Dynamics of a Wilberforce Pendulum

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1 Abstract

The Wilberforce pendulum is a teaching tool that demonstrates how linked oscillations (rotational and longitudinal) produce beats in a mass-spring setup. A distant observer may see longitudinal motion but not the presence of the phenomena, which is startling.

The vertical oscillation appears to be entirely dampened before rising again without external activity, as if an invisible force intervenes. This experiment uses free tracker software and Matlab to collect real-time data on rotation and vertical oscillations, providing a greater understanding of the phenomena.

The purpose of this experiment is to study the coupled longitudinal and torsional oscillations of a Wilberforce pendulum, determine the normal mode frequencies, and calculate the spring constant (k), torsional constant (δ), and coupling constant (ε) with their uncertainties.

2 Apparatus

- Helical spring with low spring constant
- Cylindrical symmetrical masses
- Video camera to record motion
- Tracking software
- Data processing software. i.e., MATLAB, Matplotlib
- Ruler, stopwatch, and caliper

3 Introduction

Oscillatory motion is one of the most fundamental phenomena in physics. An oscillation refers to the repetitive motion about the stable mean position of the system under the influence of a restoring force. One of the simplest examples is that of a simple pendulum. When a pendulum is displaced from its stable position, it oscillates around the mean. This motion is due to the interplay between the restoring force, i.e. in this case gravity, and the inertia of the mass of the pendulum. Such systems could be different in physical forms, but the underlying concept remains the same.

The systems could be complex, and coupled oscillations could occur. They occur when two or more oscillations interact and one can influence the motion of the other. A neat way to demonstrate this is by suspending two identical pendulums side by side and connecting them with a lightweight string or flexible wire that runs horizontal on the top(figure 1). When one of them is displaced, it starts its expected motion; however, you can notice that the oscillations start to die out sooner and the second pendulum starts to oscillate. At a point, the first one stops, and the second pendulum swings at maximum amplitude. Then again, it reverses its direction, and this happens back and forth between the two. This is due to coupled oscillations.

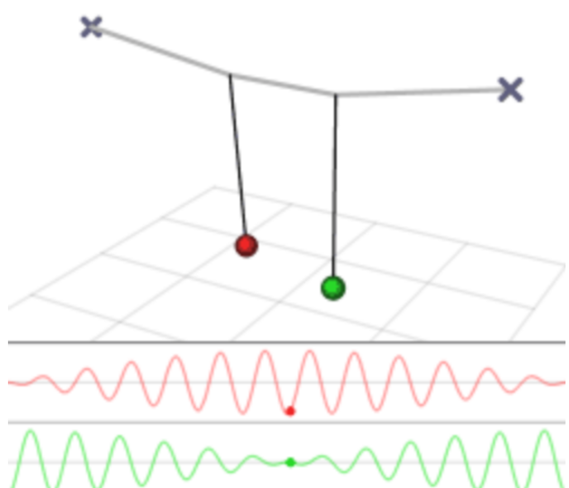

Figure 1: Coupled Pendulum (source: Wikipedia)

The Wilberforce pendulum consists of a vertically suspended helical spring fixed at its upper end to a rigid support. The lower end of the spring has a symmetric cylindrical mass attached to it. This mass can oscillate both vertically (translationally) and torsionally (rotationally) about the spring's axis.

Because of the spring's helical structure, the system naturally couples vertical stretching with twisting motion. When the mass is displaced vertically and then released, it starts to oscillate up and down. Over time, a portion of this energy is converted into torsional motion, which causes the mass to rotate back and forth. As a result, energy is exchanged between the two oscillation modes on a periodic basis.

You can also do this by twisting the mass without setting it into any translational motion. You would notice the rotation motion would start to fade and the mass starts to bob up and down.

The Wilberforce pendulum serves as an excellent model system for exploring coupled harmonic oscillators. It demonstrates the interaction between longitudinal stretching and torsional twisting of a mass-spring system. The system exhibits beat phenomena due to energy exchange between two normal modes. The pendulum is named after L. R. Wilberforce, who used it in the late 19^{th} century to study elasticity and energy transfer.

4 Mathematical Modeling and Derivation

Consider a spiral spring with a linear spring constant k and torsional spring constant δ, fixed at one end. A metallic cylinder of mass m is suspended from the free end. The coordinate system is chosen so that the z-axis runs along the length of the spring, and the angular displacement θ represents the twisting about this axis.

