Homework 2

Part 1: Linear Regression

In [1]:
```python
import os
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
```

In [2]:
```python
path = os.getcwd() + '/'
file_name_str = 'dataset1.csv'
data_df = pd.read_csv(path + file_name_str)
print('\n\n      Data')
data_df.head()
```

```
      Data
```

Out[2]:

	x	y
0	6.1101	17.5920
1	5.5277	9.1302
2	8.5186	13.6620
3	7.0032	11.8540
4	5.8598	6.8233

In [3]:
```python
x_series = data_df['x']
y_series = data_df['y']
print(f'Shape of the x: {x_series.shape}')
print(f'Shape of the y: {y_series.shape}')
```

```
Shape of the x: (97,)
Shape of the y: (97,)
```

- **A. Linear Regression with one variable:**

  Consider the attached file dataset1.txt. The first column of the data file shows the input data (x), and the second column shows each samples' output value (y).

- - 1. What is the cost function J(θ) equation for linear regression?
- - - Answer:

$$\textbf{LSE}\ \text{Approach}\ \begin{cases} J(\theta) = \left(\frac{1}{2}\right) \sum_{i=1}^{m} (\mathbf{X}^{(i)}\theta - \mathbf{y}^{(i)})^2 \\ J(\theta) = \left(\frac{1}{2}\right)\ (\mathbf{X}\theta - \mathbf{y})^T(\mathbf{X}\theta - \mathbf{y}) \\ J(\theta) = \left(\frac{1}{2}\right)\ \|\mathbf{X}\theta - \mathbf{y}\|_2^2 \end{cases}$$

- - 2. Fit a linear regression model on your data using:

- - - a. Closed-form solution calculated by **MSE** Method
- - - - Answer:

$$\theta^* = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$

By implementing in Python:

$$\theta^* = \begin{pmatrix} -3.89578088 \\ 1.19303364 \end{pmatrix}$$

In [4]:
```python
x_ndarray = np.array(x_series)
y_ndarray = np.array(y_series)
x_vect_ndarray = np.reshape(x_ndarray, newshape=(x_ndarray.shape[0], -1))
y_vect_ndarray = np.reshape(y_ndarray, newshape=(y_ndarray.shape[0], -1))

x_mat_ndarray = np.concatenate((np.ones(shape=(x_vect_ndarray.shape[0], 1)), x_vect_ndarray), axis=1)
m = len(x_mat_ndarray)
print(f'Shape of X: {x_mat_ndarray.shape}\n')

theta_opt = np.linalg.inv(x_mat_ndarray.T @ x_mat_ndarray) @ x_mat_ndarray.T @ y_vect_ndarray
print(f'\u03B8 = \n{theta_opt}\n')
```

```
Shape of X: (97, 2)

θ =
[[-3.89578088]
 [ 1.19303364]]
```

In [5]:
```python
x_mat_ndarray[: 5]
```

Out[5]:
```
array([[1.    , 6.1101],
       [1.    , 5.5277],
       [1.    , 8.5186],
       [1.    , 7.0032],
       [1.    , 5.8598]])
```

- b. Gradient Descent method in online (stochastic) mode (1500 iterations)

  Answer:

In [6]:
```python
def SGD(x: np.ndarray, y: np.ndarray, iteration: int=1500, learning_rate: float=0.01) -> (np.ndarray, np.nda

    m, n = x.shape
    y = np.reshape(y, newshape=(m, 1))
    theta_init = np.zeros(shape=(n, 1))
    theta = theta_init
    J_theta = np.zeros(shape=(iteration*m, 1))
    count = 0
    for it in range(iteration):

        d_theta = np.zeros(shape=(n, 1))
        for i in range(m):

            for j in range(n):
                d_theta[j][0] = (1/m)*(x[i] @ theta - y[i]) * x[i][j]

#             d_theta /= (1/m)
            theta_new = theta - (learning_rate*d_theta)
            theta = theta_new
            J_theta[count][0] = (1/(2*m)) * ((x @ theta - y).T @ (x @ theta - y))
            count += 1

    return (theta, J_theta)
```

In [7]:
```python
theta_opt1, J_theta1 = SGD(x_mat_ndarray, y_vect_ndarray)
```

In [8]:
```python
print(f'Result of the Learning by using SGD algorithm: \n\n \u03B8 = \n{theta_opt1}')
```

```
Result of the Learning by using SGD algorithm:

 θ =
[[-3.58838901]
 [ 1.12366721]]
```

- c. Gradient descent method in batch mode (1500 iterations)

  Answer:

  We have

$$\frac{\partial J(\theta)}{\partial \theta_j} = (\frac{1}{m}) \sum_{i=1}^{m} (\mathbf{X}^{(\mathbf{i})}\theta - \mathbf{y}^{(\mathbf{i})})\ \mathbf{X}_j^{(i)}$$

In [9]:
```python
def BGD(x: np.ndarray, y: np.ndarray, iteration: int=1500, learning_rate: float=0.01) -> (np.ndarray, np.nda

    m, n = x.shape
    y = np.reshape(y, newshape=(m, 1))
    theta_init = np.zeros(shape=(n, 1))
    theta = theta_init
    J_theta = np.zeros(shape=(iteration, 1))
    for it in range(iteration):


        d_theta = np.zeros(shape=(n, 1))
        for j in range(n):

            for i in range(m):
                d_theta[j][0] += ((x[i] @ theta) - y[i][0]) * x[i][j]
            d_theta[j][0] /= m

        theta_new = theta - (learning_rate*d_theta)
        theta = theta_new
        J_theta[it][0] = (1/(2*m)) * ((x @ theta - y).T @ (x @ theta - y))

    return (theta, J_theta)
```

In [10]:
```python
theta_opt2, J_theta2 = BGD(x_mat_ndarray, y_vect_ndarray)
```

In [11]:
```python
print(f'Result of the Learning by using BGD algorithm: \n\n \u03B8 = \n{theta_opt2}\n')
```

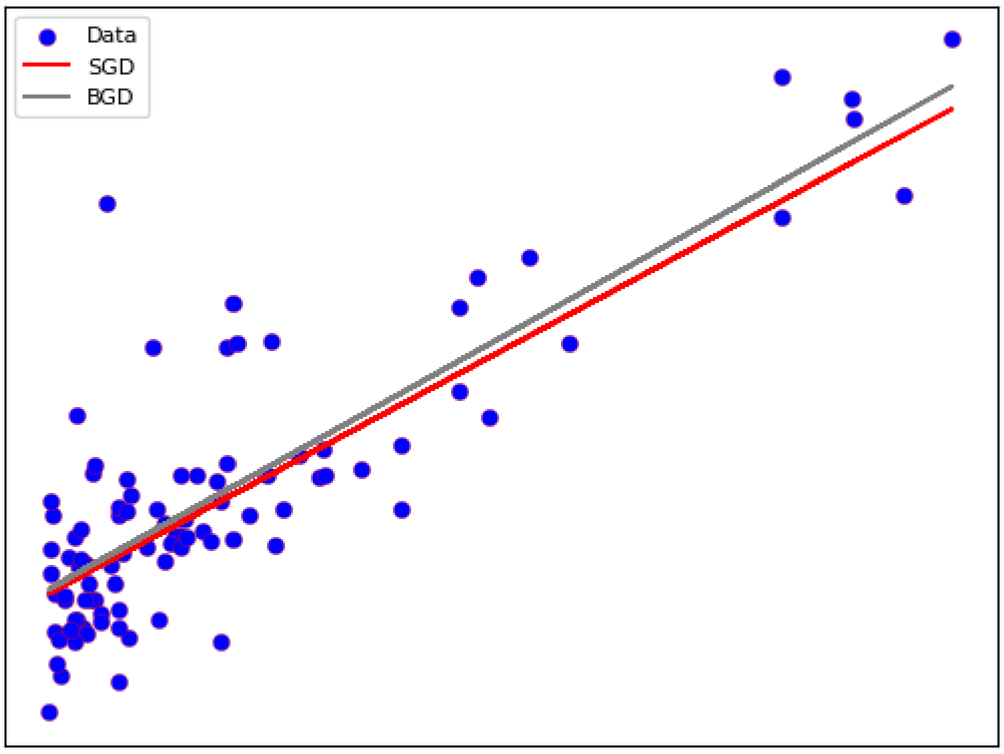
```
Result of the Learning by using BGD algorithm:

 θ =
[[-3.63029144]
 [ 1.16636235]]
```

In [12]:
```python
x = x_mat_ndarray
y = y_vect_ndarray
m = x.shape[0]
theta = theta_opt
J_train_closed_form = (1/(2*m))*((x @ theta - y).T @ (x @ theta - y))
print(f'Result of the Cost Function for Closed-form method: \n\nJ(\u03B8) = {J_train_closed_form}\n')
```

```
Result of the Cost Function for Closed-form method:

J(θ) = [[4.47697138]]
```

- - 3. Plot the dataset and superimpose the fitted models using the three above methods.

