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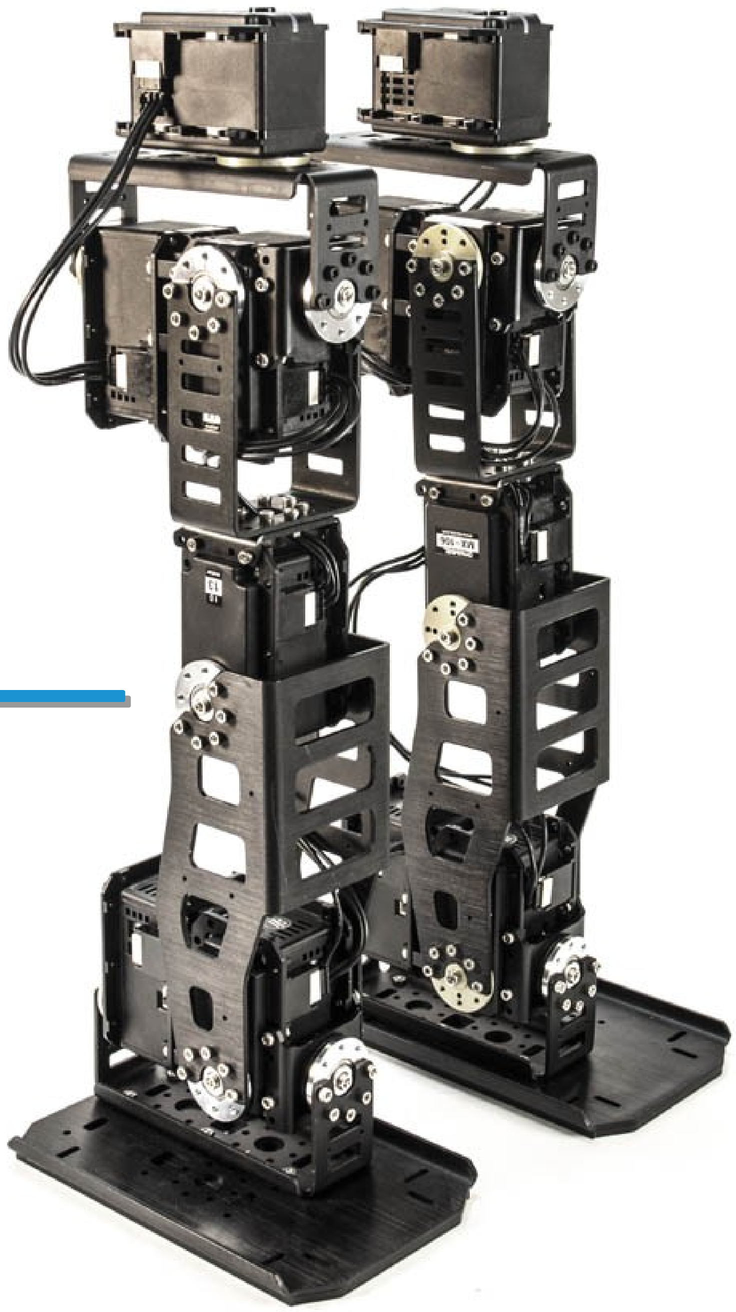
ROCO318 Mobile and Humanoid Robots



Linear Algebra

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Lecture Content

- Determinant of a matrix
- Rank of a matrix

Determinant

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (ad - bc)$$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei + bfg + cdh - ceg - bdi - afh$$

Rank

- The **rank of a matrix** is defined as (a) the maximum number of linearly independent column vectors in the **matrix** or (b) the maximum number of linearly independent row vectors in the **matrix**.

- If the determinant of a matrix is equal to zero than the rank of the matrix is lower than the number of rows/columns of the matrix
  - Either a row or a column can be obtained by combining the remaining rows/columns.

Determinant

- For a 2x2 matrix

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

Determinant

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$$A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$|A| = 1 * 2 - 0 * 3 = 2$$

Determinant

- For a 2x2 matrix

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$|A| = \boxed{1 * 2} - 0 * 3 = 2$$

Determinant

- For a 2x2 matrix

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$|A| = \boxed{1 * 2} - \boxed{0 * 3} = 2$$

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Determinant

- For a 2x2 matrix

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$|A| = \boxed{1 * 2} - \boxed{0 * 3} = 2$$

$$\text{rank(A)} = 2$$

Determinant

- For a 3x3 matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

Determinant

- For a 3x3 matrix

$$A = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \end{array}$$

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Determinant

- For a 3x3 matrix

$$A = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \end{array}$$

$$|A| = +1 * det\left(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\right) - 1 * det\left(\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\right) + 1 * det\left(\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\right)$$

Determinant

- For a 3x3 matrix

$$
\begin{array}{cc}
 & \begin{array}{ccc} + & - & + \end{array} \\
A = & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}
\end{array}
$$

$$
|A| = \boxed{+1 * det\left(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\right)} - 1 * det\left(\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\right) + 1 * det\left(\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\right)
$$

Determinant

- For a 3x3 matrix

$$A = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \end{array}$$

$$|A| = +1 * det\left(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\right) \boxed{-1 * det\left(\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\right)} + 1 * det\left(\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\right)$$

Determinant

- For a 3x3 matrix

$$A = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \end{array}$$

$$|A| = +1 * det\left(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\right) - 1 * det\left(\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\right) \boxed{+ 1 * det\left(\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\right)}$$