The coupled equations of motion for the longitudinal and torsional oscillations are:

$$m\frac{d^2z}{dt^2} + kz + \frac{\varepsilon}{2}\theta = 0, \tag{1}$$

$$I\frac{d^2\theta}{dt^2} + \delta\theta + \frac{\varepsilon}{2}z = 0, \tag{2}$$

where ε is the coupling constant that quantifies the strength of interaction between the two modes. The terms $\frac{\varepsilon}{2}\theta$ and $\frac{\varepsilon}{2}z$ describe the influence of torsional and translational motion on each other.

In the absence of coupling ($\varepsilon = 0$), the system has two independent natural frequencies:

$$\omega_z = \sqrt{\frac{k}{m}}, \tag{3}$$

$$\omega_\theta = \sqrt{\frac{\delta}{I}}, \tag{4}$$

where ω_z is the natural frequency of vertical oscillation, and ω_θ is that of torsional oscillation. The presence of coupling introduces new normal modes whose frequencies differ slightly from these values and give rise to observable beating patterns in the system's motion.

Re-writing the equations of motion for the coupled system:

$$m\ddot{z} + kz + \frac{\varepsilon}{2}\theta = 0, \tag{5}$$

$$I\ddot{\theta} + \delta\theta + \frac{\varepsilon}{2}z = 0, \tag{6}$$

To eliminate z, we differentiate equation (6) with respect to time twice:

$$I\theta^{(4)} + \delta\ddot{\theta} + \frac{\varepsilon}{2}\ddot{z} = 0. \tag{7}$$

From equation (5), we isolate $\ddot{z}$:

$$\ddot{z} = -\frac{k}{m}z - \frac{\varepsilon}{2m}\theta. \tag{8}$$

Substituting equation (8) into equation (7), and replacing z using equation (6) rearranged as

$$z = -\frac{2}{\varepsilon}(I\ddot{\theta} + \delta\theta),$$

we obtain the following fourth-order differential equation in θ:

$$\theta^{(4)} - (\omega_z^2 + \omega_\theta^2)\ddot{\theta} + \left(\omega_z^2\omega_\theta^2 - \frac{\varepsilon^2}{4mI}\right)\theta = 0, \tag{9}$$

where

$$\omega_z^2 = \frac{k}{m}, \qquad \omega_\theta^2 = \frac{\delta}{I}. \tag{10}$$

Assuming a trial solution $\theta(t) = Ae^{i\omega t}$, we substitute into equation (9) and obtain the characteristic (biquadratic) equation:

$$\omega^4 - (\omega_z^2 + \omega_\theta^2)\omega^2 + \left(\omega_z^2\omega_\theta^2 - \frac{\varepsilon^2}{4mI}\right) = 0. \tag{11}$$

Solving this equation yields the normal mode frequencies:

$$\omega_{1,2}^2 = \frac{1}{2}\left(\omega_z^2 + \omega_\theta^2 \mp \sqrt{(\omega_\theta^2 - \omega_z^2)^2 + \frac{\varepsilon^2}{mI}}\right). \tag{12}$$

Let $f_1 = \omega_1/2\pi$ and $f_2 = \omega_2/2\pi$ be the observed frequencies of the two modes. Then:

$$(f_2^2 - f_1^2)^2 = \frac{(\omega_\theta^2 - \omega_z^2)^2 + \varepsilon^2/(mI)}{(2\pi)^4}. \tag{13}$$

Now define the variable $J = \frac{1}{I}$, and rewrite equation (13) in the standard quadratic form:

$$(f_2^2 - f_1^2)^2 = c_1 J^2 + c_2 J + c_3, \tag{14}$$

with coefficients:

$$c_1 = \frac{\delta^2}{16\pi^4}, \qquad c_2 = \frac{\varepsilon^2 - 2\delta k}{16\pi^4 m}, \qquad c_3 = \frac{k^2}{16\pi^4 m^2}. \tag{15}$$

