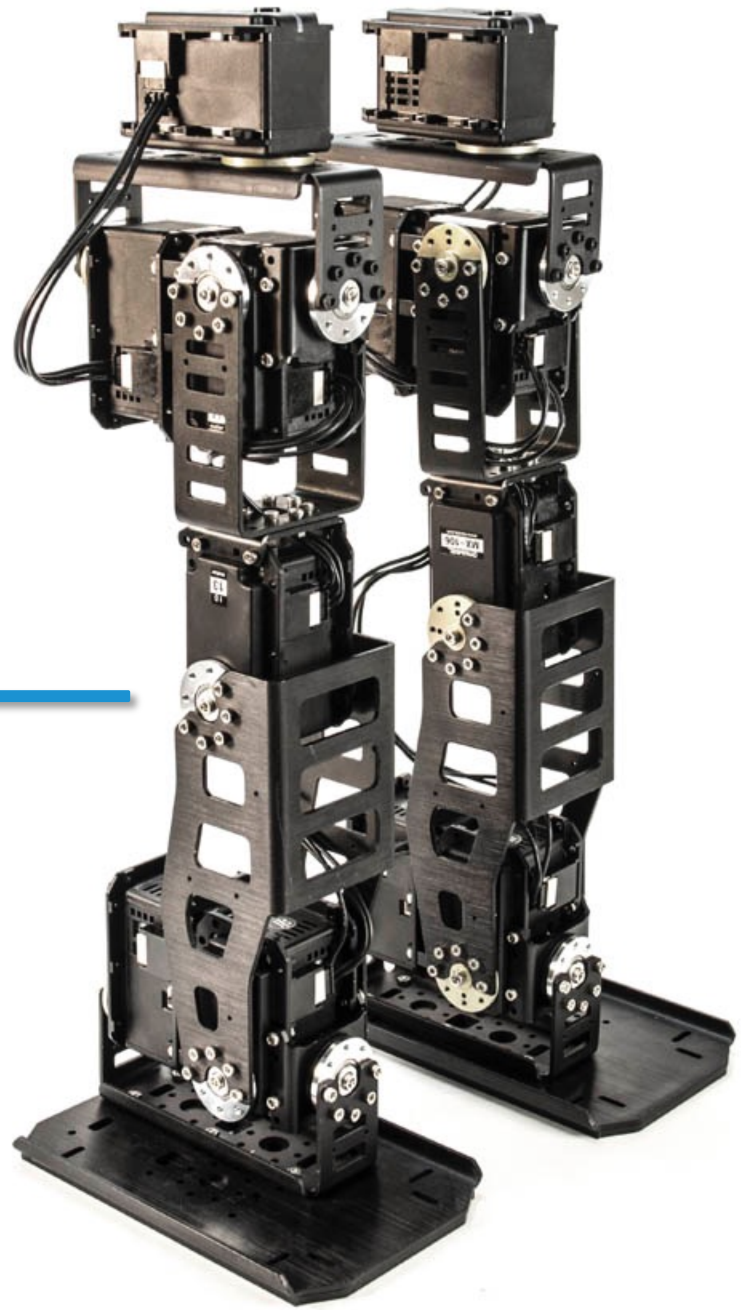
ROCO318
Mobile and Humanoid Robots

Introduction to Kinematics



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Lecture Content

- Forward kinematics
  - Reference frames
    - Links and joints
  - Homogeneous transformations
    - Translation and rotations of reference frames
  - Exercise:
    - Planar kinematic chain
  - Non-planar motions
    - Revolute joints
  - Exercise:
    - Two links planar kinematic chain
  - Denavit-Hartenberg
- Inverse kinematics
  - Geometric IK
    - Exercise:
      - Two links planar kinematic chain
  - Redundant IK
  - Numeric
    - Newton method or Jacobian inverse
      - Exercise:
        - Two links planar kinematic chain

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Notation used throughout the lecture

- Scalars:

$$c_1 \quad a \quad b \quad \alpha$$

- Vectors:

$$\mathbf{a} = \begin{bmatrix} 1 & 2 & \dots & 6 \end{bmatrix} \in \mathbb{R}^{1\times 6}$$

$$\mathbf{w} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}^{\boxed{\top}} \in \mathbb{R}^{3\times 1}$$

transpose operator

- Matrices:

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

- Functions

$$\mathbf{x} = f(\mathbf{q})$$

Our variable convention for robotics

- Joint angles (usually in radians):

$$\mathbf{q} = \begin{bmatrix} q_1 & q_2 & \dots & q_n \end{bmatrix}^\top$$

- Pose of an end-effector – position and orientation - (SI convention, thus in meters):

$$\mathbf{x} = \begin{bmatrix} x & y & z & \alpha & \beta & \gamma \end{bmatrix}^\top$$

- Velocities and accelerations:

$$\dot{\mathbf{q}} \quad \ddot{\mathbf{q}}$$

- Motor torques

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_1 & \tau_2 & \dots & \tau_n \end{bmatrix}^\top$$

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Our variable convention for robotics

- Time dependent variables:

$$\mathbf{q}_{1\ldots T} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \ldots & \mathbf{q}_T \end{bmatrix}$$

  - Discrete time

$$\mathbf{q}_t = \begin{bmatrix} q_t^{[1]} & q_t^{[2]} & \ldots & q_t^{[n]} \end{bmatrix}^\top \in \mathbb{R}^n \qquad \forall t \in \mathbb{N}_0$$

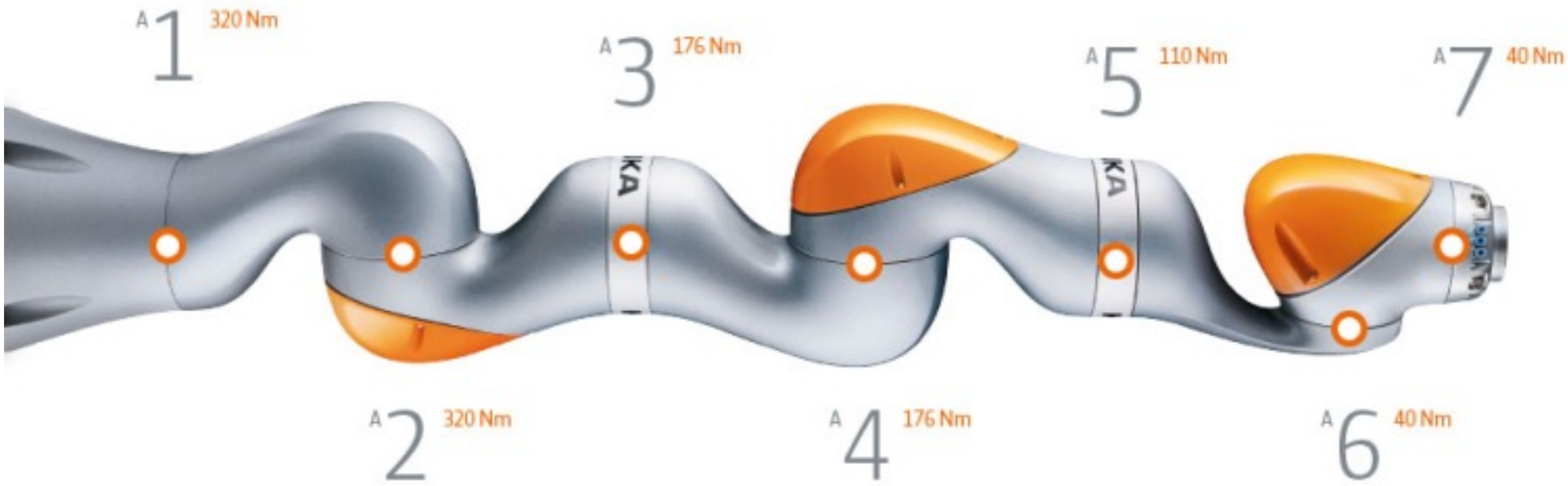
  - Continuous time

$$\mathbf{q}(t) = \begin{bmatrix} q_1(t) & q_2(t) & \ldots & q_n(t) \end{bmatrix}^\top \in \mathbb{R}^n \quad \forall t \in \mathbb{R}^+$$

Kinematics

- Involve determining the possible movements of a robot, without considering the forces and torques acting on the robot (the response to forces and torques is the subject of dynamics).

- Forward Kinematics (or direct kinematics)
  - determine the pose of, say, an end-effector (e.g. a hand), given the joint angles of the robot.

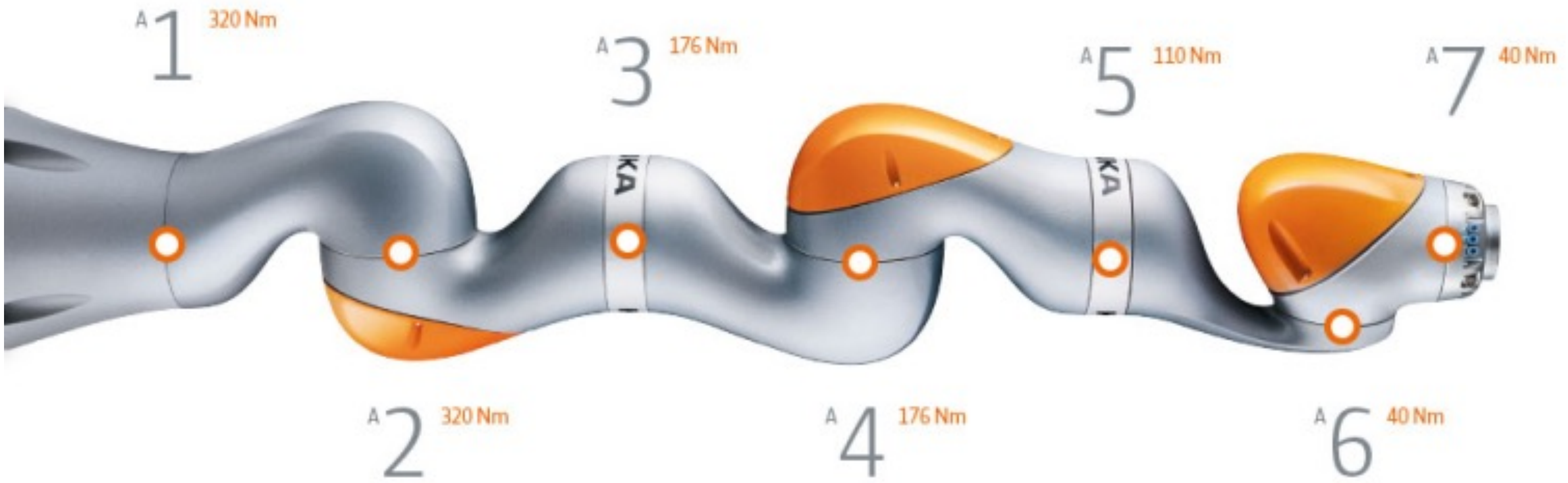


$$\mathbf{x} \in \mathbb{R}^{6\times1} = f_{\text{fwd kin}}(\mathbf{q} \in \mathbb{R}^{7\times1})$$

Kinematics

- Involve determining the possible movements of a robot, without considering the forces and torques acting on the robot (the response to forces and torques is the subject of dynamics).

- Inverse Kinematics
    - determine the joint angles given the pose of the end-effector



$$\mathbf{x} \in \mathbb{R}^{6\times1} \underset{\text{inv kin}}{\mapsto} \mathbf{q} \in \mathbb{R}^{7\times1}$$

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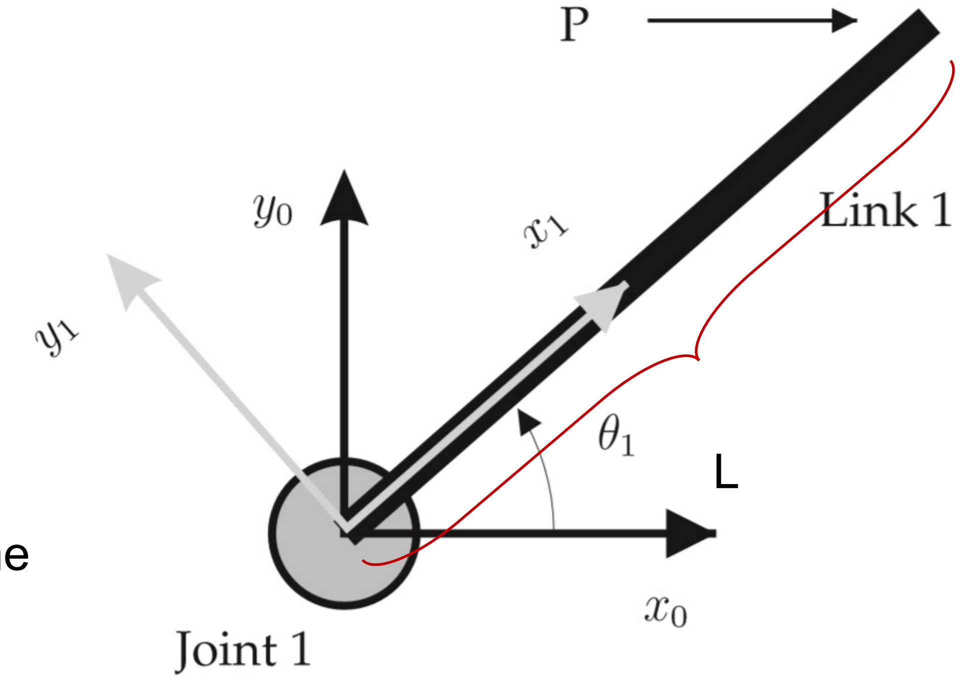
Forward Kinematics

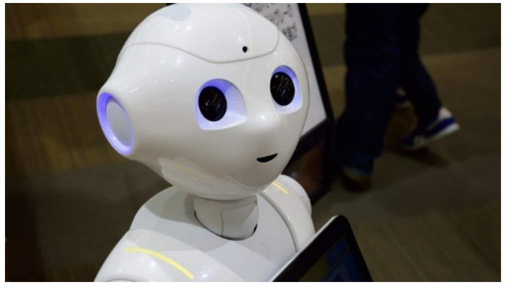
- Humanoid robots are typically built from a set of links, joined with revolute joints that can rotate (a part of) the robot around a given axis.

- Frame 0, with coordinate axes marked x_0 and y_0 is called the reference frame or the inertial frame. Frame 1, with coordinate axes marked x_1 and y_1 is attached to the link. Thus, when joint 1 rotates, link 1 follows.

- Consider now the problem of expressing the orientation of frame 1 with respect to frame 0. As an example, consider the position vector of the end point P of link 1. In frame 1, the link extends along the x_1-axis, and the position vector for the point P thus equals (L,0).

- What are the coordinates of point P in frame 0?





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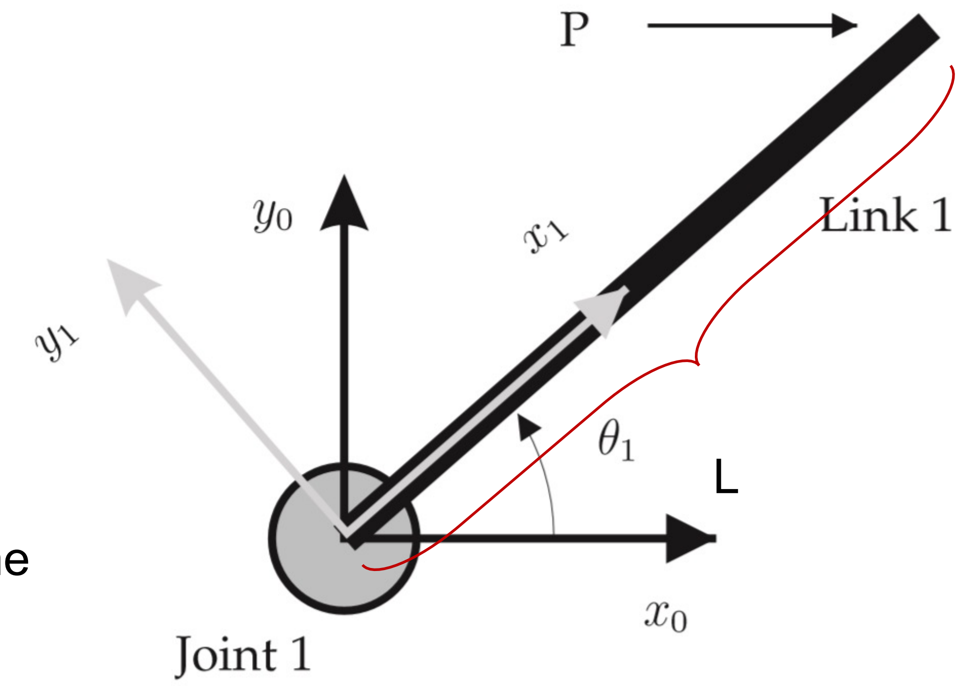
- What are the coordinates of point P in frame 0?

$$x_P = L\cos\theta_1$$

$$y_P = L\sin\theta_1$$



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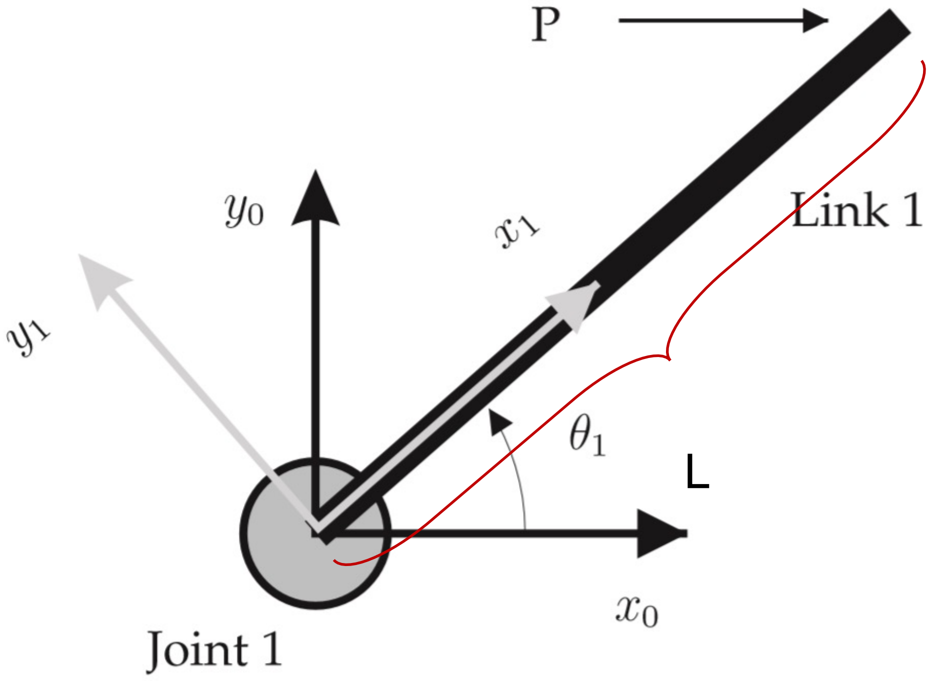

