NATIONAL INSTITUTE OF TECHNOLOGY CALICUT

Department of Mathematics

MA2003D Mathematics IV

Winter Semester 2018-19

Assignment 2 (Module 1)

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1. Find the change-of-basis matrix P from the usual basis E of $\mathbb{R}^2$ to a basis S, the change-of-basis matrix Q from S back to E, and the coordinates of $v = (a, b)$ relative to S, for the following bases S:

   (a) $S = \{(1, 2), (2, 3)\}$

   (b) $S = \{(1, 0), (1, 1)\}$

   (c) $S = \{(1, 1), (1, -1)\}$

   (d) $S = \{(1, 2), (2, 1)\}$

2. Let V be the space of $n \times n$ real matrices over $\mathbb{R}$ and let B be a fixed $n \times n$ real matrix. Which of the following functions on V are linear

   (a) $T(A) = A^T$

   (b) $T(A) = BA$

   (c) $T(A) = A^2$

   (d) $T(A) = A + B$

   (e) $T(A) = AB + BA$

   (f) $T(A) = A^T B$

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Find $T(x, y)$ if $T(1, 2) = (1, 3)$ and $T(2, 1) = (3, 1)$. Use this idea to prove that a linear transformation $T : V \to W$ is deterimined by its action on the elements of any basis of V.

4. Find a linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose image is spanned by $(1, 2, 3)$ and $(4, 5, 6)$.

5. Find a linear mapping $S : \mathbb{R}^4 \to \mathbb{R}^3$ whose kernel is spanned by $(1, 2, 3, 4)$ and $(0, 1, 1, 1)$.

6. Let F and G be the linear operators on $\mathbb{R}^2$ defined by $F(x, y) = (x + y, x - y)$ and $G(x, y) = (0, y)$. Find formulas defining the linear operators:

   (a) $F + G$

   (b) $5F - 3G$

   (c) FG

   (d) GF

   (e) F^2

   (f) G^2

7. Give an example of a non-linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T^{-1}(0, 0) = \{(0, 0)\}$ but T is not one-one.

8. Find the kernal space and range space of linear transformation $T : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ defined by $T(p(x)) = p'(x)$.

9. For each linear map T, find a basis and the dimension of the kernel and the image of T:

   (a) $T(x, y, z) = (x + y + z, 2x + 2y + 2z)$

   (b) $T(x, y, z) = (x + y, y + z)$

   (c) $T(x, y) = (x + y, 0)$

   (d) $T(x, y) = (x + y, x - y, 2x - 3y)$

10. Find a basis and the dimension of the null-space (kernel) of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x - 2y + z, 2x - 4y + 2z)$. Extend the basis you obtained for the null-space to a basis of the vector space $\mathbb{R}^3$.

11. Check which of the following linear operator T on $\mathbb{R}^2$ are invertible and find a formula for T^{-1} if invertible

   (a) $T(x, y) = (x + y, x - y)$

   (b) $T(x, y) = (x + y, 2x + 2y)$

   (c) $T(x, y) = (x + y, 0)$

   (d) $T(x, y) = (y, x)$

12. Verify Rank-Nullity Theorem for the linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ by $T(x, y, z) = (x + y, 2z, 0, z)$.

13. Let V be the real vector space of all even polynomials with real coefficients and degree at most 6. Let $T\colon V \to V$ be defined by $T(p(x)) = \frac{d^2}{dx^2}(p(x))$. Verify rank-nully theorem for T.

14. For each of the following linear transformation T on $\mathbb{R}^2$, find the matrix A representing T (relative to the standard basis of $\mathbb{R}^2$

    (a) T is the rotation in $\mathbb{R}^2$ counterclockwise by 45^o

    (b) T is the reflection in $\mathbb{R}^2$ about the line $y = x$.

15. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(x, y, z) = (2x - 3y + z, -2x + 5z)$. Find the matrix for T relative to the standard basis of $\mathbb{R}^2$.

16. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation whose matrix representation relative to the standard basis is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

    Find a basis for the kernel and a basis for the image of T.

17. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(x, y, z) = (x + y, y + z, x + z)$. Find the matrix $[T]_{\mathcal{B}_1,\mathcal{B}_2}$ representing T relative to the following pairs of basis.

    (a) $\mathcal{B}_1 = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}, \mathcal{B}_2 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

    (b) $\mathcal{B}_1 = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}, \mathcal{B}_2 = \{(1, 1, -1), (1, -1, 1), (-1, 1, 1)\}$

18. Let $T : \mathcal{P}^3 \to \mathcal{P}^2$ be a linear transformation defined by $T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2$. Find the matrix $[T]_{\mathcal{B}_1,\mathcal{B}_2}$ representing T relative to the following pairs of basis.

    (a) $\mathcal{B}_1 = \{1, x, x^2, x^3\}, \mathcal{B}_2 = \{1 + x, 1 - x, x^2\}$

    (b) $\mathcal{B}_1 = \{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3\}, \mathcal{B}_2 = \{1, x, x^2\}$

19. Consider the linear operator T on $\mathbb{R}^2$ defined by $T(x, y) = (2x + 3y, x - 4y)$ and the following bases of $\mathbb{R}^2$. $S = \{(1, 2), (2, 3)\}, S' = \{(1, 0), (1, 1)\}$.

    (a) Find the matrix A representing T relative to the basis S.

    (b) Find the matrix B representing T relative to the basis S' .

    (c) Find the change-of-basis matrix P from S to S'.

    (d) How are A and B related?

20. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(x, y, z) = (x + y, y + z, x + z)$. Find the matrix P so that $[T]_{\mathcal{B}_1} = P^{-1}[T]_{\mathcal{B}_2}P$ for the following pairs of basis.

    (a) $\mathcal{B}_1 = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}, \mathcal{B}_2 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

    (b) $\mathcal{B}_1 = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}, \mathcal{B}_2 = \{(1, 1, -1), (1, -1, 1), (-1, 1, 1)\}$

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