In [13]:
```python
x = x_vect_ndarray
y = y_vect_ndarray
SGD_model = theta_opt1[0][0] + theta_opt1[1][0] * x
BGD_model = theta_opt2[0][0] + theta_opt2[1][0] * x
plt.figure()
plt.scatter(x, y, linewidths=0.2, c='blue', edgecolors='red', label='Data')
plt.plot(x, SGD_model, label='SGD', c='red')
plt.plot(x, BGD_model, label='BGD', c='Gray')
plt.xticks([]), plt.yticks([]), plt.legend(fontsize=8)
plt.show()
```



- - 4. Use each estimated parameter θ (for each method) to predict the output for x = 6.2, 12.8, 22.1, 30.

In [14]:
```python
x_for_pred = np.array([[6.2], [12.8], [22.1], [30]])
y_pred_closed_form = theta_opt[0][0] + theta_opt[1][0] * x_for_pred
y_pred_SGD = theta_opt1[0][0] + theta_opt1[1][0] * x_for_pred
y_pred_BGD = theta_opt2[0][0] + theta_opt2[1][0] * x_for_pred
```

In [15]:
```python
predictions_df = pd.DataFrame({'Closed-form': y_pred_closed_form.ravel(), \
                               'SGD': y_pred_SGD.ravel(), 'BGD': y_pred_BGD.ravel()})

print('\n\n Results of the Predictions')
predictions_df
```

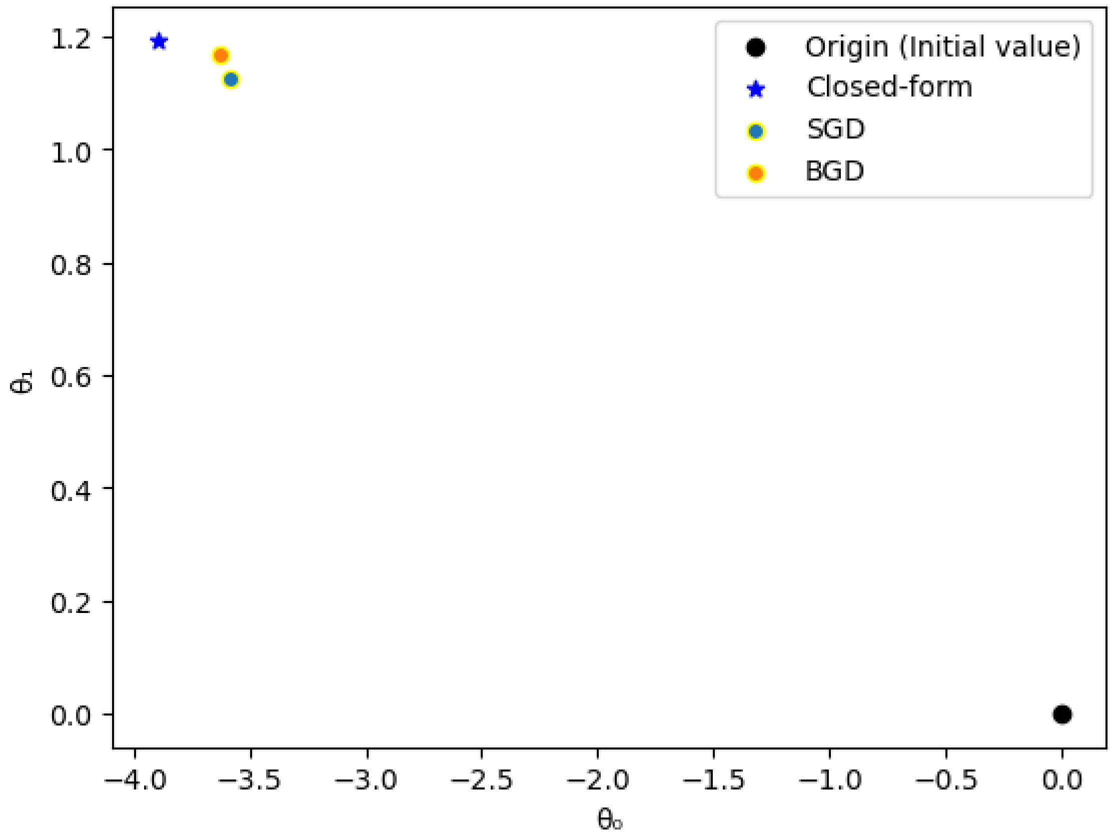
```
 Results of the Predictions
```

Out[15]:

	Closed-form	SGD	BGD
0	3.501028	3.378348	3.601155
1	11.375050	10.794551	11.299147
2	22.470263	21.244656	22.146317
3	31.895228	30.121627	31.360579

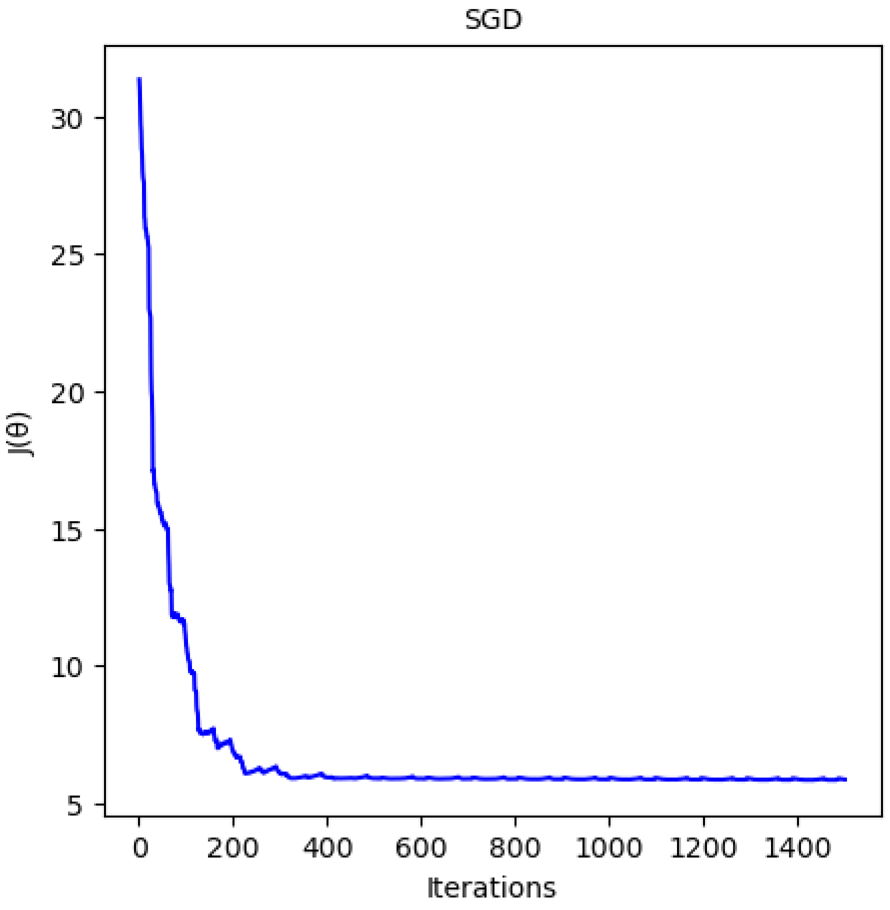
- ▪ 5. Compare the parameter θ estimated by each method by plotting them in one figure.

In [16]:
```python
edge_color = 'yellow'
plt.figure()

plt.scatter([0], [0], c='black', label='Origin (Initial value)')
plt.scatter(theta_opt[0], theta_opt[1], label='Closed-form', c='blue', marker='*')
plt.scatter(theta_opt1[0], theta_opt1[1], edgecolor=edge_color, label='SGD')
plt.scatter(theta_opt2[0], theta_opt2[1], edgecolor=edge_color, label='BGD')
plt.xlabel('\u03B8\u2080'), plt.ylabel('\u03B8\u2081')
plt.legend()
plt.show()
```



