# Rake Demodulation

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**Necessary Functions:**

[1]:
```
import numpy as np
from termcolor import colored
import matplotlib.pyplot as plt
```

[2]:
```
def an_Generator(num_messages: int, k: int=1) -> np.array:

    np.random.seed(0)
    out_seq = np.random.randint(low=0, high=2, size=(1, num_messages*k),␣
 ↪dtype=int)
    out_seq = out_seq.flatten()

    return out_seq
```

For $BPSK$ Modulation:

$$s(t) \;=\; Re[\;g(t)\;e^{j\frac{2\pi(m-1)}{M}}\;e^{j2\pi f_c t}\;] \;=\; Re[\;g(t)\;e^{j2\pi(f_c t+\frac{m-1}{M})}\;] \;=\; Re[\;g(t)\;e^{j2\pi(f_c t+\phi(m))}\;]$$

$$\begin{cases} \phi(m=2) = \pi \\ \phi(m=1) = 0 \end{cases} \Rightarrow a_n \rightarrow m_{array} \Rightarrow \begin{cases} a_n : 0 \rightarrow m : 2 \\ a_n : 1 \rightarrow m : 1 \end{cases}$$

[3]:
```
def m_Generator(an: np.array) -> np.array: # In is a list of ms

    In = -(an - 2)
    In.dtype = int

    return In
```

For rectangular pulse shpe we will have:

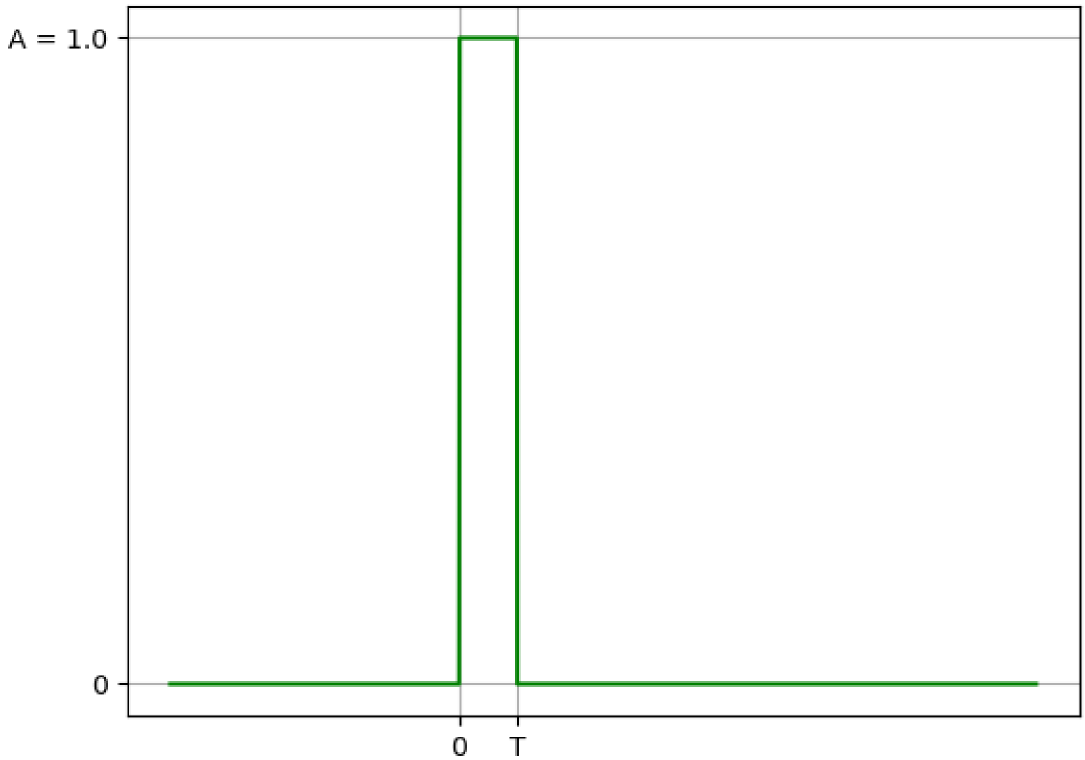
$$\varepsilon_g = \int_{-\infty}^{+\infty} g^2(t)\, dt = \int_0^T A^2\, dt = A^2 T \Rightarrow \varepsilon_g = A^2 T \Rightarrow A = \sqrt{\frac{\varepsilon_g}{T}}$$

