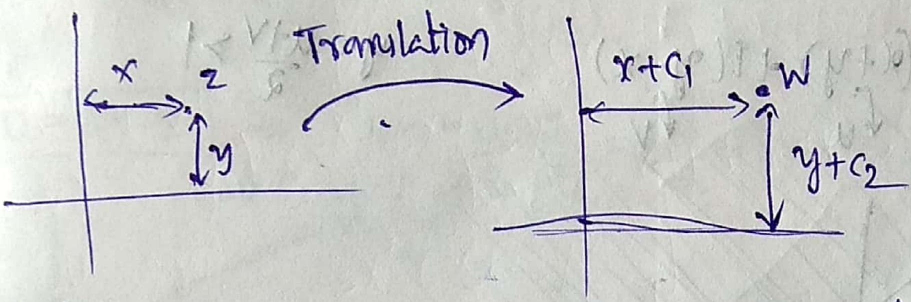
$u(x,y) = x^2 - y^2$; $v(x,y) = -y/x^2+y^2$ (done) ✓ → 18

1/2/19 **Mapping of Transformations:**

① **Translation:** For a point $c \in \mathbb{C}$ The translation by c in the mapping $W = z+c$

$$z \longmapsto W = z+c$$

if $z = x+iy$ and $c = c_1 + ic_2 \Rightarrow W = (x+c_1) + i(c_2+y)$



② **Rotation:** For a point $a \in \mathbb{C}$ The rotation by a in the mapping

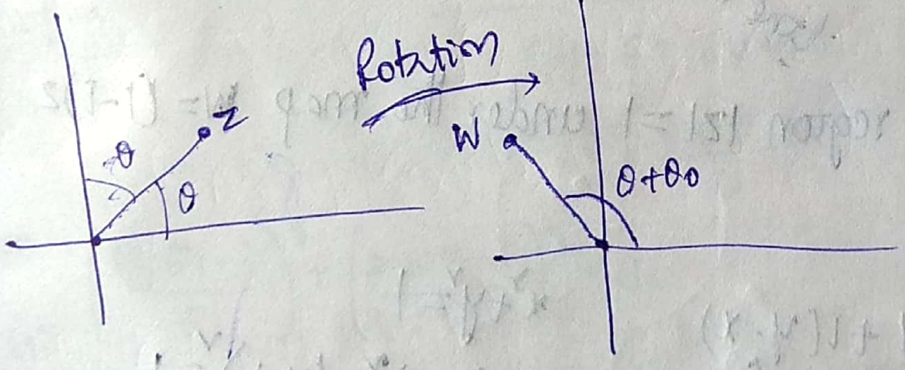
$W = az$ $\quad (a = r_0 e^{i\theta_0})$ $\quad r_0 = |a|$

if $z = re^{i\theta} \Rightarrow W = rr_0 e^{i(\theta+\theta_0)}$

$|a| = 1$ Just a rotation

$|a| < 1$ dilation

$|a| > 1$ expansion.



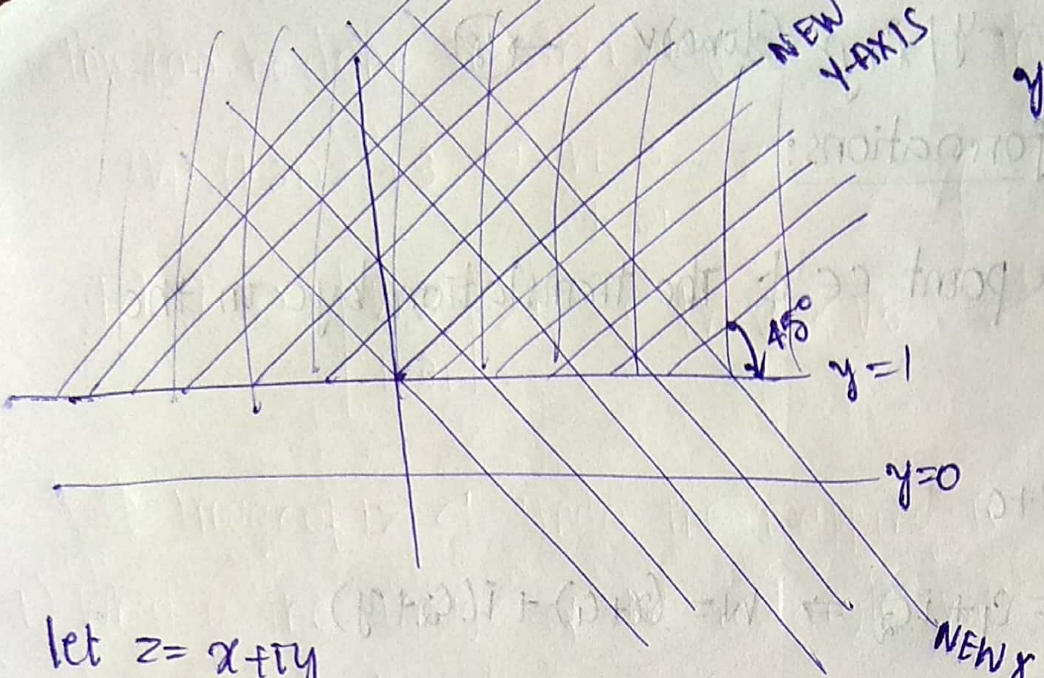
$$z \longmapsto z+c \longrightarrow az+ac$$

$$z \longmapsto az+b$$

(Q): Find the image of the region $y > 1$ (Im(z)) under the map.

$W = (1-i)z$

$|a| = \sqrt{2}$ so, it is expansion Rotation.



$y>1$

$-\pi/4$ anticlockwise

Rotation of the axis 45° in clockwise direction.

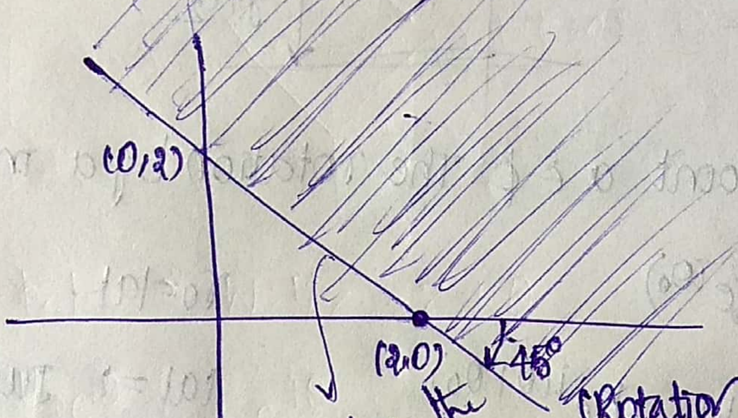
let $z = x+iy$

$W = (x+iy)(1-i) = \underbrace{(x+y)}_{u} + i\underbrace{(y-x)}_{v}$

$y = \frac{u+v}{2} > 1$

$\underline{\underline{u+v>2}}$

$\Rightarrow$



Rotation & Expansion $(|1-i|=\sqrt{2})$

Excluding the line $u+v=2$

(Rotation) of axes (x) by 45° clkwise

Q) Find the image of the region $|z|=1$ under the map $W=(1-i)z$

$z = x+iy$

$W = (x+iy)(1-i) = \underbrace{x+y}_{u} + i\underbrace{(y-x)}_{v}$

$x^2+y^2=1$

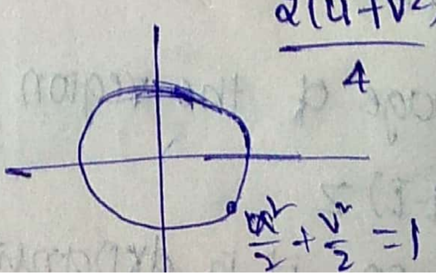
$\Rightarrow \left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{-2}\right)^2 = 1$

$x = \frac{u-v}{2}$; $y = \frac{u+v}{2}$

$\frac{2(u^2+v^2)}{4} = 1$

$\boxed{u^2+v^2=2}$

A circle of radius of $\sqrt{2}$.



Q) consider the mapping $w = (1-2i)z+3$: Find the image ① of the curve $x=-1$ ② $|z| < 1$

$z = x+iy$

$$(1-2i)(x+iy)+3 = (x+2y)+3+i(y-2x) = \underbrace{(x+\overset{2y+3}{\cancel{3}})}_{u} + i\underbrace{(y-2x)}_{v}$$

①

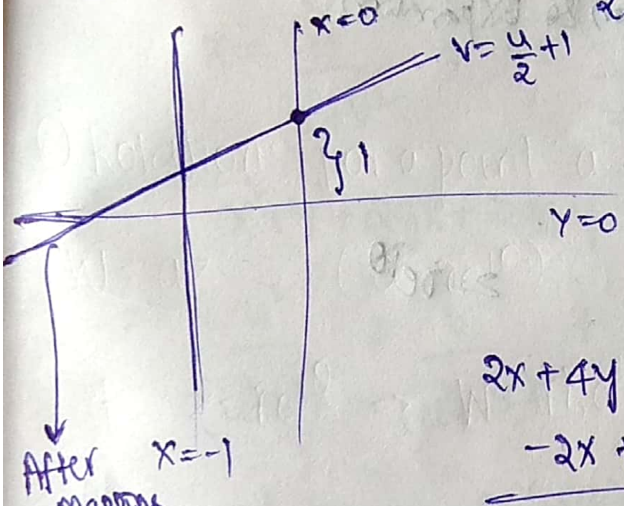
$$\left.\begin{aligned} x+2y+3 &= u \\ -2x+y &= v \\ -4x+2y &= 2v \end{aligned}\right\}$$

$5x+3 = u-2v$

$x = \dfrac{u-2v-3}{5} = -1 \Rightarrow u-2v = -2$

$2v-u-2=0$

$\boxed{u = 2v-2} \Rightarrow v = \dfrac{u+2}{2} = \dfrac{u}{2}+1$ $\quad (y = mx+c$, $c=1$; $m = 1/2)$.



