Simulation Exercise with Solution

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Solution Q1:

Monte-Carlo approximation for the integral

$$\int_0^\infty \frac{\theta^6 \exp^{-\theta^2-\theta^3-2\theta}}{1+3\theta^2} d\theta$$

We aim to write this integral as $\int f(\theta)g(\theta)$ where $g(\theta)$ is a known density that is easy to simulate from. Then, we draw a large number of samples from $g(\theta)$, evaluate $f(\theta)$ for each one of those samples and take the mean, which will be the *Monte-Carlo* approximation for the above mentioned integral.
Approach 1: We use $Gamma(7, 2)$ as $g(\theta)$.

Now,

$$g(\theta) = \frac{2^7}{\Gamma(7)} \exp^{-2\theta} \theta^{7-1}$$

Thus our $f(\theta)$ will become,

$$f(\theta) = \frac{\Gamma(7)}{2^7} \frac{\exp^{-\theta^2-\theta^3}}{1+3\theta^2} d\theta$$

Approach 2: We use $exp(2)$ as $g(\theta)$.

Now,

$$g(\theta) = 2e^{-2\theta}$$

Thus our $f(\theta)$ will become,

$$f(\theta) = \frac{1}{2} \frac{\theta^6 \exp^{-\theta^2-\theta^3}}{1+3\theta^2} d\theta$$

```
Output:

 Actual result:  0.00359857
 Monte-Carlo approximation using Gamma(7, 2) PDF:  0.00362067
 Monte-Carlo approximation using Exponential(2) PDF: 0.00361638
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Solution Q2:

Monte-Carlo approximation for the integral

$$\int_{1.5}^{6} \frac{\log \theta}{2+\theta^2} d\theta$$

As per procedure mentioned in *solution-1*, we use $unif(1.5, 6)$ as $g(\theta)$. Now,

$$g(\theta) = \frac{1}{6-1.5} = \frac{1}{4.5}$$

Thus our $f(\theta)$ will become,

$$f(\theta) = 4.5\frac{\log(\theta)}{2+\theta^2}$$

```
Output:

