if $B = \{\alpha_1, \alpha_2 \cdots \alpha_n\}$ then

$$T\alpha_j = \sum_{i=1}^{n} a_{ij}\,\alpha_i.$$

$[v]_{B_1} = P[v]_{B_2}$

$[T]_{B_2} = P[T]_{B_1}P^{-1}$

$$A = (a_{ij})_{n\times n}.$$

$\rightarrow$ if $B_1 = \{\alpha_1, \alpha_2 \cdots \alpha_n\}$.

$B_2 = \{\beta_1, \beta_2 \cdots \beta_n\}$. are two basis of V, what is the relation ship b/w $[T]_{B_1}$ & $[T]_{B_2}$.

we will use Base changing rule

Let 'P' be the matrix $(P_{ij})_{n\times n}$ Such that

$$[v]_{B_1} = P[v]_{B_2},\ \forall\, v \in V.$$

$$[Tv]_{B_1} = [T]_{B_1}\,[v]_{B_1}$$

$$[Tv]_{B_2} = [T]_{B_2}\cdot[v]_{B_2}.$$

$P[Tv]_{B_2}$. $[Tv]_{B_1} = P[Tv]_{B_2}$.

$$[Tv]_{B_1} = P\,[T]_{B_2}\cdot[v]_{B_2}.$$

$$[Tv]_{B_2} = P^{-1}[T]_{B_1}\,[v]_{B_1} \Rightarrow P^{-1}[T]_{B_1}\cdot P\,[v]_{B_2}.$$

So $[T]_{B_2} = P^{-1}[T]_{B_1}\cdot P$

So $[T]_{B_1}$ & $[T]_{B_2}$ are similar

$\longrightarrow$ $T \in L(V)$.

$B_1, B_2 \longrightarrow$ basis (ordered)

Ex:- (i) $[T]_B = 0 \Rightarrow T = 0_{(V)}$

(ii) $[T]_B = I \Rightarrow T = I_{(V)}$

(iii) $[T]_B = CI \Rightarrow T = CI_{(V)}$

(iv) $[T]_B = A \Rightarrow [kT]_B = kA\ (\checkmark)$

Q) Given $T \in L(V)$, is it possible to find a b 'B' so that $[T]_B$ is diagonal. [diagonalisation]

$\downarrow$

Q) Given matrix 'A', ~~Given~~ is it possible to find a diag matrix 'B' Such that A & B are similar.

(($B = P^{-1}AP$) for some P).

Ans:- [Diagonalisation].

$\rightarrow$ $[T]_{B_1} = A$, $[T]_{B_2} = B$, diagnol.

as $B = P^{-1}AP$.

Let $T \in L(V)$. Let $B = \{\alpha_1, \alpha_2 \cdots \alpha_n\}$, $[T]_B$ is diagnol means that

$$[T]_B = \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{bmatrix}, \quad c_j \in F.$$

$$\left\{ \begin{array}{l} [T]_B = A\,[v]_B. \\ \begin{bmatrix} c_1 & & \\ & c_2 & \\ & & \ddots \\ & & & c_n \end{bmatrix} = A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \end{array} \right\} \longrightarrow \text{wrong.}$$

$$\left. \begin{array}{l} T\alpha_1 = c_1\alpha_1 \\ T\alpha_2 = c_2\alpha_2. \\ \vdots \\ T\alpha_n = c_n\alpha_n. \end{array} \right\} \text{then } \alpha_1, \alpha_2 \cdots \alpha_n \text{ are eigen vectors of } T.$$

If 'α' is eigen vector means $T(\alpha) = c\alpha$ for $\alpha \neq 0$.

So basis 'B' contains eigen vectors.

Now question will be is it possible to find a basis consisting eigen vectors.

1/2/19

Def:- Let 'λ' be an eigen value of $T \in L(V)$ then

$\{ u \in V : T_v = \lambda_v \}$ is called eigen space corresponding for the eigen value 'λ'.

$\longrightarrow$ The eigen space is subspace of 'V'.

