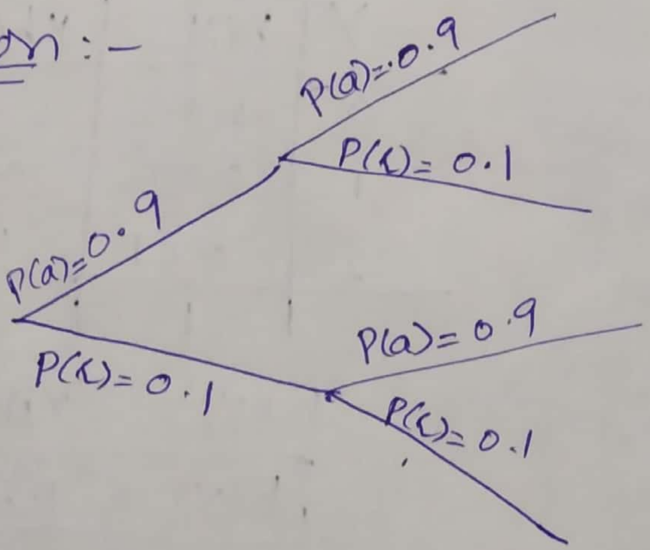
Question

Test two integrated circuits one after another. On each test the possible outcomes are 'a' (acceptable) and 'r' (Reject). Assume that all circuits are acceptable with probability 0.9 and the outcomes of successive tests are independent. Count the number of acceptable circuits X and count the number of tests, Y before observing the first reject. Draw the diagram for the experiment and find the joint PMF of X and Y. Also find $cov(X,Y)$.

Solution:-



$P(a \cap a) = 0.81$
$X = 2,\ Y = 2$

$P(a \cap r) = 0.09$
$X = 1,\ Y = 1$

$P(r \cap a) = 0.09$
$X = 1,\ Y = 0$

$P(r \cap r) = 0.01$
$X = 0,\ Y = 0$

$S = \{aa, ar, ra, rr\}$

X: Number of acceptable circuits

Y: Number of tests before observing first reject.

$$f(x,y) = \begin{cases} 0.81 & x=2, y=2 \\ 0.09 & x=1, y=1 \\ 0.09 & x=1, y=0 \\ 0.01 & x=0, y=0 \\ 0 & \text{otherwise} \end{cases}$$

X \ Y	0	1	2	g(x)
0	0.01	0	0	0.01
1	0.09	0.09	0.	0.18
2	0	0	0.81	0.81
h(y)	0.1	0.09	0.81	1

As $cov(X,Y) = E(XY) - E(X)E(Y)$

X	g(x)	xg(x)
0	0.01	0
1	0.18	0.18
2	0.81	1.62
	1	1.18

Y	h(y)	yh(y)
0	0.1	0
1	0.09	0.09
2	0.81	1.62
	1	1.71

$E(X) = \Sigma x g(x)$

$\boxed{E(X) = 1.18}$

$E(Y) = \Sigma y h(Y)$

$\boxed{E(Y) = 1.71}$

$E(XY) = \Sigma\Sigma xy f(x,y)$

$= (1\times1)(0.09) + (2\times2)(0.81) + 0$

$= 0.09 + 3.24$

$\boxed{E(XY) = 3.33}$

$Cov(X,Y) = E(XY) - E(X)\cdot E(Y)$

$= 3.33 - (1.18)\cdot(1.71)$

$\boxed{Cov(X,Y) = 1.3122}$

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Question:-

A firm sends out two kinds of promotional facsimiles. One kind contains only text and requires 40 seconds to transmit each page, the other kind contains grayscale pictures that takes 60 seconds per page, faxes can be 1, 2, 3 page long. Let random variable L represents the length of a fax in pages, $S_L = \{1, 2, 3\}$. Let the random variable T represents the time to send each page, $S_T = \{40, 60\}$. After observing many faxes transmissions, the firm derives the following probability models for L and T.

	t = 40 sec	t = 60 sec
l = 1 page	0.15	0.1
l = 2 pages	0.3	0.2
l = 3 pages	0.15	0.1