 Actual result:  0.3734025
 Monte-Carlo approximation using Uniform(1.5, 6) PDF: 0.3737958
```

Solution 3:

Metropolis-Hastings Simulation to draw sample where target density is:

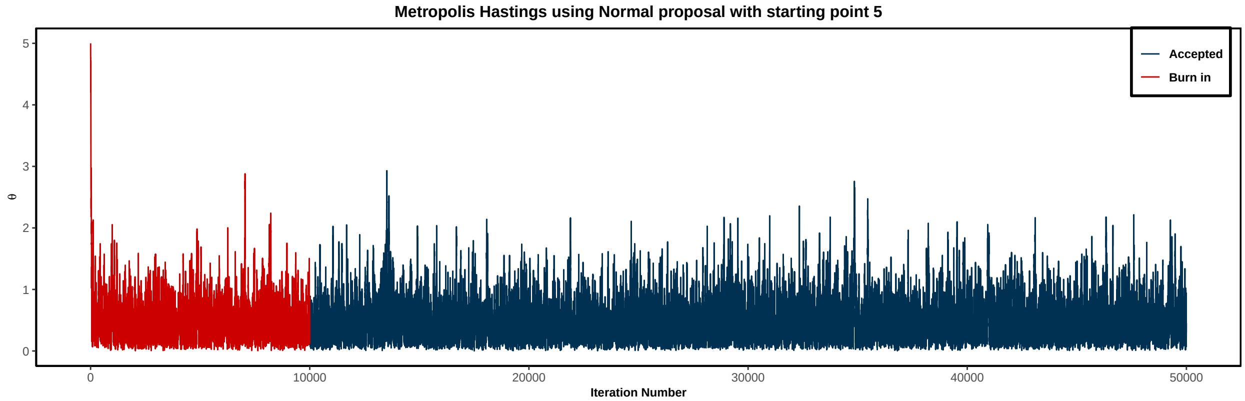
$$\theta \sim Gamma(2, 4)$$

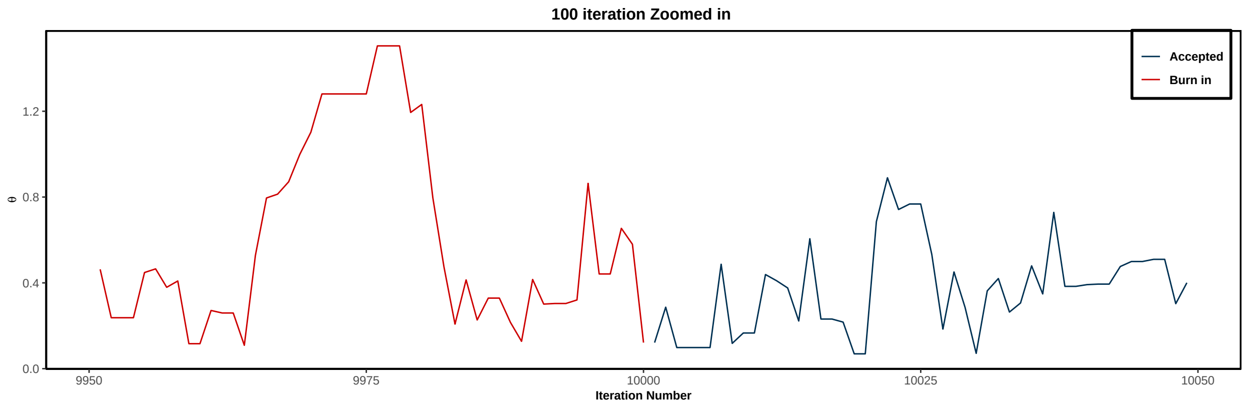
From target distribution True mean is 0.5