[4]:
```
def g_Rect_Generator(t_array: np.array, Eg: float, T: float) -> np.array:  # Eg:␣
↪Energy of the g(t)

    A = np.sqrt(Eg / T)

    g_t = np.zeros_like(t_array)
    g_t[(0 <= t_array) & (t_array <= T)] = A


    return g_t
```

[5]:
```
Eg = 1
Eg = float(Eg)
T = 1 # T is Symbol Duration.
T = float(T)
A = np.sqrt(Eg / T)
t_array = np.arange(start=-5, stop=10, step=0.01)
g_t_array = g_Rect_Generator(t_array=t_array, Eg=Eg, T=T)
```

[6]:
```
color='green'
x_tick_positions_list = [0, T]
x_tick_labels_list = ['0', 'T']
```

```
y_tick_positions_list = [0, A]
y_tick_labels_list = ['0', 'A = ' + str(A)]
plt.plot(t_array, g_t_array, color=color)
plt.xticks(ticks=x_tick_positions_list, labels=x_tick_labels_list)
plt.yticks(ticks=y_tick_positions_list, labels=y_tick_labels_list)
plt.grid(True)
plt.show()
```




**BPSK Modulation:**

For $BPSK$ Modulation:

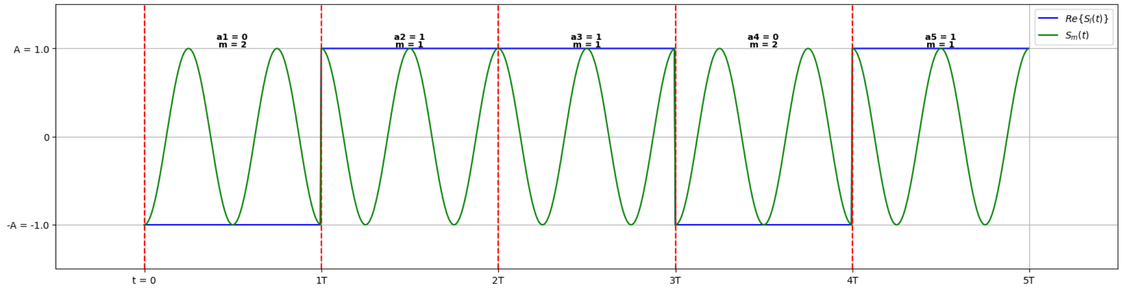
$$s_m(t) \ = \ g(t)\cos 2\pi[f_c t + \frac{m-1}{M=2}]$$

[7]:
```
def S_m_Generator(an: np.array, Eg: float, T: float, fc: float=1e3, M: int=2) ->
 ↪np.array:

    if fc != 0:
        step = 0.01 * (1/fc) # step << 1/fc
    else:
        step = 0.01

    t_array = np.arange(start=0, stop=T, step=step)
    m_array = m_Generator(an=an)
    g_t_array = g_Rect_Generator(t_array=t_array, Eg=Eg, T=T)
    s_m_list = []
    s_l_list = []
    t_list = []
    fixed_exp = np.exp(1j*2*np.pi*fc*t_array)
    for i, m in enumerate(m_array):

        variable_phase = (m - 1) / M
        s_l_symbol = g_t_array * np.exp(1j*2*np.pi*(m-1)/M)
        s_m_symbol = np.real(s_l_symbol*fixed_exp)
        s_l_list.extend(s_l_symbol)
        s_m_list.extend(s_m_symbol)
        t_list.extend(i*T + t_array)

    s_l_array = np.array(s_l_list)
    s_m_array = np.array(s_m_list)
    T_array = t_array

    return s_l_array, s_m_array, np.array(t_list), T_array, step
```

