④ $\int_{|z|=2} \frac{2z-1}{z(z+1)(z-3)} dz$  $z=0$; $z\neq 3$; $z=-1$ ✓
✓ ↳ Not inside.

Mapping, Integration, Laurent & Taylor Series for T2

12/8/19 Evaluation of Real integrals:

Type-I: $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$ where $F(\cos\theta, \sin\theta)$ is rational $f^n$ of $\sin\theta, \cos\theta$. such integrals can be reduced into complex line integrals by the substitution $z = e^{i\theta}$

$$z = e^{i\theta} \Rightarrow dz = i e^{i\theta}.d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

if $\theta$ varies from $0 \to 2\pi$ then $z : \to |z| = 1$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z+1/z}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z-1/z}{2i}.$$

$$\Rightarrow \int_0^{2\pi} F(\cos\theta, \sin\theta) d\theta = \oint_{|z|=1} F\left(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\right). \frac{dz}{iz}$$

For Ex: Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

$$\int_{|z|=1} \frac{1}{2+\frac{z+1/z}{2}} \cdot \frac{dz}{iz} = \int \frac{2}{4+z+\frac{1}{z}} \cdot \frac{dz}{iz} = \int \frac{2\cancel{z}}{4z+z^2+1} \cdot \frac{dz}{i\cancel{z}}$$

$$= \frac{2}{i} \int \frac{dz}{z^2+4z+1}$$

$$I = \frac{2}{i} \int_{|z|=1} \frac{dz}{z^2+4z+1} = \frac{2}{i} \oint_{|z|=1} \frac{dz}{(z+2)^2-3}$$

Singularities: $\frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm \sqrt{12}}{2} = \underline{-2 \pm \sqrt{3}}$

Singularity inside $|z|=1$ is $-2+\sqrt{3}$

$$I = \frac{2}{i} \int \frac{1/z-(-\sqrt{3}-2)}{z-(-2+\sqrt{3})} \cdot dz \qquad f(z) = \frac{1}{z+2+\sqrt{3}}$$

$$f(-2+\sqrt{3}) = \frac{1}{2\sqrt{3}}$$

$$= \frac{2}{\cancel{i}} \times 2\pi \cancel{i} \times f(-2+\sqrt{3}) = 4\pi \times \frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

Q2) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_{|z|=1} \frac{1}{5+\cancel{2}4\left(\frac{z-1/z}{2i}\right)} \cdot \frac{dz}{iz}$

$$= \int_{|z|=1} \frac{1}{5+\frac{2z}{i}-\frac{2}{iz}} \cdot \frac{dz}{iz} = \int \frac{\cancel{iz}}{5iz+2z^2-2} \cdot \frac{dz}{\cancel{iz}}$$

$$I = \int_{|z|=1} \frac{1}{2z^2+5iz-2} \cdot dz = \int_{|z|=1} \frac{1}{2z^2+5iz+2i^2} dz$$

$$2z^2+4iz+iz+2i^2 = 2z(z+2i)+i(z+2i)$$
$$= (2z+i)(z+2i)$$

$z = -i/2, -2i$ are singularities

$z = -i/2$ is singularity inside $|z|=1$

$$I = \int \frac{1/z+2i}{(2z+i)} \, dz \qquad f(z) = 1/z+2i$$

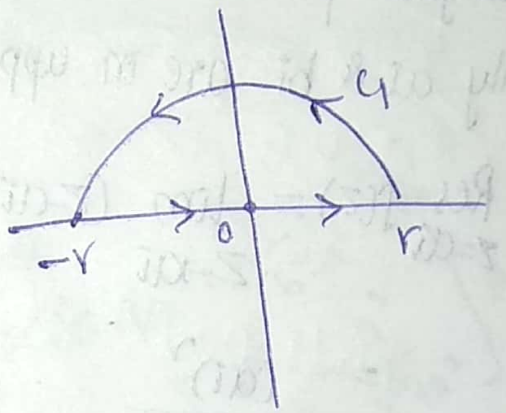
$$f(-i/2) = \frac{1}{z+2i} = \frac{1}{3i/2} = \frac{2}{3i}$$

$$= \frac{1}{2} \int \frac{1/z+2i}{z+i/2} \, dz$$

$$= \frac{1}{\cancel{2}} \times 2\pi \cancel{i} \times \frac{\cancel{2}}{3\cancel{i}} = 2\pi/3. \Rightarrow \int_0^{2\pi} \frac{d\theta}{4\sin\theta+5} = \frac{2\pi}{3}$$

13/3/19 Type-II: $\int_{-\infty}^{\infty} f(x)\,dx.$ $f(x) = \frac{g(x)}{h(x)}$ degree $(h(x)) \geq 2 + \deg(g(x))$

case I: $C = [-r, r] + C_1$



$$\int_C f(x)\,dx = 2\pi i \,[\text{sum of Residues lie in upper } 1/2 \text{ of } C_1]$$

