31/1/19 If $T \in L(V)$ and B_1 and B_2 ordered basis of V, then $[T]_{B_2}$ can be written as $P^{-1}[T]_{B_1}P$ for some invertible matrix P.

Ex: if $[T]_B = 0$, Then $T=0$

if $[T]_B = I$ Then $T = I_d$

Q) Given a $T \in L(V)$, Is it possible to find a basis B of V such that $[T]_B$ is diagnal. (Diagonalisation of Matrices)

Q) Given a matrix (square) Q, Is it possible to find a matrix P so that $P^{-1}QP$ is diagnal?

$$\underbrace{P^{-1} \overbrace{Q}^{[T]_{B_1}} P}_{[T]_{B_2}}$$

→ ~~Let B_1, B_2~~

→ Let $B = \{v_1, v_2 -- v_n\}$ be an ordered basis of V if $[T]_B = \begin{bmatrix} c_1 & 0 & 0 & 0 \\ \vdots & c_2 & \cdots & - \\ \vdots & \vdots & \ddots & c_n \end{bmatrix}$ is diagonal, then $Tv_i = c_i v_i$, $i = 1, 2, \cdots n$

Def: A non-zero vector v is said to be an eign vector of $T \in L(V)$ and $\exists$ a scalar "c" such that $Tv = cv$

Q) Given $T \in L(V)$, is it possible to get a Basis B whose elements are eign vectors.

Def: If $T \in L(V)$ ~~and~~ if λ is an eigenvalue of T, then $\{v : Tv = \lambda v\}$ is called the eigen space corresponding to λ.

Called a subspace of V.

Ex: The eigen space is called a subspace of V.

Theorem: If $v_1, v_2 .. v_k$ are eigen values of $T \in L(V)$ coresponding to distinct eigen value $\lambda_1, \lambda_2 - \lambda_k$ respectively, then $\{v_1, v_2 - v_k\}$ is

Linearly independent.

Proof: Suppose $v_1, v_2, v_3 \ldots v_k$ are linearly dependent. Let $m \in N$ so that $\{v_1, v_2 \ldots v_m\}$ is linearly independent But $\{v_1, \ldots v_m, v_{m+1}\}$ are linearly dependent.

Hence there exists scalars $c_1, c_2 \ldots c_{m+1}$ (not all zero) such that

$$c_1 v_1 + c_2 v_2 \ldots c_m v_m + c_{m+1} v_{m+1} = 0 \longrightarrow (1)$$

Also By Applying T to (1) we get:

$$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \ldots c_m \lambda_m v_m + c_{m+1} \lambda_{m+1} v_{m+1} = 0 \longrightarrow (2)$$

$\lambda_{m+1}(1) - (2)$: $c_1(\lambda_{m+1} - \lambda_1) v_1 + \cdots + c_m(\lambda_{m+1} - \lambda_m) v_m = 0$

$\Rightarrow c_1(\lambda_{m+1} - \lambda_1) = c_2(\lambda_{m+1} - \lambda_2) = c_3(\lambda_{m+1} - \lambda_3) \ldots = c_m(\lambda_{m+1} - \lambda_m) = 0$

(Since $\{v_1, v_2, v_3 \ldots v_m\}$ are linearly independent)

$\Rightarrow c_1 = c_2 = c_3 \ldots c_m = 0$

(Since $\lambda_{m+1} \neq \lambda_1, \lambda_2 \ldots \lambda_m$, Given that they are distinct)

(1) $\Rightarrow c_{m+1} \cdot v_{m+1} = 0$

$\Rightarrow c_{m+1} = 0$, ($\because$ v_{m+1} is an eign vector $\Rightarrow v_{m+1} \neq 0$)

$\times$ So our Assumption that $v_1, v_2 \ldots v_k$ are linearly independent is wrong. So, $v_1, v_2 \ldots v_k$ are linearly independent.

* **Invariant subspace:** $T \in L(V)$ a subspace W of V is said to be T invariant if $T(v) = w$ $\forall$ $v \in W$

↳ Invariant

For every element of W, Applying T on it, doesn't change w.

Proof: If w is T-invariant, we can restrict T to w to get an operator $T|W \in L(w)$ $\forall$ $v \in W$, $T|W(v) = T(v)$

→ Is an Eigen space a T-invariant subspace? Ans: ✓ But vice versa is not true

Ex: $T(x,y) = (-y, x)$ $\quad T: R^2 \to R^2 \quad B = \{(1,0),(0,1)\}$

$[T]_B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ✓ invertible $\qquad -y, x = (c_1, c_2)$

$0, 1 = c_1, c_2$

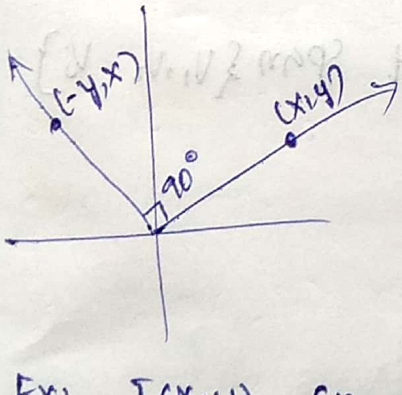
$-1, 0 = c_1, c_2$

Find all eigen vectors of T

$T(x,y) = (\lambda x, \lambda y) = (-y, x)$ $\qquad \lambda y = x ; -y = \lambda x \Rightarrow \cancel{\lambda = -1}$ / $x = 0, y = 0$

But $(0,0)$ is not an Eigen vector.

$\therefore$ This T operator doesn't have even single eigen vector.



$\left.\begin{matrix} y/x \\ -x/y \end{matrix}\right\} -1$ (⊥ar) $\quad$ This Transformation (operator) rotates the plane i/p

But for eigen vectors ⇒ o/p should be expansion/compression of given vectors.