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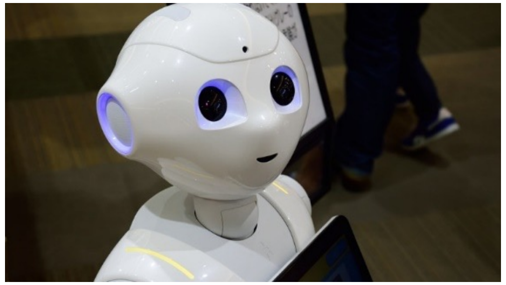
- An arbitrary point (x_1, y_1) in frame 1 can be expressed in the coordinate system of frame 0 as



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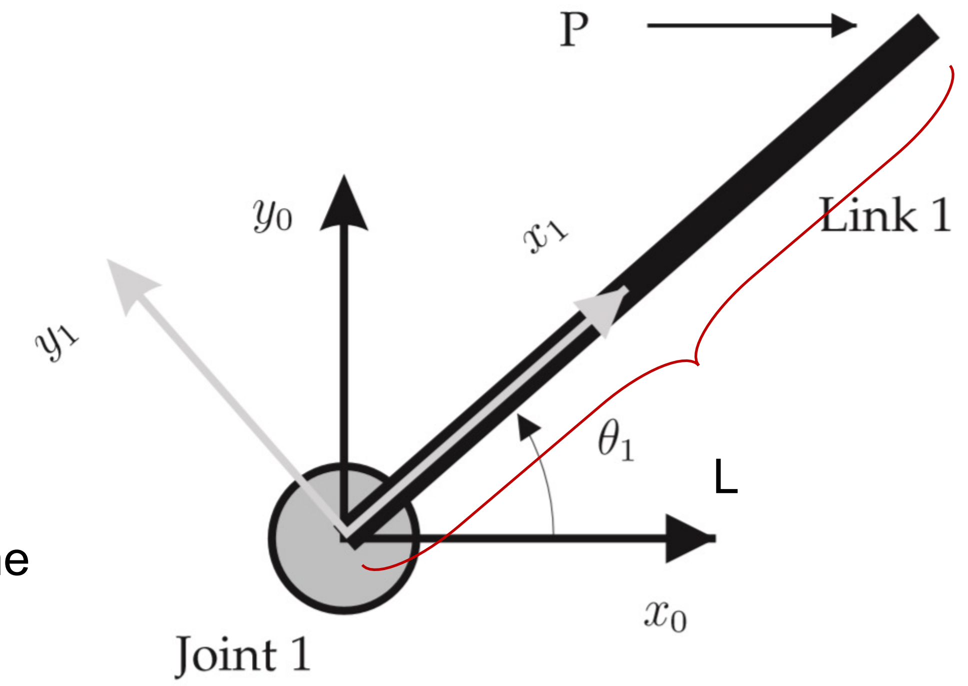


- An arbitrary point (x_1, y_1) in frame 1 can be expressed in the coordinate system of frame 0 as

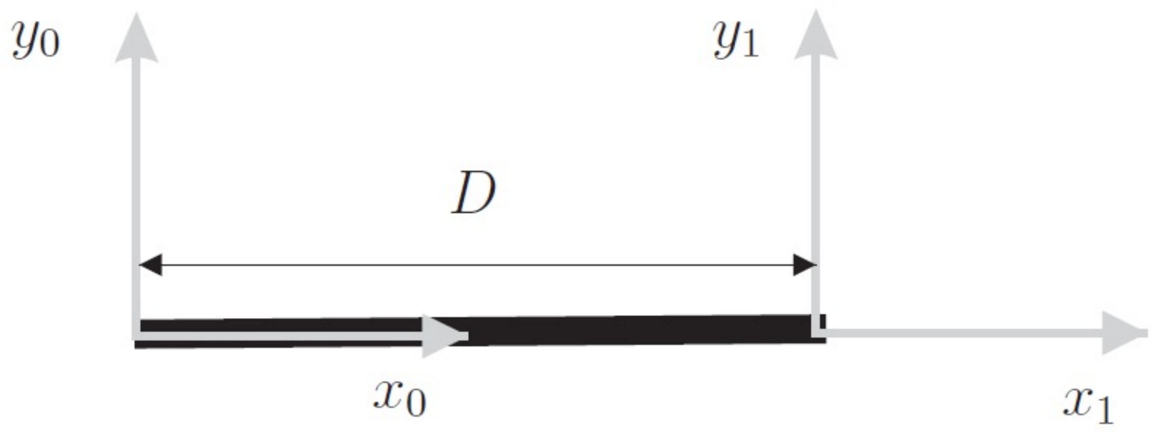
$$x_0 = x_1 \cos\theta_1 - y_1 \sin\theta_1$$
$$y_0 = x_1 \sin\theta_1 + y_1 \cos\theta_1$$

$$\longrightarrow \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Forward Kinematics: Homogeneous Transformations

- Typically, a humanoid robot has several revolute joints, which may be both translated and rotated relative to each other. In order to express the relation between different coordinate frames in robotics one normally uses homogeneous transformations that can handle both translations and rotations.

- Translation



- one coordinate system translated a distance D along a link parallel to the x_0-axis (which coincides with the x_1-axis).

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{pmatrix}$$

(t_x, t_y, t_z) translation vector relating frame 0 to frame 1

- The homogeneous transformation matrix T has sixteen elements.
- The first three rows in the rightmost column define the translation (if any) relating two frames.
- Any point (x_1, y_1, z_1) in the coordinate system of frame 1 can be expressed in the coordinate system of frame 0 through the transformation.

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Forward Kinematics: Homogeneous Transformations

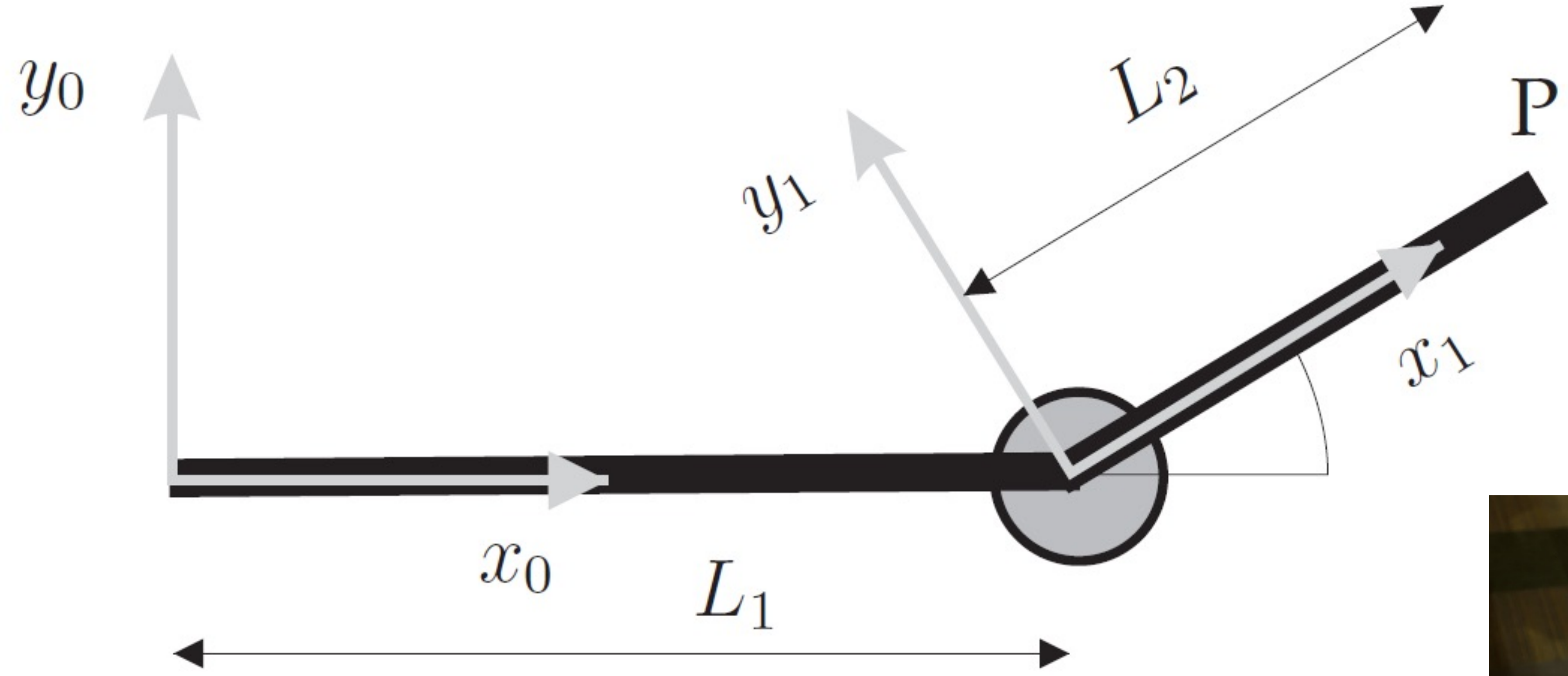
- Rotations around the x1, y1 and z1 (by an angle θ) are given by

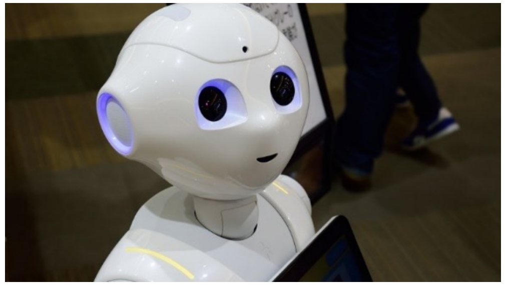
$$\mathrm{RotX}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathrm{RotY}(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{RotZ}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise: Planar Kinematic Chain

- The robot has a single joint, which is translated a distance L_1 along the x_0-axis.





- Questions?
  - What are the coordinates of the end point P, expressed in the reference frame 0?
  - What are the coordinates of P in frame 1?
  - Which is the coordinate transformation (homogeneous matrix), combining the translation (L_1, 0, 0) with a rotation around the z_0-axis?

Solution: Planar Kinematic Chain

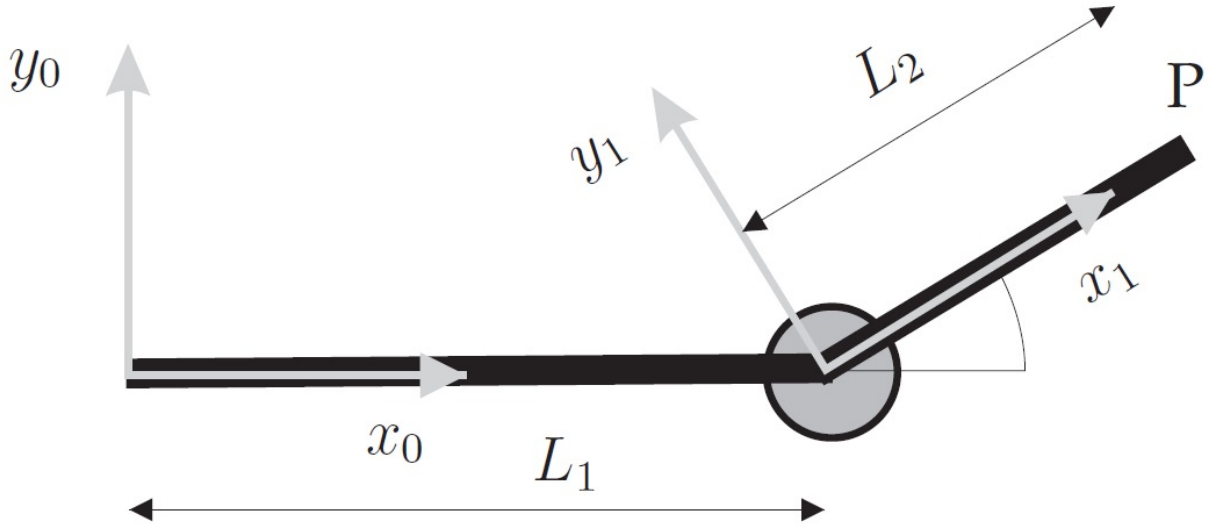
- Coordinates of the end point P, expressed in the reference frame 0

$$x_0 = L_1 + L_2 \cos\theta$$

$$y_0 = L_2 \sin\theta$$

- Coordinates of the end point P, expressed frame 1

$$\mathbf{P} = \begin{bmatrix} L_2 & 0 & 0 \end{bmatrix}$$



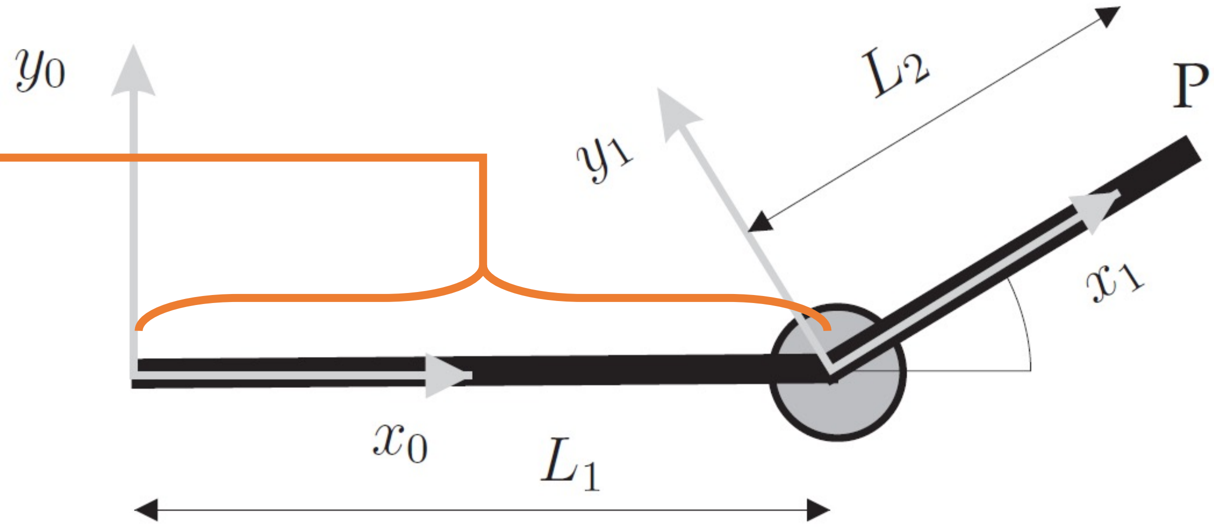
- Coordinate transformation (homogeneous matrix), combining the translation (L_1, 0, 0) with a rotation around the z_0-axis?

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & L_1 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} L_2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} L_1 + L_2\cos\theta \\ L_2\sin\theta \\ 0 \\ 1 \end{pmatrix}$$

Solution: Planar Kinematic Chain

- Coordinates of the end point P, expressed in the reference frame 0

$$x_0 = \boxed{L_1} + L_2 \cos\theta$$
$$y_0 = L_2 \sin\theta$$



- Coordinates of the end point P, expressed frame 1

$$\mathbf{P} = \begin{bmatrix} L_2 & 0 & 0 \end{bmatrix}$$

- Coordinate transformation (homogeneous matrix), combining the translation (L_1, 0, 0) with a rotation around the z_0-axis?

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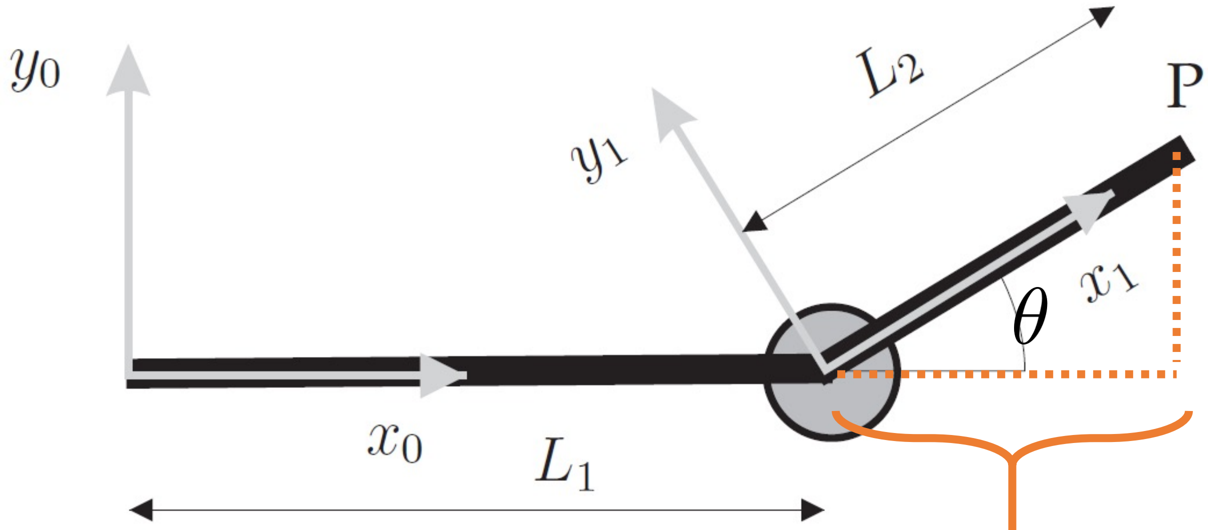

- Coordinates of the end point P, expressed frame 1

$$\mathbf{P} = \begin{bmatrix} L_2 & 0 & 0 \end{bmatrix}$$

- Coordinate transformation (homogeneous matrix), combining the translation (L_1, 0, 0) with a rotation around the z_0-axis?

