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CSc 8830: Computer Vision

Assignment 3 (Motion Tracking Equation & Lucas-Kanade Algorithm) - April 5, 2024

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1 Deriving Motion Tracking Equation

Step 1: Given two consecutive frames $I(x, y, t)$ and $I(x+\Delta x, y+\Delta y, t+\Delta t)$, where (x, y) represents the pixel coordinates in the image plane and t represents time, we want to estimate the motion between these frames.

Step 2: Taylor Series Expansion - Assume that the motion between frames is small. We can approximate the intensity value in the second frame around the point (x, y) in the first frame using a Taylor series expansion:

$$I(x + \Delta x, y + \Delta y, t + \Delta t) \approx I(x, y, t) + \frac{\partial I}{\partial x}\Delta x + \frac{\partial I}{\partial y}\Delta y + \frac{\partial I}{\partial t}\Delta t$$

Step 3: Temporal Derivative - Since $\Delta t = 1$ (assuming frames are captured at unit time intervals), we can rewrite the equation as:

$$I(x + \Delta x, y + \Delta y, t + 1) \approx I(x, y, t) + \frac{\partial I}{\partial x}\Delta x + \frac{\partial I}{\partial y}\Delta y + \frac{\partial I}{\partial t}$$

Step 4: Approximation of Gradient Terms - We consider the motion $(\Delta x, \Delta y)$ to be small, so the gradient terms $\frac{\partial I}{\partial x}$ and $\frac{\partial I}{\partial y}$ can be approximated by their values at (x, y) in the first frame. Let's denote them as I_x and I_y respectively.

Step 5: Substitute the gradient terms into the equation:

$$I(x + \Delta x, y + \Delta y, t + 1) \approx I(x, y, t) + I_x\Delta x + I_y\Delta y + \frac{\partial I}{\partial t}$$

Step 6: This equation represents the motion tracking equation, which relates the intensity values between two consecutive frames with the motion parameters $(\Delta x, \Delta y)$ and the temporal derivative $\frac{\partial I}{\partial t}$.

Step 7: To compute the motion function estimates, various motion estimation techniques such as optical flow methods (e.g., Lucas-Kanade method) or block matching algorithms can be used. These techniques analyze the intensity changes between frames to estimate the motion parameters $(\Delta x, \Delta y)$.

Step 8: Once we have the motion estimates, we can use them to track objects or analyze the motion patterns in the video sequence.

2 Derivation of Lucas-Kanade Algorithm for Affine Motion

We have the affine motion model:

$$u(x, y) = a_1x + b_1y + c_1$$

$$v(x, y) = a_2 x + b_2 y + c_2$$

Step 1: Objective Function: The objective function E to minimize is the sum of squared differences between the observed image intensity gradients ∇I and the predicted gradients $\nabla I'$ using the affine motion model:

$$E = \sum_{(x,y)} [\nabla I(x, y) - \nabla I'(x, y)]^2$$

Step 2: Affine Motion Model: We model the image gradients under affine motion as:

$$\nabla I'(x, y) = \begin{bmatrix} \frac{\partial I}{\partial x} \\ \frac{\partial I}{\partial y} \end{bmatrix}$$

We need to express the gradients $\frac{\partial I}{\partial x}$ and $\frac{\partial I}{\partial y}$ in terms of the affine motion parameters $a_1, b_1, c_1, a_2, b_2, c_2$.

Step 3: Computing Gradients: We compute the image gradients $\frac{\partial I}{\partial x}$ and $\frac{\partial I}{\partial y}$ for each pixel. Let's denote $I(x, y)$ as the intensity of pixel at position (x, y) in the image.
The image gradients are calculated using central differences:

$$\frac{\partial I}{\partial x} = I(x + 1, y) - I(x - 1, y)$$
$$\frac{\partial I}{\partial y} = I(x, y + 1) - I(x, y - 1)$$

Step 4: Formulating Equations: Substituting the expressions for $\frac{\partial I}{\partial x}$ and $\frac{\partial I}{\partial y}$ into the affine motion model, we get:

$$\nabla I'(x, y) = \begin{bmatrix} I(x + 1, y) - I(x - 1, y) \\ I(x, y + 1) - I(x, y - 1) \end{bmatrix}$$
$$= \begin{bmatrix} I_x & I_y \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

Step 5: Solving Equations: We need to solve for the affine transformation parameters $(a_1, b_1, c_1, a_2, b_2, c_2)$ by minimizing the objective function E. This is typically done using least squares estimation.
The least squares solution can be obtained by forming the normal equations and solving them. The normal equations are given by:

$$A^T A x = A^T b$$

where A is the design matrix and b is the target vector.

Let's denote $A = \begin{bmatrix} x & y & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & 1 \end{bmatrix}$ and $x = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \end{bmatrix}$.

Step 6: Warp Second Frame: Once we have obtained the estimated motion parameters $(a_1, b_1, c_1, a_2, b_2, c_2)$, we apply them to warp the second frame to align with the first frame.

This completes the procedure for performing the Lucas-Kanade algorithm for motion tracking when the motion is known to be affine.