Ex: $T(x,y) = (x+2y, 2x+y)$ $\qquad x+2y = \lambda x \rightarrow 2x+4y = 2\lambda x$

$T(x,y) = (\lambda x, \lambda y)$ $\qquad 2x+y = \lambda y \quad 2x+y = \lambda y$

$y = \dfrac{2\lambda x - \lambda y}{3}$ ⇒ $(3+\lambda)y = 2\lambda x$; $3x = 2\lambda y - \lambda x$ $\quad \dfrac{2\lambda}{3+\lambda} = \dfrac{3+\lambda}{2\lambda}$

$(3+\lambda)x = 2\lambda y$

$2\lambda = 3+\lambda$; $2\lambda = -(3+\lambda)$

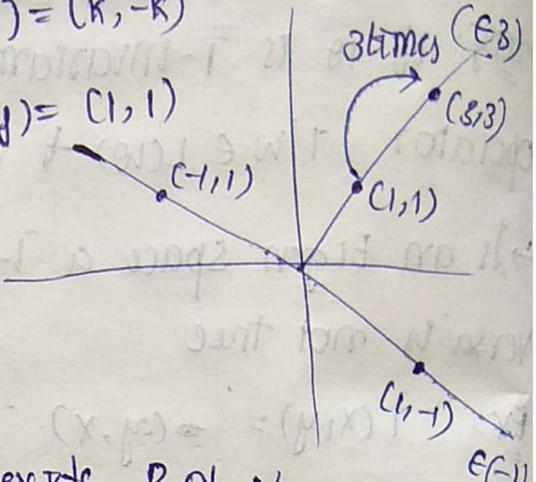
$\boxed{\lambda = 3}$; $\boxed{\lambda = -1}$

$T(x,y) = (x+2y,\ 2x+y) = (-x,-y) \Rightarrow (x,y) = (k,-k)$

& $(x+2y,\ 2x+y) = (3x,3y) \Rightarrow (x,y) = (1,1)$

$T(1,1) = 3(1,1)$; $T(1,-1) = -1(1,-1)$

Diagnalisible → Eigenvector vector should. Exist



* Upper triangulisable: $T \in L(V)$ if there exists B of V such that $[T]_B$ is upper triangular.

let $T \in L(V)$ then following are equivalent.

i) T is upper triangulisable [$\exists$ a basis such that $[T]_B$ is upper triangular]

ii) $\exists$ a basis $B = \{v_1, v_2 - v_n\}$ of V such that $T(v_j) \in \text{span}\{v_1, v_2 \ldots v_j\}$ for each v_j.

iii) $\exists$ a basis $B = \{v_1, v_2 - v_n\}$ of V such that $\text{span}\{v_1, v_2 - v_j\}$ is T-invariant for each j.

proof: Discussed Tomorrow

Continued on Next page

7/2/19 Theorem: let $T \in L(V)$. Then

① T is a upper ~~triangular~~ triangnalisable (That is there exits a Basis $B \ni [T]_B$ is upper triangle)

② $\exists$ a Bases $B = \{v_1, v_2 \ldots v_m\}$ Such that $T_{v_j} \in$ span $\{v_1, v_2 \ldots v_j\}$ for each j.

③ There exists a basis $B = \{v_1, v_2 \ldots v_n\}$ of V such that span $\{v_1, v_2 - v_j\}$ is T-invariant for each j.

proof: ① Assume (1). let $B = \{v_1, v_2 \ldots v_m\}$ be a Basis of $V \ni [T]_B = (a_{ij})_{n \times n}$ whose $a_{ij} = 0$ if $i > j$ =

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & a_{32} & \cdots \\ 0 & 0 & 0 & \cdots a_{nn} \end{bmatrix}$$

$$\Rightarrow T_{v_j} = \sum_{i=1}^{n} a_{ij} v_j = a_{1j} v_1 + a_{2j} v_2 + a_{3j} v_3 \cdots + a_{nj} v_n$$

for $j = 1, 2 \ldots n$

$$\Rightarrow T_{v_j} \in \text{span}(v_1, v_2 \ldots v_j) \Rightarrow ②$$

② Assum ②

let $v \in$ span $\{v_1, v_2 \ldots v_k\}$

$$\Rightarrow v = \sum_{i=1}^{n} c_i v_i \Rightarrow T_v = \sum_{i=1}^{k} c_i T(v_i)$$

For each $i = 1, 2, \ldots k$, we know that $T.v_i \in \text{span}(v_1, v_2 - v_i)$
$\subset \text{span}(v_1, v_2 \ldots v_k)$

Here Tv; being a linear combination of $Tv_1, Tv_2, \ldots Tv_k$ also belongs to span $[v_1, v_2 \ldots v_k]$.

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That is span $\{v_1, v_2 \dots v_k\} \Rightarrow Tv \in$ span $\{v_1, v_2 \dots v_k\}$ Hence ③ follows.

③ Assume ③.

Since $v_j \in$ span $\{v_1, v_2 \dots v_j\}$, by (3)

$Tv_j \in$ span of $v_1, v_2, \dots v_j \Rightarrow Tv_j = \sum_{i=1}^{j} a_{ij} v_i$

$$\Rightarrow [T]_B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a \\ 0 & a_{22} & \cdots & \\ 0 & 0 & \cdots & \\ & & & \end{bmatrix} \Rightarrow \text{upper Triangle ①}$$

<u>NOTATION</u>: If $p(x) = a_0 + a_1 x - a_n x^n$ is a polynomial. ax $T \in L(V)$, Then $p(T)$ denotes the operator, $a_0 I + a_1 T + a_2 T^2 \dots + a_n T^n$. Since $(p.q)(T) = p(T) \cdot q(T)$, The above sentence makes some sense.

<u>Theorem</u>: Let V be a complex vector space and Let $T \in L(V)$. Then T has atleast one eign value (vector)

<u>proof</u>: Let dim $V = n$ let u be a non-zero vector in V. consider the set $\{u, Tu, T^2u, \dots T^n u\} \subset V$

Assume That $u, Tu, T^2u \dots T^n u$ are distinct,
if these elements are distinct & nonzero, since $\# \{u, Tu, \dots T^n u\} = n+1 > n$ so, This set $\{u, Tu, T^2u \dots T^n u\}$ is linearly dependent.

so, $\exists$ complex scalars $c_0, c_1 \dots c_n$; not all zero's.

