NATIONAL INSTITUTE OF TECHNOLOGY CALICUT

Department of Mathematics

MA2003D Mathematics IV

Winter Semester 2018-19

Assignment 1 (Module 1)

---

1. **In $\mathbb{R}^2$ over $\mathbb{R}$, define $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + |b_2|)$ and $\alpha(a, b) = (\alpha a, \alpha b)$. Is $\mathbb{R}^2$ a vectorspace with respect to these operations? Justify your answer.**

2. **In $\mathbb{R}^2$ over $\mathbb{R}$, define $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$ and $\alpha(a, b) = (\alpha b, \alpha a)$. Is $\mathbb{R}^2$ a vectorspace with respect to these operations? Justify your answer.**

3. **Prove /disprove that the following are examples of vector spaces under usual (natural) addition and scalar multiplication.**

   (a) $\mathbb{R}^3$ **over** $\mathbb{R}$

   (b) $\mathbb{C}$ **over** $\mathbb{R}$

   (c) $\mathbb{R}$ **over** $\mathbb{C}$

   (d) $\mathbb{R}$ **over** $\mathbb{Q}$

   (e) $\mathcal{P} = \{$**Polynomials with real coefficients**$\}$ **over** $\mathbb{R}$.

   (f) $\mathcal{P}_n = \{p \in \mathcal{P} :$ **degree of** $p \leq n\}$ **over** $\mathbb{R}$.

   (g) $\mathbb{R}$ **over** $\mathbb{Q}$

   (h) $\mathcal{P}'_n = \{p \in \mathcal{P} :$ **degree of** $p = n\}$ **over** $\mathbb{R}$.

4. **Find non trivial subspaces for all vector spaces in the above exercises.**

5. **Let V be the vector space $\mathbb{R}^3$ over $\mathbb{R}$. Examine whether the following are subspaces or not.**

   (a) $W = \{(a, b, c) \in V : a \geq 0\}$

   (b) $W = \{(a, b, c) \in V : a, b, c \in \mathbb{Z}\}$

   (c) $W = \{(a, b, c) \in V : a \leq b \leq c\}$

   (d) $W = \{(a, b, c) \in V : a - b + 2c = 0\}$

   (e) $W = \{(a, b, c) \in V : 7a - 5b + 2c = 3\}$

   (f) $W = \{(a, b, c) \in V : a^2 - 2b + 3c = 0\}$

   (g) $W = \{(a, b, c) \in V : A[a \;\; b \;\; c \;\;]^T = 0$ **for some 3 X 3 matrix** $A\}$

6. **Let V be the vector space of all 2×2 matrices with real entries . Determine whether W is a subspace of V or not, where**

   (a) W **consists of all matrices with non zero determinant.**

   (b) W **consists of all matrices whose determinant is zero.**

   (c) W **consists of all matrices whose trace is zero.**

   (d) W **consists of all matrices A such that $A^2 = A$.**

   (e) W **consists of all diagonal matrices.**

   (f) W **consists of all symmetric matrices.**

   (g) W **consists of all skew-symmetric matrices.**

   (h) W **is the set of all matrices of the form** $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$, **where a, b, c are any 3 real numbers.**

7. Let V be the vectorspace of all continuous and differentiable real valued function defined on $\mathbb{R}$. Verify whether the following subsets of V are subspaces or not. Justify your answers.

   (a) W_1 is the set of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(2) = 0$ and $f(0) = 2$.

   (b) W_2 is the set of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(1) = 0$ and $f'(2) = 0$.

   (c) W_3 is the set of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ which are solutions of the differentiable equation $3\frac{df}{dx} + 2f = 0$.

8. Prove or disprove: (a) The set of all polynomials $p(x) \in \mathcal{P}_3$ such that $p(-x) - p(x) = 0$ is a subspace of $\mathcal{P}_3$.
   (b) The set of all polynomials $p(x) \in \mathcal{P}_3$ such that $p(-x) + p(x) = 0$ is a subspace of $\mathcal{P}_3$.

9. In the vector space $\mathbb{R}^3$ over $\mathbb{R}$, let $u = (1, 2, 3), v = (3, 1, 5), w = (3, -4, 7)$. Prove that the subspace S spanned by u and v and the subspace T spanned by u, v and w are the same.

10. Is the vector $(3, -1, 0, -1)$ an element in the subspace of $\mathbb{R}^4$ over $\mathbb{R}$ spanned by the vectors $(2, -1, 3, 2), (-1, 1, 1, -3)$ and $(1, 1, 9, -5)$.

11. Prove that the polynomials $1, 2 - x, 3 + x^2, 4 - x^3$ span the vector space $\mathcal{P}_3$.

12. If x, y, z are linearly independent vectors in a vector space V then prove that $x+y, y+z, z+x$ are also linearly independent.

13. Under what condition on a, the vectors $(1+a, 1-a)$ and $(1-a, 1+a)$ in $\mathbb{R}^2$ over $\mathbb{R}$ are linearly independent?

14. Find a if the vectors $(1, -1, 3), (1, 2, -3), (a, 0, 1)$ are linearly dependent.

15. In questions 5 and 6, find the dimension of W, if W is a subspace.

16. V_1 and V_2 are subspaces of $\mathbb{R}^4$ over $\mathbb{R}$ given by $V_1 = \{(a, b, c, d); b - 2c + d = 0\}, V_2 = \{(a, b, c, d); a = d, b = 2c\}$. Find a basis and dimension of V_1, V_2 and $V_1 \cap V2$.

17. Let V be a vector space over the field of scalars $\mathbb{F}$. Then prove the following results:

   (a) If W is a subset of V, such that W spans V, then for any set W' containing W, span (W') $= V$.

   (b) If W is a linearly independent subset of V and W' is a subset of W, then W' is also linearly independent.

   (c) If W is a linearly independent subset of V, then $W \cup \{x\}$ is linearly independent if and only if $x \notin \text{span}(W)$

   (d) If W is a subset of V such that W spans V, then there exist a subset W' of W such that W' is linearly independent and span $(W') = V$.

   (e) If V is spanned by a set containing n elements, any subset of V containing more than n elements is linearly dependent.

   (f) If dimension of $V =$ n, then any linearly independent set containing n elements is a basis of V.

   (g) If dimension of $V =$ n, then any set W that spans V and contains n elements is a basis of V.

   (h) If dimension of V is finite, and W is a linearly independent subset of V, then there exists a set W' such that $W \subset W'$ and W' is a basis of V.

*****