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & L_1 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} L_2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} L_1 + L_2\cos\theta \\ L_2 \sin\theta \\ 0 \\ 1 \end{pmatrix}$$

Solution: Planar Kinematic Chain

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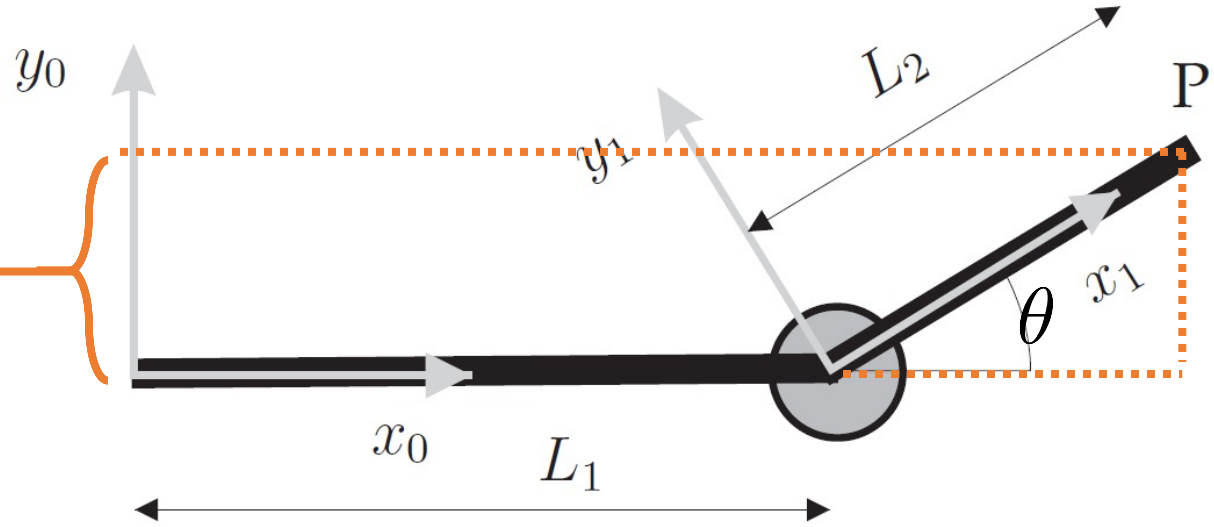

- Coordinates of the end point P, expressed frame 1

$$\mathbf{P} = \begin{bmatrix} L_2 & 0 & 0 \end{bmatrix}$$

- Coordinate transformation (homogeneous matrix), combining the translation (L_1, 0, 0) with a rotation around the z_0-axis?

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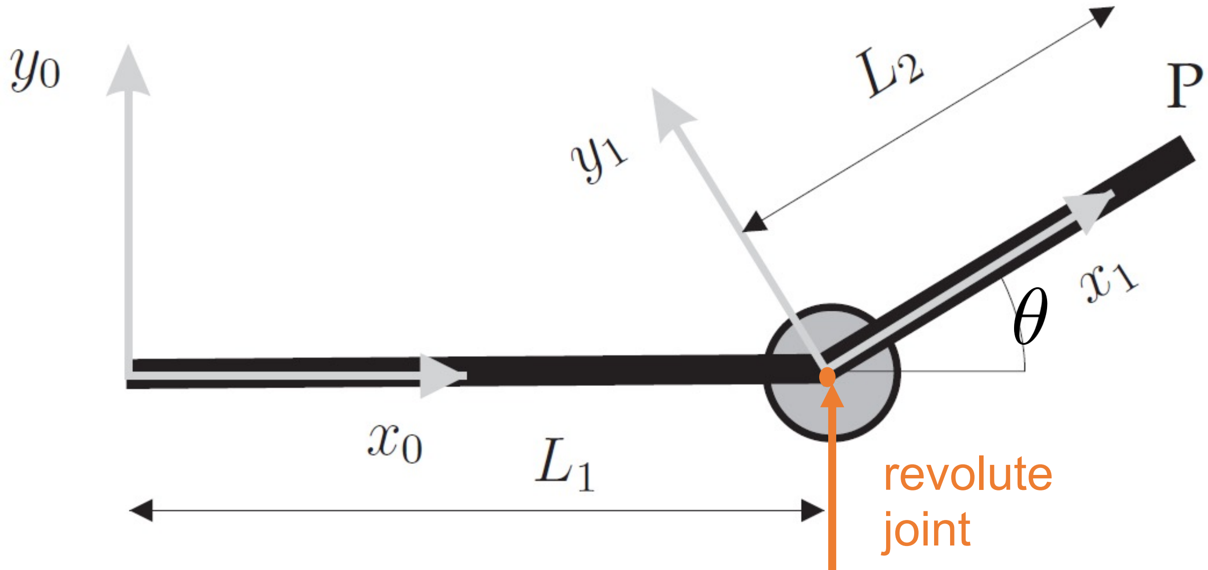

- Coordinate transformation (homogeneous matrix), combining the translation (L_1, 0, 0) with a rotation around the z_0-axis?

rotation of frame 1 around z-axis of frame 0

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & L_1 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} L_2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} L_1 + L_2\cos\theta \\ L_2 \sin\theta \\ 0 \\ 1 \end{pmatrix}$$

Solution: Planar Kinematic Chain

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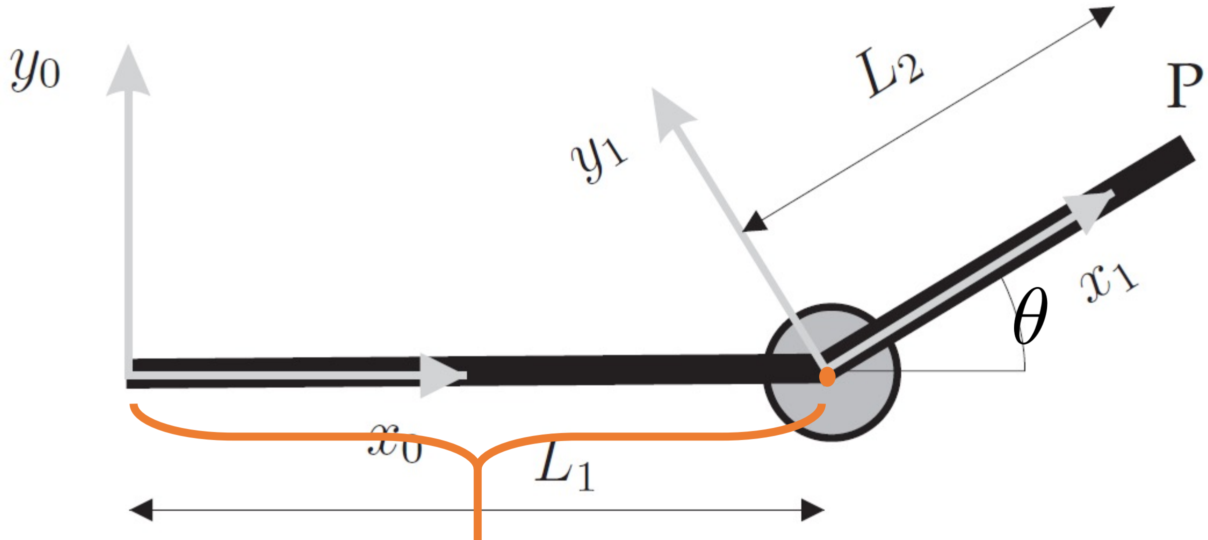


- Coordinates of the end point P, expressed frame 1

$$\mathbf{P} = \begin{bmatrix} L_2 & 0 & 0 \end{bmatrix}$$

- Coordinate transformation (homogeneous matrix), combining the translation (L_1, 0, 0) with a rotation around the z_0-axis?

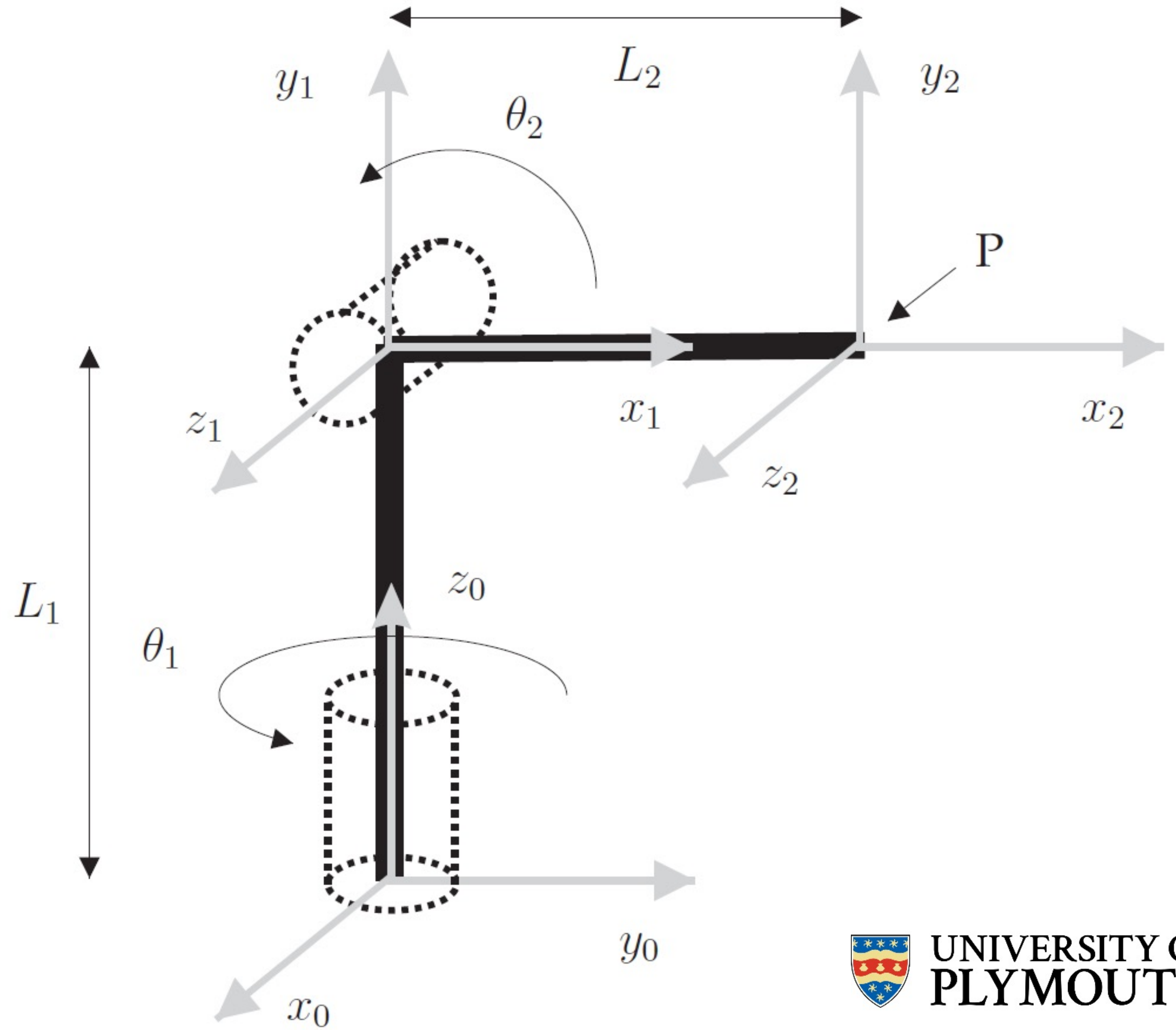
displacement of frame 1 wrt frame 0

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & L_1 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} L_2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} L_1 + L_2\cos\theta \\ L_2 \sin\theta \\ 0 \\ 1 \end{pmatrix}$$

Non-planar Motions: Two revolute joints

- Transformations (rotations and translations) can be represented as a product of homogeneous transformation matrices.
  - systematic way of representing translations and rotations.

- Two revolute joints.
  - the placement of the coordinate axes is such that the rotation of the joints occurs around the z axes in the various frames
  - in frame 2 the position of the end point P equals (0, 0, 0) and, when $\theta_1 = \theta_2 = 0$, the corresponding position in frame 0 equals $(0, L_2, L_1)$.



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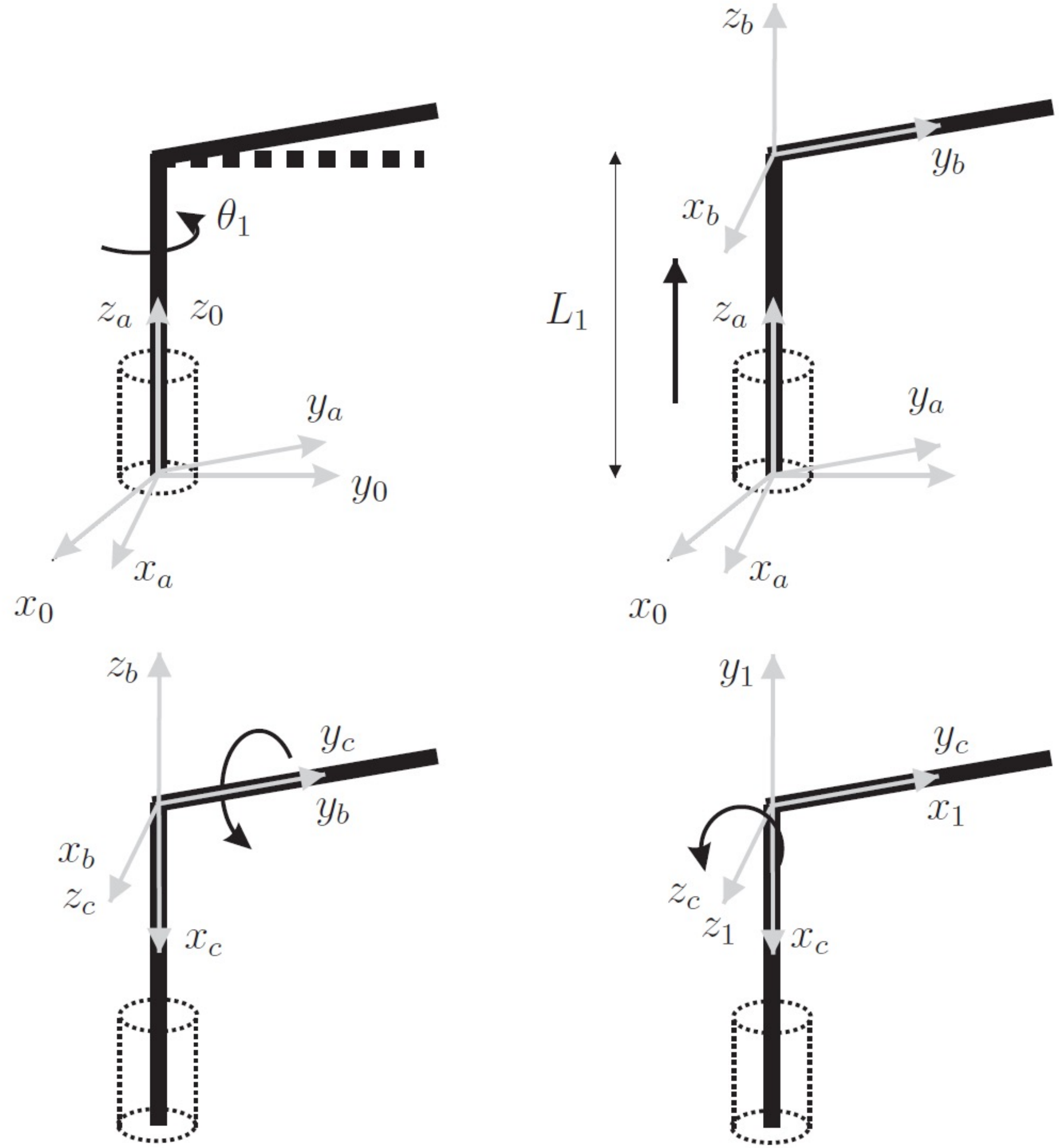
Non-planar Motions: Two revolute joints

- Consider the transformations connecting frame 1 to frame 0 (the reference frame).
  - Four steps
    - *a*, *b*, and *c* are used to denote the intermediate frames