$\Rightarrow c_0 x + c_1 T x + c_2 T^2 x \dots + c_n T^n x = 0$

v, $p(T)v = 0$ where $p(x) = c_0 + c_1 x + \dots c_n x^n$ is a non zero complex polynomial

$a_1, a_2, \dots a_n \in \mathbb{C}$

v: $p(x) = c_n (x - a_1)(x - a_2) \dots (x - a_n)$

Here $c_n(T-a_1I)(T-a_2I)\cdots(T-a_nI)u=0$

* If $(T-a_nI)u=0$, a_n is an eigen value since $u\neq 0$.

If not, if $(T-a_{n-1}I)(T-a_nI)u=0$, a_{n-1} is eigen value since $(T-a_nI)u\neq 0$.

If not, if $(T-a_{n-1}I)(T-a_{n-2}I)(T-a_nI)u=0$, a_{n-2} is eigen value since $(T-a_nI)(T-a_{n-1}I)u\neq 0$.

⋮

Combining like this, at least one of the numbers $a_1, a_2, \ldots a_n$ is an eigen value

(or) If all $(T-a_iI)$ are injective, then $p(T)=c_n(T-a_nI)\cdots(T-a_1I)$ is also injective ✳ ($\because p(T)u=0$ for $u\neq 0$)

11/2/19 If $\dim V=n$, then for any $u\neq 0$, the set $\{u, Tu, T^2u, \ldots T^nu\}$ is linearly dependent.

Theorem: Let V be a complex vector space as $T\in L(V)$. Then $\exists$ a Basis B of $V \ni [T]_B$ is upper triangular

proof: We prove that by using induction of the dimension of V

① If $\dim V=1$, the result follows trivially (Because every 1×1 matrix is a upper triangular matrix)

② Assume the result for any $T\in L(U)$ if $\dim(V)\leq k$

Consider $T\in L(V)$ for a complex vector space W of dimension $k+1$.

Since W is a complex vectorspace. T has an eigen vector space, say

λ and

consider $W' = \text{Range}(T-\lambda I)$

Since $0 \neq u \in N(T-\lambda I) \Rightarrow \dim R(T-\lambda I) \leq k$

But $v \in W'$ Then $Tv = (T-\lambda I)v + \lambda v \in W'$

Since $(T-\lambda I) \in \text{Range}(T-\lambda I) = W'$ and $\lambda v \in W'$ $\because v \in W'$ & W' is a sub space

Hence W is a T invariant subspace of elementary at most k

Hence we can use the induction hypothesis for $T/W' : W' \to W'$ to get a Basis $B' = \{v_1, v_2 \ldots v_l\}$, $(l \leq k)$, such that $[T/W']$ is upper triangular.

$T/W'\, v_j = Tv_j \in \text{span}\{v_1, v_2 \ldots v_j\}$ for $j = 1, 2 \ldots l$

$\Rightarrow$ Extend this basis to a Basis

$B = \{v_1, v_2 \ldots v_{k+1}\}$ of W'. We know that $Tv_j \in \text{span}\{v_1, v_2 \ldots v_j\}$

for $j = 1, 2 \ldots l$

For $j > l$, $Tv_j = (T-\lambda I)v_j + \lambda v_j$

$\in \text{span}(v_1, v_2, \ldots v_l, v_j\} \subset \text{span}\{v_1, v_2 \ldots v_l, v_{l+1}, v_j\}$

Because, $(T-\lambda I)v_j \in W' = \text{span}(v_1, v_2 \ldots v_l)$. Hence $[T]_B$ is upper triangular

Corollary: Let A be an $n \times n$ complex matrix. Then $\exists$ an invertible $n \times n$ complex matrix P such that $P^{-1}AP$ is upper triangular.

Theorem: Let $T \in L(V)$. If $[T]_B$ is upper triangular, with $\mu_1, \mu_2, \mu_3 \ldots \mu_n$ base of the diagonal entries of $[T]_B$, then T is invertible iff $\mu_j \neq 0$, for $j = 1, 2 \ldots n$.

proof: ① Assume $\mu_l = 0$ for some $j \in \{1, 2 \ldots n\}$. Let $B = \{v_1, v_2 \ldots v_n\}$

Since $[T]_B$ is uppertriangular $\subset$ span $\{v_1, v_2 \ldots v_j\}$ is T-invariant for each $j = 1, 2 \ldots n$.

$\Rightarrow T(\text{span}\{v_1, v_2 \ldots v_{l-1}\}) \subset \text{span}\{v_1, v_2 \ldots v_{l-1}\}$

Since $\mu_l = 0$, $\ni \ldots \in$. $Tv_l \in \text{span}\{v_1, v_2, \ldots v_{l-1}\}$

Hence if $W_j = \text{span}\{v_1, v_2 \ldots v_j\}$, $T(W_l) \subset W_{l-1}$ that is

$T|W_l : W_l \rightarrow W_{l-1}$.

Since $\dim W_{l-1} < \dim W_l$, $\eta(T|W_l) \geq 1$ ($\dim W_l = l$, $\dim(R(T)) \leq \dim W_{l-1} = l-1$)

$\Rightarrow \dim N(T|W_l) = L$

Hence there exists $u \in W_l \ni u \neq 0$, $T|W_l(u) = 0 \Rightarrow T(u) = 0$, $u \neq 0$

$\Rightarrow$ T is not one-one $\Rightarrow$ T is not invertible.

14/2/18. **Theorem:** let $T \in L(V)$ and let B be a basis of V $\ni [T]_B$ is uppertriangular Then T is invertible iff all the diagonal entries of $[T]_B$ are non-zero.

proof: Let $B = \{v_1, v_2 \ldots v_n\}$ if $W_j = \text{span}\{v_1, v_2 \ldots v_n\}$ Then

$T(W_j) \subset W_j$

If the k^{th} diagonal entry is zero, then $Tv_k \in \text{span}\{v_1, v_2 \ldots v_{k-1}\}$

$\Rightarrow T(W_k) \subset W_{k-1}$

$\Rightarrow$ if $u \in W_k$, $u \neq 0 \ni T|W_k^{(u)} = 0 \Rightarrow T(u) = 0$

$\Rightarrow$ T is not one-one $\Rightarrow$ T is not invertible.

$$\begin{bmatrix} a_1 & & & \\ 0 & a_2 & & \\ 0 & 0 & \ddots & \\ 0 & 0 & 0 & a_n \end{bmatrix} \begin{bmatrix} \cdot \end{bmatrix} = 0$$

Conversely Assume that T is not invertible. Here $\exists u \in V$ such that $u \neq 0$ $T(u) = 0$.

