quad 0 < x < 2.$$

$$\therefore \quad f(y|x) = \frac{f(x,y)}{f(x)} = \frac{\frac{3}{16\sqrt{2}} \left(y\sqrt{x} + x\sqrt{y}\right)}{\frac{1}{4}\left(x + \frac{3}{2\sqrt{2}}\sqrt{x}\right)}$$

$$= \frac{3}{4\sqrt{2}} \quad \frac{y\sqrt{x} + x\sqrt{y}}{x + \frac{3}{2\sqrt{2}}\sqrt{x}} \quad ; \quad \begin{matrix} 0 < y < 2 \\ 0 < x < 2. \end{matrix}$$

solution : Q3 (3) :

As we can see from the full conditional distributions, $F^{-1}(.)$ cannot be easily computed. Thus, we are going to perform rejection sampling method to draw samples within gibbs.

Now, we can rewrite the full conditionals in the following form :

$$f(x|y) = \frac{3}{4\sqrt{2}} \cdot \frac{y\sqrt{x} + x\sqrt{y}}{y + \frac{3}{2\sqrt{2}}\sqrt{y}}$$

$$= \frac{3}{4\sqrt{2}} \quad \frac{1}{y + \frac{3}{2\sqrt{2}}\sqrt{y}} \left(y\sqrt{x} + x\sqrt{y} \right)$$

since, y is given or known for $f(x|y)$, it can be treated as a constant.

let, $$c_x = \frac{3}{4\sqrt{2}} \quad \frac{1}{y + \frac{3}{2\sqrt{2}}\sqrt{y}}$$

$$\therefore \quad f(x|y) = c_x \left(y\sqrt{x} + x\sqrt{y} \right)$$

Now, lets get the maximum of $f(x|y)$, to get this,

set, $\frac{\partial}{\partial x} f(x|y) = 0$

$$\Rightarrow \quad C_x \left[y \cdot \frac{1}{2\sqrt{x}} + \sqrt{y} \right] = 0$$

$$\Rightarrow \quad C_x \sqrt{y} \left[\frac{\sqrt{y}}{2\sqrt{x}} + 1 \right] = 0$$

$$\Rightarrow \quad \frac{\sqrt{y}}{2\sqrt{x}} = -1 \Rightarrow y = 4x \Rightarrow x = \frac{y}{4}$$

since, $0 < y < 2$ so, $\max\{f(x|y)\} = \frac{2}{4} = \frac{1}{2}$.

lets assume an envelope distribution $g(w) \sim \text{unif}(0,2)$.

then, $m\, g(w) \geq f(x|y)$; $g(w) = \frac{1}{2}$ [PDF of uniform]

$$\Rightarrow \quad m \cdot \max\{g(w)\} \geq \max\{f(x|y)\}$$

$$\Rightarrow \quad m \cdot \frac{1}{2} \geq \frac{1}{2}$$

Thus, $m \geq 1$.

Next, we will ~~d~~ be doing the following steps:

i) Generate z from uniform $(0,2)$

ii) Calculate the ratio $R = \dfrac{f_{x|y}(x=z)}{m \cdot g(z)}$

$$R = \frac{f_{x|y}(x=z)}{1 \times \frac{1}{2}} = 2 \cdot f_{x|y}(x=z)$$

iii) if $u < 2 \cdot f_{x|y}(x=z)$ then accept z, otherwise reject where $u \sim unif(0,1)$. Here, if we accept z, it will be a draw ~~from~~ from the full conditional of x.

Now, we can do similar things for $f(y|x)$ as both $f(x|y)$ and $f(y|x)$ are symmetric.

so, generate $z \sim uniform(0,2)$, then calculate

$$R = \frac{f_{y|x}(y=z)}{m \cdot g(z)} = 2 \cdot f_{y|x}(y=z)$$

then, accept z as a draw ~~from~~ from $f_{y|x}(y|x)$ if $u < 2 f_{y|x}(y=z)$ where $u \sim uniform(0,1)$

otherwise, reject it.

so far we have,

$$\cancel{f_{X|Y}(x|y)} \quad f(x|y) = C_x \left(y\sqrt{x} + x\sqrt{y} \right)$$

and, $$f(y|x) = C_y \left(y\sqrt{x} + x\sqrt{y} \right)$$

where, C_x and C_y are known constant for $f(x|y)$ and $f(y|x)$ respectively. Both function looks similar as X and Y are symmetric.

Now we can perform gibbs sampling by the following steps:

we will use the order $X \to Y$

step 1: initialize (x^0, y^0)

step 2: sample $x' \sim f(x|Y = y^0)$ $\quad y' \sim f(y|X = x')$

using rejection algorithm.

in general: $x^{(i)} \sim f(x|Y = y^{(i-1)})$; $y^{(i)} \sim f(y|X = x^{(i)})$

step 3: Repeat step 2

Solution : Q4 -

Given that,

$$x_1, \ldots, x_n \overset{iid}{\sim} N(\mu, \sigma^2)$$

and $f(\mu) \cdot f(\sigma^2) \propto \frac{1}{\sigma^2}$

we need to find the conditional posterior of σ^2.

lets calculate the likelihood first.

$$L(\mu, \sigma^2 | \underset{\sim}{x}) = \prod_{i=1}^{n} f(x_i | \mu, \sigma^2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x_i-\mu)^2\right\}$$

$$\propto (\sigma^2)^{-n/2} \cdot \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}$$

[ignoring constant term.

we know that,

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posterior $\propto$ likelihood $\times$ prior.

$$\therefore \quad f(\sigma^2 | \mu, \underset{\sim}{x}) \propto L(\mu, \sigma^2 | \underset{\sim}{x}) \, f(\mu) \cdot f(\sigma^2)$$

$$\propto (\sigma^2)^{-n/2} \cdot \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\} \cdot \frac{1}{\sigma^2}.$$

$$f(\sigma^2 \mid \mu, \underset{\sim}{x}) \propto (\sigma^2)^{-n/2} \cdot \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\} \cdot (\sigma^2)^{-1}$$

$$\propto (\sigma^2)^{-n/2-1} \cdot \exp\left\{-\frac{1}{\sigma^2} \cdot \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2}\right\}.$$

As we can see, the above expression is the kernel of the inverse-gamma PDF.

Thus,

$$\cancel{\sigma^2 \mid \mu, \underset{\sim}{x} \sim IG\left(\alpha = \frac{n}{2}, \beta = \frac{1}{2}\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{}\right)}$$

$$\sigma^2 \mid \mu, \underset{\sim}{x} \sim IG\left(\alpha = \frac{n}{2}, \beta = \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\right)$$

[if ~~x~~ $y \sim IG(\alpha, \beta)$ then,

$$f(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \exp\left\{-\frac{\beta}{y}\right\} \cdot y^{-\alpha-1}$$]

Now, here, $n = 5$; $\mu = 2$.

$$\therefore \sum_{i=1}^{5}(x_i-\mu)^2 = 3.25.$$

$$\therefore \sigma^2 \mid \mu = 2, \underset{\sim}{x} = (1.5, 2.6, 1.0, 3.0, 2.8) \sim IG\left(\frac{5}{2}, \frac{13}{8}\right)$$