1. full conditional for β:

prior on β:

$\beta \sim MVN_{p+1}(\mu_\beta, \Sigma_\beta)$.

$Y \sim MVN_n(X\beta + \theta, \underbrace{\sigma^2 I_n}_{\Sigma})$.

$\pi(\beta|\cdots) \propto f(D|\cdots)\,\pi(\beta)$

$$\propto \frac{1}{(2\pi)^{n/2} \cdot |\Sigma|^{1/2}} \cdot \exp\left\{-\frac{1}{2}(Y - X\beta - \theta)^T \Sigma^{-1} (Y - X\beta - \theta)\right\}.$$

$$\frac{1}{(2\pi)^{(p+1)/2} \cdot |\Sigma_\beta|^{1/2}} \cdot \exp\left\{-\frac{1}{2}(\beta - \mu_\beta)^T \Sigma_\beta^{-1} (\beta - \mu_\beta)\right\}.$$

$$\propto \exp\left\{-\frac{1}{2}(y - X\beta - \theta)^T \cdot (\sigma^2 I_n)^{-1} (Y - X\beta - \theta)\right\}.$$

$$\exp\left\{-\frac{1}{2}(\beta - \mu_\beta)^T \Sigma_\beta^{-1} (\beta - \mu_\beta)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma^2}\left(-Y^T X\beta - \beta^T X^T Y + \beta^T X^T X\beta + \beta^T X^T \theta + \cancel{\beta X^T} + \theta^T X\beta\right)\right.\right.$$

$$\left.\left. + \left(\beta^T \Sigma_\beta^{-1} \beta - \beta^T \Sigma_\beta^{-1} \mu_\beta - \mu_\beta^T \Sigma_\beta^{-1} \beta\right)\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma^2}\left(\beta^T X^T X\beta - 2Y^T X\beta + 2\theta^T X\beta\right) + \left(\beta^T \Sigma_\beta^{-1}\beta - 2\mu_\beta^T \Sigma_\beta^{-1}\beta\right)\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\beta^T\left(\frac{X^T X}{\sigma^2} + \Sigma_\beta^{-1}\right)\beta - 2\left(\frac{Y^T X}{\sigma^2} - \frac{\theta^T X}{\sigma^2} + \mu_\beta^T \Sigma_\beta^{-1}\right)\beta\right]\right\}$$

in general,

$$\propto \exp\left\{-\frac{1}{2}\left[\beta^T\left(X^T \Sigma^{-1} X + \Sigma_\beta^{-1}\right)\beta - 2\left(Y^T \Sigma^{-1} X - \theta^T \Sigma^{-1} X + \mu_\beta^T \Sigma_\beta^{-1}\right)\beta\right]\right\}$$

Now, comparing the exponent term with. $\theta^T A \theta - 2b^T\theta$

we get, $A = (X^T \Sigma^{-1} X + \Sigma_\beta^{-1})$

and, $b = X^T \Sigma^{-1} Y - X^T \Sigma^{-1}\theta + \Sigma_\beta^{-1}\mu_\beta$ | Dispersions are symmetric]

Note: $\Sigma = \sigma^2 I_n$

$\Sigma_\beta = 10^4 \times I_3$.

$\therefore\ \beta|\cdots \sim MVN_{p+1}(A^{-1}b.\ A^{-1})$

2. Full conditional for σ^2 : prior on $\sigma^2 \sim IG(a_0 = 2.5, b_0 = 2.5)$.

$$\pi(\sigma^2|\cdots) \propto f(D|\cdots)\,\pi(\sigma^2)$$

$$\propto (\sigma^2)^{-n/2} \exp\left\{-\frac{1}{\sigma^2}\frac{(y-X\beta-\theta)^T(y-X\beta-\theta)}{2}\right\} \exp\left\{-\frac{b_0}{\sigma^2}\right\}.(\sigma^2)^{-a_0-1}$$

$$\propto (\sigma^2)^{-(n/2+a_0)-1} \cdot \exp\left\{-\frac{1}{\sigma^2}\left[\frac{(y-X\beta-\theta)^T(y-X\beta-\theta)}{2} + b_0\right]\right\}$$

so,

$$\sigma^2|\cdots \sim IG\left(\frac{n}{2}+a_0,\ \frac{(y-X\beta-\theta)^T(y-X\beta-\theta)}{2} + b_0\right)$$

3. Full conditional for θ: $\theta \sim MVN(0, \underbrace{\tau^2 R}_{\Sigma_\theta})$; $y \sim MVN(X\beta + \theta, \Sigma = \sigma^2 I_n)$

$$\pi(\theta|\ldots) \propto f(D|\ldots)\, \pi(\theta|\tau^2, \lambda)$$

$$\propto |\Sigma|^{-1/2} \exp\left\{-\frac{1}{2}(y - X\beta - \theta)^T \Sigma^{-1}(y - X\beta - \theta)\right\} . |\Sigma_\theta|^{-1/2} .$$

$$\exp\left\{-\frac{1}{2}(\theta - 0)^T \Sigma_\theta^{-1}(\theta - 0)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left(-\underline{y^T\Sigma^{-1}\theta} + \beta^T X^T \Sigma^{-1}\theta - \underline{\theta^T\Sigma^{-1}y} + \theta^T\Sigma^{-1}X\beta + \theta^T\Sigma^{-1}\theta + \theta^T\Sigma_\theta^{-1}\theta\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\theta^T(\Sigma^{-1} + \Sigma_\theta^{-1})\theta - 2\left(y^T\Sigma^{-1} - \beta^T X^T\Sigma^{-1}\right)\theta\right]\right\}$$

comparing the exponent term with $\theta^T A \theta - 2b^T\theta$, we get.