② $|z| = -1$

$x = \dfrac{u-2v-3}{5}$; $y = \dfrac{2u+v-6}{5}$

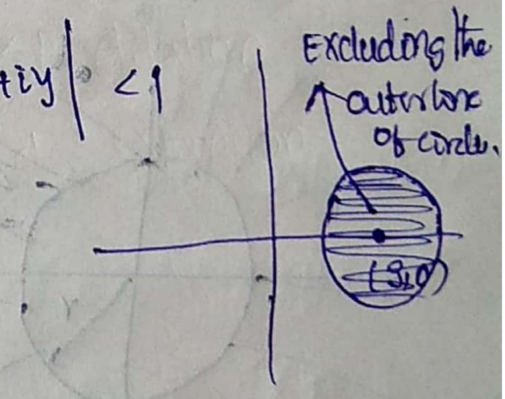
$2x+4y+6 = 2u$

$-2x+y = v$

$5y+6 = v+2u$

$y = \dfrac{2u+v-6}{5}$

$|x+iy| < 1$



$\Rightarrow \left(\dfrac{u-2v-3}{5}\right)^2 + \left(\dfrac{2u+v-6}{5}\right)^2 < 1$

$\Rightarrow (u-2v)^2+\cancel{9}-6(u-2v)+(2u+v)^2+\cancel{36}-12(2u+v) < \overset{20}{\cancel{25}}$

$\Rightarrow u^2+4v^2-\cancel{4uv}-6u+\cancel{12v}+4u^2+v^2+\cancel{4uv}+20-24u-\cancel{12v} < 0$

$5u^2+5v^2-30u+20 < 0$

$u^2+v^2-6u+4 < 0 \Rightarrow u^2-6u+9+v^2 < 5$

$\boxed{(u-3)^2+v^2 < 5}$

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This can be done on other way:

$W = (1-2i)z + 3 \Rightarrow \dfrac{W-3}{1-2i} = z$

$|z| < 1 \Rightarrow \left|\dfrac{W-3}{1-2i}\right| < 1 \Rightarrow |W-3| < \sqrt{5}$

This is a circle centered at $(3,0)$ and radius $\sqrt{5}$.

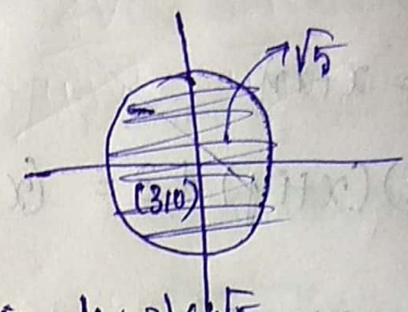
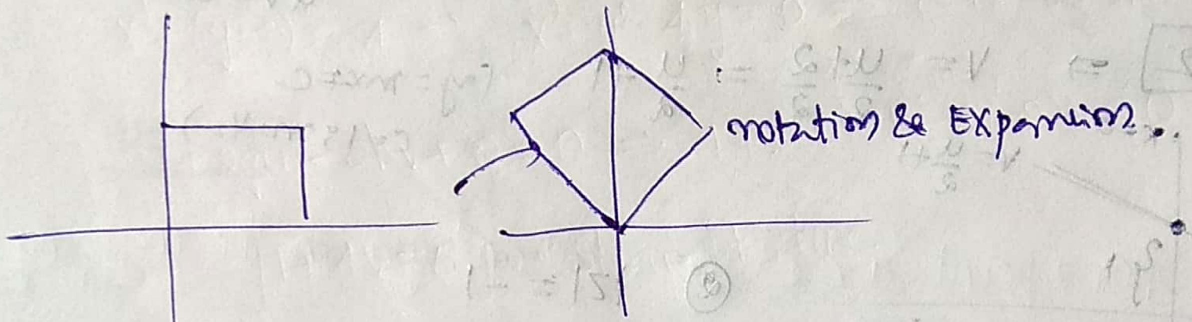


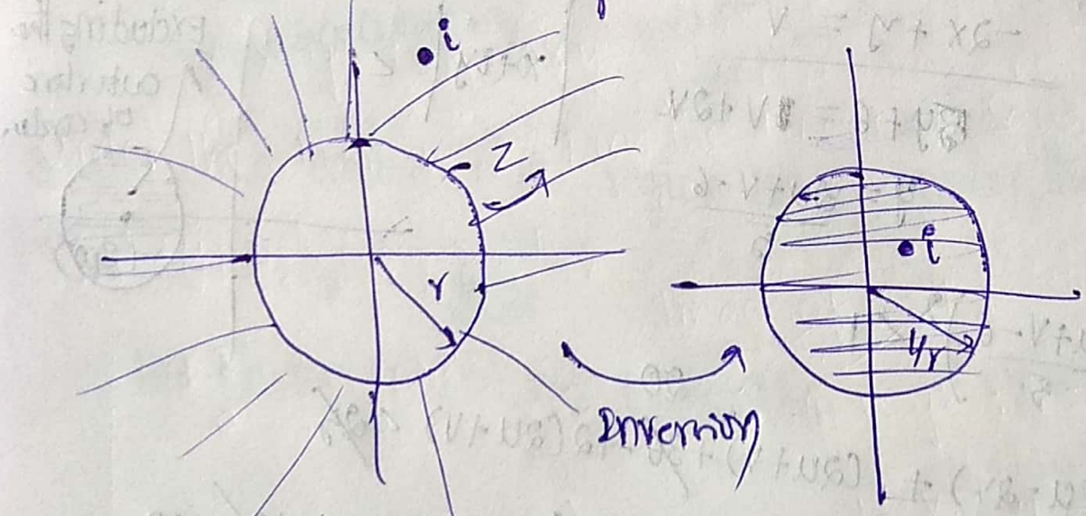
Image of $|z|<1$ under the mapping is $|W-3| < \sqrt{5}$.

E) What is the image of the rectangle with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ under the mapping $W = (1+i)z$



③ Inversion: $z \longmapsto W = 1/z \quad z \neq 0 \qquad z = re^{i\theta}$

$W = 1/z = 1/re^{i\theta} = \frac{1}{r}e^{-i\theta}$



→ if point z is inside circle r the image of it (inversion) lies outside the circle of radius $1/r$.

→ And vice versa.

Q) what is the image of the circle $|z|=1$ under the mapping $w=1/z$

$|w| = |1/z| \Rightarrow |z| = \frac{1}{|w|} \Rightarrow |w| = 1$

Q) $|z|<1 \Rightarrow |w|>1$

Q) show that the mapping $w=1/z$ transforms a circle into a circle / straight line.

5/2/19

$z = x+iy$

$u+iv = \frac{1}{x+iy} \Rightarrow x+iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$

$x = \frac{u}{u^2+v^2}$; $y = \frac{-v}{u^2+v^2}$

$x^2+y^2+2gx+2hy+c=0$ General Eqn of a circle

$\Rightarrow \frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + 2g\left(\frac{u}{u^2+v^2}\right) - 2h\left(\frac{v}{u^2+v^2}\right) + c = 0$

$\Rightarrow u^2+v^2 + 2gu(u^2+v^2) - 2hv(u^2+v^2) + c(u^2+v^2)^2 = 0$

$\Rightarrow 1+2gu-2hv+c(u^2+v^2)=0$

if $c=0$ it maps to a straight line.

if $c\neq 0$ it maps to a circle

Q) what is the image of a circle $|z-2i|=2$ under the mapping $w=1/z$

$w = \frac{1}{z}$

$z = x+iy$

$|x+iy-2i|=2$

$x = \frac{u}{u^2+v^2}$; $y = \frac{-v}{u^2+v^2}$

$x^2+(y-2)^2 = 4$

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{v}{u^2+v^2}+2\right)^2 = 4$$

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \cancel{4} + \frac{4v}{u^2+v^2} = \cancel{4} \Rightarrow \frac{4v}{\cancel{u^2+v^2}} = \frac{-1}{\cancel{u^2+v^2}}$$

$\boxed{v = -1/4}$ straight line

NOTE: In the Equation $x^2+y^2+2gx+2fy+c=0$ if $c=0$ then it maps to straight line under $w = 1/z$.

$$x^2+(y-2)^2 = 4 \Rightarrow x^2+y^2-4y=0 \Rightarrow c=0 \text{ so straight line}$$

$$1+2gu-2hv+0=0 \Rightarrow 1+2(0)u-2(-2)v=0$$

$\boxed{1+4v=0}$ → straight line

Q) $w = z^2$

$$u+iv = (x+iy)^2 = x^2-y^2+2xyi$$

$$u = x^2-y^2 \;;\; v = 2xy$$

Find the image of Square region Bounded by under the Tran. $w=z^2$

$x=1, y=2, x=2, y=1.$

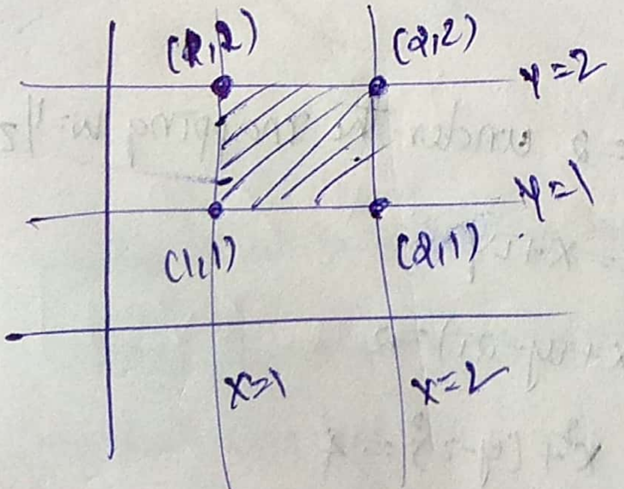


Image of $x=1$;

$$u = 1-y^2 \;;\; v = 2y$$

$$1-\left(\frac{v}{2}\right)^2 = u \Rightarrow 1-u = \frac{v^2}{4}$$

$\boxed{v^2 = 4-4u}$ } a parabola.

