Convex Optimization in Python

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I. INTRODUCTION

This article contains some of the exercises and examples from the book **"S. Boyd and L. Vandenberghe, Convex Optimization. Cambridge: Cambridge University Press, 2004"** that I have coded in Python3. The code can be downloaded from **"https://github.com/Shahrokh-Hamidi/Convex-Optimization"**.

II. CHEBYSHEV CENTER

```
#Chebyshev Center
import numpy as np
import matplotlib.pyplot as plt
import scipy
import cvxpy as cp


n = 2
px = np.array([0, .5, 2, 3, 1])
py = np.array([0, 1, 1.5, .5, -.5])

px = np.hstack((px, px[0]))
py = np.hstack((py, py[0]))

px_diff = px[1:] - px[:-1]
py_diff = py[1:] - py[:-1]

px_avg = 0.5*(px[1:] + px[:-1])
py_avg = 0.5*(py[1:] + py[:-1])


A = []
for i in range(0, len(px)-1):
    p = np.array([px_diff[i], py_diff[i]])
    p = p/np.linalg.norm(p)
    A.append([-p[1], p[0]])

A = np.array(A)

b = []
for i in range(0, len(px)-1):
    p = np.array([px_avg[i], py_avg[i]])
    b.append(A[i,:].dot(p))

#plt.plot(px, py)

m = A.shape[-1]
x = cp.Variable(m)
r = cp.Variable(1)

constr = []
```

```
constr += [A@x + r <= b]

prob = cp.Problem(cp.Maximize(r), constr)
prob.solve(verbose = False)



x = x.value
r = r.value


#---- Display

N = 100

theta = np.linspace(0,2*np.pi, N).reshape(1,-1)

x1 = r*np.cos(theta).squeeze() + x[0]
x2 = r*np.sin(theta).squeeze() + x[1]


plt.plot(px, py)
plt.plot(x1, x2, 'r--')
plt.plot(x[0], x[1], 'ko')
plt.title('Chebyshev Center')
plt.axis('off')
plt.show()
```

Fig. 1 illustrates the result of the code for the *Chebyshev Center* code.

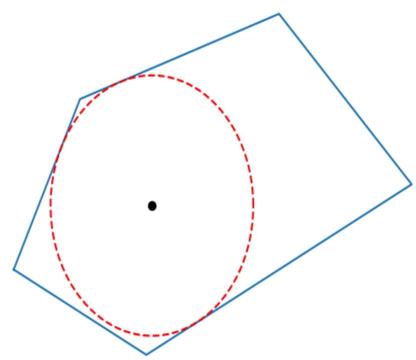

Fig. 1. The result for the *Chebyshev Center*.

III. Chebyshev center of a 2D polyhedron

```
#Chebyshev center of a 2D polyhedron
import numpy as np
import matplotlib.pyplot as plt
import scipy
import sys
import cvxpy as cp

#%matplotlib qt



a1 = np.array([ 2, 1])
a2 = np.array([ 2, -1])
a3 = np.array([-1,  2])
```

```
a4 = np.array([-1, -2])
b = np.ones(4)


m = 2
c = cp.Variable(m)
r = cp.Variable(1)

constr = []

constr += [a1@c + np.linalg.norm(a1,2)*r <= b[0]]
constr += [a2@c + np.linalg.norm(a2,2)*r <= b[1]]
constr += [a3@c + np.linalg.norm(a3,2)*r <= b[2]]
constr += [a4@c + np.linalg.norm(a4,2)*r <= b[3]]

prob = cp.Problem(cp.Maximize(r), constr)
prob.solve(verbose = False)



c = c.value
r = r.value


#---- Display

N = 100

theta = np.linspace(0,2*np.pi, N).reshape(1,-1)
x = np.linspace(-2,2,100)




x1 = r*np.cos(theta).squeeze() + c[0]
x2 = r*np.sin(theta).squeeze() + c[1]

plt.plot(x, b[0]/a1[1] - a1[0]*x/a1[1], 'k')
plt.plot(x, b[1]/a2[1] - a2[0]*x/a2[1], 'k')
plt.plot(x, b[2]/a3[1] - a3[0]*x/a3[1], 'k')
plt.plot(x, b[3]/a4[1] - a4[0]*x/a4[1], 'k')
plt.plot(x1, x2, 'r--')
plt.plot(c[0], c[1], 'bo')
plt.title('Chebyshev center of a 2D polyhedron')
plt.ylim(-1,1)

plt.show()
```

Fig. 2 illustrates the result of the code for the *Chebyshev center of a 2D polyhedron* code.

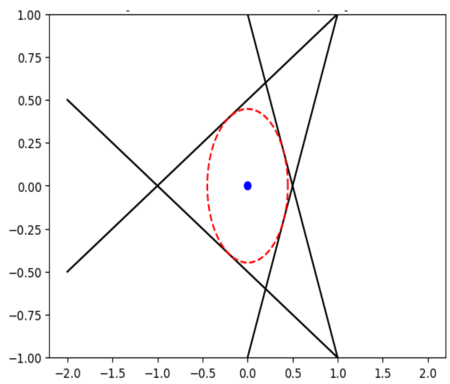

Fig. 2. The result for the *Chebyshev center of a 2D polyhedron*.

IV. FASTEST MIXING MARKOV CHAIN ON A GRAPH

```
#fastest mixing Markov chain on a graph
import numpy as np
import matplotlib.pyplot as plt
import scipy
import sys
import cvxpy as cp

#%matplotlib qt


n = 5
E = np.array([[0, 1, 0, 1, 1],
              [1, 0, 1, 0, 1],
              [0, 1, 0, 1, 1],
              [1, 0, 1, 0, 1],
              [1, 1, 1, 1, 0]])


P = cp.Variable((n,n), symmetric = True)



constr = []
constr += [P@np.ones(n) == np.ones(n)]
constr += [P >= 0]
constr += [P[E==0] == 0]

cost = cp.norm(P - np.ones((n,n))/n)
prob = cp.Problem(cp.Minimize(cost), constr)
prob.solve(verbose = True)


print(f'the optimal value is {np.round(P.value,2)}')
```

V. FITTING A CONVEX FUNCTION TO GIVEN DATA

```
#Fitting a convex function to given data
import numpy as np
import matplotlib.pyplot as plt
import scipy
import sys
import cvxpy as cp
```

```
#%matplotlib qt



lambda_ = 1

yns = np.array([[ 5.2057354 ],
       [ 5.16852954],
       [ 4.46931747],
       [ 3.16764149],
       [ 3.21867268],
       [ 2.75587742],
       [ 1.63606279],
       [ 0.72527756],
       [ 0.24583927],
       [-0.58044829],
       [-0.87676552],
       [-0.82548372],
       [-0.79731423],
       [-0.05948396],
       [-0.04975525],
       [ 0.70500264],
       [ 0.82600211],
       [ 0.1403059 ],
       [ 0.51054544],
       [ 0.38582234],
       [ 0.83860086],
       [ 0.41632982],
       [ 0.81154682],
       [ 0.23060127],
       [ 0.84177419],
       [ 0.34454159],
       [ 0.37408514],
       [ 0.86597229],
       [ 0.2120701 ],
       [ 0.71788999],
       [ 0.80995603],
       [ 1.06910934],
       [ 0.64850169],
       [ 1.09248439],
       [ 0.76143045],
       [ 1.2122857 ],
       [ 1.17728916],
       [ 0.84659501],
       [ 0.95866895],
       [ 1.82113178],
       [ 1.80159358],
       [ 1.63543887],
       [ 1.77429526],
       [ 2.52647668],
       [ 2.65227764],
       [ 3.75011515],
       [ 4.05642222],
       [ 4.62476811],
       [ 4.91230273],
       [ 5.80689459],
       [ 7.02609346]])

u = np.array([[0.  ],
       [0.04],
       [0.08],
       [0.12],
       [0.16],
       [0.2 ],
```

```
       [0.24],
       [0.28],
       [0.32],
       [0.36],
       [0.4 ],
       [0.44],
       [0.48],
       [0.52],
       [0.56],
       [0.6 ],
       [0.64],
       [0.68],
       [0.72],
       [0.76],
       [0.8 ],
       [0.84],
       [0.88],
       [0.92],
       [0.96],
       [1.  ],
       [1.04],
       [1.08],
       [1.12],
       [1.16],
       [1.2 ],
       [1.24],
       [1.28],
       [1.32],
       [1.36],
       [1.4 ],
       [1.44],
       [1.48],
       [1.52],
       [1.56],
       [1.6 ],
       [1.64],
       [1.68],
       [1.72],
       [1.76],
       [1.8 ],
       [1.84],
       [1.88],
       [1.92],
       [1.96],
       [2.  ]])


n = len(yns)
y = cp.Variable((n,1))
g = cp.Variable((n,1))


constr = []

constr += [y@np.ones((1,n)) >=  np.ones((n,1))@y.T + cp.multiply((np.ones((n,1))@g.T), (u@np.
                                               ones((1,n)) - np.ones((n,1))@u.T))]

cost = cp.norm(y - yns)
prob = cp.Problem(cp.Minimize(cost), constr)
prob.solve(verbose = False)



y = y.value
```

```
plt.figure()
plt.plot(u, yns, 'o', mfc= 'none')
plt.plot(u, y, 'b')
plt.xlabel('u', fontsize = 16)
plt.ylabel('y', fontsize = 16)
ax = plt.gca()
ax.xaxis.set_tick_params(labelsize=12)
ax.yaxis.set_tick_params(labelsize=12)
plt.title('Fitting a convex function to given data')
plt.xlim(-0.5,2.5)
plt.ylim(-1,8)
plt.grid()
plt.show()
```