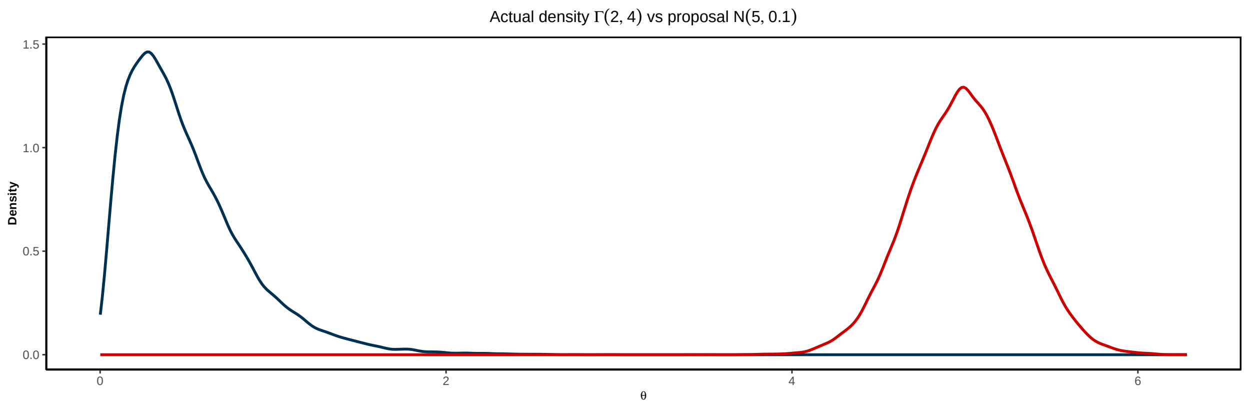
- Approach 1: Normal distribution as a proposal with variance = 0.1 and $\theta_0 = 5$

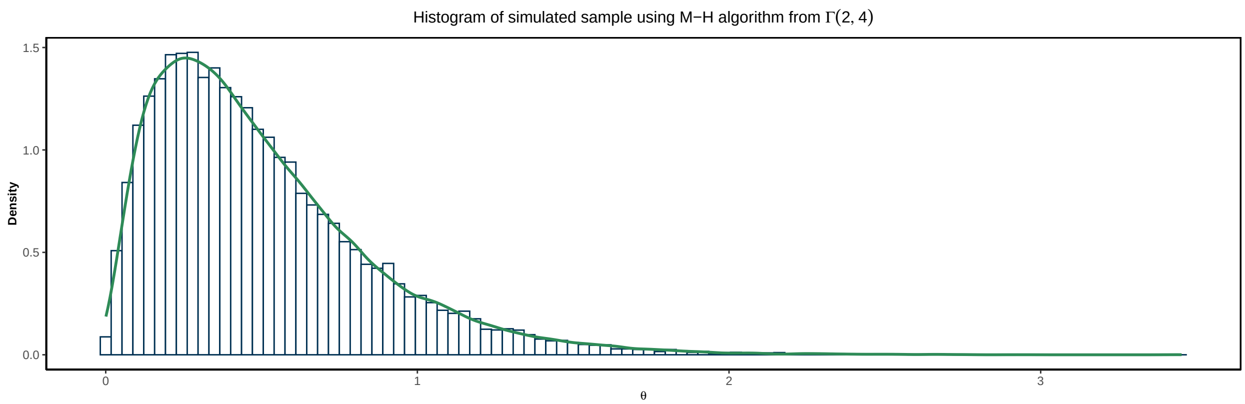
```
Acceptance Rate: 0.66786
Estimated Mean: 0.4978995
Difference from True Mean: 0.002100459
```

- Approach 1: Normal distribution as a proposal with variance $= 0.1$ and $\theta_0 = 5$