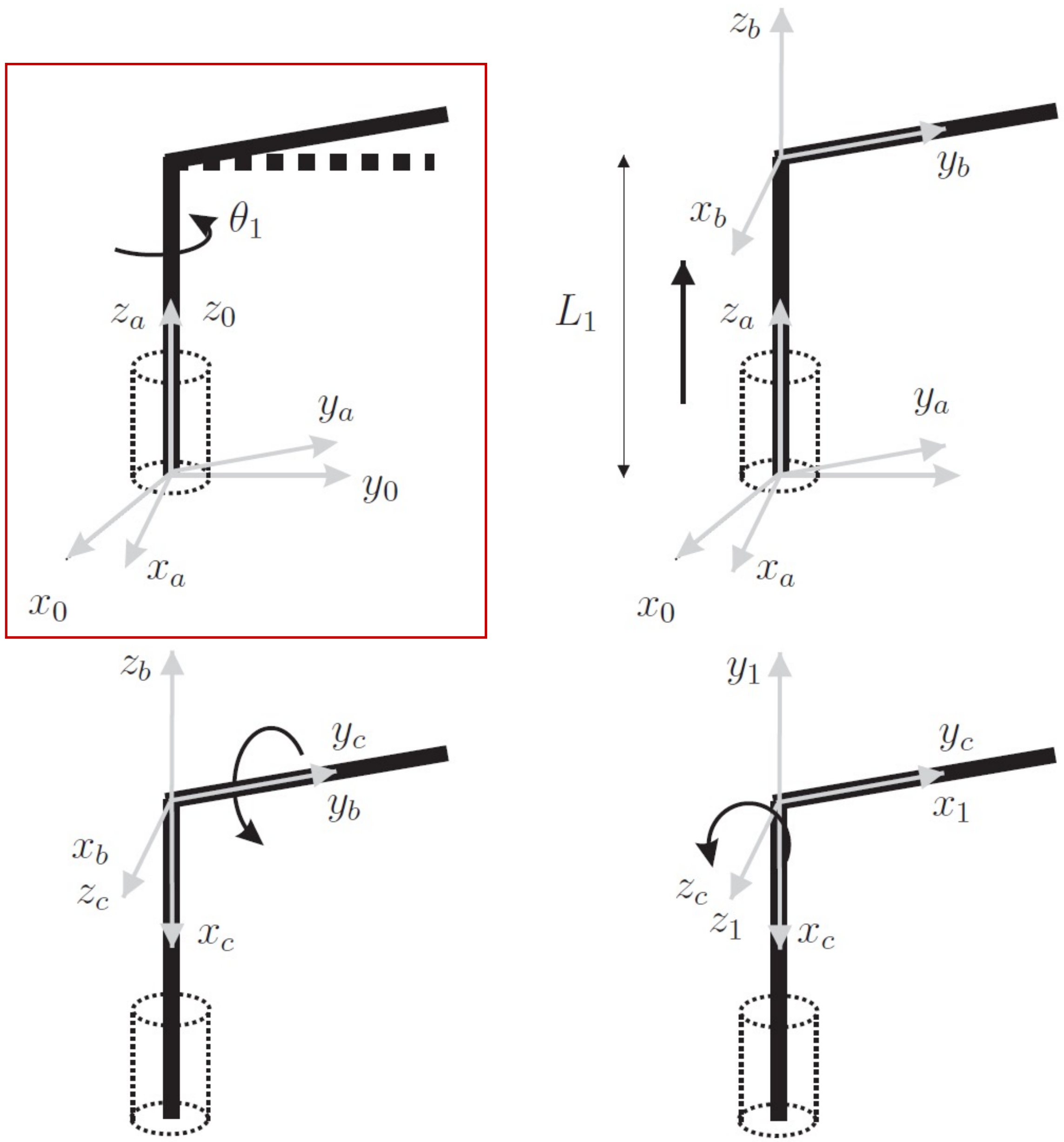


1. The first part of the transformation involves the rotation of the coordinate frame (by an angle θ_1) around the z_0 axis.
2. The second transformation involves a translation along the z_a axis (which coincides with the z_0 axis)
3. The two final steps consist of rotating the coordinate frame to place the z_1 axis.

Non-planar Motions: Two revolute joints

- Transformation relating frame 0 to frame a
  - Joint 1 rotates the robot around the z_0 axis.

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_a \\ y_a \\ z_a \\ 1 \end{pmatrix} = \begin{pmatrix} x_a\cos\theta_1 - y_a\sin\theta_1 \\ x_a\sin\theta_1 + y_a\cos\theta_1 \\ z_a \\ 1 \end{pmatrix}.$$



Non-planar Motions: Two revolute joints

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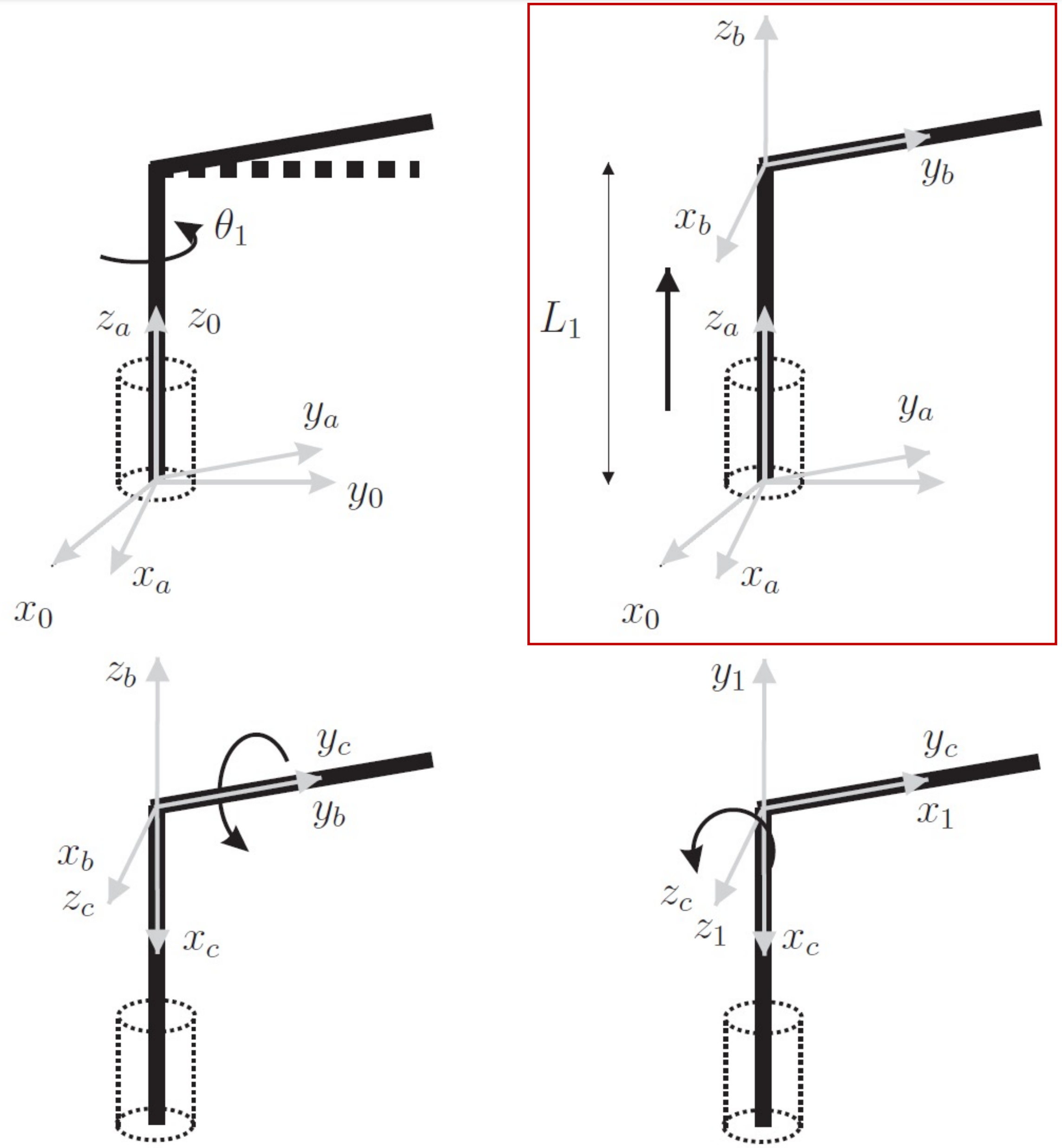
- Transformation relating frame a to frame b
  - Translation L_1 length units along the z_a axis

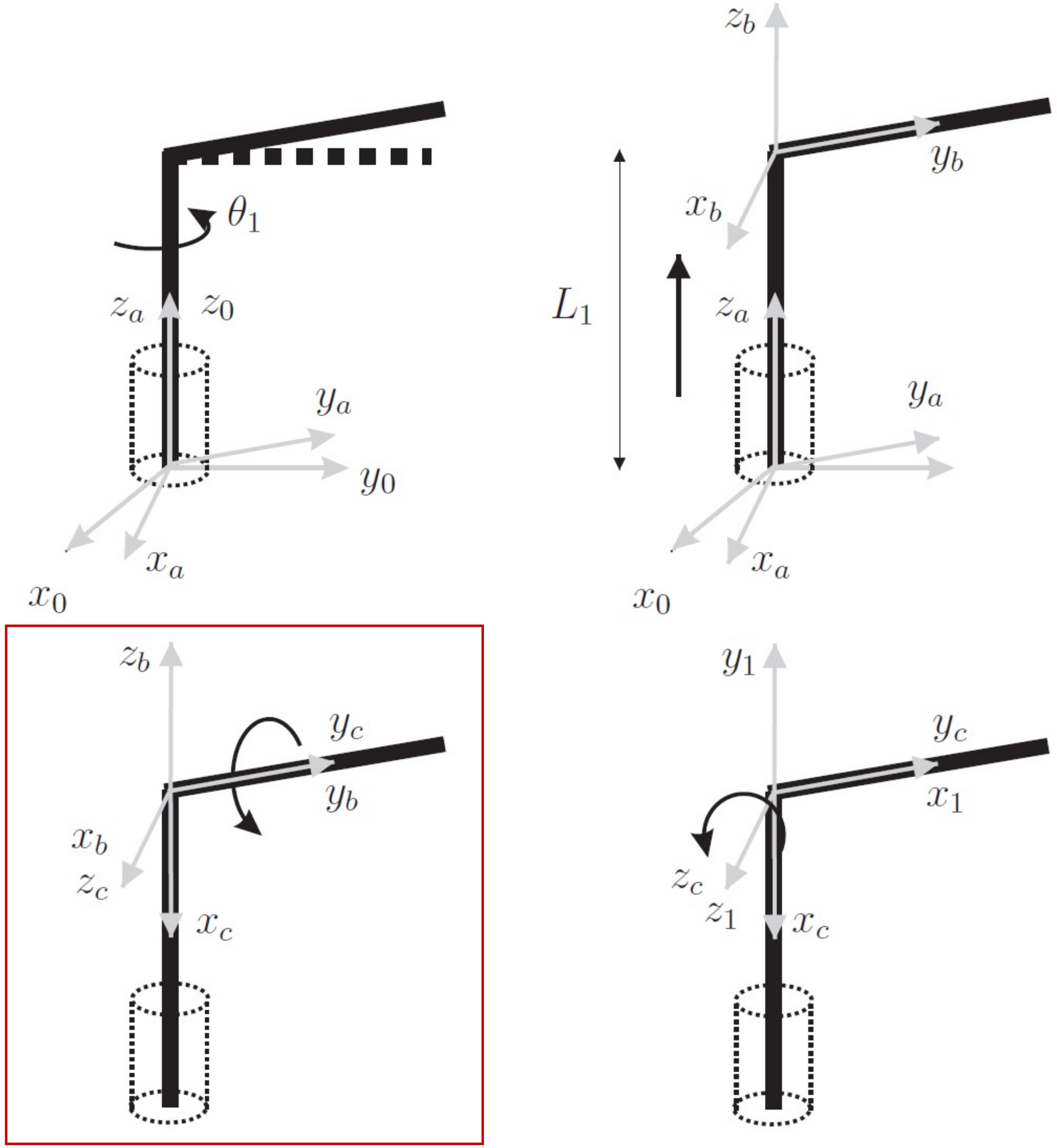
$$\begin{pmatrix} x_a \\ y_a \\ z_a \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \\ 1 \end{pmatrix} = \begin{pmatrix} x_b \\ y_b \\ z_b + L_1 \\ 1 \end{pmatrix}$$



Non-planar Motions: Two revolute joints

- Transformation relating frame b to frame c
  - the coordinate system is rotated 90 degrees around the y_b axis (note the direction of rotation).

$$\begin{pmatrix} x_b \\ y_b \\ z_b \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_c \\ y_c \\ z_c \\ 1 \end{pmatrix} = \begin{pmatrix} z_c \\ y_c \\ -x_c \\ 1 \end{pmatrix}$$



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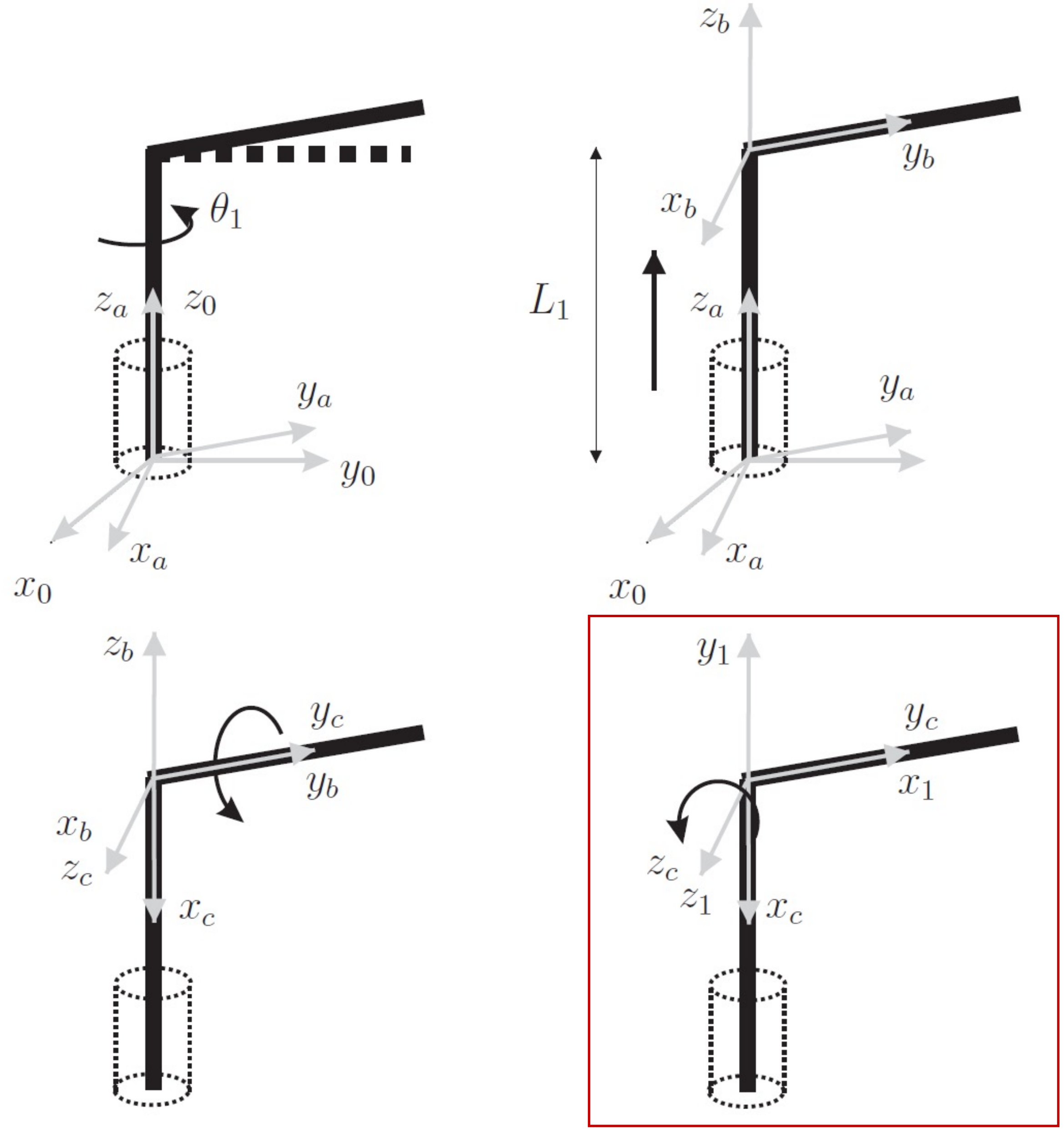
- Transformation relating frame c to frame 1
  - the coordinate system is rotated 90 degrees around the z_c axis

$$\begin{pmatrix} x_c \\ y_c \\ z_c \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{pmatrix} = \begin{pmatrix} -y_1 \\ x_1 \\ z_1 \\ 1 \end{pmatrix}$$

