Braid Groups and Knot Theory

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Abstract

In this report we aim to emphasize the strong connection between the study of braids and links. We will focus on three theorems in particular. The first we will look at is Alexander's theorem. This theorem shows us that given any oriented link we can manipulate it in such a way that it can be represented by a braid. We quickly realise that in fact every link must be represented by an entire class of braids. The second theorem proposed by Markov seeks to find the relation between all braids which are contained in this class. Finally we will look at the Burau representation of the braid group which gives rise to an invariant of oriented links.

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1 Introduction

The goal of this report is to give an introduction to braid theory and its strong connection to the study of knots. By a mathematical braid we refer to an object almost identical to a braid you would weave in to somebody's hair. The only difference being that we consider strands of hair being twisted in any order and not just the repetitive right then left motion we are all comfortable with. In a similar vein a mathematical knot is almost identical to what we would initially envision. The difference here being that after tying the knot we fuse the ends of the string together so that it cannot be undone. Braids and knots have been of significant interest to people for thousands of years however notable study into these fields has taken place relatively recently. In particular an explicit definition of the braid group was only given by Emil Artin in 1925.

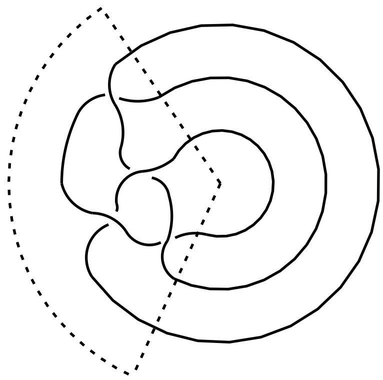

Figure 1. *Connecting the ends of a braid*

On first inspection a clear link between braids and knots may already be apparent. One may notice that if we connect the ends of the strands in a braid we are left with some collection of knots. From this realisation several questions naturally arise: can we obtain any knot in this way?; is the braid unique?; if not can we find the others? In this report we will answer all of these questions. We will look at three main theorems regarding these questions.

Before we seek to answer these questions we will begin by giving a more formal definition of braids and knots. Following this we will give an overview of some important points from both knot and braid theory necessary to appreciate the following sections.

In the following sections the basic definitions are stated with reference to Sections 1 and 2 of Murasugi's book [4]. The proof of Reidemeister's theorem comes from a set of notes written by Roberts [5]. The remainder of the section is written with reference to Section 1.4 of Kassel and Turaev's book [2] as well as notes from MathWorld [8].

1.1 Knots and Links

We will begin by looking at knots, links and ideas of equivalence between them.

Definition 1.1 (Knot)**.** *A* ***knot*** *can be thought of as an entwined closed curve in $\mathbb{R}^3$. This curve may not intersect itself at any point. In addition to this any given knot may also be given an orientation. Some examples of valid knots are shown in Figure 2.*

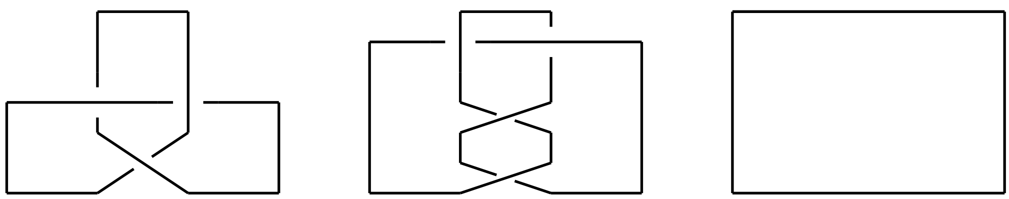

Figure 2. *Example knots: trefoil, figure eight and (trivial) unknot*

Definition 1.2 (Link)**.** *A* ***link*** *is a collection of entwined closed ordered curves in $\mathbb{R}^3$. Once again these curves may not intersect. In other words a link is an object made entirely of knots. Components of the link may be given orientation. Some examples of valid links are shown in Figure 3. Note that the components of a link need not be connected.*

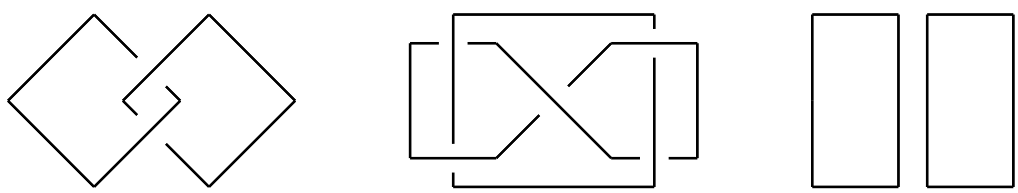

Figure 3. *Example links: Hopf link, Whitehead link and the (trivial) unlink*

In addition to these definitions we will consider our knots and links to be composed entirely of straight edges. This does not affect any properties of the knots but will make it easier to explain various elements later.

1.1.1 Knot and Link Diagrams

As all knots and links exist in $\mathbb{R}^3$ it is necessary for us to define a way in which we can accurately represent a knot or link on the plane. In fact the method for representing a link is the one we have already used in the previous figures but we will define this properly now.

To define these representations we must first consider what a 'good' projection of a knot onto the plane looks like. There are three conditions such a projection must meet. First, the projection must not include any triple crossing points as there is no clean way to accurately represent the crossings if this is the case. Secondly, no part of the curve can meet tangentially, again due to the fact that crossing information could not properly be represented. Finally we cannot have one edge obscuring anymore than one point on another edge. By this condition we mean that we cannot have two edges mapped to the same line on the plane or have a single edge mapped to a single point on the plane. In order to represent the crossing information we will break the line that passes underneath leaving a gap for the line that travels over the top to pass through.

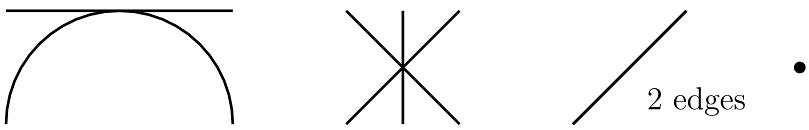

Figure 4. *'Bad' projections*

Definition 1.3 (Link Diagram)**.** *A **link diagram** is a 'good' projection of a given link on to the plane which also provides information on crossings.*

1.1.2 Equivalence of Knots and Links

Before we define what it is for two knots to be equivalent we must first illustrate the elementary knot moves. These are transformations on the knot itself in $\mathbb{R}^3$ and not on its corresponding diagram. In total there are four elementary knot moves. The first move states, given any edge AB, of a knot we may choose some point C along the edge and add a vertex there. The second move allows us to take an edge, AB, of the knot along with some point C that does not lie on the knot itself and replace AB with the edges AC and BC as long as the knot does not intersect with the triangle and its interior formed by ABC. We will call the second move the triangle move. The remaining two moves are the inverses of the first two. We may now define what it is for two knots to be equivalent.

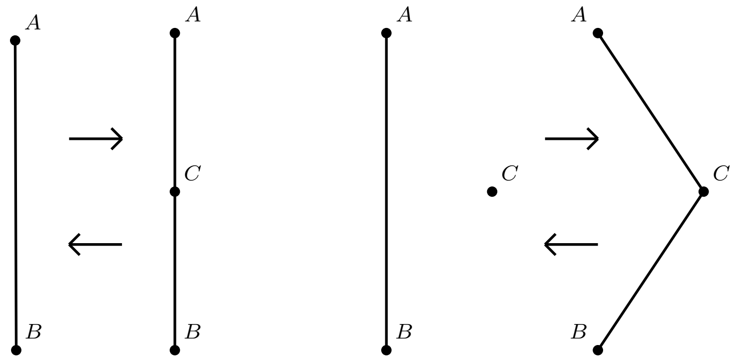

Figure 5. *The elementary knot moves*

Definition 1.4 (Equivalence of Knots)**.** *We say that two knots K and K' are* ***equivalent*** *if we can deform one into the other through a finite series of elementary knot moves.*

As for the equivalence of links we will simply extend Definition 1.4.

Definition 1.5 (Equivalence of links)**.** *We say that two links L and L' are* ***equivalent*** *if we can deform one into the other through a finite series of elementary knot moves.*

So we see that two links are equivalent only if the knots within the links are equivalent.

1.1.3 The Reidemeister Moves

The Reidemeister moves are a set of transformations on the link diagram. Generally speaking there are six moves. These consist of three moves and their inverses. The first we shall call Ω_1 shown below in Figure 6. This move removes a twist in the knot.

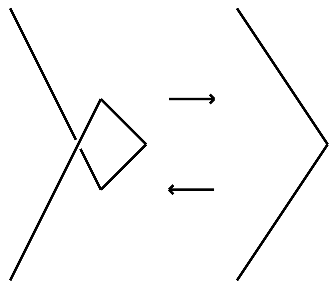

Figure 6. Ω_1 *type moves*

The second move we will call Ω_2. This move separates two overlapping edges of the diagram. As links may have orientation this gives two alternate oriented versions of the move, two where the left and right edges are both oriented in the same manner and two more when they are oriented in opposite directions.

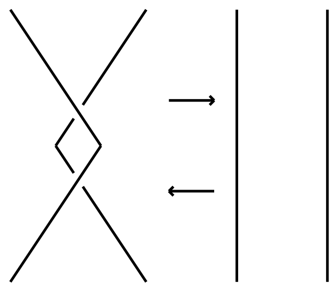

Figure 7. Ω_2 *type moves*

The third move type we will call Ω_3. This move passes an edge under a crossed pair of edges. Like the previous move there are oriented versions of this move. There are quite a few of these so we will not list them but they are obtained in a similar way to those for Ω_2.

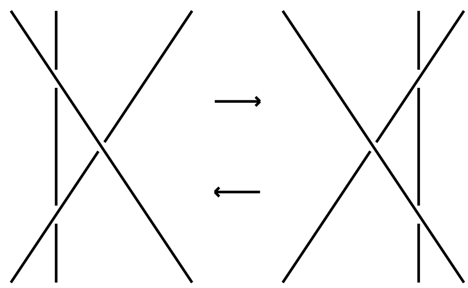

Figure 8. Ω_3 *type move*

Finally we will introduce a move that we will call Ω_0 or the triangle move. This move is not one of the Reidemeister moves however it is also very useful. This move is a diagrammatic version of the triangle elementary move. It is always possible to draw a diagram of the elementary triangle move with Ω_0. If an edge of the triangle is not visible in the diagram then we can perturb the point of the triangle slightly in $\mathbb{R}^3$ so that it is now visible.

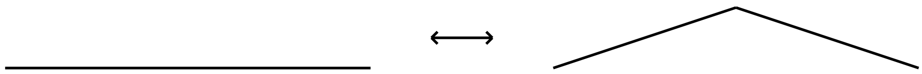

Figure 9. Ω_0 *type move*

Now that we know of the Reidemeister moves we can look at Reidemeister's theorem. This is an important theorem that will allow us to discuss equivalence of knots in terms of their diagram instead of comparing knots directly in three dimensions. It is not entirely obvious why there are three Reidemeister moves and why they are the ones shown above, the proof of Reidemeister's theorem shows exactly why we have these particular moves.

Theorem 1.6 (Reidemeister's Theorem)**.** *Two knots K and K' are equivalent if and only if their corresponding diagrams D and D' can be transformed into one another through a finite series of Reidemeister moves.*

Sketch of Proof. Let us first consider the statement that two knots K and K' are equivalent if their corresponding diagrams D and D' can be transformed into one another through a finite series of Reidemeister moves. It is fairly simple to convince ourselves of this. If D can be transformed into D' then both K and K' are represented by the same diagram and so they must be equivalent.

The second part of the claim is less trivial. We now consider the statement that if two knots K and K' are equivalent then their diagrams can be related through a finite series of Reidemeister moves. As K and K' are equivalent we can form a series of knots $K = K_1, K_2, ..., K_n = K'$ such that each K_i differs from K_{i-1} by exactly one triangle move. A single triangle move could cause a number of crossings with strands along the way so we will restrict the size of the triangle move we can perform so that the triangle contains one of the types shown in Figure 10. The five diagrams in the Figure do not fully capture all outcomes we might obtain. The bulk of a full proof would require classification of all possible types of triangles like these. Here we will consider just these few as an illustrative aid to visualise what a complete proof would look like.

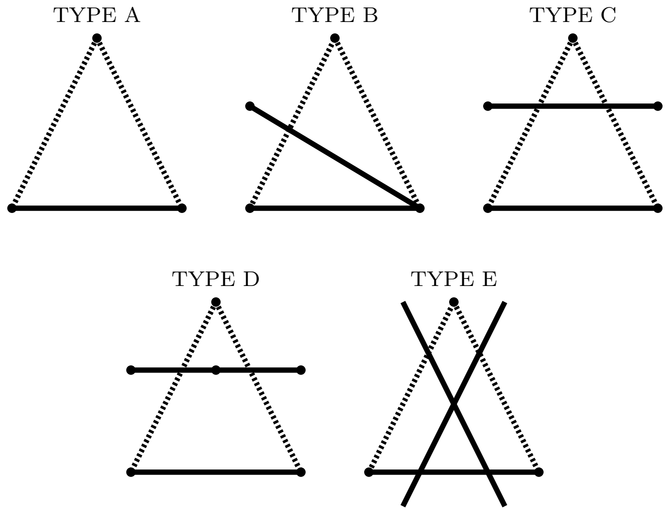

Figure 10. *The possible triangle moves*

Type A is merely a triangle move and nothing else, Type B is a triangle move where the edge of the triangle and the edge passing through the triangle are incident. Types C and D are very similar to one another the only difference is that in D we have a vertex within the triangle. Finally Type E is a triangle with a crossing pair of edges in the center. Given this series of knots we can form a corresponding series $D = D_1, D_2, ..., D_n = D'$ by taking the diagram of the corresponding K_i. Each D_i differs from D_{i-1} by one Ω_0 move. The diagrams of these small triangle moves will look like the diagrams shown in Figure 10. Notice that these moves each resemble one of the Reidemeister moves. Type B resembles Ω_1, C and D resemble Ω_2 and E, Ω_3. Thus we can transform one diagram into the other through a series of Reidemeister moves. □

1.2 Braids

In this section we give a formal definition of a braid and introduce the braid group. As with we did with knots we will also define some notion of equivalence between braids.

Definition 1.7 (Braid)**.** *A* ***braid*** *can be thought of as an entwined collection of curves (or strands) in* $\mathbb{R}^3$ *however there are a couple of rules that these curves must follow. The first rule restricts curves from returning in the direction that they originated from. For example, we will generally consider braids travelling from the left to the right. In this case the curve must be continuously travelling to the right and may not travel left. Our second rule will not allow any of the curves to intersect with one another. Some examples of valid braids are shown below in Figure 11. The example braids in Figure 11 each consist of four strands and so they shall be known as* 4*-braids but in general a braid of* n *strands will be called an* n*-braid.*

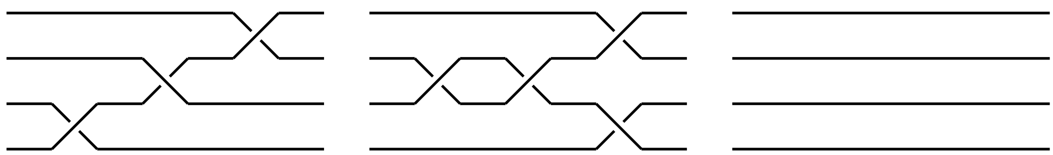

Figure 11. *Example braids: The trivial braid shown on the right*

1.2.1 Braid Diagrams

As braids exist in $\mathbb{R}^3$ we need to define a way in which we can accurately represent a braid on the plane. We will define this representation in a similar way to the diagrams we use for knots and links. As before, we will say that a 'good' projection is one that contains no tangential crossings, triple intersection points or obscured edges. We will also use the same method as before to convey crossing information.

Definition 1.8 (Braid Diagram)**.** *A* ***braid diagram*** *is a 'good' projection of a given braid onto the plane which also provides information on crossings.*

1.2.2 Braid σ Notation

Now that we have a notion of what a braid is it would be helpful to establish some notation that we can use to formally express a given braid. To do this we will begin by assigning labels to the strands of the braid. Given an n-braid we will label the strands from 1 to n in an increasing fashion starting from the bottom strand of the braid and working up to the top as shown in Figure

12. We will also label the points that the strands originate and terminate 1 to n in the same way. This means that a strand that originates at position 1 will not necessarily terminate in position 1, for example strand 1 of the first braid in Figure 11 originates at position 1 but terminates in position 4.

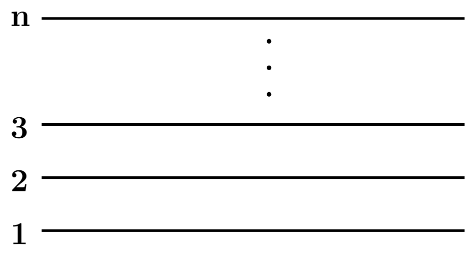

Figure 12. *Braid labelling*

If we look at strand 1 we see that there are only three possible moves that it could make. Either it will pass over strand 2, pass under strand 2 or not cross at all. We will denote the move where strand 1 passes over strand 2 as σ_1 and when it passes under we shall use σ_1^{-1}. If we look at the rest of the strands in the same way it is clear we can obtain similar notation for them as well. Notice that the path of strand n is entirely determined by the $n-1$th strand and so we will only require $n-1$ σ_i and σ_i^{-1}s. In Figure 13 we illustrate σ_i and σ_i^{-1}. After each σ_i we will write the next relative to the new position of the strands and not their original positions. This notation can now be used to express any braid. For example the first braid in Figure 11 is given by $\sigma_3\sigma_2^{-1}\sigma_1$.

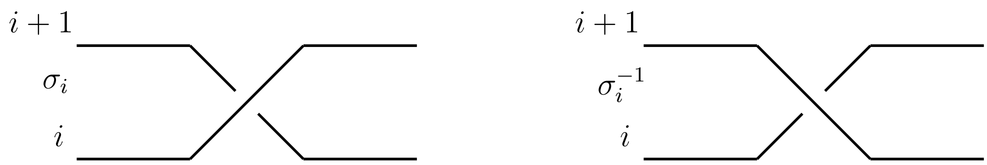

Figure 13. *Diagram showing σ_i and σ_i^{-1}*

1.2.3 The Braid Group

In this section we will look at how the set of all n-braids can be used to form a group. First we must define the operation under which we shall form the group. Consider two n-braids, X and Y. Let us now say that a braid runs over some interval $I = [0, 1]$ so that the braid begins at 0 and ends at 1.

Definition 1.9 (Composition of braids)**.** *The* ***composition of two braids*** *say X and Y will be given by concatenating the two braids. In other words $X \circ Y$ will be obtained by taking the ends of the strands of X and 'gluing' them to the beginning of the strands in Y such that strands in the same position are connected to one another. The resultant braid must also run over the same interval I. This means that the portion of the interval taken up by the braids X and Y is halved.*

For example let $X = \sigma_1\sigma_3\sigma_2^{-1}$ and $Y = \sigma_1\sigma_2\sigma_3^{-1}$. Figure 14 shows $X \circ Y = \sigma_1\sigma_2\sigma_3^{-1}\sigma_1\sigma_3\sigma_2^{-1}$. This is precisely the braid you would obtain if you concatenated the σ notation for X to the end of that for Y.

