Mathematical Expectation

Expected value or mean of a random Variable:-

The expected value of a random variable 'x' is defined by $E(x) = \mu$.

⇒ It is the value to which the mean tends as the length of the sequence gets larger and larger or tends to infinity.

⇒ Let 'x' be a discrete random variable with values $x_1, x_2 \dots x_n, \dots$ having corresponding probabilities $f(x_1), f(x_2) \dots f(x_n) \dots$ such that $\Sigma f(x) = 1$ then the mathematical expectation or the expected value of x is defined as

$$E(x) = x_1 f(x_1) + x_2 f(x_2) + \dots + x_n f(x) + \dots$$

$$E(x) = \sum_{i=1}^{\infty} x_i f(x_i)$$

⇒ If the random variable is continuous with p.d.f $f(x)$, then

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$$

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Example 7.10 What is the mathematical expectation of the number of heads when 3 fair coins are tossed.

Let X: Number of heads
X: 0, 1, 2, 3

x	$f(x)$	$xf(x)$
0	1/8	0
1	3/8	3/8
2	3/8	6/8
3	1/8	3/8
		12/8

$$E(X) = \sum x f(x)$$
$$= \frac{12}{8} = 1.5$$
$$\boxed{E(X) = 1.5}$$

Average number of heads 1.5 means that if a person tosses 3 coins over and over again will on the average gets 1.5 heads per toss.

Variance of Random Variable:-

$$Var(X) = E(X^2) - [E(X)]^2$$
$$= \sum x^2 f(x) - \left[\sum x f(x)\right]^2$$

x	$f(x)$	$xf(x)$	$x^2f(x)$
0	1/8	0	0
1	3/8	3/8	3/8
2	3/8	6/8	12/8
3	1/8	3/8	9/8
		12/8	24/8

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \Sigma x^2 f(x) - [\Sigma x f(x)]^2$$

$$= \frac{24}{8} - (12/8)^2$$

$$= 3 - (1.5)^2$$

$$\boxed{\text{Var}(X) = 0.75}$$

Example 7.15 If the continuous r.v X has p.d.f.

$$f(x) = \frac{3}{4}(3-x)(x-5) \qquad 3 \le x \le 5$$

$$= 0 \qquad \text{Elsewhere.}$$

Find Mean, variance and S.D of X.

Solution:-

As $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$

So

$$E(x) = \int_{-\infty}^{\infty} x \cdot \left[\frac{3}{4}(3-x)(x-5)\right] dx$$

$$E(x) = \frac{3}{4}\int_3^5 x(3-x)(x-5)\,dx$$

$$= \frac{3}{4}\int_3^5 (-x^3+8x^2-15x)\,dx$$

$$= \frac{3}{4}\left[-\frac{x^4}{4}+\frac{8x^3}{3}-\frac{15x^2}{2}\right]_3^5$$

$$E(x) = \frac{3}{4}\left[\left(-\frac{5^4}{4}+\frac{8(5^3)}{3}-\frac{15(5^2)}{2}\right)-\left(-\frac{3^4}{4}+\frac{8(3)^3}{3}-\frac{15(3^2)}{2}\right)\right]$$

$$= \frac{3}{4}\left[\left(-\frac{125}{12}+\frac{63}{4}\right)\right]$$

$$= \frac{3}{4}\left(\frac{64}{12}\right) = 4$$

$$\boxed{E(x) = 4}$$

As $var(x) = E(x^2) - [E(x)]^2$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \frac{3}{4}\int_3^5 x^2(3-x)(x-5)\,dx$$

$$= \frac{3}{4}\int_3^5 (-x^4+8x^3-15x^2)\,dx$$

$$= \frac{3}{4}\left[-\frac{x^5}{5}+\frac{8x^4}{4}-\frac{15x^3}{3}\right]_3^5$$

$$= \frac{3}{4}\left[-\frac{1}{5}(3125)+2(625)-5(125)-\left\{-\frac{1}{5}(243)+2(81)-5(27)\right\}\right]$$

$$= \frac{3}{4}\left(\frac{108}{5}\right)$$

$$\boxed{E(x^2) = \frac{81}{5}}$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= \frac{81}{5} - (4)^2$$

$$= 1/5 = 0.2$$

$$\boxed{Var(x) = 0.2}$$

$$S.D(x) = \sqrt{0.2}$$

$$\boxed{S.D(x) = 0.447}$$

Expectation of function of random variable :-

Let $H(x)$ be a function of random variable x then

$$E[H(x)] = \sum H(x_i) f(x_i)$$ Discrete random variable

$$E[H(x)] = \int_{-\infty}^{\infty} H(x_i) f(x_i)\, dx$$ continuous random variable

(Walpole)

Example 4.4

x	$f(x)$	$H(x) = 2x-1$	$H(x)f(x)$
4	1/12	$2(4)-1 = 7$	7/12
5	1/12	9	9/12
6	1/	11	11/4
7	1/	13	13/4
8	1/6	15	15/6
9	1/6	17	17/6
			12.67

$$E[H(x)] = E(2x-1) = \sum H(x)f(x)$$
$$= 12.67$$

Ex(7-14):- Let x be a r.v with p.d.f

$$f(x) = 2(x-1) \qquad 1 < x < 2$$
$$= 0 \qquad \text{else where.}$$

Find the Expected values of $H(x) = 2x-1$ and $H(x) = x^2$

Solution

$$H(x) = 2x-1$$

$$E[H(x)] = E(2x-1)$$

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$$E(2X-1) = \int_{-\infty}^{\infty} (2x-1) f(x)\, dx$$

$$= \int_{1}^{2} (2x-1) \cdot 2(x-1)\, dx$$

$$= 2\int_{1}^{2} (2x-1)(x-1)\, dx$$

$$= 2\int_{1}^{2} (2x^2-3x+1)\, dx$$

$$= 2\left[\frac{2x^3}{3} - \frac{3x^2}{2} + x\right]_1^2$$

$$= 2\left[\left(\frac{16}{3} - 6 + 2\right) - \left(\frac{2}{3} - \frac{3}{2} + 1\right)\right]$$

$$= 2\left[\frac{4}{3} - \frac{1}{6}\right]$$

$$\boxed{E(2X-1) = \frac{7}{3}}$$

ii) $H(X) = x^2$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{1}^{2} x^2\, 2(x-1)\, dx$$

$$= 2\int_{1}^{2} x^2(x-1)\, dx$$

$$= 2\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^2$$

$$= 2\left[\left(4 - 8/3\right) - \left(1/4 - 1/3\right)\right]$$

$$= 2\left[4/3 + 1/12\right] = 17/6$$