- Put all together to complete the transformation relating frame 1 to frame 0



Non-planar Motions: Two revolute joints

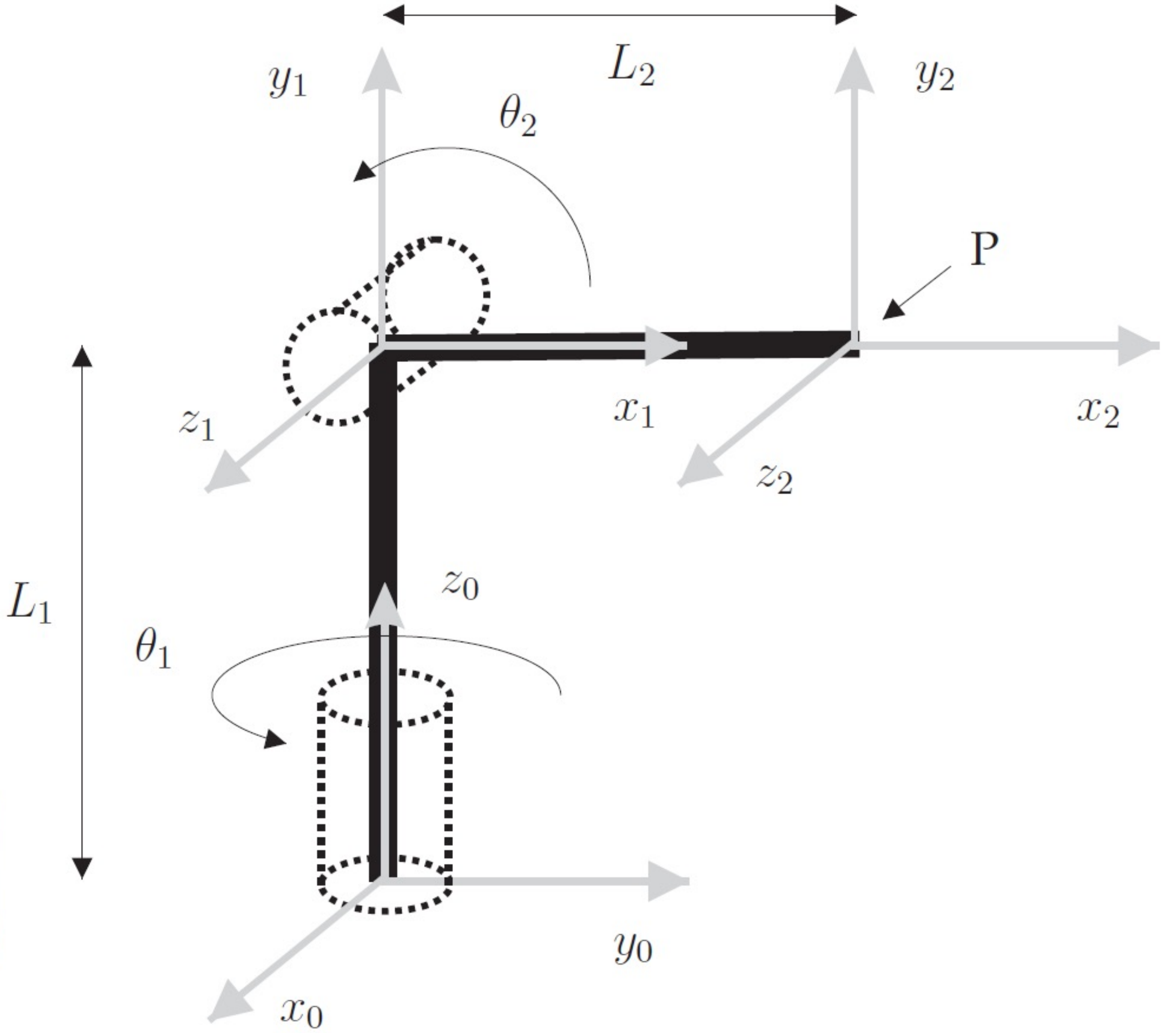
- Transformation relating frame 2 to frame 1
  - introduce a temporary frame d which is obtaine
    rotating frame 1 by an angle θ_2 around the z_1 a

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_d \\ y_d \\ z_d \\ 1 \end{pmatrix}$$

- The relation between frame d and frame 2 is a simple translation (of a distance L_2) along the x_d axis (which coincides with the x_2 axis).

$$\begin{pmatrix} x_d \\ y_d \\ z_d \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{pmatrix} = \begin{pmatrix} x_2 + L_2 \\ y_2 \\ z_2 \\ 1 \end{pmatrix}$$

- Put all together to complete the transformation relating frame 2 to frame 0



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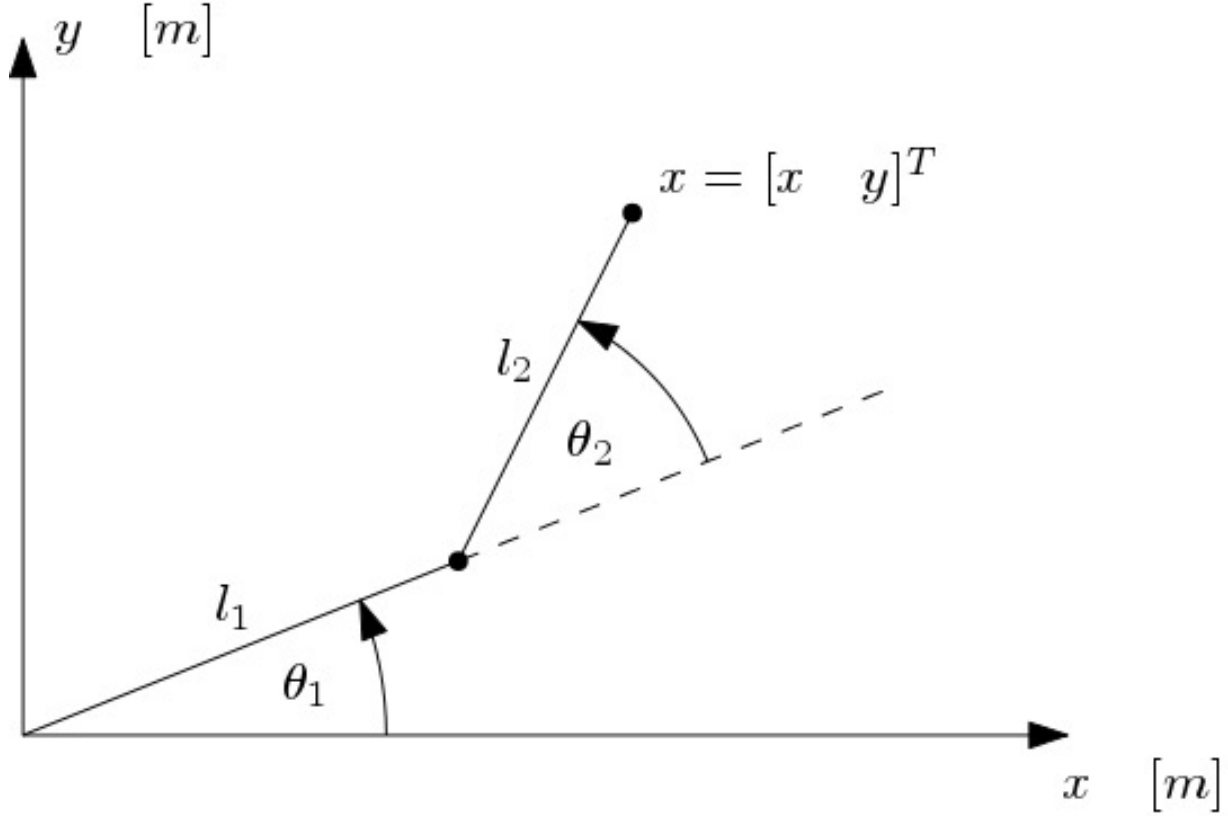
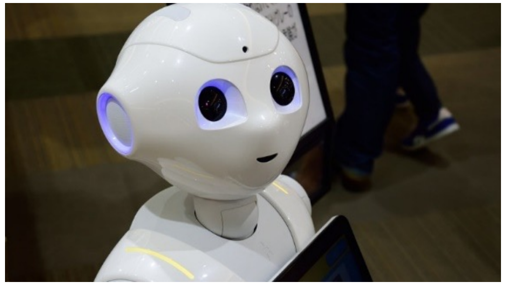
Exercise: Two Links Planar Kinematic Chain

- The robot has two revolute joints.





- Questions?
  - What are the coordinates of the end point x, expressed in the reference frame?
  - What are the coordinates of x in the second frame?
  - Which is the coordinate transformation (homogeneous matrix), combining the two rotations?

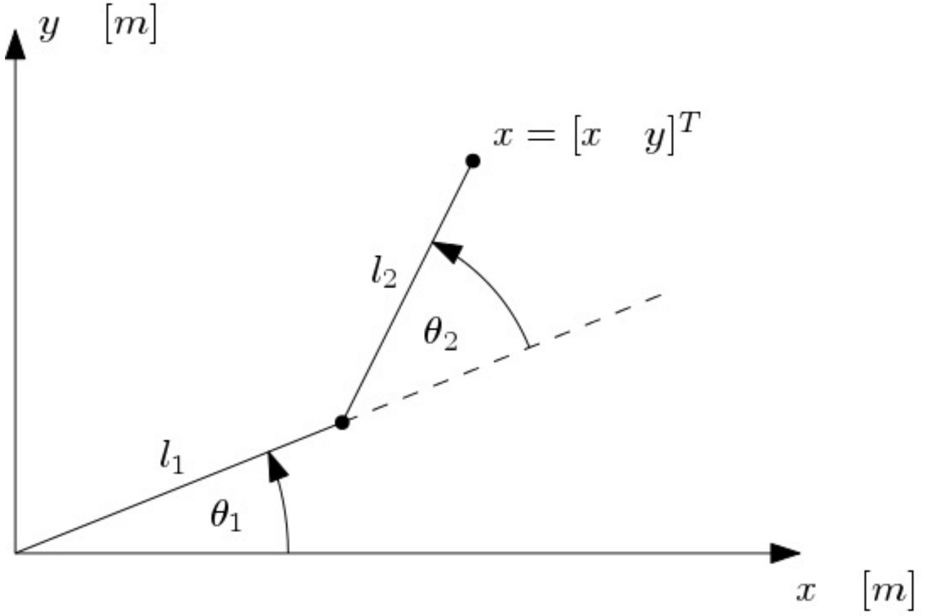
Solution: Planar Kinematic Chain

- Coordinates of the end point P, expressed in the reference frame

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{bmatrix}$$

- Coordinates of the end point x, expressed frame 2

$$\mathbf{x} = \begin{bmatrix} l_2 & 0 & 0 \end{bmatrix}$$



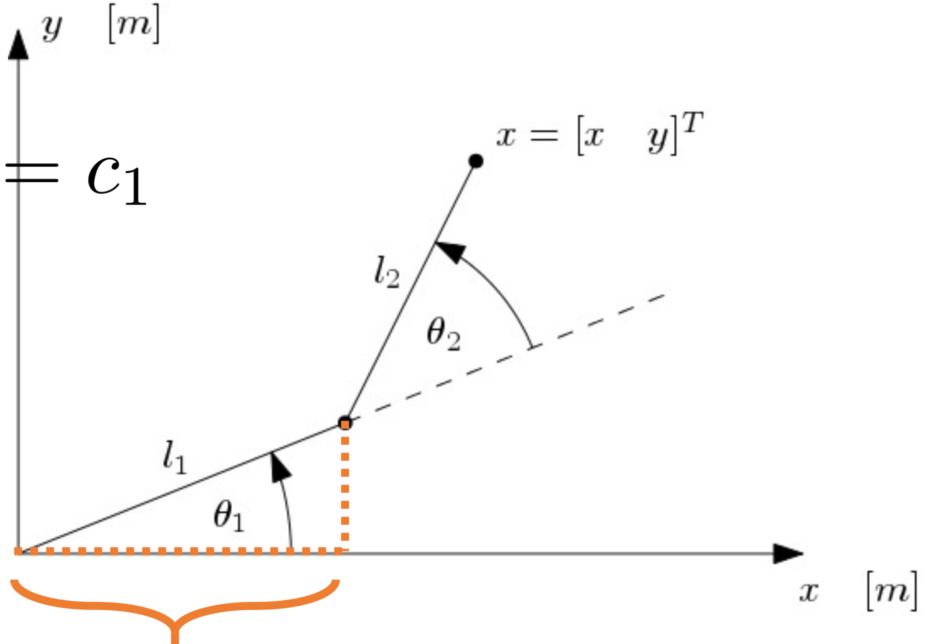
Solution: Planar Kinematic Chain

- Coordinates of the end point P, expressed in the reference frame

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \boxed{l_1 c_1} + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{bmatrix}$$

shorthand

$$\cos(\theta_1) := c_1$$



- Coordinates of the end point x, expressed frame 2

$$\mathbf{x} = \begin{bmatrix} l_2 & 0 & 0 \end{bmatrix}$$

Solution: Planar Kinematic Chain

- Coordinates of the end point P, expressed in the reference frame

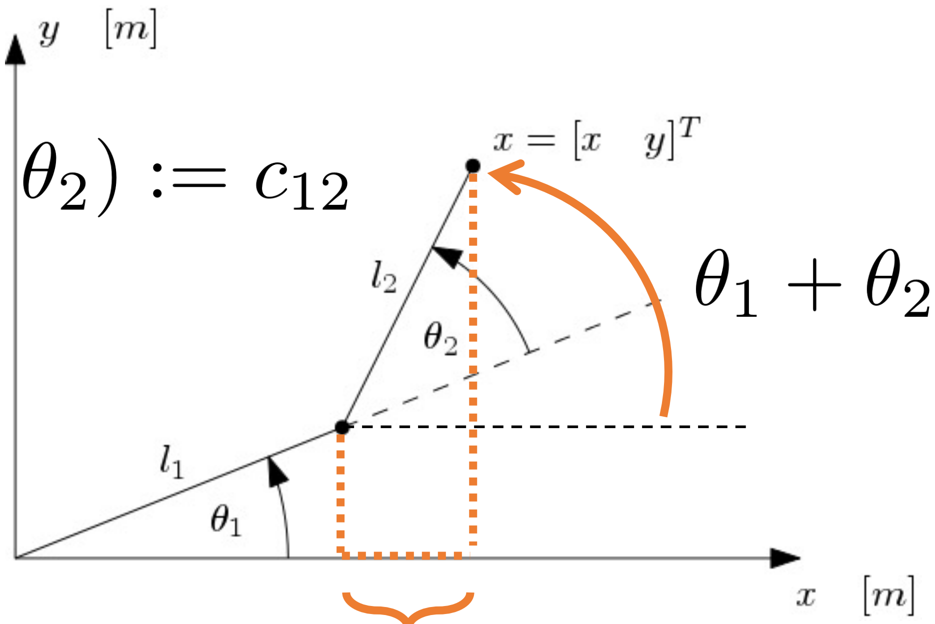
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 c_1 + \boxed{l_2 c_{12}} \\ l_1 s_1 + l_2 s_{12} \end{bmatrix}$$

shorthand

$$\cos(\theta_1 + \theta_2) := c_{12}$$

- Coordinates of the end point x, expressed frame 2

$$\mathbf{x} = \begin{bmatrix} l_2 & 0 & 0 \end{bmatrix}$$



Solution: Planar Kinematic Chain

- Coordinates of the end point P, expressed in the reference frame

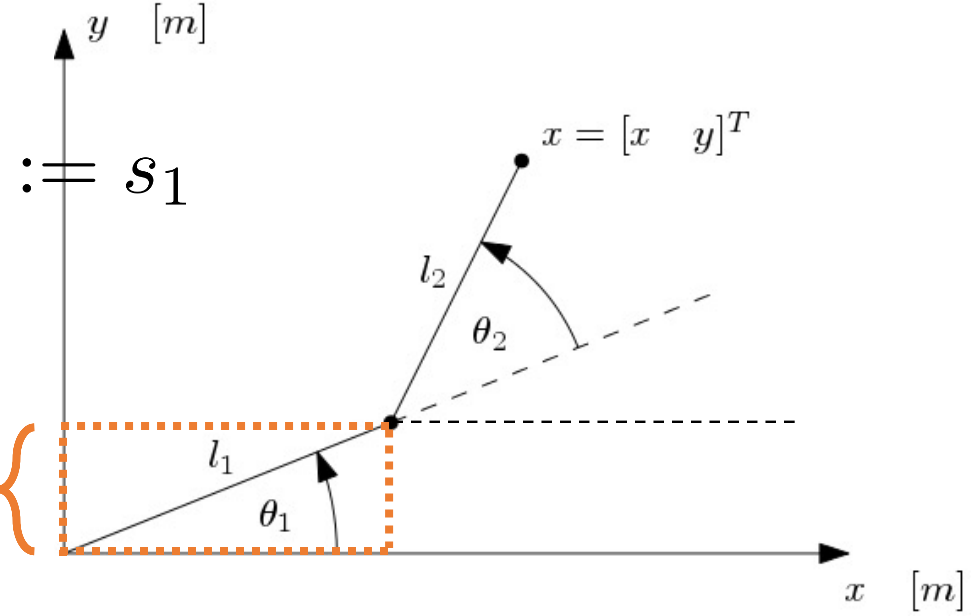
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ \boxed{l_1 s_1} + l_2 s_{12} \end{bmatrix}$$

shorthand

$$\sin(\theta_1) := s_1$$

- Coordinates of the end point x, expressed frame 2

$$\mathbf{x} = \begin{bmatrix} l_2 & 0 & 0 \end{bmatrix}$$



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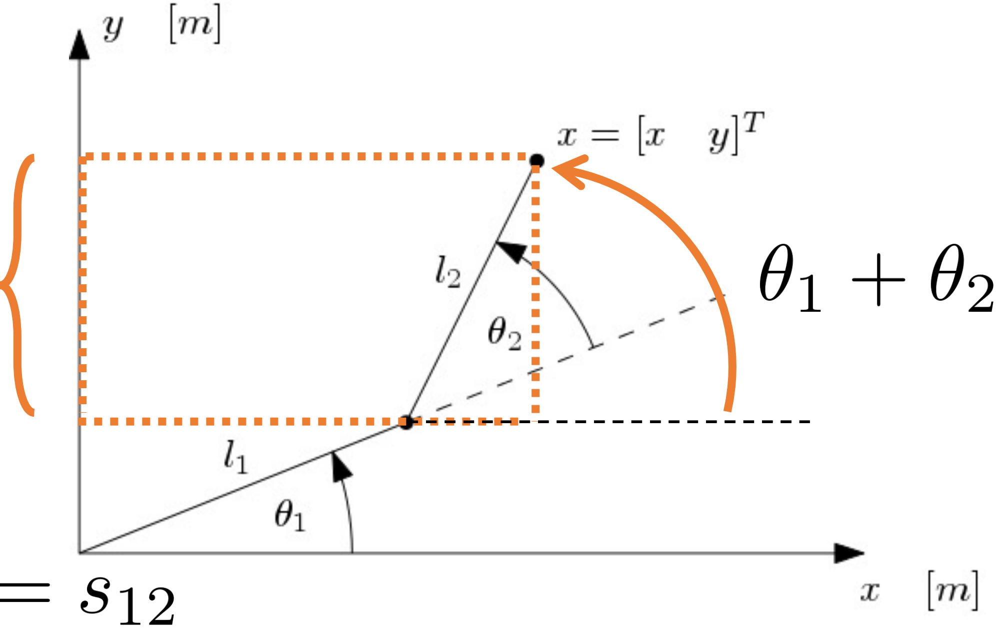
Solution: Planar Kinematic Chain

- Coordinates of the end point P, expressed in the reference frame

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + \boxed{l_2 s_{12}} \end{bmatrix}$$



