JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 41, 753–758 (1973)

A Generalization of Hartman's Linearization Theorem*

KENNETH J. PALMER

Department of Mathematics, The University, Southampton S09 5NH, England

Submitted by N. Levinson

1. INTRODUCTION

Suppose A is an $n \times n$ matrix none of whose eigenvalues has zero real part. Then one formulation of Hartman's linearization theorem (see [1], Chapter 9) says that if f is a C^1-small map of Euclidean n-space R^n into itself there is a homeomorphism of R^n onto itself sending the solutions of the differential equation,

$$x' = Ax + f(x),$$

onto the solutions of the linear differential equation,

$$x' = Ax.$$

Grobman proves a similar theorem in [2], and he only requires that f be small Lipschitzian. In this paper we generalize this theorem to nonautonomous differential equations where the linear equation is assumed to have an exponential dichotomy.

2. STATEMENT OF THE THEOREM

If x is in R^n we denote its norm by $|x|$ and if A is an $n \times n$ matrix we denote its operator norm by $|A|$. (Although we are concerned only with R^n here, our theorem is true with only minor changes when R^n is replaced by an arbitrary Banach space.) Suppose that $A(t)$ is a matrix function defined and continuous for all t on the real line R. Then we say that the linear differential equation,

$$x' = A(t)\, x, \tag{1}$$

* Work supported by the Science Research Council.

has an exponential dichotomy if it has a fundamental matrix $X(t)$ such that

$$\begin{aligned} |X(t)PX^{-1}(s)| &\leqslant Ke^{-\alpha(t-s)} \qquad \text{for } s \leqslant t, \\ |X(t)(I-P)X^{-1}(s)| &\leqslant Ke^{-\alpha(s-t)} \qquad \text{for } s \geqslant t, \end{aligned} \tag{2}$$

where P is a projection ($P^2 = P$) and K, α are positive constants. Then if $f(t)$ is a bounded continuous vector function the inhomogeneous linear equation,

$$x' = A(t)x + f(t),$$

has a unique bounded solution $x(t)$ given by

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)f(s)\,ds - \int_{t}^{\infty} X(t)(I-P)X^{-1}(s)f(s)\,ds, \tag{3}$$

and an elementary calculation yields the inequality

$$|x(t)| \leqslant \frac{2K}{\alpha} \sup_{-\infty<t<\infty} |f(t)|, \tag{4}$$

for all t in R.

When $A(t)$ is a constant matrix A, (1) has an exponential dichotomy if and only if none of the eigenvalues of A has zero real part.

Now we are ready to state our

THEOREM. *Suppose $A(t)$ is a continuous matrix function such that the linear equation* (1) *has a fundamental matrix $X(t)$ satisfying* (2). *Suppose $f(t, x)$ is a continuous function of $R \times R^n$ into R^n such that*

$$|f(t,x)| \leqslant \mu, \qquad |f(t,x_1) - f(t,x_2)| \leqslant \gamma\,|x_1 - x_2|,$$

for all t, x, x_1, x_2. Then if

$$4\gamma K \leqslant \alpha,$$

there is a unique function $H(t, x)$ of $R \times R^n$ into R^n satisfying

(i) *$H(t, x) - x$ is bounded in $R \times R^n$,*

(ii) *if $x(t)$ is any solution of the differential equation*

$$x' = A(t)x + f(t,x), \tag{5}$$

then $H[t, x(t)]$ is a solution of (1).

Moreover H is continuous in $R \times R^n$ and

$$|H(t,x) - x| \leqslant 4K\mu\alpha^{-1},$$

for all t, x. For each fixed t, $H_t(x) = H(t, x)$ is a homeomorphism of R^n. $L(t, x) = H_t^{-1}(x)$ is continuous in $R \times R^n$ and if $y(t)$ is any solution of (1) *then $L[t, y(t)]$ is a solution of* (5).

3. Proof of the Theorem

For the proof of the theorem we require a result which we state without proof since it is well known in principle (see, for example, [1, pp. 441–442]).