Rank:- Every element of eigen space except the zero vector is a eigen vector of T corresponding to eigen value 'λ'.

if $B = \{ v_1, v_2 \ldots v_n \}$ an

$[T]_B$ is diagnol, $[T]_B = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \lambda_n \end{bmatrix}$

then $T v_i = \lambda_i v_i$

for any $v \in V$, $v = \sum c_i v_i$

$T v = \sum c_i \lambda_i v_i$

$\sum (c_i v_i) \xrightarrow{T} \sum \lambda_i c_i v_i$

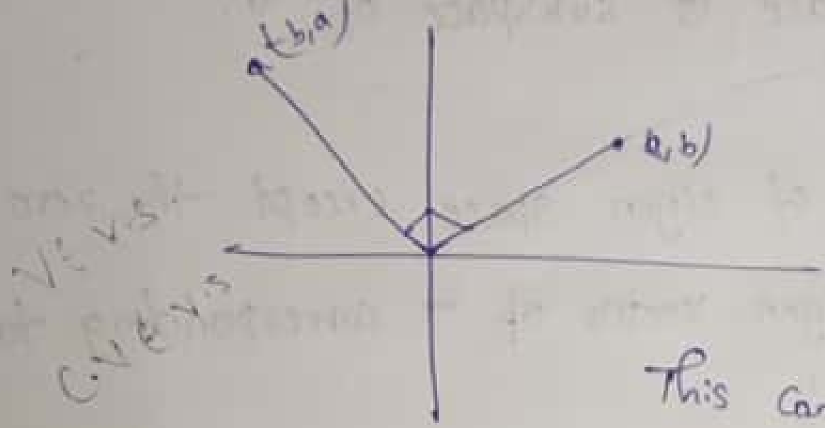
$V = \sum v_i \longrightarrow \sum \lambda_i v_i$

$\uparrow$

$E(\lambda_i) \longrightarrow$ Eigen Space eg:- $T \in L(\mathbb{R}^3)$, by

$$T(x,y) = (y,x)$$

$$[T]_{\{e_1,e_2\}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



This can't be diagnolised becz. $T(v) = cv$ means with same angle it should increase its length but here rotation is happening.

Th:- Let $T \in L(V)$ are $v_1, v_2 \ldots v_k$ be eigen vectors of T corresponding to distinct eigen values $\lambda_1, \lambda_2, \ldots \lambda_k$ respectively. Then $\{v_1, v_2 \ldots v_k\}$ is linearly independent.

$$T(v_1) = \lambda_1 v_1,\ T(v_2) = \lambda_2 v_2, \ldots T(v_k) = \lambda_k v_k$$

if $c_1 v_1 + c_2 v_2 + \cdots c_n v_n = 0$, $c_1 = c_2 \cdots c_n = 0$.

$T(v_1)+T(v_2)+\cdots T(v_k) = \lambda_1 v_1+\lambda_2 v_2+\cdots \lambda_k v_k = 0.$

$\lambda_1 \neq \lambda_2 \neq \cdots \lambda_k.$

$\lambda_1 v_1 \neq \lambda_2 v_2 \neq \cdots + \lambda_k v_k$

Proof :- If $\{v_1, v_2 \cdots v_k\}$ is L.D, let 'l' be the smallest +ve integer, such that $\{v_1, v_2 \cdots v_l\}$ is L.I, but $\{v_1, v_2 \cdots v_m, v_{l+1}\}$ is L.D.