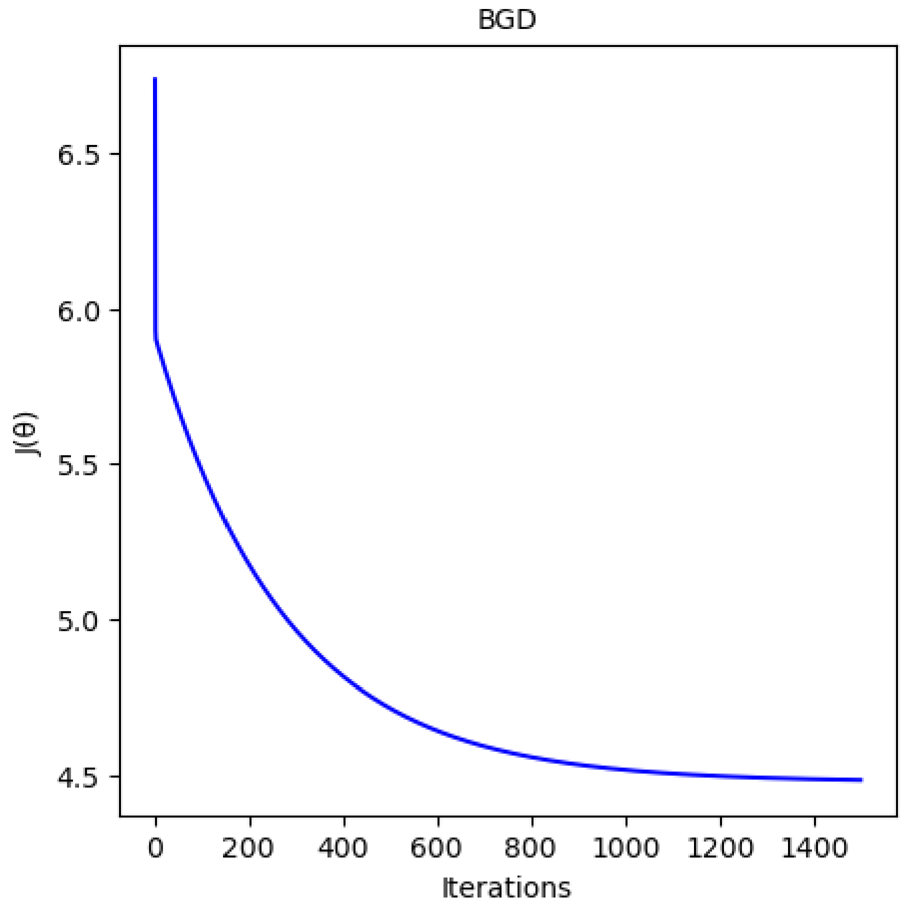
- ▪ 6. Plot the cost function (θ) along the epochs (plot both online & batch methods on one figure using hold on command).

In [17]:
```
iterations1 = np.arange(1, len(J_theta1) + 1)
plt.figure(figsize=(5, 5))
plt.plot(iterations1[: 1500], J_theta1[: 1500], c='blue'), plt.title('SGD', fontsize=10), \
plt.xlabel('Iterations'), plt.ylabel('J(\u03B8)')
plt.show()
print(f'\nFinal value of the Cost Function for SGD method:\n\nJ(\u03B8) = {J_theta1[-1]}\n')
```



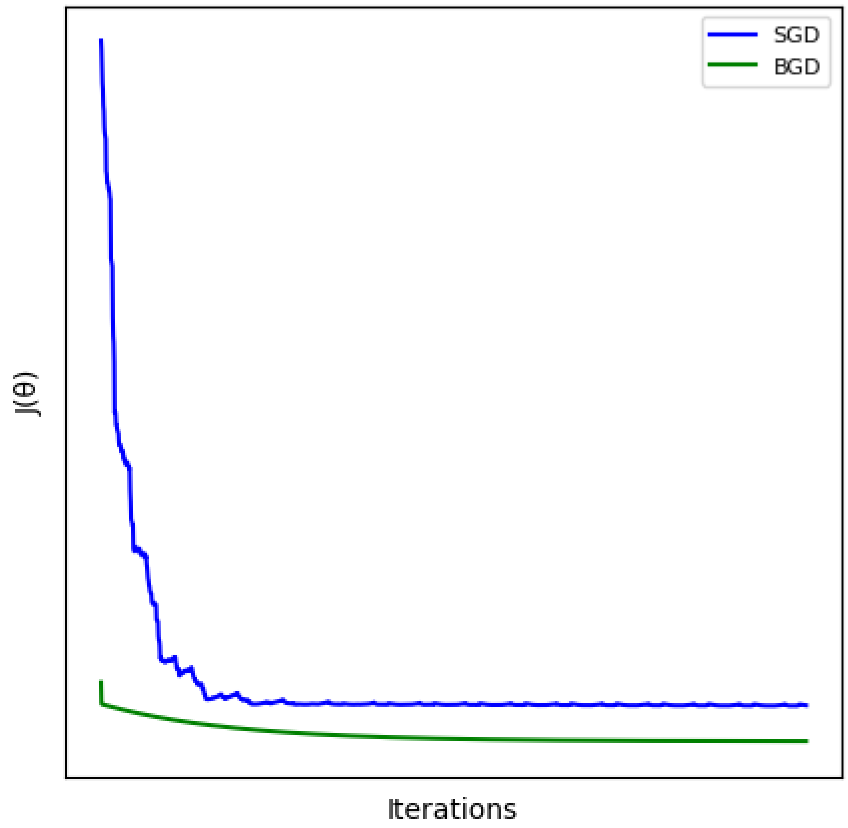

```
Final value of the Cost Function for SGD method:

J(θ) = [4.54607324]
```

In [18]:
```python
iterations2 = np.arange(1, len(J_theta2) + 1)
plt.figure(figsize=(5, 5), )
plt.plot(iterations2, J_theta2, c='blue'), plt.title('BGD', fontsize=10), plt.xlabel('Iterations'), \
plt.ylabel('J(\u03B8)')
plt.show()
print(f'\nFinal value of the Cost Function for BGD method:\n\nJ(\u03B8) = {J_theta2[-1]}\n')
```




```
Final value of the Cost Function for BGD method:

J(θ) = [4.48338826]
```

In [19]:
```python
iterations1 = np.arange(1, len(J_theta1) + 1)
iterations2 = np.arange(1, len(J_theta2) + 1)

plt.figure(figsize=(5, 5))
plt.plot(iterations1[0: len(J_theta2)], J_theta1[0: len(J_theta2)], label='SGD', c='blue'), \
plt.xlabel('Iterations'), plt.ylabel('J(\u03B8)')
plt.plot(iterations2, J_theta2, label='BGD', c='g'), plt.xlabel('Iterations'), \
plt.ylabel('J(\u03B8)'), plt.xticks([]), plt.yticks([]), plt.legend(fontsize=8)
plt.show()
```



- 7. Which type of G.D. (online\batch) do you prefer here? Why?
- Answer:

  BGD, because its convergence speed is higher (per iteration) and because we have all the data at the beginning, there is n

  to use the online method.

- A

- B. Multiple variable Regression:

In [20]:
```
file_name_train = 'train.csv'
file_name_test = 'test.csv'

data_train = pd.read_csv(path + file_name_train)
data_test = pd.read_csv(path + file_name_test)

features_name_list = ['age', 'gender', 'bmi', 'children', 'smoker', 'region']
target_name_list = ['charges']
X_train_df = data_train[features_name_list]
y_train_df = data_train[target_name_list]
X_test_df = data_test[features_name_list]
y_test_df = data_test[target_name_list]
```

In [21]:
```
print('\n                 Training Data:')
data_train.head()
```

```
                 Training Data:
```

Out[21]:

	age	gender	bmi	children	smoker	region	charges
0	19	female	27.900	0	yes	southwest	16884.92400
1	18	male	33.770	1	no	southeast	1725.55230
2	28	male	33.000	3	no	southeast	4449.46200
3	33	male	22.705	0	no	northwest	21984.47061
4	32	male	28.880	0	no	northwest	3866.85520

In [22]:
```
print(f'Shape of the Train Data: {data_train.shape}')
```

```
Shape of the Train Data: (1000, 7)
```

In [23]:
```
print('\n                  Test Data:')
data_test.head()
```

```
                  Test Data:
```

Out[23]:

	age	gender	bmi	children	smoker	region	charges
0	30	male	22.99	2	yes	northwest	17361.76610
1	24	male	32.70	0	yes	southwest	34472.84100
2	24	male	25.80	0	no	southwest	1972.95000
3	48	male	29.60	0	no	southwest	21232.18226
4	47	male	19.19	1	no	northeast	8627.54110

In [24]:
```
print(f'Shape of the Test Data: {data_test.shape}')
```

```
Shape of the Test Data: (150, 7)
```

- - Preprocessing for Train Data and Test Data:

In [25]:
```
def int_encoder(data_df: pd.DataFrame, series_name: str) -> np.ndarray:
    new_df = data_df.copy()
    x = np.array(new_df[series_name])
    values = np.unique(x)
    out = np.zeros(x.shape, dtype=int)
    values_dict = {}
    for i, value in enumerate(values):
        values_dict[value] = int(i + 1)
        out[x == value] = i + 1
    new_df[series_name] = out
    print(f'{series_name}: {values_dict}')
    return new_df
```

In [26]:
```python
def one_hot_encoder(data_df: pd.DataFrame, series_name: str) -> pd.DataFrame:
    df1 = data_df.copy()
    series_shape = df1[series_name].shape
    features_list = list(df1.columns)
    features_list.remove(series_name)
    df2 = data_df.copy()
    df2 = df2[features_list]
    values_list = list(np.unique(df1[series_name]))
    values_dict = {}
    for value in values_list:
        new_name = series_name + '_' + value
        values_dict[new_name] = np.zeros(series_shape, dtype=int)

    for value in values_list:
        new_name = series_name + '_' + value
        for i in range(len(df1[series_name])):
            if df1[series_name][i] == value: values_dict[new_name][i] = 1

        df2[new_name] = values_dict[new_name]

    return df2
```

Note: To encode sex and smoking variables, apply integer encoding, and for region variable, apply one-hot encoding (OHE).