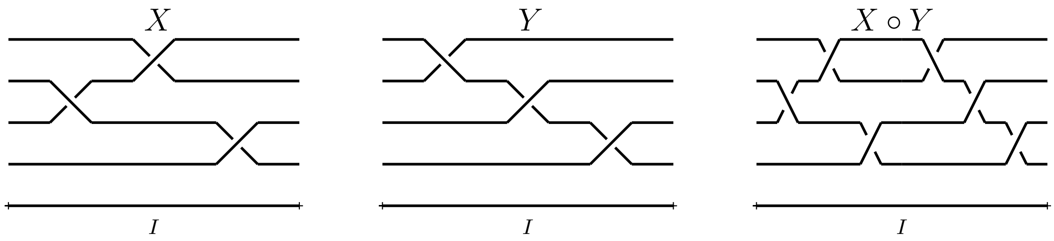

Figure 14. *Composition of two braids*

Now that the composition of two n-braids has been defined let us consider why this forms a group. First of all the operation is clearly associative. The identity for an n-braid will consist of n strands that never interact with one another as shown on the right of Figure 11. The inverse of a given braid also exists. Notice that the braid $\sigma_i\sigma_i^{-1}$ does indeed give us the identity braid. With this we can quite easily obtain the inverse of any given braid.

For this group there are only two relations. The first relation states $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$. This relation is quite easy to see. Without loss of generality assume that $i < j$ and $|i-j| > 1$, then σ_i will not interact with σ_j as it can only interact with the strand one ahead of it on the braid. The second relation is $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. This relation is slightly harder to visualise. Figure 15 is a diagram of both sides of the relation. Consider strand 3 of the braid on the right. If we take the strand and slide it to the right we can make the crossing of the strand under strands 1 and 2 occur after the final crossing of the braid as strands 1 and 2 are disjoint from 3. Strand 1 and 2 remain on top of 3 and their terminal destinations are unchanged so the braid is in essence the same as it was before. In addition to this, the new braid is identical to the one on the left. This relation will become clearer when we discuss the equivalence of braids in Section 1.2.5.

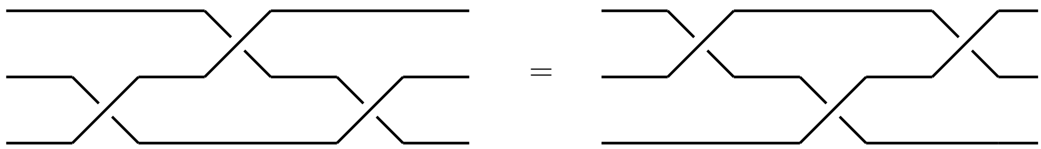

Figure 15. *Second braid relation*

This is all the information we need to describe the braid group on n strands which we will denote B_n.

$$B_n = \langle \sigma_1, ..., \sigma_{n-1} | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1 \rangle$$

Notice that this group looks very similar to the fundamental group for some space. Indeed B_n is the fundamental group for the configuration space which we will discuss in the next section.

1.2.4 Configuration Space

The ordered configuration space on distinct n-tuples is defined as follows:

$$\mathcal{F}_n(\mathbb{C}) = \{(z_1, z_2, ..., z_n) \in \mathbb{C} \times ... \times \mathbb{C} \mid z_i \neq z_j \text{ if } i \neq j\}.$$

On first inspection of this object, a simple visualisation is not immediately apparent. To start you may begin by considering a single element of this set. This is composed of n points sitting on the plane. Now imagine these points moving around the plane. The points are free to move as they please given that they never meet. If we take a 'snapshot' of these points as they move around the plane and stack them on top of one another we obtain a picture like the one below. This is a nice way in which we can visualise the configuration space.

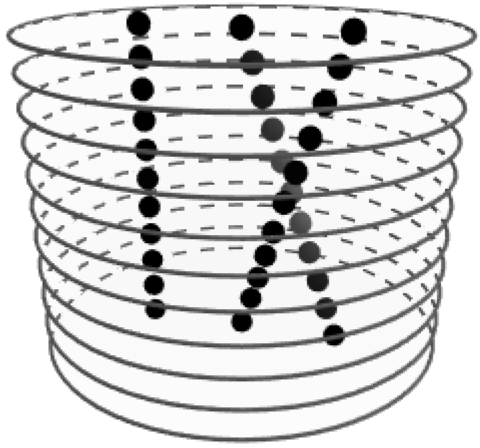

Figure 16. *Configuration space visualisation*

We see that there is an action of the symmetric group, S_n on $\mathcal{F}_n(\mathbb{C})$. This action takes the coordinates and permutes them. From this action we see that the quotient gives us the idea of an unordered configuration space:

$$\mathcal{C}_n(\mathbb{C}) = \mathcal{F}_n(\mathbb{C})/S_n = \{\{z_1, z_2, ..., z_n\} \in \mathbb{C} \times ... \times \mathbb{C} \mid z_i \neq z_j \text{ if } i \neq j\}.$$

The fundamental group of this space, $\pi_1(\mathcal{C}_n(\mathbb{C}))$, is the braid group on n strands. The fundamental group of the ordered configuration space $\pi_1(\mathcal{F}_n(\mathbb{C}))$ is known as the pure braid group on n strands denoted PB_n. This is the subset of braids in B_n in which strands that originate at position i also terminate in position i.

1.2.5 Equivalence of Braids

As we did for knots and links we will define some notion of equivalence between two braids.

Definition 1.10 (Equivalence of braids)**.** *Two braids β and β' in $\mathbb{R}^3$ are **equivalent** if we can transform one into the other through a finite series of elementary knot moves.*

Notice that here a triangle move on β may transform it in such a way that it is no longer a braid by breaking the rules set in definition 1.7. This kind of transformation is accepted in the intermediate steps from β to β'. Let us now consider equivalence of the braid diagram. Recall the Reidemeister moves Ω_2 and Ω_3. We can use these two moves on a braid as we would on a knot, Ω_1 is not of any use as it breaks the rules of definition 1.7. We will call these the braid-like Reidemeister moves and use the notation Ω_2^{br} and Ω_3^{br} when referring to braids. Notice that with a braid we do not need to consider oriented versions of the moves, the orientation of strands is always the same. If the braid diagram runs in a direction other than from left to right we may rotate it in $\mathbb{R}^2$ so that it does.

Definition 1.11 (Equivalence of braid diagrams)**.** *Given to braids β and β' we say that their diagrams D and D' are* ***equivalent*** *if we can transform one into the other through a finite series of the braid-like Reidemeister moves and their inverses.*

2 Alexander's Theorem

In this section we will discuss and prove a theorem presented by J. W. Alexander. This theorem gives a nice way to relate braids to links. The theorem states that every oriented link is equivalent to a closed braid. We will first look at what is meant by a closed braid before proving the theorem.

The content in this section is made primarily with reference to sections 2.3 and 2.4 of Kassel and Turaev's book [2]. A more detailed description of Seifert's algorithm as well as more detail on Seifert surfaces in general can be found in Section 5 of Rolfsen's book [6]. Section 2.3.2 was written with reference to both [2] and Birman and Brendle [1]. The Seifert surface renders are created with seifertView by Jarke J. van Wijk [7].

2.1 Braid Closure

The closure of a braid is a way in which we can obtain a link from a given braid. In order to close a given braid we will add a series of edges connecting position i at the start of the braid to position i at the end of the braid for all i from 1 to n as illustrated in Figure 17. If we have a braid β we will denote the closure of the braid $\hat{\beta}$.

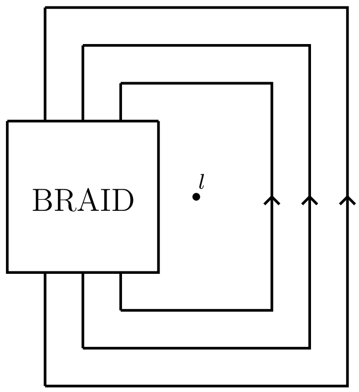

Figure 17. *Closure of a braid*

When closing braids we will generally perform the operation in such a way that the orientation of the resultant link runs counterclockwise. Note that this choice of orientation does not really matter in $\mathbb{R}^3$ as we can simply flip the edges over to the other side of the braid. It is only when we begin to look at diagrams in $\mathbb{R}^2$ that this convention will become meaningful. Notice that all of the strands revolve around some center point we will call l. Say we choose one point, v, of the closed braid and draw a line connecting it to l. If we then move v along the closed braid following its orientation we see that the connecting line rotates positively (that is counterclockwise) about l for the entirety of the closed braid. This is of course the case for any braid due to the rules we imposed on the braid while defining it.

2.2 Proof of Alexander's Theorem

Theorem 2.1 (Alexander's Theorem). *Any oriented link in $\mathbb{R}^3$ is equivalent to a closed braid.*

Before starting the proof we will first define terminology that will be useful. We will begin by drawing our link in $\mathbb{R}^3$ such that it does not intersect with the line $l = (0, 0, z)$. The choice of l is arbitrary but we choose the z-axis for ease. Choose an edge, AB, of the link and draw a line from the origin to a point v on AB. Now if we move v along AB following its orientation and the line connecting it to the origin rotates positively around l we will call AB a positive edge otherwise we will call it negative. It is possible that a chosen edge results in no rotation whatsoever just a lengthening or contraction of the edge. In this case we can simply perturb one of AB's endpoints slightly. It would be best for the upcoming proof to do this in such a way that the edge becomes positive. Consider the same edge, AB, once more. If we can choose some point c on l such that the triangle and its interior formed by ABc does not intersect with the link then we call AB an accessible edge.

Proof. As we remarked in the last section, all edges of a closed braid travel about some centre point and so they are all positive. This means that if we have some link whose edges are all positive then we can find a braid whose closure is equivalent to it. So in order to show that every knot can be represented by some closed braid it suffices to show given a negative edge AB, we can replace it with a sequence of positive edges such that the paths from A to B are equivalent.

Let us first consider a negative edge AB that is accessible. To form a series of edges that are all positive find a point c on l such that the triangle and its interior formed by ABc does not intersect with the link. Now choose some point C that is just beyond this point such that the triangle ABC does not meet l. Replace the initial negative edge with AC and BC. These new edges are both positive.

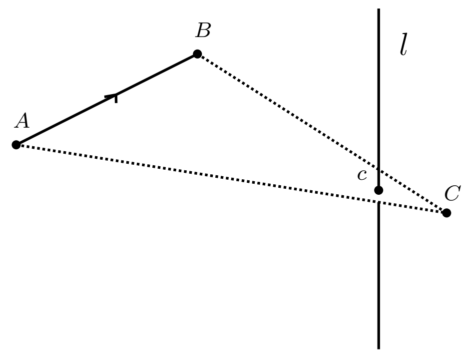

Figure 18. *Transformation of accessible negative edge*

All that is left to consider is the case when the edge AB is not accessible. Here we see that although the entire edge is not accessible we can divide the edge into subsections where all of the subsections are accessible. Now we just need to follow the same process we would for an accessible edge for each subsection.

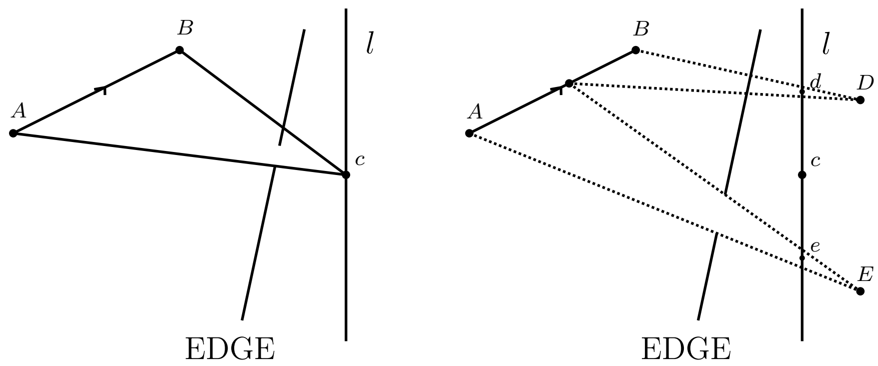

Figure 19. *Example of a non-accessible edge and division into subsections*

After we perform these operations on all of the negative edges we will be left with only positive ones which means they all continuously travel around l and so we have a closed braid.

□

An Illustrative Example of the Proof

Let us take a closer look at the method of the proof. Consider the trefoil knot oriented as shown in Figure 20. If we are careful about how we position our knot for example if we were considering rotation about i rather than l all edges would already be positive however we will consider l to be as shown in the Figure. The line l is shown by a point however we are still considering the knot to be in $\mathbb{R}^3$ we are just viewing the picture from above.

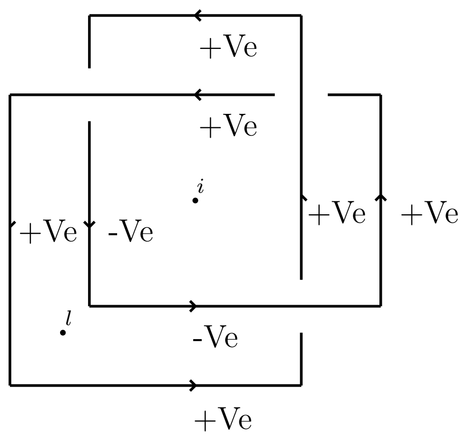

Figure 20. *Trefoil example*

We see that there are two negative edges in this case. Take the horizontal edge. This is clearly accessible as there are no lines crossing over the top of it. We can pick a z value as high as necessary for point c so that we ensure there are no crossings with other edges of the trefoil. Now we will simply choose some point C beyond this and replace the old edge with two new edges that connect the endpoints of the old edge to C.

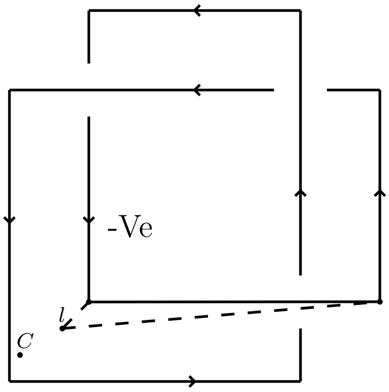

Figure 21. *Trefoil example: step 1*

Now let us consider the remaining negative edge. In a similar way to before this edge is also accessible as there are no edges that pass under it. Now we can choose a z value as low as necessary to ensure there will be no intersections with the rest of the knot. Again we will choose some point C_2 beyond our chosen z and replace the edge as before.

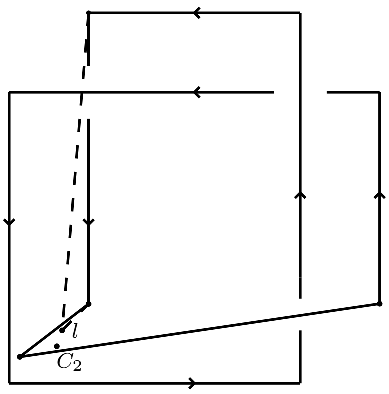

Figure 22. *Trefoil example: step 2*

This process has left us with a knot whose edges are all positive. We see that all edges are travelling around l. This means we must have a closed braid. If we cut the knot in a straight line out from l we can see what the corresponding braid looks like. The dashed line in Figure 23 is the plane we will cut across.

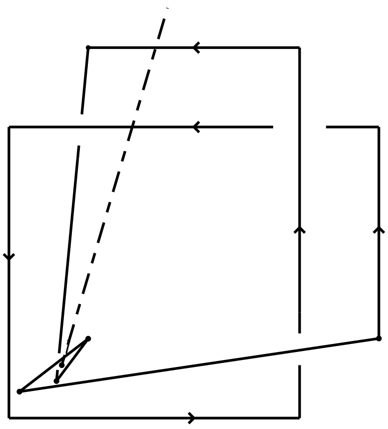

Figure 23. *Trefoil in closed braid form*

The braid shown on the left of Figure 24 is the result of cutting the knot. Notice that if we close this braid again the strands on the top can be untwisted with an Ω_1 type Reidemesister move. If we then shrink the strand down so that it is now in position 2 we will be left with the braid on the right. The braid on the right is exactly the braid we would have obtained if we focused on the line i rather than l.

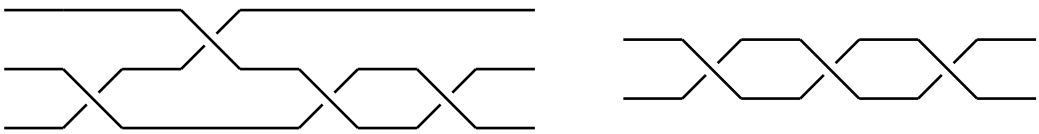

Figure 24. *Braids representing the trefoil*

It is clear that although every link can be represented as a closed braid these braids are not unique. In Section 3 we will look at theorem proposed by Markov which gives us insight in to whether two braids will have equivalent closures or not.

2.3 An Algorithm For Finding Closed Braids

Although we could use the method described in the proof of Alexander's theorem to find a closed braid, it is rather tedious and time consuming. Here we look at the Yamada-Vogel algorithm. Given some knot or link this algorithm allows us to form a surface which is bounded by it whilst also being in closed braid form. First however, we will look at Seifert's algorithm.

2.3.1 Seifert's Algorithm

Consider a crossing, x, in an oriented knot or link diagram D. Locally every crossing has the appearance of a two braid with one crossing. We call a smoothing of x the operation which takes this 2-braid and replaces it with the trivial 2-braid. In addition to this we will draw a line connecting the two strands which we will label $(+)$ if we have a σ_1^{-1} crossing and $(-)$ for σ_1. If we smooth every crossing in D we will be left with a collection of circles connected by labelled edges. These are the Seifert circles of D. We will call the smoothed diagram the Seifert picture of D.

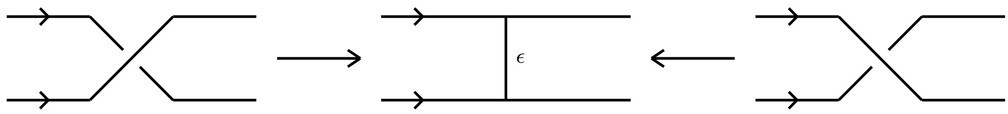

Figure 25. *Smoothing of σ_1 and σ_1^{-1}*

Let us now take these circles and consider them to be disks in $\mathbb{R}^3$. If we have a disk contained in another then we will elevate it so that they do not intersect. Next we will replace each of the labelled edges with strips that have been twisted. If the edge is labelled $(+)$ we will twist the strip in such a way that its boundary minus the short edges resembles σ_1^{-1}, if it is $(-)$ we twist the strip in such a way that its boundary resembles σ_1. In this way we have obtained a surface whose boundary is the knot which D describes. This is Seifert's algorithm. Figure 26 shows an example of the algorithm performed on the trefoil.