Here $\exists$ scalar $c_1, c_2 \cdots c_{l+1}$, not all zero, such that

$$\sum_{i=1}^{l+1} c_i v_i = 0. \quad \text{—(1)} \qquad v_m = d_1 v_1 + \cdots d_{l+1} v_{l+1}$$

$$\sum_{i=1}^{l+1} c_i T(v_i) = 0.$$

$$\sum_{i=1}^{l+1} c_i \lambda_i v_i = 0. \quad \text{—(2)}$$

λ_{l+1} (1) $-$ (2).

$$\sum_{i=1}^{l} c_i(\lambda_{l+1} - \lambda_i) v_i = 0.$$

As $v_1, \cdots v_l$ is L.I

$\forall\, i = 1$ to l $\quad c_i(\lambda_{l+1} - \lambda_i) = 0.$

$c_i = 0(\checkmark)$ (or) $\lambda_i = \lambda_{l+1}$ (x)

~~So they are L.I~~

Sub in ①

$\Rightarrow c_{l+1} V_{l+1} = 0.$

$C_{l+1} = 0 \Rightarrow V_{l+1} \neq 0$ because eigen vector

So we proved c_1 to c_{l+1} all are '0' so it is Contradiction.

5/2/19

Def:- Let $T \in L(V)$. Then Subspace W of V, said to be T-invariant if

$T(v) \in W$ if $v \in W$.

eg:- Every eigen space $E(\lambda)$ of T is a T-invariant Subspace of V.

Rank:- If W is a invariant subspace of V, then T Can be restricted to W

$T|_W \in L(W)$ ⓞ

$T|_W(u) = T(u)$ if $u \in W$.

Let $T \in L(V)$. If 'T' is diagnalisable

* $\exists$ a basis B of V $\ni [T]_B$ is diagnol

* $\exists$ a basis B consisting of eigen vectors of T.

* 'V' Can be written as the direct sum T-invariant Subspaces $W_1, W_2 \ldots W_k$.

$$(V = W_1 \oplus W_2 \oplus \cdots \oplus W_k).$$

and $T/_{W} = C_i I$

and $T/_{W_i} = \lambda_i I$. Then any $v \in V$ Can be written uniquely as $v = u_1 + u_2 + \cdots u_k$ where $u_i \in W_i$

let $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_k$ be the distinct eigen values of T & let $W_1, W_2, \cdots W_k$ be the Corresponding eigen Spaces

$$B = \{\underbrace{\alpha_1}_{\substack{\lambda_1 \\ W_1}}, \underbrace{\alpha_2}_{\substack{\lambda_2 \\ W_2}} \cdots \underbrace{\alpha_{n-1}, \alpha_n}_{\substack{\lambda_k \\ W_k}}\}.$$

$$\text{If } v = \underbrace{C_1\alpha_1 + C_2\alpha_2}_{W_1} + \underbrace{\cdots}_{W_2} \underbrace{C_n\alpha_n}_{W_k}.$$

Let $V = \mathbb{R}^2$

$W_1 = \{ (x, 0) : x \in \mathbb{R} \}$

$W_2 = \{ (0, y) : y \in \mathbb{R} \}$

Then $V = W_1 \oplus W_2$. but not union of $W_1 \oplus W_2$.

eg :- $T(x,y) = (x+3y, 3x+y)$.

$T(x,y) = \lambda (x,y)$.

$(x+3y, 3x+y) = (\lambda x, \lambda y)$.

$x+3y = \lambda x \quad , \quad 3x+y = \lambda y$.

$x(\lambda-1) - 3y = 0$.

$-3x + (\lambda-1) y = 0$.

$3x(\lambda-1) - 9y = 0$.

$-3x(\lambda-1) + (\lambda-1)^2 y = 0$.

$(\lambda-1)^2 \cancel{y} = 9y$

$\lambda - 1 = +3 \qquad \lambda - 1 = -3$.

$\lambda = 4 \qquad \lambda = -2$

$T(1,1) = 4(1,1)$

$T(1,-1) = -2(1,-1)$

$\{(1,1)(1,-1)\}$ form a basis of eigen vectors.