Calculate

i) $E(L)$,

ii) $E(T)$

iii) $var(L)$

(iv) $var(T)$

(v) $E(LT)$

(vi) $cov(LT)$

(vii) $\rho_{L,T}$

Solution:-

L \ T	40	60	g(L)
1	0.15	0.1	0.25
2	0.0	0.2	0.5
3	0.15	0.1	0.25
h(T)	0.6	0.4	1

$E(L) = \Sigma L\, g(L)$

i) $E(L) = (1 \times 0.25) + (2 \times 0.5) + (3 \times 0.25)$

$$\boxed{E(L) = 2}$$

(ii) $E(T) = \Sigma T\, h(T)$

$= (40 \times 0.6) + (60 \times 0.4)$

$$\boxed{E(T) = 48}$$

(iii) $E(L^2) = \Sigma L^2 g(L)$

$= (1^2 \times 0.25) + (2^2 \times 0.5) + (3^2 \times 0.25)$

$E(L^2) = 4.5$

$var(L) = E(L^2) - [E(L)]^2$

$= 4.5 - (2)^2$

$= 0.5$

(iv) $E(T^2) = \Sigma T^2 h(T)$

$= (40^2 \times 0.6) + (60^2 \times 0.4)$

$= 960 + 1440$

$E(T^2) = 2400$

$var(T) = E(T^2) - [E(T)]^2$

$$\mathrm{var}(T) = E(T^2) - [E(T)]^2$$
$$= 2400 - (48)^2$$
$$\boxed{\mathrm{Var}(T) = 96}$$

(v) $E(LT) = \sum\sum LT\, f(L,T)$

$$= (40 \times 0.15) + (80 \times 0.3) + (120 \times 0.15)$$
$$+ (60 \times 0.1) + (120 \times 0.2) + (180 \times 0.1)$$
$$= 6 + 24 + 18 + 6 + 24 + 18$$
$$\boxed{E(LT) = 96}$$

(vi) $\mathrm{cov}(L,T) = E(LT) - E(L)E(T)$

$$= 96 - (2 \times 48)$$
$$\boxed{\mathrm{cov}(L,T) = 0}$$

(vii) $\rho_{(L,T)} = \dfrac{\mathrm{cov}(x,y)}{\sqrt{\mathrm{var}(x)\cdot \mathrm{var}(y)}} = 0$

Moment Generating Function :-

The moment generating function (m.g.f) ussally denoted by $M_0(t)$, of a random variable x about the origin if it exists, is defined as the expected value of the r-v e^{tx}, where t is a real variable lying in neighbourhood of zero.

$$M_0(t) = E(e^{tx}) = \sum_{i=1}^{\infty} e^{tx_i} f(x_i)$$ if x is discrete r.v.

$$M_0(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$ if x is Continuous r.v.

MGF of binomial distribution :-

$$M_0(t) = E(e^{tx})$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x} \qquad \because e^{tx} = (e^t)^x$$

$$= (q + pe^t)^n$$

we get the moments by differentiating $M_0(t)$ once, twice etc w.r.t t and putting $t=0$.

MGF of Continuous uniform distribution.

$$M_0(t) = E[e^{tx}] \qquad \text{as } f(x) = \int_a^b \frac{1}{b-a}\,dx$$

$$= \int_a^b e^{tx} \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\int_a^b e^{tx}\,dx$$

$$= \frac{1}{b-a}\left[\frac{1}{t}e^{tx}\right]_a^b$$

$$M_0(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$$

MGF of Exponential distribution:-

$$M_0(t) = E[e^{tx}]$$

As $f(x) = \lambda e^{-\lambda x}$ for $x > 0$

$$M_0(t) = \int_0^\infty e^{tx}\lambda e^{-\lambda x}\,dx$$

$$= \lambda\int_0^\infty e^{-(\lambda - t)x}\,dx$$

$$= \lambda\left[\frac{-e^{-(\lambda-t)x}}{\lambda - t}\right]_0^\infty$$

$$= \frac{\lambda}{\lambda - t} \quad \text{for } t < \lambda$$

Characteristic Function :-

The m.g.f does not exist for many probability distributions. we then use another function, called the characteristic function (c.f). The characteristic function of a r.v X denoted by $\phi(t)$, is defined as the expected value of the r.v e^{itx} i.e

$$\phi(t) = E(e^{itx})$$

$$\phi(t) = \sum e^{itx} f(x) \quad \text{(Discrete)}$$

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx \quad \text{(continuous)}$$

where t is a real number and $i = \sqrt{-1}$, the imaginary unit.

$\Rightarrow$ The characteristic function always exists because $|e^{itx}| = 1$ for all t, and hence may be defined for every probability distribution.

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