- Coordinates of the end point x, expressed frame 2

$$\mathbf{x} = [l_2 \quad 0 \quad 0]$$

shorthand

$$\sin(\theta_1 + \theta_2) := s_{12}$$

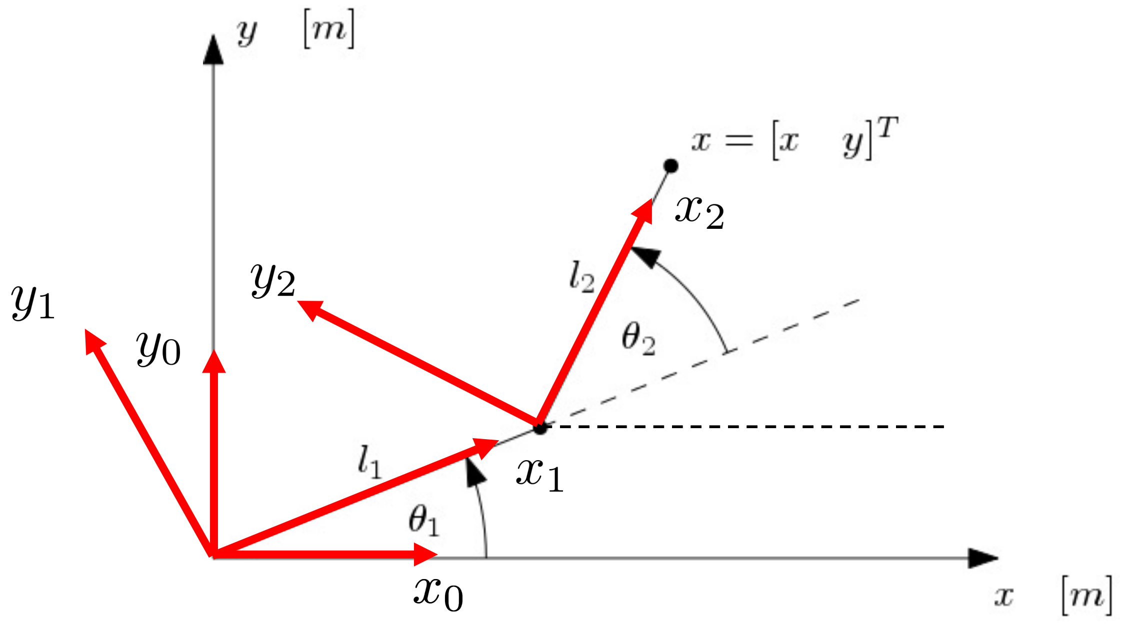
Solution: Planar Kinematic Chain

- Coordinates of the end point P, expressed in the reference frame

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{bmatrix}$$

- Coordinates of the end point x, expressed frame 2

$$\mathbf{x} = \begin{bmatrix} l_2 & 0 & 0 \end{bmatrix}$$



- Coordinate transformation (homogeneous matrix), combining the two rotations
  - Transformation relating frame 0 (inertial frame) to frame 1

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 c_1 - y_1 s_1 \\ x_1 s_1 + y_1 c_1 \\ z_1 \\ 1 \end{bmatrix}$$

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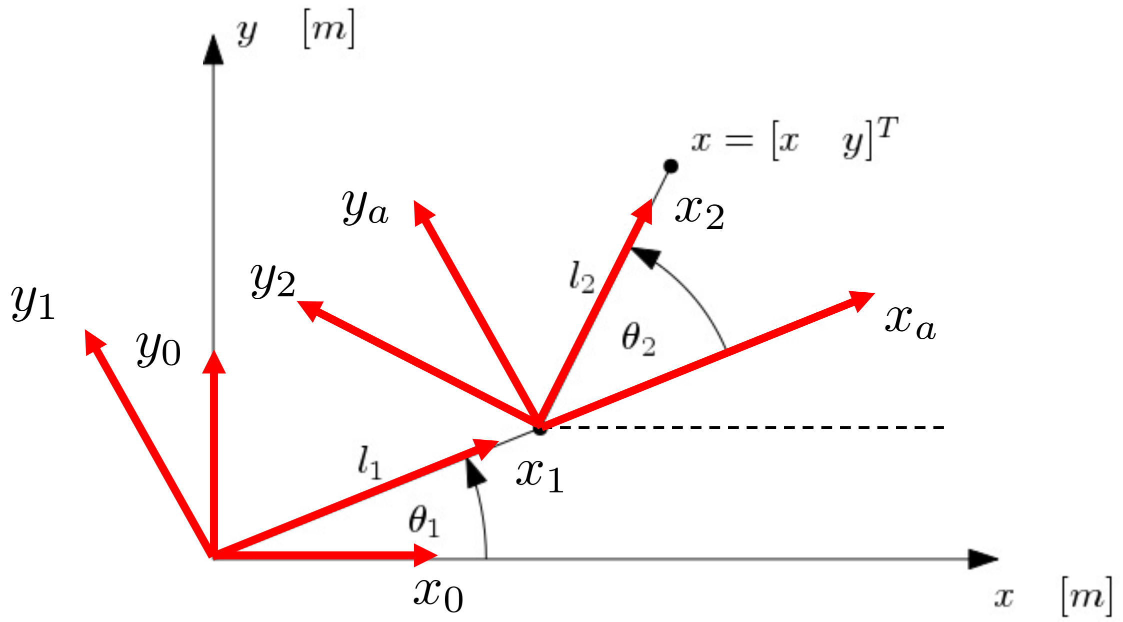
Solution: Planar Kinematic Chain

- Coordinates of the end point P, expressed in the reference frame

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{bmatrix}$$

- Coordinates of the end point x, expressed frame 2

$$\mathbf{x} = \begin{bmatrix} l_2 & 0 & 0 \end{bmatrix}$$



- Coordinate transformation (homogeneous matrix), combining the two rotations
  - Transformation relating frame 1 to frame a (temporary introduced)
    - Translation l_1 length units along the x_1 axis

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_a \\ y_a \\ z_a \\ 1 \end{bmatrix} = \begin{bmatrix} x_a + l_1 \\ y_a \\ z_a \\ 1 \end{bmatrix}$$

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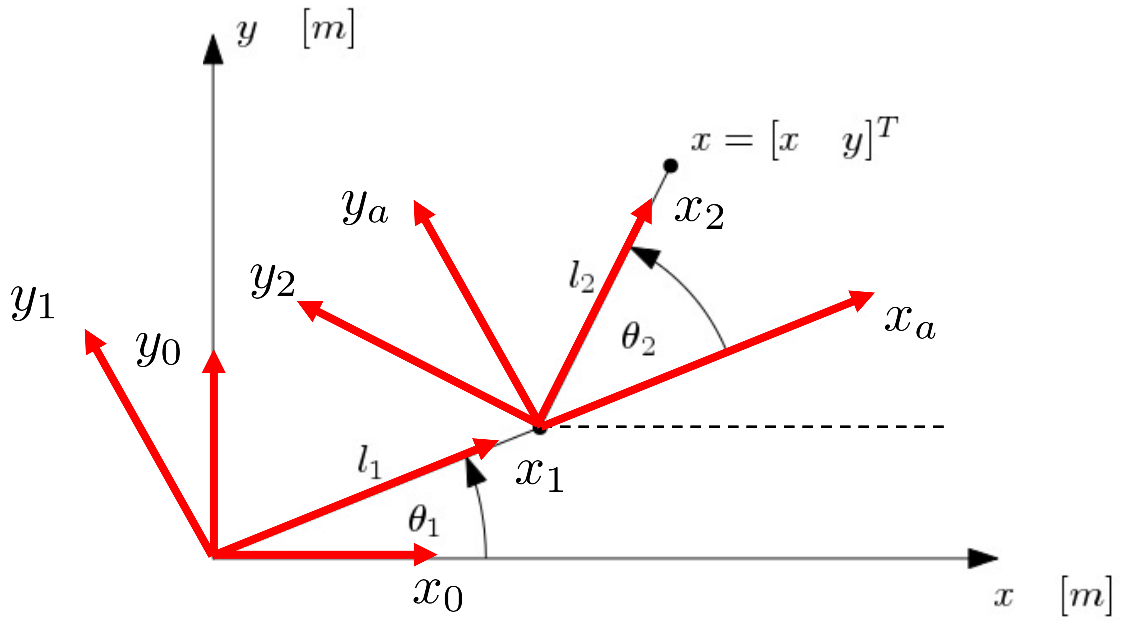
Solution: Planar Kinematic Chain

- Coordinates of the end point P, expressed in the reference frame

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{bmatrix}$$

- Coordinates of the end point x, expressed frame 2

$$\mathbf{x} = \begin{bmatrix} l_2 & 0 & 0 \end{bmatrix}$$



- Coordinate transformation (homogeneous matrix), combining the two rotations
  - Transformation relating frame a to frame 2

$$\begin{bmatrix} x_a \\ y_a \\ z_a \\ 1 \end{bmatrix} = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 2 \end{bmatrix} = \begin{bmatrix} x_2 c_2 - y_2 s_2 \\ x_2 s_2 + y_2 c_2 \\ z_2 \\ 1 \end{bmatrix}$$

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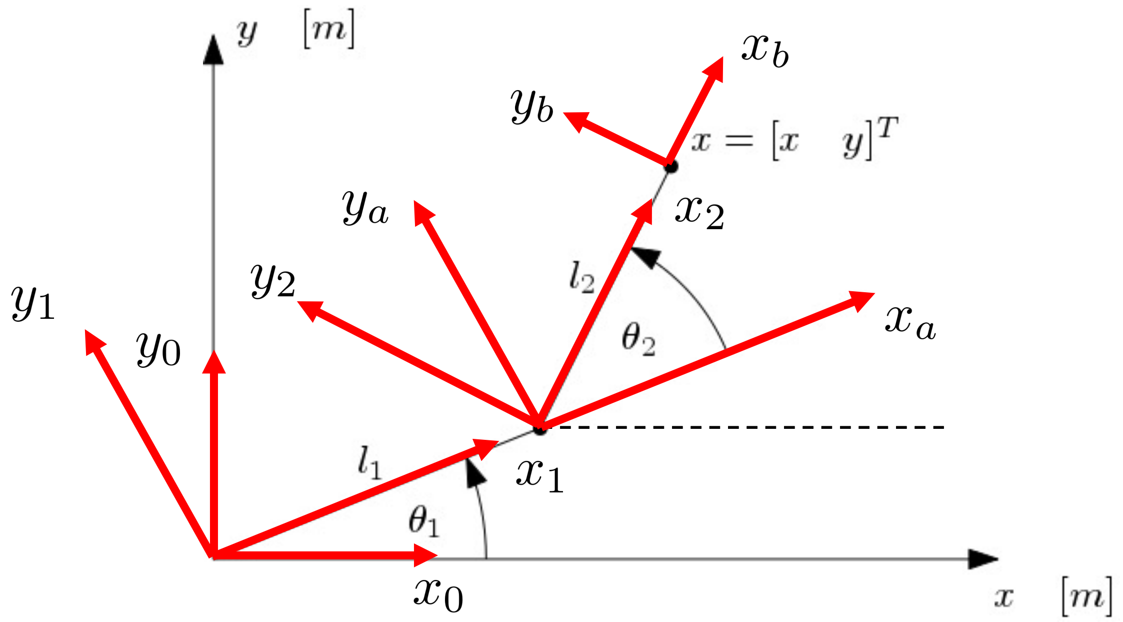
Solution: Planar Kinematic Chain

- Coordinates of the end point P, expressed in the reference frame

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{bmatrix}$$

- Coordinates of the end point x, expressed frame 2

$$\mathbf{x} = \begin{bmatrix} l_2 & 0 & 0 \end{bmatrix}$$



- Coordinate transformation (homogeneous matrix), combining the two rotations
  - Transformation relating frame 2 to frame b (temporary introduced)

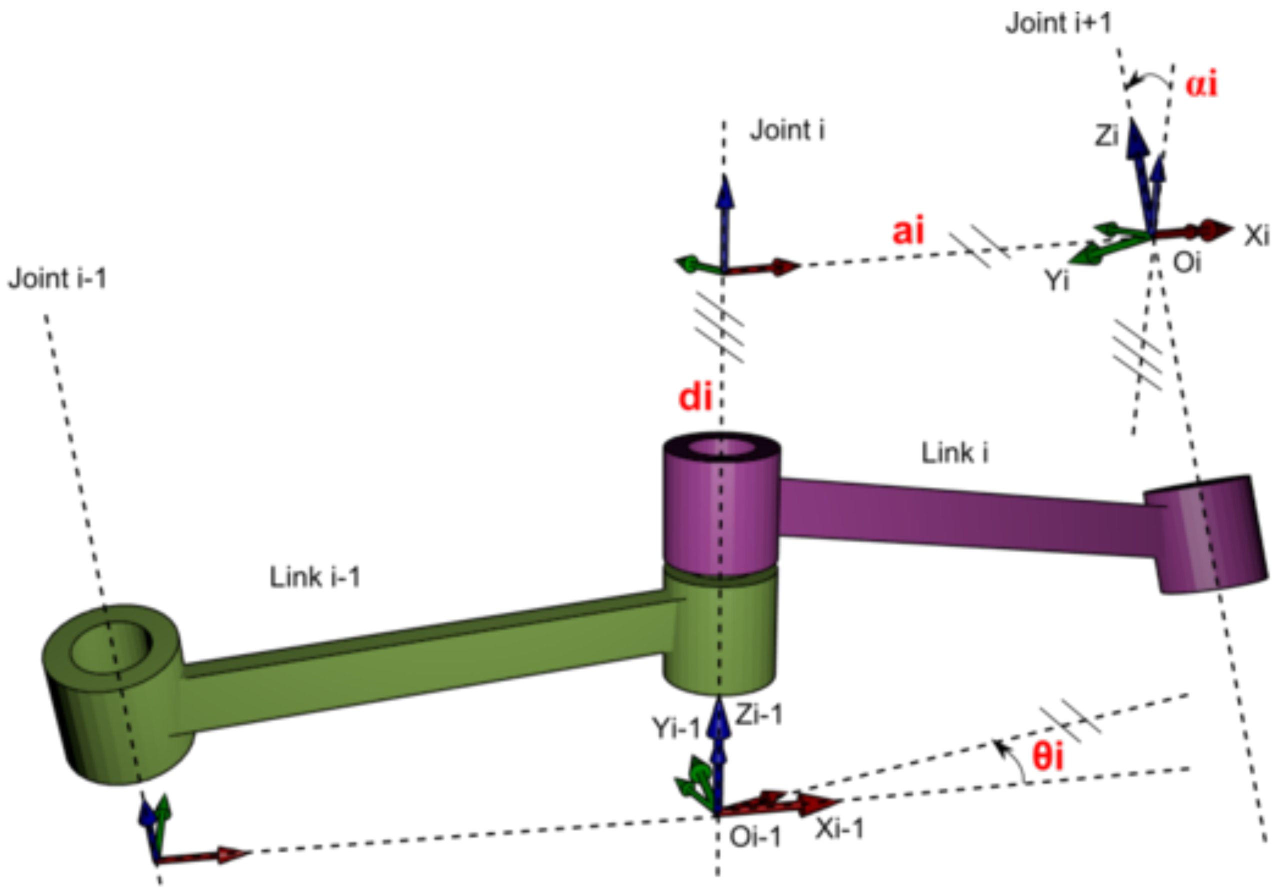
$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_b \\ y_b \\ z_b \\ 1 \end{bmatrix} = \begin{bmatrix} x_b + l_2 \\ y_b \\ z_b \\ 1 \end{bmatrix}$$

- Put all together to complete the transformation relating frame b to frame 0
  - To obtain the initial relation the point must coincide with the origin of frame b

Denavit-Hartenberg

- Commonly used convention for selecting frames of reference in robotics applications
  - Systematic general method

- Four parameters associated with a particular convention for attaching reference frames to the links of a spatial kinematic chain
  - d - offset along previous z to the common normal
  - θ - angle about previous z, from old x to new x
  - r - length of the common normal (aka a, but if using this notation, do not confuse with α). Assuming a revolute joint, this is the radius about previous z.
  - α angle about common normal, from old z axis to new z axis



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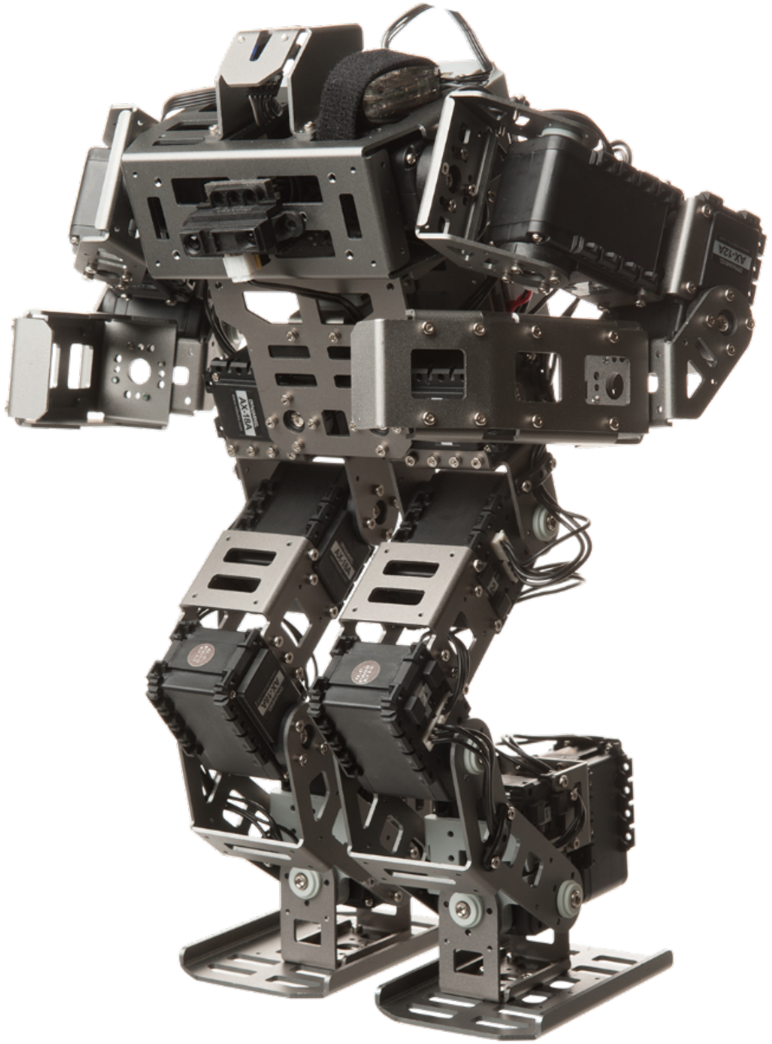
Denavit-Hartenberg