$$A = (\Sigma^{-1} + \Sigma_\theta^{-1}) \;;\; b = \cancel{\Sigma^{-1}y - \Sigma^{-1}X\beta}.$$
$$\Sigma^{-1}y - \Sigma^{-1}X\beta.$$

$\therefore \theta|\ldots \sim MVN(A^{-1}b, A^{-1})$

$\Sigma = \sigma^2 I_n.$

$\Sigma_\theta = \tau^2 R.$

where, $R_{ij} = e^{-\lambda d(s_i, s_j)}$

4. Full conditional for τ^2 ; prior on $\tau^2 \sim IG(a_\tau = 2.5, b_\tau = 2.5)$

$\theta \sim MVN(0, \Sigma_\theta = \tau^2 R)$

$|\Sigma_\theta|^{-1/2} = |\tau^2 R|^{-1/2}$

$= \left((\tau^2)^n \cdot |R| \right)^{-1/2}$

$= (\tau^2)^{-n/2} \cdot |R|^{-1/2}$.

$$\pi(\tau^2 | \ldots) \propto f(D|\ldots)\, \pi(\theta | \tau^2, \lambda) \cdot \pi(\tau^2)$$

$$\propto |\Sigma|^{-1/2} \cdot \exp\left\{ -\frac{1}{2}(Y - XB - \theta)^T \Sigma^{-1} (Y - XB - \theta) \right\} \cdot$$

$$|\Sigma_\theta|^{-1/2} \cdot \exp\left\{ -\frac{1}{2} (\theta - 0)^T \Sigma_\theta^{-1} (\theta - 0) \right\} \cdot$$

$$(\tau^2)^{-a_\tau - 1} \cdot \exp\left\{ -\frac{1}{\tau^2} b_\tau \right\}$$

$$\propto (\tau^2)^{-n/2} \cdot |R|^{-1/2} \cdot \exp\left\{ -\frac{1}{\tau^2} \left[\frac{\theta^T R^{-1} \theta}{2} + b_\tau \right] \right\} \cdot (\tau^2)^{-a_\tau - 1}$$

$$\propto (\tau^2)^{-(n/2 + a_\tau) - 1} \cdot \exp\left\{ -\frac{1}{\tau^2} \left[\frac{\theta^T R^{-1} \theta}{2} + b_\tau \right] \right\}$$

$$\therefore \quad \tau^2 | \ldots \sim IG\left(\frac{n}{2} + a_\tau ,\ \frac{\theta^T R^{-1} \theta}{2} + b_\tau \right)$$

for τ^2: $\theta \sim MVN(0, \tau^2 R)$
$\hookrightarrow \Sigma_\theta$.

$$|\Sigma_\theta|^{-1/2} = |\tau^2 \underset{n\times n}{R}|^{-1/2} = ((\tau^2)^n \cdot |R|)^{-1/2} = (\tau^2)^{-n/2} \cdot |R|^{-1/2}.$$

FCD of λ:

$\to \lambda \sim \text{Unif}(a_\lambda, b_\lambda)$

$$\pi(\lambda|\cdots) \propto \underline{f(D|\cdots)} \cdot \pi(\theta|\tau^2,\lambda) \cdot \pi(\lambda)$$

$$\propto \cancel{(\tau^2)^{-n/2} \cdot |R|}^{-1/2}$$

$$\propto |\Sigma_\theta|^{-1/2} \cdot \exp\left\{-\frac{1}{2}\theta^T \Sigma_\theta^{-1}\theta\right\} \cdot \frac{1}{b_\lambda - a_\lambda}$$

[Likelihood $Y \sim MVN(X\beta + \theta, \sigma^2 I_n)$ is λ free, therefore ignored]

$$\propto (\tau^2)^{-n/2} \cdot |R|^{-1/2} \cdot \exp\left\{-\frac{1}{2\tau^2}\theta^T R^{-1}\theta\right\} \cdot \frac{1}{b_\lambda - a_\lambda}.$$

$$\propto |R|^{-1/2} \cdot \exp\left\{-\frac{1}{2\tau^2} \cdot \theta^T R^{-1}\theta\right\}.$$

[if all values of λ possess same probability]

Acceptance ratio:

$$P_A = \left(\frac{|R^*|}{|R^{(i)}|}\right)^{-1/2} \cdot \exp\left\{-\frac{1}{2\tau^2}\theta^T\left(R^{*-1} - R^{(i)-1}\right)\theta\right\}$$

$$\Rightarrow \ln(P_A) = -\frac{1}{2}\left(\ln|R^*| - \ln|R^{(i)}|\right) - \frac{1}{2\tau^2}\theta^T\left(R^{*-1} - R^{(i)-1}\right)\theta.$$

if $\ln(\text{runif}) < \ln(P_A)$

accept λ^*, otherwise $\lambda^{(i+1)} = \lambda^{(i)}$

Here, $R^*_{ij} = e^{-\lambda^* d(s_i, s_j)}$

and $R^{(i)}_{ij} = e^{-\lambda^{(i)} d(s_i, s_j)}$