Parabola with centre $(1,0)$

$$v^2 = -4(u-1)$$

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Image of $x=2$;

$u = 4-y^2$; $v = 2y$

$u = 4-\left(\frac{v}{2}\right)^2 \Rightarrow 16-v^2 = 4u$

$v^2 = 16-4u$

$v^2 = -4(u-4)$ ↗ parabola with centre (4,0)

Image of $y=1$;

$u = x^2-1$; $v = 2x$

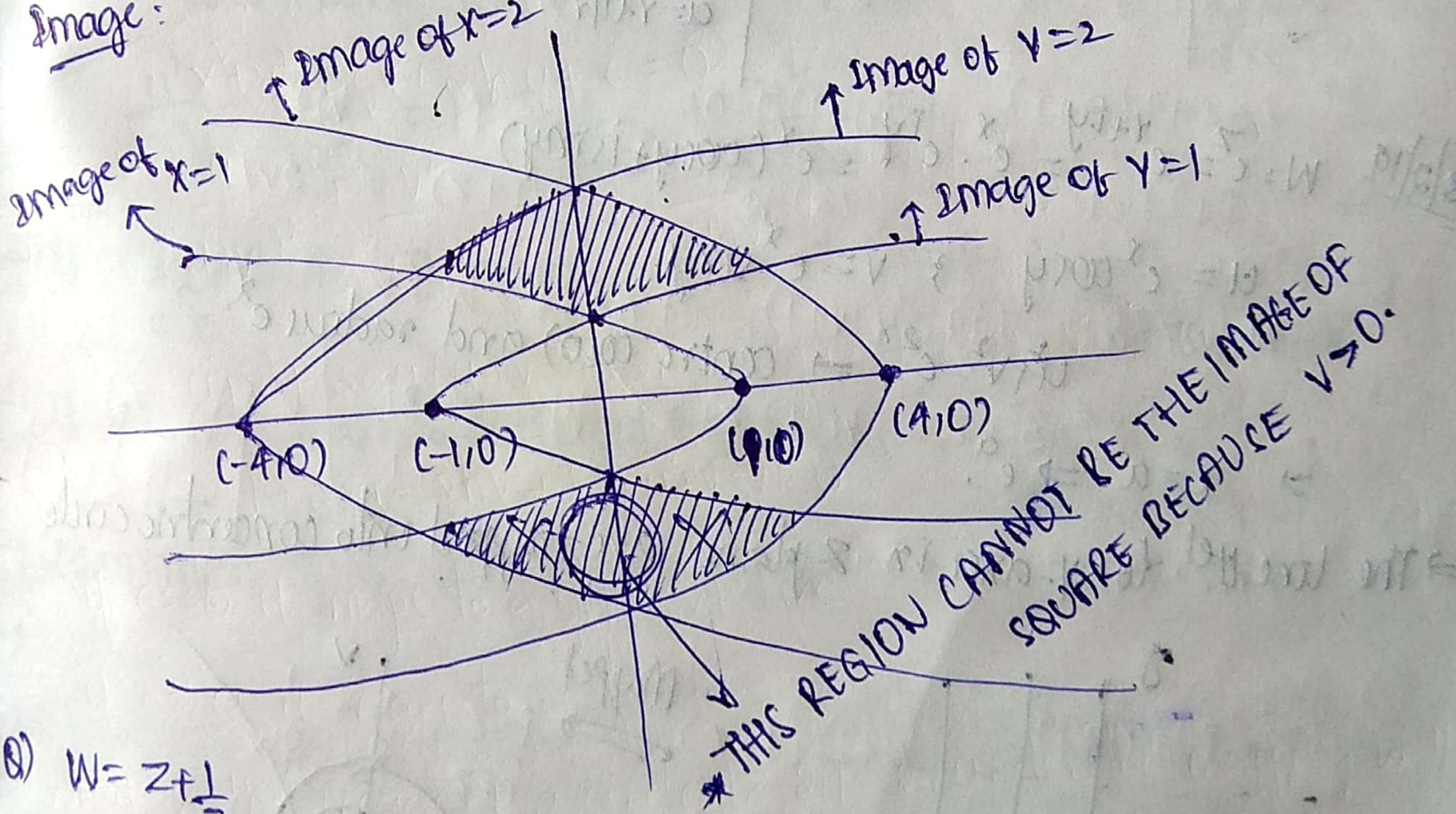
$u = \left(\frac{v}{2}\right)^2 - 1 \Rightarrow v^2 = 4(u+1)$ ↗ parabola with centre (0,-1)

Image of $y=2$;

$u = x^2-4$; $v = 2x$

$u = \left(\frac{v}{2}\right)^2 - 4 \Rightarrow v^2 = 4(u+4)$ ↗ parabola with centre (0,-4)

Image:



Q) $W = z + \frac{1}{z}$

$z = re^{i\theta}$ $\quad W = re^{i\theta} + \frac{1}{r}e^{-i\theta} = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$

$= \left(r\cos\theta + \frac{1}{r}\cos\theta\right) + i\left(r\sin\theta - \frac{1}{r}\sin\theta\right)$

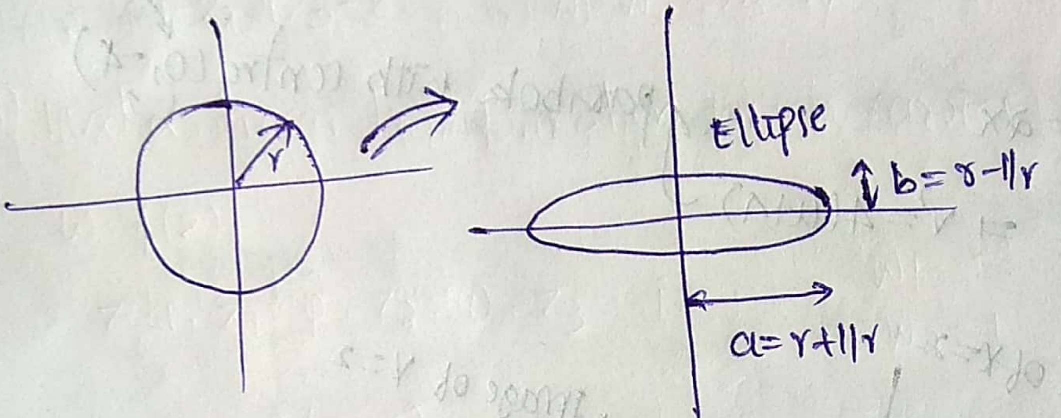
$= u + iv$

Q) under the mapping, $w = z + 1/z$. Find the image of circle $|z| = r$ and line $\theta = \theta_0$.

$|z| = r$

$w = \cos\theta\left(r + \frac{1}{r}\right) + i\sin\theta\left(r - \frac{1}{r}\right) = u + iv$

$\Rightarrow \cos^2\theta + \sin^2\theta = 1 \Rightarrow \frac{u^2}{(r+1/r)^2} + \frac{v^2}{(r-1/r)^2} = 1$ Ellipse with $a = r + 1/r$; $b = r - 1/r$.



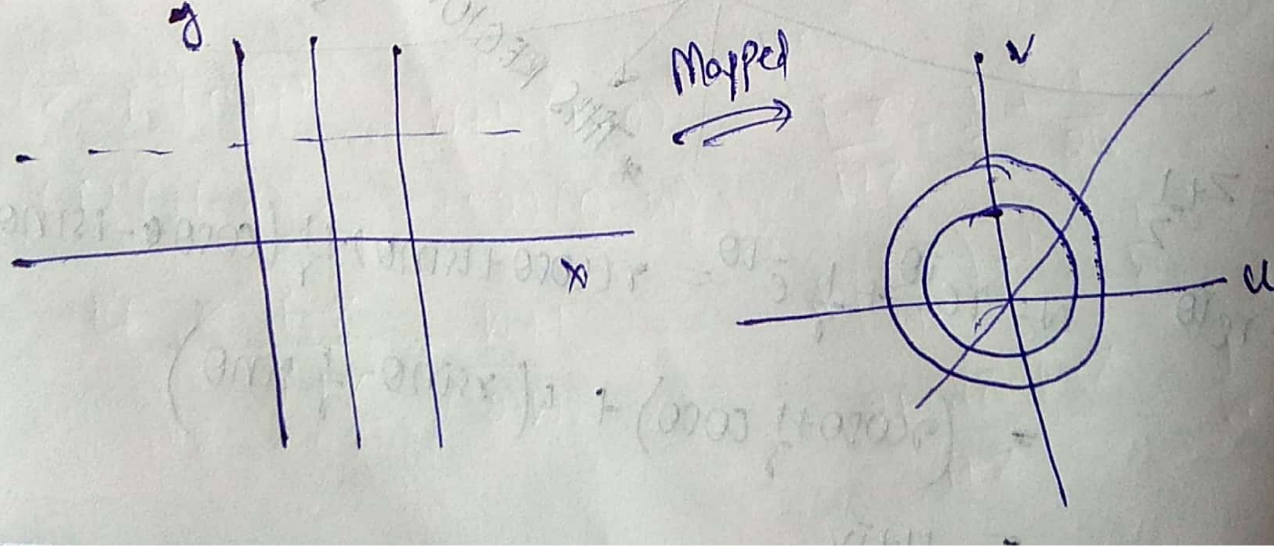
8/2/19. $w = e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$

$u = e^x \cos y$; $v = e^x \sin y$

$u^2 + v^2 = e^{2x} \Rightarrow$ centre $(0,0)$ and radius e^x

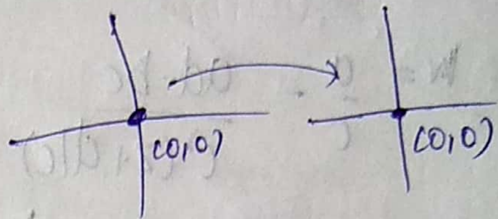
$\Rightarrow x = a \Rightarrow e^a$.

$\Rightarrow$ The line ‖el to y-axis in z-plane are mapped onto concentric circle



*) $w = \sin z = \sin(x+iy) = \sin x \cos(iy) + \sin(iy)\cos x$

$u = \sin x \cdot \cosh y$; $v = \cos x \sinh y$.



$u^2 = \sin^2 x \cosh^2 y$ $v^2 = \cos^2 x \sinh^2 y$

$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1$; $\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1$

For x=constant, it maps to different Hyperboles.