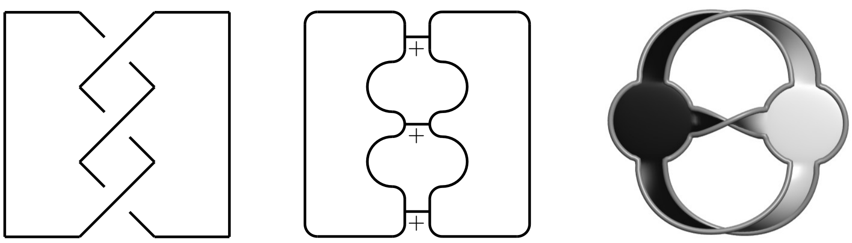

Figure 26. *Example of Seifert's algorithm[7]*

2.3.2 Yamada-Vogel Algorithm

First we will introduce some new terminology. We say that two Seifert circles are compatible if either, one is not contained inside the other and they have opposite orientation, or if one is contained inside the other and they have the same orientation otherwise they are incompatible. After obtaining the Seifert picture of a link diagram, D, we will denote the number of Seifert circles $n(D)$ and the number of pairs of incompatible circles $h(D)$, the latter is also known as the height of D. If the height of D is zero then D is in closed braid form. It is easy to convince ourselves of this. If we were to smooth any n-braid we would be left with the trivial braid. Now if we closed this braid we would be left with n circles all oriented in the same direction nested within one another and so they are compatible which implies $h(D)$ is 0.

Next we will create a graph that represents D which we will call $|D|$. This graph is constructed by ignoring the crossing information of D. The vertices of $|D|$ will correspond to the crossing points in D. The edges will be formed by taking a union of the connected edges in D which connect the crossings. Unlike a standard graph it is possible for us to have a circular edge if there are no crossings in the diagram. The faces of this graph will be that of a standard graph given our definition of the vertices and edges. Figure 27 is an example of $|D|$ for the three twist knot.

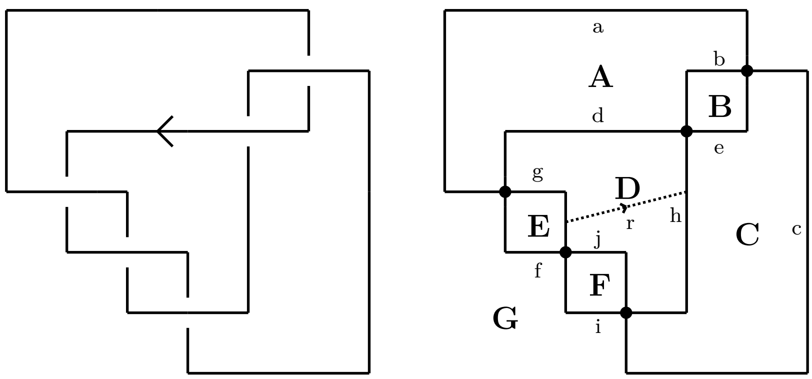

Figure 27. *Diagram D and corresponding graph $|D|$*

We say that a face in $|D|$ is adjacent to an edge if it is one of the edges that circle the face. For example face **B** in Figure 27 is adjacent to both b and e. We say that a face is adjacent to a Seifert circle if one of the edges that forms the circle is adjacent to the face. In our example face **D** is adjacent to all four Seifert circles whereas face **B** is adjacent only to S_1 and S_2. The Seifert picture of D is shown in Figure 28. We say that a face is defect if it is adjacent to two edges which are included in two different Seifert circles which are incompatible. If we have a defect face we may draw an oriented line between the two edges such that the line does not originate or terminate at the end points of either edge, we will call this a reduction arc. In the diagram above we see that face **D** is adjacent to g and h which are included in circles S_2 and S_3 which are incompatible with one another and so **D** is defect. Thus we form a reduction arc between g and h which we will call r.

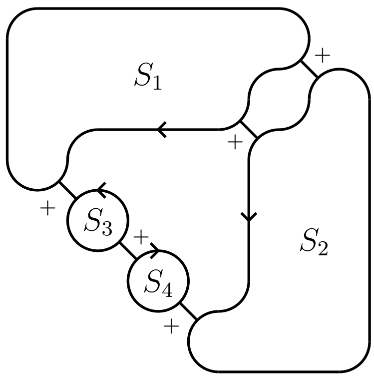

Figure 28. *Seifert picture of the three twist knot*

If we can form a reduction arc in $|D|$ then we can perform a Reidemeister Ω_2 move along the reduction arc in D stretching the edge at the base of r over the top of the other. In this way we add an additional two crossings to the diagram. In this context we will call such a move a bending of D. The inverse of this move we will call a tightening. Figure 29 shows an example of a bending.

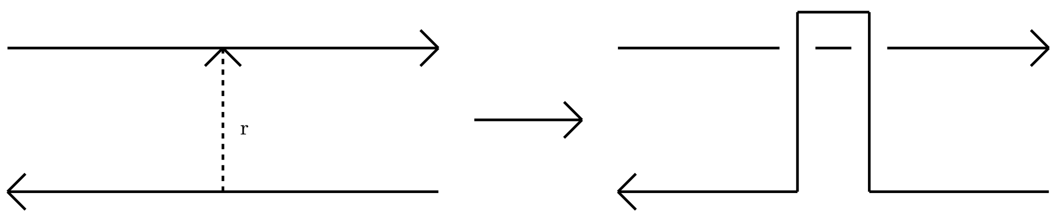

Figure 29. *A bending of D*

Let us now consider the smoothing obtained after a bending of D. We see that a channel is created between the two old Seifert circles forming a larger Seifert circle which contains another circle formed by the two new crossings we added to D. These circles will now be oriented in the same direction in a compatible manner. So it seems that $h(D)$ will have decreased by one.

Notice that the Seifert picture corresponds exactly to the graph $|D|$, where the vertices correspond to the labelled lines connecting the Seifert circles, the circles themselves are the edges, and finally the empty space between the labelled lines and circles are the faces. In this way we can work directly on the Seifert picture forming reduction arcs between pairs of incompatible circles as long as the arc can be drawn without any intersection. For example a reduction arc can be draw from S_3 to S_2 on Figure 28. A bending performed on D affects the Seifert circles in the manner shown in Figure 30.

Both Figures 29 and 30 depict a bending along r such that one edge is pulled over the top of the other. In general this does not need to be the case. A bending that pulls the strand underneath is valid as well. The one difference between these two different bendings is that the labelled edges connecting circles will have opposite signs, the circles themselves will remain unchanged. In some cases it can be useful for us to use one these bending types over the other so we will not set any convention on which to use.

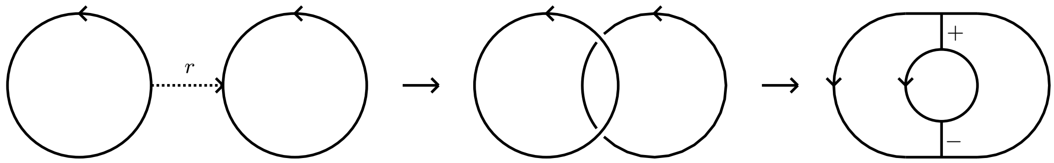

Figure 30. *A bending of D on its Seifert circles*

We will now prove two results which will make it clear that we will always be able to perform a bending of D along some reduction arc if the height is greater than zero. The first lemma shows that the height does in fact decrease by one after such a bending. This lemma implies that we can only perform a finite number of bendings that follow a reduction arc on D before $h(D) = 0$.

Lemma 2.2. *If we have a diagram D such that we can perform a bending along a reduction arc to obtain another diagram D' then $n(D') = n(D)$ and $h(D') = h(D) - 1$.*

Proof. The proof of the first claim we have already observed. Let us call the two Seifert circles concerned in the bending of D, S_1 and S_2. We see that after bending S_1 and S_2 a channel connecting the two is formed creating a new circle we will call S_∞. Another new circle is also formed on the inside of S_∞ formed by the two new crossings introduced by the bending which we will call S_0. Notice that this bending has no impact on any other part of the knot and so the smoothing of crossings throughout the braid remain unchanged meaning the total number of Seifert circles remains unchanged.

For the second claim we will count the number of incompatible pairs in both cases. Let us say that the number of Seifert circles inside S_1 is d_1 and the number inside S_2 is d_2. We will say that the number of Seifert circles on the outside of both S_1 and S_2 is d and finally we will say that the number of incompatible pairs of D that do not concern either S_1 or S_2 is h. Let us first consider the sum of $h(D)$. We see that the number of incompatible pairs contributed by the circles inside S_1 is simply d_1. This is because such a circle is either incompatible with S_1 or S_2, it cannot be both. This can be seen in Figure 31. In a similar way the circles inside S_2 contribute d_2 to the sum. Next if there are circles outside of S_1 and S_2 such that they are incompatible with S_1 then they are necessarily incompatible with S_2 as both S_1 and S_2 are oriented in the same direction. This gives a contribution of $2d$. Finally S_1 and S_2 are incompatible with each other giving a contribution of 1. This covers all possibilities relating to S_1 and S_2 so the total $n(D) = h + d_1 + d_2 + 2d + 1$.

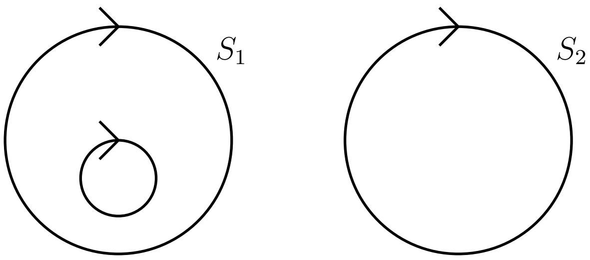

Figure 31. *Example of a circle in S_1*

We will compute the sum of $h(D')$ in the same way. As before if we consider the circles that were inside of S_1 which will now be contained in S_∞ we see that they will either be incompatible with S_0 or S_∞ and not both. This is also true of the circles that were in S_2 but are now in S_∞ giving us a contribution of $d_1 + d_2$. All circles that were outwith S_1 and S_2 are also outwith S_∞ and so if one of these circles was incompatible with S_1 it will also be incompatible with S_0 and S_∞ as they will all now have the same orientation giving us a contribution of $2d$. This covers all outcomes concerning S_0 and S_∞ and so we have $h(D') = h + d_1 + d_2 + 2d = h(D) - 1$.

□

The following lemma shows that if the height of D is greater than zero it is always possible to form a reduction arc. In combination with our previous result this will show that we can always obtain a diagram of height zero through a finite number of bendings. This fact is the basis for the Yamada-Vogel algorithm.

Lemma 2.3. *Given an oriented diagram D with $h(D) > 0$ then there exists a defect face in $|D|$.*

Proof. Let us begin by defining some terminology. Consider the Seifert picture. As we mentioned before the faces of a Seifert picture are the sections bounded by the labelled edges and or the Seifert circles. Let us assign some orientation to one of these faces. We see that the the orientation induced on the Seifert circles that are adjacent to the face either correspond to the actual orientation of the circle or they are opposite. We shall call a circle whose actual orientation matches the induced one positive. If the induced orientation is opposite we shall call it negative. As an example Figure 32 shows a face with clockwise orientation. In the Figure we see that the induced orientation on circles S_1 and S_4 matches their actual orientation so they are positive whereas the induced orientation on S_2 and S_3 is opposite to their actual orientation and so they are negative.

If a face is adjacent to either two positive or two negative circles then the face must be defect and a reduction arc can be formed. Notice that if we have a face adjacent to three or more circles then it is necessarily defect as there are only two options for the orientations of the circles. This leaves us with the case where each face is adjacent to only two circles.

Let us now assume that $h(D) > 0$. If both of the circles adjacent to the face are both positive or negative we are done so let us assume that they are opposite. Let us now leave the face we are centered on by crossing any of the labelled edges adjacent to the face. As this edge must also be adjacent to the new face the new face must also be adjacent to the same two Seifert circles. This would imply that every face is adjacent to these two Seifert circles and therefore the height must be zero. This is a contradiction to our assumption, thus there must exist a defect face somewhere in D.

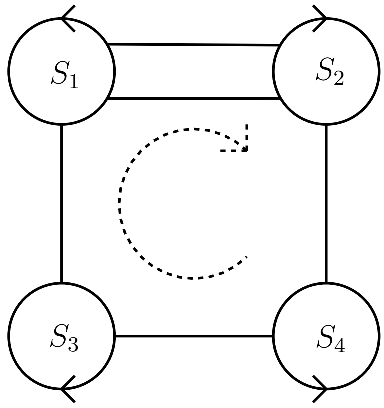

Figure 32. *Simplified Seifert picture of the three twist knot*

□

The Algorithm

We will now describe the steps of the Yamada-Vogel algorithm. We begin by checking the height of D. If $h(D) > 0$ then we can form a reduction arc between two of the Seifert circles. We then bend D along the reduction arc. This means that $h(D)$ has been reduced by one by Lemma 2.2. We repeat this process until $h(D) = 0$ which we know is always possible by Lemmas 2.2 and 2.3. When we reach this point there are two possibilities. We could have one series of circles $S_1, ..., S_n$ all oriented in the same direction where S_i is nested inside S_{i+1} for all $i \in \{1, ..., n-1\}$, in this case we are done. The other possibility is that we have two of these series of nested circles where the circles of each series are oriented in opposite directions. In this case we introduce a new move. We will choose one of the series of circles and isotope the edge of D it corresponds to in $S^2 = \mathbb{R}^2 \cup \{\infty\}$, pushing a point in the edge past the point at infinity. This creates a loop which circles the entire knot. In this way the circle has become the outermost circle of the other series. If there are more circles in the series we repeat the process. We will call this an ∞ move, an example of this is shown in Figure 33. The algorithm is now complete.

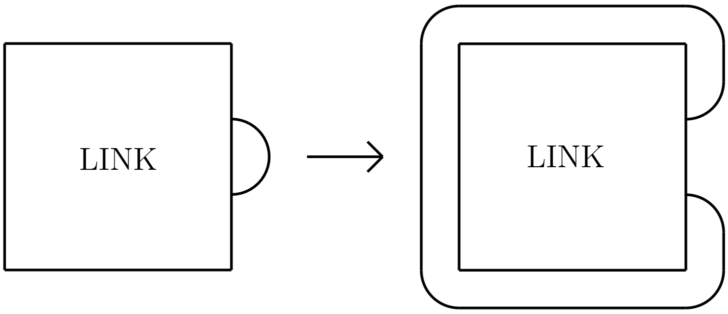

Figure 33. *An ∞ move on D*

2.3.3 An Example of the Yamada-Vogel Algorithm

Consider once again the three twist knot and its Seifert picture shown in Figures 27 and 28. We see that S_3 is incompatible with S_2 and that S_4 is incompatible with S_1 so $h(D) = 2$. We noted that we could form a reduction arc between S_2 and S_3 so let us begin with this bending. In Figure 34 we show the result of this bending. The diagram on the left shows the effects of the bending on the the original diagram D and the diagram on the right shows the corresponding Seifert picture of D.

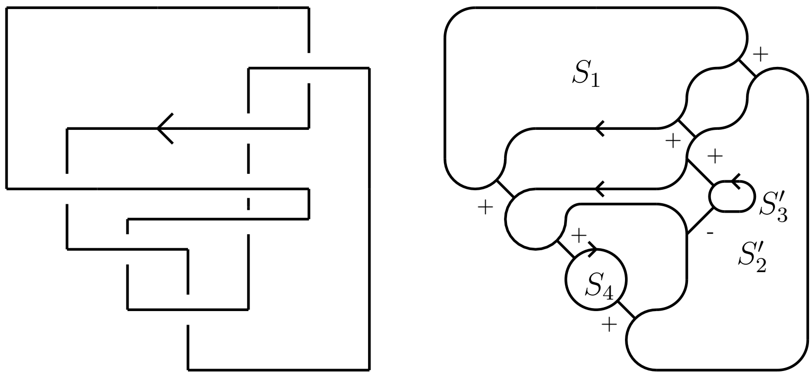

Figure 34. *Yamada-Vogel algorithm: step 1*

The height of D is now 1. If we look at the smoothed diagram we see that we can form a reduction arc from S_4 to S_1.

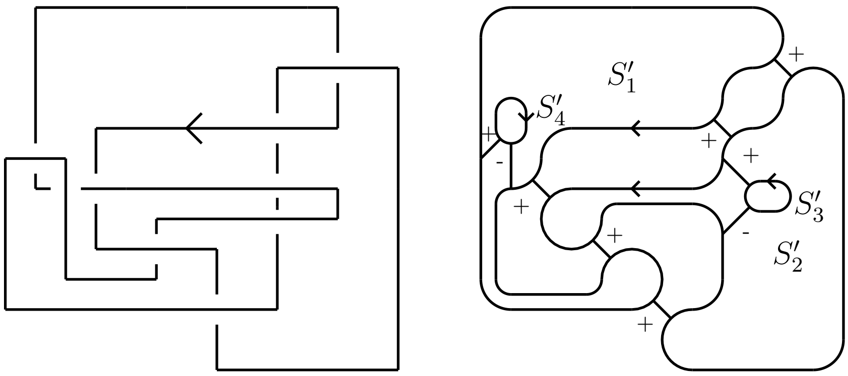

Figure 35. *Yamada-Vogel algorithm: step 2*

The height of D is now 0. We see that we are left with two series of nested circles so we will take the series on the right and perform ∞ moves on them. Figure 36 is the final diagram we obtain on completion of the algorithm.

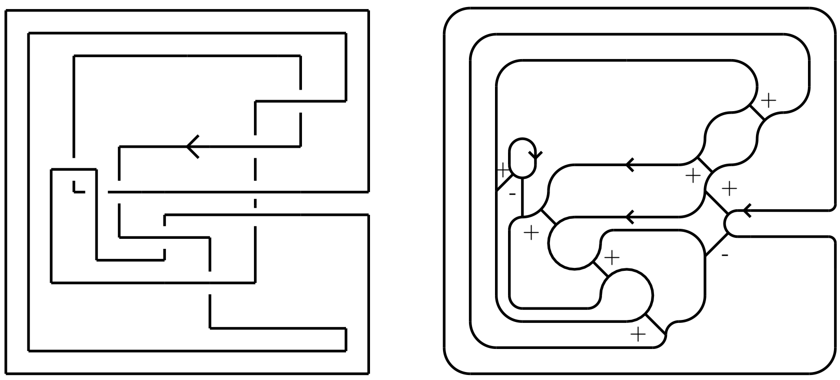

Figure 36. *Yamada-Vogel algorithm: step 3*

We can now deform the Seifert circles, making sure to preserve the position of the labelled edges relative to one another, so that they are perfectly circular as shown in Figure 37. In this way we can clearly view the braid which is being represented. Each circle represents a strand of the braid so in general the algorithm will leave us with an $n(D)$-braid. Let us label the circles from 1 to n from left to right dependant on the orientation of the diagram. To read off the braid from the diagram we will draw a line from the centre of the circles out to the edge of the outermost circle and rotate the line about the centre. As the line crosses a (+) edge we record σ_i^{-1} where the i is the number of the left most circle that the edge is connected to. As the line crosses a $(-)$ edge we record σ_i where the i is the number of the left most circle connected to the edge once again. The example below gives us $\sigma_2^{-1}\sigma_3\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1}$ when starting the reading from the arrows indicating orientation.