Lemma 1. *Suppose $A(t)$ is a continuous matrix function such that the linear equation* (1) *has a fundamental matrix $X(t)$ satisfying* (2). *Suppose $h(t, x, \eta)$ is a continuous vector function defined for all t in R, x in R^n and η in some Banach space F such that*

$$| h(t, x, \eta)| \leqslant \mu, \qquad | h(t, x_1, \eta) - h(t, x_2, \eta)| \leqslant \gamma \mid x_1 - x_2 \mid,$$

for all t, x, x_1, x_2, η. Then if

$$4\gamma K \leqslant \alpha,$$

the differential equation,

$$x' = A(t)\, x + h(t, x, \eta),$$

has, for each η in F, a unique bounded solution $x(t) = \chi(t, \eta)$. Moreover $\chi(t, \eta)$ is continuous in $R \times F$ and

$$| \chi(t, \eta)| \leqslant 2K\mu\alpha^{-1},$$

for all t, η.

The proof of our main theorem will be straightforward once we have proved the following.

Lemma 2. *Suppose $A(t)$ is a continuous matrix function such that the linear equation* (1) *has a fundamental matrix $X(t)$ satisfying* (2). *Suppose $f(t, x)$ is a continuous function of $R \times R^n$ into R^n such that*

$$| f(t, x)| \leqslant \mu, \qquad | f(t, x_1) - f(t, x_2)| \leqslant \gamma \mid x_1 - x_2 \mid,$$

for all t, x, x_1, x_2. Suppose g is another continuous function satisfying the same conditions as f. Then if

$$4\gamma K \leqslant \alpha,$$

there is a unique function $H(t, x)$ of $R \times R^n$ into R^n satisfying

(i) *$H(t, x) - x$ is bounded in $R \times R^n$,*

(ii) *if $x(t)$ is any solution of* (5), *then $H[t, x(t)]$ is a solution of the differential equation*

$$x' = A(t)\,x + g(t, x). \tag{6}$$

Moreover H is continuous in $R \times R^n$ and

$$|\,H(t, x) - x\,| \leqslant 4K\mu\alpha^{-1},$$

for all t, x.

Proof. The idea of the proof is to take the unique solution $x(t)$ of (5) such that $x(\tau) = \xi$. Using Lemma 1 there is a unique solution $y(t)$ of (6) such that $y(t) - x(t)$ is bounded. Then define $H(\tau, \xi)$ as $y(\tau)$.

So let $x(t, \tau, \xi)$ denote the unique solution $x(t)$ of (5) such that $x(\tau) = \xi$. Put

$$h(t, z, (\tau, \xi)) = g[t, x(t, \tau, \xi) + z] - f[t, x(t, \tau, \xi)].$$

Then

$$|\,h(t, z, (\tau, \xi))| \leqslant 2\mu, \qquad |\,h(t, z_1, (\tau, \xi)) - h(t, z_2, (\tau, \xi))| \leqslant \gamma\,|\,z_1 - z_2\,|,$$

for all $t, z, z_1, z_2, (\tau, \xi)$. It follows from Lemma 1 that the differential equation,

$$z' = A(t)\,z + h(t, z, (\tau, \xi)), \tag{7}$$

has for each (τ, ξ) a unique bounded solution $z(t) = \chi(t, (\tau, \xi))$. Moreover $\chi(t, (\tau, \xi))$ is continuous in (t, τ, ξ) and

$$|\,\chi(t, (\tau, \xi))| \leqslant 4K\mu\alpha^{-1},$$

for all t, τ, ξ. We define

$$H(\tau, \xi) = \xi + \chi(\tau, (\tau, \xi)),$$

for all τ, ξ. Then $H(\tau, \xi)$ is continuous in $R \times R^n$ and

$$|\,H(\tau, \xi) - \xi\,| \leqslant 4K\mu\alpha^{-1}.$$

Now let $x(t)$ be any solution of (5). Then

$$H[t, x(t)] = x(t) + \chi(t, (t, x(t))).$$

$\chi(s, (t, x(t)))$ is the unique bounded solution of the differential equation

$$dz/ds = A(s)\,z + h[s, z, (t, x(t))].$$

But

$$h[s, z, (t, x(t))] = h[s, z, (0, x(0))]$$

since $x(s, t, x(t)) = x(s, 0, x(0))$ from the uniqueness of the solutions of (5). So