For y=constant, it maps to different Ellipse.

$(0,0) \to (1,0)$

$x \geq 0 \to 1 \leq \cosh x \leq \infty$

$R \longrightarrow (1,\infty)$

$y=k$, $k \neq n\pi/2$

$\frac{u^2}{\cos^2 k} - \frac{v^2}{\sin^2 k} = 1$

$k \neq$

$x=k$ $\frac{u^2}{\cosh^2 k} + \frac{v^2}{\sinh^2 k} = 1$

* $w = \cosh z = \cos(iz) = \cos(i(x+iy))$

$u = \cosh x \cdot \cos y$; $v = \sinh x \sin y$.

$u^2 = \cosh^2 x \cos^2 y$; $v^2 = \sinh^2 x \sin^2 y$

$\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1$; $\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 x} = 1$

→ if x=constant $=k$, it maps to different Ellipses i.e $\frac{u^2}{\cosh^2 k} + \frac{v^2}{\sinh^2 k} = 1$

→ if $y=k$, it maps to different Hyperbolas i.e $\frac{u^2}{\cos^2 k} - \frac{v^2}{\sin^2 k} = 1$

→ $(0,0)$ maps to ⇒ $u=1$; $v=0$ $(1,0)$

→

Bilinear Transformations: $w = \frac{az+b}{cz+d}$, $ad-bc \neq 0$.

$$w = \frac{a}{c} - \frac{ad-bc}{c^2(z+d/c)}$$

Cross ratio or Mobius Transformation:

$w = f(z)$

$z_1, z_2, z_3 \to w_1, w_2, w_3$

$$\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

Q) Find a linear Transformation which carries $i, 0, -i \to -1, 1, 0$.

$w = f(z)$

$z_1 = i$, $z_2 = 0$, $z_3 = -i$ $\quad w_1 = -1$, $w_2 = 1$, $w_3 = 0$

$$\frac{(z-i)(0+i)}{(i)(-i-z)} = \frac{(w+1)(1-0)}{(+2)(w)} \Rightarrow \frac{z-i}{+(i+z)} = \frac{w+1}{+2w}$$

$w(2z-2i) = (i+z)(w+1) = \cancel{iw+i+zw+z}$

$2z-3i-z = w(i+z) \Rightarrow \cancel{\boxed{w = \frac{z-3i}{z+i}}}$ ✓ $\boxed{w = \frac{z+i}{i-3z}}$

$\cancel{2wz} - 2iw = \cancel{iw} + i + z\cancel{w} + z$

$$\frac{(z-i)(0+i)}{(i)(-i-z)} = \frac{(w+1)(1-0)}{(-2)(0-w)}$$

$$\frac{i-z}{i+z} = \frac{w+1}{(-2)w}$$

$$\frac{i-z}{i+z} = \frac{1}{2} + \frac{1}{2w}$$

$$2\left(\frac{i-z}{i+z}\right) - 1 = \frac{1}{w}$$

$$\frac{2i-2z-i-z}{i+z} = \frac{1}{w}$$

Q) $(0,1,\infty)$ → ① $(0,1,2)$. If ∞ is there then replace it by $1/z$.
② $(0,i,\infty)$

$z_1=\infty$
$1/z_1=0$

$$\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(z-z_1)(z_2-1/z_3)}{(z_1-z_2)(\frac{1}{z_3}-z)}$$ $z_3=0$

$$= \frac{(z-z_1)(z_2z_3-1)}{(z_1-z_2)(1-zz_3)} = \frac{z_1-z}{z_1-z_2} = \frac{0-z}{-1} = z$$

$$z = \frac{(w-0)(\cancel{1}-2)}{(\cancel{-1})(2-w)} \Rightarrow w = \frac{z}{2-w} \quad \frac{2-w}{w} = \frac{1}{2}$$ $\boxed{w=\frac{2z}{z+1}}$

$\frac{2}{w} = \frac{z+1}{2}$

case i) $w = 2z/z+1$

(-1)

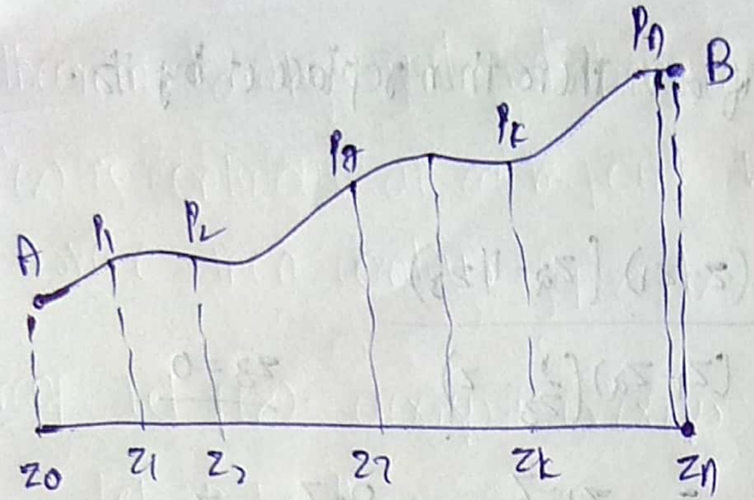
case ii :

$$z = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(w-w_1)(w_2w_3-1)}{(w_1-w_2)(1-ww_3)} = \frac{(w)(-1)}{+i}$$

$\boxed{z=-iw}$ ⇒ $w = iz$

Q) complex Integration:

Let $f(z)$ be a continous function of the complex variable $z=x+iy$ defined at all points of a curve c having end points from A to B. Divide the curve c into n parts at $A=P_0(z_0), P_1(z_1), \ldots \ldots P_n(z_n)=B$. Let $\delta z_n = z_i - z_{i-1}$, $f(z)$ be any point on the arc (P_{i-1}, P_i). Then Limit of $\sum_{i=1}^{n} f(z)\,\delta z_i$ as $\begin{cases} \infty \leftarrow n \\ \delta z_i \to 0 \end{cases}$ if exists is called the line integral of $f(z)$ along the curve c and denoted By $\int_c f(z)dz$.



if consider large intervals then some the area will be missing. so inorder to have them, we make small division i.e $\delta z \to 0$

If the points $P_0 \dots P_n$ are points such that c is a closed curved. then the integral is called contour and denoted by $\oint_c f(z)dz$

If $f(z) = u(x,y) + iv(x,y)$ and $dz = d(x+iy) = dx + idy$

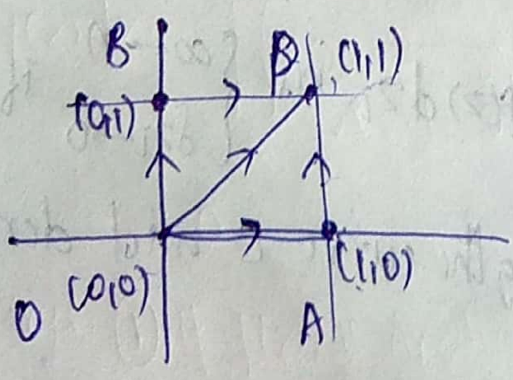
$$\int_c f(z)dz = \int (u(x,y) + iv(x,y))(dx+idy)$$

$$= \int_c udx - vdy + i\int_c vdx + udy$$

Remark: The value of integral depends on the path of integration (values) unless the f^n is analytic.

Q) Find $\int_0^{1+i} (x-y+ix^2)dz$ ① Along the straight line $z=0$ to $z=1+i$

② Along the real axis from $z=0$ to $z=1$.



Ans: ① $x=y$ (OP)

$dz = dx + i dy = (1+i)dx = (1+i)dy$

$$\int_0^1 (x - x + ix^2)(1+i)\, dx = \int_0^1 (ix^2)(1+i)\, dx = \left(\frac{ix^3}{3}\right)_0^1 + \left(-\frac{x^3}{3}\right)_0^1$$

$$= \left(\frac{i-1}{3}\right)(1) = \frac{i-1}{3}$$

② Along real axis $\Rightarrow y=0$

$dz = dx$

$$\int_0^{1} (x + ix^2)\, dx = \left(\frac{x^2}{2} + \frac{ix^3}{3}\right)_0^{1} = \frac{1}{2} + \frac{i}{3}$$

Along $x=1 \Rightarrow$ $\cancel{\int_0^1 (-y)i\,dy = \left(\frac{-iy^2}{2}\right)_0^1 = -\frac{i}{2}}$

$$\rightarrow i\int_0^1 (1+i-y)\,dy = \left(y + iy - \frac{y^2}{2}\right)_0^1 i = \left(1 + i - \frac{1}{2}\right) i = -1 + \frac{i}{2}$$

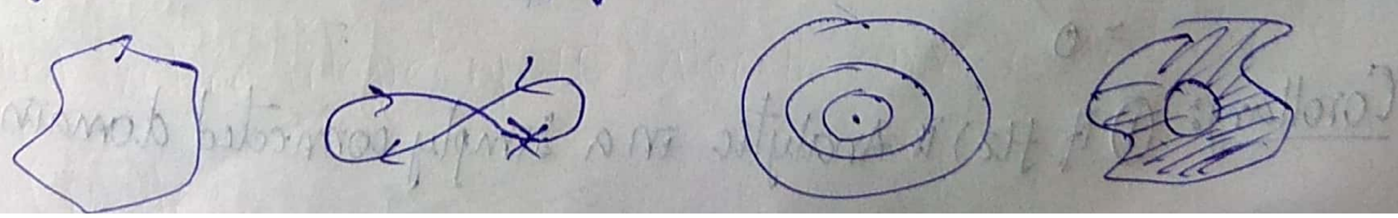
12/2/19

Simply & Multiply:

→ A curve is called simple closed curve if it doesn't close (cross) itself. A curve which close itself is called a multiple curve.

→ A region is called simply connected if every closed curve in the rgn encloses points of the region only.

→ A region which is not simply connected is called multiply region.