In [27]:
```python
X_train_pre_df = int_encoder(X_train_df, 'gender')
X_train_pre_df = int_encoder(X_train_pre_df, 'smoker')
X_train_pre_df = one_hot_encoder(X_train_pre_df, 'region')
print('\n\n                                            Encoded Training Data')
X_train_pre_df
```

```
gender: {'female': 1, 'male': 2}
smoker: {'no': 1, 'yes': 2}


                                            Encoded Training Data
```

Out[27]:

	age	gender	bmi	children	smoker	region_northeast	region_northwest	region_southeast	region_southwest
0	19	1	27.900	0	2	0	0	0	1
1	18	2	33.770	1	1	0	0	1	0
2	28	2	33.000	3	1	0	0	1	0
3	33	2	22.705	0	1	0	1	0	0
4	32	2	28.880	0	1	0	1	0	0
...	...	...	...	...	...	...	...	...	...
995	39	1	23.275	3	1	1	0	0	0
996	39	1	34.100	3	1	0	0	0	1
997	63	1	36.850	0	1	0	0	1	0
998	33	1	36.290	3	1	1	0	0	0
999	36	1	26.885	0	1	0	1	0	0

1000 rows × 9 columns

In [28]:
```python
X_test_pre_df = int_encoder(X_test_df, 'gender')
X_test_pre_df = int_encoder(X_test_pre_df, 'smoker')
X_test_pre_df = one_hot_encoder(X_test_pre_df, 'region')
print('\n\n                                    Endcoded Test Data')
X_test_pre_df
```

```
gender: {'female': 1, 'male': 2}
smoker: {'no': 1, 'yes': 2}


                                    Endcoded Test Data
```

Out[28]:

	age	gender	bmi	children	smoker	region_northeast	region_northwest	region_southeast	region_southwest
0	30	2	22.990	2	2	0	1	0	0
1	24	2	32.700	0	2	0	0	0	1
2	24	2	25.800	0	1	0	0	0	1
3	48	2	29.600	0	1	0	0	0	1
4	47	2	19.190	1	1	1	0	0	0
...	...	...	...	...	...	...	...	...	...
145	52	2	32.775	3	1	0	1	0	0
146	60	2	32.800	0	2	0	0	0	1
147	20	1	31.920	0	1	0	1	0	0
148	55	2	21.500	1	1	0	0	0	1
149	42	2	34.100	0	1	0	0	0	1

150 rows × 9 columns

In [29]:
```python
# Min_Max_Scaler:
def min_max_scaler(x: pd.DataFrame) -> pd.DataFrame:
    scalered_df = x.copy()
    features_list = list(x.columns)
    for feature in features_list:
        scalered_df[feature] = (scalered_df[feature] - scalered_df[feature].min())/(scalered_df[feature].max
                                                                                   scalered_df[feature]

    return scalered_df
```

In [30]:
```python
# Min_Max_Scaler for Train Data:
X_train_pre_df = min_max_scaler(X_train_pre_df)
print('\n\n                                    Normalized Training Data')
X_train_pre_df
```

```
                                    Normalized Training Data
```

Out[30]:

	age	gender	bmi	children	smoker	region_northeast	region_northwest	region_southeast	region_southwest
0	0.021739	0.0	0.346891	0.0	1.0	0.0	0.0	0.0	1.0
1	0.000000	1.0	0.517432	0.2	0.0	0.0	0.0	1.0	0.0
2	0.217391	1.0	0.495061	0.6	0.0	0.0	0.0	1.0	0.0
3	0.326087	1.0	0.195962	0.0	0.0	0.0	1.0	0.0	0.0
4	0.304348	1.0	0.375363	0.0	0.0	0.0	1.0	0.0	0.0
...	...	...	...	...	...	...	...	...	...
995	0.456522	0.0	0.212522	0.6	0.0	1.0	0.0	0.0	0.0
996	0.456522	0.0	0.527019	0.6	0.0	0.0	0.0	0.0	1.0
997	0.978261	0.0	0.606915	0.0	0.0	0.0	0.0	1.0	0.0
998	0.326087	0.0	0.590645	0.6	0.0	1.0	0.0	0.0	0.0
999	0.391304	0.0	0.317403	0.0	0.0	0.0	1.0	0.0	0.0

1000 rows × 9 columns

In [31]:
```
# Min_Max_Scaler for Test Data:
X_test_pre_df = min_max_scaler(X_test_pre_df)
print('\n\n                                  Normalized Test Data')
X_test_pre_df
```

Normalized Test Data

Out[31]:

	age	gender	bmi	children	smoker	region_northeast	region_northwest	region_southeast	region_southwest
0	0.260870	1.0	0.161519	0.4	1.0	0.0	1.0	0.0	0.0
1	0.130435	1.0	0.436668	0.0	1.0	0.0	0.0	0.0	1.0
2	0.130435	1.0	0.241145	0.0	0.0	0.0	0.0	0.0	1.0
3	0.652174	1.0	0.348824	0.0	0.0	0.0	0.0	0.0	1.0
4	0.630435	1.0	0.053840	0.2	0.0	1.0	0.0	0.0	0.0
...	...	...	...	...	...	...	...	...	...
145	0.739130	1.0	0.438793	0.6	0.0	0.0	1.0	0.0	0.0
146	0.913043	1.0	0.439501	0.0	1.0	0.0	0.0	0.0	1.0
147	0.043478	0.0	0.414565	0.0	0.0	0.0	1.0	0.0	0.0
148	0.804348	1.0	0.119297	0.2	0.0	0.0	0.0	0.0	1.0
149	0.521739	1.0	0.476339	0.0	0.0	0.0	0.0	0.0	1.0

150 rows × 9 columns

In [32]:
```
m_train = X_train_pre_df.shape[0]
X_train_new_df = pd.DataFrame({'x0': np.ones((m_train, )), 'age': X_train_pre_df['age'], \
                               'gender': X_train_pre_df['gender'], \
'bmi': X_train_pre_df['bmi'], 'children': X_train_pre_df['children'], \
'smoker': X_train_pre_df['smoker'], 'region_northeast': X_train_pre_df['region_northeast'], \
'region_northwest': X_train_pre_df['region_northwest'], 'region_southeast': X_train_pre_df['region_southeast
'region_southwest': X_train_pre_df['region_southwest']})
print('\n\n                              Final Preprocessed Training Data')
X_train_new_df
```

Final Preprocessed Training Data

Out[32]:

	x0	age	gender	bmi	children	smoker	region_northeast	region_northwest	region_southeast	region_southwest
0	1.0	0.021739	0.0	0.346891	0.0	1.0	0.0	0.0	0.0	1.0
1	1.0	0.000000	1.0	0.517432	0.2	0.0	0.0	0.0	1.0	0.0
2	1.0	0.217391	1.0	0.495061	0.6	0.0	0.0	0.0	1.0	0.0
3	1.0	0.326087	1.0	0.195962	0.0	0.0	0.0	1.0	0.0	0.0
4	1.0	0.304348	1.0	0.375363	0.0	0.0	0.0	1.0	0.0	0.0
...	...	...	...	...	...	...	...	...	...	...
995	1.0	0.456522	0.0	0.212522	0.6	0.0	1.0	0.0	0.0	0.0
996	1.0	0.456522	0.0	0.527019	0.6	0.0	0.0	0.0	0.0	1.0
997	1.0	0.978261	0.0	0.606915	0.0	0.0	0.0	0.0	1.0	0.0
998	1.0	0.326087	0.0	0.590645	0.6	0.0	1.0	0.0	0.0	0.0
999	1.0	0.391304	0.0	0.317403	0.0	0.0	0.0	1.0	0.0	0.0

1000 rows × 10 columns

In [33]:
```
m_test = X_test_pre_df.shape[0]
X_test_new_df = pd.DataFrame({'x0': np.ones((m_test, )), 'age': X_test_pre_df['age'], \
'gender': X_test_pre_df['gender'], 'bmi': X_test_pre_df['bmi'], 'children': X_test_pre_df['children'], \
'smoker': X_test_pre_df['smoker'], 'region_northeast': X_test_pre_df['region_northeast'], \
'region_northwest': X_test_pre_df['region_northwest'], 'region_southeast': X_test_pre_df['region_southeast']
'region_southwest': X_test_pre_df['region_southwest']})
print('\n\n                                             Final Preprocessed Test Data')
X_test_new_df
```

```
Final Preprocessed Test Data
```