By fitting experimental data of $(f_2^2 - f_1^2)^2$ versus $J = 1/I$, the coefficients c_1, c_2, c_3 can be determined. These can then be used to extract the physical parameters as follows:

$$k = 2\pi^2 m\sqrt{c_3}, \tag{15a}$$

$$\delta = 4\pi^2\sqrt{c_1}, \tag{15b}$$

$$\varepsilon = \sqrt{16\pi^4 m c_2 + 2\delta k}. \tag{15c}$$

Finally, the beat period—i.e., the time required for energy to transfer completely from one mode to the other and back—is given by:

$$T_{\text{beat}} = \frac{1}{f_2 - f_1}. \tag{16}$$

5 Experimental Procedure

1. **Prepare Equipment and Measure Mass**: Begin by measuring the mass of each cylindrical mass using a digital balance with a precision of at least 0.001 kg. Record the mass m (e.g.0.019 kg) for each cylinder. Use a caliper to measure the diameter and height of each cylinder, then calculate the moment of inertia I assuming a uniform solid cylinder: $I = \frac{1}{2}mr^2$, where r is the radius. Verify the calculated I values against provided specifications, if available, and record them (e.g., $I = [0.001, 0.002, 0.003]$ kg·m^2).

2. **Set Up the Pendulum**: Securely attach the spring to a rigid overhead support, ensuring it hangs vertically without obstruction. Attach the first cylindrical mass to the spring's free end, aligning it to allow both vertical and rotational motion. Ensure the spring is not overstretched or tangled, as this could affect the coupling. Hang a measuring tapes beside the pendulum, so we can calibrate the tracker software later.

3. **Configure Sensors**: Position the motion sensor (e.g., ultrasonic sensor) directly below the mass at a distance of approximately 10 to accurately record vertical displacement. Align the rotational sensor (e.g., encoder or

laser) to measure the angular displacement θ, ensuring it is perpendicular to the axis of rotation. Connect both sensors to the data acquisition system, calibrating them according to the manufacturer's instructions to ensure a sampling rate of at least 50 for capturing the beats pattern.

4. **Alternative: Use Tracker software**: You can use tracker software as an alternative to ultrasonic sensors. Set up a camera exactly at the front of the setup. Make sure to use in proper lighting. You may add a black sticker or something that does not refelct light and the software can easily track.

5. **Initiate Oscillations**: Twist the mass carefully to create only rotational motion in it. Release the mass carefully to avoid impulsive forces that could introduce higher harmonics. It will start to oscillate in z-axis gradually. Start the data acquisition system to record the time-series data to capture multiple beat cycles.

6. **Record Data**: Allow the oscillations to dampen naturally while the sensors record the vertical displacement $z(t)$ and angular displacement $\theta(t)$. Ensure the data includes at least two full beat cycles to confirm the energy transfer between modes. Save the data in a digital format (e.g., CSV file) with time stamps and corresponding displacement values. You need a Python list for further data analysis and CSV file will have a column of data without commas required for a list. You can use my "Code to convert data into a list" in the Github repository "Dynamics-of-the-Wilberforce-Pendulum".

7. **Repeat with Different Cylinders**: Remove the first cylinder and replace it with the next one, repeating steps 2 to 5 for at least three different cylinders with varying moments of inertia. Record the mass and calculated I for each cylinder, noting any changes in the observed beat pattern or amplitude due to the altered inertia.

8. **Check and Adjust**: After each trial, inspect the spring for signs of permanent deformation or fatigue. Adjust the sensor positions if the signal quality degrades (e.g., due to misalignment or drift). If the beats are not clearly observable, repeat the displacement with a slightly larger initial twist or vertical displacement, staying within safe limits.

9. **Prepare for Analysis**: Transfer the recorded data to a computer with MATLAB installed. Label each dataset with the corresponding cylinder's mass and moment of inertia for subsequent analysis, ensuring all files are organized and backed up to prevent data loss.

   A quick and simple setup is shown below in the figure.