[8]:
```
num_symbols = 5
an = an_Generator(num_messages=num_symbols)
m_array = m_Generator(an=an)
T = 1
T = float(T)
Eg = 1
Eg = float(Eg)
A = np.sqrt(Eg / T)
fc = 2
```

```
fc = float(fc)
s_l_t_array, s_m_t_array, t_array, T_array, step = S_m_Generator(an=an, Eg=Eg,␣
↪T=T, fc=fc)
```

[9]:
```
color1 = 'green'
color2 = 'red'
color3 = 'black'
color4 = 'blue'
plt.figure(figsize=(20, 5))
plt.plot(t_array, np.real(s_l_t_array), color=color4, label='$Re\{S_{l}(t)\}$')
plt.plot(t_array, s_m_t_array, color=color1, label='$S_{m}(t)$')
y_tick_positions_list = [-A, 0, A]
y_tick_labels_list = ['-A = ' + str(-A), '0', 'A = ' + str(A)]
x_tick_labels_list = ['t = 0']
x_tick_positions_list = [0]
for i in range(num_symbols):
    plt.axvline(x=i*T, color=color2, linestyle='--')
    x_tick_positions_list.append((i + 1)*T)
    x_tick_labels_list.append(str(int((i + 1)*T)) + 'T')
    plt.text((i + 0.42)*T, A + 0.02, f'm = {m_array[i]}', fontsize=9,␣
↪color=color3, fontweight='bold')
    plt.text((i + 0.41)*T, A + 0.1, f'a{i + 1} = {an[i]}', fontsize=9,␣
↪color=color3, fontweight='bold')

plt.grid(True)
plt.xticks(ticks=x_tick_positions_list, labels=x_tick_labels_list)
plt.yticks(ticks=y_tick_positions_list, labels=y_tick_labels_list)
plt.axis([-0.5*T, (num_symbols + 0.5)*T, -(A + 0.5), (A + 0.5)])
plt.legend()
plt.show()
```



**Tapped-Delay-Line Channel Model:**

When the $c_n(t)$ are Gaussian random processes, they are statistically independent.

$$\Rightarrow\ c_n(t) \sim \mathcal{N}(\mu,\sigma^2), (iid)\ \Rightarrow \begin{cases} Re\{c_n(t)\} \sim \frac{\sqrt{2}}{2}\mathcal{N}(\mu,\sigma^2) \\ Im\{c_n(t)\} \sim \frac{\sqrt{2}}{2}\mathcal{N}(\mu,\sigma^2) \end{cases} \xrightarrow{if\ \sigma^2=1,\ \mu=0} \begin{cases} Re\{c_n(t)\} \sim \frac{\sqrt{2}}{2}\mathcal{N}(0,1) \\ Im\{c_n(t)\} \sim \frac{\sqrt{2}}{2}\mathcal{N}(0,1) \end{cases}$$

$$\mu_{c_n(t)} = \mu_{Re\{c_n(t)\}} + j\mu_{Im\{c_n(t)\}}$$

$$\sigma^2_{c_n(t)} = \sigma^2_{Re\{c_n(t)\}} + \sigma^2_{Im\{c_n(t)\}}$$

[10]:
```
def c_t_Generator(n_t: int, mode: str, u: complex=0, sigma: float=1) -> dict: #␣
↪sigma is standard deviation, n_t is the size of t_array

    u_Re, u_Im = u.real, u.imag
    c_dict = {}
    if mode == 'static':
        np.random.seed(0)

    elif mode == 'dynamic':
        pass


    random_array = np.random.randn(2, n_t) # randn is standard distribution
    c_Re = (np.sqrt(2) / 2) * sigma * random_array[0] + u_Re
    c_Im = (np.sqrt(2) / 2) * sigma * random_array[1] + u_Im
```

```python
    c_t_array = c_Re + 1j*c_Im

    return c_t_array
```