Fig. 3 illustrates the result of the code for the *Fitting a convex function to given data* code.

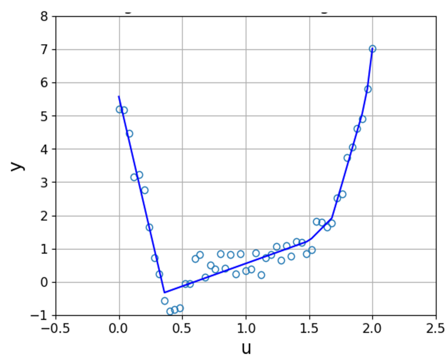

Fig. 3. The result for the *Fitting a convex function to given data*.

VI. Matrix fractional minimization using SOCP

```
#Matrix fractional minimization using SOCP
import numpy as np
import matplotlib.pyplot as plt
import scipy
import sys
import cvxpy as cp

#%matplotlib qt





A = np.array([[-0.432564811528221,  1.06676821135919, 0.815622288876143,  -2.17067449430526,
                    0.428183273045163,  0.623233851138494,  0.
                    781181617878392,  1.47247993441992],
[-1.66558437823810, 0.0592814605236054, 0.711908323500893,  -0.0591878245211912,  0.
                    895638471211752,  0.799048618147778,  0.
                    568960645723274,  0.0557438318378432],
[0.125332306474831, -0.0956484054836690,  1.29024975493248, -1.01063370647425,  0.
                    730957338429453,  0.940889940727780,  -0.
                    821714291696256, -1.21731745370455],
[0.287676420358549, -0.832349463650023, 0.668600505682040,  0.614463048895481,  0.
                    577857346330798,  -0.992091735543795, -0.
                    265606851332549, -0.0412271336864321],
[-1.14647135068146, 0.294410816392640,  1.19083807424337, 0.507740785341986,  0.
                    0403140316184403, 0.212035152165055,  -1.
                    18777701646980,  -1.12834386432023],
```

```
[1.19091546564300,  -1.33618185793780,  -1.20245711477394,  1.69242987019052, 0.
                                              677089187597305,  0.237882072875579,  -2.
                                              20232071732344,  -1.34927754310249],
[1.18916420165210,  0.714324551818952,  -0.0197895577687705,  0.591282586924176,  0.
                                              568900205200723,  -1.00776339167827,  0.
                                              986337391002023,  -0.261101623061621],
[-0.0376332765933176, 1.62356206444627, -0.156717298831981, -0.643595202682526, -0.
                                              255645415631965, -0.742044752133604, -0.
                                              518635066344746, 0.953465445504819],
[0.327292361408654, -0.691775701702287, -1.60408556200116,  0.380337251713910,  -0.
                                              377468955522361, 1.08229495315533, 0.
                                              327367564080834,  0.128644430046645],
[0.174639142820925, 0.857996672828263,  0.257304234677490,  -1.00911552434079,  -0.
                                              295887110003557, -0.131499702945274, 0.
                                              234057012847185,  0.656467513885396],
[-0.186708577681439,  1.25400142160253, -1.05647292808148,  -0.0195106695302893,  -1.
                                              47513450585526,  0.389880489687039,  0.
                                              0214661388790945, -1.16781936472664],
[0.725790548293303, -1.59372957644748,  1.41514148587234, -0.0482207891453123,  -0.
                                              234004047656033, 0.0879871065797930, -1.
                                              00394446674772,  -0.460605179506150],
[-0.588316543014189,  -1.44096443190102,  -0.805090404196880, 4.31918416255450e-05, 0.
                                              118444837054121,  -0.635465225479316, -0.
                                              947146064738541, -0.262439952838333],
[2.18318581819710,  0.571147623658178,  0.528743010962225,  -0.317859451247688, 0.
                                              314809043395056,  -0.559573302196241, -0.
                                              374429195029166, -1.21315206849391],
[-0.136395883086596,  -0.399885577715363, 0.219320672667622,  1.09500373878749, 1.
                                              44350824434982, 0.443653489503667,  -1.
                                              18588621380853,  -1.31943699810954],
[0.113931313520810, 0.689997375464345,  -0.921901624355539, -1.87399025764096,  -0.
                                              350974738327742, -0.949903798547645, -1.
                                              05590292352369,  0.931217514995436]])


b  =np.array([0.0112448963841337,-0.645145815691170,0.805728793112376,0.231626010780437,-0.
                                              989759671682004,1.33958570061039,0.
                                              289502034538413,
              1.47891705768128,1.13802801285837,-0.684138585136340,-1.29193604496594,-0.
                                                            0729262762636467,-0.
                                                            330598879892764,-0.
                                                            843627639154800,
              0.497769664182782,1.48849047090348])


B  = np.array([[-0.546475894767623, 0.469383119866330,  0.288807018730340,  1.95738475514751,
                                               -0.321004692181292, 1.22744798900972, 0.
                                              485497707312810,  -1.27050020370838],
                [-0.846758163883060,  -0.903566942617776, -0.429303004551915, 0.
                                                              504542353592166,  1.
                                                              23655565160192, -0.
                                                              696204800032889, -0.
                                                              00500507375553139, -1.
                                                              66360645282977],
                [-0.246336528084900,  0.0358796387294769, 0.0558011901764726, 1.
                                                              86452902048530, -0.
                                                              631279656725146, 0.
                                                              00752448652301445,  -0.
                                                              276217859354759, -0.
                                                              703554261536755],
                [0.663024145855908, -0.627531219966832, -0.367873566740638, -0.
                                                              339811777414964, -2.
                                                              32521112888377,  -0.
                                                              782893044378287, 1.
                                                              27645247367439, 0.
```

```
                                280880488523302],
 [-0.854197374468980,  0.535397954249106,  -0.464973367171118, -1.
                                139779402313 23,  -1.
                                23163653332502,  0.
                                586938559214431,  1.
                                86340061318454, -0.
                                541209329916194],
 [-1.20131481533904, 0.552883517423822,  0.370960583848952,  -0.
                                211123483380258, 1.
                                05564838790246, -0.
                                251207374568882, -0.
                                522559301636399, -1.
                                33353072973639],
 [-0.119869428057387,  -0.203690479567358, 0.728282931551495,  1.
                                19024493625120, -0.
                                113223989369025, 0.
                                480135822842601,  0.
                                103424446937315,  1.
                                07268626789014],
 [-0.0652940148415865, -2.05432468055661,  2.11216016977150, -1.11620875778561
                                ,  0.379223622685033,  0.
                                668155034433641,  -0.
                                807649130897181, -0.
                                712085452494356],
 [0.485295555916544, 0.132560731417280,  -1.35729774309675,  0.635274134747122
                                ,  0.944199726747308,  -0.
                                0783211962734119,  0.
                                680438583748946,  -0.
                                0112855612306856],
 [-0.595490902619476,  1.59294070376602, -1.02261014433421,  -0.
                                601412126269725, -2.
                                12042668822421,  0.
                                889172618412599,  -2.
                                36458984794158,  -0.
                                000817029195695836],
 [-0.149667743824475,  1.01841178862471, 1.03783419871876, 0.551184711824902,
                                  -0.644678915541937, 2.
                                30928748595239, 0.
                                990114872049490,  -0.
                                249436284695434],
 [-0.434751931152533,  -1.58040249930316,  -0.389799548476831, -1.
                                09984045471081,  -0.
                                704301728433609, 0.
                                524638679771098,  0.
                                218899120881177,  0.
                                396575318711652],
 [-0.0793302230234206, -0.0786619193594521,  -1.38126562401984,  0.
                                0859905932937184, -1.
                                01813721639907,  -0.
                                0117873239513068,  0.
                                261662460161402,  -0.
                                264013354922243],
 [1.53515226612215,  -0.681656860002363, 0.315542632772365,  -2.00456332159079
                                ,  -0.182081868411385, 0.
                                913140817761371,  1.
                                21344449497535, -1.
                                66401087693059],
 [-0.606482859277266,  -1.02455305742903,  1.55324256851535, -0.
                                493087917659697, 1.
                                52101323900559, 0.
                                0559406788884020, -0.
                                274666986456781, -1.
                                02897509954380],
 [-1.34736267385024, -1.23435347798426,  0.707893884632476,  0.462048011799193
                                ,  -0.0384387638867116,  -1.
```

```
10706989482601,  -0.
133134450813529, 0.
243094700224565]])
```

```python
def Form_I():

    x = cp.Variable(n)

    constr = []

    constr += [x >= 0]

    cost = cp.matrix_frac(A@x + b, np.eye(m) + (B@cp.diag(x))@B.T)
    prob = cp.Problem(cp.Minimize(cost), constr)
    prob.solve(verbose = False)


    x = x.value
    print(f'Form_I Objective : {cost.value}')




def Form_II():

    x = cp.Variable(n)
    Y = cp.Variable((n,n))

    constr = []

    constr += [x >= 0]
    constr += [Y == cp.diag(x)]

    cost = cp.matrix_frac(A@x + b, np.eye(m) + (B@Y)@B.T)
    prob = cp.Problem(cp.Minimize(cost), constr)
    prob.solve(verbose = False)


    x = x.value
    print(f'Form_II Objective : {cost.value}')


def Form_III():

    x = cp.Variable(n)
    v = cp.Variable(m)
    w = cp.Variable(n)

    constr = []

    constr += [x >= 0]
    constr += [v + B@w == A@x + b]

    cost = cp.sum_squares(v) + cp.matrix_frac(w, cp.diag(x))
    prob = cp.Problem(cp.Minimize(cost), constr)
    prob.solve(verbose = False)


    x = x.value
    print(f'Form_II Objective : {cost.value}')
```

```
def Form_IV():

    x = cp.Variable(n)
    v = cp.Variable(m)
    w = cp.Variable(n)
    Y = cp.Variable((n,n))


    constr = []

    constr += [x >= 0]
    constr += [v + B@w == A@x + b]
    constr += [Y == cp.diag(x)]
    cost = cp.sum_squares(v) + cp.matrix_frac(w, Y)
    prob = cp.Problem(cp.Minimize(cost), constr)
    prob.solve(verbose = False)


    x = x.value
    print(f'Form_II Objective : {cost.value}')


m, n = A.shape

Form_I()
Form_II()
Form_III()
Form_IV()
```