Out[33]:

	x0	age	gender	bmi	children	smoker	region_northeast	region_northwest	region_southeast	region_southwest
0	1.0	0.260870	1.0	0.161519	0.4	1.0	0.0	1.0	0.0	0.0
1	1.0	0.130435	1.0	0.436668	0.0	1.0	0.0	0.0	0.0	1.0
2	1.0	0.130435	1.0	0.241145	0.0	0.0	0.0	0.0	0.0	1.0
3	1.0	0.652174	1.0	0.348824	0.0	0.0	0.0	0.0	0.0	1.0
4	1.0	0.630435	1.0	0.053840	0.2	0.0	1.0	0.0	0.0	0.0
...	...	...	...	...	...	...	...	...	...	...
145	1.0	0.739130	1.0	0.438793	0.6	0.0	0.0	1.0	0.0	0.0
146	1.0	0.913043	1.0	0.439501	0.0	1.0	0.0	0.0	0.0	1.0
147	1.0	0.043478	0.0	0.414565	0.0	0.0	0.0	1.0	0.0	0.0
148	1.0	0.804348	1.0	0.119297	0.2	0.0	0.0	0.0	0.0	1.0
149	1.0	0.521739	1.0	0.476339	0.0	0.0	0.0	0.0	0.0	1.0

150 rows × 10 columns

In [250]:
```
#      X_test_new_df              ---------          X_train_new_df
# ------------------------->>    |  Model  |    <<---------------------------
#        (For_Prob)              ---------            (Back_Prob)
```

- **Q :** Why is OHE better at encoding regional features compared to integer encoding?
- ◦ Answer:

  Because there is no ordering relationship between regional features.

> **Note:** set the basis function for feature bmi to x^2. (use bmi^2 instead of bmi feature).

- 1. Calculate w by closed-form solution calculated by MSE method.
- ◦ Answer:

$$\mathbf{w}^* = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$

By implementing in Python:

$$\mathbf{w}^* = \begin{pmatrix} 1.115780 \\ 0.196793 \\ -0.004687 \\ 0.201581 \\ 0.031830 \\ 0.370975 \\ -0.482970 \\ -0.566448 \\ -0.496574 \\ -0.426065 \end{pmatrix}$$

In [34]:
```
X_train_ndarray = np.array(X_train_new_df)

y_train_normal_df = min_max_scaler(y_train_df)
y_train_ndarray = np.array(y_train_normal_df)

y_test_normal_df = min_max_scaler(y_test_df)
y_test_ndarray = np.array(y_test_normal_df)
```

In [35]:
```
X_train_new_sett_ndarray = X_train_ndarray.copy()
X_train_new_sett_ndarray[:, 3] = X_train_ndarray[:, 3]**2
y_train_new_sett_ndarray = y_train_ndarray

X_test_new_sett_ndarray = np.array(X_test_new_df)
X_test_new_sett_ndarray[:, 3] = X_test_new_sett_ndarray[:, 3]**2
y_test_new_sett_ndarray = y_test_ndarray
```

In [36]:
```
w_opt = np.linalg.inv(X_train_new_sett_ndarray.T @ X_train_new_sett_ndarray) @ \
X_train_new_sett_ndarray.T @ y_train_new_sett_ndarray
pd.DataFrame({'W': w_opt.ravel()}, index=['w0', 'w1', 'w2', 'w3', 'w4', 'w5', 'w6', 'w7', 'w8', 'w9'])
```

Out[36]:

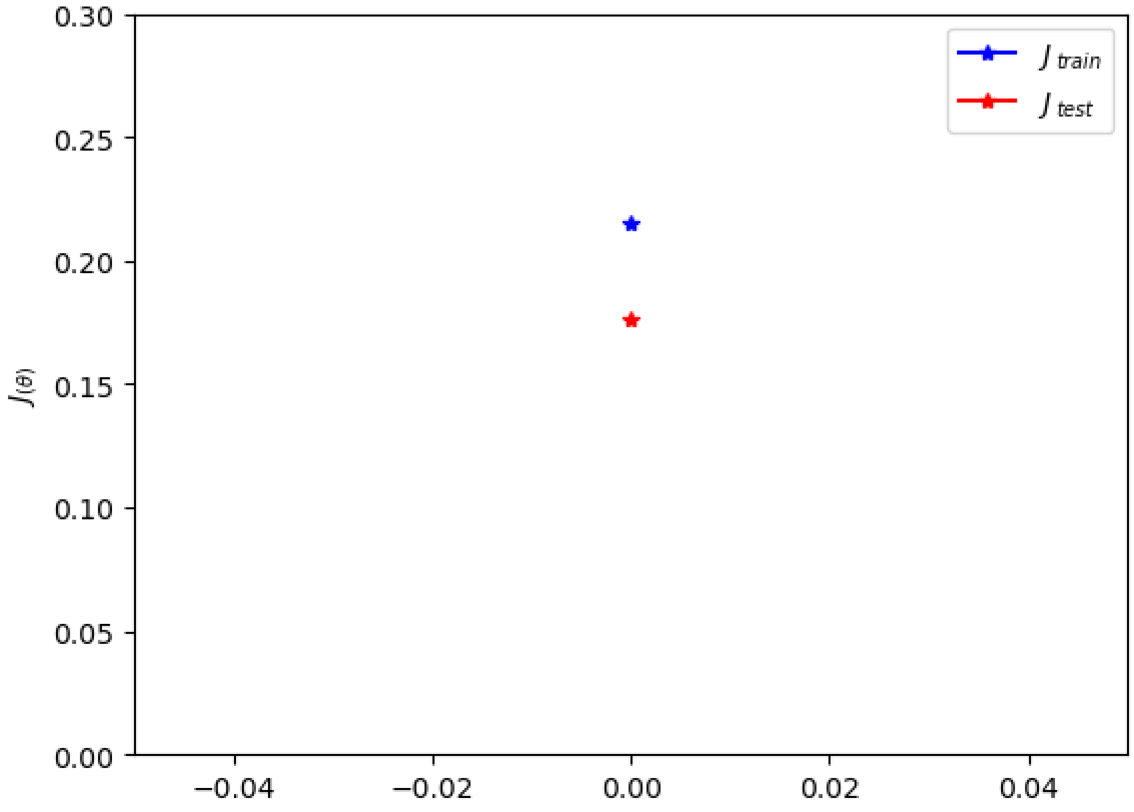
	W
w0	1.115780
w1	0.196793
w2	-0.004687
w3	0.201581
w4	0.031830
w5	0.370975
w6	-0.482970
w7	-0.566448
w8	-0.496574
w9	-0.426065

In [37]:
```
X_train = X_train_new_sett_ndarray.copy()
y_train = y_train_new_sett_ndarray.copy()
m_train = X_train.shape[0]

X_test = X_test_new_sett_ndarray.copy()
y_test = y_test_new_sett_ndarray.copy()
m_test = X_test.shape[0]

w_opt = np.linalg.inv(X_train.T @ X_train) @ X_train.T @ y_train
J_train = (1/(2*m_train))*np.linalg.norm(X_train @ w_opt - y_train)**2
J_test = (1/(2*m_test))*np.linalg.norm(X_test @ w_opt - y_test)**2
```

In [40]:
```
print(f'\nJ(\u03B8) for train:\n\n{J_train}')
print(f'\n\n\nJ(\u03B8) for test:\n\n{J_test}')
```

```
J(θ) for train:

0.21493123965683714



J(θ) for test:

0.17653574740460437
```

In [38]:
```
plt.figure()
plt.plot(J_train, marker='*', c='blue', label='$J_{\; train}$')
plt.plot(J_test, marker='*', c='red', label='$J_{\; test}$'), plt.ylabel('$J_{(\\theta)}$')
plt.xlim([-0.05, 0.05]), plt.ylim([0, 0.3])
plt.legend()
plt.show()
```



In [39]:
```
print('\n\n                         X_Train[:10, :4] \n\n\n')
X_train_new_sett_ndarray[:10, :4]
```

```
                         X_Train[:10, :4]
```

Out[39]:
```
array([[1.        , 0.02173913, 0.        , 0.1203336 ],
       [1.        , 0.        , 1.        , 0.26773559],
       [1.        , 0.2173913 , 1.        , 0.2450854 ],
       [1.        , 0.32608696, 1.        , 0.03840097],
       [1.        , 0.30434783, 1.        , 0.1408975 ],
       [1.        , 0.2826087 , 0.        , 0.08073391],
       [1.        , 0.60869565, 0.        , 0.25790581],
       [1.        , 0.41304348, 0.        , 0.11713019],
       [1.        , 0.41304348, 1.        , 0.1623795 ],
       [1.        , 0.91304348, 0.        , 0.08239335]])
```

In [40]:
```
print('\n\n                        X_Test[:10, :4] \n\n\n')
X_test_new_sett_ndarray[:10, :4]
```

```


                        X_Test[:10, :4] 