Let $u = c_1v_1 + c_2v_2 + \cdots c_kv_k$. Where $1 \leq k \leq n$ & $c_k \neq 0$.

$0 = T(u) = c_1 T(v_1) + c_2 T(v_2) \cdots c_k T(v_k)$ k and c_k are non-zero.

Since $c_k \neq 0$,

$$T(v_k) = -\frac{c_1}{c_k} T(v_1) - \frac{c_2}{c_k} T(v_2) - \cdots \frac{c_{k-1}}{c_k} T(v_k).$$

This implies $T(v_k) \in W_{k-1} = \text{span}\{v_1, v_2 \cdots v_{k-1}\}$

Since, $T(v_j) \in W_j$, $j = 1, 2 \cdots (k-1)$ and $W_1 \subset W_2 \cdots \subset W_{k-1}$

$$[T v_k]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{k-1} \\ 0 \\ 0 \\ \vdots \end{bmatrix} \Rightarrow$$ The k^{th} Diagonal entries of $[T]_B$ is zero.

Corollary: Let $T \in L(V)$ and let B be a basis of V such that $[T]_B$ is uppertriangular. Then the eigenvalues of T are precisely the diagonal entries of the $[T]_B$.

$T v_j = \sum_{i=1}^{n} a_{ij} v_i$

proof: if $[T]_B = A$, Then $[T - \lambda I]_B = A - \lambda I$

Identity operator Identity Matrix

$(T - \lambda I) v_j = \sum_{i=1}^{n} a_{ij} v_j - \lambda v_j$

$= \sum^{n} a_{ij}$

λ is an Eigen value of T if and only if $(T - \lambda I)$ is not invertible iff $[T - \lambda I]_B$ has 0 in its diagonal if and only if $(A - \lambda I)$ has a zero in its diagonal if and only if λ appears in the diagonal of A $A = [T]_B$. $\therefore$ The diagonal values of $[T]_B$ are precisely the eigen values of T provided that representation is upper Δ luar matrix.

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Corollary: Let $T \in L(V)$. If $[T]_B$ is upper triangular and if all the diagonal entries of $[T]_B$ are distinct, then T is diagonalizable.

Since all the diagonal entries are distinct so, we have all the n eigen values distinct and independent. So it forms a basis.

If $p(x) = a_0 + a_1 x \cdots + a_n x^n$ is a real polynomial (i.e $a_i \in \mathbb{R}$) only if

if $(\alpha + i\beta)$ is a root of p, then $(\alpha - i\beta)$ is also a root of p

i.e $(x - (\alpha + i\beta))(x - (\alpha - i\beta)) = (x-\alpha)^2 + \beta^2 = x^2 - 2\alpha x + \alpha^2 + \beta^2$ is a factor

$\Rightarrow$ of $p(x)$ for $((x-\alpha)^2 + \beta^2)(Q(x)) = p(x)$ i.e $\alpha^2 - 4\beta < 0$.

i.e Any real polynomial, can be written as the product linear factor $(x - a_i)$ and Quadratic factor with $x^2 + a_j x + b_j \mid a_j^2 - 4b_j < 0$

If $(T - a_j I)u = 0$ for $u \neq 0$, a_j is an Eigen ~~vector~~ value of T, if u is an eigen vector.

if $(T^2 + a_j T + b_j I)u = 0$, then $T^2 u = -a_j Tu - b_j u \in \operatorname{span}\{u, Tu\}$ Here $W = \operatorname{span}\{u, Tu\}$ is T-invariable $cu = c_1 u + c_2 Tu$,

$Tv = c_1 Tu + c_2 T^2 u = c_1 Tu + c_2(-a_j Tu - b_j u) \in W$.

If $T \in L(V)$, V is a real vector space then V has T-invariant subspace of Dimension atmost 2.

18/2/19 Def: Let V be a vector space over F ($\mathbb{R}$ or $\mathbb{C}$). Then an inner product on V is a function $\langle \cdot \rangle ; V \times V \to F$ satisfying

a. $(b \times c)$

① $\langle cu_1 + u_2, v \rangle = c\langle u_1, v \rangle + \langle u_2, v \rangle$ (for any $c \in F$ and

$u_1, u_2, v \in V$) (~~for~~ ~~any~~ linearity in 1st variable)

ii) $\langle u, v \rangle = \overline{\langle v, u \rangle}$ for any $u, v \in V \to$ conjugate symmetry

iii) $\langle u, u \rangle \geq 0 \ \forall u \in V$ and +ve

$\langle u, u \rangle = 0$ iff $u = 0$

Remark: $\langle u, cv_1 + v_2 \rangle = \bar{c} \langle u, v_1 \rangle + \langle u, v_2 \rangle$ (TRUE)

proof:

$\langle u, cv_1 + v_2 \rangle = \overline{\langle cv_1 + v_2, u \rangle} = \bar{c}\,\overline{\langle v_1, u \rangle} + \overline{\langle v_2, u \rangle}$

$= \bar{c} \langle u, v_1 \rangle + \langle u, v_2 \rangle$ ✓✓

~~Note~~

Eg: ① $V = \mathbb{R}^n$ $\quad \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$, the standard inner product on $\mathbb{R}^n$

② $V = \mathbb{C}^n$ $\quad (a+ib)(c+id)$

$ac - bd + i(ad + bc)$

~~$\langle z, w \rangle = \sum_{i=1}^{n} z_i w_i$~~

$\langle z, w \rangle = \sum_{i=1}^{n} z_i \bar{w}_i$ is the standard inner product on $\mathbb{C}^n$

③ $V = \mathbb{C}$ over $\mathbb{R}$

$\langle z, w \rangle = \text{Im}(z\bar{w})$ $\quad c \in \mathbb{R}$

$\langle cz_1 + z_2, w \rangle = \text{Im}((cz_1 + z_2)\bar{w}) = \text{Im}(cz_1\bar{w} + z_2\bar{w})$

$= c\,\text{Im}(z_1\bar{w}) + \text{Im}(z_2\bar{w})$

$c\langle z_1, v \rangle = c(\text{Im}(z_1\bar{w}))$

$\langle z_2, v \rangle = \text{Im}(z_2\bar{w})$

TRUE : ①

$\langle u, u \rangle = \text{Im}(u\bar{u}) = 0$ where $u = x + iy$

need not necessarily 0 : so, prop ③ violation

$\Rightarrow \langle z, w \rangle = \text{Im}(z\bar{w})$ cannot be a inner product on V.

