محمد امانلو ۸۱۰۱۰۰۰۱۴ تکلیف ۶ آمار مهندسی

1

$Z = X+Y \qquad T = aX + \frac{1}{2}Y \qquad E[X] = E[Y] = 0 \qquad var(X) = var(Y) = \sigma^2$

$$\rho = \frac{1}{2} \longrightarrow \frac{cov(Z,T)}{\sqrt{var(Z)var(T)}} = \frac{E\left[(Z-E[Z])(T-E[T])\right]}{\sqrt{var(Z)var(T)}} = \frac{1}{2}$$

$$E[Z] = E[X+Y] = E[X]+E[Y] = 0$$

$$E[T] = E[aX+\tfrac{1}{2}Y] = aE[X]+\tfrac{1}{2}E[Y] = 0$$

$$var(Z) = E[Z^2] - E[Z]^2 = E[Z^2] = E[X^2+Y^2+2XY] = E[X^2]+E[Y^2]+2\underbrace{E[X]E[Y]}_{0}$$

$$= var(X) + \underbrace{E[X]^2}_{0} + var(Y) + \underbrace{E[Y]^2}_{0} = \sigma^2+\sigma^2 = 2\sigma^2$$

$$var(T) = E[T^2] - E[T]^2 = E[T^2] = E\left[a^2X^2+\tfrac{1}{4}Y^2+aXY\right] = a^2E[X^2]+\tfrac{1}{4}E[Y^2]+a\underbrace{E[X]E[Y]}_{0}$$

$$= a^2\, var(X) + \frac{1}{4}\, var(Y) = a^2\sigma^2+\frac{1}{4}\sigma^2 = (a^2+\tfrac{1}{4})\sigma^2$$

$$\rightarrow \rho = \frac{E[(Z-0)(T-0)]}{\sqrt{2\sigma^2(a^2+\frac{1}{4})\sigma^2}} = \frac{E[ZT]}{\sigma^2\sqrt{2a^2+\frac{1}{2}}} = \frac{E\left[aX^2+\frac{1}{2}Y^2+(a+\frac{1}{2})XY\right]}{\sigma^2\sqrt{2a^2+\frac{1}{2}}} = \frac{aE[X^2]+\frac{1}{2}E[Y^2]+(a+\frac{1}{2})E[XY]}{\sigma^2\sqrt{2a^2+\frac{1}{2}}}$$

$$= \frac{a\, var(X)+\frac{1}{2}var(Y)+0}{\sigma^2\sqrt{2a^2+\frac{1}{2}}} = \frac{(a+\frac{1}{2})\sigma^2}{\sigma^2\sqrt{2a^2+\frac{1}{2}}} = \frac{a+\frac{1}{2}}{\sqrt{2a^2+\frac{1}{2}}} = \frac{1}{2} \longrightarrow \boxed{a = -1 \pm \frac{\sqrt{3}}{2}}$$

3

$Y \sim u(0,1) \qquad X \sim u(0,1)$

$$Z = (-2\ln X)^{\frac{1}{2}} \cos(2\pi Y)$$

$$W = (-2\ln X)^{\frac{1}{2}} \sin(2\pi Y) \qquad // \text{ تغییر متغیر}$$

$$\begin{cases} \frac{W}{Z} = \tan(2\pi Y) \\ \cos^2 x + \sin^2 x = 1 \longrightarrow \frac{Z^2}{-2\ln X} + \frac{W^2}{-2\ln X} = 1 \longrightarrow Z^2+W^2 = -2\ln X \end{cases} \longrightarrow \begin{cases} X = e^{-\frac{1}{2}(Z^2+W^2)} \\ Y = \frac{1}{2\pi}\tan^{-1}\left(\frac{W}{Z}\right) \end{cases}$$

$$J = \begin{vmatrix} \frac{\partial X}{\partial Z} & \frac{\partial X}{\partial W} \\ \frac{\partial Y}{\partial Z} & \frac{\partial Y}{\partial W} \end{vmatrix} = \begin{vmatrix} -Ze^{-\frac{1}{2}(Z^2+W^2)} & -We^{-\frac{1}{2}(Z^2+W^2)} \\ -\frac{1}{2\pi}\frac{W}{W^2+Z^2} & \frac{1}{2\pi}\frac{Z}{Z^2+W^2} \end{vmatrix} = -\frac{e^{-\frac{1}{2}(Z^2+W^2)}}{2\pi}$$