```

Out[40]:
```
array([[1.        , 0.26086957, 1.        , 0.02608834],
       [1.        , 0.13043478, 1.        , 0.1906786 ],
       [1.        , 0.13043478, 1.        , 0.05815081],
       [1.        , 0.65217391, 1.        , 0.1216782 ],
       [1.        , 0.63043478, 1.        , 0.0028987 ],
       [1.        , 0.23913043, 1.        , 0.16742915],
       [1.        , 0.2173913 , 1.        , 0.11504957],
       [1.        , 0.63043478, 1.        , 0.0958384 ],
       [1.        , 0.15217391, 1.        , 0.04754599],
       [1.        , 0.7173913 , 1.        , 0.0876858 ]])
```

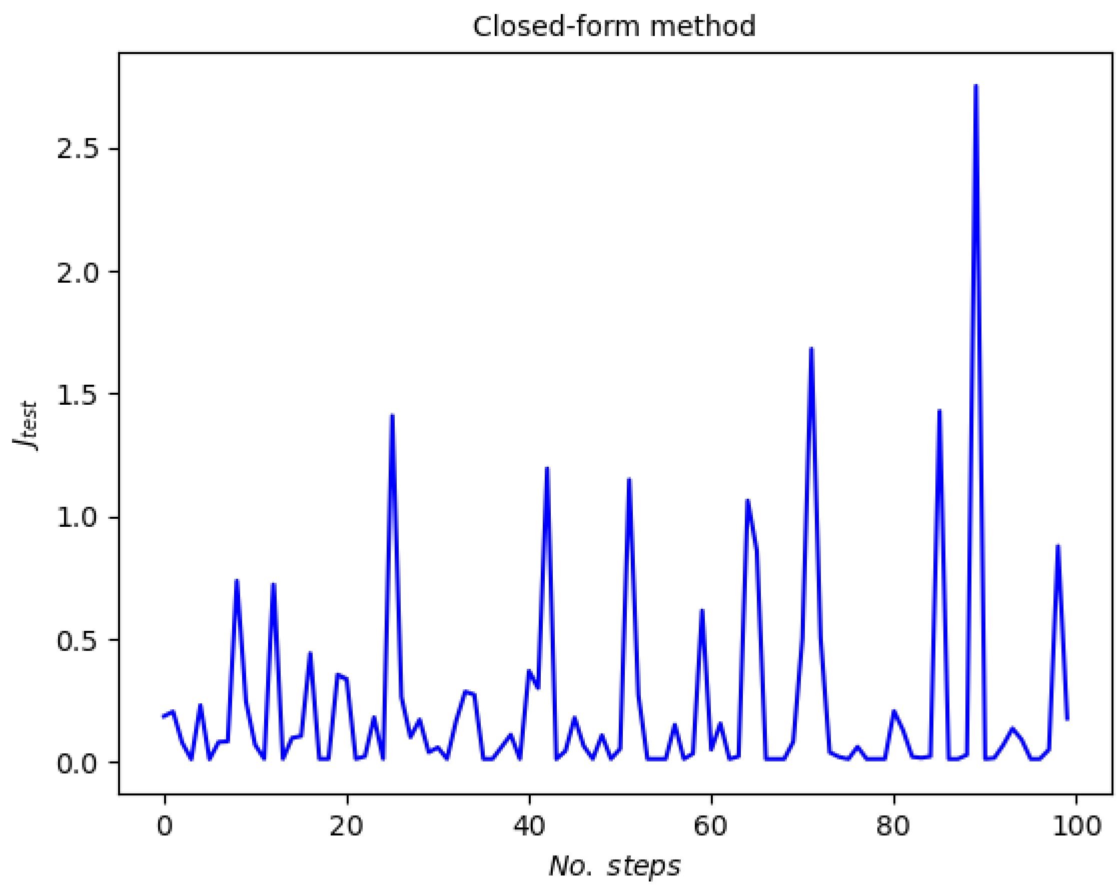
- - 2. At first, assume that the total training data is 10, then gradually increase the training data (with step size equals to 10) until reach out to 1000. In each step, report the test error (MSE) in a diagram.

In [44]:
```
X_test = X_test_new_sett_ndarray
y_test = y_test_new_sett_ndarray
m_test = X_test.shape[0]
eps = 1e-9
n = X_train_new_sett_ndarray.shape[1]
I_n = np.ones((n, n))
J_test1_list = []

for i in range(1, 101):
    try:
        X_train = X_train_new_sett_ndarray[: 10*i]
        y_train = y_train_new_sett_ndarray[: 10*i]

        w_trained = np.linalg.inv(X_train.T @ X_train) @ X_train.T @ y_train

        J_closed_form = np.linalg.norm(X_test @ w_trained - y_test)
        J_closed_form = (J_closed_form**2) / (2*m_test)
        J_test1_list.append(J_closed_form)
    except:
        w_trained = np.linalg.inv((X_train.T @ X_train) + eps*I_n) @ X_train.T @ y_train
        J_closed_form = np.linalg.norm(X_test @ w_trained - y_test, ord=2)
        J_closed_form = (J_closed_form**2) / (2*m_test)
        J_test1_list.append(J_closed_form)

J_test1_ndarray = np.array(J_test1_list)
```

In [45]:
```
J_test1_ndarray.shape
```

Out[45]:
```
(100,)
```

In [46]:
```
J1_closed_form = J_test1_ndarray
levels = np.arange(len(J1_closed_form))
plt.plot(levels, J1_closed_form, c='blue'), plt.title('Closed-form method', fontsize=10), \
plt.ylabel('$J_{test}$'), \
plt.xlabel('$No.\; steps$')
plt.show()
```



In [333]:
```
print(f'\nFinal J(\u03B8) (test error) for Closed-form method is: \n\n{J1_closed_form[-1]}')
```

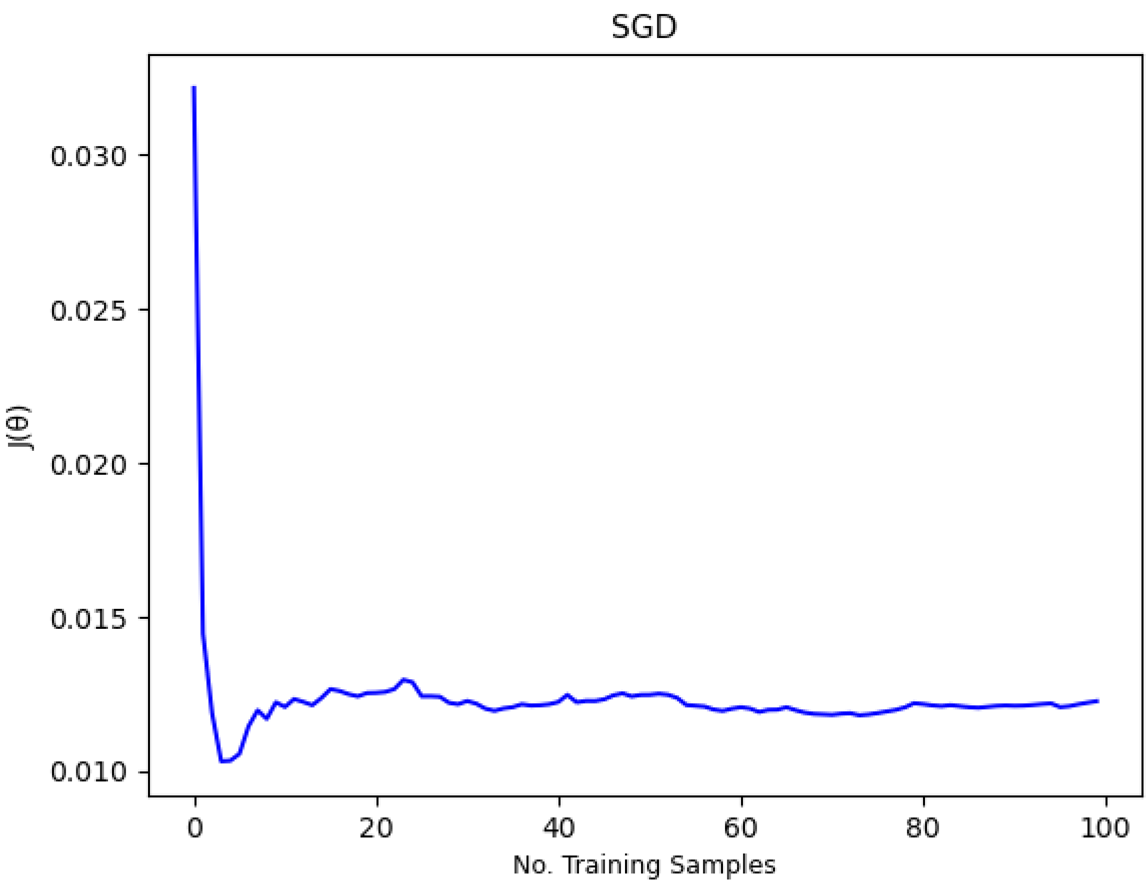
```
Final J(θ) (test error) for Closed-form method is:

0.17653574740460434
```

