Moment Generating Function :-

The moment generating function (m.g.f) ussally denoted by $M_0(t)$, of a random variable x about the origin if it exists, is defined as the expected value of the r.v e^{tx}, where t is a real variable lying in neighbourhood of zero.

$$M_0(t) = E(e^{tx}) = \sum_{i=1}^{\infty} e^{tx} f(x_i) \quad \text{if } x \text{ is discrete r.v.}$$

$$M_0(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \quad \text{if } x \text{ is continuous r.v}$$

Moment Generating function of binomial distribution:-

$$M_0(t) = E(e^{tx})$$

$$f(x) = {}^nC_x p^x q^{n-x}$$

$$M_0(t) = \sum_{x=0}^{n} e^{tx}\, {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} (e^t)^x \cdot p^x\, {}^nC_x q^{n-x}$$

$$M_0(t) = \sum_{x=0}^{n} (pe^t)^x \cdot {}^nC_x q^{n-x}$$

$$= (pe^t)^0\, {}^nC_0 q^{n-0} + (pe^t)^1\, {}^nC_1 q^{n-1} + (pe^t)^2\, {}^nC_2 q^{n-2} + \dots + (pe^t)^n\, {}^nC_n q^0$$

$$= q^n + (pe^t) n q^{n-1} + \frac{n(n-1)}{2!} (pe^t)^2 q^{n-2} + \dots + (pe^t)^n$$

$$M_0(t) = (q + pe^t)^n$$

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We get the moments by differentiating $M_0(t)$ once, twice etc w.r.t t and putting $t=0$

first moment $\mu_1' = M_0'(t) = \frac{d}{dt} M_0(t) \Big|_{t=0}$

$$E(X) = \mu_1' = \frac{d}{dt}(q+pe^t)^n \Big|_{t=0}$$

$$= n(q+pe^t)^{n-1} pe^t \Big|_{t=0}$$

$$= n(q+pe^0)^{n-1} pe^0$$

$$= n(q+p)^{n-1} p$$

$$= np(1) \qquad \because q+p=1$$

$$\boxed{M_0'(t) = \mu_1' = np}$$

Second Moment

$$\mu_2' = \frac{d^2}{dt^2} M_0(t) \Big|_{t=0}$$

$$= \frac{d}{dt}\left[n(q+pe^t)^{n-1} pe^t\right] \Big|_{t=0}$$

$$= np \frac{d}{dt}\left[(q+pe^t)^{n-1} e^t\right]_{t=0}$$

$$\mu_2' = np\left[(q+pe^t)^{n-1} e^t + (n-1)(q+pe^t)^{n-2} pe^t . e^t\right] \Big|_{t=0}$$

$$= np\left[(q+pe^t)^{n-1} e^t + (n-1)p(q+pe^t)^{n-2} e^{2t}\right] \Big|_{t=0}$$

$$= np\left[(q+pe^0)^{n-1} e^0 + (n-1)p(q+pe^0)^{n-2} e^0\right]$$

$$= np\left[(q+p)^{n-1} + (n-1)p(q+p)^{n-2}\right]$$

$$= np\left[(1)^{n-1} + (n-1)p(1)^{n-2}\right]$$

$$= np\left[1 + (n-1)p\right]$$

$$= np\left[1 + np - p\right] = np\left[np - p + 1\right]$$

$$E(x^2) = \mu_2' = np\left[p(n-1) + 1\right]$$

$$var(x) = \mu_2' - (\mu_1')^2$$

$$= np\left[p(n-1)+1\right] - (np)^2$$

$$= n(n-1)p^2 + np - n^2p^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$var(x) = npq$$

Moment generating function of Exponential dist :-

$$f(x) = \lambda e^{-\lambda x} \quad \text{for} \quad x > 0$$

$$M_o(t) = E[e^{tx}]$$

$$M_o(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{(t-\lambda)x} dx$$

$$= \lambda \left[\frac{e^{(t-\lambda)x}}{t-\lambda} \right]_0^\infty$$

$$= \lambda \left[\frac{e^{(t-\lambda)x}}{t-\lambda} \right|_0^\infty$$

$$= \frac{\lambda}{t-\lambda} \frac{1}{e^{(\lambda-t)x}} \Big|_0^\infty$$

$$= \frac{\lambda}{t-\lambda} \left[\frac{1}{\infty} - \frac{1}{1} \right]$$

$$= \frac{\lambda}{t-\lambda} (0-1)$$

$$= \frac{-\lambda}{t-\lambda} = \frac{\lambda}{\lambda-t}$$

$$E(x) = \mu_1' = M_o'(t) = \frac{d}{dt} \frac{\lambda}{\lambda - t} \Big|_{t=0}$$

$$= \frac{d}{dt} \lambda (\lambda - t)^{-1}$$

$$= -\lambda(\lambda-t)^{-2}(-1)$$

$$= \frac{\lambda}{(\lambda-t)^2} \Rightarrow \frac{\lambda}{(\lambda-0)^2}$$

$$= \frac{\lambda}{\lambda^2}$$

$$\boxed{E(x) = \frac{1}{\lambda}}$$

$$E(x^2) = \mu_2' = \frac{d}{dt}\left[\frac{\lambda}{(\lambda-t)^2}\right]_{t=0}$$

$$= \frac{d}{dt}\left[\lambda(\lambda-t)^{-2}\right]$$

$$= -2\lambda(\lambda-t)^{-3}(-1)$$

$$= 2\lambda(\lambda-t)^{-3}$$

$$= \frac{2\lambda}{(\lambda-t)^3} \Rightarrow \frac{2\lambda}{(\lambda-0)^3} \Rightarrow \frac{2\lambda}{\lambda^3}$$

So $= \frac{2}{\lambda^2}$

$$\text{var}(x) = \mu_2' - \mu_1'^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\boxed{\text{var}(x) = \frac{1}{\lambda^2}}$$

Characteristic Function :-

The m.g.f does not exist for many probability distributions. We then use another function, called the characteristic function (c.f). The characteristic function of a r.v x denoted by $\phi(t)$, is defined as the expected value of the r.v e^{itx} i.e

$$\phi(t) = E(e^{itx})$$

$$\phi(t) = \Sigma e^{itx} f(x) \qquad \text{(Discrete)}$$

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx} f(x) dx \qquad \text{(continuous)}$$

where t is a real number and $i = \sqrt{-1}$, the imaginary unit.

$\Rightarrow$ The characteristic function always exists because $|e^{itx}| = 1$ for all t, and hence may be defined for every probability distribution.