Theorem: Cauchy's - Integral Theorem: If $f(z)$ is an Analytic f^n and $f'(z)$ is continous at each point with in and on a single closed curve c then contour $\oint_c f(z) \cdot dz = 0$

proof: Let $f(z) = u(x,y) + iv(x,y)$

let R be the region bounded by the curve c.

$$\oint_c f(z)dz = \oint_c u(x,y)dz + \oint_c iv(x,y)dz = \oint_c (u(x,y)+iv(x,y))(dx+idy)$$

$$= \oint udx - vdy + i\oint vdx + udy$$

Green's Theorem: $\int Mdx + Ndy = \iint_R \left(\frac{dN}{dx} - \frac{dM}{dy}\right) dx\, dy.$

$$\oint_c f(z)dz = \oint_c udx - vdy + i\oint_c vdx + udy$$

Since $f'(z)$ is continuous, u_x, u_y, v_x, v_y are continous on "R" region.

Then by green's Theorem:

$$\oint_c f(z)dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + i\iint_R \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$$

Given $f(z)$ is Analytic, $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$$\oint_c f(z)dz = \iint_R \cancelto{0}{-\frac{\partial v}{\partial x} - \left(-\frac{\partial v}{\partial x}\right)} + i\iint_R \cancelto{0}{\frac{\partial u}{\partial x} - \left(\frac{\partial u}{\partial x}\right)}$$

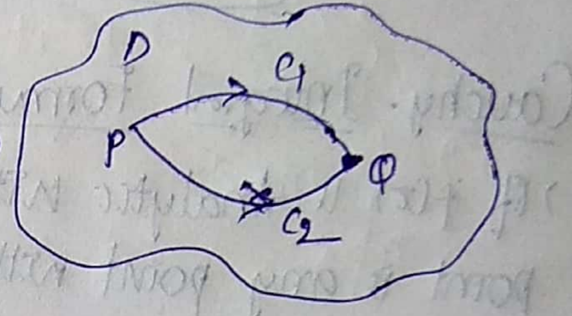
$$= 0$$

Corollary: ① If $f(z)$ is Analytic in a simply connected domain D, Then the integral of $f(z)$ is independent of path in D.

proof: $f(z)$ is Analytic in a simply connected domain 'D', then by cauchy-integral theorem.

$$\oint_C f(z)dz = 0 \text{ for all closed curve in } D.$$

Let P, Q be any two distinct points in D and C_1 and C_2 are two paths from P to Q.



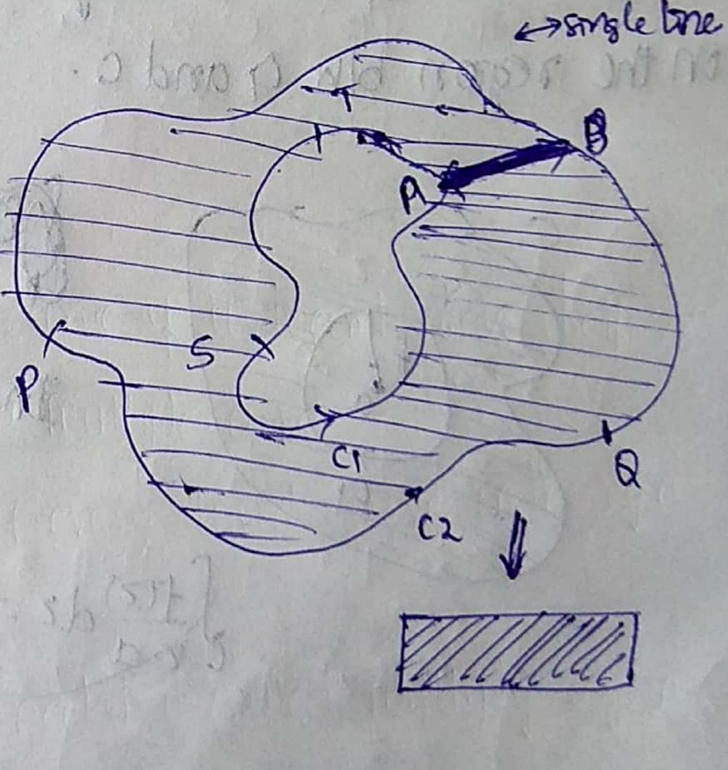
$\Rightarrow$ $(C_1 - C_2)$ is a closed ~~counter~~ contour in C then

$$\oint_{C_1 - C_2} f(z)dz = 0 \Rightarrow \oint_{C_1} f(z)dz + \oint_{-C_2} f(z)dz = 0$$

$$\Rightarrow \oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

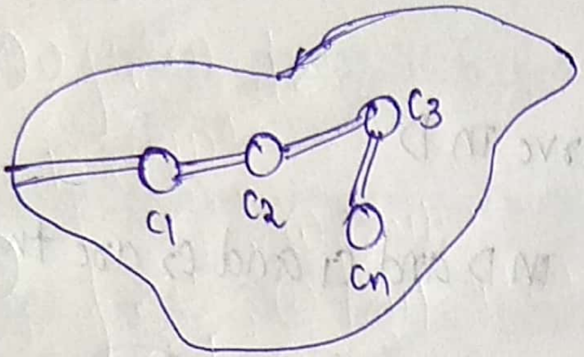
ii) if $f(z)$ is Analytic in a region bounded by two simple closed curves C_1, C_2 then:

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$



Let BA be a crosscut joining the curves C_1, C_2. Then the region becomes simply connected region.

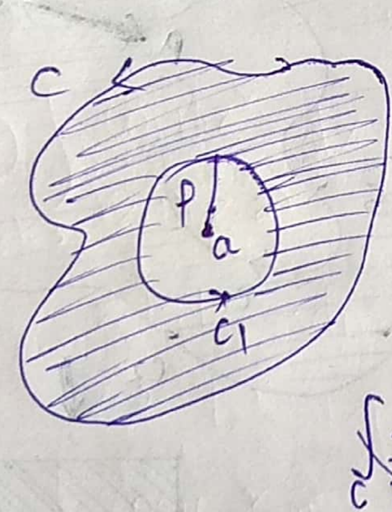
→ If a closed curve "c" contains non-intersecting closed curves $c_1, c_2 \dots c_n$. Then by introducing cross cuts $\oint_{C} f(z)dz = \oint_{C_1} f(z)dz + \dots + \oint_{C_n} f(z)dz$



Cauchy-Integral Formula:

→ If $f(z)$ is Analytic within and on a closed curve and a is an point is any point within C then $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$.

Proof: consider the function $f\left(\frac{z}{z-a}\right)$ $\left(\frac{f(z)}{z-a}\right)$ which is Analytic at every pt except at $z=a$. Draw a circle C_1 with "a" as centre and radius "ρ" such that c_1 lies entirely inside C. Thus $f\left(\frac{z}{z-a}\right)$ $\left(\frac{f(z)}{z-a}\right)$ is analytic in the region b/w c_1 and c.



$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz$$

In $C_1 \to z = a + \rho e^{i\theta}$

$$dz = i\rho e^{i\theta}$$

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(a+\rho e^{i\theta})}{\rho e^{i\theta}}\, i\rho e^{i\theta}\, d\theta$$

$$= \int_0^{2\pi} f(\rho + a e^{i\theta})\, i\, d\theta$$

$$= i\int_0^{2\pi} f(\rho + a e^{i\theta})\, d\theta$$

$\frac{f(z)}{z-a}$ → Analytic b/w C_1 & C In curve C_1 → $z = a + \rho e^{i\theta}$

$dz = \rho i e^{i\theta} d\theta$

$$\int_C \frac{f(z)}{z-a} dz = \int_{C_1} \frac{f(z)}{z-a} dz$$

$$\int_{C_1} \frac{f(z)}{z-a} \cdot dz = \int_{C_1} \frac{f(a+\rho e^{i\theta})}{\cancel{\rho e^{i\theta}}} \cdot \cancel{\rho i e^{i\theta}} \cdot d\theta = i \int_0^{2\pi} f(a+\rho e^{i\theta}) \cdot d\theta$$

As $\rho \to 0 \Rightarrow f(a+\rho e^{i\theta}) = f(a)$

$$\Rightarrow \int_{C_1} \frac{f(z)}{z-a} dz = i \int_0^{2\pi} f(a) d\theta = 2\pi i f(a).$$

$$\Rightarrow \boxed{f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz.}$$

How to find $f'(a)$. Differentiation wrt a.

$$f'(a) = \frac{1}{2\pi i} \int_C f(z) \cdot \left(\frac{-1(-1)}{(z-a)^2} \right) dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz.$$

$$f''(a) = \frac{2!}{2\pi i} \int \frac{f(z)}{(z-a)^3} dz$$

In general, $\boxed{f^{n}(a) = \frac{n!}{2\pi i} \int \frac{f(z)}{(z-a)^{n+1}} dz.}$

15/2/19

1. $\int_{|z|=1} (x^2 - y^2 + 2ixy) dz = \int_{|z|=1} (x+iy)^2 dz = \int_{|z|=1} z^2 dz = \left(\frac{z^3}{3}\right)$

2. $\int_C \frac{e^{-z}}{z+1} dz$ (a) $|z|=2$

(b) $|z|=1/2$

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③ $\int_{|z|=3} \frac{\cos \pi z^2}{(z-1)(z-2)} dz$ ⑥ $\oint_C \frac{\cos z}{(z-\pi i)^2} dz$, $\pi i \in C$

④ $\int_{|z|=2} \frac{e^{2z}}{(z+1)^4} dz$ ⑦ $\oint_C \frac{z^4-3z^2+6}{(z+i)^3} dz$, $-i \in C$

⑤ $\int_C \frac{z^2+1}{z^2-1}$ ⓐ $|z-1|=1$ ⓑ $|z+1|=1$ ⓒ $|z-i|=1$ ⑧ $\oint \frac{e^z}{(z-1)^2(z^2+4)} dz$, $1 \in C$ $\pm 2i \in C$

⑨ $\int_{|z|=2} \tan z \cdot dz$

Answers:

② $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$ $\int \frac{f(z)}{z-a} dz = 2\pi i f(a)$

In this $f(z) = e^{-z}$ and $a = -1$ $= 2\pi i e' = 2\pi e i$

$f(a) = e^{-(-1)} = e$ $\Rightarrow \int \frac{e^{-z}}{z+1} dz = 2\pi e i$

③ $\int_{|z|=3} \frac{\cos \pi z^2}{(z-1)(z-2)} dz$ ~~z-1=t~~ ~~dz=dt~~ ~~t(t-1)~~ $(t+1)^2$ ~~t^2+2t+1~~