[11]:
```python
def z_t_Generator(n_t: int, mode: str, u: complex=0, sigma: float=1) -> np.array:
 ↪ # n_t is the size of the t_array, sigma is standard deviation

    if mode == 'static':
        np.random.seed(0)

    elif mode == 'dynamic':
        pass

    u_Re, u_Im = u.real, u.imag
    z_t_Re = (np.sqrt(2) / 2) * sigma * np.random.randn(n_t) + u_Re
    z_t_Im = (np.sqrt(2) / 2) * sigma * np.random.randn(n_t) + u_Im
    z_t_complex = z_t_Re + 1j * z_t_Im

    return z_t_complex
```

For received low pass signal we have:

$$r_l(t) = [\sum_{k=1}^{L} c_k(t) s_{li}(t - \frac{k}{W})] + z(t) \quad \overset{W=\frac{1}{T}}{\Rightarrow} \quad [\sum_{k=1}^{L} c_k(t) s_{li}(t - kT)] + z(t) = v_i(t) + z(t), \quad 0 \leq t \leq T, \quad i = 1, 2$$

Note:

$T:$ *Symbol Duration*

$W = \frac{1}{T_s}$ : *Sampling Frequency*

[12]:
```python
def Symbol_Right_Shifter(s_l_array: np.array, k: int) -> np.array:

    n_T = len(s_l_array)
```

```
    s_l_long_array = s_l_array
    if k == 0:
        s_l_shifted_array = s_l_array

    else:
        s_l_shifted_array = np.concatenate((np.zeros(k), s_l_array[:-k]))

    return s_l_shifted_array
```

[13]:
```
def v_i_t_Generator(s_l_t: np.array, L: int) -> np.array:

    n_T = len(s_l_t)
    c_k_t = c_t_Generator(n_t=n_T, mode='static')
    out_array = np.zeros(n_T, dtype=complex)
    for k in range(1, L+1):

        s_l_t_k_nT = Symbol_Right_Shifter(s_l_array=s_l_t, k=k)
        out_array += c_k_t * s_l_t_k_nT

    return out_array
```

[14]:
```
def r_l_t_Generator(v_i_t_array: np.array) -> np.array:

    n_t = len(v_i_t_array)
    z_t_array = z_t_Generator(n_t=n_t, mode='static')
    r_l_t_array = v_i_t_array + z_t_array

    return r_l_t_array
```

**Rake Demodulator:**

Note:

We assume that the $c_k(t)$ is known and with this assumption, we will have the following:

[15]:
```
num_symbols = 5
an = an_Generator(num_messages=num_symbols)
m_array = m_Generator(an=an)
T = 1
T = float(T)
Eg = 1
Eg = float(Eg)
fc = 2
fc = float(fc)
s_l_t_array, s_m_t_array, t_array, T_array, step = S_m_Generator(an=an, Eg=Eg,␣
↪T=T, fc=fc)
```

[16]:
```
M = 2
m1 = 1
m2 = 2
L = 10
g_t = g_Rect_Generator(t_array=T_array, Eg=Eg, T=T)
s_l_1 = g_t * np.exp(1j*2*np.pi*(m1 - 1)/M)
s_l_2 = g_t * np.exp(1j*2*np.pi*(m2 - 1)/M)
v_1 = v_i_t_Generator(s_l_t=s_l_1, L=L)
v_2 = v_i_t_Generator(s_l_t=s_l_2, L=L)
r_l_1 = r_l_t_Generator(v_i_t_array=v_1)
r_l_2 = r_l_t_Generator(v_i_t_array=v_2)
```

$$U_m = Re[\ \int_0^T r_l(t) v_m^*(t)\ dt\ ] = Re[\ \sum_{k=1}^{L} \int_0^T r_l(t) c_k^*(t) s_m^*(t - \frac{k}{W})\ ]$$