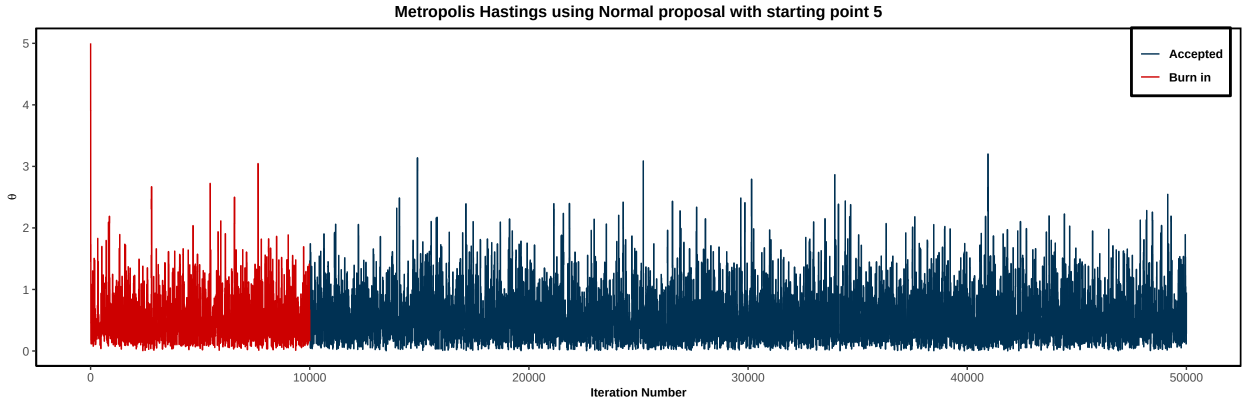


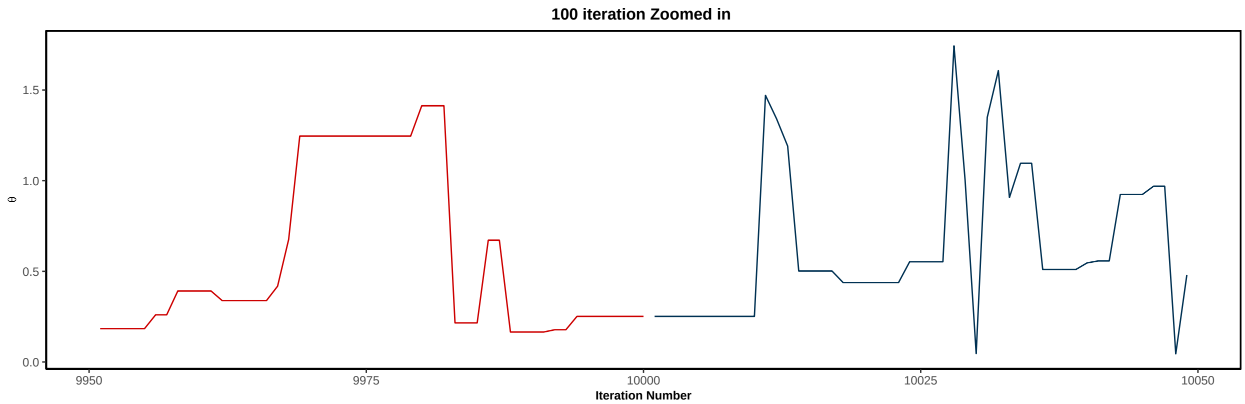


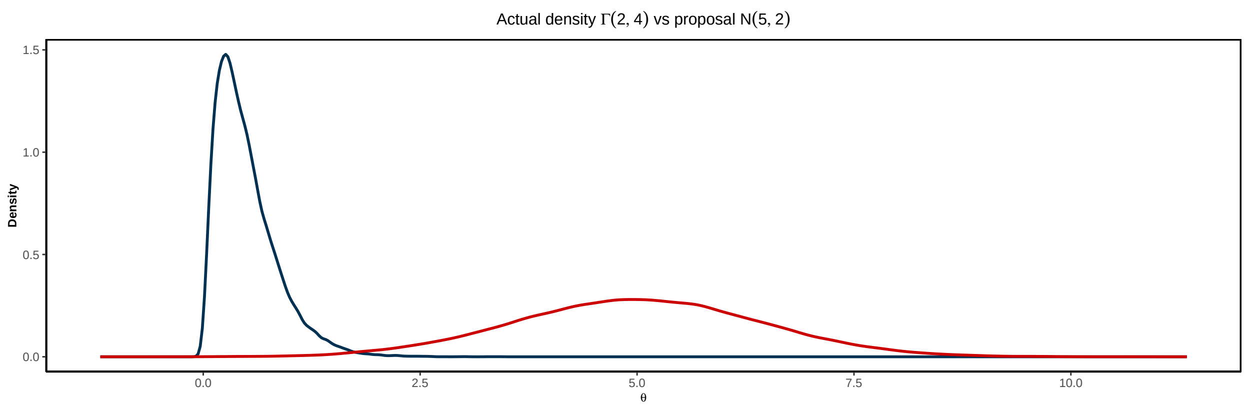


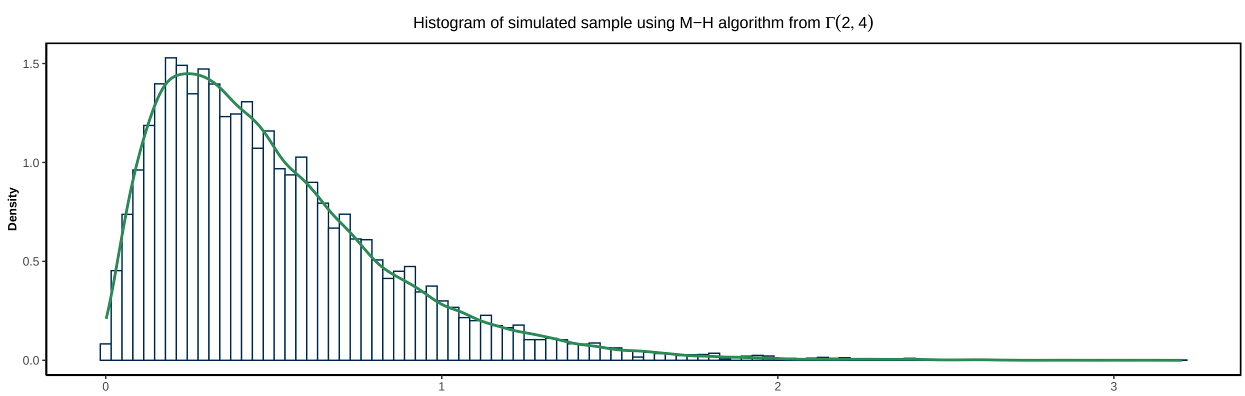


- Approach 2: Normal distribution as a proposal with variance = 2 and $\theta_0 = 5$









- Approach 2: Normal distribution as a proposal with variance = 2 and $\theta_0 = 5$