Inverse Kinematics

- In simple cases, closed-form solutions can be found.
  - However, this is rare: typically, the joint angles must be found using some numerical method.
- Unlike the forward problem, the inverse problem does not always have a unique solution.
- Methods:
  - Analytic (using Denavit-Hartenberg inverse transformations)
  - Geometric (using trigonometric identities)
  - Numeric (based on optimization)
    - Newton method or Jacobian inverse
    - Gradient method or Jacobian transpose
  - Machine Learning (Neural Networks, Gaussian Processes)



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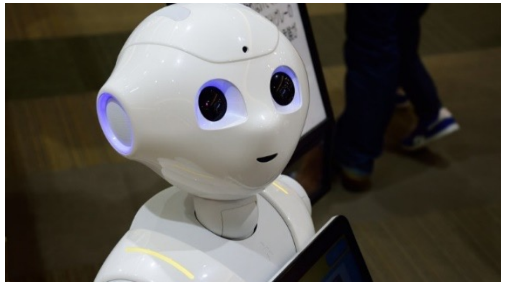
Exercise: Geometric Inverse Kinematics

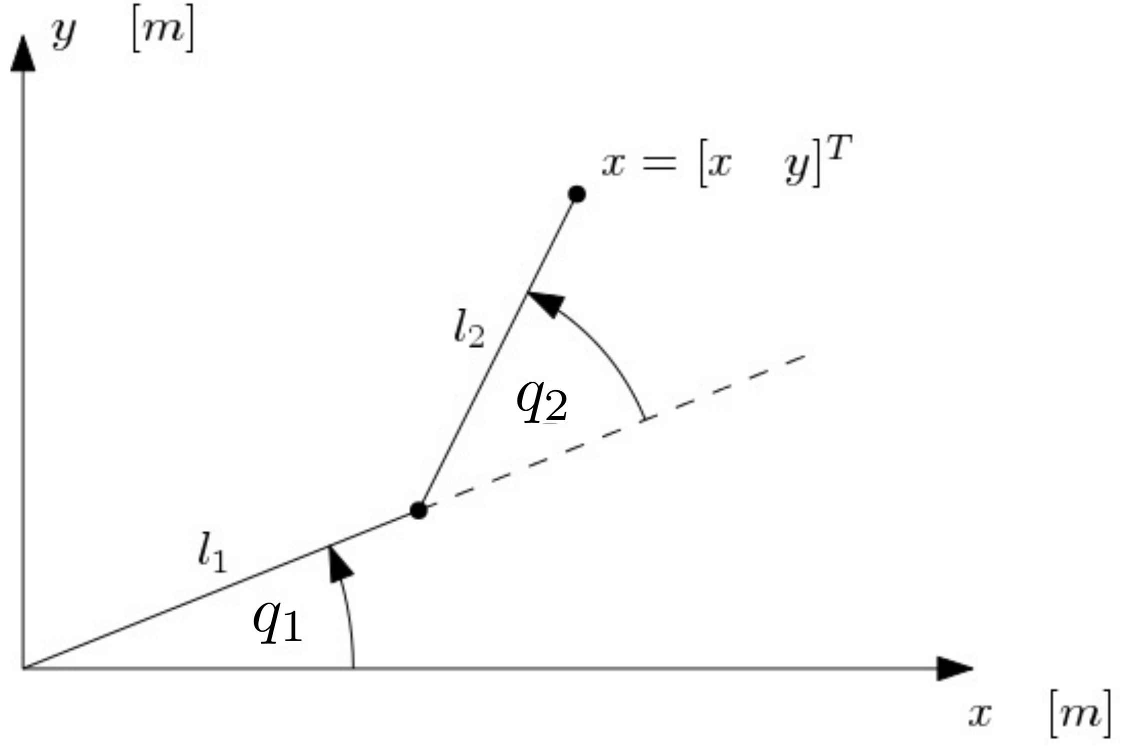
- Given

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{bmatrix}$$

- Compute

$$\mathbf{q} = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^\top \in \mathbb{R}^2$$





shorthand

$$cos(q_1) := c_1$$
$$cos(q_1 + q_2) := c_{12}$$

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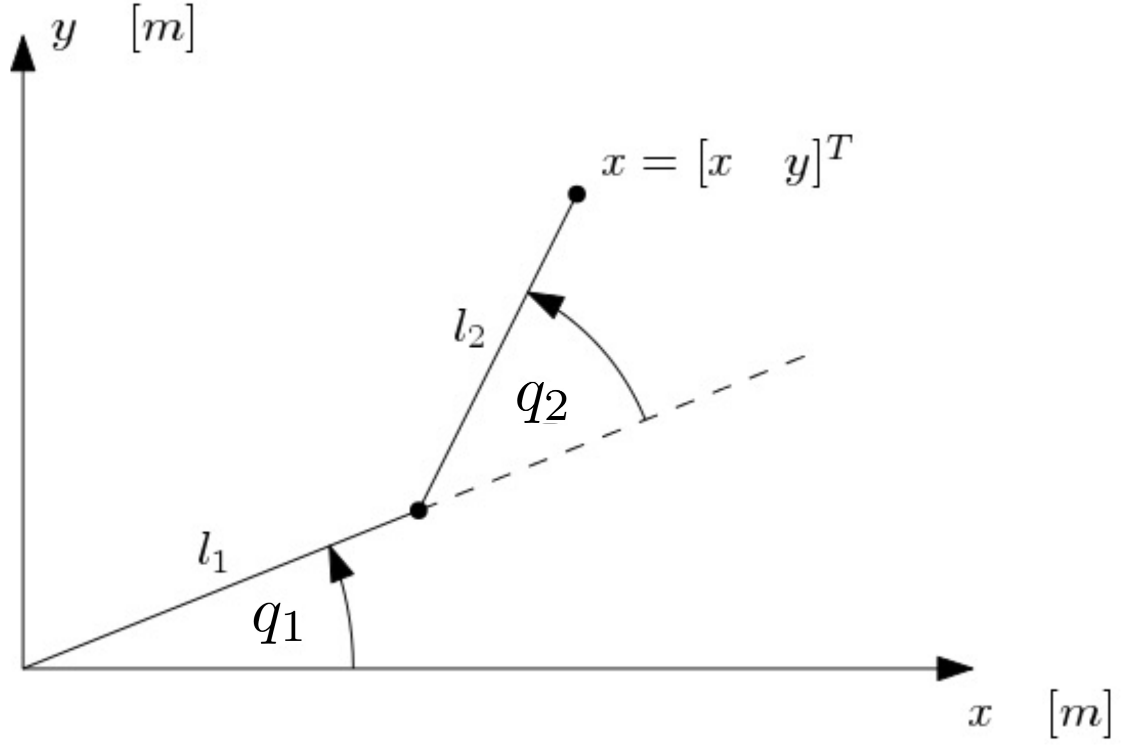
Exercise: Geometric Inverse Kinematics

- Given

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{bmatrix}$$

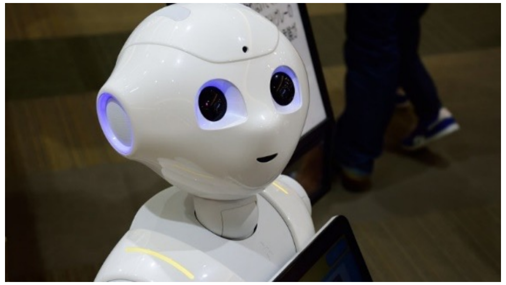
- Compute

$$\mathbf{q} = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^\top \in \mathbb{R}^2$$



shorthand

$$cos(q_1) := c_1$$
$$cos(q_1 + q_2) := c_{12}$$



- Hint:
  - Use the relation below

$$x^2 + y^2$$

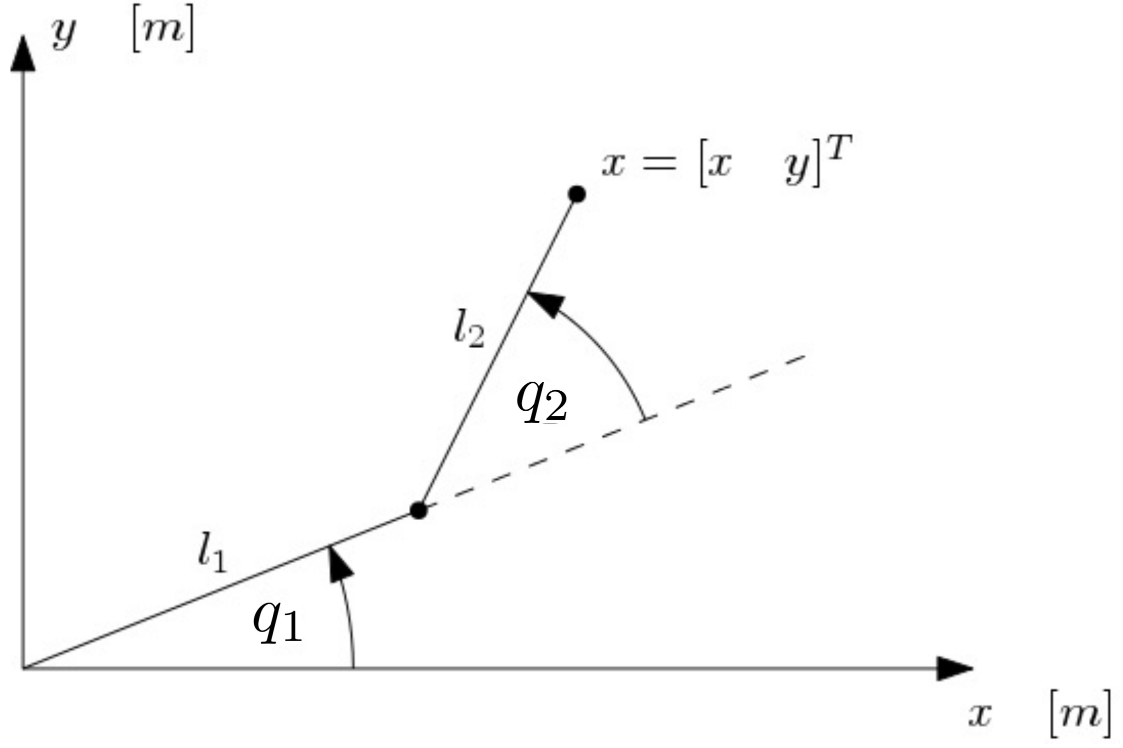
Exercise: Geometric Inverse Kinematics

$$x^2 + y^2 = l_1^2 + l_2^2 + 2l_1 l_2 c_2$$

$$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}$$

$$q_2 = atan2(\pm\sqrt{1 - c_2^2}, c_2)$$

$$q_1 = atan2(y, x) - atan2(l_2 s_2, l_1 + l_2 c_2)$$



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Exercise: Geometric Inverse Kinematics

$$x^2 + y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 c_2$$

$$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$

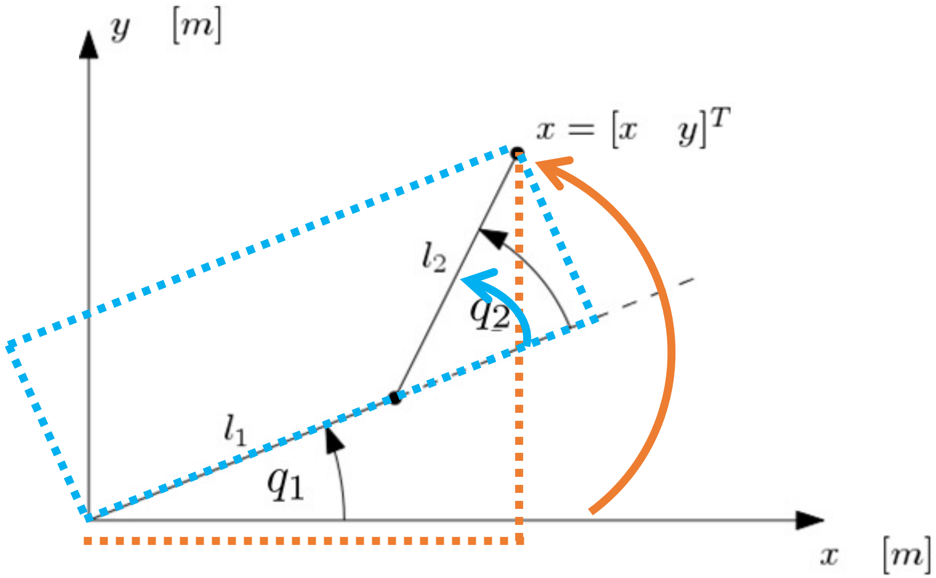
$$q_2 = atan2(\pm\sqrt{1 - c_2^2}, c_2)$$