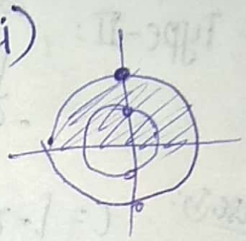
(No singularities on the real axis)

Q) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx$, $a>0, b>0$

$$f(x) = \frac{x^2}{(x^2+a^2)(x^2+b^2)} = \frac{g(x)}{h(x)}$$

degree of $g(x) = 2$
degree of $h(x) = 4$

$$f(z) = \int_C \frac{z^2}{(z^2+a^2)(z^2+b^2)}\, dz = 2\pi i \left( \text{sum of Residues lies in upper half of } C_1 \right)$$



singular points: $z = \pm ai ; \pm bi$

Only $ai$ & $bi$ are in upper half of $C_1$

$$\Rightarrow \operatorname*{Res}_{z=ai} f(z) = \lim_{z\to ai} (z-ai) f(z) = \lim_{z\to ai} \cancel{(z-ai)} \left( \frac{z^2}{\cancel{(z-ai)}(z+ai)(z^2+b^2)} \right)$$

$$= \frac{(ai)^2}{((ai)^2+b^2)(z+ai)} = \frac{-a^2}{(2ai)(b^2-a^2)}$$

$$= \frac{-a}{(2i)(b^2-a^2)} = \frac{-ai}{2(a^2-b^2)}$$

$$\Rightarrow \operatorname*{Res}_{z=bi} f(z) = \lim_{z\to bi} (z-bi) f(z) = \lim_{z\to bi} \cancel{(z-bi)} \left( \frac{z^2}{\cancel{(z-bi)}(z+bi)(z^2+a^2)} \right)$$

$$= \frac{(bi)^{\cancel{2}}}{(a^2+(bi)^2)(2\cancel{b}i)}$$

$$= \frac{bi}{2(a^2-b^2)}$$

① + ②

$$\Rightarrow \frac{(b-a)i}{2(a^2-b^2)} = \frac{\cancel{(b-a)}i}{2\cancel{(a-b)}(a+b)}$$

$$= \frac{-i}{2(a+b)}$$

$$\cancel{2}\pi i \left( \frac{-i}{\cancel{2}(a+b)} \right)$$

$$= \pi/a+b$$

① $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$ $f(x) = 1/1+x^2$

Singularities $= -i, i$ only singularity having in upper 1/2

$2\pi i$ (sum of Residues lies in upper 1/2 of a)

$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{(1+ix)(1-ix)} dx$ $m=1$

$\lim_{z=i} f(z) = \lim_{z\to i} \cancel{(z-i)} \left(\frac{1}{\cancel{(z-i)}(z+i)}\right) = \frac{1}{2i}$

$2\pi i \left(1/2i\right) = \pi$

$\int_{-\pi/2}^{\pi/2} \frac{1}{\cancel{1+\tan^2\theta}} \cdot \cancel{\sec^2\theta} \cdot d\theta$ $x = \tan\theta$ $dx = \sec^2\theta \, d\theta$

$= \int_{-\pi/2}^{\pi/2} d\theta = (\theta)_{-\pi/2}^{\pi/2} = \pi$

$\frac{b\pi - a\pi}{b^2-a^2} = \frac{\pi}{a+b}$

$\int_{-\infty}^{\infty} \left(\frac{1}{b^2-a^2}\right)\left[\frac{b^2}{x^2+b^2} - \frac{a^2}{x^2+a^2}\right] dx$

$x/b = \tan\theta$ $dx = b\sec^2\theta \, d\theta$

$\frac{1}{(b^2-a^2)}\left[\int_{-\pi/2} \frac{1}{\left(\frac{x}{b}\right)^2+1} dx - \int \frac{1}{\left(\frac{x}{a}\right)^2+1} dx\right]$

$\int b\, d\theta$

$\cancel{\int \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx}$

$\cancel{\int_{-\pi/2}^{\pi/2} \frac{\tan^2\theta}{(a^2+\tan^2\theta)(b^2+\tan^2\theta)} \cdot \sec^2\theta \, d\theta}$ $\frac{1}{(b^2-a^2)}$

$k_1 = b^2$

$k_2 = -a^2$

$= \cancel{\frac{1}{a^2}\int \frac{x^2}{\left(\left(\frac{x}{a}\right)^2+1\right)(x^2+b^2)} dx}$

$\frac{(a^2-b^2)x^2}{(x^2+a^2)(x^2+b^2)}$ $\frac{a^2x^2 + b^2a^2 + b^2x^2 + a^2b^2}{(x^2+a^2)(x^2+b^2)}$

$\frac{k_1}{(x^2+b^2)} + \frac{k_2}{(x^2+a^2)} = \frac{k_1(x^2+a^2) + k_2(x^2+b^2)}{(x^2+a^2)(x^2+b^2)}$

$k_1a^2 - k_2b^2 = 0$

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Type-2: Improper integral.