$$f_{XY}(x,y) = \frac{f_{ZW}(z,w)}{|J(z,w)|} \longrightarrow f_{ZW}(z,w) = f_{XY}(x,y)\,|J(z,w)|$$

$$X,Y \sim U(0,1) \longrightarrow f_{XY}(x,y) = 1 \longrightarrow f_{ZW}(z,w) = |J(z,w)| = \frac{e^{-\frac{1}{2}(z^2+w^2)}}{2\pi}$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{ZW}(z,w)\,dw = \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}(z^2+w^2)}}{2\pi}\,dw = \frac{e^{-\frac{1}{2}z^2}}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}w^2}\,dw = \frac{e^{-\frac{1}{2}z^2}}{2\pi}\sqrt{2\pi} = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$$

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$S \leftarrow$ امتیاز

$D \leftarrow$ تاس

$C \leftarrow$ سکه

$$E[S] = 2E[D|C=H]P(C=H) + \frac{1}{2}E(D|C=T)P(C=T)$$

$$= 2\times\frac{1}{2}\times\frac{1}{6}(1+2+3+4+5+6) + \frac{1}{2}\times\frac{1}{2}\times\frac{1}{6}(1+2+3+4+5+6)$$

$$= \frac{35}{8} = 4.375$$

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X: تعداد درخواست های ارسالی به سرور اول

$$X \sim Bin(2000, N)$$

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$$f_{N|X}(n|x=2000) = \frac{P_{X|N}(x|n)\,f_N(n)}{P_X(x)} = 2001\,n^{2000} \qquad 0<n<1$$

$$P_{X|N}(x=2000|n) = \binom{2000}{2000} n^{2000}(1-n)^0 = n^{2000}$$

$$f_N(n) = \begin{cases} 1 & 0<n<1 \\ 0 & oth \end{cases}$$

$$P_X(x=2000) = \int_0^1 n^{2000}\,dn = \frac{n^{2001}}{2001}\Big|_0^1 = \frac{1}{2001}$$

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$$E[N|X=2000] = \int_0^1 n f_{N|X}(n|x=2000)\,dn = \int_0^1 \frac{n}{B(2001,1)} n^{2000}\,dn = \int_0^1 n^{2001}\,2001\,dn = 2001\,\frac{n^{2002}}{2002}\Big|_0^1$$

$$= \frac{2001}{2002}$$

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$$P(x_2=2000|x_1=2000) = \int_0^1 n^{2000}\times f_{N|X}(n|x_1=2000)\,dn = \int_0^1 n^{2000}(2001\,n^{2000})\,dn$$

$$= 2001\int_0^1 n^{4000}\,dn = 2001\,\frac{n^{4001}}{4001}\Big|_0^1 = \frac{2001}{4001}$$

$$Z = \frac{X}{Y}$$

(2)

if $z \geq 1$: $P(Z \leq z) = P(X \leq zY) = \frac{1}{a^2} \int_0^a \int_0^a X_{x \leq zy} \, dy \, dx = \frac{1}{a^2} \int_0^a \min(a, zy) \, dy$

$$= \frac{1}{a^2}\left(\int_0^{\frac{a}{z}} yz \, dy + \int_{\frac{a}{z}}^a a \, dy \right) = \frac{1}{a^2}\left(\frac{a^2}{2z} + a^2 - \frac{a^2}{z} \right) = 1 - \frac{1}{2z}$$

if $z < 1$: $P(Z \leq z) = P(X \leq zY) = \frac{1}{a^2} \int_0^a \int_0^a X_{x \leq zy} \, dy \, dx = \frac{1}{a^2} \int_0^a zy \, dy = \frac{z}{2}$

if $w \geq 1$: $P(W \leq w) = P(\frac{1}{w} \leq Z \leq w) = P(Z \leq w) - P(Z < \frac{1}{w}) = 1 - \frac{1}{2w} - \frac{1}{2w} = 1 - \frac{1}{w}$