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③ $V = \mathbb{C}$ over $\mathbb{R}$

① st. is satisfied on the previous one do.

② nd: $Re(w\bar{z}) = Re(\overline{\bar{w}z}) \Rightarrow$ conjugate of $Re(\bar{w}z)$

$\langle u,u\rangle = 0 = x^2+y^2 \rightarrow$ If and only if $u=0$

So, ⊛ is an Innerproduct on V
$\langle u,v\rangle$

④ $V = M_n(\mathbb{C})$, the space of $n\times n$ complex matrices.

$\langle A,B\rangle = tr\,(A B^*)$ $\qquad A^* = \overline{(A^{\text{Transpose}})}$

① $\langle cu_1+u_2, v\rangle =$ ~~Transpo~~ $Trace\,((cu_1+u_2)v^*)$

$= Trace\,(cu_1 * v^*) + Trace\,(u_2 * v^*)$

$c\in\mathbb{R}$

$= c\,Trace\,(u_1 * v^*) + Trace\,(u_2^* v^*)$
multiplication

② $\langle A,B\rangle = trace\,(AB^*)$

$\langle B,A\rangle = Trace\,(BA^*) = Trace\,((AB^*)^*)$

$=$ conjugate of $\langle A,B\rangle$ $\qquad$ $Trace(A^*)$ is conjugate of $Trace(A)$.

③ Obvious.

So, This ~~forms~~ is a ~~linear~~ inner product on V.

⑤ $V = C([0,1]) = \{f : [0,1]\to\mathbb{C},\ f \text{ is continuous}\}$

$$\langle f,g\rangle = \int_0^1 f(x)\cdot\overline{g(x)}\cdot dx.$$

i) $\langle cf_1+f_2, g\rangle = \int_0^1 cf_1(x)\cdot\overline{g(x)}\cdot dx + \int_0^1 f_2\,\overline{g(x)}\cdot dx$

$= c\langle f_1, g\rangle + \langle f_2, g\rangle$

ii) $\langle f, g\rangle = \int_0^1 f(x)\,\overline{g(x)}\cdot dx = \overline{\langle g, f\rangle} = \int_0^1 \overline{g(x)\cdot\overline{f(x)}}\cdot dx$

$= \int_0^1 g(x)\cdot f(x)\cdot dx = \langle f, g\rangle$

if $g(x) \geq 0, \forall [0,1]$; $\int_0^1 g(x)dx = 0 \Rightarrow g(x) = 0, \forall x \in [0,1]$;

(prove it on freetime).

Def: i(i) $\langle f, g\rangle = \int_0^1 f(x)\cdot\overline{g(x)}\cdot dx$

Def: A vectorspace V with an inner product defined on it is called as inner product space.

Def: If V is an innerproduct space, $u, v \in V$ then u, v are said to orthogonal if $\langle u, v\rangle = 0$.

$\hookrightarrow (u \perp v)$

Ex: If $u \perp v, \forall v \in V$, then $u = 0$.

$\Updownarrow$ u also belongs to $V \Rightarrow \langle u, u\rangle = 0$

$\langle u, v\rangle = 0$ ~~[illegible]~~ $\Rightarrow \boxed{u=0}$

Def: Let V be an vectorspace. A norm on V is a function $\|\cdot\|$: $V \to [0, \infty)$ such that

1. $\|cv\| = |c|\,\|v\|, \forall c \in \mathbb{F}, u \in V$
2. $\|u+v\| \leq \|u\| + \|v\|$
3. $\|u\| = 0$ if and only if $u = 0$.

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Def: A vectorspace V with a norm defined on it is called a normed linear space

Exercise: If V is an inner product space, V is a normed linear space with $\|v\| = \langle v, v\rangle^{1/2}$

① $\|cv\| = \langle cv, cv\rangle^{1/2} = (|c|^2 \langle v, v\rangle)^{1/2}$ ③ $\|u\| = 0$ then $u = 0$

④ $\|u\|$ is not negitive

$= |c| \langle v, v\rangle^{1/2} = |c| \|v\|$

② $\|u+v\| = (\langle u+v, u+v\rangle)^{1/2}$

$= (\langle u,u\rangle + \langle v,v\rangle + \langle u,v\rangle + \langle v,u\rangle)^{1/2}$

$= (\|u\|^2 + \|v\|^2 + \langle u,v\rangle + \langle v,u\rangle)^{1/2}$

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$\langle u,u\rangle^{1/2} = \|u\|$

Theorem: if $v \perp u$, then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ (pythogorans)

$\|u+v\|^2 = \langle u+v, u+v\rangle$

$= \langle u,u\rangle + \langle v,v\rangle + \langle u,v\rangle + \langle v,u\rangle$

$= \|u\|^2 + \|v\|^2 + \langle u,v\rangle + \langle v,u\rangle$

Since $v \perp u \Rightarrow \langle u,v\rangle = \langle v,u\rangle = 0$

$\|u+v\|^2 = \|u\|^2 + \|v\|^2 \Rightarrow \therefore \|u+v\|^2 = \|u\|^2 + \|v\|^2$ if $u \perp v$

* Given u, v in V, an in inner product space, write u as $u_1 + u_2$ where $u_1 \perp u_2$ and $v \perp u_2$

$u = u_1 + u_2$

Since $u_1 \perp u_2 \Rightarrow \|u_1+u_2\|^2 = \|u_1\|^2 + \|u_2\|^2 \longrightarrow ①$

$v \perp u_2 \Rightarrow \|v+u_2\|^2 = \|v\|^2 + \|u_2\|^2 \longrightarrow ②$

$\Rightarrow ① - ② \rightarrow \|u_1+u_2\|^2 - \|v+u_2\|^2 = \|u_1\|^2 - \|v\|^2$