$$\chi(s, (t, x(t))) = \chi(s, (0, x(0))),$$

for all s. In particular

$$\chi(t, (t, x(t))) = \chi(t, (0, x(0))).$$

Thus

$$H[t, x(t)] = x(t) + \chi(t, (0, x(0))).$$

Then, differentiating, we find that $H[t, x(t)]$ is a solution of (6).

So H satisfies (i) and (ii). Suppose that $K(t, x)$ is another function satisfying (i) and (ii). Then, for all τ and ξ, $K[t, x(t, \tau, \xi)]$ is a solution of (6). Put $z(t) = K[t, x(t, \tau, \xi)] - x(t, \tau, \xi)$. Differentiating, we find that $z(t)$ is a solution of (7). Moreover $z(t)$ is bounded because K satisfies (i). Then we must have $z(t) = \chi(t, (\tau, \xi))$. So, for all τ and ξ,

$$K[t, x(t, \tau, \xi)] = x(t, \tau, \xi) + \chi(t, (\tau, \xi)).$$

Taking $t = \tau$,

$$K(\tau, \xi) = \xi + \chi(\tau, (\tau, \xi)) = H(\tau, \xi).$$

So H is unique and the proof of the lemma is complete.

Proof of Theorem. To prove the theorem we use a device, originally used by Moser, which was also used by Pugh [3]. Applying Lemma 2, there is a unique $H(t, x)$ such that

(i) $H(t, x) - x$ is bounded,

(ii) if $x(t)$ is any solution of (5), then $H[t, x(t)]$ is a solution of (1).

Moreover H is continuous and $| H(t, x) - x | \leqslant 4K\mu\alpha^{-1}$. Applying the lemma in the reverse direction, there is a unique $L(t, x)$ such that

(iii) $L(t, x) - x$ is bounded,

(iv) if $y(t)$ is any solution of (1), then $L[t, y(t)]$ is a solution of (5).

L is continuous and

$$| L(t, x) - x | \leqslant 4K\mu\alpha^{-1}.$$

Put $J(t, x) = L[t, H(t, x)]$. If $x(t)$ is a solution of (5), then $H[t, x(t)]$ is a solution of (1) and so $L[t, H[t, x(t)]]$, i.e., $J[t, x(t)]$, is a solution of (5). Also

$$\begin{aligned} | J(t, x) - x | &= | L[t, H(t, x)] - x | \\ &\leqslant | L[t, H(t, x)] - H(t, x)| + | H(t, x) - x | \\ &\leqslant 8K\mu\alpha^{-1}. \end{aligned}$$

Now in Lemma 2, when $f = g$ the function $H(t, x) = x$ for all t and x must be the unique function satisfying (i) and (ii). But we have just shown that $J(t, x)$ satisfies (i) and (ii) when $f = g$. So we must have $J(t, x) = x$ for all t and x. Hence

$$L[t, H(t, x)] = x,$$

for all t and x. A similar argument shows that

$$H[t, L(t, x)] = x,$$

for all t and x. So H and L are inverses of each other for each fixed t and so they are both homeomorphisms for each fixed t. This completes the proof of the theorem.

References

1. P. Hartman, "Ordinary Differential Equations," John Wiley, New York, 1964.
2. D. M. Grobman, The topological classification of the vicinity of a singular point in n-dimensional space, *Math. USSR-Sb.* **56** (1962), 77–94.
3. C. Pugh, On a theorem of P. Hartman, *Amer. J. Math.* **91** (1969), 363–367.