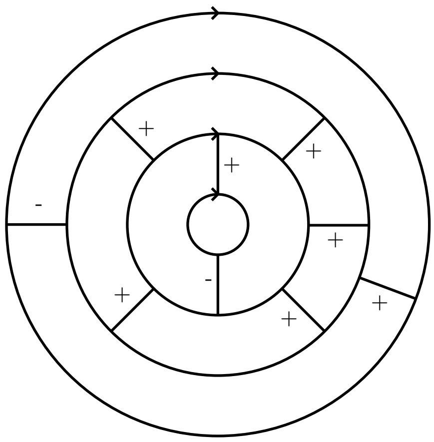

Figure 37. *Result of the Yamada-Vogel algorithm*

The final result of this algorithm is a surface in $\mathbb{R}^3$ in which the series of nested circles are in fact stacked disks connected by twisted strips as described before in the section on Seifert's algorithm. The boundary of this surface is the knot or link that D represents. These are known as Seifert surfaces. Seifert surfaces are fairly hard to visualise due to the fact that they are not common structures in nature, Figure 38 is the Seifert surface for the three twist knot we obtained through the Yamada-Vogel algorithm. Like before we see that this surface is in no way unique. The closure of the braid $\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_1^{-1}$ for example also gives us the three twist knot. We could quite easily given this information form a Seifert surface composed of three disks and six twisted strips.

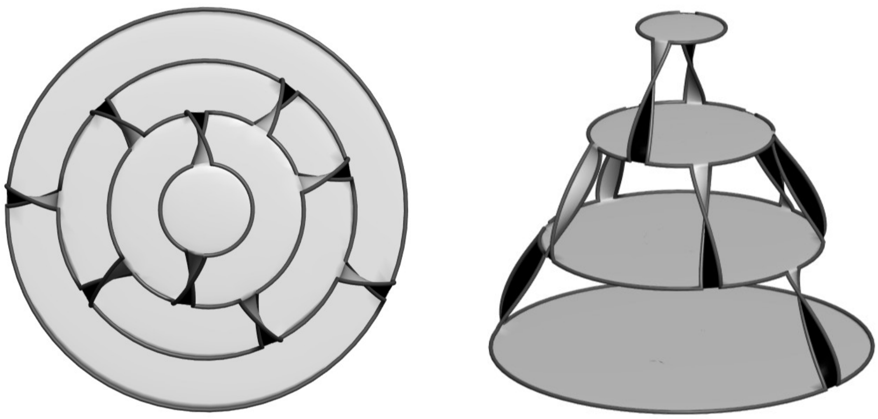

Figure 38. *Seifert surface of the three twist knot[7]*

In the next section we will look at the equivalence of closed braids further.

3 Markov's Theorem

In this section we look at a theorem proposed by Markov which establishes a relationship between braids which close to give the same link. Before looking at Markov's theorem we will first begin by introducing several new ideas and definitions necessary to prove the theorem.

The following sections on Markov's theorem are made with reference to the proof presented in sections 2.5-2.7 of Kassel and Turaev's book [2].

3.1 Markov's Moves

As we noted at the end of the last section a link does not have a unique braid closure. So we might consider what we could do in order to find an equivalent closure. One might begin by taking two given braids, $\alpha, \beta \in B_n$ and forming an expression such as, $\beta\alpha\alpha^{-1}$. Closure of this braid will of course be equivalent to the closure of β as these two braids are themselves equivalent. Notice that if we take the closure of this braid it is possible for us to take a collection of the σ_i's from the end of the braid and move them to the top. In this way we can take α^{-1} from the bottom of the braid to the top giving us a braid whose closure is not trivially equivalent to β. This is the first Markov move which we will denote $M1$. More formally we will will define $M1$ to be the move taking β to $\alpha\beta\alpha^{-1}$ where $\alpha, \beta \in B_n$. The inverse of $M1$ is just another $M1$ move in which we conjugate by the inverse of α instead of α itself.

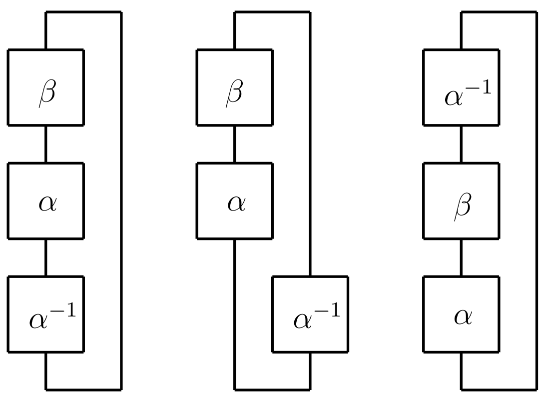

Figure 39. *Visualisation of Markov move $M1$*

Another method through which we could obtain an equivalent closure involves adding an additional strand to the braid. If we then cross this additional strand with the one next to it at the end of the braid (or at the start) we will obtain a braid whose closure is equivalent to the original. A good way to think of this move is to imagine we take a knot and twist one section over by π and then pull this new loop back over the rest of the knot, this is the second Markov move. A visualisation of this is shown in Figure 40. We will call this move $M2$ and say that it takes β to $\sigma_n^{\epsilon 1}\iota(\beta)$ where $\epsilon = \pm$ and $\iota(\beta)$ is the inclusion of β from B_n in to B_{n+1} by adding an extra $n+1$th strand to β. We will also consider the inverse of this map which removes the crossing involving the $n+1$th strand and then removes the strand altogether from the braid. The move $M2$ is also known as stabilisation and its inverse destabilisation.

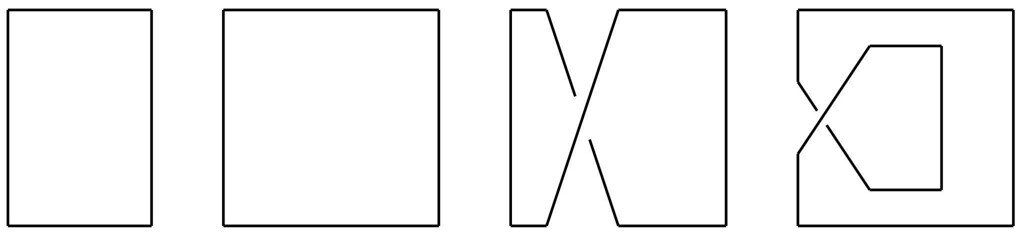

Figure 40. *Visualisation of Markov move $M2$*

We will now define some notion of equivalence for braid closures.

Definition 3.1 (M-equivalence)**.** *We will say that two braids $\beta \in B_n$ and $\beta' \in B_m$ are **M-equivalent** if we can transform one braid into the other through a finite sequence of Markov moves and their inverses.*

3.2 Expanding Notation

In this section we will define some new σ moves. Unlike the moves we have already defined the new moves will affect more than two strands at once. The first move we will define takes a collection of strands and passes them all over another collection of strands as σ_i would do for two strands.

Definition 3.2 ($\sigma^{\epsilon}_{m,n}$)**.** *We say that $\sigma^{+}_{m,n}$ is the move that takes the strands in positions 1 to m and passes them over the top of the next n strands in the braid in such a way that a strand in position $i \in \{1, \dots, m\}$ ends in position $n+i$ and strands in position $i \in \{m+1, \dots, m+n\}$ end in position $i-m$. In* *a similar way we define $\sigma^-_{m,n}$ to be the passing of m strands under n strands. An example is shown in Figure 41.*

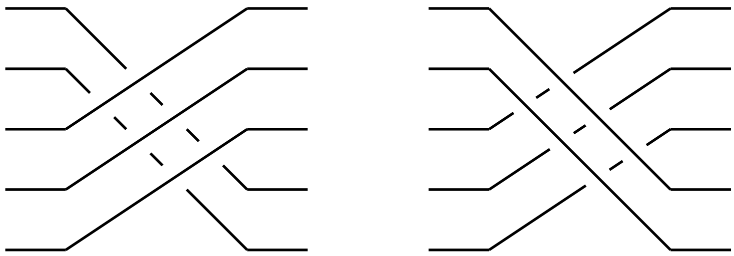

Figure 41. $\sigma^+_{3,2}$ *and* $\sigma^-_{3,2}$

The next move we will introduce takes a given braid and flips it over. To do this first consider the braid $\Delta_n = (\sigma_1\sigma_2 \ldots \sigma_{n-1})(\sigma_1\sigma_2 \ldots \sigma_{n-2}) \ldots (\sigma_1\sigma_2)(\sigma_1)$. This braid adds a twist by π to the braid so that a strand in position i ends in position $n+1-i$. The inverse of Δ_n will therefore be a twist by $-\pi$. A visualisation of these braids on a macro level is shown in Figure 42.

Figure 42. *Visualisation of* Δ_n *and* Δ_n^{-1}

Definition 3.3 (Braid flip)**.** *We will denote the* ***flipping of a braid*** β *by* π *as* $\bar{\beta} = \Delta_n \beta \Delta_n^{-1}$.

To see that the definition is indeed correct consider the composition of the two diagrams in Figure 42 with β in the centre. This will give us a diagram like the one on the left of Figure 43. If we were to pull the ends of this strip we see that the braid would uncoil and any part of the braid between the two twists will now be facing in the opposite direction.

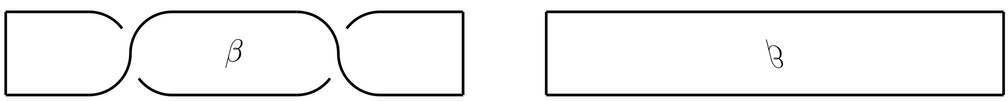

Figure 43. *Visualisation of* $\bar{\beta}$

The following lemma is fairly trivial but we state it explicitly as it appears in some of the proofs to come.

Lemma 3.4. *If we have to braids β, β' of any size that are M-equivalent then the braids $\bar{\beta}, \bar{\beta}'$ must also be M-equivalent.*

Proof. This claim is quite easy to see, a braids closure will of course be the same if it is rotated by π. In particular $\bar{\beta}$ is M-equivalent to β by $M1$, β to β' by assumption and β' to $\bar{\beta}'$ again by $M1$.

□

The final new move we will define takes two braids $\alpha \in B_n$ and $\beta \in B_m$ and joins them together so that we have a new braid in B_{m+n}.

Definition 3.5 (Tensor product)**.** *Given two braids $\alpha \in B_n$ and $\beta \in B_m$ the **tensor product** $\alpha \otimes \beta \in B_{m+n}$ is the braid formed by placing β above α. In other words the first strand of β becomes strand $n+1$ of $\alpha \otimes \beta$ and so on. A diagram of this is shown on the left of Figure 44.*

Given these three new moves it would be useful to develop a new type of diagram. Let us use our definition of the tensor product as an example. We will represent this diagrammatically as shown on the left of Figure 44.

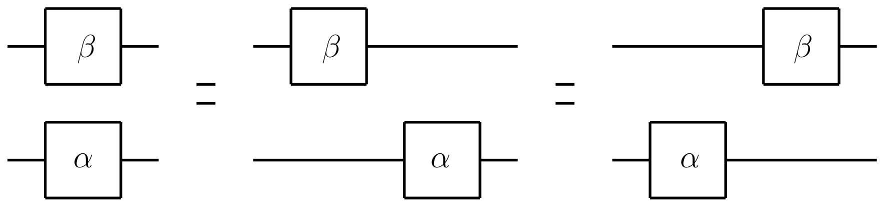

Figure 44. *Tensor product:* $\alpha \otimes \beta = (\alpha \otimes 1_m)(1_m \otimes \beta) = (1_n \otimes \beta)(\alpha \otimes 1_m)$

The single strands in this diagram actually represent a collection of strands. For example the line going into α is the braid 1_n where 1_i is the trivial braid on i strands. To express these diagrams notationally we use the same method as before. In particular above we have $(1_n \otimes 1_m) \circ (\alpha \otimes \beta) \circ (1_n \otimes 1_m)$. As before these expressions are read from right to left but the contents of the brackets are read from left to right. One may also read the expressions from left to right the whole way through which will result in a diagram that is the mirror to the method we shall use. A good way to visualise the braid from the notation is to stack the brackets on top of one another. In this way we can clearly see how the braid will be constructed. Notice in our example that α and β are disjoint so they can move freely past each other. The figure above shows this clearly but it is also easy to see it from the notation if we use this stacking method shown below.

$$\begin{pmatrix} 1_n \otimes 1_m \\ \alpha \otimes \beta \\ 1_n \otimes 1_m \end{pmatrix} \to \begin{pmatrix} \alpha \otimes 1_m \\ 1_n \otimes \beta \\ 1_n \otimes 1_m \end{pmatrix} \to \begin{pmatrix} \alpha \otimes 1_m \\ 1_n \otimes 1_m \\ 1_n \otimes \beta \end{pmatrix}$$

Diagrams and notation like this will be used heavily throughout the following section so it is important to become comfortable with visualising these types of braid.

3.3 Ghost Braids

Here we will introduce the idea of a ghost braid. These ghost braids preserve M-equivalence when composed with other braids in a particular manner. We will use these as a tool to show the M-equivalence of one particular pair of braids discussed at the end of this section. This pair will be very important in proving Markov's theorem. Let us begin with the definition.

Definition 3.6 (Ghost Braid). *Consider the braids $\beta \in B_{m+n}$, $\beta' \in B_{n+m}$, $\mu \in B_{n+k}$, and $\mu' \in B_{k+n}$ where $m \geqslant 0$, $n \geqslant 1$ and $k \geqslant 0$. We will say that μ is **n right ghost** if $(\beta \otimes 1_k)(1_m \otimes \mu)$ is M-equivalent to β. In a similar fashion we will say that μ' is **n left ghost** if $(1_k \otimes \beta')(\mu' \otimes 1_m)$ is M-equivalent to β'. We write $\mu \equiv 1_n$ as it is n right ghost and $\mu' \equiv' 1_n$ as it is n left ghost. The braids are shown in Figure 45*

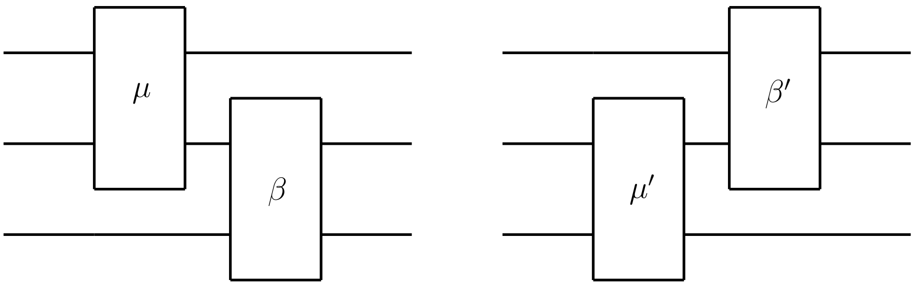

Figure 45. *Example braids M-equivalent to β and β'*

Given this definition of a ghost braid we now introduce an operation that allows us to insert a ghost braid into a given braid.

Definition 3.7 ($M(\mu)$ and $M'(\mu')$)**.** *Consider the braids $\alpha, \beta \in B_{m+n}$, $\rho \in B_m$, $\rho' \in B_n$, $\mu \in B_{n+k}$, and $\mu' \in B_{k+m}$ where $\mu \equiv 1_n$ and $\mu' \equiv' 1_m$.* $\boldsymbol{M(\mu)}$ *is an operation which takes $\beta(\rho \otimes 1_n)\alpha$ to $(\beta \otimes 1_k)(\rho \otimes \mu)(\alpha \otimes 1_k)$. Similarly we will define* $\boldsymbol{M'(\mu')}$ *to take $\beta(1_m \otimes \rho')\alpha$ to $(1_k \otimes \beta)(\mu' \otimes \rho')(1_k \otimes \alpha)$. A diagram of $M(\mu)$ is shown in Figure 46, $M(\mu')$ has a similar diagram but μ' and k are added underneath the m strand and ρ' is in between α and β on the n strand.*

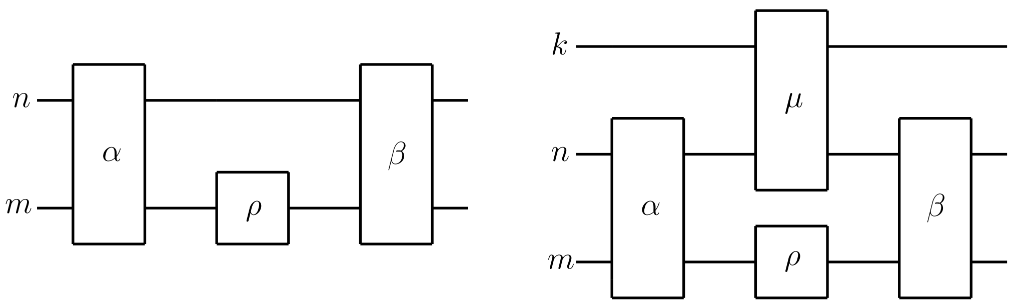

Figure 46. *Diagram of $M(\mu)$*

Notice that both $M(\mu)$ and $M'(\mu')$ preserve the M-equivalence of the braid. To see this consider the braid on the right of Figure 46. If we conjugate by α we can treat treat $\alpha\beta(\rho \otimes 1_n)$ as β from Definition 3.6 and so we can remove μ and 1_k. We then conjugate by α once more and we are left with the braid on the left of Figure 46. In particular:

$$\begin{aligned}
\beta(\rho \otimes 1_n)\alpha \underset{M1}{\sim} & \alpha\beta(\rho \otimes 1_n) \\
& \sim (\alpha\beta(\rho \otimes 1_n) \otimes 1_k)(1_m \otimes \mu) \\
& = (\alpha \otimes 1_k)(\beta \otimes 1_k)(\rho \otimes 1_{n+k})(1_m \otimes \mu) \\
& = (\alpha \otimes 1_k)(\beta \otimes 1_k)(\rho \otimes \mu) \\
& \underset{M1}{\sim} (\beta \otimes 1_k)(\rho \otimes \mu)(\alpha \otimes 1_k)
\end{aligned}$$

where the second equivalence is the addition of μ to the braid which by definition preserves the M-equivalence. For M' we form an analogous argument:

$$\begin{aligned}\beta(1_m \otimes \rho')\alpha \underset{M1}{\sim} & \alpha\beta(1_m \otimes \rho') \\ & \sim (\alpha\beta(1_m \otimes \rho') \otimes 1_k)(\mu' \otimes 1_n) \\ & = (1_k \otimes \alpha)(1_k \otimes \beta)(1_{m+k} \otimes \rho')(\mu' \otimes 1_n) \\ & = (1_k \otimes \alpha)(1_k \otimes \beta)(\mu' \otimes \rho') \\ & \underset{M1}{\sim} (1_k \otimes \beta)(\mu' \otimes \rho')(1_k \otimes \alpha).\end{aligned}$$

It is useful to think of $M(\mu)$ as a special form of stabilisation. Our definition of an $M2$ move only allows us to add or remove a crossing at the end of our braid. The move $M(\mu)$ allows us to add a crossing in to the middle of the strand essentially conjugating some braid adding a crossing onto the new strand and then reversing the conjugation. In the following series of lemmas we will discuss an important property of ghost braids as well as give an example of a ghost braid which will be needed in our proof of Markov's theorem.

Lemma 3.8. *If $\mu \equiv 1_n$ then $\bar{\mu} \equiv' 1_n$, where $\mu \in B_{n+k}$ with $n > 0$ and $k \geqslant 0$.*

Proof. Take some braid $\beta \in B_{m+n}$ where $m \geqslant 0$. From the definition of left ghost we must show that the braid $(1_k \otimes \beta)(\bar{\mu} \otimes 1_m)$ is M-equivalent to β. If we conjugate this expression by Δ_{m+n+k} we obtain the braid $(\bar{\beta} \otimes 1_k)(1_m \otimes \mu)$. By assumption this braid is M-equivalent to $\bar{\beta}$. Once more by conjugation of $\bar{\beta}$ by Δ_{m+n} we see that it is M-equivalent to β and so $\bar{\mu}$ is n left ghost.