$\Rightarrow \|u\|^2 - \|v+u_2\|^2 = \|u_1\|^2 - \|v\|^2 \rightarrow$

$\Rightarrow \|u\|^2 + \|v\|^2 = \|u_1\|^2 + \|v+u_2\|^2 = \|u-u_2\|^2 + \|v+u_2\|^2$

$= \langle u-u_2, u-u_2\rangle + \langle v+u_2, v+u_2\rangle$

$2\|u_2\|^2$

$= \|u\|^2 + \|u_2\|^2 - \langle u, u_2\rangle - \langle u_2, u\rangle$

$=$

$+ \|v\|^2 + \|u_2\|^2 + \langle v, u_2\rangle + \langle u_2, v\rangle$

Remark: $\left\langle u - \frac{\langle u,v\rangle}{\langle v,v\rangle} v, v \right\rangle = 0$

$\hookrightarrow u_2$

Take $u_2 = u - \frac{\langle u,v\rangle}{\langle v,v\rangle} . v$ and $u_1 = \frac{\langle u,v\rangle}{\langle v,v\rangle} . v$ so that $u_1+u_2 = u$
& $u_2 \perp v$.

$\langle u_2, u_1\rangle = \left\langle u - \frac{\langle u,v\rangle}{\langle v,v\rangle} . v, \frac{\langle u,v\rangle}{\langle v,v\rangle} . v \right\rangle$

$= \frac{\langle u,v\rangle}{\langle v,v\rangle} \langle u,v\rangle - \frac{\langle u,v\rangle}{\langle v,v\rangle} \langle v,v\rangle = 0.$

$= \frac{\overline{\langle u,v\rangle}}{\langle v,v\rangle} \langle u,v\rangle - \frac{\overline{\langle u,v\rangle}}{\langle v,v\rangle} . \frac{\langle u,v\rangle}{\langle u,v\rangle} \langle v,v\rangle$

prop 2 of norm proof: $\|u+v\| \le \|u\| + \|v\|$

$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + \langle u, v\rangle + \langle v, u\rangle$

WKT, $\langle v, u\rangle = \overline{\langle u, v\rangle}$ (congugate to each other)

$\cancel{2}\,\text{Re}\langle u, v\rangle \le \cancel{2}\,\|u\|\,\|v\|$

from Cauchy-Schwarz inequality:

$|\langle u, v\rangle| \le \|u\| \cdot \|v\| \Rightarrow \text{Re}\langle u, v\rangle \le \|u\| \cdot \|v\|$

so, $\|u+v\|^2 \le (\|u\| + \|v\|)^2 \Rightarrow \|u+v\| \le \|u\| + \|v\|$

Cauchy-Schwarz inequality proof:

$|\langle u, v\rangle| \le \|u\| \cdot \|v\|$

$\times$(proof: Given u, v. write $u = u_1 + u_2$ where $u_1 \perp u_2$ & $u_2 \perp v$.

$\langle u, v\rangle = \langle u_1 + u_2, v\rangle$)$\times$

$\langle u, v\rangle \langle v, u\rangle \le \langle u, u\rangle . \langle v, v\rangle$

$0 \le \langle u - v, \cancel{u} - v\rangle = \langle u, u\rangle - \langle u, v\rangle - \langle v, v\rangle + \langle v, v\rangle$

$\Rightarrow \langle u, v\rangle + \langle v, u\rangle \le \langle u, u\rangle + \langle v, v\rangle$

$\Rightarrow \|u+v\|^2 = \langle u+v, u+v\rangle = \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle$
$+ \langle v, v\rangle$

$\le \|u\|^2 + \|v\|^2 + \|u\|^2 + \|u\|^2$

— ⊗ - no result we will prove later question

Theorem: In an inner product space, if $\|u\| = \langle u,u\rangle^{1/2}$, then $\|u\|$ is a norm on V.

proof: since $\langle u,u\rangle \geq 0$, $\|u\| \geq 0$

① $\|ku\| = \langle ku, ku\rangle^{1/2} = (k\cdot \bar{k}\langle u,u\rangle)^{1/2} = |k|\langle u,u\rangle^{1/2}$

$= |k|\cdot\|u\|$.

② $\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + \langle u,v\rangle + \langle v,v\rangle + \langle v,u\rangle$

$= \|u\|^2 + \|v\|^2 + \langle u,v\rangle + \langle v,u\rangle$

$= \|u\|^2 + \|v\|^2 + 2Re(\langle u,v\rangle)$

$\leq \|u\|^2 + \|v\|^2 + 2|\langle u,v\rangle| \leq \|u\|^2 + \|v\|^2 + 2\|u\|\cdot\|v\|$

$\leq (\|u\| + \|v\|)^2$

Since $\|u+v\|$, $\|u\| + \|v\|$ are +ve $\Rightarrow$ $\|u+v\| \leq \|u\| + \|v\|$ ✓

③ $\|u\| = 0 \Leftrightarrow \langle u,u\rangle = 0 \Leftrightarrow u = 0$

So, The three properties of norm are proved.

Now let's try to prove cauchy's inequality:

Given u, v, write $u = u_1 + u_2$ such that $u_1 \parallel u_2$ & $u_2 \perp v$.

$u = u_1 + u_2$ such that $u_1 \perp u_2$

$\|u\|^2 = \|u_1 + u_2\|^2 = \langle u_1 + u_2, u_1 + u_2\rangle = \langle u_1, u_1\rangle + \langle u_2, u_2\rangle + \langle u_1, u_2\rangle + \langle u_2, u_1\rangle$

$= \cancel{\|u_1\|^2 + \|u_2\|^2 + 2\|u_1\|\cdot\|u_2\|}$

$= \|u_1\|^2 + \|u_2\|^2 \Rightarrow \|u_1 + u_2\|^2 = \|u_1\|^2 + \|u_2\|^2 \geq \|u_1\|^2$

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$$\|u\|^2 \geq \|u_1\|^2 = \left\| \frac{\langle u,v\rangle}{\langle v,v\rangle} \cdot v \right\|^2 = \left| \frac{\langle u,v\rangle}{\langle v,v\rangle} \right|^2 \|v\|^2$$

NOTE: $\|kv\|^2$

$$\Rightarrow \|u\|^2 \geq \frac{|\langle u,v\rangle|^2}{|\,\|v\|^2|^2} \cdot \cancel{\|v\|^2}$$

$$\Rightarrow \|u\|^2 \cdot \|v\|^2 \geq |\langle u,v\rangle|^2 \Rightarrow |\langle u,v\rangle| \leq \|u\| \cdot \|v\|$$

Hence proved the Cauchy's ineq.