VII. Maximum determinant PSD matrix completion

```
#Maximum determinant PSD matrix completion
import numpy as np
import matplotlib.pyplot as plt
import scipy
import sys
import cvxpy as cp

#%matplotlib qt


n = 4


A = cp.Variable((n,n), PSD = True)


constr = []

constr += [A[0,0] == 3]
constr += [A[1,1] == 2]
constr += [A[2,2] == 1]
constr += [A[3,3] == 5]
constr += [A[0,1] == 0.5]
constr += [A[0,3] == 0.25]
constr += [A[1,2] == 0.75]

cost = -cp.log_det(A)
prob = cp.Problem(cp.Minimize(cost), constr)
prob.solve(verbose = True)
```

```
print(f'A = {A.value}')
```

VIII. Polynomial Fitting

Fig. 1 illustrates the result of the code for the *Polynomial Fitting* code.

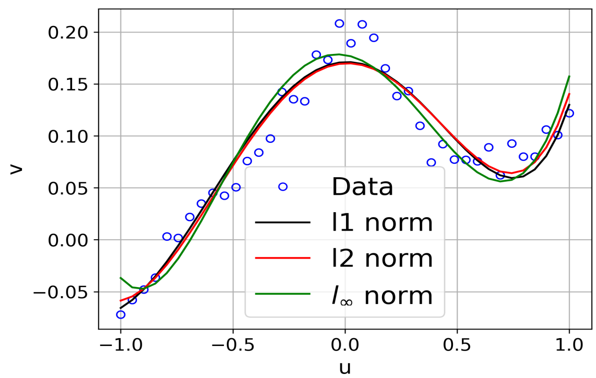

Fig. 4. The result for the *Polynomial Fitting*.

IX. Quadratic smoothing

```
#Quadratic smoothing
import numpy as np
import matplotlib.pyplot as plt
import cvxpy as cp

#%matplotlib qt


n = 4000
t = np.arange(0, n)

signal = 0.5*np.sin((2*np.pi/n)*t)*np.sin(0.01*t)

noisy_signal = signal + np.random.normal(0,0.1,n)


lambda_ = 500
D = np.zeros((n,n))

for i in range(n-1):
    D[i, i:i+2] = [-1,1]



x = cp.Variable(n)

cost = cp.norm(noisy_signal - x, 2) + lambda_*cp.norm(D@x,2)
prob = cp.Problem(cp.Minimize(cost))
prob.solve()

denoised_signal = x.value


plt.subplot(311)
```

```
plt.plot(t, signal, 'k', lw = 2,  label = 'signal')
plt.xlabel('x', fontsize = 12)
plt.ylabel('y', fontsize = 12)
ax = plt.gca()
ax.xaxis.set_tick_params(labelsize=12)
ax.yaxis.set_tick_params(labelsize=12)
plt.grid()
plt.legend()
plt.title('Quadratic Smoothing', fontsize = 16)



plt.subplot(312)
plt.plot(t, noisy_signal, label = 'noisy_signal')
plt.xlabel('x', fontsize = 12)
plt.ylabel('y', fontsize = 12)
ax = plt.gca()
ax.xaxis.set_tick_params(labelsize=12)
ax.yaxis.set_tick_params(labelsize=12)
plt.grid()
plt.legend()



plt.subplot(313)
plt.plot(t, denoised_signal, 'r', label = 'denoised_signal')
plt.xlabel('x', fontsize = 12)
plt.ylabel('y', fontsize = 12)
ax = plt.gca()
ax.xaxis.set_tick_params(labelsize=12)
ax.yaxis.set_tick_params(labelsize=12)
plt.grid()
plt.legend()


plt.show()
```

Fig. 5 illustrates the result of the code for the *Quadratic smoothing* code.

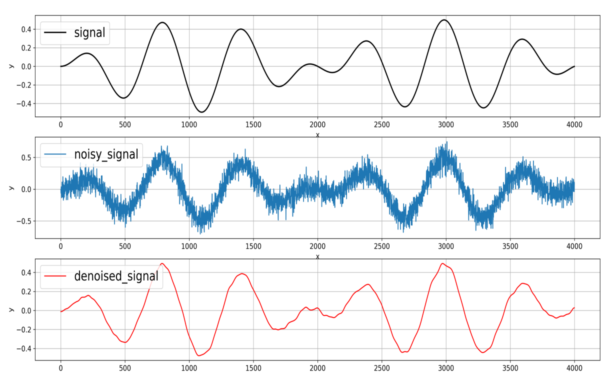

Fig. 5. The result for the *Quadratic smoothing*.