$$W_1 = E(4) = \{(u,u) : u \in \mathbb{R}\}$$

$$W_2 = E(-2) = \{(u,-u), u \in \mathbb{R}\}$$ b

the eigen spaces and

$$\mathbb{R}^2 = V = W_1 \oplus W_2.$$

$$(x,y) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right) + \left(\frac{x-y}{2}, \frac{y-x}{2}\right)$$

$$T(x,y) = T\left(\frac{x+y}{2}, \frac{x+y}{2}\right) + T\left(\frac{x-y}{2}, \frac{y-x}{2}\right)$$

$$= 4\left(\frac{x+y}{2}, \frac{x+y}{2}\right) + (-2)\left(\frac{x-y}{2}, \frac{y-x}{2}\right)$$

ie $V = \mathbb{R}^2 = W_1 \oplus W_2$

$$T = T/_{W_1} + T/_{W_2}$$

$$= 4 I_{W_1} + -2 I_{W_2}$$

eg:- $T(x,y) = (-y,x)$

Not diagnolisable.

Def:- Let $T \in \mathcal{L}(V)$. Then 'T' is said to be upper triangulisable if $\exists$ a basis of V such that

$[T]_B$ is upper triangular.

Th:- Let $T \in \mathcal{L}(V)$. Then TFAE

1) $\exists$ a basis B of V $\ni [T]_B$ is upper triangular.

2) $\exists$ a basis $B = \{\alpha_1, \alpha_2 .. \alpha_n\}$ of V $\ni$

$T\alpha_j \in \text{Span}\{\alpha_1, \alpha_2 \dots \alpha_j\}$ for each $j = 1, 2, \dots n$.

3) $\exists$ a basis $B = \{\alpha_1, \dots \alpha_n\}$ of V $\ni$ $\text{span}\{\alpha_1, \alpha_2 \dots \alpha_j\}$ is T-invariant for $j = 1, 2 \dots n$.

Proof:- Asume ① i.e $\exists B = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ Such that

$$[T]_B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & & a_{1n} \\ 0 & a_{22} & a_{23} & & a_{2n} \\ 0 & 0 & a_{33} & & a_{3n} \\ 0 & 0 & 0 & & a_{4n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & & a_{nn} \end{bmatrix}$$

$\Rightarrow T\alpha_1 = a_{11}\alpha_1$

$T\alpha_2 = a_{12}\alpha_1 + a_{22}\alpha_2 \dots$

$T\alpha_j \in \text{span}\{\alpha_1, \alpha_2 \cdots \alpha_j\}$. $\Rightarrow$ ② point ①

If ② is true

$\Rightarrow T\alpha_j \in \text{Span}\{\alpha_1, \alpha_2 \cdots \alpha_j\}$.

Let $v \in \text{span}\{\alpha_1, \alpha_2 \cdots \alpha_k\}$.

$$v = \sum_{i=1}^{k} c_i \alpha_i \Rightarrow T(v) = \sum_{i=1}^{k} c_i T(\alpha_i).$$

$$T(v) = c_1 T(\alpha_1) + c_2 T(\alpha_2) + \cdots c_k T(\alpha_k).$$

$\in \text{span}(\alpha_1) \subset \text{span}(\alpha_1 \cdots \alpha_k)$

$\in \text{span}(\alpha_2) \subset \text{span}(\alpha_1, \cdots \alpha_n)$

For each $T(\alpha_i) \in \text{Span}\{\alpha_1, \alpha_2 \cdots \alpha_k\}$ for $i = 1, \cdots k$

$$T(v) \in \text{Span}\{\alpha_1, \alpha_2 \cdots \alpha_k\}.$$

$\text{Span}\{\alpha_1, \alpha_2 \cdots \alpha_k\}$ is T-invariant.

If ③ is true. then

$T\alpha_1 \in \text{span}(\alpha_1) \Rightarrow T\alpha_1 = c_1 \alpha_1$.

$T\alpha_2 \in \text{Span}\{\alpha_1, \alpha_2\}$..

→ If 'A' is a $n \times n$ matrix, we can define $P(A)$ for any polynomial p.