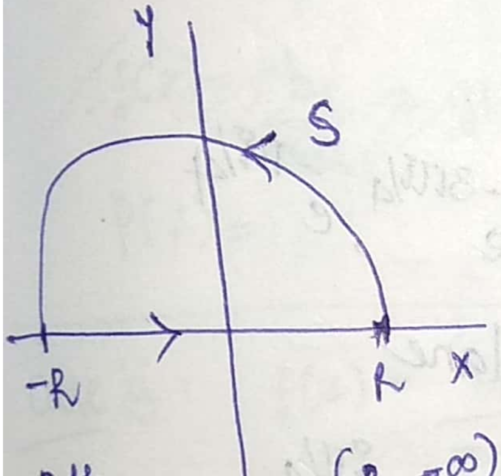
$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to -\infty}\int_a^0 f(x)dx + \lim_{b\to\infty}\int_0^b f(x)dx$$ if Both Limits Exists.

→ can be written as $\lim_{R\to\infty}\int_{-R}^{R} f(x)dx$ (cauchy principal value)

& can be written as $\text{pr.v.}\int_{-\infty}^{\infty} f(x)dx$.

case i): $f(x) = \dfrac{p(x)}{q(x)}$ where $\deg(q(x)) - \deg(p(x)) \geq 2$

&




Path C.

$$\oint_C f(z)dz = \int_S f(z)dz + \int_{-R}^{R} f(x)dx$$

(Here $-R$ to $R$ real axis so, $f(x) = f(z)$ & $dz = dx$).

$$\int_{-R}^{R} f(x)dx = \oint_C f(z)dz - \int_S f(z)dz$$

$\oint_C f(z)dz$ → $2\pi i$ (sum of residues in upper ½ plane in which $f(z)$ is pole).

From residue theorem,

$$\oint_z f(z)dz = 2\pi i \sum \operatorname{Res} f(z).$$

Consider, $\int_S f(z)dz$ ; $z = Re^{i\theta}$

$R = \text{const}$,

$z$ ranges along $S$,

$\theta \to 0$ to $\pi$.

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Since $p(z)$ is atleast two units higher than deg.

$\deg(p(x) - q(x)) \geq 2$

$\Rightarrow \quad |f(z)| < k/|z|^2 \quad (|z| = R > R_0)$ $k$ & $z$ are sufficiently large

$$\left| \int_S f(z)dz \right| < \frac{k}{|z|^2} \cdot \pi R = \frac{k}{R^2} \cdot \pi R = \frac{k\pi}{R}$$

Since $R \to \infty$, $\left| \int_S f(z)dz \right| = 0$

$\Rightarrow \oint_C f(z)dz = \int_C f(x)dx$ ~~hence proved~~

Ex: $\displaystyle\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

$\Rightarrow \displaystyle\frac{1}{2}\int_{-\infty}^{\infty} \frac{dz}{1+z^4} \Rightarrow$ poles: $\underbrace{e^{i\pi/4}, e^{3\pi i/4}}_{\text{upper } 1/2 \text{ plane}}, e^{-3\pi i/4}, e^{-\pi i/4}$

$$\operatorname*{Res}_{z=e^{i\pi/4}} f(z) = \left(\frac{z - e^{i\pi/4}}{z^4+1}\right)_{e^{i\pi/4}} = -\frac{1}{4}e^{i\pi/4}$$

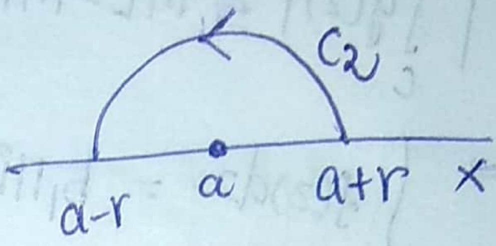
$$\operatorname*{Res}_{z=e^{i3\pi/4}} f(z) = \frac{1}{4}e^{-i\pi/4}$$

$$\Rightarrow \frac{e^{-i\pi/4} - e^{i\pi/4}}{4} = \pi/\sqrt{2} \Rightarrow \pi/2\sqrt{2} = \int_0^\infty \frac{dx}{1+x^4}$$

case ii : If there are poles on the real axis:

If $f(x)$ has a simple pole on real axis at $z=a$, then

$$\lim_{r\to 0}\int_{C_2} f(z)\,dz = \pi i \operatorname*{Res}_{z=a} f(z)$$




$f(x) = p(x)/q(x)$

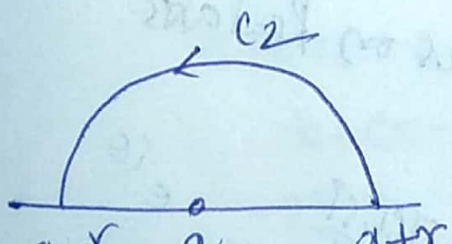
$q(x)$ is analytic on $C_2$ : $C_2 : z = a + re^{i\theta}$ $0 \le \theta \le \pi$.

$$\int_{C_2} f(z)\,dz = \int_0^{\pi} \frac{b_1}{re^{i\theta}} \cdot ire^{i\theta} \cdot d\theta + \int_{C_2} q(z)\,dz = b_1\pi i + \int_{C_2} q(z)\,dz$$

$$\cancel{f(z) = \frac{b_1}{z-a} + g(z)}.$$

$p(z) =$

case ii : $f(z) = \dfrac{b_1}{z-a} + g(z)$ $\quad b_1 = \operatorname*{Res}_{z=a} f(z)$.