X. ROBUST REGRESSION USING THE L1 L2 HUBER PENALTY

```
#Robust regression using the l1 l2 huber penalty
import numpy as np
import matplotlib.pyplot as plt
import cvxpy as cp
```

```python
#%matplotlib qt

def DATA_gen():

    x = np.linspace(-10, 10, 30)

    y = x + np.random.normal(0, 2, len(x))
    data = np.hstack((x.reshape(-1,1), y.reshape(-1,1)))
    data = np.vstack((data, np.array([-10, 10])))
    data = np.vstack((data, np.array([10, -10])))

    return data




def cvx_opt(penalty):

    m = cp.Variable(1)
    b = cp.Variable(1)

    if penalty == '12':
        cost = cp.norm(data[:,1] - (m*data[:,0] + b), 2)

    if penalty == '11':
        cost = cp.norm(data[:,1] - (m*data[:,0] + b), 1)

    if penalty == 'huber':
        cost = cp.sum(cp.huber(data[:,1] - (m*data[:,0] + b), M = 4))

    prob = cp.Problem(cp.Minimize(cost))
    prob.solve()

    b = b.value
    m = m.value

    return m, b




data = DATA_gen()

penalty = '12'
m2, b2 = cvx_opt(penalty)

penalty = '11'
m1, b1 = cvx_opt(penalty)

penalty = 'huber'
mh, bh = cvx_opt(penalty)


x = np.linspace(-10, 10, 100)
plt.plot(x, m2*x+b2, 'k--', label = '12 norm')
plt.plot(x, m1*x+b1, label = '11 norm')
plt.plot(x, mh*x+bh, label = 'huber')
plt.plot(data[:,0], data[:,1], 'ko', mfc = 'none')
plt.legend()

plt.title('Robust regression using the 11, 12, and huber penalty')

plt.xlabel('x', fontsize = 16)
plt.ylabel('y', fontsize = 16)
```

```
ax = plt.gca()
ax.xaxis.set_tick_params(labelsize=12)
ax.yaxis.set_tick_params(labelsize=12)
plt.grid()
plt.show()
```

Fig. 6 illustrates the result of the code for the Robust regression using the l1 l2 huber penalty code.

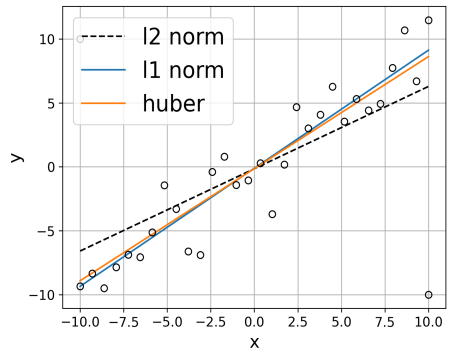

Fig. 6. The result for the *Robust regression using the l1 l2 huber penalty*.

XI. SIMPLE QP WITH INEQUALITY CONSTRAINTS

```
#simple QP with inequality constraints
import numpy as np
import matplotlib.pyplot as plt
import scipy
import sys
import cvxpy as cp




P = np.array([[13, 12, -2], [12, 17, 6], [-2, 6, 12]])
q = np.array([[-22, -14.5, 13]])
r = 1
m = 3
#x_star = [1;1/2;-1];



x = cp.Variable(m)


constr = []

constr += [x <= 1]
constr += [x >= -1]

cost = 0.5*cp.quad_form(x, P) + q@x + r
prob = cp.Problem(cp.Minimize(cost), constr)
prob.solve(verbose = True)


x = x.value

print(f'x_optimal : {x}')
```

XII. Total variation reconstruction example

```
#Total variation reconstruction example
import numpy as np
import matplotlib.pyplot as plt
import cvxpy as cp

#%matplotlib qt


n = 4000
t = np.arange(0, n)

signal = 0.5*np.sin(0.05*(2*np.pi/n)*t + np.pi/4)


for i in range(len(signal)):
    if n//4 <= i <= n//2:
        signal[i] = -signal[i]

    if  3*n//4 <= i:
        signal[i] = - signal[i]

noisy_signal = signal + np.random.normal(0,0.1,n)


lambda_ = 1
D = np.zeros((n,n))

for i in range(n-1):
    D[i, i:i+2] = [-1,1]



x = cp.Variable(n)

cost = cp.norm(noisy_signal - x, 2) + lambda_*cp.norm(D@x,1)
prob = cp.Problem(cp.Minimize(cost))
prob.solve()

denoised_signal = x.value


plt.subplot(311)
plt.plot(t, signal, 'k', lw = 2,  label = 'signal')
plt.xlabel('x', fontsize = 12)
plt.ylabel('y', fontsize = 12)
plt.ylim(-0.6, 0.6)
ax = plt.gca()
ax.xaxis.set_tick_params(labelsize=12)
ax.yaxis.set_tick_params(labelsize=12)
plt.grid()
plt.legend()
plt.title('Total variation reconstruction example', fontsize = 16)



plt.subplot(312)
plt.plot(t, noisy_signal, label = 'noisy_signal')
plt.xlabel('x', fontsize = 12)
plt.ylabel('y', fontsize = 12)
plt.ylim(-0.6, 0.6)
ax = plt.gca()
ax.xaxis.set_tick_params(labelsize=12)
ax.yaxis.set_tick_params(labelsize=12)
```

```
plt.grid()
plt.legend()



plt.subplot(313)
plt.plot(t, denoised_signal, 'r', label = 'denoised_signal')
plt.xlabel('x', fontsize = 12)
plt.ylabel('y', fontsize = 12)
ax = plt.gca()
plt.ylim(-0.6, 0.6)
ax.xaxis.set_tick_params(labelsize=12)
ax.yaxis.set_tick_params(labelsize=12)
plt.grid()
plt.legend()


plt.show()
```

Fig. 7 illustrates the result of the code for the *total variation reconstruction example* code.

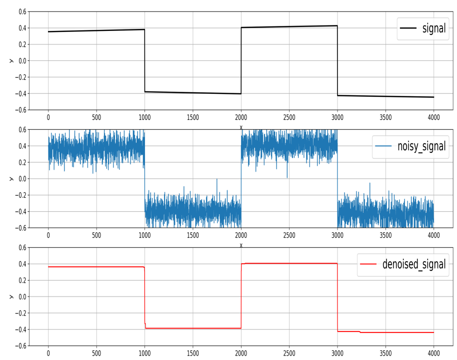

Fig. 7. The result for the *total variation reconstruction example*.

XIII. Maximum Entropy Distribution

```
#Maximum Entropy Distribution
import cvxpy as cp
import numpy as np
import scipy as scipy
import matplotlib.pyplot as plt


#%matplotlib qt


n = 100
u = np.linspace(-1,1,n)

A = np.array([[u], [-u], [u**2], [-u**2], [3 * ( u**3 ) - 2 * u], [-3 * ( u**3 ) + 2 * u], [
                              ((u < 0)*1).tolist()]]).squeeze()
b = np.array([0.1, 0.1, 0.5, -0.5, -0.2, 0.3, 0.4])


p = cp.Variable(n)
constr = []
constr += [A@p <= b]
constr += [cp.sum(p) ==1]
constr += [p >= 0]
```

```
cost = cp.sum(cp.entr(p))
prob = cp.Problem(cp.Maximize(cost), constr)

prob.solve()

p = p.value

plt.plot(u, p)
plt.xlim(-1.3, 1.3)
plt.ylim(0, 0.05)
plt.xlabel('$u_i$', fontsize = 16)
plt.ylabel('$ Prob( X == u_i ) $', fontsize = 16)
#plt.rc('legend',fontsize=20) # using a size in points
#plt.rc('legend',fontsize='medium') # using a named size
ax = plt.gca()
ax.xaxis.set_tick_params(labelsize=14)
ax.yaxis.set_tick_params(labelsize=14)
plt.grid()
plt.legend()
#
plt.title('Maximum Entropy Distribution', fontsize = 16, color = 'k')
plt.tight_layout()
plt.show()
```

Fig. 8 illustrates the result of the code for the *Maximum Entropy Distribution* code.

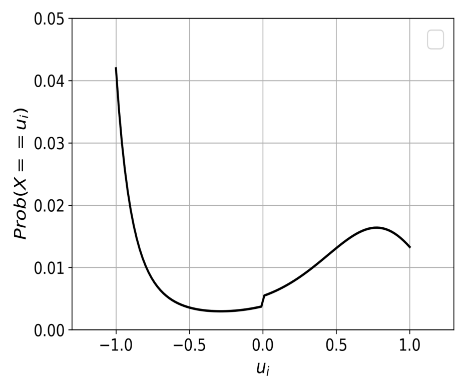

Fig. 8. The result for the *Maximum Entropy Distribution*.