Similary if $T \in \mathcal{L}(V)$, and $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n$

then $P(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_n T^n \in \mathcal{L}(V)$

Also $(pq)(T) = P(T) \cdot q(T)$

– If $P(x,y) = a_{00} + a_{10}x + a_{01}y + a_{11}xy + \cdots$ is a polynomial is 2 variables, $T, S \in \mathcal{L}(V)$.

$$P(T,S) = a_{00} I + a_{10} T + a_{11} S + a_{11} TS + \cdots$$

make sense iff T & S are commutative

<u>Theorem</u>:- If V is a <u>complex</u> vector space and $T \in \mathcal{L}(V)$, then 'T' has an eigen value.

<u>Proof</u>:- Let dim of $V = n$. Then the set

Let $u \in V, u \neq 0$.

$$\{u, Tu, Tu^2, \ldots Tu^n\}$$ is L.D.

Then $\exists$ $c_0, c_1, \ldots c_n$ ~~where~~ $\in \mathbb{C}$ not all '0' such that

$$c_0 u + c_1 Tu + \cdots c_n T^n u = 0.$$

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$\Rightarrow$ $p(T)u = 0$ for $p(x) = c_0 + c_1x + \cdots c_nx^n$, a complex polynomial of degree atleast '1'.

(there is atleast one c_j such that $c_j \neq 0$ and if $j=0$ then it means $c_0u = 0$ (x) because $u \neq 0$. Hence $c_j \neq 0$ for some $j \geq 1$).

Since 'p' is a complex polynomial, we can factorise it to get

$$p(x) = c_n(x-a_1)(x-a_2)\cdots(x-a_n),\ a_i \in \mathbb{C}$$

$$p(T)u = c_0(T-a_1I)(T-a_2I)\cdots(T-a_nI)u = 0.$$

If all of the operators $T-a_1I, T-a_2I \cdots T-a_nI$ are injective, then their product is also injective. Since their product kills a non-zero vector u; at least one of them is non-injective.

ie:- at least one of the elements in $\{a_1, a_2 \cdots a_n\}$ is eigen value of T.

Th:- If 'V' is a complex vector space and $T \in \mathcal{L}(V)$, $\exists$ a basis B of V $\ni$ $[T]_B$ is upper triangular.

Proof :- By induction on the dim of vector space

1) If $\dim(V) = 1$, then the resut $\rightarrow$ obvious

2) Assume the result for all operators on any vector space whose dimension is $\leq k$.

Let $\dim(V) = k+1$, $T \in \mathcal{L}(V)$

By previous theorem, ~~$\mathcal{B}$~~ 'T' has atleast one eigen value

ie $Tv_1 = \lambda_1 v_1$ for some $v_1 \in V$, $v_1 \neq 0$, $\lambda_1 \in \mathbb{C}$

$T - \lambda_1 I \in \mathcal{L}(V)$

$v_1 \in N(T - \lambda_1 I) \Rightarrow N(T - \lambda_1 I) \geq 1$

$r(T - \lambda_1 I) \leq k$

w_1 = Range of ~~$T\lambda_1 B$~~ $(T - \lambda_1 I)$ is a subspace of V whose dimension is atmost 'k'.

If $u \in \omega_1$;

$$Tu = (T-\lambda_1 I)u + \lambda_1 u \in \omega_1$$

Since $(T-\lambda_1 I)u \in \text{Range}(T-\lambda I) = \omega_1$

$\lambda \mu \in \omega_1$ [$\because u \in \omega_1$]

So ω_1 is T-invariant subspace of 'V'!

Th:- Let 'V' be a Complex vector space $T \in L(V)$. Then $\exists$ a basis B of V Such that $[T]_B$ is upper triangleular.

Proof:- ~~Pg~~ ;

We can apply induction hypothesis to $T|_{\omega_1} : W_1 \to W_1$ to get a basis $B_1 = \{\alpha_1, \alpha_2 \ldots \alpha_l\}$ of ω_1 $(l \leq k)$.

$\exists$ $[T/\omega_1]_{B_1}$ is upper triangular.

ie, $T|_{\omega_1}(\alpha_j) = T(\alpha_j) \in \text{span}(\alpha_1, \alpha_2 \ldots \alpha_j)$

for $j = 1, 2 \ldots l$.