```
Acceptance Rate: 0.2449
Estimated Mean: 0.5086692
Difference from True Mean: 0.008669169
```

Solution 5:

Metropolis-Hastings Simulation to draw samples from density:

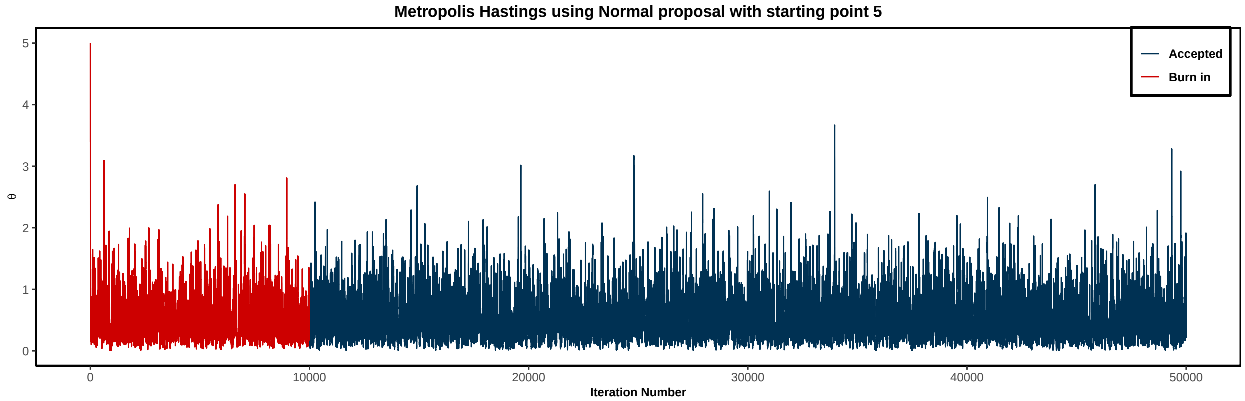
$$\theta \sim Gamma(2, 4)$$

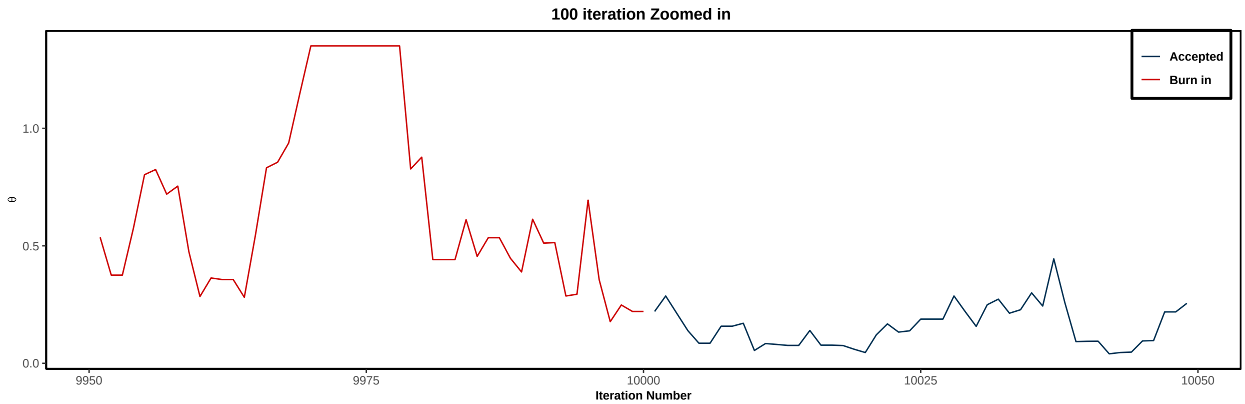
From target distribution True mean is 0.5