$g(z)$ is analytic on $C_2$

$C_2 : z = a + re^{i\theta}$ $\quad 0 \le \theta \le \pi$

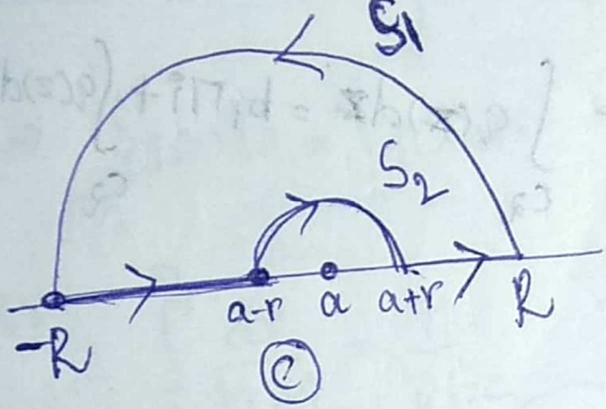
$$\int_{C_2} f(z)\,dz = \int_{C_2} \frac{b_1}{z-a}\,dz + \int_{C_2} g(z)\,dz$$

$$= \int_0^{\pi} \frac{b_1}{re^{i\theta}} \cdot ire^{i\theta}\,d\theta + \int_{C_2} g(z)\,dz = b_1\pi i + \int_{C_2} g(z)\,dz$$

$|g(z)| \le M$

$\Rightarrow \left| \int_C g(z) \right| \le ML = M\pi r \Rightarrow \left| \int_C g(z) \right| \xrightarrow{r \to 0} 0$

$\Rightarrow \int_{C_2} f(z)dz = b_1 \pi i = \pi i(\Sigma$ sum of Residues on Real axis).




$\oint_C f(z)dz =$

$\int_{-\infty}^{\infty} f(x)dx = \oint f(z)dz$

direction　　direction

$$\int_{-\infty}^{\infty} f(x)dx = \oint_C f(z)dz - \int_{S_1}^{0} f(z)dz + \int_{S_2} f(z)dz$$

$$= 2\pi i \Sigma \text{Res } f(z) + \pi i \Sigma \text{Res}(f(z))$$

poles in upper 1/2 other than real axis.　　poles on Real axis

Ex: $\int_0^{\infty} \frac{\cos x}{1+x^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi \cos i}{2}$ $1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \;= \frac{\pi}{2e}$　$e^{i\theta}$

Poles: $i, -i$, $i$

$\cancel{2\pi i} \lim_{z \to i} \frac{\cancel{(z-i)} \cos z}{\cancel{(z-i)}(z+i)} = \frac{\cos i}{2i}$

$= \pi \cos i$

② $\int_C \frac{\cancel{(z+1)}(z+2)}{z^2\cancel{(z+1)}}\,dz \quad |z| = 1.5$

$= \int_C \frac{1}{z} + \frac{2}{z^2}\,dz = \int_{|z|=1.5} \frac{z+2}{z^2}\,dz = 2\pi i\,(\text{Res of } 0)$

Singularities: $z = 0$ of order 2.

$f(z) = \frac{z+2}{z^2}$

$\underset{z=0}{\text{Res}}\, f(z) = \frac{1}{1!} \lim_{z\to 0} \left\{ \frac{d}{d} (z-a)^2 f(z) \right\}$

$= \lim_{z\to 0} \frac{d}{dz} \cancel{(z)^2} \cdot \frac{(z+2)}{\cancel{z^2}} = \lim_{z\to 0} \frac{d}{dz}(z+2) = 1$

$\int_C \frac{z^2+3z+2}{z^2(z+1)}\,dz = 2\pi i(1) = 2\pi i$

Assignment-3

① i) $\int \frac{3z+2}{z(z-1)(z-2)}\,dz \quad |z| = 3$

Singularities: $z=0,\ z=1;\ z=2$ lies inside $|z|=3$

$= 2\pi i\,(\sum \text{Residues})$

$\text{Res}(0) = \lim_{z\to 0} \cancel{(z-0)} \cdot \frac{3z+2}{\cancel{z}(z-1)(z-2)} = \frac{2}{2} = 1$

$$\text{Res}(1) = \lim_{z\to 1} \cancel{(z-1)} \frac{3z+2}{z\cancel{(z-1)}(z-2)} = \frac{5}{(1)(-1)} = -5$$

$$\text{Res}(2) = \lim_{z\to 2} \cancel{(z-2)} \frac{3z+2}{z(z-1)\cancel{(z-2)}} = \frac{8}{(2)(1)} = 4$$

