NATIONAL INSTITUTE OF TECHNOLOGY CALICUT
Department of Mathematics
MA2002 Mathematics IV-Tutorial sheet II - Winter Semester 2018-2019
(Common to All Branches)

Part A – Analytic Functions

1. Prove that $f(z)=|z|^2$ is continuous every where but differentiable nowhere except at origin.
2. If a function f(z) is analytic, show that it is independent of $\bar{z}$.
3. If the analytic function f(z) = u + iv is expressed in terms of polar co-ordinates, show that $\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r}=\frac{-1}{r}\frac{\partial u}{\partial \theta}$. Also show that its real and imaginary parts are solutions of Laplace equation in polar co-ordinates given by $\frac{\partial^2 \phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}=0$.
4. If f(z) is analytic prove the following.

   (i) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\ln|f'(z)|=0$ (ii) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2$
5. Determine whether the following functions are analytic or not. If analytic find its derivative

   (i) $f(z) = (z^2-2)\, e^{-x}(\cos y - i \sin y)$ (ii) $f(z) = \log z$

   (iii) $f(z) = \cos x.\cosh y - i \sin x.\sinh y$ (iv) $f(z) = \sinh z$

   (v) $f(z) = e^{3z}$ (vi) $f(z) = \cos z$ (viii) $f(z) = z\bar{z}$
6. *Given the following functions, show that* C-R equations are not sufficient for *differentiability* at the point specified.

   (a) $f(z)=\sqrt{|xy|}$ at $z=0$ (b) $f(z)=\begin{cases}\frac{xy}{x^2+y^2} & z\neq 0\\ 0 & z=0\end{cases}$ at z = 0.
7. Check for analyticity the following functions

   (a) $\frac{i}{z^5}$ (b) f(z) = cosx . coshy + i sinx . sinhy
8. If f(z) and $\overline{f(z)}$ are analytic in a region D, show that f(z) is constant in that region.
9. Prove that an analytic function whose real part is constant is a constant function
10. Determine the constants a and b such that the function, $f(z)=(x^2+ay^2-2xy)+i(bx^2-y^2+2xy)$ is analytic. Also find its derivative.
11. Using CR equations prove that if u+iv and v+iu are analytic, then u and v are constants.
12. If f(z) is analytic in D, then f(z) is a constant if

    (i) $|f(z)|$ is constant (ii) $f'(z)=0$
13. Show that if a function f(z) = u + iv analytic in a domain R and if u and v have continuous second order partial derivatives, then u and v satisfy the Laplace Equation. i.e. $\nabla^2 u=0$ and $\nabla^2 v=0$.
14. Find an analytic function whose imaginary part is $3x^2y-y^3$ and which vanishes at z = 0.
15. Check whether $f(x+iy)=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}(y/x)$; $(x^2+y^2\neq 0)$ is analytic. If so find $f'(z)$.

16. Determine whether the following functions are harmonic. If so, find the corresponding analytic function f(z) = u + iv

(a) $u = \dfrac{xy}{x^2+y^2}$ (b) $u = e^{2x}(x\cos 2y)$ (c) $v = (x^2 - y^2)^2$

(d) $v = -e^{-x}\sin y$ (e) $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$

(f) $v = e^{-x}[2xy \cos y + (y^2 - x^2)\sin y]$ (g) $u = \sin x. \text{Cushy}$

17. Find the analytic function f(z) = u + iv if

(a) $u - v = (x-y)(x^2+2xy+y^2)$ (b) $u+v = \dfrac{x}{x^2+y^2}, f(1) = 1$

(c) $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$, where $f(\frac{\pi}{2}) = 0$

(d) $u - v = \dfrac{e^{y} - \cos x + \sin x}{\cosh y - \cos 2x}$, where $f(\frac{\pi}{2}) = \dfrac{3-i}{2}$

(e) $u + v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

18. Prove that u(x, y) = $x^2 - y^2$ and v(x, y) = $\dfrac{-y}{x^2+y^2}$ are both harmonic but u + iv is not analytic.

19. If f(z) is analytic show that $\left\{\dfrac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\dfrac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$

20. Check whether $f(z) = \begin{cases} \dfrac{x^2}{(x^2+y^2)^{1/2}}, & z \neq 0 \\ 0 & z = 0 \end{cases}$ is continuous or not at the origin.