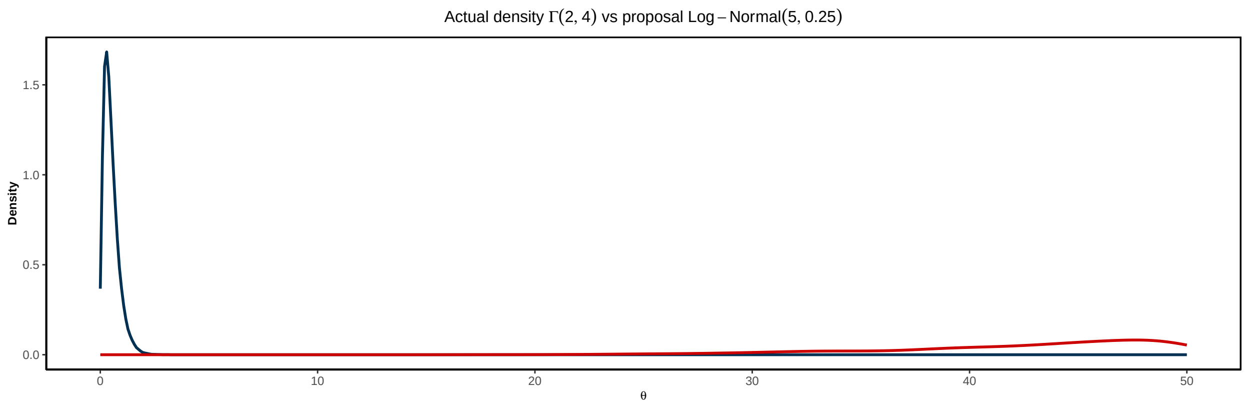
- Approach 1: Log-Normal distribution as a proposal with variance = 0.25 and $\theta_0 = 5$

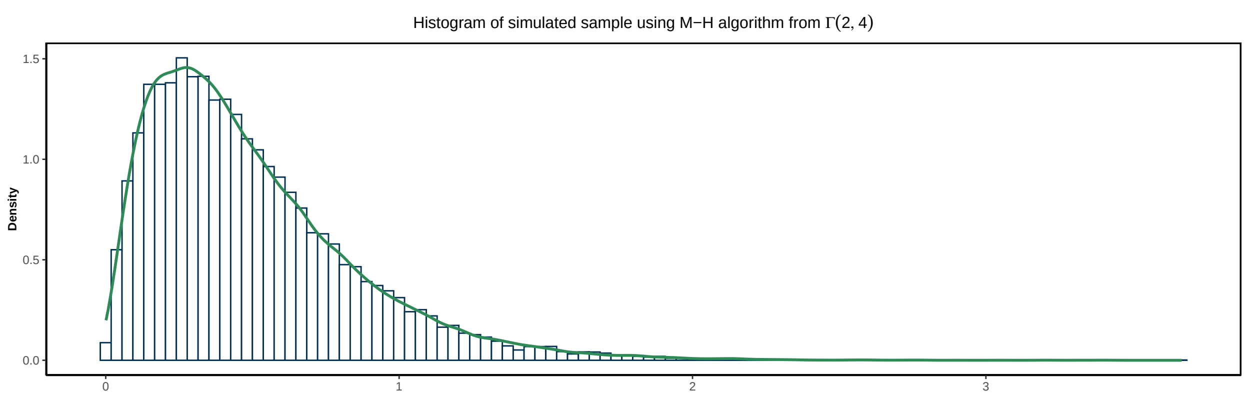
```
Acceptance Rate: 0.7917
Estimated Mean: 0.4979935
Difference from True Mean: 0.002006515
```

- Approach 1: Log-Normal distribution as a proposal with variance $= 0.25$ and $\theta_0 = 5$