$$= 2\pi i(1-5+4) = 0$$

(ii) $\displaystyle\int \frac{z^2+3z+2}{z^2(z-1)}dz \quad |z|=1.5$

$z=0$ of order 2
$z=1$ of order 1

$$\text{Res}(0) = \lim_{z\to 0} \frac{d}{dz}\cancel{(z^2)}\left(\frac{z^2+3z+2}{\cancel{z^2}(z-1)}\right)$$

$$= \lim_{z\to 0}\left(\frac{(z-1)(2z+3)-(z^2+3z+2)}{(z-1)^2}\right)$$

$$= \lim_{z\to 0}\left(\frac{z^2-2z-5}{(z-1)^2}\right) = \frac{-5}{1} = -5$$

$$\text{Res}(1) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{z^2+3z+2}{z^2} = \frac{6}{1} = 6$$

$$I = 2\pi i(-5+6) = 2\pi i$$

(iii) $\displaystyle\int \frac{4z^3+2z}{z^4-z^3}dz \quad |z|=2$

$z^3(z-1)$
$z=1$ of order 1
$z=0$ of order 3

$$\text{Res}(0) = \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\cancel{(z)^3}\left(\frac{4z^3+2z}{\cancel{z^3}(z-1)}\right)$$

$$= \lim_{z\to 0}\frac{d^2}{dz^2}\left(\frac{2z^3+z}{z-1}\right) = -8$$

$2\pi i(6-8)$
$= \underline{-4\pi i}$

$$\text{Res}(1) = \lim_{z\to 1}\cancel{(z-1)}\, 4z^3+2z/z^3\cancel{(z-1)} = \frac{6}{1} = 6$$

$$\frac{d^2}{dz^2}\left(\frac{2z^3+z}{z-1}\right) \Rightarrow \frac{d}{dz}\left(\frac{2z^3+z}{z-1}\right) = \frac{(z-1)(6z^2+1)-(2z^3+z)}{(z-1)^2}$$

$$= \frac{4z^3-6z^2-1}{(z-1)^2}$$

$$\frac{d}{dz}\left(\frac{4z^3-6z-1}{(z-1)^2}\right) = \frac{(z-1)^2(12z^2-6)-2(z-1)(4z^3-6z-1)}{(z-1)^4}$$

$z=0 \Rightarrow \dfrac{(-6)-2(-1)(-1)}{1} = -8$

$I = -4\pi i$

$\sqrt{\frac{1}{4}+\frac{7}{4}} = \sqrt{2}$

④ $\displaystyle\int_C \frac{dz}{z^2(z^2+z+2)}$ $\quad |z| = 1/2$

$z=0$; $z = \dfrac{-1\pm\sqrt{1-8}}{2} = \dfrac{-1\pm\sqrt{7}\,i}{2}$

$$\lim_{z\to0}\frac{d}{dz}\left(\frac{1}{\cancel{z^2}(z^2+z+2)}\cdot \cancel{z^2}\right) = \lim_{z\to0}\frac{-1}{(z^2+z+2)^2}(2z+1)$$

$$= -\frac{1}{4}$$

$\Rightarrow 2\pi i(-1/4) = \dfrac{\pi i}{2} = -\dfrac{\pi i}{2}$

⑤ $\displaystyle\int_C \frac{e^z\,dz}{z^3+z}$, $C: |z| = 2$

$z(z^2+1)$ $\quad z = 0, i, -i$

$2\pi i(\Sigma \text{Res})$

$2\pi i\left(1 - \dfrac{e^i}{2} + \dfrac{e^{-i}}{2}\right).$

$\displaystyle\lim_{z\to0}\frac{\cancel{(z)}e^z}{\cancel{z}(z^2+1)} = 1$

$\displaystyle\lim_{z\to i}\frac{\cancel{(z-i)}e^z}{z\cancel{(z-i)}(z+i)} = \frac{e^i}{-2}$

$= -e^i/2$

$\displaystyle\lim_{z\to -i}\frac{\cancel{(z+i)}e^z}{z\cancel{(z+i)}(z-i)} = \frac{e^{-i}}{2}$

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$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$2\pi i\left(1 - \frac{1}{2}(2i\sin 1)\right)$

$$\boxed{I = 2\pi\sin 1 + 2\pi i}$$

⑥ —

② i) $\displaystyle\int_0^{2\pi} \frac{d\theta}{3\cos\theta + 5} = \pi/2$

$$\int_{|z|=1} \frac{1}{3\left(\frac{z+1/z}{2}\right)+5} \cdot \frac{dz}{iz} = \int_{|z|=1} \frac{2z}{3z^2+3+10z} \cdot \frac{dz}{i\cancel{z}}$$

$$= \frac{2}{i}\int_{|z|=1} \frac{1}{3z^2+10z+3} \cdot dz$$

$3z^2+9z+z+3$

$(3z)(z+3)+1(z+3)$

$(z+3)(3z+1)$

$z=-1/3$, $z=-3$

$$= \frac{2}{i}\frac{(2\pi i)}{3} \lim_{z\to -1/3} \frac{\cancel{(z+1/3)}}{\cancel{(z+1/3)}(z+3)}$$

$$= \frac{4\pi}{3} \lim_{z\to -1/3}\left(\frac{1}{z+3}\right) = \left(\frac{4\pi}{3}\right)\left(\frac{3}{8}\right) = \frac{\pi}{2}.$$