□

Lemma 3.9. *Let $\theta_n^\epsilon = \Delta_n^{\epsilon 2}$ where $n \geqslant 1$ and $\epsilon = \pm$. Now we will define*

$$\mu_{n,\epsilon} = (1_n \otimes \theta_n^{-\epsilon})\sigma_{n,n}^\epsilon = \sigma_{n,n}^\epsilon(\theta_n^{-\epsilon} \otimes 1_n) \in B_{2n}.$$

This means that

$$\bar{\mu}_{n,\epsilon} = (\theta_n^{-\epsilon} \otimes 1_n)\sigma_{n,n}^\epsilon = \sigma_{n,n}^\epsilon(1_n \otimes \theta_n^{-\epsilon}) \in B_{2n}.$$

In addition to this $\mu_{n,\epsilon} \equiv 1_n$, $\bar{\mu}_{n,\epsilon} \equiv 1_n$, $\mu_{n,\epsilon} \equiv' 1_n$ and $\bar{\mu}_{n,\epsilon} \equiv' 1_n$

Proof. To prove that $\bar{\mu}_{n,\epsilon} = (\theta_n^{-\epsilon} \otimes 1_n)\sigma_{n,n}^\epsilon = \sigma_{n,n}^\epsilon(1_n \otimes \theta_n^{-\epsilon})$ we need to show that $\bar{\theta}_n^\epsilon = \theta_n^\epsilon$. To do this we just need to write out the expression fully,

$\bar{\theta}_n^\epsilon = \Delta_n \Delta_n^{\epsilon 2} \Delta_n^{-1} = \Delta_n^{\epsilon 2} = \theta_n^\epsilon$. So we see that $\bar{\mu}_{n,\epsilon}$ is indeed the rotation of $\mu_{n,\epsilon}$ by π about its centre.

For the second claim of the proof we see that due to lemma 3.8 it suffices to show just that $\mu_{n,\epsilon}$ and $\bar{\mu}_{n,\epsilon}$ are n right ghost as this will imply that they are also left ghost. We begin by taking the expressions $(\beta \otimes 1_n)(1_m \otimes \mu_{n,\epsilon})$ and $(\beta \otimes 1_n)(1_m \otimes \bar{\mu}_{n,\epsilon})$ where $\beta \in B_{m+n}$. If these expressions are equivalent to β then the claim is proven. Now we will show that these expressions are equivalent to one another. The following series shows this,

$$\begin{aligned}
(\beta \otimes 1_n)(1_m \otimes \mu_{n,\epsilon}) &= (\beta \otimes 1_n)(1_m \otimes 1_n \otimes \theta_n^{-\epsilon})(1_m \otimes \sigma_{n,n}^\epsilon) \\
&= (\beta \otimes \theta_n^{-\epsilon})(1_m \otimes \sigma_{n,n}^\epsilon) \\
&\underset{M1}{\sim} (1_m \otimes \sigma_{n,n}^\epsilon)(\beta \otimes \theta_n^{-\epsilon}) \\
&= (1_m \otimes \sigma_{n,n}^\epsilon)(1_{m+n} \otimes \theta_n^{-\epsilon})(\beta \otimes 1_n) \\
&= (1_m \otimes \bar{\mu}_{n,\epsilon})(\beta \otimes 1_n) \\
&\underset{M1}{\sim} (\beta \otimes 1_n)(1_m \otimes \bar{\mu}_{n,\epsilon}).
\end{aligned}$$

We will choose the term $(1_m \otimes \sigma_{n,n}^\epsilon)(\beta \otimes \theta_n^{-\epsilon})$ from the expressions above and show that it is equivalent to β. In order to show that it is equivalent to β we will form an inductive argument on n. Let us consider the base case where $n = 1$. We see that $\theta_1^{-\epsilon} = 1_1$ and $1_m \otimes \sigma_{1,1}^\epsilon = \sigma_{m+1}^{\epsilon 1}$. Substituting these in to $(1_m \otimes \sigma_{n,n}^\epsilon)(\beta \otimes \theta_n^{-\epsilon})$ we obtain $\sigma_{m+1}^\epsilon(\beta \otimes 1_1)$ which is a Markov move on β therefore it is M-equivalent to it. Now consider some $n > 1$ and assume that the expression is M-equivalent to β for all values less than this. We begin by taking $(1_m \otimes \sigma_{n,n}^\epsilon)(\beta \otimes \theta_n^{-\epsilon})$ and splitting the $\theta_n^{-\epsilon}$ term from β. We then slide $\theta_n^{-\epsilon}$ through and past $\sigma_{n,n}^\epsilon$. This leaves us with the braid $(1_m \otimes \theta_n^{-\epsilon} \otimes 1_n)(1_n \otimes \sigma_{n,n}^\epsilon)(\beta \otimes 1_n)$ shown on the left of Figure 47. In this diagram we have labelled the crossings that are dependent on ϵ. The diagrams in the remainder of the proof correspond to $\epsilon = +$ however the argument is identical for both cases.

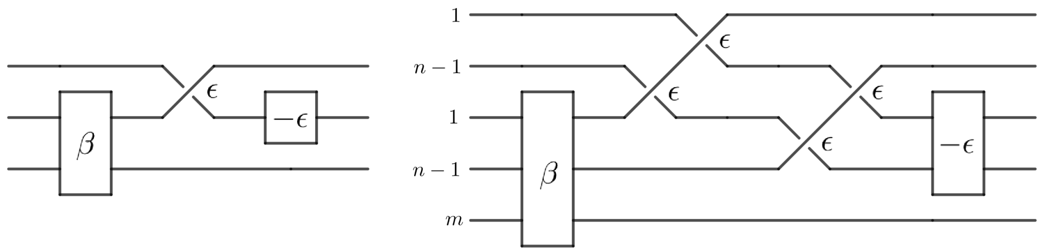

Figure 47. $(1_m \otimes \theta_n^{-\epsilon} \otimes 1_n)(1_n \otimes \sigma_{n,n}^{\epsilon})(\beta \otimes 1_n) =$
$(1_m \otimes \theta_n^{-\epsilon} \otimes 1_n)(1_m \otimes \sigma_{n-1,n}^{\epsilon} \otimes 1_1)(1_{m+n-1} \otimes \sigma_{1,n}^{\epsilon})(\beta \otimes 1_n)$

Let us now redraw this diagram splitting the two collections of n strands in to collections of $n-1$ and 1 strands. A diagram of this is shown in Figure 47. From this new diagram it is clear we can conjugate to obtain the braid $(1_{m+n-1} \otimes \sigma_{1,n}^{\epsilon})(\beta \otimes 1_n)(1_m \otimes \theta_n^{-\epsilon} \otimes 1_n)(1_m \otimes \sigma_{n-1,n}^{\epsilon} \otimes 1_1)$. To visualise this imagine closing the diagram of the braid and sliding the $\theta_n^{-\epsilon}$ term and the two crossings to its left around the link until they are on the left of β. Notice that we can destabilise (or take an $M2^{-1}$ move on) this new braid so that we have a braid in B_{m+2n-1} shown in Figure 48.

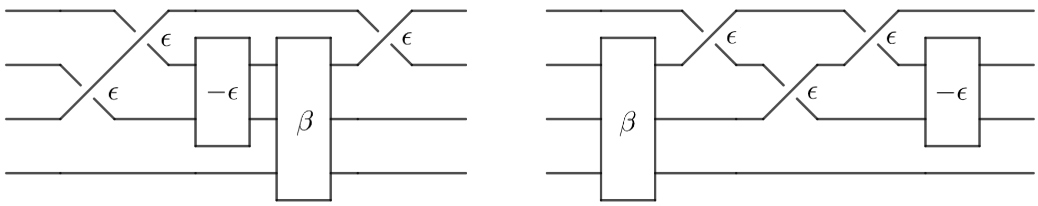

Figure 48. $(1_{m+n-1} \otimes \sigma_{1,n-1}^{\epsilon})(\beta \otimes 1_{n-1})(1_m \otimes \theta_n^{-\epsilon} \otimes 1_{n-1})(1_m \otimes \sigma_{n-1,n}^{\epsilon}) \underset{M1}{\sim}$
$(1_m \otimes \theta_n^{-\epsilon} \otimes 1_{n-1})(1_m \otimes \sigma_{n-1,n}^{\epsilon})(1_{m+n-1} \otimes \sigma_{1,n-1}^{\epsilon})(\beta \otimes 1_{n-1})$

We can now reverse the conjugation we performed earlier. In doing so we obtain the braid on the right of Figure 48. In this diagram we see that the middle two collections of strands are disjoint from the last so we can take the crossing on the left and slide it under the other two so that it is performed just before $\theta_n^{-\epsilon}$. This leaves us with a $\theta_n^{-\epsilon}\sigma_{1,n-1}^{\epsilon}$ term in our expression. By considering the visualisation of this term in Figure 49 it is clear that it is equal to $\sigma_{1,n-1}^{-\epsilon}(1_1 \otimes \theta_{n-1}^{-\epsilon})$.

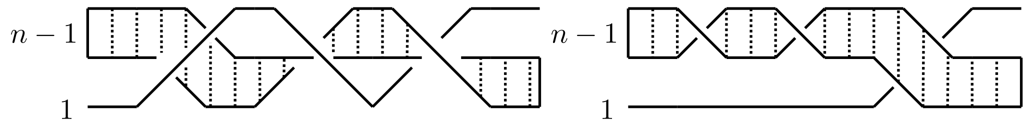

Figure 49. *Visualisation of* $\theta_n^{-\epsilon}\sigma_{1,n-1}^{\epsilon}$

We will now substitute this identity in to the braid on the right of figure 48. This will leave us with the diagram on the left of Figure 50.

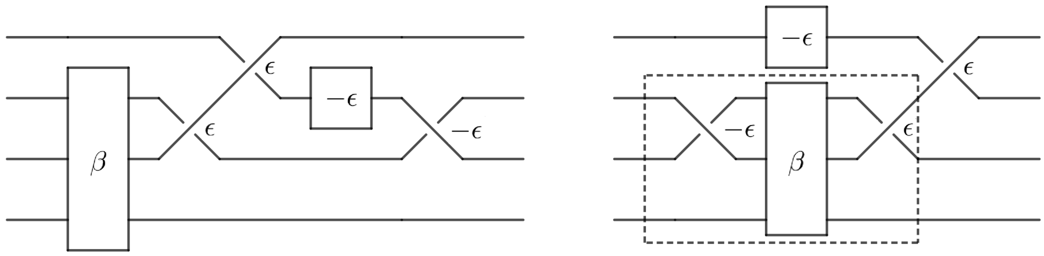

Figure 50.

$$(1_m \otimes \sigma_{1,n-1}^{-\epsilon} \otimes 1_{n-1})(1_{m+1} \otimes \theta_{n-1}^{-\epsilon} \otimes 1_{n-1})(1_m \otimes \sigma_{n-1,n}^{\epsilon})(\beta \otimes 1_{n-1}) \sim$$
$$(1_{m+1} \otimes \sigma_{n-1,n-1}^{\epsilon})(1_m \otimes \sigma_{n-1,1}^{\epsilon} \otimes 1_{n-1})(\beta \otimes \theta_{n-1}^{-\epsilon})(1_m \otimes \sigma_{1,n-1}^{-\epsilon} \otimes 1_{n-1})$$

The diagram to the right of Figure 50 is obtained by first sliding $\theta_{n-1}^{-\epsilon}$ left along the braid until it is in line with β. We then conjugate the braid by $(1_m \otimes \sigma_{1,n-1}^{-\epsilon} \otimes 1_{n-1})$ sliding the final crossing around the closure of the braid until it is to the left of β. We will call the braid contained on the inside of the dashed box β', this leaves us with the expression $(1_{m+1} \otimes \sigma_{n-1,n-1}^{\epsilon})(\beta' \otimes \theta_{n-1}^{-\epsilon})$. Notice that by the inductive hypothesis this is expression is equivalent to β'. On closer inspection of β' it is clear that it is a conjugate of β and so the expression is M-equivalent to β by $M1$. By induction $\mu_{n,\epsilon}$ and $\bar{\mu}_{n,\epsilon}$ are n right ghost which implies the rest of the lemma.

□

The proof of the previous lemma does indeed clearly show that the statement is true however there is a far more intuitive way to view $\mu_{n,\epsilon}$ which will give us a much clearer understanding of why it is such a versatile ghost braid. The key is to consider the 'braid' shown in Figure 51. In this Figure the $\epsilon = +$. If the sign was opposite then we we form the loop in the opposite direction.

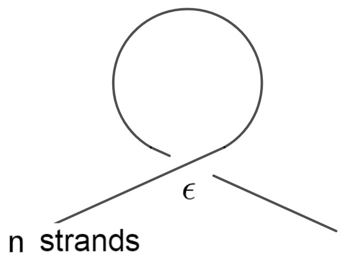

Figure 51. *A 'braid' to consider*

With some thought we notice that if we pulled this braid apart from its ends the loop is in fact equivalent to θ_n^+ for $\epsilon = +$ and similarly it is equivalent to θ_n^- for $\epsilon = -$.

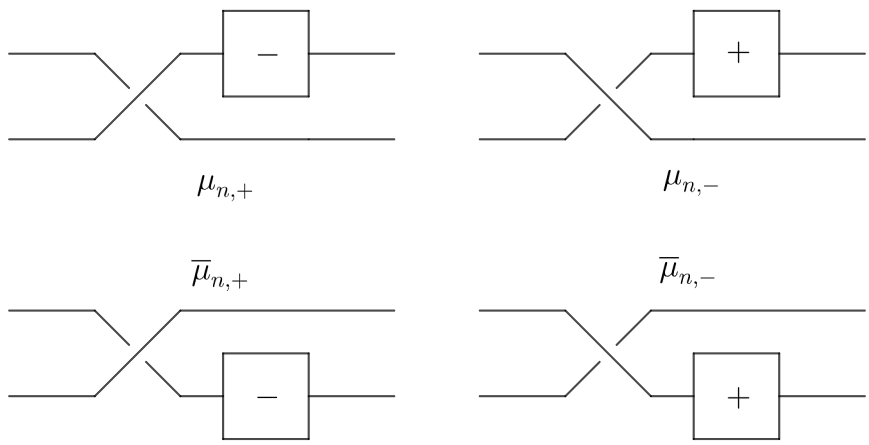

Figure 52. *All configurations of* $\mu_{n,\epsilon}$

Let us now consider all the configurations of $\mu_{n,\epsilon}$ shown in Figure 52. It is clear that if these braids are both right and left ghost then if we close either the top or the bottom strand on each braid and pull apart the 'braid' from the ends as before we should be left with the trivial braid. If this is not the case the braids are not ghost braids. If this is unclear one may redraw our definitions of right and left ghost substituting μ with one of the braids in Figure 52. From our previous realisation we see that these 'braids' are indeed equivalent to the trivial braid. In each case where we have a θ_n^ϵ the corresponding loop formed by the closure is equivalent to $\theta_n^{-\epsilon}$.

Lemma 3.10. *Given two braids $\beta \in B_{m+r}$ and $\gamma \in B_{m+n}$ where $m, n \geqslant 0$ and $r > 0$, we form the braid:*

$$\alpha_\epsilon = (\beta \otimes 1_n)(1_m \otimes \sigma_{n,r}^\epsilon)(\gamma \otimes 1_r)(1_m \otimes \sigma_{r,n}^{-\epsilon}),\ \epsilon = \pm.$$

The M-equivalence of α_ϵ and $\bar{\alpha}_\epsilon$ does not depend on ϵ.

Proof. We will first prove that this is true for α_ϵ Consider the outer diagrams of Figure 53. These show α_+ and α_- respectively. If we set $m = 0$ it is clear that γ and β will have the freedom to move along their respective strands and so the crossings between them can be undone. This means that $\alpha_\epsilon = \beta \otimes \gamma$. If we set $n = 0$ we see that both the σs disappear as there are no longer any strands to cross over and so $\alpha_\epsilon = \beta(\gamma \otimes 1_r)$. If both m and n are 0 then the claim is trivial.

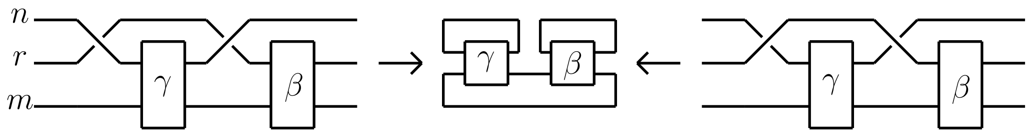

Figure 53. *Closure of α_+ and α_-*

We will now show that α_+ is M-equivalent α_- when both m and n are greater than 0. Intuitively we see that this should be true. If we take the closure of α shown in Figure 53, we see that the resultant link is not affected by ϵ. We begin by taking α_+ and performing a triangle move passing the collection of r strands over the top of the collection of n just after γ in the diagram. Recall the definition of $M(\mu)$ and $\mu_{n,\epsilon}$. We will now perform an $M(\mu_{n,-})$ move on the braid in between the two new crossings introduced by our triangle move. Figure 54 shows these two operations.

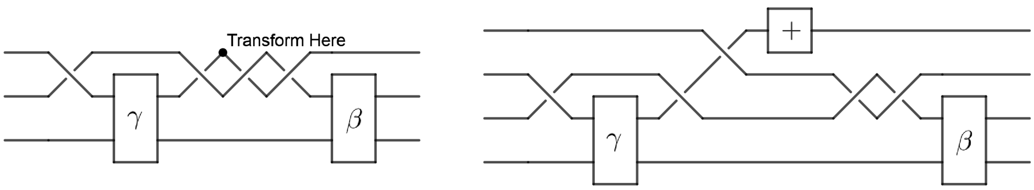

Figure 54. *$M(\mu_{n,-})$ on modified α_+*

Notice that the two crossings to the left of β in Figure 54 can be pushed left along the braid until they are between the first crossing and γ. We can also slide θ_n^+ and the crossing just before it past β. In doing this we see that we can conjugate by $(1_n \otimes \theta_n^+)\sigma_{n,n}^-$. A diagram of this is shown in Figure 55.

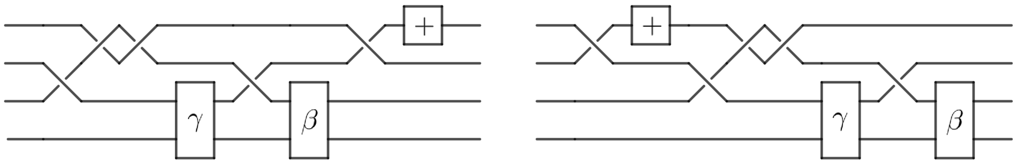

Figure 55. *Conjugation by* $(1_n \otimes \theta_n^+)\sigma_{n,n}^-$

Observe now that we can change the strand on which θ_n^+ lies. To do this we simply slide it to the right down the strand so that it is the last component of the braid. We can then conjugate by $1_{m+r+n} \otimes \theta_n^+$. This moves it onto the second collection of n strands in the braid. We now push the two crossings to the left of γ underneath the strands that θ_n^+ now lies on. This leaves us with the braid shown on the left of Figure 56.