[17]:
```
def Um_Generator(r_l_t: np.array, T_array: np.array, Eg: float, T: float, L:␣
↪int, step: float, mode: str) -> dict:

    n_t = len(T_array)
    n_r = len(r_l_t)
    if n_t != n_r:
        raise ValueError(f'Shape of the r_l_t and T array must be equal, r_l_t.
↪shape = {r_l_t.shape}, T_array.shape = {T_array.shape}')

    g_t = g_Rect_Generator(t_array=T_array, Eg=Eg, T=T)
    M, m1, m2 = 2, 1, 2
    s_l_1 = g_t * np.exp(1j*2*np.pi*(m1 - 1)/M)
```

```python
    s_l_2 = g_t * np.exp(1j*2*np.pi*(m2 - 1)/M)
    c_k_t = c_t_Generator(n_t=n_t, mode=mode)
    v_1 = v_i_t_Generator(s_l_t=s_l_1, L=L)
    v_2 = v_i_t_Generator(s_l_t=s_l_2, L=L)

    U_dict = {}
    U1 = (step * (r_l_t * np.conj(v_1)).sum()).real
    U2 = (step * (r_l_t * np.conj(v_2)).sum()).real
    U_dict['U1'], U_dict['U2'] = U1, U2

    return U_dict
```

[18]:
```python
mode = 'static'
Um_for_r_l_1 = Um_Generator(r_l_t=r_l_1, T_array=T_array, Eg=Eg, T=T, L=L,␣
 ↪step=step, mode=mode)
Um_for_r_l_2 = Um_Generator(r_l_t=r_l_2, T_array=T_array, Eg=Eg, T=T, L=L,␣
 ↪step=step, mode=mode)
```

[19]:
```python
print(f'\n{colored(f"When we transmit the s1,2 and receive receive r1,2:",␣
 ↪"blue", attrs=["bold"])}\n')
U1, U2 = Um_for_r_l_1['U1'], Um_for_r_l_1['U2']
print(f'\n {colored(f"r1: U1 = {U1: 0.2f}, U2 = {U2: 0.2f}", "black",␣
 ↪attrs=["bold"])} \n')

U1, U2 = Um_for_r_l_2['U1'], Um_for_r_l_2['U2']
print(f'\n {colored(f"r2: U1 = {U1: 0.2f}, U2 = {U2: 0.2f}", "black",␣
 ↪attrs=["bold"])} \n')
```

```
When we transmit the s1,2 and receive receive r1,2:


 r1: U1 =  102.81, U2 = -102.81


 r2: U1 = -83.95, U2 =  83.95
```

**Conclusion** :

As we can see demodulation can be done correctly.

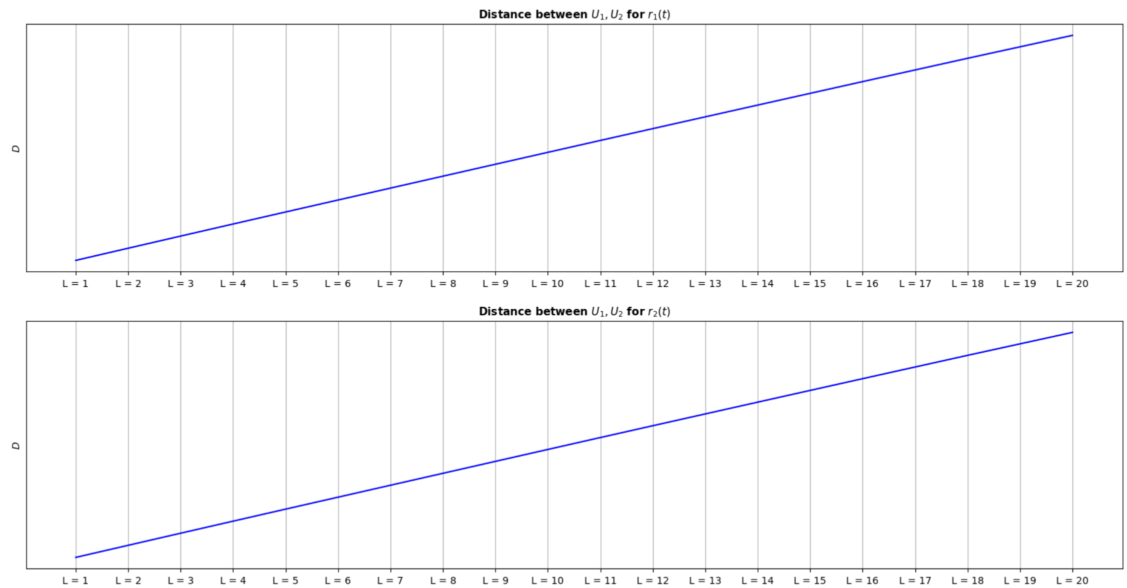
**Exploring the L (No. of taps or fingers):**