$5-\cancel{4}^{2}\left(\frac{z+1/z}{\cancel{2}}\right)$

$5-2z-\frac{2}{z}$

ii) $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5-4\cos\theta} = \int_0^{2\pi} \frac{2\cos^2\theta - 1}{5-4\cos\theta}\, d\theta$

$$2\left(\frac{z+\frac{1}{z}}{2}\right)^2 - 1 = \frac{\cancel{2}}{\cancel{4}_2}\left(\frac{z^2+1}{z}\right)^2 - 1 = \frac{(z^2+1)^2 - 2z^2}{2z^2}$$

$$= \frac{z^4+1}{2z^2}$$

$5-\cancel{4}^{2}\left(\frac{z+\frac{1}{z}}{\cancel{2}}\right)$

$5-2z-\frac{2}{z}$

$5z - 2z^2 - 2$

$$\frac{5z-2z^2-2}{z}$$

$$\int_{|z|=1} \frac{\frac{z^4+1}{2z^2}}{\frac{5z-2z^2-2}{\cancel{z}}} \cdot \frac{dz}{iz} = \int \frac{z^4+1}{(2z)(5z-2z^2-2)} \cdot \frac{dz}{iz}$$

$$4z+z-2z^2-2 = (2z)(2-z)$$

$$= \cancel{(2z)(2z-1)} \quad -1(2-z)$$

$z=1/2$, $z=2$, $z=0$

$$= (2z-1)(2-z)$$

$2\pi i$ (sum of $0, 1/2$ Residue)

$$\lim_{z\to 0} (z^2) \cdot \left(\frac{z^4+1}{(2z)(5z-2z^2-2)}\right) = \frac{1}{2(-2)} = -\frac{1}{4} = -\frac{12}{48}$$

$$\lim_{z\to 1/2} (z-1/2)\left(\frac{z^4+1}{(2z)(2z-1)(2-z)}\right) = \frac{(17/16)}{(1)(2)(3/2)} = \frac{17}{48}$$

$$(2\pi i)\left(\frac{5}{48}\right) \cdot \frac{1}{i} =$$

$\frac{5}{48} \to \frac{5}{24}$

---

$$\int_{|z|=1} \frac{z^4+1}{(2z^2)} \times \frac{\cancel{2z}}{5z-2z^2-2} \frac{dz}{iz} = \frac{1}{2i}\int \frac{z^4+1}{(z^2)(5z-2z^2-2)}$$

$$= \left(\frac{1}{2i}\right)\int \frac{z^4+1}{-(z^2)(2z-1)(z-2)} \cdot dz$$

$z=0$ of order 2 ; $z=1/2$ of order 1

$$\lim_{z\to 0} \frac{d}{dz}(z^2)\cdot\left(\frac{z^4+1}{z^2(5z-2z^2-2)}\right) = \frac{(5z-2z^2-2)(4z^3) - (z^4+1)(5-4z)}{(5z-2z^2-2)^2}$$

$$= \frac{(-1)(1)(5)}{4} = -\frac{5}{4}$$

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$$\lim_{z\to 1/2} (z-1/2)\left(\frac{z^4+1}{(z^2)(z-2)(\frac{1-2z}{-2})}\right) = \frac{(17/16)}{(1/4)(-3/2)(-2)} = \frac{17}{(4)(3)} = \frac{17}{12}$$

$$\cancel{\frac{17}{12}-\frac{5}{4} = \frac{17}{12}-\frac{15}{12} = \frac{2}{12}}$$

$$\Rightarrow \cancel{(2\pi i)\left(\frac{1}{2i}\right)\left(\frac{2}{12}\right) = \frac{\pi}{6}}$$

$$I = \left(\frac{1}{2i}\sum \text{sum of residues}\right)2\pi i = (\pi)\left(\frac{17}{12}-\frac{5}{4}\right) = \pi/6$$

(ii) $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{5+4\cos\theta}$

$5+4\cos\theta = 5+\frac{4}{2}\left(z+\frac{1}{z}\right)$

$$\sin^2\theta = \left(\frac{z-\frac{1}{z}}{2i}\right)^2 = \left(\frac{z^2-1}{2zi}\right)^2 = \frac{z^4-2z^2+1}{-4z^2} \qquad = \frac{5z+2z^2+2}{z}$$

$$\int_{|z|=1}\left(\frac{z^4-2z^2+1}{-4z^2}\right)\left(\frac{z}{5z+2z^2+2}\right)\cdot\frac{dz}{iz}$$

$$2z^2+4z+z+2 \qquad (2z+1)(z+2)$$

$$\left(\frac{1}{i}\right)\left(-\frac{1}{4i}\right)\int_{|z|=1}\frac{(z^4-2z^2+1)}{(z^2)(2z+1)(z+2)}\cdot dz \Rightarrow (2\pi i)\left(-\frac{1}{4i_2}\right)\left(\frac{-2/4}{2}\right) = \pi/4$$

