Due 23:59 Dec 6 (Tuesday). There are 100 points in this assignment.
Submit your answers (**must be typed**) in pdf file to CourSys `https://coursys.sfu.ca/2022fa-cmpt-705-x1/`. Submissions received after 23:59 of Dec 6 will get penalty of reducing points: 20 and 50 points deductions for submissions received at $[00:00, 00:10]$ and $(00:10, 00:30]$ of Dec 6, respectively; no points will be given to submissions after $00:30$ of Dec 6.

1. (Chapter 10 Problem 1 of the text book) 20 points

   The Hitting Set problem is defined as follows: Let $A = \{a_1, .., a_n\}$ and $B_1, .., B_m$ be a collection of subsets of A, a subset H of A is a hitting set for $B_1, .., B_m$ is $H \cap B_i \neq \emptyset$ for every $1 \leq i \leq m$. The Hitting Set problem is that given A and $B_1, .., B_m$, and integer $k > 0$, whether there is a hitting set of size k or not. The problem is NP-complete. Assume that $|B_i| \leq c$ for some constant $c > 0$. Give an algorithm that solves the problem with a running time of $O(f(c, k) \cdot p(n, m))$, where $p(n, m)$ is a polynomial in n and m, and $f(c, k)$ is an arbitrary function depending only on c and k, not on n or m.

   Answer

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2. (Chapter 10 Problem 3 of the text book) 15 points

   Give a digraph G of $V(G) = \{v_1, .., v_n\}$, we want to decide if G has a Hamiltonian path from v_1 to v_n. Give an $O(2^n p(n))$ time algorithm to solve the Hamiltonian path problem in G, where $p(n)$ is a polynomial in n.

   Answer

3. (Chapter 10 Problem 5 of the text book) 15 points

   A dominating set D of a graph G is a subset of $V(G)$ such that for every node u of G, either $u \in D$ or u is adjacent to a node $v \in D$. The minimum dominating set problem in G is to find a minimum dominating set D of G (the problem is NP-complete). Give a polynomial time algorithm for the minimum dominating set problem in a tree.

Answer

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4. (15 points)

   Below is an algorithm to create a random permutation of n elements in an array $A[1..n]$.
   RANDOM-PERMUTE(A, n)
   for $i = 1$ **to** n
   select j uniformly at random from $\{i, i+1, .., n\}$ and exchange $A[i]$ with $A[j]$;

   Prove that the algorithm creates every permutation of $A[1..n]$ with probability$1/n!$.

Answer

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5. (20 points)

   Given a graph G, we consider the following problem: color the nodes of G using k colors; an edge $e = \{u, v\}$ is called satisfied if u and v are colored by different colors; and we want to color the nodes to satisfy as many edges as possible.

   (a) (14 points) A randomized algorithm RA for the maximization problem is as follows: for every node v of G, select one of the k colors independently with probability $1/k$ and assign the color to v. Prove that the expected number of edges satisfied by RA is $(1 - 1/k)m$, where m is the number of edges in G. Assume $(1 - 1/k)m$ is an integer, prove that the probability that RA satisfies at least $(1 - 1/k)m$ edges is at least $1/m$.

   (b) (6 points) Assume $(1 - 1/k)m$ is an integer. Give a Monte Carlo algorithm which satisfies at least $(1-1/k)m$ edges with probability at least $1-1/m$. Give a Las Vegas algorithm which satisfies at least $(1 - 1/k)m$ edges.

Answer

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6. (Chapter 13 Problem 11 of the text book) 15 points

   There are k machines and k jobs. Each job is assigned to one of the k machines independently at random (with each machine equally likely).

   (a) Let $N(k)$ be the expected number of machines that do not receive any jobs, so $N(k)/k$ is the expected fraction of machines with no job. What is the limit $\lim_{k\to\infty} N(k)/k$? Give a proof of your answer.

   (b) Suppose that machines are not able to queue up excess jobs, so if the random assignment of jobs to machines sends more than one job to a machine M, then M will do the first of the jobs it receives and reject the rest. Let $R(k)$ be the expected number of rejected jobs; so $R(k)/k$ is the expected fraction of rejected jobs. What is $\lim_{k\to\infty} R(k)/k$? Give a proof of your answer.

   (c) Now assume that machines have slightly larger buffers; each machine M will do the first two jobs it receives, and reject any additional jobs. Let $R_2(k)$ denote the expected number of rejected jobs under this rule. What is $\lim_{k\to\infty} R_2(k)/k$? Give a proof of your answer.

Answer