- - 3. Implement batch gradient descent and stochastic gradient descent to solve this problem. Report the train and test error in e

In [158]:
```
def SGD2(x: np.ndarray, y: np.ndarray, iteration: int=1500, learning_rate: float=0.01) -> (np.ndarray, float
    m, n = x.shape
    theta_init = np.zeros(shape=(n, 1))
    theta = theta_init
    for it in range (iteration):
        d_theta = np.zeros(shape=(n, 1))
        for i in range(m):
            for j in range(n):
                d_theta[j][0] = (x[i] @ theta - y[i]) * x[i][j]
            d_theta /= m
            theta_new = theta - (learning_rate*d_theta)
            theta = theta_new

    J_theta = (1/(2*m)) * ((x @ theta - y).T @ (x @ theta - y))
    return (theta, J_theta)
```

In [160]:
```
X_test = X_test_new_sett_ndarray
y_test = y_test_new_sett_ndarray
m_test = X_test.shape[0]
J_test2 = []
for i in range(1, 101):

    X_train = X_train_new_sett_ndarray[: 10*i]
    y_train = y_train_new_sett_ndarray[: 10*i]
    w_trained, _ = SGD2(X_train, y_train)
    J = np.linalg.norm((X_test @ w_trained - y_test))**2
    J_test2.append(J)


J_test2_ndarray = np.array(J_test2)
```

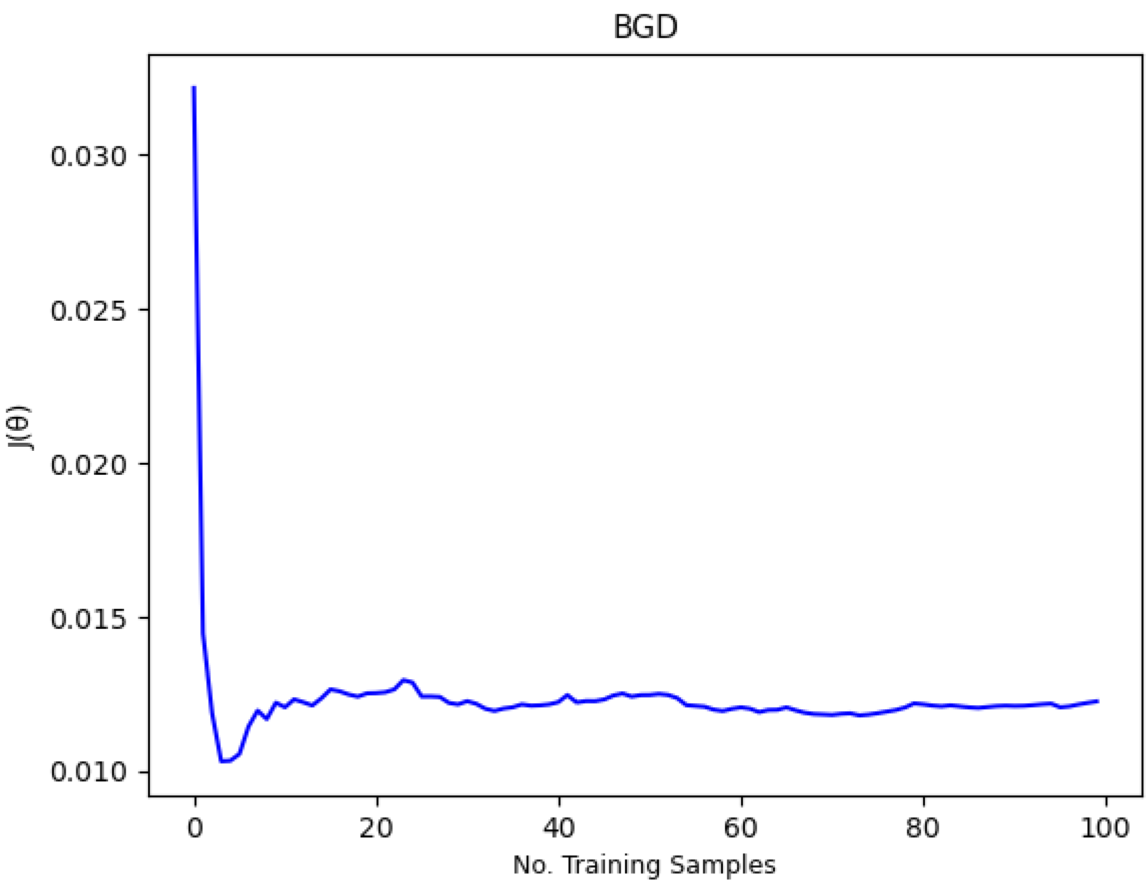
In [161]:
```
J2 = J_test2_ndarray/(2*m_test)

levels = range(len(J2))
plt.plot(levels, J2, c='blue'), plt.title('SGD', fontsize=11)
plt.xlabel('No. Training Samples', fontsize=9), plt.ylabel('J(\u03B8)')
plt.show()
```



In [162]:
```
def BGD2(x: np.ndarray, y: np.ndarray, iteration: int=1500, learning_rate: float=0.01) -> (np.ndarray, float

    m, n = x.shape
    theta_init = np.zeros(shape=(n, 1))
    theta = theta_init
    for it in range(iteration):


        d_theta = np.zeros(shape=(n, 1))
        for j in range(n):

            for i in range(m):
                d_theta[j][0] += (x[i] @ theta - y[i]) * x[i][j]
            d_theta[j][0] /= m

        theta_new = theta - (learning_rate*d_theta)
        theta = theta_new

    J_theta = (1/(2*m)) * ((x @ theta - y).T @ (x @ theta - y))
    return (theta, J_theta)
```

In [163]:
```
X_test = X_test_new_sett_ndarray
y_test = y_test_new_sett_ndarray
m_test = X_test.shape[0]
J_test3 = []
for i in range(1, 101):

    X_train = X_train_new_sett_ndarray[: 10*i]
    y_train = y_train_new_sett_ndarray[: 10*i]
    w_trained, _ = BGD2(X_train, y_train)
    J = np.linalg.norm((X_test @ w_trained - y_test))**2
    J_test3.append(J)


J_test3_ndarray = np.array(J_test3)
```

In [164]:
```
J3 = J_test3_ndarray/(2*m_test)

levels = range(len(J3))
plt.plot(levels, J3, c='blue'), plt.title('BGD', fontsize=11)
plt.xlabel('No. Training Samples', fontsize=9), plt.ylabel('J(\u03B8)')
plt.show()
```



- 4. Now, add a L2 regulator to the cost function and solve the problem using closed-form. Specify the best value for regularize parameter (λ) between $10^{-4}, 10^{-3}, \ldots, 1, \ldots, 10^{3}, 10^{4}$. Plot both train and test errors concerning the logarithmic amount of λ.

- Answer:

For **MSE** aproach we have:

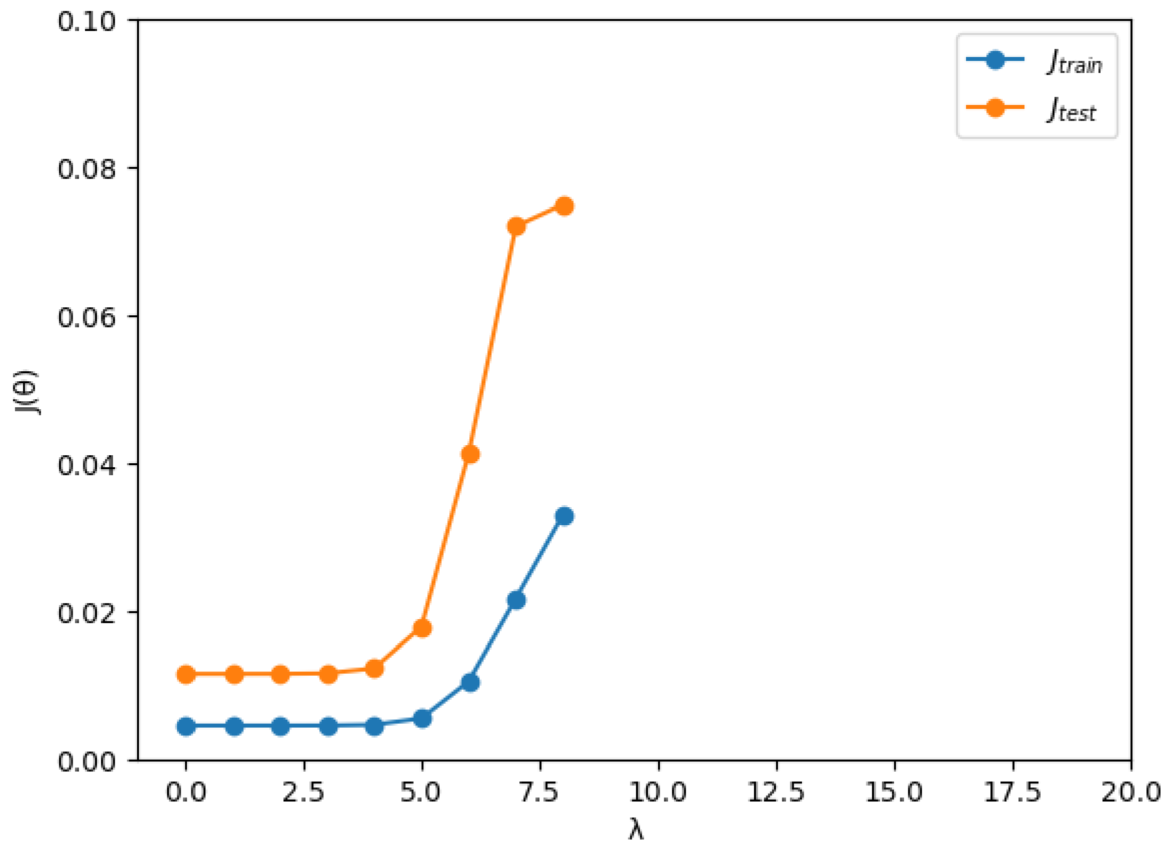
$$J_{level}(\mathbf{w}) = \frac{1}{2m_{level}} \ \{ \ \|\mathbf{X}_{level}\mathbf{w} - \mathbf{y}_{level}\|_2^2 + \lambda\|\mathbf{w}\|_2^2 \ \} \qquad \text{that} \qquad level = \begin{cases} \text{train} \\ \text{test} \end{cases}$$