Singularities: $0, -1/2$ → order 1 inside $|z|=1$
↳ order 2

$$\lim_{z\to 0}\frac{d}{dz}\left(\frac{z^4-2z^2+1}{(2z+1)(z+2)}\right) = \lim_{z\to 0}\frac{(z+2)(2z+1)(2)(z^2-1)(2z) - (z^4-2z^2+1)(2z+4+2z+1)}{((2z+1)(z+2))^2}$$

$$= -\frac{5}{4} \;;\; \lim_{z\to -1/2}\frac{(z+1/2)(z^2-1)^2}{2(z+1/2)(z^2)(z+2)} = \frac{9/16}{2(1/4)(3/2)} = \frac{3}{4}$$

iv) R

v) $$\int_0^{2\pi} \frac{d\theta}{15\sin^2\theta+1}$$

$1225$ $\quad 25$

$5\times 4$

$$15\left(\frac{z-1/z}{2i}\right)^2+1 = 15\left(\frac{z^2-1}{2iz}\right)^2+1 = \frac{15(z^2-1)^2-4z^2}{-4z^2}$$

$$\int_{|z|=1} \frac{(-4z^2)}{15(z^2-1)^2-4z^2}\cdot\frac{dz}{i\cancel{z}} = \frac{-4}{i}\int_{|z|=1}\frac{z}{15z^4-34z^2+15}\cdot dz$$

$$15z^4-34z^2+15 = 15z^4-25z^2-9z^2+15$$

$$= (5z^2)(3z^2-5)-3(3z^2-5) = (5z^2-3)(3z^2-5)$$

$z^2=3/5$ ; $z^2=\cancel{5}/3 \Rightarrow z=\pm\sqrt{\frac{3}{5}}$

$\frac{3}{5}-5$

$= \frac{-16}{5}$

$$\lim_{z\to\sqrt{\frac{3}{5}}} \frac{(1/5)z}{\cancel{(z^2-\frac{3}{5})}(3z^2-5)}\cancel{(z-\sqrt{3/5})} = \frac{(1/5)\cancel{\sqrt{\frac{3}{5}}}}{\left(2\cancel{\sqrt{\frac{3}{5}}}\right)\left(\frac{-16}{\cancel{5}}\right)} = -\frac{1}{32}$$

$z+\sqrt{\frac{3}{5}}$

$$\lim_{z\to\sqrt{\frac{3}{5}}} \frac{(1/5)z}{\cancel{(z^2-3/5)}(3z^2-5)}\cancel{(z+\sqrt{3/5})} = \frac{(1/5)\left(-\cancel{\sqrt{\frac{3}{5}}}\right)}{\left(-2\cancel{\sqrt{\frac{3}{5}}}\right)\left(-\frac{16}{5}\right)} = \frac{-1}{32}$$

$z-\sqrt{\frac{3}{5}}$

$$\Rightarrow (\cancel{2}\pi i)\left(\frac{-1}{\cancel{16}}\right)\left(\frac{-4}{\cancel{i}}\right) = \pi/2$$

vi) $$\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+4)} \Rightarrow x=\pm i,\ \pm 2i$$

upper ½ $\quad i, 2i$

$$\frac{1}{(z+2i)(z^2+1)} = \frac{1}{(-3)(4i)} = \frac{-1}{12i}$$

$$\lim_{\cancel{x}\to i}\frac{(\cancel{x}-i)}{(\cancel{x}^2+1)(\cancel{x}^2+4)} = \frac{(1)}{(z+i)(z^2+4)} = \frac{1}{(2i)(3)} = \frac{1}{6i}$$

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$\Rightarrow 2\pi i\left(\frac{-1}{12i}+\frac{1}{6i}\right)=\frac{2\pi i}{12i}=\pi/6$

$x=\pm 4i$   $4i$ in upper 1/2)

$\boxed{z=4i \text{ of order } 2}$

vii) $\int_0^\infty \frac{x^2dx}{(x^2+16)^2}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+16)^2}$

$\cancel{\lim_{z\to 4i}\frac{(z-4i)z^2}{z^2-4i)}}$

$\lim_{z\to 4i}\frac{d}{dz}\frac{\cancel{(z-4i)^2}z^2}{\cancel{(z-4i)^2}(z+4i)^2}$   $\lim_{z\to 4i} = \frac{(2z)(z+4i)^2-2(z+4i)(z^2)}{(z+4i)^4}$

$=\frac{\cancel{2z^3}+16iz^2-32z-\cancel{2z^3}-8iz^2}{(z+4i)^4}=\frac{8iz^2+32zi^2}{}$   $\frac{(8zi)\cancel{(z+4i)}}{\cancel{(z+4i)^4}}$   $(z+4i)^3$