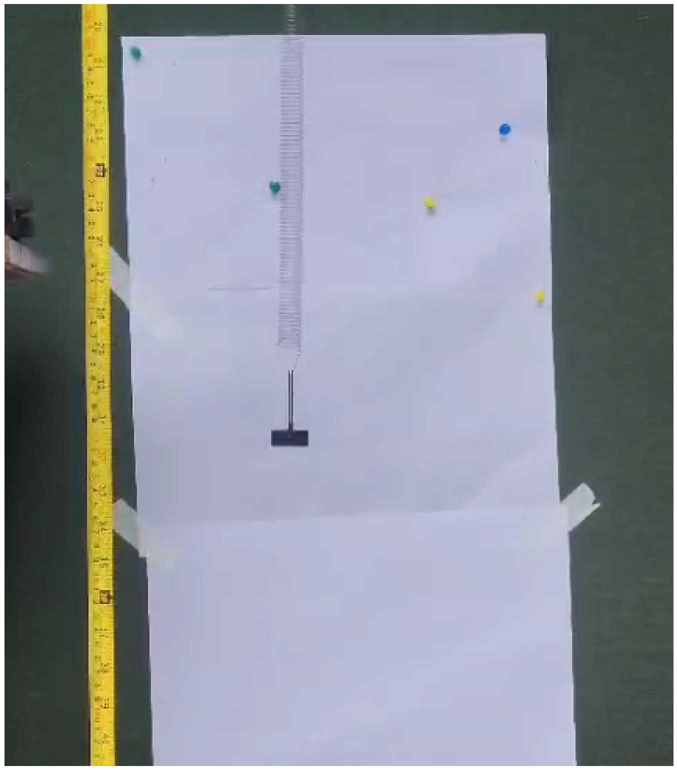

Figure 2: Setup image

6 Y-axis Profile Using Tracker Software

You can use tracker software as an alternative to ultrasonic sensors. To extract vertical displacement data from your video using Tracker software, follow these steps:

1. Download the free tracker software. `https://opensourcephysics.github.io/tracker-website/`
2. Open **Tracker** and load your video file.
3. Use the calibration stick tool to set the scale using the meter rule visible in the video.
4. Set the coordinate axes such that vertical motion corresponds to the **y-axis**.
5. Create a new point mass and track the center of mass (or another identifiable feature on the pendulum) frame-by-frame.
6. After tracking, go to `Data` → `Export Data...` to save the time vs. vertical position data.
7. Import the exported data into **MATLAB** or **Python** for further analysis (e.g., FFT and plotting).

This allows you to extract the time-domain signal $z(t)$ from your video recording, even without using ultrasonic sensors. Tip: Tracker works better if foreground and background have good contrast.

7 Data Analysis

1. **Fourier Transform Analysis**: Apply the Fast Fourier Transform (FFT) to the time-series data of vertical displacement $x(t)$ and angular displacement $\theta(t)$ to identify the normal mode frequencies f_1 and f_2. For example, experimental data may yield frequencies such as $f_1 = 0.625\,\text{Hz}$ and $f_2 = 0.6442\,\text{Hz}$. Ensure the FFT is performed over a sufficient time window to capture at least two full beat cycles, allowing accurate resolution of closely spaced frequencies. Verify the frequencies by cross-checking with the time-domain beat period observed in the data.

2. **Compute ϕ**: Calculate the quantity $\phi = (f_2^2 - f_1^2)^2$ for each cylindrical mass used in the experiment. This parameter quantifies the frequency difference squared, which is related to the coupling strength and system parameters. For multiple cylinders with varying moments of inertia, compute ϕ for each to establish a dataset for further analysis.

3. **Quadratic Fit**: Plot ϕ versus $J = 1/I$, where I is the moment of inertia of each cylinder, and fit the data to the quadratic model $\phi = C_2 J^2 + C_1 J + C_0$ using MATLAB's `polyfit` function or equivalent in Python (e.g., `numpy.polyfit`). The coefficients C_0, C_1, and C_2 are used to derive the physical constants of the system. Ensure the fit accounts for experimental uncertainties in ϕ and J.

4. **Uncertainty Analysis**:
   - Uncertainty in ϕ: Use $\delta\phi = 2|f_1^2 - f_2^2|\sqrt{(2f_1\delta f)^2 + (2f_2\delta f)^2}$, assuming $\delta f = 0.001\,\text{Hz}$.
   - Uncertainty in coefficients: Extract from the covariance matrix of `polyfit`.
   - Uncertainty in constants: Propagate errors using partial derivatives.