~~z-1=t => dz=dt~~ $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

$z-2 = t-1$ ~~|z|=|t+1|=3~~

$\int_{|z|=3} \frac{\cos \pi z^2}{z-2} dz - \int_{|z|=3} \frac{\cos \pi z^2}{z-1} dz =$ ~~f~~cos $\int$ $2\pi i z^2$

$4\pi e i$

~~f(2)=~~

$f(z) = \cos\pi z^2$

$$\int \frac{f(z)}{z-a}\,dz = 2\pi i f(a) = 2\pi i \cos(4\pi) = 2\pi i e \qquad a=2$$

$$\int \frac{f(z)}{z-a}\,dz = 2\pi i f(a) = 2\pi i \cos(\pi) = -2\pi i e \qquad a=1$$

$4\pi e i$

$(z+1)^4 = t$

$4(z+1)^3\,dz = dt$

~~dto~~

④ $$\int_{|z|=2} \frac{e^{2z}}{(z+1)^4}\,dz$$

compare with $f^{n}(a) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-a)^{n+1}}\,dz$

$\boxed{n=3}$

$$f^{3}(a) = \frac{3!}{2\pi i}\int \frac{e^{2z}}{(z+1)^4}\,dz$$

$a=-1$; $f(z)=e^{2z}$

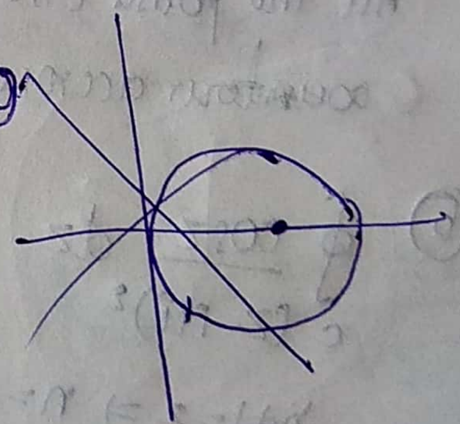
$f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z} \Rightarrow f'''(z) = 8e^{2z}$

$$\frac{f^{3}(a)\cdot 2\pi i}{3!} = \frac{8e^{2z}\cdot 2\pi i}{3!} = \int_{|z|=2} \frac{e^{2z}}{(z+1)^4}\,dz$$

$$\int_{|z|=2} \frac{e^{2z}}{(z+1)^4}\,dz = \frac{16\pi i e^{2z}}{3!}$$

ⓐ

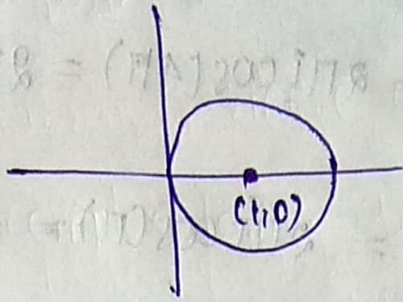


⑤ $$\int \frac{z^2+1}{z^2-1}\,dz = \int \frac{(z^2+1)}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)dz$$

$$= \frac{1}{2}\left[\int \frac{z^2+1}{z-1}\,dz - \int \frac{z^2+1}{z+1}\,dz\right]$$

⑤ $\int \frac{z^2+1}{z^2-1} dz$

ⓐ $|z-1|=1$



$f(z) = \frac{z^2+1}{z+1}$

$f(1) = 1$

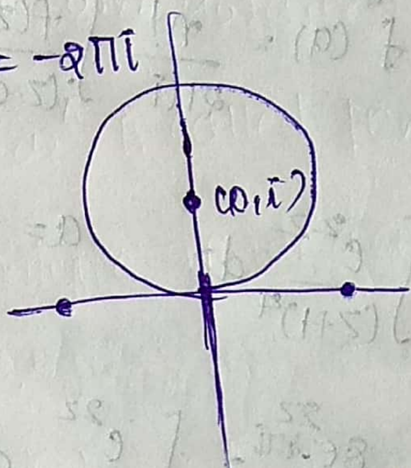
$$\int_{|z-1|=1} \frac{\left(\frac{z^2+1}{z+1}\right)}{z-1} dz = 2\pi i f(a) = 2\pi i$$

$f(z) = \frac{z^2+1}{z-1}$

ⓑ $|z+1|=1$

$f(-1) = \frac{2}{-2} = -1$

$$\int_{|z+1|=1} \frac{\left(\frac{z^2+1}{z-1}\right)}{z+1} \cdot dz = 2\pi i f(a) = -2\pi i$$



ⓒ $|z-i|=1$

$\Rightarrow$

All the points $(-1,0)$, $(1,0)$ are outside the curve $|z-i|=1$ so, the contour across $|z-i|=1 = 0$.

⑥ $\oint_c \frac{\cos z}{(z-\pi i)^2} dz$, $\pi i \in$ curve.

$n+1=2 \Rightarrow n=2$

$a = \pi i$

$f(z) = \cos z \Rightarrow f'(z) = -\sin z$

$f'(a) = \frac{1}{2\pi i} \int \frac{f(z)}{(z-a)^2} dz$

~~$f(a) = \cos \pi i$~~

$$\int \frac{\cos z}{(z-a)^2} dz = 2\pi i(-\sin \pi i)$$

(7) $\oint_C \frac{z^4 - 3z^2 + 6}{(z+i)^3} dz$ $-i \in C$

$n+1 = 3 \Rightarrow n = 2$

$f(z) = z^4 - 3z^2 + 6$

$f''(a) = \frac{2!}{2\pi i} \int \frac{f(z)}{(z-a)^3} dz$ $a = -i$

$f'(z) = 4z^3 - 6z \Rightarrow f''(z) = 12z^2 - 6$

$f''(a) = 12(-i)^2 - 6 = -12 - 6 = -18$

$\int \frac{f(z)}{(z-a)^3} dz = \frac{-36\pi i}{2} = -18\pi i.$

(8) $\oint \frac{e^z}{(z-1)^2(z^2+4)} dz$, $1 \in C$, $\pm 2i \in C$

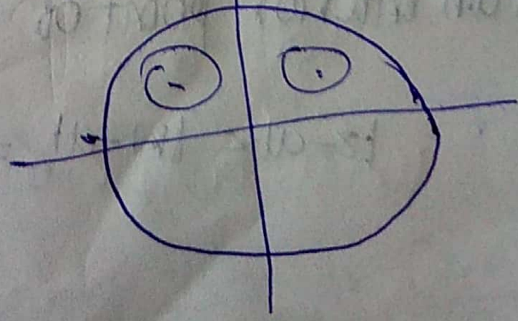
$a = 1$; $f(z) = \frac{e^z}{z^2+4}$

$\oint \frac{\frac{e^z}{(z^2+4)}}{(z-1)^2} dz$ $f(z) = \frac{e^z}{z^2+4}$

$f'(z) = \frac{(z^2+4)(e^z) - e^z(2z)}{(z^2+4)^2} \Rightarrow f'(1) = \frac{(5)e - 2e}{25} = \frac{3e}{25}$

$f'(a) = \frac{1}{2\pi i} \int \frac{f(z)}{(z-a)^2} dz \Rightarrow 2\pi i f'(a) = \int \frac{f(z)}{(z-a)^2} dz = \frac{6\pi e i}{25}$

(9) $\int_{|z|=2} \tan z \cdot dz = \int_{|z|=2} \frac{\sin z}{\cos z} \cdot dz$ → Not defined at $\pi/2, -\pi/2$ so consider small circles.



$= 2\pi i f(\pi/2) + 2\pi i f(-\pi/2)$

$= 0.$

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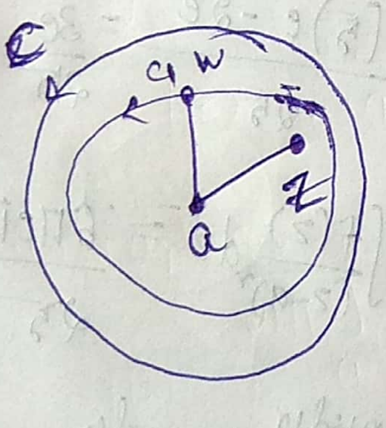
19/2/19 **Series of Complex forms:** A series of the form $(a_1+ib_1)+(a_2+ib_2)+\cdots(a_n+ib_n)-\cdots$ where $a_1, b_1, a_2, b_2 \cdots \in \mathbb{R}$ is called a series of complex forms. and can be expressed as summation $\sum_{j=1}^{\infty} a_j + i\sum_{j=1}^{\infty} b_j$. This series converges if $\sum_{j=1}^{\infty} a_j$ & $\sum_{j=1}^{\infty} b_j$ converge.

Power Series: A series of the form $a_0+a_1(z-a)+a_2(z-a)^2+\cdots a_n(z-a)^n+\cdots = \sum_{n=0}^{\infty} a_n(z-a)^n$ is called power series in $(z-a)$.