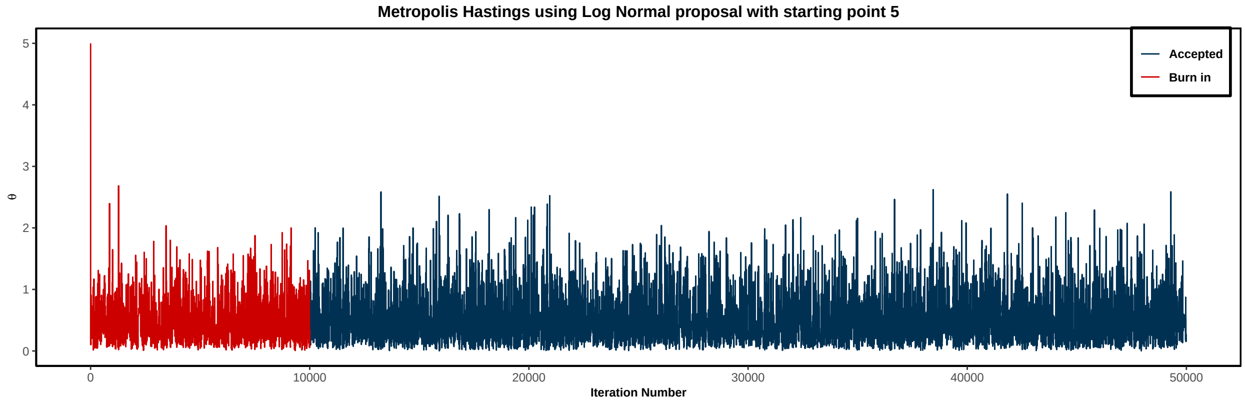


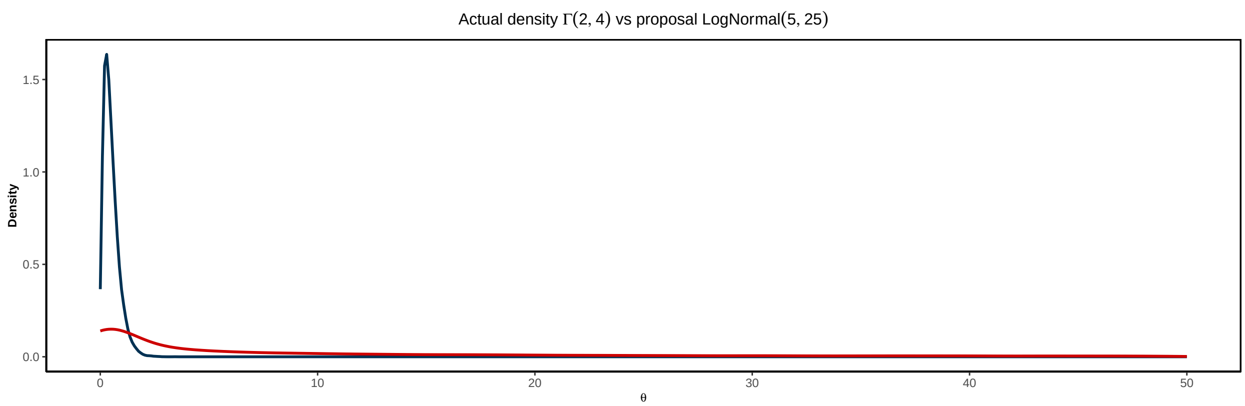


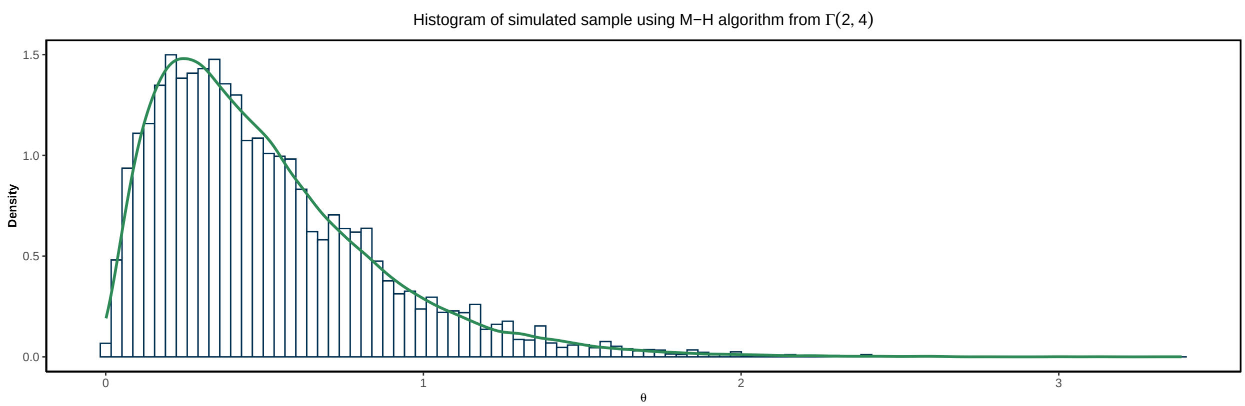




- Approach 2: Log Normal distribution as a proposal with variance = 25 and $\theta_0 = 5$







- Approach 2: Log Normal distribution as a proposal with variance = 25 and $\theta_0 = 5$