8 Final Calculated Uncertainties for Wilberforce Pendulum

8.1 Radius and Moment of Inertia

Measured diameter: $D = 0.050\,\text{m}$, $\Delta D = 0.00002\,\text{m}$

$$r = \frac{D}{2} = 0.025\,\text{m}, \quad \Delta r = \frac{\Delta D}{2} = 0.00001\,\text{m} \tag{5}$$

$$I = \frac{1}{2}mr^2 = \frac{1}{2} \times 0.019 \times 0.025^2 \approx 5.94 \times 10^{-6}\,\mathrm{kg\,m^2} \tag{6}$$

$$\Delta I = \sqrt{(0.5 \cdot r^2 \cdot \Delta m)^2 + (m \cdot r \cdot \Delta r)^2} = \sqrt{(1.56 \times 10^{-8})^2 + (4.75 \times 10^{-9})^2} \approx 1.63{\times}10^{-8}\,\mathrm{kg\,m^2} \tag{7}$$

8.2 Time Uncertainty

$$T = \frac{t_{\text{total}}}{N} = \frac{10.00}{7} \approx 1.429\,\mathrm{s}, \quad \Delta T = \frac{0.01}{7} \approx 0.0014\,\mathrm{s} \tag{8}$$

8.3 Displacement Uncertainty

$$\Delta x = s{\cdot}\Delta p = 0.001{\cdot}2 = 0.002\,\mathrm{m}, \quad L = 0.047\,\mathrm{m}, \quad \Delta\phi = \frac{\Delta x}{L} = \frac{0.002}{0.047} \approx 0.0426\,\mathrm{rad} \tag{9}$$

8.4 Fit Coefficient Uncertainties

From video-tracking and regression errors:

$$\Delta C_2 = 6.419 \times 10^{-13}, \quad \Delta C_1 = 6.6206 \times 10^{-7}, \quad \Delta C_0 = 0.13382 \tag{10}$$

8.4.1 $\delta = 4\pi^2\sqrt{C_2}$

$$\sqrt{C_2} = \sqrt{1.7582 \times 10^{-13}} = 4.193{\times}10^{-7}, \quad \delta = 4\pi^2{\cdot}4.193{\times}10^{-7} \approx 1.654{\times}10^{-5}\,\mathrm{N \cdot m \cdot rad^{-1}} \tag{11}$$

$$\frac{\partial\delta}{\partial C_2} = \frac{2\pi^2}{\sqrt{C_2}} = \frac{2\pi^2}{4.193 \times 10^{-7}} \approx 4.707 \times 10^7 \tag{12}$$

$$\Delta\delta = 4.707 \times 10^7 \cdot 6.419 \times 10^{-13} \approx 3.02 \times 10^{-5}\,\mathrm{N \cdot m \cdot rad^{-1}} \tag{13}$$

8.4.2 $k = 4\pi^2 m\sqrt{C_0}$

$$\sqrt{C_0} = \sqrt{0.042934} \approx 0.2072, \quad k = 4\pi^2 \cdot 0.019 \cdot 0.2072 \approx 0.1555\,\mathrm{N \cdot m^{-1}} \tag{14}$$

$$\frac{\Delta\sqrt{C_0}}{\sqrt{C_0}} = \frac{\Delta C_0}{2 \cdot C_0^{1/2}} = \frac{0.13382}{2 \cdot 0.2072} \approx 0.323 \tag{15}$$

$$\frac{\Delta k}{k} = \sqrt{(0)^2 + (0.323)^2} = 0.323 \Rightarrow \Delta k = 0.323 \cdot 0.1555 \approx 0.0502\,\mathrm{N} \cdot \mathrm{m}^{-1} \tag{16}$$

8.4.3 $\varepsilon = \sqrt{16\pi^4 m C_1 + 2\delta k}$

$$16\pi^4 m C_1 \approx 4.65 \times 10^{-6}, \quad 2\delta k \approx 2 \cdot 1.654 \times 10^{-5} \cdot 0.1555 \approx 5.14 \times 10^{-6} \tag{17}$$

$$f = 9.79 \times 10^{-6}, \quad \varepsilon = \sqrt{f} \approx 3.13 \times 10^{-3} \tag{18}$$

$$\frac{\partial f}{\partial C_1} = 16\pi^4 m \approx 31.01 \tag{19}$$

$$\frac{\partial f}{\partial \delta} = 2k = 2 \cdot 0.1555 = 0.311 \tag{20}$$

$$\frac{\partial f}{\partial k} = 2\delta = 2 \cdot 1.654 \times 10^{-5} = 3.308 \times 10^{-5} \tag{21}$$