Taylor's Series: If $f(z)$ is analytic inside a circle C with center at "a", for all z inside c,

$$f(z) = f(a) + (z-a)f'(a) + f''(a)\frac{(z-a)^2}{2!} + \cdots + \frac{(z-a)^n}{n!}f^n(a) + \cdots$$



proof:

Let z be any point inside the circle "c". Draw a circle C_1 centered at a and radius smaller than that of "C" such that z is an interior point of C_1. Let w be any point on C_1, Then

$$|z-a| < |w-a| \Rightarrow \left|\frac{z-a}{w-a}\right| < 1$$

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$$\Rightarrow \frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)}\left[\frac{1}{1-\left(\frac{z-a}{w-a}\right)}\right], \left|\frac{z-a}{w-a}\right| < 1$$

$$= \frac{1}{(w-a)}\left[1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\cdots\left(\frac{z-a}{w-a}\right)^n+\cdots\right]$$

Multiply both sides by $\frac{1}{2\pi i} f(w)$, $\int$ wrt w over C_1.

$$\Rightarrow \frac{1}{2\pi i}\left(\frac{1}{w-z}\right)f(w) = \frac{1}{2\pi i}\frac{f(w)}{(w-a)}\left[1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2 \cdots\right]$$

$$\Rightarrow \int_{C_1}\frac{1}{2\pi i}\left(\frac{1}{w-z}\right)f(w)\,dw = \int_{C_1}\frac{1}{2\pi i}\frac{f(w)}{(w-a)}\left[1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\cdots\right]dw$$

$$\Rightarrow f(z) = \int_{C_1}\frac{1}{2\pi i}\frac{f(w)}{w-a}\,dw + \int_{C_1}\frac{1}{2\pi i}\frac{f(w)}{w-a}\cdot\left(\frac{z-a}{w-a}\right)dw +$$

Cauchy integral formula

$f(w)$ is analytic inside the curve C_1

$$\int_{C_1}\frac{1}{2\pi i}\frac{f(w)}{w-a}\left(\frac{z-a}{w-a}\right)^2 dw + \int_{C_1}\frac{1}{2\pi i}\frac{f(w)}{w-a}\left(\frac{z-a}{w-a}\right)^3 dw + \cdots$$

$$= f(a) + (z-a)\int\frac{1}{2\pi i}\frac{f(w)}{(w-a)^2}\,dw + (z-a)^2\int_{C_1}\frac{1}{2\pi i}\frac{f(w)}{(w-a)^3}\,dw$$

$$+ (z-a)^3\int\frac{1}{2\pi i}\frac{f(w)}{(w-a)^4}\,dw + \cdots$$

$$= f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \cdots$$

(Since $\int \frac{1}{2\pi i}\left(\frac{f(w)}{w-a}\right)dw = f(a)$ & $\int \frac{1}{2\pi i}\frac{f(w)}{(w-a)^{n+1}}\,dw = \frac{(z-a)^n}{n!}$)

Therefore, $f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots$

Corollary 1: substitute $z = a+h$ in ~~ty~~ taylor's series,

$$f(a+h) = f(a) + f(h)f'(a) + f''(a)\frac{h^2}{2!} + \cdots f^{n}\frac{h^n}{n!} + \cdots$$

corollary 2: If $a=0$ in taylor's series.

$$f(z) = f(0) + (z)f'(0) + \frac{z^2}{2!}f''(0) + \cdots + \frac{z^n}{n!}f^{n}(0)$$

→ Maclorian's series.

Laurent's series:

If $f(z)$ is Analytic, inside and on the boundary of the Anular region R bounded by two ~~centeric~~ concentric circles C_1 and C_2 of radii r_1 and r_2 respectively where $r_1 > r_2$ strictly having centre at 'a' then for all $z \in R$,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} \cdots$$

$$a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^{n+1}}\,dw, \quad n = 0,1,2 \cdots$$

$$a_{-n} = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{(w-a)^{-n+1}}\,dw, \quad n = 1,2,3 \cdots$$

$f(z)$ is Analytic on shaded region.

Remark 1: $a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw \neq \frac{f^n(a)}{n!}$ (inside C_2 $f(w)$ is not Analytic)

Because in C_1 $f(w)$ is not Analytic (inside).

Remark 2: If $f(z)$ is analytic inside C_1 then $a_{-n} = 0$ and

$$a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw = \frac{f^n(a)}{n!}$$

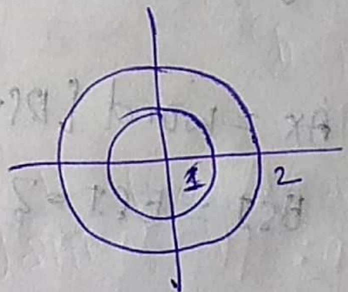
22/2/19 Express the $f(z)$ in terms of taylor / Laurent series.

$$f(z) = -\frac{1}{(z-1)(z-2)}$$

ⓐ $|z|<1$

ⓑ $1<|z|<2$

ⓒ $|z|>2$



ⓐ $|z|<1$

The only points where $f(z)$ is not analytic is 1 and 2. Hence $f(z)$ is Analytic in the region $|z|<1$. Hence $f(z)$ can be expressed in taylor's series.

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

ⓐ $f(z) = \frac{1}{z-1} - \frac{1}{z-2}$

$|z|<1$ $\quad |z/2|<1$

$$= \left(\frac{-1}{1-z}\right) - \frac{1}{2\left(\frac{z}{2}-1\right)} = \left(\frac{-1}{1-z}\right) + \frac{1}{2(1-z/2)}$$

$$= -[1+z+z^2 - \cdots + z^n + \cdots] + \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2 + \cdots + \left(\frac{z}{2}\right)^n + \cdots\right]$$

$$= \sum_{n=0}^{\infty}\left(-z^n + \frac{1}{2}\left(\frac{z}{2}\right)^n\right) = \sum_{n=0}^{\infty}\left[\left(\frac{1}{2^{n+1}}-1\right)z^n\right]$$

Address of A[j][k] = B + W x [N * (j - lr) + (k - lc)]

k - column subscript, W - storage size of one element.

lr - lower unit of row ; lc - lower unit of column

(If not given lr = 0, lc = 0), m - no. of rows ; n - no. of colms.

B - Base address

If starting address is 1000 then it is 2 byte

Address of A[j][k] = B + W * (m * (k - lc) + j - lr)

A(2)(1) = 1000 + 2 * [2 * (2) + 1] = 1010

A(1)(0) = 1000 + 2 * (3 * 0 + 1) = 1002

AL = Byte [DS · ESI]   AX = word [DS · ESI]

ESI = ESI 1   ESI = ESI 2

EAX = dword [DS · ESI]

ESI = ESI 4

~~Cmpsx~~

Continuation :

(b) $|z| > 1 \Rightarrow \left|\frac{1}{z}\right| < 1$

$|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{z(1-1/z)} - \frac{1}{2\left(\frac{z}{2}-1\right)}$$

$$= \frac{-1}{z(1-1/z)} + \frac{1}{2(1-z/2)}$$

$|1/z| < 1$ $|z/2| < 1$ Apply taylors series

$$f(z) = -\frac{1}{z}\left(1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\cdots+\left(\frac{1}{z}\right)^n\cdots\right)+\frac{1}{2}\left(1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2\cdots+\left(\frac{z}{2}\right)^n+\cdots\right)$$

$$= \sum_{n=0}^{\infty} -\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1}\cdot z^n$$

ⓒ $|z|>2$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z\left(1-\frac{1}{z}\right)} - \frac{1}{z\left(1-\frac{2}{z}\right)}$$

$\Rightarrow \left|\frac{2}{z}\right| < 1$

$\Rightarrow \left|\frac{1}{z}\right| < 1$

$\left|\frac{1}{z}\right|<1$; $\left|\frac{2}{z}\right|<1$

$$= \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}\cdots+\frac{1}{z^n}\cdots\right) - \frac{1}{z}\left(1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2\cdots\right)$$

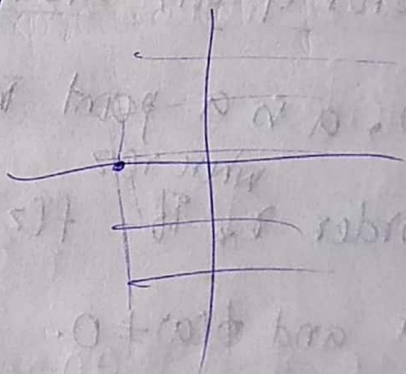
$$f(z) = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty}\frac{1-2^n}{z^{n+1}}$$

Q) Find the Laurent series for $f(z) = \dfrac{1}{(z+1)(z+2)}$ about $z=-2$.

ⓐ
$$f(z) = \frac{1}{(z+1)(z+2)}$$

$$= \frac{1}{z+1} - \frac{1}{z+2}$$

$$= \frac{1}{(z+2)-1} - \frac{1}{z+2}$$



Function $f(z)$ is analytic, inside and on the boundary of region $z=-1$ to $z=-2$ bounded by two concentric circles.

So, it is centered at 0. $\Rightarrow a=0$

It can be expressed as laurent series. (DO IT AS HOMEWORK)

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Q) Expand $\underline{\cos z}$ in taylor series about $z = \pi/4$.

$f(z) = \cos z$ $\quad$ $f'(z) = -\sin z$ $\quad$ $f'(\pi/4) = -1/\sqrt{2}$

$f(\pi/4) = 1/\sqrt{2}$ $\quad$ $f''(z) = -\cos z$ $\quad$ $f''(\pi/4) = -1/\sqrt{2}$

$f'''(z) = \sin z$ $\quad$ $f'''(\pi/4) = 1/\sqrt{2}$

$$f(z) = \frac{1}{\sqrt{2}} + (z-\pi/4)\cdot\left(\frac{-1}{\sqrt{2}}\right) + \frac{(z-\pi/4)^2}{2!}\left(\frac{-1}{\sqrt{2}}\right) + \frac{(z-\pi/4)^3}{3!}\left(1/\sqrt{2}\right) ---$$

$$= \frac{1}{\sqrt{2}}\left[1 - (z-\pi/4) - \frac{(z-\pi/4)^2}{2!} + \frac{(z-\pi/4)^3}{3!} --- \right]$$

HW: $f(z) = \dfrac{\sin z}{z-\pi}$ about $\underline{z = \pi}$

$f(z)$ is not analytic at $z = \pi$.

Zeros of an Analytic function: Let $f(z)$ is an Analytic f^n in region D, a is a point in region D then a is said to be a zero of order r, if $f(z) = (z-a)^r \phi(z)$ where $r \in \mathbb{Z}$ where $\phi(z)$ is Analytic at a and $\phi(a) \neq 0$.

① $f(z) = \sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} ---$

$$= z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} ---\right) \qquad \underline{\phi(0) = 1} \neq 0 \checkmark$$

$(z-0)^1 \phi(z)$

0 is the zero of order 1.