$=\frac{(8i)(4i)}{(4i)^3\times 8}=\frac{-4}{-64i}=\frac{1}{16i}$

$\frac{(2\pi i)}{2}\left(\frac{1}{16i}\right)=\pi/16$ ✓

viii) $\int_{-\infty}^{\infty}\frac{3x^2+2}{(x^2+4)(x^2+9)}dx=\int_{-\infty}^{\infty}\frac{3z^2+2}{(z^2+4)(z^2+9)}dz$   $z=\pm 3i$, $z=\pm 2i$, $3i, 2i$ in upper 1/2

$\lim_{z\to 2i}\cancel{(z-2i)}\frac{3z^2+2}{\cancel{(z^2+4)}(z^2+9)}=\frac{-10}{(4i)(5)}=-\frac{1}{2i}=\frac{-3}{6i}$

$z+2i$

$\lim_{z\to 3i}\cancel{(z-3i)}\frac{3z^2+2}{(z^2+4)\cancel{(z^2+9)}}=\frac{\cancel{-25}\,5}{\cancel{(-5)}(6i)}=\frac{5}{6i}$

$z+3i$

$\Rightarrow 2\pi i\left(\frac{2}{\cancel{6}i\,3}\right)=\frac{2\pi}{3}$ ✓

$i, 2i$ in upper 1/2 plane 1/6i

ix) $\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+4)}=\int_{-\infty}^{\infty}\frac{dz}{(z^2+1)(z^2+4)}$

$\lim_{z\to i}(x-i)\left(\frac{1}{(x^2+1)(x^2+4)}\right)=\frac{1}{3(2i)}$

$\lim_{z\to 2i}(x-2i)\left(\frac{1}{(x^2+1)(x^2+4)}\right)=\frac{1}{(-3)(4i)}=-1/12i$

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$I = (2\pi i)\left(\frac{1}{12i}\right) = \pi/6$

x) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^3}$ $x = 2i$ on upper 1/2 plane of order 3.

$\frac{1}{2!} \lim_{z\to 2i} \frac{d^2}{dz^2} \cancel{(z-2i)^3} \cdot \frac{1}{\cancel{(z^2+4)^3}\ (z+2i)^3}$

$(-3)(z+2i)^{-4}$

$(-4)(-3)(z+2i)^{-5}$

$\cancel{\frac{1}{2!} \lim_{z\to 2i} \frac{6}{(z+2i)^4}} = \cancel{\left(\frac{1}{2}\right)\left(\frac{3}{4\ 5i}\right)}$

$(-3)(z+2i)^{-4}$

$(12)(z+2i)^{-5} = \frac{12}{(4i)^5} = \frac{\cancel{12}^{3}}{\cancel{2\times 1024}_{256}\, i} \times \cancel{2}\pi \cancel{i} = \frac{3\pi}{256}$.

xi) $\int_{-\infty}^{\infty} \frac{(2x^2+3)dx}{(x^2+9)^2}$ $\boxed{x = 3i}$ of order 2.

$\cancel{\lim_{z\to 3i} \frac{d}{dz} (z-3i)^2 \left(\frac{2z^2+3}{(z^2+9)^2\ (z+3i)^2}\right)}$

$\cancel{\frac{(z+3i)^2 (4z) - (2z^2+3)(2z+6i)}{(z+3i)^4}}$

$= \frac{(6i)^2(12i) - (-15)(12i)}{64} = \frac{180i - 432i}{64}$

$= -252$

36
127
432
180
252

$\lim_{z\to 3i} \frac{d}{dz} \cancel{(z-3i)} \left(\frac{2z^2+3}{\cancel{(z-3i)^2}(z+3i)^2}\right)$

$-\frac{1}{12} - \frac{i}{3} + \frac{5i}{36}$

$\lim_{z\to 3i} \left[\frac{(4z+3)}{(z+3i)^2} + \frac{(2z^2+3)(-2)}{(z+3i)^3}\right]$

$-\frac{1}{12} + \frac{7i}{36}$

$= \frac{3+12i}{-36} + \frac{(-2)(-15)}{-\cancel{216}\, i}$ 
$\cancel{108}\ 36$

$2\pi i\left(\frac{7i-3}{36}\right) =$

(xii) $\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+4)}\,dx = \int_{-\infty}^{\infty} \frac{z\sin z}{(z^2+4)}\,dz$ $\qquad e^{i\theta} = \cos\theta + i\sin\theta$

$z = 2i$ of order $\underline{1}$

$\lim_{z\to 2i} \cancel{(z-2i)} \frac{z\sin z}{\cancel{z^2+4}_{\,z+2i}} = \frac{(\cancel{2i})(\sin(2i))}{\cancel{4i}\,2} = \frac{\sin(2i)}{2}$

$\cancel{e^{i} = \cos i + i\sin i}$

$\cancel{e^{-i} = \cos i - i\sin i}$

$(\cancel{2}\pi i)\left(\frac{\sin(2i)}{\cancel{2}}\right) = (2\pi i \underline{\sin i})(\cos i)$

$= (\pi)(i\sin 2i)$

$\cos i = \frac{1}{e}$

$= \frac{\pi}{e^2}$

$e^{-2} = e^{i\theta} \quad \theta = 2i$

$= \cos\theta + i\sin\theta$