$$\Longrightarrow \qquad \underbrace{\mathbf{w}^* \ = (\ \mathbf{X}^T\mathbf{X} + \lambda\mathbf{I_n} \ )^{-1} \ \mathbf{X}^T\mathbf{y}}_{\textbf{Redge Regression}}$$

In [41]:
```python
lambda_ndarray = np.logspace(start=-4, stop=4, num=9)
# lambda_ndarray = [0]
print(f'\u03BB values: {lambda_ndarray}')

X_train = X_train_new_sett_ndarray.copy()
y_train = y_train_new_sett_ndarray.copy()
X_test = X_test_new_sett_ndarray.copy()
y_test = y_test_new_sett_ndarray.copy()
m_train, n = X_train.shape
m_test = X_test.shape[0]
I_n = np.identity(n)

J_test_list = []
J_train_list = []
for lambda_value in lambda_ndarray:
    w_star = np.linalg.inv(X_train.T @ X_train + (lambda_value * I_n)) @ X_train.T @ y_train
    J_train_float = (np.linalg.norm(X_train @ w_star - y_train)**2) + (lambda_value*(np.linalg.norm(w_star)*
    J_test_float = (np.linalg.norm(X_test @ w_star - y_test)**2) + (lambda_value*(np.linalg.norm(w_star)**2)
    J_train_list.append(J_train_float)
    J_test_list.append(J_test_float)


J_train_ndarray = np.array(J_train_list)
J_train_ndarray /= (2*m_train)

J_test_ndarray = np.array(J_test_list)
J_test_ndarray /= (2*m_test)

print(f'\n\n\n J(\u03B8) values for train: \n\n{J_train_ndarray}')
print(f'\n\n\nJ(\u03B8) values for test: \n\n{J_test_ndarray}')
```

```
λ values: [1.e-04 1.e-03 1.e-02 1.e-01 1.e+00 1.e+01 1.e+02 1.e+03 1.e+04]



 J(θ) values for train:

[0.0045274  0.00452749 0.00452847 0.00453825 0.00463485 0.00550644
 0.01049786 0.02169809 0.03305775]



J(θ) values for test:

[0.01151603 0.01151672 0.01152354 0.01159162 0.01225848 0.01787743
 0.04131668 0.07203825 0.0749632 ]
```

In [42]:
```python
plt.figure()
plt.plot(J_train_ndarray, marker='o', label='$J_{train}$')
plt.plot(J_test_ndarray, marker='o', label='$J_{test}$')
plt.xlabel('\u03BB'), plt.ylabel('J(\u03B8)'), plt.axis([-1, 20, 0, 0.1])
plt.legend()
plt.show()
```



-
  -
    - Answer:

      $\lambda^* = 10^{-4}$

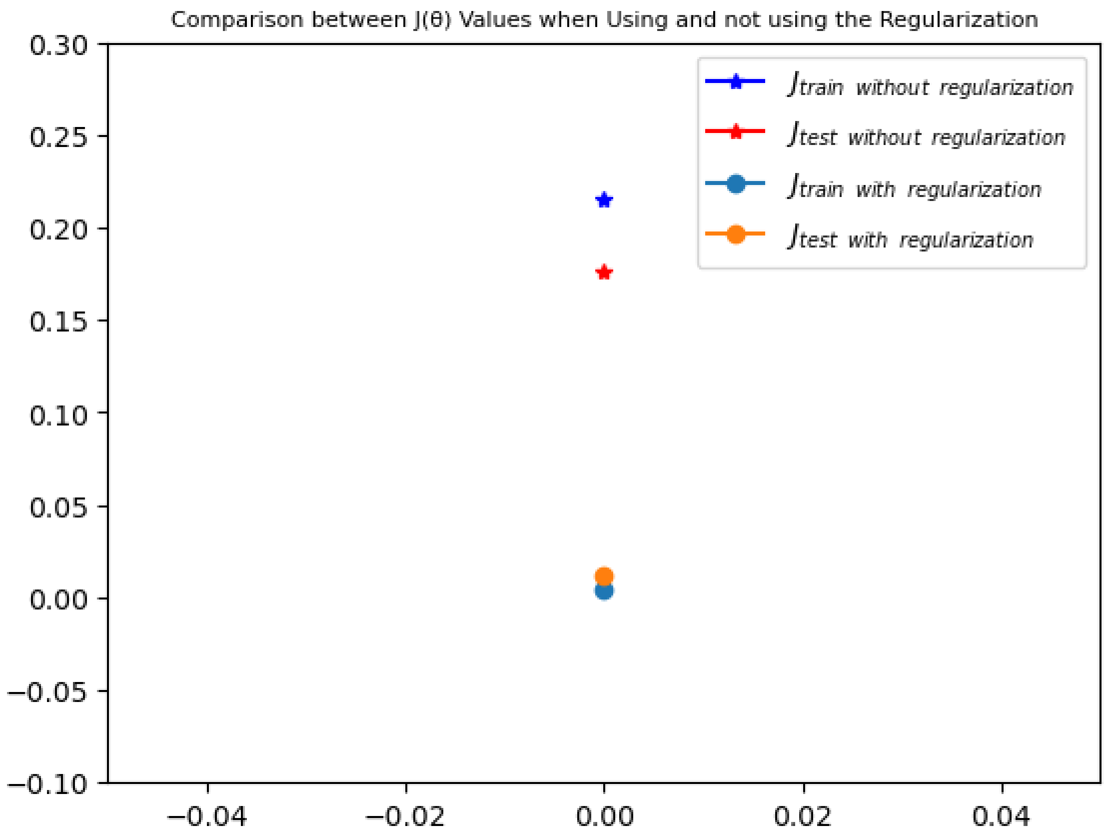
- ▪ 5. Compare the results with and without regularization.

In [54]:
```
results_dict = {'\u03BB': [0, 1e-4], 'Train Error': [0.215, 0.0045], 'Test Error': [0.176, 0.012]}
results_df = pd.DataFrame(results_dict, )
results_df
```

Out[54]:

	λ	Train Error	Test Error
0	0.0000	0.2150	0.176
1	0.0001	0.0045	0.012

In [52]:
```
J_train_without_reg = [0.21493123965683714]
J_test_without_reg = [0.17653574740460437]
J_train_with_reg = [0.0045274]
J_test_with_reg = [0.011510603]

plt.figure()
plt.plot(J_train_without_reg, label='$J_{train\;\;without\;\;regularization}$', marker='*', c='blue')
plt.plot(J_test_without_reg, label='$J_{test\;\;without\;\;regularization}$', marker='*', c='red')
plt.plot(J_train_with_reg, label='$J_{train\;\;with\;\;regularization}$', marker='o')
plt.plot(J_test_with_reg, label='$J_{test\;\;with\;\;regularization}$', marker='o')
plt.xlim([-0.05, 0.05]), plt.ylim([-0.1, 0.3]), plt.title("Comparison between J(\u03B8) \
Values when Using and not using the Regularization", fontsize=8)
plt.legend()
plt.show()
```



- ▪ ◦ Answer:
  By using the best value for the Regularization parameter (λ^*), both the testing and training errors are reduced, moreover, they are closer to each other, which shows the ability to further generalize the model.

- ☑ **B**