trigonometric rules used

$$c_{12} = c_1 c_2 - s_1 s_2$$

$$s_{12} = s_1 c_2 - c_1 s_2$$

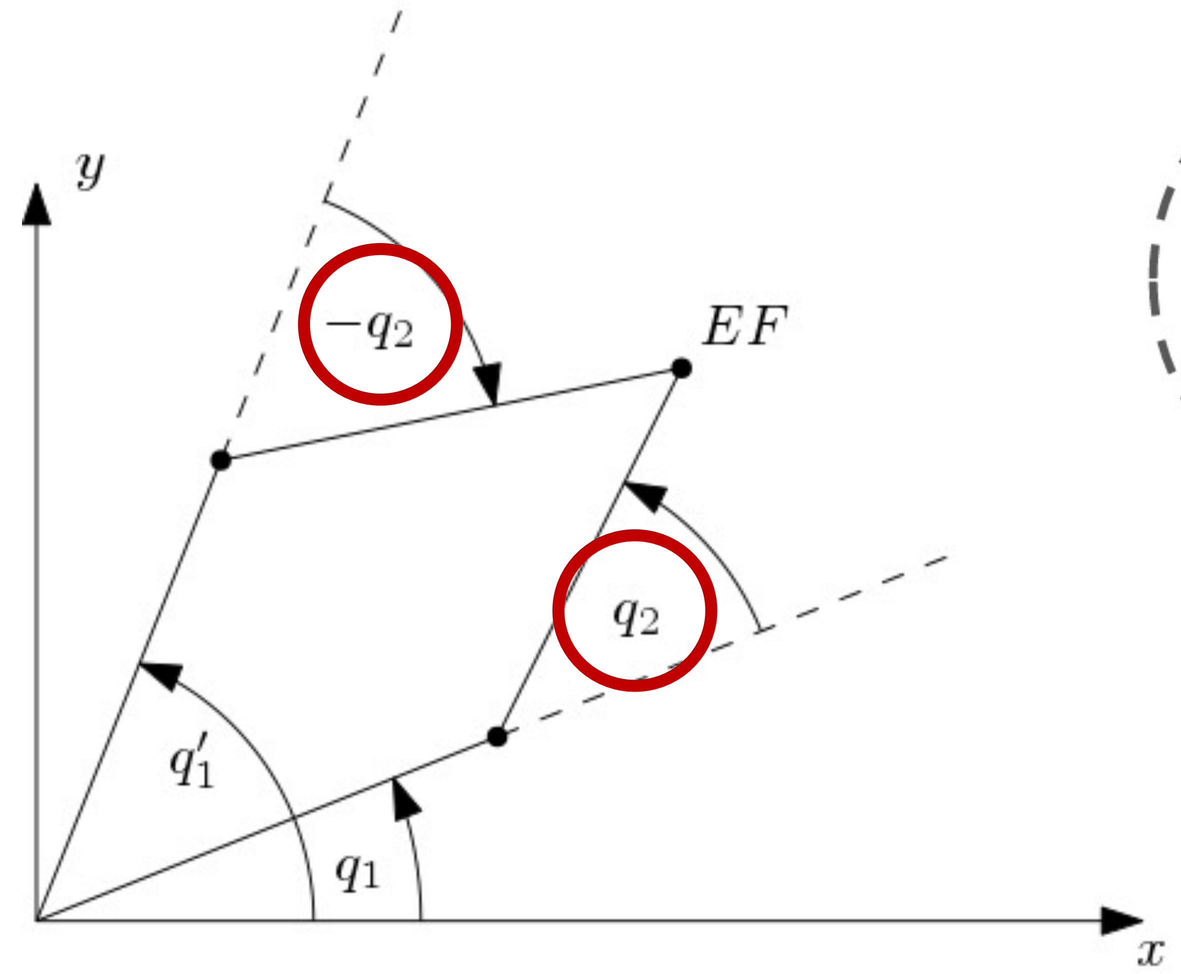
$$c_1^2 + s_1^2 = 1$$

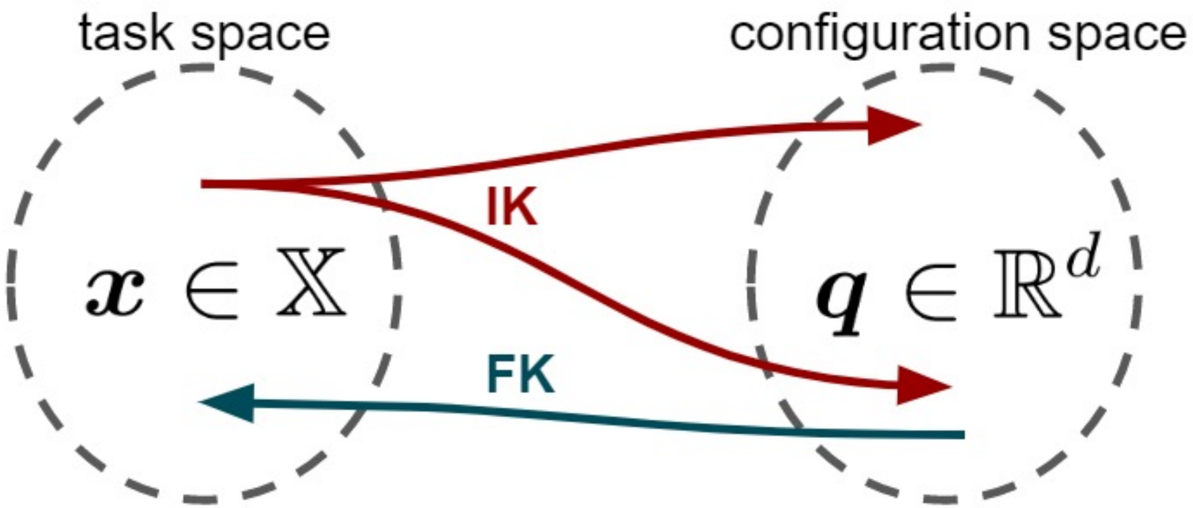


$$q_1 = \boxed{atan2(y, x)} - \boxed{atan2(l_2 s_2, l_1 + l_2 c_2)}$$

Redundant Inverse Kinematics

$$q_2 = atan2(\pm\sqrt{1 - c_2^2}, c_2)$$





Numeric Inverse Kinematics

- Given the forward kinematics

$$\mathbf{x} = f(\mathbf{q})$$

- Compute incremental updates

$$\mathbf{q}(t+1) = \mathbf{q}(t) + \boxed{\eta\Delta}\mathbf{q}$$

step size

- Via velocity kinematics

$$\dot{\mathbf{x}} = \frac{d}{dt} f(\mathbf{q})$$

Newton's Inverse Kinematics

- Given the forward kinematics

$$\mathbf{x} = f(\mathbf{q})$$

- Compute incremental updates

$$\mathbf{q}(t+1) = \mathbf{q}(t) + \eta\Delta\mathbf{q}$$

- Via velocity kinematics

$$\dot{\mathbf{x}} = \frac{d}{dt} f(\mathbf{q})$$

- Compute the velocity kinematics

$$\dot{\mathbf{x}} = \frac{d}{dt} f(\mathbf{q})$$

$$= \frac{d}{dq} f(\mathbf{q}) \frac{d}{dt}\mathbf{q} = \underbrace{J(\mathbf{q})}_{\text{Jacobian}}\dot{\mathbf{q}}$$

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Newton's Inverse Kinematics

- Given the forward kinematics

$$\mathbf{x} = f(\mathbf{q})$$

- Compute incremental updates

$$\mathbf{q}(t+1) = \mathbf{q}(t) + \eta\Delta\mathbf{q}$$

- Via velocity kinematics

$$\dot{\mathbf{x}} = \frac{d}{dt} f(\mathbf{q})$$

- Compute the velocity kinematics

$$\dot{\mathbf{x}} = \frac{d}{dt} f(\mathbf{q})$$

$$= \frac{d}{dq} f(\mathbf{q}) \frac{d}{dt}\mathbf{q} = J(\mathbf{q})\dot{\mathbf{q}}$$

Jacobian

- Apply an incremental solution

$$\dot{\boldsymbol{q}} = J^{-1}(\boldsymbol{q})\dot{\boldsymbol{x}}$$

$$\Delta\mathbf{q} = J^{-1}\Delta\mathbf{x}$$

$$\mathbf{q}(t+1) = \mathbf{q}(t) + \eta\Delta\mathbf{q}$$

Newton's Inverse Kinematics

- Given the forward kinematics

$$\mathbf{x} = f(\mathbf{q})$$

- Compute incremental updates

$$\mathbf{q}(t+1) = \mathbf{q}(t) + \eta\Delta\mathbf{q}$$

- Via velocity kinematics

$$\dot{\mathbf{x}} = \frac{d}{dt}f(\mathbf{q})$$

- Compute the velocity kinematics

$$\dot{\mathbf{x}} = \frac{d}{dt}f(\mathbf{q})$$

$$= \frac{d}{dq}f(\mathbf{q})\frac{d}{dt}\mathbf{q} = J(\mathbf{q})\dot{\mathbf{q}}$$

Jacobian

- Apply an incremental solution

$$\dot{\boldsymbol{q}} = \boxed{J^{-1}}(\boldsymbol{q})\dot{\boldsymbol{x}}$$

$$\Delta\mathbf{q} = J^{-1}\Delta\mathbf{x}$$

$$\mathbf{q}(t+1) = \mathbf{q}(t) + \eta\Delta\mathbf{q}$$

- Issue
  - Inverting J is usually not possible

$$J \in \mathbb{R}^{3\times n} \quad J \in \mathbb{R}^{6\times n}$$

Newton's Inverse Kinematics

- Given the forward kinematics

$$\mathbf{x} = f(\mathbf{q})$$

- Compute incremental updates

$$\mathbf{q}(t+1) = \mathbf{q}(t) + \eta\Delta\mathbf{q}$$

- Via velocity kinematics

$$\dot{\mathbf{x}} = \frac{d}{dt} f(\mathbf{q})$$

- Compute the velocity kinematics

$$\dot{\mathbf{x}} = \frac{d}{dt} f(\mathbf{q})$$

$$= \frac{d}{dq} f(\mathbf{q}) \frac{d}{dt}\mathbf{q} = J(\mathbf{q})\dot{\mathbf{q}}$$

Jacobian

- Apply an incremental solution

$$\dot{\boldsymbol{q}} = \boxed{J^{-1}}(\boldsymbol{q})\dot{\boldsymbol{x}}$$

$$\Delta\mathbf{q} = J^{-1}\Delta\mathbf{x}$$

$$\mathbf{q}(t+1) = \mathbf{q}(t) + \eta\Delta\mathbf{q}$$

- Issue
  - Inverting J is usually not possible

$$J \in \mathbb{R}^{3\times n} \quad J \in \mathbb{R}^{6\times n}$$

- Solution
  - Pseudo-Inverse

$$\Delta\boldsymbol{q} = \boxed{(J^T J + \lambda I)^{-1} J^T}\Delta\boldsymbol{x}$$

regularized pseudo inverse

Exercise: Newton's Inverse Kinematics

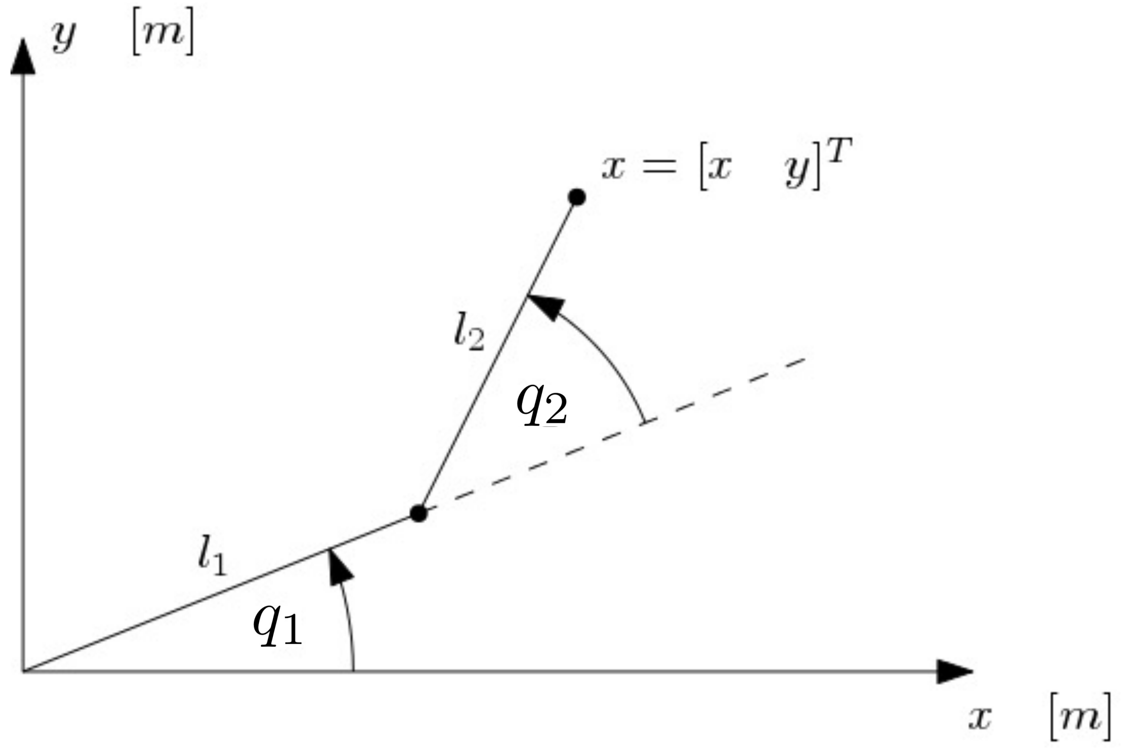
- Given

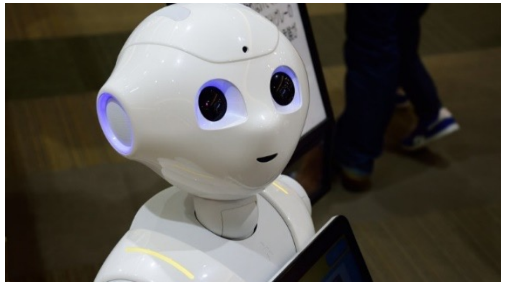
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{bmatrix}$$

- Compute

$$\dot{\mathbf{x}} = J(\mathbf{q})\dot{\mathbf{q}}$$

- Write the pseudocode of the incremental algorithm computing the solution





shorthand

$$cos(q_1) := c_1$$
$$cos(q_1 + q_2) := c_{12}$$

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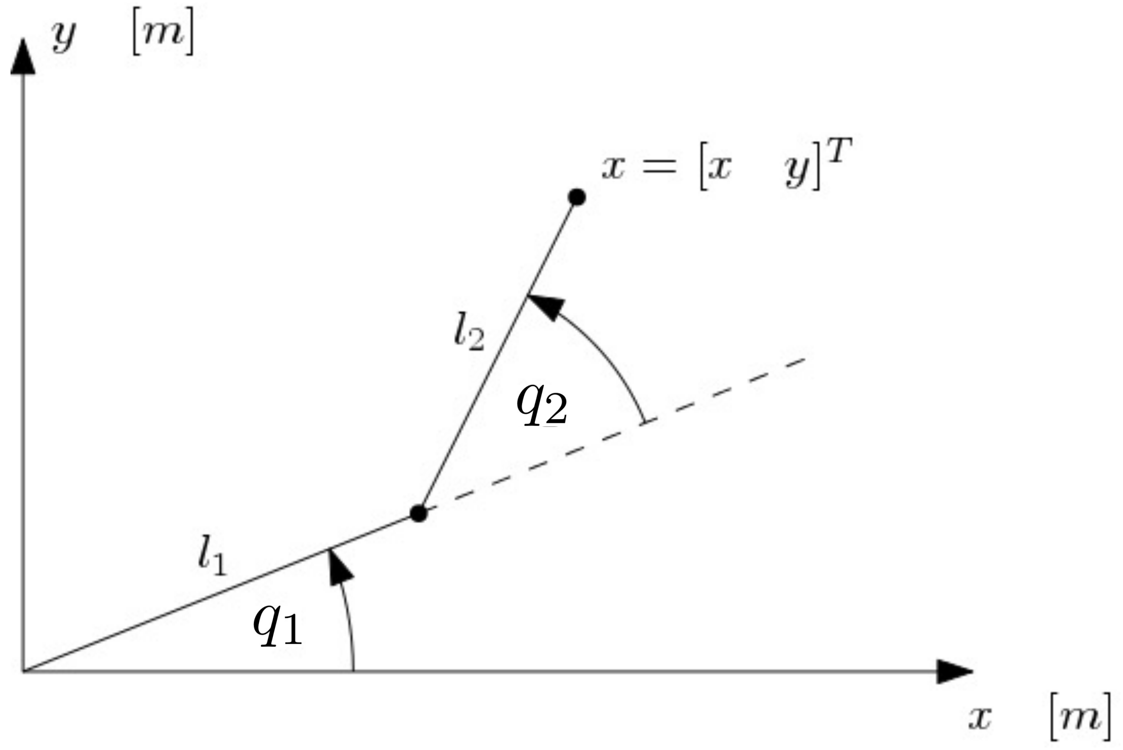
Exercise: Newton's Inverse Kinematics

- Given

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{bmatrix}$$

- Compute

$$\dot{\mathbf{x}} = J(\mathbf{q})\dot{\mathbf{q}}$$



- Solution:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}}_{\text{Jacobian}} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

shorthand

$$cos(q_1) := c_1$$
$$cos(q_1 + q_2) := c_{12}$$

- Algorithm:

$$\mathbf{q}(t+1) = \mathbf{q}(t) + \eta\Delta\mathbf{q}$$ with $$\Delta\mathbf{q} = J^{-1}\Delta\mathbf{x}$$

Newton's Inverse Kinematics: Incremental Algorithm

- Forward kinematics

$$\mathbf{x} = f(\mathbf{q})$$

depend by $\mathbf{q}$

- Velocity kinematics

$$\dot{\mathbf{x}} = \frac{d}{dt} f(\mathbf{q}) = \frac{d}{dq} f(\mathbf{q}) \frac{d}{dt}\mathbf{q} = \boxed{J(\mathbf{q})}\dot{\mathbf{q}}$$

Newton's Inverse Kinematics: Incremental Algorithm

- How to compute the Jacobian
  - Three revolute joints arm

$$J(\mathbf{q}) = \begin{bmatrix} \frac{\partial q_1}{\partial x} & \frac{\partial q_2}{\partial x} & \frac{\partial q_3}{\partial x} \\ \frac{\partial q_1}{\partial y} & \frac{\partial q_2}{\partial y} & \frac{\partial q_3}{\partial y} \\ \frac{\partial q_1}{\partial z} & \frac{\partial q_2}{\partial z} & \frac{\partial q_3}{\partial z} \\ \frac{\partial q_1}{\partial \alpha} & \frac{\partial q_2}{\partial \alpha} & \frac{\partial q_3}{\partial \alpha} \\ \frac{\partial q_1}{\partial \beta} & \frac{\partial q_2}{\partial \beta} & \frac{\partial q_3}{\partial \beta} \\ \frac{\partial q_1}{\partial \gamma} & \frac{\partial q_2}{\partial \gamma} & \frac{\partial q_3}{\partial \gamma} \end{bmatrix}$$

Newton's Inverse Kinematics: Incremental Algorithm

- How to compute the Jacobian
  - Three revolute joints arm

$$J(\mathbf{q}) = \begin{bmatrix} \frac{\partial q_1}{\partial x} & \frac{\partial q_2}{\partial x} & \frac{\partial q_3}{\partial x} \\ \frac{\partial q_1}{\partial y} & \frac{\partial q_2}{\partial y} & \frac{\partial q_3}{\partial y} \\ \frac{\partial q_1}{\partial z} & \frac{\partial q_2}{\partial z} & \frac{\partial q_3}{\partial z} \\ \frac{\partial q_1}{\partial \alpha} & \frac{\partial q_2}{\partial \alpha} & \frac{\partial q_3}{\partial \alpha} \\ \frac{\partial q_1}{\partial \beta} & \frac{\partial q_2}{\partial \beta} & \frac{\partial q_3}{\partial \beta} \\ \frac{\partial q_1}{\partial \gamma} & \frac{\partial q_2}{\partial \gamma} & \frac{\partial q_3}{\partial \gamma} \end{bmatrix}$$

first joint

second joint

third joint

robot state

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Newton's Inverse Kinematics: Incremental Algorithm

- Forward kinematics

$$\mathbf{x} = f(\mathbf{q})$$

depend by $\mathbf{q}$

- Velocity kinematics

$$\dot{\mathbf{x}} = \frac{d}{dt} f(\mathbf{q}) = \frac{d}{dq} f(\mathbf{q}) \frac{d}{dt} \mathbf{q} = \boxed{J(\mathbf{q})} \dot{\mathbf{q}}$$

- Inverse velocity kinematics

$$\dot{\boldsymbol{q}} = J^{-1}(\boldsymbol{q}) \dot{\boldsymbol{x}}$$

- Inverse control algorithm

$$\Delta \mathbf{q} = J^{-1} \Delta \mathbf{x}$$

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Incremental Algorithm: Basic Idea

- Short distance movements
  - How short?

$$\Delta \mathbf{x}$$

target position of the end-effector

current end-effector position

- Smooth movements
- Useful when you are very far from the target position
- Re-planning the trajectory due to an obstacle
  - Detected by a perceptual system (camera, sonar, imu, range finder, infrared)

Incremental Algorithm: Pseudo-code

while $\|\mathbf{x}^d - \mathbf{x}(t)\|^2 < \epsilon$ do

1. Compute the Jacobian $J(\mathbf{q}(t))$ given the configuration of the joints $\mathbf{q}(t)$ at time t

2. Compute the inverse $J^{-1}(\mathbf{q}(t))$ of the Jacobian

3. Compute

$$\Delta\mathbf{q}(t) = J^{-1}(\mathbf{q}(t))\Delta\mathbf{x}$$

with $\Delta\mathbf{x} = \|\mathbf{x}^d - \mathbf{x}(t)\|^2$

NOT THE SMARTEST SOLUTION

4. Compute

$$\mathbf{q}(t+1) = \mathbf{q}(t) + \eta\Delta\mathbf{q}$$

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