XIV. Analytical Center of a Polytope

```
#Analytical Center of a Polytope
import numpy as np
import matplotlib.pyplot as plt
import scipy

import cvxpy as cp


#%matplotlib qt

n = 2
px = np.array([0, .5, 2, 3, 1])
py = np.array([0, 1, 1.5, .5, -.5])

px = np.hstack((px, px[0]))
py = np.hstack((py, py[0]))
```

```
px_diff = px[1:] - px[:-1]
py_diff = py[1:] - py[:-1]

px_avg = 0.5*(px[1:] + px[:-1])
py_avg = 0.5*(py[1:] + py[:-1])


A = []
for i in range(0, len(px)-1):
    p = np.array([px_diff[i], py_diff[i]])
    p = p/np.linalg.norm(p)
    A.append([-p[1], p[0]])

A = np.array(A)

b = []
for i in range(0, len(px)-1):
    p = np.array([px_avg[i], py_avg[i]])
    b.append(A[i,:].dot(p))

#plt.plot(px, py)

m = A.shape[-1]
x = cp.Variable(m)

constr = []
constr += [A@x <= b]

prob = cp.Problem(cp.Minimize(-cp.sum(cp.log(b - A@x))), constr)
prob.solve(verbose = False)



xp = x.value



#-------- Display

plt.plot(px, py, 'k')
plt.plot(xp[0], xp[1], 'ro')
plt.title('Analytical Center of a Polytope')
plt.axis('off')
plt.show()
```

Fig. **??** illustrates the result of the code for the *Analytical Center of a Polytope* code.

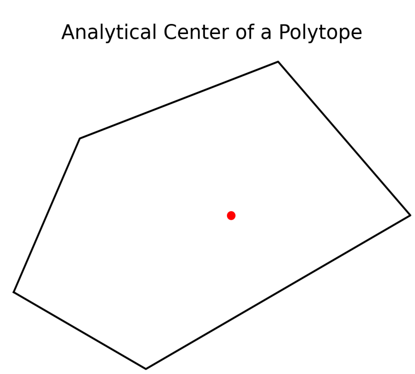

Fig. 9. The result for the *Analytical Center of a Polytope*.

XV. Bounding correlation coefficients

```
#Bounding correlation coefficients
import numpy as np
import matplotlib.pyplot as plt
import scipy
import sys
import cvxpy as cp


#%matplotlib qt



m = 4
p = cp.Variable((m,m), PSD = True)
constr = []
constr += [p[0][1] <= 0.9]
constr += [0.6 <= p[0][1]]
constr += [p[0][2] <= 0.9]
constr += [0.8 <= p[0][2]]
constr += [p[1][3] <= 0.7]
constr += [0.5 <= p[1][3]]
constr += [p[2][3] <= -0.4]
constr += [-0.8 <= p[2][3]]
for i in range(m):
    constr += [p[i][i] == 1.]


prob = cp.Problem(cp.Maximize(p[0][3]), constr)
prob.solve(verbose = False)



print(f'p.value = {p.value}')
```

XVI. Euclidean projection on a halfspace

```
#Euclidean projection on a halfspace
import numpy as np
import matplotlib.pyplot as plt
import scipy
import sys
import cvxpy as cp


#%matplotlib qt

n = 2
px = np.array([-5, 5])
py = np.array([-10, 10])



px_diff = px[1:] - px[:-1]
py_diff = py[1:] - py[:-1]

px_avg = 0.5*(px[1:] + px[:-1])
py_avg = 0.5*(py[1:] + py[:-1])


A = []
for i in range(0, len(px)-1):
```

```
    p = np.array([px_diff[i], py_diff[i]])
    p = p/np.linalg.norm(p)
    A.append([-p[1], p[0]])

A = np.array(A)


b = []
for i in range(0, len(px)-1):
    p = np.array([px_avg[i], py_avg[i]])
    b.append(A[i,:].dot(p))

x0 = [-5, 5]
m = 2
x = cp.Variable(m)

constr = []
constr += [A@x <= b]

prob = cp.Problem(cp.Minimize(cp.norm(x - x0, 2)), constr)
prob.solve(verbose = False)



xp = x.value



#-------- Display

plt.plot(px, py, 'b', lw = 2)
plt.plot(xp[0], xp[1], 'k*', markersize = 10)
plt.plot(x0[0], x0[1], 'ro')
plt.xlim(-10,10)
plt.ylim(-10,10)
plt.title('Euclidean projection on a halfspace')
#plt.axis('off')
plt.show()
```

Fig. 10 illustrates the result of the code for the *Euclidean projection on a halfspace* code.

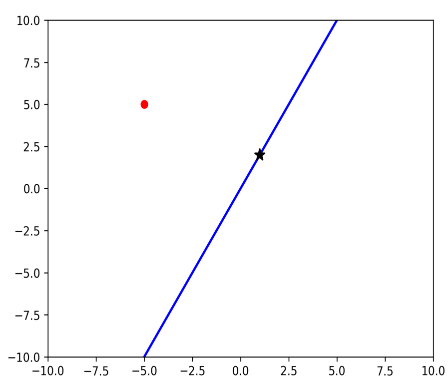

Fig. 10. The result for the *Euclidean projection on a halfspace*.

XVII. Euclidean projection on nonnegative orthant

```
#Euclidean projection on nonnegative orthant
import numpy as np
import matplotlib.pyplot as plt
```

```
import scipy

import cvxpy as cp


%matplotlib qt



x0 = np.array([5, -5])
m = 2
x = cp.Variable(m)

constr = []
constr += [x >= 0]

prob = cp.Problem(cp.Minimize(cp.norm(x - x0)), constr)
prob.solve(verbose = False)



xp = x.value



#-------- Display


plt.plot(xp[0], xp[1], 'k*')
plt.plot(x0[0], x0[1], 'ro')
plt.xlim(-10,10)
plt.ylim(-10,10)
#plt.axis('off')
```

Fig. 11 illustrates the result of the code for the *Euclidean projection on nonnegative orthant* code.

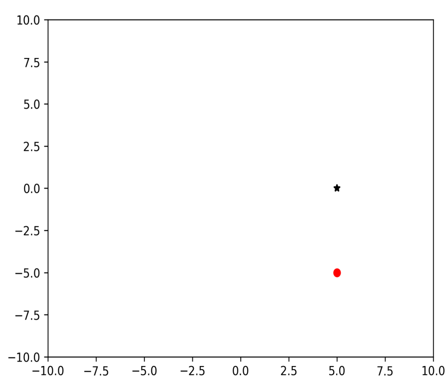

Fig. 11. The result for the *Euclidean projection on nonnegative orthant*.

XVIII. LINEAR DISCRIMINATION

```
#linear discrimination
import numpy as np
import matplotlib.pyplot as plt
import scipy
import sys
import cvxpy as cp
```

```
#%matplotlib qt



x = np.random.multivariate_normal([-3,-3], [[1,0],[0,1]],  10)
y = np.random.multivariate_normal([3,3], [[1,0],[0,1]],  10)



a = cp.Variable((1,x.shape[-1]))
b = cp.Variable(1)


constr = []

for i in range(x.shape[0]):
    constr += [a@x[i,:].reshape(-1,1) - b >= 1]
    constr += [a@y[i,:].reshape(-1,1) - b <= -1]
cost = 0
prob = cp.Problem(cp.Minimize(cost), constr)
prob.solve(verbose = False)


# -------  Display

z = np.linspace(-5,5,100)
plt.plot(z, (b.value - a.value.squeeze()[0]*z)/a.value.squeeze()[1])
plt.plot(x[:,0], x[:,1], 'k*')
plt.plot(y[:,0], y[:,1], 'ro')
plt.title('Linear Discrimination')
plt.show()
```

Fig. 12 illustrates the result of the code for the *linear discrimination* code.

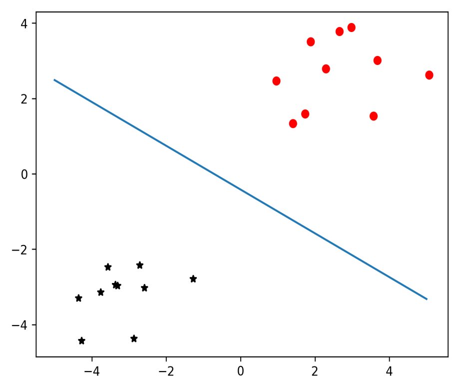

Fig. 12. The result for the *linear discrimination*.