NOTE:

$$\|k \cdot v\| = (\text{mod } k) \cdot \|v\|$$

$$\left\| \frac{\langle u,v\rangle}{\langle v,v\rangle} \cdot v \right\|^2 = \left| \frac{\langle u,v\rangle}{\langle v,v\rangle} \right|^2 \cdot \|v\|^2$$

Def: A set $\{v_1, v_2, \ldots, v_n\}$ is said to be orthogonal set if $\langle v_i, v_j\rangle = 0$ if $i \neq j$. Then ~~the~~ set $\{v_1, v_2, \ldots, v_n\}$ is said to be an orthonormal set if (1) it is orthogonal and if normal

(2) $\|v_i\|^2 = \langle v_i, v_i\rangle = 1, \forall i$

In short, Orthonormal set is $\{v_1, v_2, \ldots, v_n\}$ such that

$$\langle v_i, v_j\rangle = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

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<u>Theorem</u>: Any orthogonal set of non-zero vectors is linearly independent.

suppose orthogonal set is $\{v_1, v_2 \ldots v_n\}$

$c_1v_1 + c_2v_2 \ldots c_nv_n = 0$ (~~if and~~ all $c_i = 0$ $i = 1, \ldots n$ then $v_1, v_2 \ldots v_n$ are linearly independent)

$\Rightarrow$ consider $\langle \sum_{i=1}^{n} c_i v_i, v_j \rangle = \langle 0, v_j \rangle = 0$. (why??)

$\langle 0, v \rangle = \langle 0+0, v \rangle = \cancel{\langle 0, v \rangle} + \langle 0, v \rangle \Rightarrow \langle 0, v \rangle = 0$. ✓

$\langle \sum_{i=1}^{n} c_i v_i, v_j \rangle = \sum_{i=1}^{n} c_i \underbrace{\langle \sum_{i=1}^{n} v_i, v_j \rangle}_{1} = 0$

(for some j)

for all values of i from 1 to n except j we have $\langle v_i, v_j \rangle = 0$

Since $v_1, v_2 \ldots v_n$ belongs orthonormal set

$\Rightarrow \langle v_j, v_j \rangle = 1$

So, $\underbrace{\langle \sum_{i=1}^{j-1} v_i, v_j \rangle}_{0} + \underbrace{\langle v_j, v_j \rangle}_{1} + \underbrace{\langle \sum_{i=j+1}^{n} v_i, v_j \rangle}_{0}$

$= \langle \sum_{i=1}^{n} v_i, v_j \rangle = 1$

So,

$c_i \langle v_j, v_j \rangle = 0$

This Becomes Zero since $\langle v_j, v_j \rangle = 1$.

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$$\left\langle \sum_{i=1}^{n} (c_i v_i), v_j \right\rangle = \langle 0, v_j \rangle = 0$$

$$\left\langle \sum_{i=1}^{n} c_i v_i , v_j \right\rangle = \langle c_1 v_1 + c_2 v_2 - c_n v_n, v_j \rangle \text{ for some } j$$

$$= \langle c_1 v_1, v_j \rangle + \langle c_2 v_2, v_j \rangle - + \langle c_j v_j, v_j \rangle +$$

$$\langle c_{j+1} v_{j+1}, v_j \rangle - - \langle c_n v_n, v_j \rangle$$

$$= c_1 \langle v_1, v_j \rangle + c_2 \langle v_2, v_j \rangle + c_3 \langle v_3, v_j \rangle - -$$

$$c_{j-1} \langle v_{j-1}, v_j \rangle + c_j \langle v_j, v_j \rangle + c_{j+1} \langle v_{j+1}, v_j \rangle$$

$$- - c_n \langle v_n, v_j \rangle$$

$$\left\langle \sum_{i=1}^{n} c_i v_i , \sum_{j=1}^{n} v_j \right\rangle = 0$$

$$= c_1 \left\langle v_1, \sum_{j=1}^{n} v_j \right\rangle + c_2 \left\langle v_2, \sum_{j=1}^{n} v_j \right\rangle + - + c_j \left\langle v_j, \sum_{j=1}^{n} v_j \right\rangle$$

Theorem: If $\{v_1, v_2 \ldots v_n\}$ is a linearly independent set in V, $\exists$ a orthonormal set $\{w_1, w_2 \ldots w_n\}$ such that

$$\text{span}\{v_1, v_2 \ldots v_j\} = \text{span}\{w_1, w_2 \ldots w_j\} \text{ for each } j = 1, 2 \ldots n$$

Proof: Refer Gram-Schmidth theorem.

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Corollary: Let V be a complex inner product space (IPS) and $T \in L(V)$. Then $\exists$ an orthonormal basis of V such that $[T]_B$ is upper triangle matrix:

→ Any linear Transformation on complex ips $[T]_B$ is upper Δ.

• Since V is a complex vectorspace, there exists a basis $B' = \{v_1, v_2 \ldots v_n\}$ such that $[T]_{B'}$ is ~~not~~ upper triangular.

$\Rightarrow Tv_j \in \text{span}\{v_1, v_2, \ldots v_j\}, \forall j = 1, 2, 3 \ldots n$

Apply gram-schmidth process to B' to get an orthonormal Basis.

$B = \{w_1, w_2 \ldots w_n\}$

$\Rightarrow \text{Span}\{w_1, w_2 \ldots w_n\} = \text{span}\{v_1, v_2 \ldots v_n\}$.

$\Downarrow$

Each w_j Belongs to $\text{span}\{v_1, v_2 \ldots v_j\}$ $\quad \left(w_j = \sum_{k=1}^{j} c_k v_k\right.$

$\Rightarrow Tw_j \in \text{span}\{v_1, v_2 \ldots v_j\}$ $\quad \Rightarrow Tw_j = \sum_{k=1}^{j} c_k T(v_k)$

↳ linear combination of $Tv_1 + Tv_2 + \ldots Tv_j$

$\Rightarrow Tw_j \in \text{span}\{v_1, v_2 \ldots v_j\} = \text{span}\{w_1, w_2 \ldots w_j\}$ for $j = 1, 2 \ldots n$

$\Rightarrow [T]_B$ is upper triangular.