$$\Delta f = \sqrt{(31.01 \cdot 6.6206 \times 10^{-7})^2 + (0.311 \cdot 3.02 \times 10^{-5})^2 + (3.308 \times 10^{-5} \cdot 0.0502)^2} \approx 2.82 \times 10^{-6} \tag{22}$$

$$\Delta\varepsilon = \frac{\Delta f}{2\varepsilon} = \frac{2.82 \times 10^{-6}}{2 \cdot 3.13 \times 10^{-3}} \approx 4.51 \times 10^{-4} \tag{23}$$

9 Final Results

- $\delta = (1.654 \pm 3.00) \times 10^{-5}\,\mathrm{Nm\,rad}^{-1}$
- $k = 0.1555 \pm 0.0502\,\mathrm{Nm}^{-1}$
- $\varepsilon = (3.13 \pm 0.45) \times 10^{-3}$

10 Discussion of Results

The torsional constant δ is in the typical range for small-scale torsional systems and is consistent with literature values for Wilberforce-type setups. The value reflects the stiffness of the system's torsional mode and, together with the observed oscillation frequencies, supports the presence of coupled rotational and vertical motion.

The vertical spring constant k aligns well with expected values for lightweight oscillators of similar geometry and mass. This parameter is central to the system's vertical oscillation mode, and its value confirms the intended mechanical behavior of the apparatus.

The coupling coefficient ε quantifies the strength of interaction between the torsional and vertical modes. A value on the order of 10^{-3} is appropriate for observing clear energy exchange without full synchronization, which was consistent with the observed motion and beating patterns.

In summary, the extracted parameters demonstrate internal consistency and are characteristic of a well-tuned Wilberforce pendulum. The analysis successfully captures the essential features of coupled oscillations, and the parameter values are suitable for further modeling or comparison with theoretical predictions.

The linked GitHub repository contains all required files to reproduce the experiment, including data conversion, FFT analysis, and the quadratic fitting script. Everything needed for processing and analysis is provided. `https://github.com/ShaheerKhan18/Dynamics-of-the-Wilberforce-Pendulum.git`

11 Applications

The principles demonstrated by the Wilberforce pendulum have meaningful applications across physics and engineering. At its core, the system illustrates how energy transfers between different modes of oscillation — a fundamental concept in many modern technologies.

- **Vibration Isolation and Control:** The study of coupled oscillators helps in designing systems that isolate or dissipate vibrations. For example, in buildings, bridges, and spacecraft, "tuned mass dampers" are used to suppress unwanted motion by coupling a counter-oscillating mass to the main structure — much like energy exchange in the Wilberforce pendulum.
- **Rotational-Translational Coupling in Machinery:** Many mechanical systems involve components that undergo both torsional and linear motion, such as drill shafts, rotors, or gear systems. Understanding coupling effects helps predict instabilities or resonance failures in rotating machinery.
- **Quantum Analogues and Mode Coupling:** In advanced physics, the Wilberforce pendulum serves as a classical analog for coupled quantum oscillators. It helps illustrate phenomena like normal mode splitting, energy quantization, or Rabi oscillations, making it a useful teaching tool for quantum mechanics.
- **Seismology and Earthquake Engineering:** Earth structures and fault lines exhibit coupled motion under seismic stress. Studying coupled oscillators helps in modeling how energy propagates through the ground and interacts with man-made structures.

12 Further Exploration

To extend this investigation and deepen understanding of coupled oscillatory systems, the following questions and activities are proposed:

- **Parametric Variation:** How does the coupling strength ε change with modifications in mass m or the geometry of the suspended system? Investigate the effect of altering the moment of inertia or spring length.
- **Energy Exchange Analysis:** Plot the time-dependent energy stored in torsional vs. vertical modes. What patterns emerge during mode beating, and how do they depend on ε?
- **Damping Effects:** Introduce controlled damping (e.g., air resistance or magnetic brakes) and study its influence on the coupling dynamics. How does damping affect the rate and completeness of energy transfer?
- **Numerical Modeling:** Develop a system of coupled differential equations to simulate the pendulum's behavior. Compare the numerical results with experimental data for both displacement and energy.

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Further Reading

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Supervisor

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