XIX. LINEAR PLACEMENT PROBLEM

```
#Linear placement problem
import numpy as np
import matplotlib.pyplot as plt
import scipy
import sys
import cvxpy as cp


#%matplotlib qt
```

```
fixed = np.array([[ 1,   1,  -1, -1,    1,   -1,  -0.2,   0.1], [ 1,  -1,  -1,  1, -0.5, -0.2,
                                                              -1,    1]]).T
M = fixed.shape[0]
N = 6

# first N columns of A correspond to free points,
# last M columns correspond to fixed points

A = np.array([[ 1,  0,  0, -1,  0,  0,    0,  0,  0,  0,  0, 0,  0, 0],
              [1,  0, -1,  0,  0,  0,    0,  0,  0,  0,  0,  0,  0,  0],
      [1,  0,  0,  0, -1,  0,    0,  0,  0,  0,  0,  0,  0,  0],
      [1,  0,  0, 0,  0,  0,   -1,  0,  0,  0,  0,  0,  0,  0],
      [1,  0,  0,  0,  0,  0,    0, -1,  0,  0,  0,  0,  0,  0],
      [1,  0, 0,  0,  0, 0,    0,  0,  0,  0, -1, 0,  0,  0],
      [1,  0,  0,  0,  0,  0,    0,  0,  0,  0,  0,  0,  0, -1],
      [0,  1, -1,  0,  0,  0,    0,  0,  0,  0,  0,  0,  0,  0],
      [0,  1,  0, -1,  0,  0,    0,  0,  0,  0,  0,  0,  0,  0],
      [0,  1,  0,  0,  0, -1,    0,  0,  0, 0,  0,  0,  0,  0],
      [0,  1,  0,  0, 0,  0,    0, -1,  0,  0,  0,  0,  0,  0],
      [0,  1,  0,  0,  0,  0,    0,  0, -1,  0,  0,  0,  0,  0],
      [0,  1,  0,  0,  0,  0,    0,  0,  0,  0,  0,  0, -1, 0],
      [0,  0,  1, -1,  0,  0,    0,  0,  0,  0,  0,  0,  0,  0],
      [0,  0,  1,  0,  0,  0,    0, -1,  0,  0,  0,  0,  0,  0],
      [0,  0,  1,  0,  0,  0,    0,  0,  0,  0, -1,  0,  0,  0],
      [0,  0,  0,  1, -1,  0,    0,  0,  0,  0,  0,  0,  0,  0],
      [0,  0,  0,  1,  0,  0,    0,  0, -1,  0,  0,  0,  0,  0],
      [0,  0,  0,  1,  0,  0,    0,  0,  0, -1,  0,  0,  0,  0],
      [0,  0,  0,  1,  0,  0,    0,  0,  0,  0,  0, -1,  0,  0],
      [0,  0,  0,  1,  0, -1,    0,  0,  0,  0,  0, -1,  0,  0] ,
      [0,  0,  0,  0,  1, -1,    0,  0,  0,  0,  0,  0,  0,  0],
      [0,  0,  0,  0,  1,  0,   -1,  0,  0,  0,  0,  0,  0,  0],
      [0,  0,  0,  0,  1,  0,    0,  0,  0, -1,  0,  0,  0,  0],
      [0,  0,  0,  0,  1,  0,    0,  0,  0,  0,  0,  0,  0, -1],
      [0,  0,  0,  0,  0,  1,    0,  0, -1,  0,  0,  0,  0,  0],
      [0,  0,  0,  0,  0,  1,    0,  0,  0,  0, -1,  0,  0,  0]])


x = cp.Variable((A.shape[-1], 2))
cost = cp.sum_squares(A@x)

constr = []
for i in range(A.shape[0]):
    constr += [x[N:,:] == fixed]

prob = cp.Problem(cp.Minimize(cost), constr)
prob.solve(verbose = False)



x = x.value

#------ Display

plt.plot(x[:N,0], x[:N,1], 'gs', label = 'free points')
plt.plot(x[N:,0], x[N:,1], 'rs', label = 'fixed points')

for i in range(A.shape[0]):
    ind = np.nonzero(A[i,:])
    for idx in ind:
        plt.plot(x[idx,0],x[idx,1], 'k-.', lw = 0.2)

plt.legend()
plt.title('Linear Placement Problem')
plt.xlim(-1.2,1.2)
```

```
plt.ylim(-1.2,1.2)
plt.show()
```

Fig. 13 illustrates the result of the code for the *Linear placement problem* code.

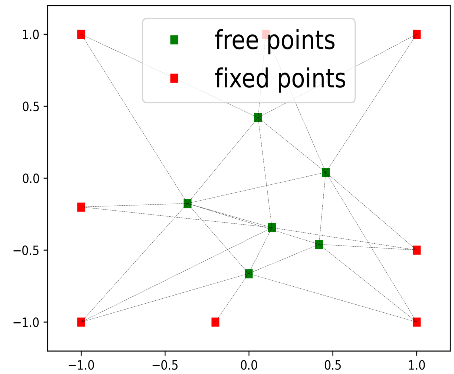

Fig. 13. The result for the *Linear placement problem*.

XX. MAX VOLUME INSCRIBED ELLIPSOID

```
#max volume inscribed ellipsoid

import numpy as np
import matplotlib.pyplot as plt
import scipy

import cvxpy as cp


n = 2
px = np.array([0, .5, 2, 3, 1])
py = np.array([0, 1, 1.5, .5, -.5])

px = np.hstack((px, px[0]))
py = np.hstack((py, py[0]))

px_diff = px[1:] - px[:-1]
py_diff = py[1:] - py[:-1]

px_avg = 0.5*(px[1:] + px[:-1])
py_avg = 0.5*(py[1:] + py[:-1])


A = []
for i in range(0, len(px)-1):
    p = np.array([px_diff[i], py_diff[i]])
    p = p/np.linalg.norm(p)
    A.append([-p[1], p[0]])

A = np.array(A)

b = []
for i in range(0, len(px)-1):
    p = np.array([px_avg[i], py_avg[i]])
    b.append(A[i,:].dot(p))

#plt.plot(px, py)

m = A.shape[-1]
```

```
X = cp.Variable((m,m), symmetric = True)
d = cp.Variable(m)

constr = []
for i in range(A.shape[0]):
    constr += [cp.norm( X@A[i,:].reshape(-1,1) , 2) + A[i,:].reshape(1,-1)@d <= b[i]]

prob = cp.Problem(cp.Maximize(cp.log_det(X)), constr)
prob.solve(verbose = False)



X = X.value
d = d.value


#---- Display

N = 100


theta = np.linspace(0,2*np.pi, N).reshape(1,-1)

ellipse = X@np.array([[np.cos(theta)], [np.sin(theta)]]).squeeze() + d.reshape(-1,1) @ np.
                                                ones((1,N))

plt.plot(px, py, 'b', lw = 2)
plt.plot(ellipse[0,:], ellipse[1,:], 'r')
plt.title('Max Volume Inscribed Ellipsoid')
plt.axis('off')
plt.show()
```

Fig. 14 illustrates the result of the code for the *max volume inscribed ellipsoid* code.

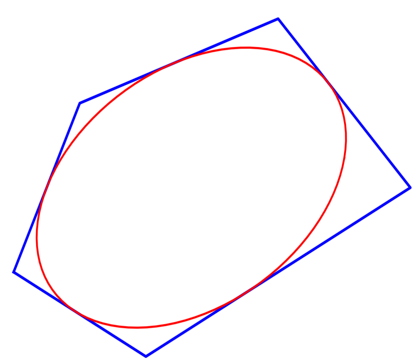

Fig. 14. The result for the max volume inscribed ellipsoid.

XXI. One Free Point Localization

```
#One free point localization
import numpy as np
import matplotlib.pyplot as plt
import scipy
import sys
import cvxpy as cp


#%matplotlib qt
```

```
p = np.random.multivariate_normal([0,0], [[10,0],[0,10]],  10)


x = cp.Variable((1,p.shape[-1]))

constr = []
cost = cp.sum(cp.norm(x - p ,1))
prob = cp.Problem(cp.Minimize(cost), constr)
prob.solve(verbose = False)



xp = x.value


# -------  Display
plt.plot(p[:,0], p[:,1], 'k*')
plt.plot(xp[:,0], xp[:,1], 'ro')
plt.title('One free point localization')
plt.show()
```

Fig. 15 illustrates the result of the code for the *One free point localization* code.

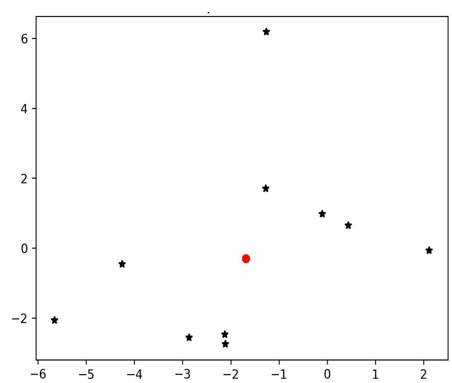

Fig. 15. The result for the *One free point localization*.