```
Acceptance Rate: 0.1868
Estimated Mean: 0.5003575
Difference from True Mean: 0.0003574558
```

solution 6:

i)

$$f(\theta) = c * \theta^3$$

to be a valid PDF, c need to be 4

ii) Simulation using Inverse CDDF method:

Since,

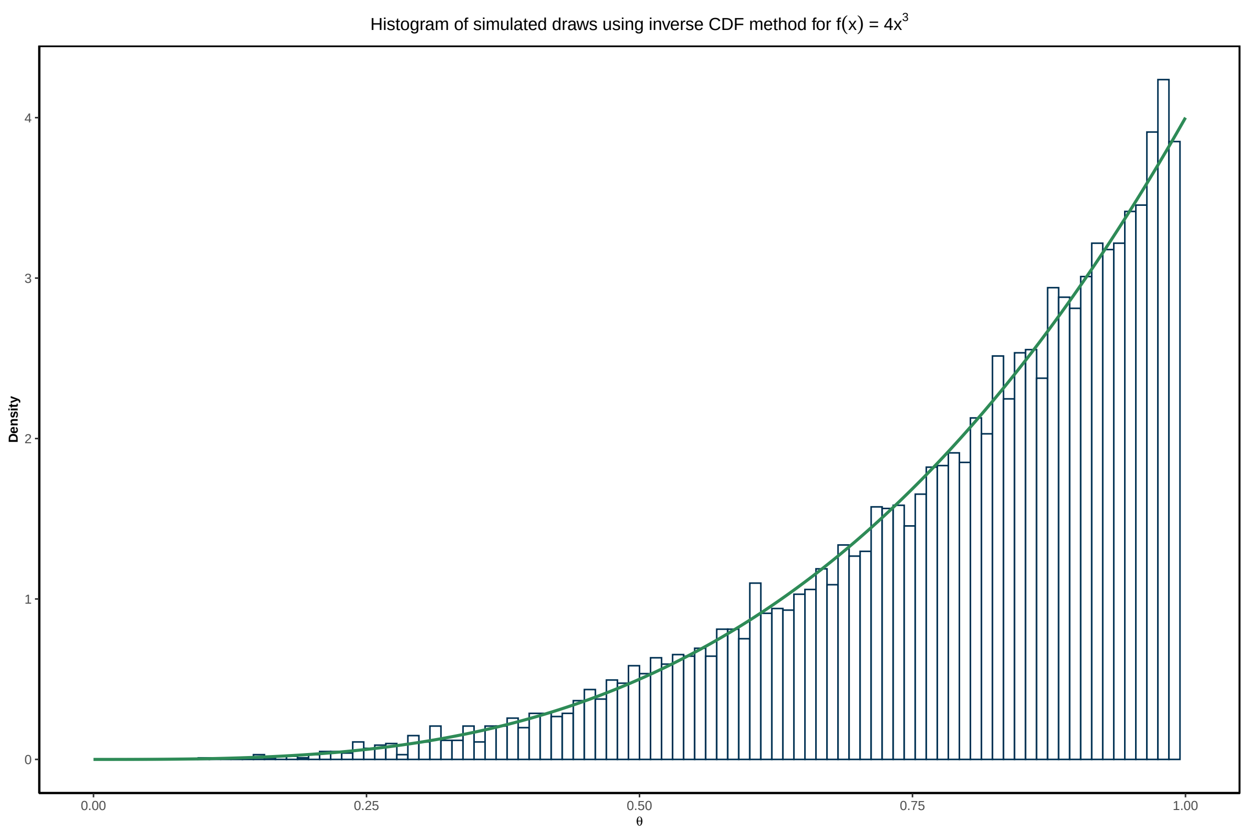
$$F_\theta(t) = t^4$$

Thus,

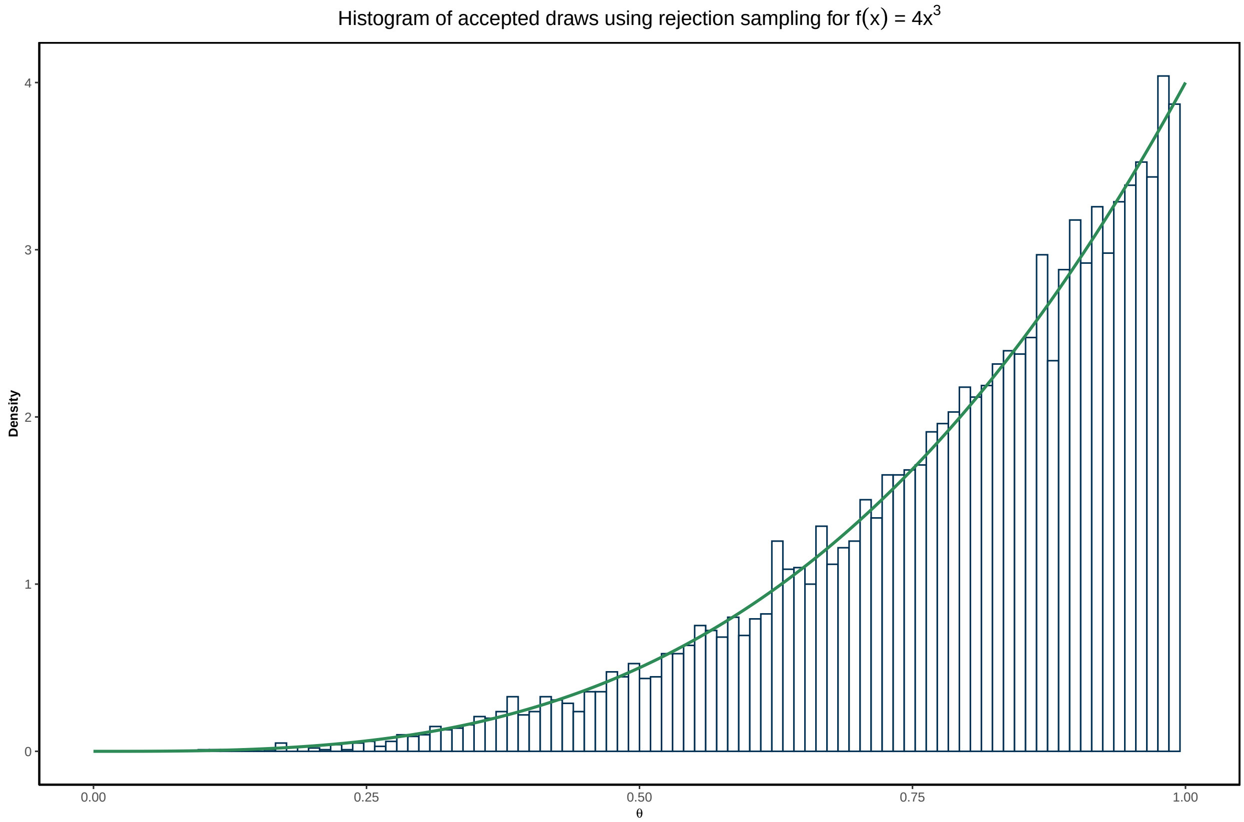
$$\theta = F^{-1}(u) = u^{\frac{1}{4}}$$

where,

$$U \sim unif(0,1)$$



iii) Simulation usin Rejection sampling:



iv) Summary results from two sets of observations

- Using Inverse CDF:

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Empirical Mean: 0.8002587
Variance: 0.02727211
60% Quantile: 0.8820321
```

- Using Rejection sampling:

```
Empirical Mean: 0.8020204
Variance: 0.02589842
60% Quantile: 0.8800872
```