Extend B' to a basis $B = \{\alpha_1, \alpha_2 \ldots \alpha_l, \alpha_{l+1} \ldots \alpha_{k+1}\}$ of V.

For $j \leq l$, $T\alpha_j \in \text{span}\{\alpha_1, \alpha_2 \ldots \alpha_j\}$

For $j > l$, $T\alpha_j = \underbrace{(T-\lambda I)\alpha_j}_{\in \omega} + \lambda \alpha_j$.

$\in \text{span}\{\alpha_1, \alpha_2, \dots \alpha_j\}$.

$\in \text{span}\{\alpha_1, \alpha_2 \dots \alpha_k, \alpha_{k+1} \dots \alpha_j\}$.

because $(T-\lambda I)\alpha_i \in \omega_1 = \text{span}\{\alpha_1, \alpha_2, \dots \alpha_k\}$.

Hence $[T]_B$ is upper triangular.

Th: Let $T \in \mathcal{L}(V)$ and let 'B' be a basis of V such that $[T]_B$ is upper triangular. Then 'T' is invertible iff all the diagnol entries of $[T]_B$ are non-zero.

Proof: Let $B = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ and let $[T]_B$ be upper triangular with diagnol entries $\mu_1, \mu_2 \dots \mu_n$.

Since it is upper triangular,

$T(\omega_j) \in \omega_j$ where

$\omega_j = \text{span}\{\alpha_1, \alpha_2 \dots \alpha_j\}$.

If for some k, $\mu_k = 0$,

then $T\alpha_k \in \text{span}\{\alpha_1, \alpha_2 \dots \alpha_{k-1}\}$.

Here $T(\omega_k) \subseteq \omega_{k-1}$

$(\because T\alpha_j \in \omega_{k-1}$ for $j = 1, \dots k)$.

ie; $T|_{W_k} : W_k \to W_{k-1}$ (44)

Since $\dim W_{k-1} < \dim W_k$..

$N(T|_{W_k}) > 0$ (by rank nullity theorem).

Here $\exists\ u \in W_k \ni T|_{W_k}(u) = 0, u \neq 0$

$\Rightarrow T(u) = 0$ for $u \neq 0$

T is not one-one so it is not invertible

Conversely, assume that T is not invertible

Hence $\exists\ u \in V \ni u \neq 0, T(u) = 0.$

Let $u = C_1V_1 + C_2V_2 + \cdots C_kV_k$ where $C_k \neq 0.$

(this is possible since $u \neq 0$) Then

$$0 = T(u) = C_1T(V_1) + C_2T(V_2) + \cdots C_kT(V_k)$$

$$\Downarrow$$

$$T(V_k) = -\frac{C_1}{C_k}T(V_1) - \frac{C_2}{C_k}T(V_2) + \cdots$$

$T(V_k)$ is linear Combination of $T(V_1), T(V_2) \cdots T(V_{k-1})$

$T(V_k) \in$ Span of $\{T(V_1), T(V_2) \cdots T(V_{k-1})\}$

$\subseteq W_{k-1}$

Since $T(v_j) \in W_j$, $W_1 \subset W_2 \cdots \subset W_{k-1}$.

ie $T(v_k) \in \text{Span}\{v_1, v_2, \ldots v_{k-1}\} \Rightarrow$ the k^{th} diagnol entry of $[T]_B$ is zero.

Cor:- Let $T \in L(V)$ and B a basis of V such that $[T]_B$ is upper Δluar. Then the eigen values of 'T' are precisely the diagnol entries of $[T]_B$.

Proof:- 'λ' is an eigen values of 'T' iff $(T-\lambda I)$ is not invertible

Since $[T-\lambda I]_B = [T]_B - \lambda [I]_B = [T]_B - \lambda I_{n\times n}$.

$T-\lambda I$ is not invertible iff a diagnol entry of $[T]_B - \lambda I$ is zero

iff

λ appears at diagnol of $[T]_B$.

ex:- $[T_1 + k T_2]_B = [T_1]_B + k[T_2]_B$.