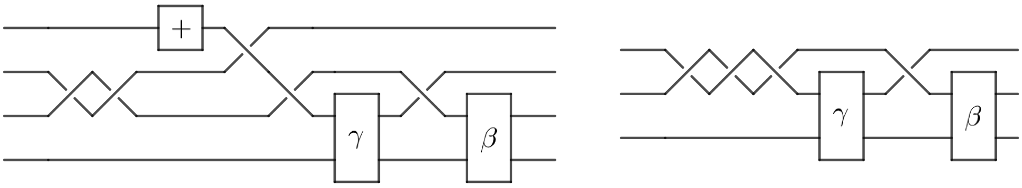

Figure 56. *Inverse* $M(\bar{\mu}_{n,-})$ *move*

We see now that we can perform an inverse $M(\bar{\mu}_{n,-})$ move on the braid. This removes the extra n strands and leaves us with the braid on the right of Figure 56. It is clear that this braid is equal to α_- and so we see that $\alpha+$ and α_- are M-equivalent.

To show that the claim is true for $\bar{\alpha}_\epsilon$ we simply apply the result of lemma 3.4. We see that $\bar{\alpha}_+ \sim \alpha_+ \sim \alpha_- \sim \bar{\alpha}_-$ as the two expressions are M-equivalent through conjugation by Δ. For clarity we see that,

$$\overline{\alpha}_\epsilon = (1_n \otimes \beta)(\sigma_{r,n}^\epsilon \otimes 1_m)(1_r \otimes \gamma)(\sigma_{n,r}^{-\epsilon} \otimes 1_m).$$

□

The next lemma is the final of this section and will be pivotal in proving Markov's theorem.

Lemma 3.11. *Given the braids $\alpha \in B_{n+r}$, $\beta \in B_{n+t}$, $\gamma \in B_{m+t}$ and $\delta \in B_{m+r}$ where $m, n \geqslant 0$, $r, t > 0$, and $\epsilon, \nu = \pm$ consider the braid:*

$$\begin{aligned}\langle \alpha, \beta, \gamma, \delta | \epsilon, \nu \rangle =& (1_m \otimes \alpha \otimes 1_t)(1_{m+n} \otimes \sigma_{t,r}^{\nu}) \\ & ...(1_m \otimes \beta \otimes 1_r)(\sigma_{n,m}^{-\epsilon} \otimes 1_{t+r}) \\ & ...(1_n \otimes \gamma \otimes 1_r)(1_{n+m} \otimes \sigma_{r,t}^{-\nu}) \\ & ...(1_n \otimes \delta \otimes 1_t)(\sigma_{m,n}^{\epsilon} \otimes 1_{r+t}) \in B_{m+n+r+t}.\end{aligned}$$

The M equivalence of $\langle \alpha, \beta, \gamma, \delta | \epsilon, \nu \rangle$ does not depend on ϵ or ν. In addition,

$$\langle \alpha, \beta, \gamma, \delta | \epsilon, \nu \rangle \sim \langle \delta, \gamma, \beta, \alpha | \epsilon, \nu \rangle.$$

Proof. We will begin by considering the independence of ϵ. This claim is merely a reformulation of the $\bar{\alpha}_\epsilon$ part of lemma 3.10. To see this reconsider the expression $(1_n \otimes \beta)(\sigma_{r,n}^{\epsilon} \otimes 1_m)(1_r \otimes \gamma)(\sigma_{n,r}^{-\epsilon} \otimes 1_m)$ from lemma 3.10. Looking at the expression below we see that this braid is of the same form as $\langle \alpha, \beta, \gamma, \delta | \epsilon, \nu \rangle$.

$$\begin{aligned}\langle \alpha, \beta, \gamma, \delta | \epsilon, \nu \rangle = & (1_m \otimes \overbrace{(\alpha \otimes 1_t)(1_n \otimes \sigma_{t,r}^{\nu})(\beta \otimes 1_r)}^{\beta \text{ of } \bar{\alpha}_\epsilon \text{ from lemma 3.10}})(\sigma_{n,m}^{-\epsilon} \otimes 1_{t+r}) \\ & (1_n \otimes \underbrace{(\gamma \otimes 1_r)(1_m \otimes \sigma_{r,t}^{-\nu})(\delta \otimes 1_t)}_{\gamma \text{ of } \bar{\alpha}_\epsilon \text{ from lemma 3.10}})(\sigma_{m,n}^{\epsilon} \otimes 1_{r+t})\end{aligned}$$

In a similar manner the independence of ν comes from the α_ϵ part of lemma 3.10. We can see that if we conjugate $\langle \alpha, \beta, \gamma, \delta | \epsilon, \nu \rangle$ by its last two factors we obtain the expression below which is clearly of the same form as α_ν and so M-equivalence is preserved.

$$\begin{aligned}\langle \alpha, \beta, \gamma, \delta | \epsilon, \nu \rangle \overset{M1}{\sim} & (\overbrace{(1_n \otimes \delta)(\sigma_{m,n}^{\epsilon} \otimes 1_r)(1_m \otimes \alpha)}^{\beta \text{ of } \alpha_\nu \text{ from lemma 3.10}} \otimes 1_t)(1_{m+n} \otimes \sigma_{t,r}^{\nu}) \\ & (\underbrace{(1_m \otimes \beta)(\sigma_{n,m}^{-\epsilon} \otimes 1_t)(1_n \otimes \gamma)}_{\gamma \text{ of } \alpha_\nu \text{ from lemma 3.10}} \otimes 1_r)(1_{m+n} \otimes \sigma_{r,t}^{-\nu})\end{aligned}$$

For the final claim it is enough for us to consider the case where $\epsilon = \nu$ due to the previous two claims. We introduce the braid,

$$\begin{aligned}\langle\langle \alpha,\beta,\gamma,\delta|\epsilon\rangle\rangle =&(\alpha\otimes\gamma)(1_n\otimes\sigma^{\epsilon}_{m,r}\otimes 1_t)(1_n\otimes\theta^{\epsilon}_{m}\otimes\sigma^{\epsilon}_{t,r})(1_n\otimes\sigma^{\epsilon}_{t,m}\otimes 1_r)\\ &(\beta\otimes\delta)(1_n\otimes\sigma^{-\epsilon}_{m,t}\otimes 1_r)(1_n\otimes\theta^{-\epsilon}_{m}\otimes\sigma^{-\epsilon}_{r,t})(1_n\otimes\sigma^{-\epsilon}_{r,m}\otimes 1_t).\end{aligned}$$

It is clear that $\langle\langle \alpha,\beta,\gamma,\delta|\epsilon\rangle\rangle \sim \langle\langle \beta,\alpha,\delta,\gamma|-\epsilon\rangle\rangle$ by $M1$. In particular if we conjugate by everything in the top line of the expression above. Next we will show that $\langle \alpha,\beta,\gamma,\delta|\epsilon,\epsilon\rangle$ is M-equivalent to $\langle\langle \alpha,\beta,\gamma,\delta|\epsilon\rangle\rangle$. If we can show that this is true then it will also imply that $\langle\langle \beta,\alpha,\delta,\gamma|-\epsilon\rangle\rangle \sim \langle \beta,\alpha,\delta,\gamma|-\epsilon,-\epsilon\rangle$. To do this take the closure of $\langle \alpha,\beta,\gamma,\delta|\epsilon,\epsilon\rangle$ where $\epsilon=+$ shown in Figure 57. This will make it easier for us to visualise the M-equivalence between the two braids. To get from the first diagram in Figure 57 to the second we perform the transformation $M(\mu_{m,-})$ on the short edge connecting δ to γ. By lemma 3.9 this does not affect the M-equivalence of the braid as $\mu_{m,-}\equiv 1_m$.

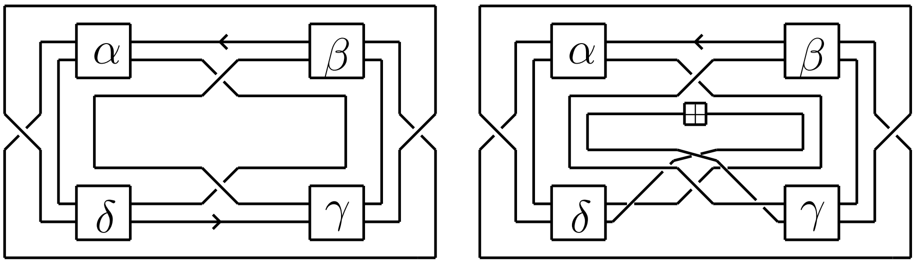

Figure 57. *Transformation by* $M(\mu_{m,-})$

Next, the diagram on the right of figure 57 and the first diagram of Figure 58 are simply equivalent as oriented links. Similarly the two diagrams in Figure 58 are also equivalent links. In this particular equivalence we have added the braid $\theta^-_m\theta^+_m$ which is equal to 1_m and so is equivalent.

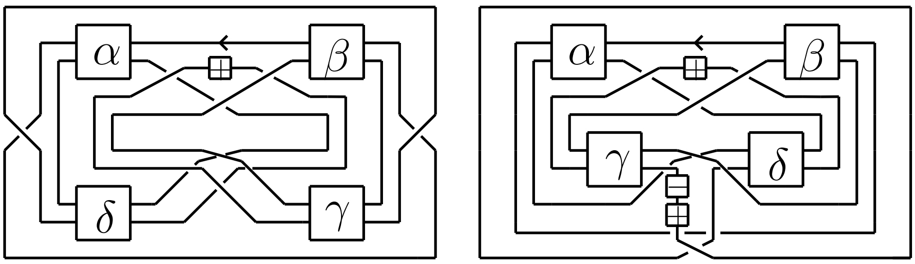

Figure 58. *Equivalence in the class of oriented links*

In order to get from the braid on the right of figure 58 to the first diagram shown in Figure 59 we perform the transformation $M'^{-1}(\bar{\mu}_{m,-})$. Again by lemma 3.9 this does not affect the M-equivalence of the braid as $\bar{\mu}_{m,-} \equiv' 1_m$. The final two braids represent equivalent links. The final link is the closure of $\langle\langle \alpha, \beta, \gamma, \delta | + \rangle\rangle$ thus it is M-equivalent to $\langle \alpha, \beta, \gamma, \delta | +, + \rangle$.

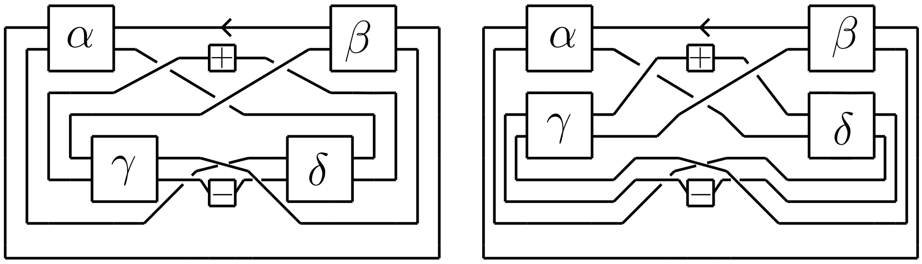

Figure 59. *Transformation by $M'^{-1}(\bar{\mu}_{m,-})$ followed by equivalence in the class of oriented links*

In order to prove this for the case where $\epsilon = -$ we form an argument in exactly the same way. The result obtained will be the mirror of the diagrams shown above. We now see that

$$\langle \alpha, \beta, \gamma, \delta | \epsilon, \epsilon \rangle \sim \langle\langle \alpha, \beta, \gamma, \delta | \epsilon \rangle\rangle \sim \langle\langle \beta, \alpha, \delta, \gamma | -\epsilon \rangle\rangle \sim \langle \beta, \alpha, \delta, \gamma | -\epsilon, -\epsilon \rangle$$

By $M1$ this is M-equivalent to $\langle \delta, \gamma, \beta, \alpha | \epsilon, \epsilon \rangle$. In particular by conjugation of $\langle \beta, \alpha, \delta, \gamma | -\epsilon, -\epsilon \rangle$ by its first four factors. Thus the claim is proven.

□

3.4 Markov's Theorem

Theorem 3.12 (Markov's Theorem)**.** *Two closed braids $\widehat{\beta}$ and $\widehat{\beta}'$ are equivalent links if and only if β and β' are M-equivalent.*

This claim on its own is rather difficult to prove, mainly due to that fact that it concerns links in $\mathbb{R}^3$. If we could reduce this claim to one in two dimensions we will have a much easier time proving it. To do this we will introduce a third 'Markov move' $M3$. This will be the operation that takes β to $\sigma_1^{\epsilon 1}(1_1 \otimes \beta)$ where $\epsilon = \pm$. This is just like an $M2$ move but the extra loop is added to the outside of the link rather than the inside. The move $M3$ is not fundamentally different to $M1$ and $M2$. The effect of $M3$ can be obtained through a composition of the other Markov moves, in particular, $\sigma_1^{\epsilon 1}(1_1 \otimes \beta) = \Delta_{n+1}^{-1} \sigma_n^{\epsilon 1}(\Delta_n \beta \Delta_n^{-1} \otimes 1_1) \Delta_{n+1}$.

To begin we will change the theorem so that it concerns only links whose edges are all positive relative to some centre axis. By Alexander's theorem it is always possible for us to draw any link in such a way. This means that it is also possible to embed this link in the solid torus as no edge can intersect with the centre axis so a circle can be drawn around it that does not intersect with the link. Therefore we can re-express the theorem in the following way. If two closed braids in the solid torus are equivalent then the corresponding braids can be related by a finite sequence of $M1$, $M2$ and $M3$ moves. To simplify this further we would like to remove $M1$ from the claim. To do this consider the following lemma.

Lemma 3.13. *Two braids $\beta \in B_n$ and $\beta' \in B_n$ whose closures are isotopic in the solid torus are conjugate to one another.*

Proof. To begin consider the effect of an isotopy of the torus on the link embedded inside. It is clear that any isotopy of the torus will be equivalent to performing a triangle move or composition of triangle moves on the link. This is in fact our definition of equivalence given in Section 1. This means that to say two links are isotopic is to say that they are equivalent with respect to the space in which they are embedded. We will use these two terms interchangeably from now on.

This means that there exists some sequence of braids with equivalent closures, $\beta \to \beta_1 \to \beta_2 \to \cdots \to \beta'$ such that each term differs by one triangle move from the one before it. We will restrict these triangle moves so that they only affect two strands at once. Larger triangle moves can be obtained as a composition of these smaller ones. As shown in Figure 60 it is enough for us to add these triangle moves to the trivial part of the braid closure. If we wanted to perform a triangle move inside β we could add the inverse elements of the braid via triangle moves to reach this interior point perform any additional triangle moves we want then add the non inverse elements back.

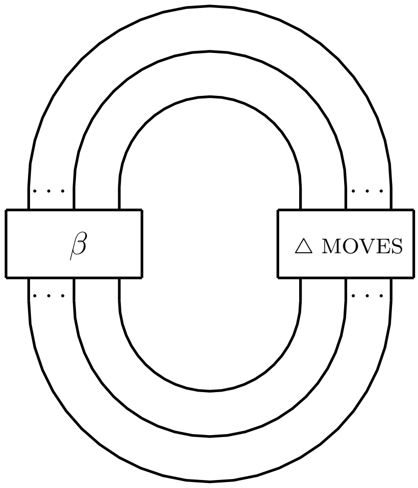

Figure 60. *Triangle moves on* $\widehat{\beta}$

Consider a triangle move on a trivial braid. Notice that as well as adding σ_i to the braid it necessarily will also add its inverse. This means that we can split the two σs and slide one up to the top of β and the other to the bottom of β.

There is no combination of triangle moves which would make this impossible. Notice that if we have a braid obtained by performing triangle moves on the trivial braid we can split it into subsections which are locally conjugate. If there is a section that is left which is not conjugate there necessarily exists another section which is the inverse of the first. Sandwiched between these two sections will be more subsections that form a sub-braid which we can view in the same way as we did the braid as a whole. Due to the finite nature of the braid eventually we will reach a sub-braid which does not contain any non-conjugate subsections.

A subsection that is conjugate is equivalent to the trivial braid and so we can 'undo' the triangle moves in one of these subsections. In this way we can freely move half the σ_i from a neighbouring locally conjugate subsection through the undone section, after which we 'redo' the triangle moves. This means that the two subsections we started with have been transformed in to a single subsection which is conjugate.

We continue this process until the sub-braid is entirely conjugate. We then repeat this until the whole braid is entirely conjugate. In this way we can split these subsections into two parts one the inverse of the other pulling them to opposite sides of the braid. This leaves us with a conjugate of β. See an example of this operation in Figure 61. Thus the claim is true.

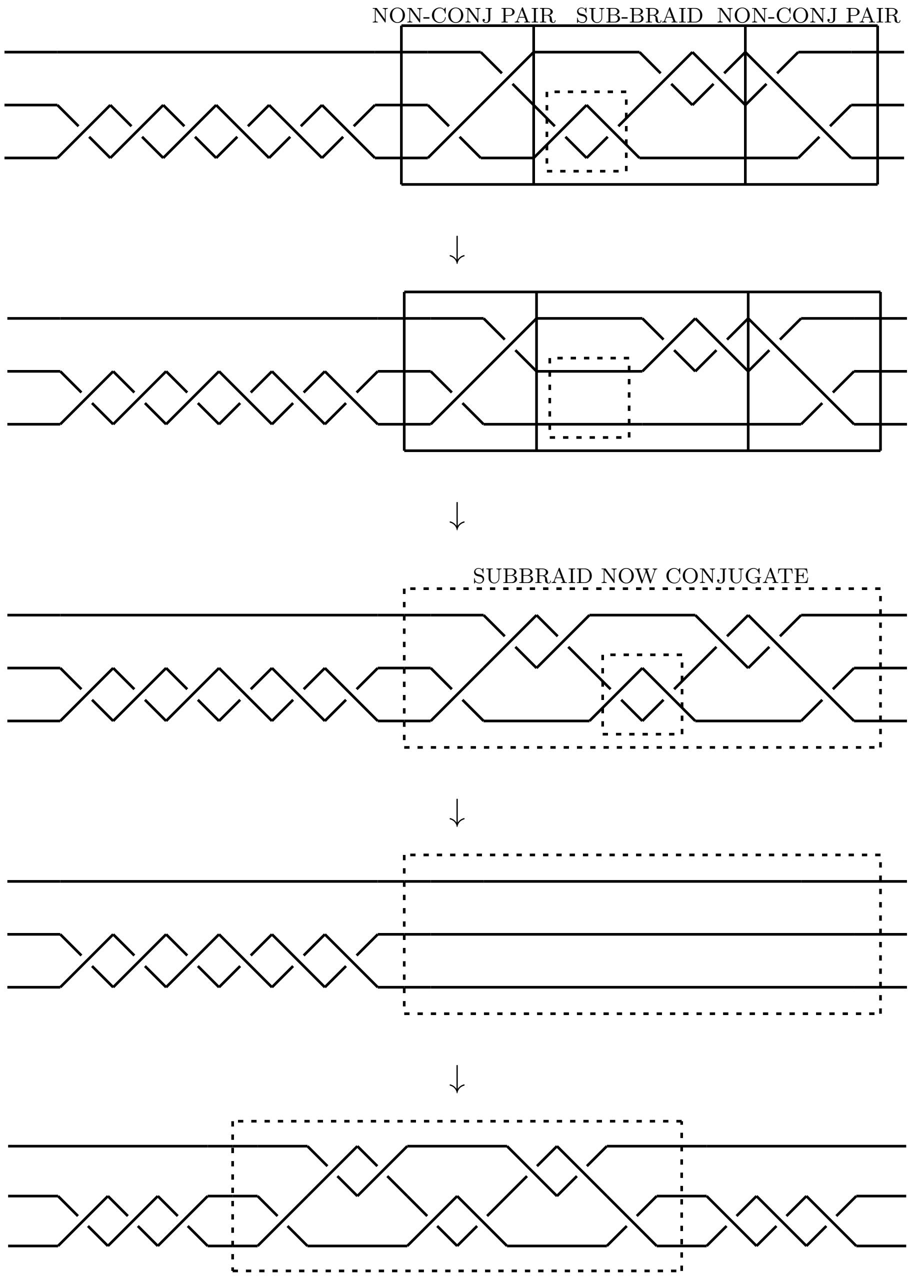

Figure 61. *The 'undoing' and 'redoing' of triangle moves*

□

With this we can replace "$M1$" in the restatement of the theorem with "isotopies in the solid torus". It is very simple to reduce this form of the theorem to one which concerns only diagrams of links. We can say that the corresponding diagrams of braids whose closures are equivalent can be related through a finite series of $M2'$, $M3'$, $(\Omega_2^{br})^{\pm 1}$, $(\Omega_3^{br})^{\pm 1}$ moves and isotopies in the annulus. Here $M2'$ and $M3'$ are merely the diagrammatic versions of $M2$ and $M3$. It is necessary for us to add the braidlike Ω moves as a triangle move is fundamentally different from an isotopy in two dimensions, in three dimensions they were equivalent.