XXII. POLYNOMIAL DISCRIMINATION

```
#polynomial discrimination
import numpy as np
import matplotlib.pyplot as plt
import scipy
import sys
import cvxpy as cp
import scipy.io

#%matplotlib qt



X = np.array([[-2.95855623e-02,  1.95754105e-01,  9.38421496e-02,
        -3.29467959e-02, -1.73989577e-01, -7.97793709e-01,
        -1.11389191e-01, -7.42729742e-01, -1.29151943e-01,
        -5.17979960e-01, -1.09758793e-01, -2.30802197e-01,
        -7.73237398e-03,  7.09756728e-02, -3.52385404e-01,
        -5.31747802e-01, -3.04772028e-01, -1.29998958e-01,
```

```
        -4.48261427e-02, -3.27872167e-03, -4.41183114e-01,
        -4.05461983e-01, -1.99804573e-01,  4.26455138e-03,
         2.62474869e-03, -3.51403372e-01, -3.96796643e-01,
        -2.97028945e-01, -1.39562473e-01, -5.57034030e-01,
        -4.39141850e-01,  8.26766799e-04, -8.98575132e-02,
         1.98784110e-01,  1.48327407e-01, -7.65239272e-01,
        -1.64675275e-01,  1.46751529e-01, -6.94701156e-02,
        -2.10891761e-01, -4.41468102e-01, -3.05412384e-01,
        -4.84305217e-02, -2.79176289e-01, -5.10060070e-01,
        -7.98311175e-02, -3.10935435e-01, -3.72311823e-01,
         2.19521351e-01, -4.94892629e-01,  1.41049092e-01,
        -3.87453856e-01,  1.34704940e-01, -3.12572867e-01,
        -6.26808596e-01, -3.25919656e-01, -6.47490319e-01,
        -1.60047770e-01, -1.02291418e-01, -2.23325151e-01,
         1.01700226e-01, -5.31054659e-01, -4.42743514e-01,
         3.72895510e-01, -2.93658107e-02],
       [-3.27235847e-01,  4.95076766e-01,  5.29947667e-02,
        -9.59639561e-03,  7.63301831e-01, -2.65502307e-01,
        -9.16862531e-02,  2.55875621e-01, -1.70749169e-01,
         2.63716029e-01,  8.63248949e-01, -5.52139923e-01,
         4.47627192e-03, -1.00834217e-01, -5.59729838e-01,
         3.16068644e-01,  6.43951955e-02,  7.36883561e-02,
         1.32683793e-01,  3.80192236e-01, -5.87038834e-01,
         8.70099909e-02,  3.12597219e-01,  2.65889204e-03,
         4.41683197e-02,  5.15480333e-01, -4.29987771e-01,
        -3.98981772e-01,  1.11941884e-01, -7.95167137e-02,
         4.92736689e-01, -8.85028346e-03, -3.67032553e-01,
         6.48520918e-01,  6.98918608e-01, -3.16115610e-01,
         2.87103155e-01, -6.41579927e-01, -1.60079670e-01,
        -5.80573323e-03,  2.21465550e-01,  7.80819944e-01,
         3.49373009e-01, -2.23253724e-01,  2.95930684e-01,
        -7.49587566e-01, -1.23183214e-01, -8.86820600e-02,
        -4.88104745e-01,  1.19707947e-01, -6.29274097e-01,
        -2.48284959e-01, -6.85654170e-01,  3.10257939e-01,
        -6.87885168e-02, -8.22196336e-01,  3.28320725e-01,
         4.06064597e-01, -1.32930751e-02, -2.41369680e-01,
        -7.44523222e-02, -5.16360847e-01, -4.14590950e-01,
        -8.18568152e-01,  4.40919531e-02]])


Y = np.array([[-1.10929648, -1.19714993, -1.46155265,  0.54850428,  1.35098877,
        -0.63978276,  1.02322679,  1.52509307,  0.61292653,  2.04638631,
         1.82855895,  1.47921071,  0.30167283,  1.73791846,  0.7515328 ,
        -1.25628825, -0.88484211,  0.53786931,  1.53502706,  1.18898028,
         0.91609724, -1.54078336,  0.16536052, -1.19906352, -1.46105915,
         1.27808057, -1.12648287,  0.3258603 ,  1.14215806,  0.28618332,
         1.47693679,  0.89203912, -0.68661538, -1.388819  ,  1.73895725,
         1.57556233,  1.28373784, -0.87911447,  1.40727042, -1.40937883,
        -1.28074118, -1.48632542,  0.38125655, -1.52045199, -1.66213125,
        -1.94551437,  0.21882479,  0.11292287, -1.41478813, -1.87993473,
        -1.00727723,  0.02675306, -0.50369109, -1.39723257, -1.22424294,
         1.30030558, -1.01635215,  1.0255251 , -0.08169464, -0.25732126,
         1.40042328, -2.00438398,  1.36791438, -0.89550151, -1.55659656,
        -0.70043184,  0.99463546,  1.61911896, -1.79102261,  0.82796362,
        -1.17658101,  0.07050389,  1.21510545, -0.26214263,  1.28184009,
        -1.80832372,  1.53041901, -0.66284338, -1.14343021,  0.8866055 ,
        -1.22024598, -0.86028255,  0.44238846, -0.39174046, -0.33583426,
        -1.0828518 ,  0.22247235,  0.06527953,  1.66426983,  0.95034167,
         0.29308652, -0.70090159, -0.7519563 , -1.04020311, -1.73012502,
         1.17321142, -0.01267105,  0.44442947,  1.43795889, -0.96444755,
        -1.61170243,  1.07293517, -1.390684  ,  1.86297062, -1.17721976,
        -0.85478808,  0.94232596,  0.90095948,  0.78233839,  0.6769013 ,
         0.57207325,  1.07070613,  0.94326805, -0.88960536, -0.70306538,
         0.49395402, -0.72786602, -1.88687969, -0.32508892,  1.30512574,
         0.33981014,  0.30344273,  0.23495848,  0.34879119,  0.29813592,
```

```
          0.7812839 ,  0.61364035,  0.38713553,  0.75876337,  0.70237945,
          0.39402689,  0.31522323,  0.39765858,  0.5236989 ,  0.74799895,
          0.28614548,  0.24846108,  0.64083345,  0.49758925,  0.48985264,
          0.28120179,  0.42798242,  0.58821225,  0.41714166,  0.30514283,
          0.62799254],
        [ 0.3852498 ,  0.48917503,  0.23041757,  1.26242449,  1.2942288 ,
          1.26090287, -1.40510591,  1.42402683, -1.4810577 ,  0.07343144,
         -0.61127391, -1.37203234, -1.17497071, -0.79741137, -1.8623418 ,
         -0.01485873,  0.84298797, -1.78331797,  0.98114637,  1.2863001 ,
         -1.61188272,  0.85716211, -1.97658857, -0.6671302 , -0.41711143,
          0.30076968, -0.14360928, -1.13532496,  1.58119757, -1.41148329,
          0.52105578, -1.80541092, -1.13739852,  1.38593001, -0.24846801,
         -1.21754355,  0.93499571,  1.89903227,  0.43621015, -0.82161215,
         -0.03838718, -0.59754699, -1.47367586,  0.88731639,  0.61900299,
          0.66901519, -1.27319787,  1.205941  , -0.87794017, -0.86446196,
         -0.49818959,  1.97003418, -1.00804074, -0.81891045,  0.4138192 ,
         -1.26725377,  0.88957589,  1.75638958,  1.23448912,  1.60107613,
         -1.42114194,  0.3921614 ,  0.43390617, -1.24962578, -0.21339644,
         -1.03449096,  0.73352648,  0.20857677,  0.35460486,  1.2492346 ,
          0.86176662,  1.3834841 , -1.54888459, -1.70639443, -0.39449858,
          1.02390852, -0.82895152,  1.15288164,  1.36591446,  1.52710936,
         -0.1884893 ,  0.72104447, -1.28836322,  1.15171548, -1.12005379,
         -1.62522906, -1.26085395, -1.13053544, -0.77038397, -1.33230553,
          1.34445451,  1.29038998, -0.8204131 ,  1.69555841, -0.93375822,
         -0.67643161, -1.66943251,  1.17823186,  0.89618357, -1.0573024 ,
         -0.7036574 ,  1.64458793,  0.92030731,  0.92614406,  1.48093904,
          0.9156026 , -1.68778436,  0.92584162, -1.55306823, -1.63437019,
          1.7825389 ,  1.02149349, -1.21931587,  1.30945838,  1.55670293,
         -1.17650937, -0.82744872,  0.11239135,  1.58052194,  0.14566471,
         -0.20296804, -0.03981107,  0.15737806, -0.06000793,  0.28542558,
          0.05675917, -0.40796816,  0.00328204, -0.25757154,  0.31645786,
          0.1985505 , -0.26469693,  0.07997021, -0.61982956,  0.47241106,
         -0.21855985, -0.22959073,  0.40903737, -0.64462399,  0.28872615,
         -0.10924733, -0.406085  ,  0.2233023 ,  0.4763915 ,  0.40876068,
         -0.59554049]])


m = 2
P = cp.Variable((m,m), PSD = True)
q = cp.Variable(m)
r = cp.Variable(1)


cost = 0

constr = []
for i in range(X.shape[-1]):
    constr += [(X[:,i].reshape(1,-1)@P)@X[:,i].reshape(-1,1) + q.T@X[:,i].reshape(-1,1) + r <
                                                            = -1]
    constr += [(Y[:,i].reshape(1,-1)@P)@Y[:,i].reshape(-1,1) + q.T@Y[:,i].reshape(-1,1) + r >
                                                            = 1]

prob = cp.Problem(cp.Minimize(cost), constr)
prob.solve(verbose = True)



P = P.value
q = q.value
r = r.value

points = np.vstack((np.linspace(-1,1,100).reshape(1,-1), np.linspace(-1,1,100).reshape(1,-1))
                                                )
discr = (points.T@P)@points + points.T@q.reshape(-1,1) + r
```

```
#plt.plot(X[0,:], X[1,:], 'ro')
#plt.plot(Y[0,:], Y[1,:], 'go')
```