Determinant

- For a 3x3 matrix

$$
\begin{array}{cc}
 & \begin{array}{ccc} + & - & + \end{array} \\
A = & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}
\end{array}
$$

$$|A| = \boxed{+1 * det\left(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\right)} - 1 * det\left(\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\right) + 1 * det\left(\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\right)$$

$$|A| = \boxed{-2} + 4 - 2$$

Determinant

- For a 3x3 matrix

$$
A = \begin{array}{c} \begin{array}{ccc} \color{blue}{+} & \color{red}{-} & \color{blue}{+} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \end{array}
$$

$$
|A| = +1 * det\left(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\right) \boxed{- 1 * det\left(\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\right)} + 1 * det\left(\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\right)
$$

$$
|A| = -2 \boxed{+ 4} - 2
$$

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Determinant

- For a 3x3 matrix

$$
A = \begin{array}{c}
\begin{array}{ccc} + & - & + \end{array} \\
\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}
\end{array}
$$

$$
|A| = +1 * det\left(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\right) - 1 * det\left(\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\right) \boxed{+ 1 * det\left(\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\right)}
$$

$$
|A| = -2 + 4 \boxed{- 2}
$$

Determinant

- For a 3x3 matrix

$$\mathrm{rank(A)} = 2$$

$$\begin{array}{cccc} & + & - & + \\ A = & \left[\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{matrix}\right. & & \left.\right] \end{array}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

- The determinant is equal to zero
  - There is at least one 2x2 sub-matrix for which the determinant is different from zero

$$|A| = +1 * det\left(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\right) - 1 * det\left(\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\right) + 1 * det\left(\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\right)$$

$$|A| = -2 + 4 - 2$$

Determinant

- For a 4x4 matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$|A| = ?$$

Determinant

- For a 4x4 matrix

$$A = \begin{array}{c} \begin{array}{cccc} + & - & + & - \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Determinant

- For a 4x4 matrix

$$A = \begin{array}{c} \begin{matrix} \quad + & \quad - & \quad + & \quad - \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$|A| = +1*det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 0*det\left(\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 1*det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 1*det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\right)$$

Determinant

- For a 4x4 matrix

$$A = \begin{array}{c} \begin{matrix} \quad + & \quad - & \quad + & \quad - \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$|A| = \boxed{+1 * det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right)} - 0 * det\left(\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 1 * det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 1 * det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\right)$$

Determinant

- For a 4x4 matrix

$$A = \begin{array}{c} \begin{array}{cccc} + & - & + & - \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$|A| = +1 * det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) \boxed{-0 * det\left(\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right)} -1 * det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) -1 * det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\right)$$

Determinant

- For a 4x4 matrix

$$A = \begin{array}{c} \begin{matrix} + & - & + & - \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$|A| = +1{*}det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 0{*}det\left(\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) \boxed{-1{*}det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right)} - 1{*}det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\right)$$

Determinant

- For a 4x4 matrix

$$A = \begin{array}{c} \begin{array}{cccc} + & - & + & - \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$|A| = +1*det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 0*det\left(\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 1*det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) \boxed{-1*det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\right)}$$

Determinant

- For a 4x4 matrix

$$
A = \begin{array}{c} \begin{array}{cccc} + & - & + & - \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}
$$

$$
|A| = +1 * det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 0 * det\left(\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 1 * det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 1 * det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\right)
$$

$$
|A| = +1 * 1 - 0 * 2 - 1 * (-1) - 1 * (-2) = 4
$$

Determinant

- For a 4x4 matrix

$$A = \begin{array}{c} \begin{array}{cccc} + & - & + & - \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$|A| = \boxed{+1 * det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right)} - 0 * det\left(\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 1 * det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 1 * det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\right)$$

$$|A| = \boxed{+1 * 1} - 0 * 2 - 1 * (-1) - 1 * (-2) = 4$$

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Determinant

- For a 4x4 matrix

$$A = \begin{array}{c} \begin{matrix} \quad + & \quad - & \quad + & \quad - \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$|A| = +1 * det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) \boxed{-0 * det\left(\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right)} -1 * det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) -1 * det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\right)$$

$$|A| = +1 * 1 \boxed{- 0 * 2} - 1 * (-1) - 1 * (-2) = 4$$

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Determinant

- For a 4x4 matrix

$$A = \begin{array}{c} \begin{array}{cccc} + & - & + & - \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$|A| = +1{*}det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 0{*}det\left(\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) \boxed{-1{*}det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right)} - 1{*}det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\right)$$

$$|A| = +1 * 1 - 0 * 2 \boxed{- 1 * (-1)} - 1 * (-2) = 4$$

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Determinant

- For a 4x4 matrix

$$A = \begin{array}{c} \begin{array}{cccc} + & - & + & - \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$|A| = +1{*}det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 0{*}det\left(\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 1{*}det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) \boxed{-1{*}det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\right)}$$

$$|A| = +1 * 1 - 0 * 2 - 1 * (-1) \boxed{-1 * (-2)} = 4$$

Determinant

- For a 4x4 matrix

$$\mathrm{rank}(\mathrm{A}) = 4$$

$$A = \begin{array}{c} \begin{array}{cccc} \color{blue}{+} & \color{red}{-} & \color{blue}{+} & \color{red}{-} \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

- There are four linearly independent rows (or columns) in the matrix

$$|A| = +1{*}det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 0{*}det\left(\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 1{*}det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) - 1{*}det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\right)$$

$$|A| = +1 * 1 - 0 * 2 - 1 * (-1) - 1 * (-2) = 4$$

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Determinant and Rank

- For a 3x3 matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A| = ?$$

Determinant and Rank

- For a 3x3 matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A| = 0$$

Determinant and Rank

- For a 3x3 matrix

$$\text{rank}(\mathrm{A}) = 2$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A| = 0$$

- The **rank of a matrix** is defined as (a) the maximum number of linearly independent column vectors in the **matrix** or (b) the maximum number of linearly independent row vectors in the **matrix**.
  - The second row is the sum of the first and the third row.