[20]:
```
num_symbols = 5
an = an_Generator(num_messages=num_symbols)
m_array = m_Generator(an=an)
T = 1
T = float(T)
Eg = 1
Eg = float(Eg)
fc = 2
fc = float(fc)
s_l_t_array, s_m_t_array, t_array, T_array, step = S_m_Generator(an=an, Eg=Eg,␣
↪T=T, fc=fc)
```

[22]:
```
mode = 'static'
L_list = list(range(1, 21))
out_r1_list = []
out_r2_list = []
L_ticks_list = []
for L in L_list:

    L_ticks_list.append(f'L = {L}')

    Um_for_r_l_1 = Um_Generator(r_l_t=r_l_1, T_array=T_array, Eg=Eg, T=T, L=L,␣
↪step=step, mode=mode)
    out_r1_list.append(Um_for_r_l_1['U1'] - Um_for_r_l_1['U2'])

    Um_for_r_l_2 = Um_Generator(r_l_t=r_l_2, T_array=T_array, Eg=Eg, T=T, L=L,␣
↪step=step, mode=mode)
    out_r2_list.append(Um_for_r_l_2['U2'] - Um_for_r_l_2['U1'])
```

[23]:
```
color = 'blue'
plt.figure(figsize=(20, 10))

plt.subplot(2, 1, 1)
plt.plot(L_list, out_r1_list, color=color)
plt.xticks(ticks=L_list, labels=L_ticks_list), plt.yticks([]), plt.
↪ylabel('$D$'), plt.title('Distance between $U_{1}, U_{2}$ for $r_{1}(t)$',␣
↪fontsize=11, fontweight="bold")
plt.grid(True)

plt.subplot(2, 1, 2)
```

```
plt.plot(L_list, out_r2_list, color=color)
plt.xticks(ticks=L_list, labels=L_ticks_list), plt.yticks([]), plt.
 ↪ylabel('$D$'), plt.title('Distance between $U_{1}, U_{2}$ for $r_{2}(t)$',␣
 ↪fontsize=11, fontweight="bold")
plt.grid(True)

plt.show()
```




**Conclusion :**

When we assumed that the $c_k(t)$ is known, as we can see increasing the $L$ (No. Taps or fingers) parameter increasesthe distance between $U_1$, $U_2$ linearly, and decreasing the error probability will be probable.

### Correlation Coefficient between Modulated Symbols $s_{m=1}, s_{m=2}$ :

[24]:
```
num_symbols = 5
an = an_Generator(num_messages=num_symbols)
```

```python
m_array = m_Generator(an=an)
T = 1
T = float(T)
Eg = 1
Eg = float(Eg)
fc = 2
fc = float(fc)
s_l_t_array, s_m_t_array, t_array, T_array, step = S_m_Generator(an=an, Eg=Eg,␣
↪T=T, fc=fc)
```

[25]:
```python
n_t = len(T_array)
s_m_t_1 = s_m_t_array[:n_t]
s_m_t_2 = s_m_t_array[n_t:2*n_t]
```

[27]:
```python
corr_coef_mat = np.corrcoef(s_m_t_1, s_m_t_2)
print(f'\n{colored(f"Correlation Coefficient Matrix = ", "blue",␣
↪attrs=["bold"])}')
print(f'\n{colored(f"{corr_coef_mat}", "black", attrs=["bold"])}')
```

```
Correlation Coefficient Matrix =

[[ 1. -1.]

 [-1.  1.]]
```

**Conclusion :**

$$\Rightarrow \rho_r = -1$$

**References:**


- **Book** : Proakis, John G. Digital Communications. 5th ed. New York: McGraw Hill, 2007.