XXIII. Robust Linear Discrimination

```
#Robust Linear Discrimination
import numpy as np
import matplotlib.pyplot as plt
import scipy
import sys
import cvxpy as cp


#%matplotlib qt




x = np.random.multivariate_normal([-2,-2], [[1,0],[0,1]],  10)
y = np.random.multivariate_normal([2,2], [[1,0],[0,1]],  10)



a = cp.Variable((1,x.shape[-1]))
b = cp.Variable(1)
t = cp.Variable(1)

constr = []

for i in range(x.shape[0]):
    constr += [a@x[i,:].reshape(-1,1) - b >= t]
    constr += [a@y[i,:].reshape(-1,1) - b <= -t]
constr += [cp.norm(a,2) <= 1]
cost = t
prob = cp.Problem(cp.Maximize(cost), constr)
prob.solve(verbose = False)

a = a.value.squeeze()
b = b.value
t = t.value
# -------  Display

z = np.linspace(-5,5,100)
plt.plot(z, (b - a[0]*z)/a[1])
plt.plot(z, (t + b - a[0]*z)/a[1], 'r--')
plt.plot(z, (-t + b - a[0]*z)/a[1], 'r--')
plt.plot(x[:,0], x[:,1], 'k*')
plt.plot(y[:,0], y[:,1], 'ro')
plt.title('Robust Linear Discrimination')
plt.show()
```

Fig. 16 illustrates the result of the code for the *Robust Linear Discrimination* code.

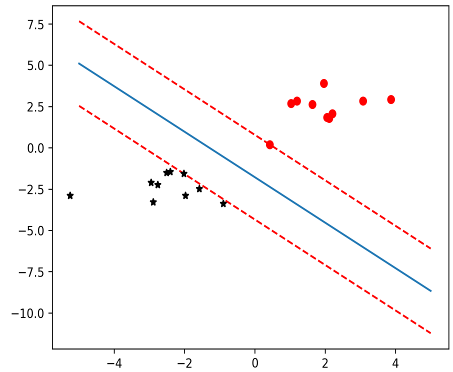

Fig. 16. The result for the *Robust Linear Discrimination*.

XXIV. Separating ellipsoids in 2D

```
#Separating ellipsoids in 2D

import numpy as np
import matplotlib.pyplot as plt
import scipy
import sys
import cvxpy as cp


#%matplotlib qt

n = 2
A = np.eye(n)
b = np.zeros(n)
C = np.array([[2, 1], [-.5, 1]])
d = np.array([-3, -3])


x = cp.Variable(n)
y = cp.Variable(n)
w = cp.Variable(n)

constr = []

constr += [cp.norm(A@x + b, 2) <= 1]
constr += [cp.norm(C@y + d, 2) <= 1]
constr += [x - y == w]
cost = cp.norm(w, 2)
prob = cp.Problem(cp.Minimize(cost), constr)
prob.solve(verbose = False)

w = w.value
x = x.value
y = y.value
# -------  Display


angle = np.linspace(0,2*np.pi,100).reshape(1,-1)
points = np.vstack((np.cos(angle), np.sin(angle)))

ellipse_1 = np.linalg.inv(A)@(points - b.reshape(-1,1))
ellipse_2 = np.linalg.inv(C)@(points - d.reshape(-1,1))

z = (x+y)/2.
m = (y[1] - x[1])/(y[0] - x[0])
```

```
m_ = (-90 + np.arctan(m)*180/np.pi)*np.pi/180
b_ = z[1] - m_*z[0]

t = np.linspace(-1.5, 2, 100)
plt.plot(t, m_*t + b_, 'm')
plt.plot([x[0], y[0]], [x[1], y[1]])
plt.plot(x[0], x[1], 'kx', markersize = 10)
plt.plot(y[0], y[1], 'rx', markersize = 10)
plt.plot(ellipse_1[0,:], ellipse_1[1,:], 'b--')
plt.plot(ellipse_2[0,:], ellipse_2[1,:], 'g--')
plt.xlim(-2,4)
plt.ylim(-2,4)
plt.title('Separating Ellipsoids in 2D')
plt.show()
```

Fig. 17 illustrates the result of the code for the *Separating ellipsoids in 2D* code.

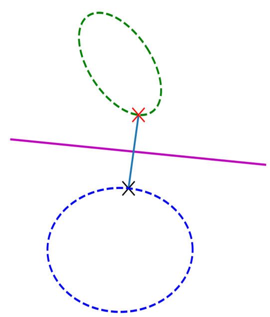

Fig. 17. The result for the *Separating ellipsoids in 2D*.

XXV. TIME SERIES ANALYSIS

```
#time series analysis


import numpy as np
import matplotlib.pyplot as plt
import cvxpy as cp
import matplotlib



#
# the data has been generated based on an AR model of order 1
# when A0 = 1, the model describes random walk
# in order to get an AR model A0 should be chosen as A0 < 1



n = 2000
y = np.zeros(n)
A0 = 1
for i in range(1,n):
    y[i] = A0*y[i-1] + np.random.normal(0,0.05)

n = len(y)


A = np.hstack((np.array([1, -2, 1]), np.zeros(n-3)))
```

```
D = []
for i in range(len(A)):
    D.append(np.roll(A,i))

D = np.array(D)



lambda_ = 10
x = cp.Variable(n)

cost = cp.norm(y - x, 2) + lambda_* cp.sum(cp.abs(D@x))       #cp.tv(x)
constr = []
prob = cp.Problem(cp.Minimize(cost), constr)
prob.solve(verbose = True)



x = x.value

plt.rcParams['savefig.dpi'] = 300

plt.plot(y, 'k-', lw = 0.3)
plt.plot(x, 'r')
matplotlib.rc('font', size=14)
plt.title('Time Series Analysis')

plt.xlabel('t', fontsize = 16)
plt.ylabel('y[t]', fontsize = 16)
matplotlib.style.use('ggplot')
plt.show()
```

Fig. 18 illustrates the result of the code for the *time series analysis* code.

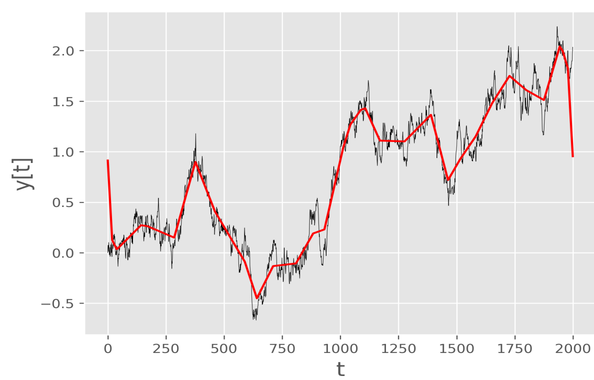

Fig. 18. The result for the *time series analysis*.

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