Def: let V be a vectorspace over F. A linear Transformation from V to F is called linear functional.

$L(V, \mathbb{F})$ is defined as V^*

Eg ① $V = \mathbb{R}^2$

ⓐ $f(x,y) = x$ ✓
ⓑ $f(x,y) = x - 3y$ ✓ } liner f'al

② $V = P$, polynomials with complex coefficients

$f(a_0 + a_1 x - a_n x^n)$ = sum of the root of f (not linear)

ⓐ $f(a_0 + \cdots a_n x^n) = \sum_{i=1}^{n} a_i$ ✓ linear f'al. $p(0)$, $p'(1)$

$= p(1)$

ⓑ ③

$V = C([0,1])$

$T(f) = f(a)$, $a \in [0,1]$

inner product space

Theorem: Let V be a finite demensional ips. if $f \in V^*$, There exists an unique vector, w in V such that $f(v) = \langle v, w \rangle$, $\forall v \in V$

(Riesz Representation Theorem)

proof: Let $\{v_1, v_2 - v_n\}$ be an orthonormal basis of V; if $u \in V$, $\exists$ $c_1, c_2, c_3 - - c_n$ such that $v = c_1 v_1 + c_2 v_2 \cdots c_n v_n$

Then $\langle v, v_j \rangle = \langle \sum_{i=1}^{n} c_i v_i, v_j \rangle = c_j \langle v_j, v_j \rangle$ 0

$= c_j$

(All $\langle v_k, v_j \rangle = 0$, $k \neq j$)

$\Rightarrow \boxed{v = \langle v, v_1 \rangle v_1 + \langle v, v_2 \rangle v_2 + \langle v, v_3 \rangle v_3 \cdots + \langle v, v_n \rangle v_n}$ *

$\Rightarrow f(v) = \sum_{j=1}^{n} \langle v, v_j \rangle f(v_j) = \langle v, w \rangle$ where w is the unique

? proof

vector, $w = \sum_{j=1}^{n} \overline{f(v_j)} \cdot v_j$

proof: if w_1 and w_2 both satisfies $f(v) = \langle v, w_1 \rangle = \langle v, w_2 \rangle$

$\Rightarrow \langle v, w_1 \rangle = \langle v, w_2 \rangle, \ \forall v \in V$

$\Rightarrow \langle v, w_1 - w_2 \rangle = 0, \ \forall v \in V$

$\Rightarrow w_1 - w_2 = 0$ (Take $v = w_1 - w_2$)

Eg: $f(x,y) = x = \langle (x,y), (1,0) \rangle$

$f(x,y) = x - 2y = \langle (x,y), (1,-2) \rangle$

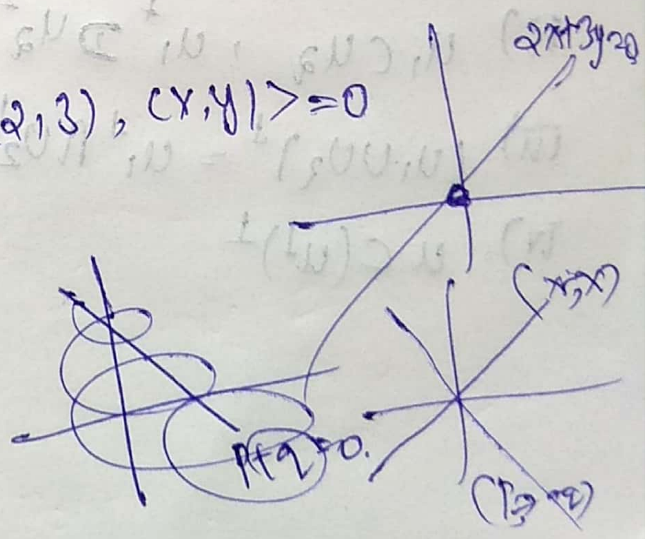
Def: Let u be a subset of an IPS. Then $u^{\perp} = \{ v \in V : \langle v, u \rangle = 0, \ \forall u \in u \}$

Eg: ① $V = \mathbb{R}^2$, $u = \{(2,3)\}$

$u^{\perp} = \{ v \in V ; \langle v, u \rangle = 0 \}$ $\quad \langle (2,3), (x,y) \rangle = 0$

$2x + 3y = 0$

$\Rightarrow u^{\perp} = \{ (x,y) : 2x + 3y = 0 \}$



② $V = \mathbb{R}^2$; $u = \{ (x, x) \}$

$\langle (x,x), (p,q) \rangle = 0 \Rightarrow xp + xq = 0 \Rightarrow \boxed{p + q = 0}$

③ $U = \{(1,2), (1,1)\}$

$V = \{(x,y)\}$ $\curvearrowright$ $y=0; x=0$

$V = \{[0,0]\}$

$x + 2y = 0$; $x + y = 0$

So, what ever be the U $U^{\perp}$ is a ~~subset~~ space of V
subset of V.

Observation: if U be any subset of V, $U^{\perp}$ is a subspace of V.

proof: $0 \in U^{\perp}$ Because $\langle 0, u \rangle = 0 \; \forall \; u \in U$

$\forall \; v_1, v_2 \in U^{\perp}, \; c \in \mathbb{F}$

$\langle cv_1 + v_2, u \rangle = c\langle v_1, u \rangle + \langle v_2, u \rangle$
$= 0$ if $u \in U$

$\Rightarrow cv_1 + v_2 \in U^{\perp}$ if $v_1, v_2 \in U^{\perp}$

Ex: (i) $\{0\}^{\perp} = V$ (all vectors) ; $V^{\perp} = \{0\}$.

(ii) $U_1 \subset U_2$, $U_1^{\perp} \supset U_2^{\perp}$??

(iii) $(U_1 \cup U_2)^{\perp} = U_1^{\perp} \cap U_2^{\perp}$

(iv) $U \subset (U^{\perp})^{\perp}$