4

A : تعداد درخواست با شروع از هسته اول

B : " " " " دوم

$$E[A] = E[B|A✓]P_A + E[B|A✗](1-P_A) \rightarrow E[B|A✓] = 2P_B + (E[A]+2)(1-P_B)$$

$$E[B|A✗] = \underbrace{1}_{\text{درخواست اول}} + E[B]$$

$$\rightarrow E[A] = \left(2P_B + (E[A]+2)(1-P_B)\right)P_A + (1+E[B])(1-P_A) =$$

$$= 2P_AP_B + P_A E[A] - P_AP_B E[A] + 2P_A - 2P_AP_B + 1 - P_B + E[B] - P_A E[B]$$

$$(1-P_A+P_AP_B)E[A] = (1-P_A)E[B] + P_A + 1$$

$$E[B] = E[A|B✓]P_B + E[A|B✗](1-P_B) \rightarrow E[A|B✓] = 2P_A + (E[B]+2)(1-P_A)$$

$$E[A|B✗] = 1 + E[A]$$

$$E[B] = \left(2P_A + (E[B]+2)(1-P_A)\right)P_B + (1+E[A])(1-P_B)$$

$$\rightarrow (1-P_B+P_AP_B)E[B] = (1-P_B)E[A] + P_B + 1$$

$$\xrightarrow{\text{جایگذاری}} (1-P_B+P_AP_B)E[B] = (1-P_B)\times\frac{(1-P_A)E[B]+P_A+1}{1-P_A+P_AP_B} + P_B + 1$$

$$\rightarrow E[B] = \frac{P_AP_B^2 - P_AP_B + 2}{2P_AP_B - P_A^2P_B - P_AP_B^2 + P_A^2P_B^2} \xrightarrow{\text{جایگذاری}} (1-P_A+P_BP_A)E[A] = (1-P_A)\frac{P_AP_B^2 - P_BP_A + 2}{2P_AP_B - P_A^2P_B - P_AP_B^2 + P_A^2P_B^2}$$

$$\longrightarrow E[A] = \frac{p_A^r p_B - p_A p_B + r}{r p_A p_B - p_A p_B^r - p_A p_B^r + p_A^r p_B^r}$$

$$p_B > p_A \longrightarrow p_A p_B^r > p_A^r p_B^r \longrightarrow E[B] > E[A]$$

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$T|N \sim Bin(n,p)$

$E[T] = E[E[T|N]] = E[Np] = p \times \frac{1}{s}$

$Var(T) = E[var(T|N)] + var(E[T|N]) = E[Np(1-p)] + var(np) = p(1-p)\times\frac{1}{s} + p^2\frac{(1-s)}{s^2}$

$= \frac{p}{s} - \frac{2p^2}{s} + \frac{p^2}{s^2}$

ب)

$M_T(t) = E[e^{tT}] = E[E[e^{tT}|N]] = E[(pe^t+q)^N] = \sum_{n=1}^{\infty} (pe^t+q)^n (1-s)^{n-1} s$

$= s\sum_{n=0}^{\infty} (pe^t+q)^{n+1}(1-s)^n = s(pe^t+q)\sum_{n=0}^{\infty}((pe^t+q)(1-s))^n = \frac{s(pe^t+q)}{1-(pe^t+q)(1-s)}$

۸ الف) ؟

$X \sim Poi(3\lambda) \quad Y \sim Poi(6\lambda) \quad Z \sim Poi(6\lambda)$

ب)

$P(X+Y | X+Y+Z = 36) = ?$

$\underbrace{P(X+Y = k \,|\, Z = 36-k)}_{\text{مستقل}} = \frac{P(X+Y=k)\,P(Z=36-k)}{P(X+Y+Z=36)} = \binom{36}{k}\left(\frac{9\lambda}{15\lambda}\right)^k\left(\frac{6\lambda}{15\lambda}\right)^{36-k}$

بنابراین $P(X+Y | X+Y+Z=36) \sim Bin(36, \frac{9}{15})$

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$E[X+Y \,|\, X+Y+Z=36] = 36 \times \frac{9}{15} = 21.6$