Finally we would like to restate this claim in terms of the Seifert picture of a diagram D. To do this let us first define a special case of the Seifert picture called the 0-diagram. An 0-diagram is the diagram which we obtain from completing the Yamada-Vogel algorithm, that is a series of nested circles such that for each circle S_{i+1} the circle S_i is contained inside it. All circles of the 0-diagram must be oriented counterclockwise. We know that this is always possible as we may perform as many ∞-moves as we need to achieve a diagram of the structure we desire. We will label the circles in the 0-diagram from 1 to $n(D)$ starting from the inner circle and working out. Consider the result of performing Ω_2^{br} and Ω_3^{br} on an 0-diagram. We see that the result is just another 0-diagram, Figure 62 shows an example of Ω_2^{br}.

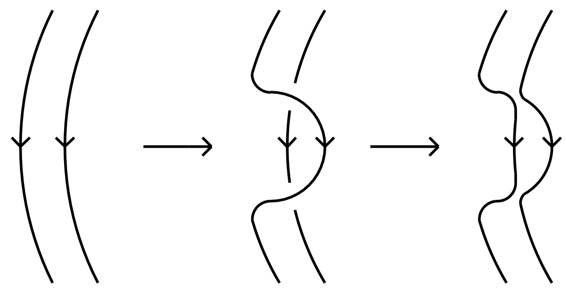

Figure 62. Ω_2^{br} *on an 0-diagram*

Let us now consider Ω_1. In general we see that in the majority of cases performing such a move on the 0-diagram will not leave us with an 0-diagram. There are however two cases in which it does. The first case occurs when Ω_1 is performed on S_1 in such a way that the loop it creates is on the inside of the circle. In this case it is quite clear that the smoothing of the crossing results in a new Seifert circle contained inside S_1 an so it is an 0-diagram. We will name the move that describes this case Ω_1^{int}.

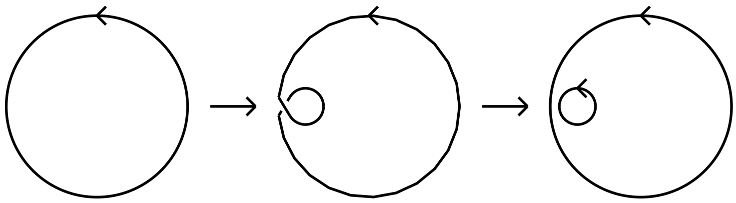

Figure 63. Ω_1^{int} *on the 0-diagram*

The second case occurs when Ω_1 is performed in such a way that the loop created is on the outside of the outermost circle, $S_{n(D)}$. After smoothing this diagram we will be left with a circle that has clockwise orientation sitting outside of $S_{n(D)}$. We then simply perform an ∞-move and we are left with an 0-diagram. We will call this Ω_1^{ext}.

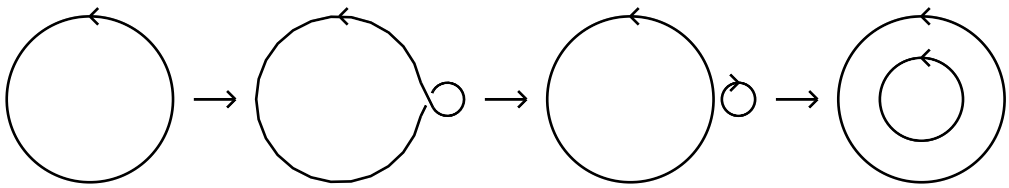

Figure 64. Ω_1^{ext} *on the 0-diagram*

Let us examine Ω_1^{int} and Ω_1^{ext} in closer detail. Recall our visualisation of the Markov move $M2$ from Figure 40. In this we visualised taking an edge of the braid and twisting it over to form a loop. Compare this to Figures 63 and 64. We see that these figures mimic exactly Figure 40, where the move Ω_1^{int} is in fact the move $M2$ and Ω_1^{ext} is the move $M3$.

We see that Ω_1^{ext} and Ω_1^{int} have the information of moves $M2$ and $M3$ encoded within them. In a similar way we can see that the braidlike Reidemeister moves encode information about the conjugacy of equivalent braid closures. We will call the moves Ω_2^{br}, Ω_3^{br}, Ω_1^{int} and Ω_1^{ext} the Ω moves for an 0-diagram. We may now form our final equivalent claim based on 0-diagrams.

Theorem 3.14 (Reformulated Markov's Theorem)**.** *Given the 0-diagrams $\mathcal{E}$ and $\mathcal{E}'$ of two equivalent links $\hat{\beta}$ and $\hat{\beta}'$ we can obtain $\mathcal{E}'$ from $\mathcal{E}$ through a finite sequence of Ω moves.*

It is not immediately apparent that this claim is equivalent to the previous one as we do not have the reliance on isotopies in S^2 included in this claim. It is not obvious that removing this preserves the equivalence of the statement but we will prove that that it is not necessary later in the section. We begin by proving that we can produce the effect of the Reidemeister moves with just Ω moves, bendings, tightenings, and isotopies in S^2.

Lemma 3.15. *We can re-express the Reidemeister moves purely in terms of Ω moves, bendings, tightenings and isotopies in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ on the 0-diagram.*

Proof. First we notice that Ω moves are only defined on the 0-diagram so we must be sure that we have an 0-diagram before using them to replace the Reidemeister moves. Let us assume that $h(D) > 0$. By lemma 2.3 we know that there exists a defect face of D. Let us choose some reduction arc such that it does not interfere with the move say M we are trying to perform. As M and the bending along the reduction arc are disjoint we can perform them in any order we choose. In this way we can perform the bending along the reduction arc then perform M finishing by performing a tightening in reverse across the reduction arc. As the two operations commute this is the same as just performing the move M. We have managed to perform our move on a diagram of lower height. We may increase the number of reduction arcs formed in order to perform the operation at height zero.

When we reach height zero it is possible that the circles are oriented clockwise. In this case we can isotope the diagram in S^2 performing as many ∞-moves as required to orient all circles counterclockwise. We then perform the move that will give us our desired result , say M'. This move is different from M as we have taken the isotopy in S^2. After we perform M', we take the inverse isotopy and we are done. We now know that we can perform any move we like at height zero on the 0-diagram. This means that we can perform any Ω move we like. In particular we can use them in the following form, $riM'i^{-1}r^{-1}$ where r is the composition of the required sequence of bendings to reduce the height to zero and i is the composition of isotopies in S^2 required to orient the diagram properly. We can now re-express the Reidemeister moves in terms of the Ω moves.

Let us begin by looking at the Ω_1 moves not covered by the Ω moves for the 0-diagram. Here there are two possibilities. Either Ω_1 occurs on the outside of one of the circles in the 0-diagram or it occurs on the inside. In the former case we will perform a series of bendings on the circle which is to be affected by Ω_1 say S_t. The first bending will take S_t and pass it under circle S_{t+1}. The second will pass it under S_{t+2} and so on until finally it passes under $S_{n(D)}$. This means that part of the edge that has been bent to the outside has now become part of the outermost circle after smoothing. At this point we perform Ω_1^{ext} on the former S_t. We may then perform a series of Ω_2^{br} and Ω_3^{br} moves to return the circle and the loop created by Ω_1^{ext} back to where they started so that the loop is contained in S_{t+1}. The left of Figure 65 shows the series of bendings required.

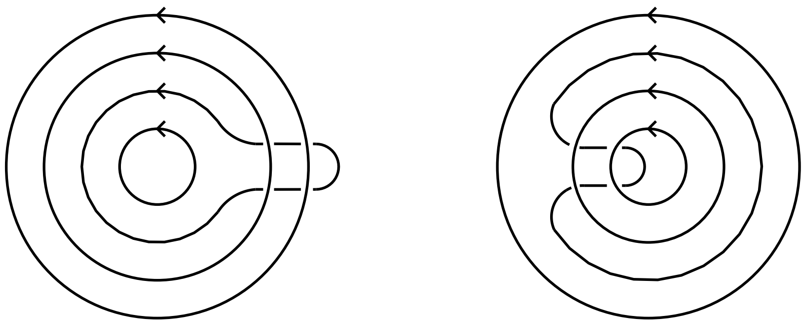

Figure 65. *Bendings required to perform* Ω_1

For the second case where Ω_1 is performed on the inside of the circle S_t, we use a very similar process but move the circle towards the center instead. As before the edge pushed inside will become part of the inner circle after smoothing. We may then perform the move Ω_1^{int} on it and then use a series of Ω_2^{br} and Ω_3^{br} moves to return the circle to its original location so that the loop is contained in S_t but outside of S_{t-1}. The process of bendings is shown on the right of Figure 65.

Next we will consider Ω_2 moves not included in the Ω moves for the 0-diagram. We see that we can perform two Ω_1 moves followed by a bending to pass the top of the loop created by the second Ω_1 move under the other circle. After this we perform a tightening to remove the two loops created by the Ω_1 moves. This process is shown more clearly in Figure 66.

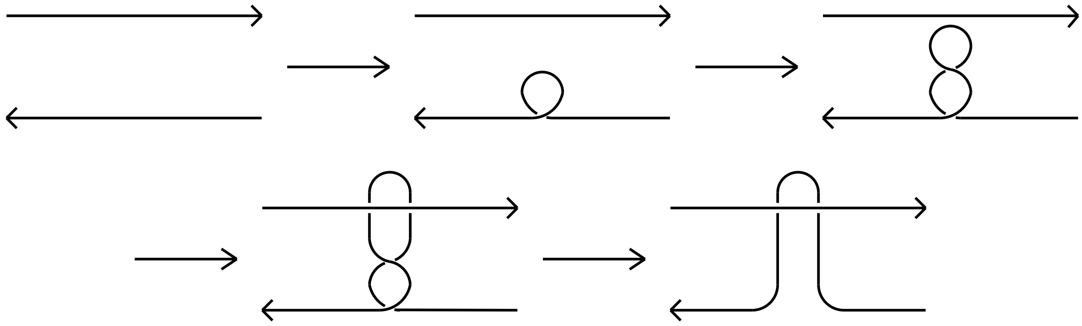

Figure 66. *Sequence of moves replacing* Ω_2

When first looking at the sequence of moves in Figure 66 one might initially think that a bending from the bottom line to the top line will suffice however this is not the case. Remember that a bending occurs along a reduction arc. There is no assurance that the two circles are in fact incompatible. When following this sequence of moves we ensure that whenever we perform a bending or tightening it follows a reduction arc. For verification of this simply redraw the smoothed version of Figure 66 and notice that if a bending is performed the height will reduce and if a tightening is performed the height will increase.

Finally we will consider the Ω_3 moves not included in the Ω moves for 0-diagrams. We will replace this move with the series of moves shown in Figure 67. The first move is an Ω_2 move which we have just shown we can use. This move twists the the strand that goes in the opposite direction of the other two in such a way that we can now perform an Ω_3^{br} move. After performing this move we simply perform an Ω_2^{-1} move and we are left with the desired result.

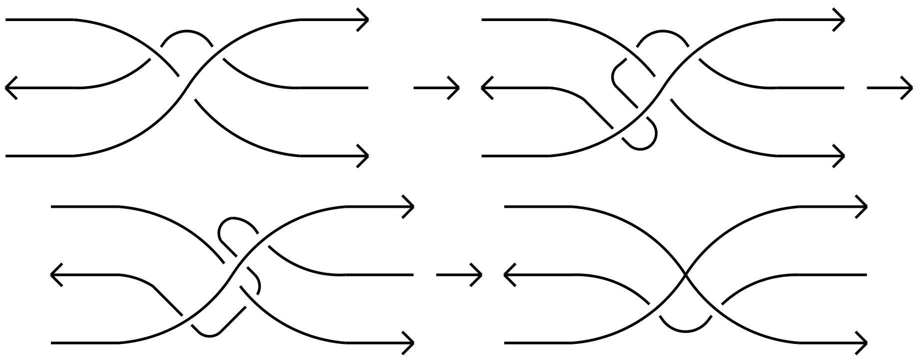

Figure 67. *Sequence of moves replacing* Ω_3

□

This means that given two equivalent 0-diagrams $\mathcal{E}$ and $\mathcal{E}'$ we can relate them through a finite series of Ω-moves, bendings, tightenings, and isotopies in S^2. This brings us closer to our goal of proving Theorem 3.14. In order to prove our claim we must also show that it is not necessary for us to take isotopies in S^2. The next series of lemmas will allow us to remove this dependency on isotopies in S^2. The next lemma looks at the case where two 0-diagrams are equivalent through a finite series of isotopies in S^2 and nothing else.

Lemma 3.16. *Consider two 0-diagrams $\mathcal{E}$ and $\mathcal{E}'$ that are equivalent through a finite series of isotopies in S^2. These two diagrams are also equivalent through a finite series of isotopies in R^2.*

Proof. Notice that no isotopy in S^2 can add an additional crossing to the 0-diagram so two isotopic diagrams must share the same number of Seifert circles. If we have two diagrams that contain one circle it is clear that they are both isotopic to a circle with counterclockwise orientation in $\mathbb{R}^2$. Any two circles in $\mathbb{R}^2$ with equal orientation are isotopic to one another and so the 0-diagrams are isotopic in $\mathbb{R}^2$.

Let us now consider two diagrams with n distinct circles. We see that each diagram is split in to $n+1$ different areas. The first $n-1$ of these are the annuli bounded by the Seifert circles. In addition to this there is one disk bounded by the innermost circles say D_i and D'_i and the final area is the disk bounded by the outermost circles say D_o and D'_o. As $\mathcal{E}$ and $\mathcal{E}'$ are isotopic there exists a set of homeomorphisms $\{F_t : S^2 \to S^2\}_{t\in I}$ where F_0 is the identity and F_1 corresponds to the transformation taking $\mathcal{E}$ to $\mathcal{E}'$.

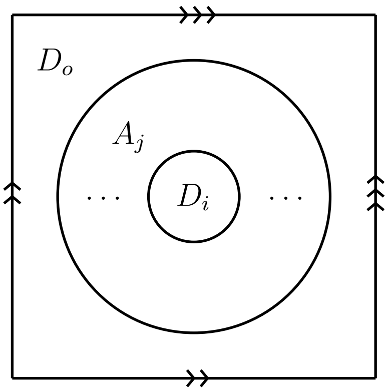

Figure 68. *The sections of S^2*

Let us now consider where D_o and D_i are mapped under this homeomorphism. Notice that the innermost circle and the disk that it bounds have the same orientation. However notice that D_o has clockwise orientation which is opposite to that of the outermost circle. This means that D_o cannot be mapped to D_i and D_i cannot be mapped to D'_o. As there are only two disks in the diagram then D_o must be mapped to D'_o and D_i to D'_i. Next notice that ∞ is in D_o and D'_o and so we can form another set of homeomorphisms such that it takes $F_1(\infty)$ to ∞ in $\mathcal{E}'$ leaving all points outside of the disk fixed. In this way we have isotoped $\mathcal{E}$ into $\mathcal{E}'$. If we restrict these homeomorphisms to $\mathbb{R}^2$ we see that $\mathcal{E}$ and $\mathcal{E}'$ are isotopic in $\mathbb{R}^2$ as well. It is not necessary for us to consider isotopy in S^2 as the point at ∞ is not used in any way during the deformation of $\mathcal{E}$ into $\mathcal{E}'$.

$\square$

If we have two diagrams that are equivalent through only a series of isotopies in $\mathbb{R}^2$ it means that they are trivially equivalent. At most the edges of the isotopic diagram are a different shape but it is impossible for such an isotopy to introduce new crossing information to the diagram. So if we have two diagrams isotopic in $\mathbb{R}^2$ we see that they are in fact the same diagram.

To prove that it is also true for cases which involve more than just isotopy in S^2 we need to introduce the idea of a local maximum. To do this consider a diagram D which contains at least one face which is defect. This means we can form two reduction arcs c and c', these may involve the same two circles. We call the sequence $\mathcal{C} \stackrel{s}{\leftarrow} D \stackrel{s'}{\rightarrow} \mathcal{C}'$ a local maximum where s and s' are the corresponding bendings of D along c and c'. An important feature to notice about this construction is that $\mathcal{C}$ and $\mathcal{C}'$ are of height one less than D. The final proof we are building up to will use an inductive argument on the maximum height in a sequence like this. To this end we will now work towards finding a way to replace a sequence like this with one whose maximum height is less than the original. First we will define $s \cdot s'$ as the number of times the corresponding reduction arcs c and c' intersect. This leads us to the following lemma.

Lemma 3.17. *Given a local maximum $\mathcal{C} \stackrel{s}{\leftarrow} D \stackrel{s'}{\rightarrow} \mathcal{C}'$ where $s \cdot s' \neq 0$ we can find an alternative sequence which starts at $\mathcal{C}$ and ends at $\mathcal{C}'$ such that*

$s_i \cdot s'_i = 0$ for all i. This sequence may contain multiple local maxima.

Proof. Consider a reduction arc c in D. We can perturb the arc either slightly to the left or right in order to obtain an new reduction arc which is disjoint from the original. If we perturb the arc to the left we will call the new arc c_l, if we perturb the arc to the right we will use c_r. Figure 69 helps illustrate this idea.

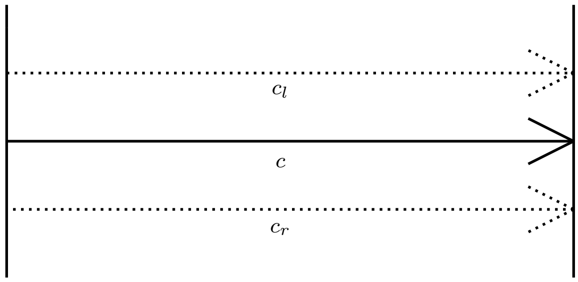

Figure 69. *Example of c_l and c_r*

We will use this notation when describing the formation of new arcs. We will prove this lemma in two steps. First we will consider the case when $s \cdot s' \geqslant 2$. We will show that it is possible in this case to obtain a new reduction arc c'' such that $s \cdot s''$ is less than the initial value. If we can do this then it will always be possible for us to create an arc that intersects c at one point only.