② $f(z) = z^2 \sin z$

$$= z^2\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \right) = z^3\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \right) = (z-0)^3\left(\phi(z)\right)$$

$$\phi(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \Rightarrow \phi(0) = 1 \neq 0.$$

Here 0 is the zero of $f(z)$ of order 3

③ $f(z) = \frac{z^3 - 1}{z^3+1} = 0$

$\Rightarrow z^3 - 1 = 0 \Rightarrow z = 1, \omega, \omega^2$

$\phi(z) = 1/z^3+1 \Rightarrow \phi(1) = 1/2$; $\phi(\omega) = 1/2$; $\phi(\omega^2) = 1/2$

$\Rightarrow$ 1 is zero of order 1 for $f(z)$ similarly ω, ω^2

④ $f(z) = (z-2)^5 (z-3)^4$ 2 is zero of order 5 & 3 is zero of order 4.

<u>Singularities of a f^n</u>:

A point "a" is called a singular point for a singularity of a f^n $f(z)$ if $f(z)$ is not analytic at point "a". For example:

1) $f(z) = 1/z-2 \Rightarrow z = 2$ is a singular point

2) $f(z) = \frac{1}{(z)(z-i)} \Rightarrow z = 0, i$ are singular points

A singular point $z=a$ of a f^n $f(z)$ is called "<u>isolated singular point</u>" if there exists a circle with centre "a", which contains no other singular points of $f(z)$.

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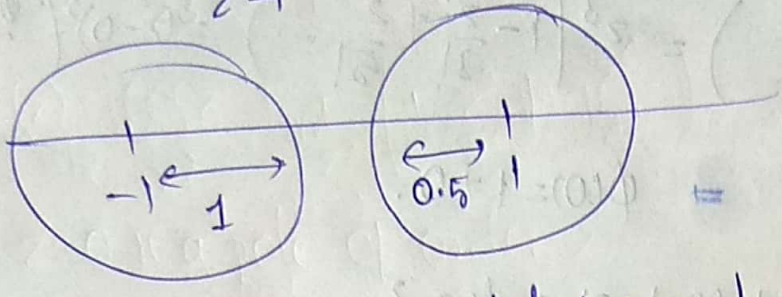
Ex: $f(z) = \frac{z}{z^2-1}$, $z = \pm 1$ are singular points.



Hence $-1, 1$ are isolated singular points of $f(z)$

$f(z) = \frac{1}{\sin \pi z}$, z = all integers. are singular points.

If we draw circle with radius 0.5 about a sing. point then we have no other singular points in the rgn of other than itself.

so, z = all integers are isolated singular points.

25/2/19 **Removable Singularity:** An isolated singularity $z=a$ is said to be removable if the principle path of $f(z)$ at $z=a$ has no terms.

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n} = 0$$

i.e if a is an removable singularity for $f(z)$ then the Laurent series expansion of $f(z)$ is given by:

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 - -$$

$$\lim_{z \to a} f(z) = a_0.$$

For Example, $f(z) = \sin z / z$ $z=0$ is an isolated singular.

$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} - - -\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} -$

so, $z=0$ is an removable singularity.

$\lim_{z \to 0} \frac{\sin z}{z} = 1$

Q) $f(z) = \frac{z - \sin z}{z^3} = \frac{1}{z^3}\left(\cancel{z} - \cancel{z} + \frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - -\right)$

Singularity: $z=0$ $= \frac{1}{3!} - \frac{z^2}{5!} - - -$

$z=0$ is an removable singularity ; $\lim_{z\to 0} f(z) = 1/6$

Def: let a be an isolated singularity of $f(z)$. Then "a" is called pole of order k if the principle part of $f(z)$ at $z=a$ has finite no. of terms.

i.e the principle part of $f(z)$ at $z=a$ is given by:

$$\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \text{——} \frac{a_{-r}}{(z-a)^r}, \quad a_{-r} \neq 0.$$

Then we say that a is a pole of order r.

→ A pole of order 1 is called simple pole

→ A pole of order 2 is called double pole

① $f(z) = \frac{e^z}{z}$ $z=0$ singular isolated.

$f(z) = \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} - -\right)\frac{1}{z} = \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \text{——}$

$= \frac{1}{(z-0)} + 1 + \frac{z}{2!} + \frac{z^2}{3!} - -$ order: 1

$z=0$ is pole of order 1.

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② $f(z) = \frac{e^z}{z^3}$ $z=0 \rightarrow$ pole of order 3

③ $f(z) = \frac{\cos z}{z^2} = \frac{1}{z^2}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} \cdots\right) = \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} \cdots$

$z=0$ is a pole of order 2.

<u>Essential Singularity</u>: Let a be an isolated singularity of $f(z)$. Then The point a is called essential singularity of $f(z)$ at $z=a$ if the principle part of $f(z)$ at $z=a$ has an infinite no. of terms.

Ex: $f(z) = e^{1/z} = 1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 \cdots$

$z=0$ is isolated singularity. and it has infinity terms.

So, $z=0$ is Essential singularity.

<u>Residue</u>: Let "a" be an isolated singularity of $f(z)$. Then the residue of $f(z)$ at "a" is defined to be the coefficient of $1/z-a$ in the laurent series expansion of $f(z)$. about point a.

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}$$

Coefficient of $(z-a)^{-1}$ in $f(z)$ is a_{-1}

$$a_{-1} = \operatorname*{Res}_{z=a} f(z) = \frac{1}{2\pi i} \oint_C f(z)\,dz$$

<u>Residue Theorem</u>: If $f(z)$ is Analytic, At all points inside and on simple closed curve C except at finite number of isolated singular points within C then, ---

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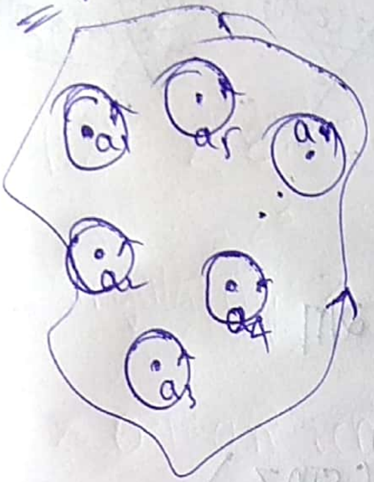
$\oint_C f(z)dz = 2\pi i$ [sum of Residues at singular points within C]

$= 2\pi i \left[\sum_{k=1}^{n} \operatorname{Res}_{z=a_k} f(z) \right]$

proof:



By Cauchy's Theorem, we can write as:

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \dots + \oint_{C_n} f(z)dz$$

$$\frac{1}{2\pi i}\oint_C f(z)dz = \frac{1}{2\pi i}\oint_{C_1} f(z)dz + \frac{1}{2\pi i}\oint_{C_2} f(z)dz + \dots + \frac{1}{2\pi i}\oint_{C_n} f(z)dz$$

$$\frac{1}{2\pi i}\oint_C f(z)dz = \operatorname{Res}_{z=a_1} f(z) + \operatorname{Res}_{z=a_2} f(z) \dots + \operatorname{Res}_{z=a_n} f(z)$$

$$\Rightarrow \oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=a_k} f(z).$$

Calculation Of Residues: If $f(z)$ has a pole of order m at $z=a$, then

$$\operatorname{Res}_{z=a} f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \left\{ \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z) \right\}$$

$z=a$ is simple pole $\Rightarrow m=1$

$$\operatorname{Res}_{z=a} f(z) = \frac{1}{0!} \lim_{z\to a} \left\{ \frac{d^0}{dz^0} (z-a) f(z) \right\} = \lim_{z\to a} [(z-a)f(z)]$$

Evaluate: $\int_{|z|=2} \frac{\sin z}{z\cos z} dz$ $\boxed{z=0,\ z=\pm\pi/2}$, $\pm 3\pi/2 \dots$ are isolated singularities

points inside $|z|=2$

$f(z) = \frac{\tan z}{z} = \frac{1}{z}(1 + \dots) = 1/z \Rightarrow z=0$ is pole of order "1"

$\Rightarrow \operatorname{Res}_{z=0} f(z) = \lim_{z\to 0} z f(z) = \cancel{z}\cdot\frac{\tan z}{\cancel{z}} = \lim_{z\to 0} \tan z = 0$

$\operatorname{Res}_{z=\pi/2} f(z) = \lim_{z\to\pi/2} (z-\pi/2) f(z) = \lim_{z\to\pi/2} (z-\pi/2)\left(\frac{\tan z}{z}\right)$

$= -2/\pi$

$\lim_{z\to\pi/2} (z-\pi/2)\left(\frac{\sin z}{z\cos z}\right)$

$= \frac{(\cos z)(z-\pi/2) + \sin z}{\cos z - z\sin z} = \frac{1}{-\pi/2} = -2/\pi$

$\operatorname{Res}_{z=-\pi/2} f(z) = \lim_{z=-\pi/2} (z+\pi/2) f(z) = \lim_{z\to-\pi/2} (z+\pi/2)\left(\frac{\sin z}{z\cos z}\right)$

$= \lim_{z\to-\pi/2} \frac{\cos z(z+\pi/2)+\sin z}{\cos z - z\sin z} = \frac{-1}{+\frac{\pi}{2}(-1)} = 2/\pi$

$\oint_{|z|=2} \frac{\sin z}{z\cos z}\,dz = 0 + \frac{2}{\pi} - \frac{2}{\pi} = 0$

② $\oint_{|z-3|=3} \frac{e^z}{(z+1)^2}\,dz$ $= 2\pi i f'(a)$ $\quad f(z) = e^z$ Since $f(z)$ is Analytic Ans wherein the curve "C" $\Rightarrow \oint_C f(z)\,dz = 0$ C-I theorem

singularity point $z=-1$

$f(a)$ $\quad f'(z) = e^z$, $f'(-1) = e^{-1}$ $\quad \Rightarrow \oint_{|z-3|=3} \frac{e^z}{(z+1)^2} = 2\pi i\, e^{-1}$

③ $\oint_{|z-3|=3} \frac{e^z}{(z-1)^2}$ $\quad z=1$ is isolated singularity.

$z=1$ pole of order 2

7) $\int_{|z|=2} \frac{2z-1}{z(z+1)(z-3)} dz$ $z=0$; $\cancel{z=3}$; $z=-1$ ✓
✓ ↳ Not inside.

Mapping , Integration , Laurent & Taylor Series for T2