Say that c and c' intersect in exactly two places. We will label these points of intersection A and B. We can assume that both c and c' are directed from A to B. This is because reversing the orientation of a reduction arc does not change the fact that it is a reduction arc, it just bends a different segment of the link. We have four possibilities. These are: 1) c' crosses A from left to right and B from right to left; 2) c' crosses A from left to right and B from left to right; 3) c' crosses A from right to left and B from right to left; 4) c' crosses A from right to left and B from left to right. Cases 3) and 4) are the mirror of 1) and 2) so we just need to consider these. For case 1) consider Figure 70.

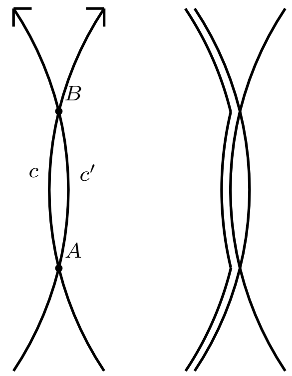

Figure 70. *Example of c'' if c' crosses A from the left and B from the right*

This clearly shows how to form c'' reducing $s \cdot s'$ from 2 to 0. For case 2) consider Figure 71. This clearly shows that we can form a reduction arc c'' reducing $s \cdot s'$ from 2 to 1. It is sufficient for us to only consider the case where $s \cdot s' = 2$. It is clear that we can follow c'_l or c'_r depending on the situation up until we reach the point where there are only two intersections left to consider and then perform the process laid out in the Figures 70 and 71.

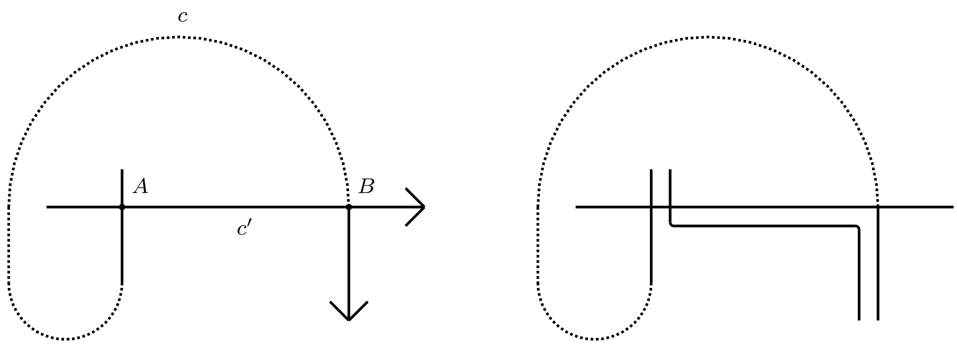

Figure 71. *Example of c'' if c' crosses A from the left and B from the left*

Let us now consider how to form c'' when $s \cdot s' = 1$. We will label the point of intersection O. The endpoints of c will be labelled A_1 and A_2 and the endpoints of c', A_3 and A_4. This labelling is shown more clearly in Figure 72. The first and simplest case would be when two of the circles are in fact the same. Without loss of generality say $S_1 = S_4$. It is clear then that S_1 and S_3 are incompatible and we can follow the arc $A_1O \cup OA_3$ perturbing it slightly so that it does not intersect with c or c'. This arc is shown in Figure 72 also. In general we will call the arc that travels from S_i to S_j in this fashion $c_{i,j}$. This means that we now need to consider the case when all the circles are distinct.

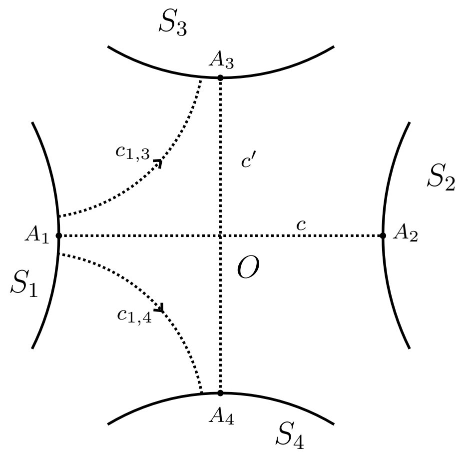

Figure 72. *Formation of* $c_{1,3}$ *and* $c_{1,4}$

Firstly if S_1 and S_3 are incompatible we can use the arc $c_{1,3}$ so we will now assume that S_1 and S_3 are compatible. Choose one of the circles say S_1 and consider the labelled edges that are connected to it. We see that there are three possibilities here: A) there could be no edges attached to the chosen circle; B) one edge attached to the chosen circle; or C) multiple edges attached to the chosen circle. The first case A) can be shown quite easily with a with a diagram, consider Figure 73.

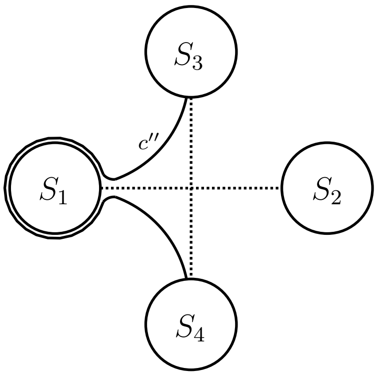

Figure 73. *The arc* c'' *in case A)*

Case B) requires a bit more thought. First assume that the labelled edge connects S_1 to S_4. To form c'' consider Figure 74. Now assume that the labelled edge connects S_1 to some other circle say S_5. Notice that as there is a labelled edge running between S_1 and S_5 they must have opposite orientations. This implies that S_5 is incompatible with S_4 as we have assumed that S_1 and S_4 are compatible. To form the arc c'' we follow $c_{4,1}$ until we reach a point just outside S_1. We then circle S_1 until we are about to intersect with the labelled edge. To complete the arc we follow the labelled edge until we intersect S_5.

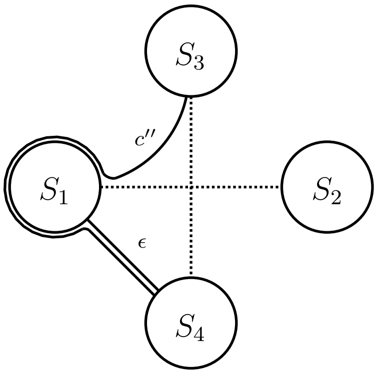

Figure 74. *The arc c'' in case B) if S_1 is connected to S_4*

For case C) let us fist assume that none of the edges connect to S_4. This means that we can use exactly the same arc as we described in case B) as any circle directly connected to S_1 is necessarily compatible with it and thus incompatible with S_4. This leaves us to consider the case when S_1 is connected to S_4. As we have already stated any to circles connected by a labelled edge are necessarily compatible. This means that any circle connected to S_1 by a labelled edge must be incompatible with S_4. To form the arc we follow along the outside of the labelled edge connecting S_1 to S_4. We then circle S_1 until we are about to intersect another labelled edge. To complete the are we then follow the edge until we intersect with the circle at the end. As our circles are arbitrarily labelled we have now covered all possibilities.

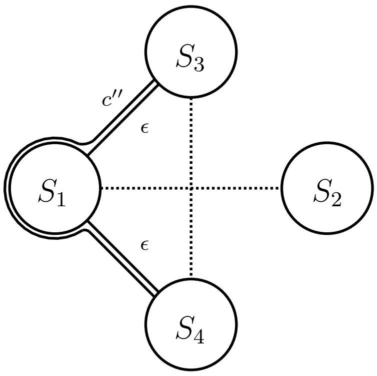

Figure 75. *Example of possible c'' in case C)*

With these new c'' we can form the following sequence for a given local maximum,

$$\mathcal{C} \overset{s}{\leftarrow} D \overset{s''}{\rightarrow} \mathcal{C}'' \overset{s''}{\leftarrow} D \overset{s'}{\rightarrow} \mathcal{C}'.$$

Such a sequence clearly meets the conditions of the lemma. This means that in general we can obtain a sequence of the form shown below,

$$\mathcal{C} = \mathcal{C}_1 \overset{s_1}{\leftarrow} D_1 \overset{s'_1}{\rightarrow} \mathcal{C}_2 \overset{s_2}{\leftarrow} \cdots \overset{s'_{m-1}}{\rightarrow} \mathcal{C}_m \overset{s_m}{\leftarrow} D_m \overset{s'_m}{\rightarrow} \mathcal{C}_{m+1} = \mathcal{C}',$$

such that $s_i \cdot s'_i = 0$.

□

Next we will present a lemma which allows us to replace the $\leftarrow D \rightarrow$ part of a local maximum with new diagrams all with lower height than D.

Lemma 3.18. *Given a local maximum $\mathcal{C} \overset{s}{\leftarrow} D \overset{s'}{\rightarrow} \mathcal{C}'$ where $s \cdot s' = 0$ we can find sequences $\mathcal{C} \rightarrow \cdots \rightarrow \mathcal{C}_*$ and $\mathcal{C}' \rightarrow \cdots \rightarrow \mathcal{C}'_*$ of bendings and isotopies in S^2 such that $\mathcal{C}_* = \mathcal{C}'_*$ or $\mathcal{C}_*$ and $\mathcal{C}'_*$ are 0-diagrams in $\mathbb{R}^2$ which can be related through a finite sequence of Ω-moves.*

Proof. We will split this proof into three sections. First we shall consider the case when the reduction arcs c and c' involve to different pairs of Seifert circles, the pairs however may have one circle in common. The second case is the one where c and c' concern the same pair of circles (S_1 and S_2) however there are reduction arcs present elsewhere. In the final and most interesting case we will consider the case where all reduction arcs concern just one pair of Seifert circles (S_1 and S_2).

Let us consider the first case. As c and c' involve different pairs of circles it is possible for us to perform both of these bendings on D without one nullifying the other. The assumption of the lemma states that $s \cdot s' = 0$. This implies that the paths of the bendings will not interact with one another thus we can perform them in any order we please. This knowledge allows us to form the following two sequences:

$$\mathcal{C} \overset{s'}{\rightarrow} D', \quad \mathcal{C}' \overset{s}{\rightarrow} D', \quad \text{where } s(s'(D)) = s'(s(D)) = D'.$$

These two sequences clearly fulfil the conditions expressed in the lemma.

A pair of sequences can be found for the second case in much the same way as the first. Consider a reduction arc c_1 that does not involve both S_1 and S_2 (the circles concerned in c and c'). By lemma 3.17 it is always possible for us to choose this c_1 such that $s_1 \cdot (s \cup s') = 0$. As before it does not matter if s_1 is performed before s and s' or after and so we can form the following two sequences:

$$\mathcal{C} \xrightarrow{s_1} \mathcal{C}_1 \xrightarrow{s'} D', \quad \mathcal{C}' \xrightarrow{s_1} \mathcal{C}_1' \xrightarrow{s} D', \quad \text{where } s_1(s'(D)) = s_1(s(D)) = D'.$$

These sequences fit the conditions of the lemma.

We will now consider the final case where all reduction arcs that do not intersect with c or c' concern the circles S_1 and S_2 (the circles that c and c' connect). We will set the orientation of c' so that it runs from S_1 to S_2. Consider S_1, S_2, c and c' on the sphere shown in Figure 76. Notice that these components split the sphere into four distinct sections. The first of these are the two disks bounded by S_1 and S_2. We will label these D_1 and D_2 respectively. To construct the other sections we see that the endpoints of c and c' split both of the two circles into two subarcs. To form the third section we take the area bounded by c, c' and a subarc from both S_1 and S_2. We choose the first subarc arbitrarily and the second subarc so that it is on the same side of c' as the first. This section shall be labelled D_3 as it is indeed a topological disk. The fourth and final section is obtained by taking the area bounded by c, c' and the two subarcs not used in the construction of D_3. This final section is also a disk which we shall label D_4.

In this way we have given ourselves a means to categorise the remaining Seifert circles on the sphere. These categories are: 1) circles in D_1 including S_1, 2) circles in D_2 including S_2, 3) circles in D_3, and 4) circles in D_4. Note that both D_1 and D_2 necessarily contain at least one circle each however there is nothing to stop D_3 and D_4 from being empty.

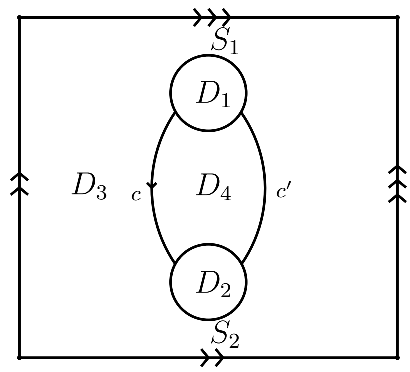

Figure 76. *The four disks in* S^2

Take the Seifert circles in D_1. Due to our assumption that all reduction arcs concern S_1 and S_2 it is not possible for there to be any reduction arcs in D_1. This would imply that all circles in D_1 are compatible with one another. As they are all compatible they must also be concentric. This last claim is easily verified by considering one Seifert circle which contains two more circles that are not concentric. In this situation it is impossible to find a combination of orientations for the circles which makes them all compatible, you will always be able to form a reduction arc. We will say that D_1 contains t concentric circles. Using the same argument we see that the circles in D_2, D_3, and D_4 must all be concentric as well. We will say that D_2 has r circles, D_3 has n, and D_4 has m. From these groups of circles we would like to find a way to reconstruct the original diagram D.

To do this we must remember that the circles are connected by labelled edges indicating the crossing type that used to be there before smoothing. We saw that these edges can be read to give us a braid when the Seifert circles are in this concentric form. If one group of circles is oriented in the opposite way to another the groups will be compatible. Groups that are compatible can be connected by a labelled edge but this is not always the case. If they are connected by one of these edges we take all the other labelled edges of the two groups and slide them around the circles so that they are in line with the edge connecting the two groups. This gives us a braid at this point where the two groups of circles are connected. If there is no such edge connecting the groups we perform the same steps as before as if the edge did exist and then we take the tensor product of the two braids we are left with. Figure 78 should make this construction clearer. Doing this we obtain four braids which we will label, $\alpha \in B_{n+r}$, $\beta \in B_{n+t}$, $\gamma \in B_{m+t}$, and $\delta \in B_{m+r}$.

The circles in D_1 and D_2 must all have the same orientation as the two groups are connected by a reduction arc. Due to our assumption that there are no reduction arcs disjoint from c and c' involving circles other than S_1 and S_2 we see that all circles in D_3 and D_4 must be oriented in the opposite direction to those in D_1 and D_2. Let us set D_1 and D_2 to have counterclockwise orientation and D_3 and D_4 to have clockwise orientation.

Let us now informally introduce the idea of a 'superbending'. Recall the definition of $\sigma^{\epsilon}_{n,m}$. We introduced this notation as a way to perform a σ move on large collections of strands. Here we introduce the superbending to perform a mass bending on these collections of strands. We simply take all the strands in one collection and pull them over all the strands in another. An example of a superbending is shown in Figure 77. Notice that the number of individual bendings required to accomplish the superbending is equal to the product of the number of strands in each collection involved in the move.

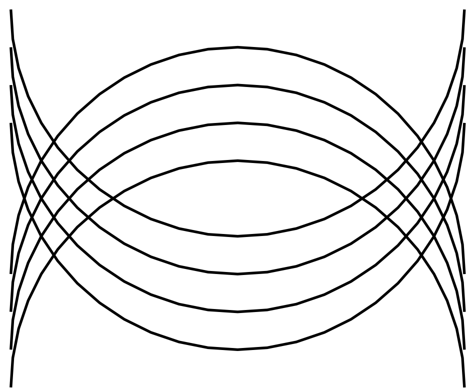

Figure 77. *A "superbending" of two groups of three concentric circles*

Consider Figure 78 showing our reconstruction of D. We would like to put this diagram into closed braid form. To do this we will need to perform two superbendings. The reduction arcs these bendings will follow are labelled c and c'' in the diagram. The first reduction arc c is just a regular bending but c'' travels to the point at ∞ and then to its destination.

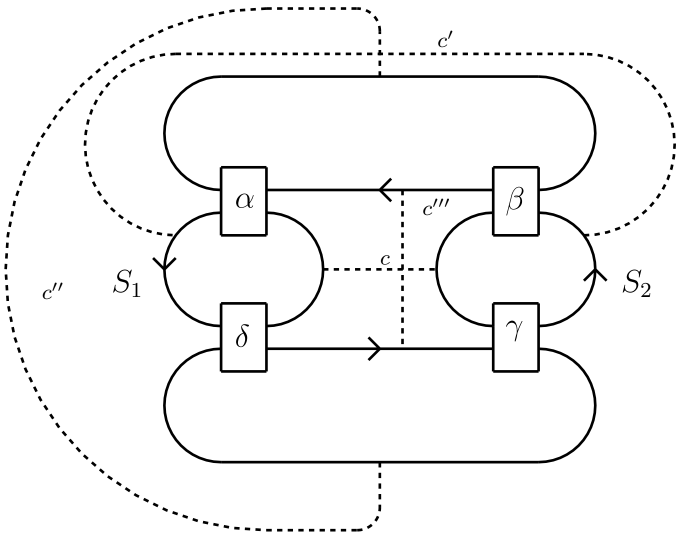

Figure 78. *Reconstruction of D from its Seifert circles*

The resultant diagram shown in Figure 79 is a very familiar one. It is in fact the closure of $\langle \alpha, \beta, \gamma, \delta | +, + \rangle$ which we first came across in Lemma 3.11. For our purposes this diagram is $\mathcal{C}_*$. The superbendings across c and c'' allow us to construct the following sequence of bendings in S^2:

$$D \xrightarrow{s} \mathcal{C} \rightarrow \overset{rt+mn-1 \text{ } bendings}{\cdots \rightarrow \cdots \rightarrow} \ldots \rightarrow \mathcal{C}_*.$$

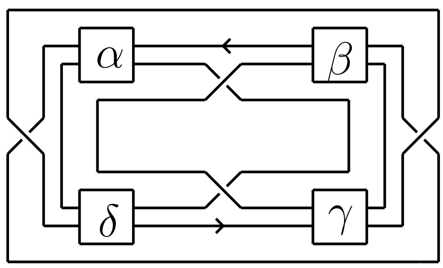

Figure 79. *Diagram of* $\mathcal{C}_*$

Consider once again Figure 78. Let us choose two different reduction arcs in the hope of obtaining another sequence. This time we will follow the reduction arcs c' and c'''. Here c' goes to the point at ∞ and then to its destination. This leaves us with the diagram shown in Figure 80 which we will call $\mathcal{C}'_{**}$. The diagram has clockwise orientation so we will perform the required number of ∞-moves on its lower strands so that it is oriented counterclockwise. This new diagram will be $\mathcal{C}'_*$. To visualise the transformation of D into $\mathcal{C}'_{**}$ it is important to notice that in D the braids α and β are travelling from right to left and the braids δ and γ travel from left to right. In $\mathcal{C}'_{**}$ the braids α and β travel from left to right and δ and γ travel from right to left. The diagram $\mathcal{C}'_*$ is in in fact the closure of $\langle \delta, \gamma, \beta, \alpha | +, + \rangle$. The bendings along c' and c''' allow us to form the following sequence:

$$D \xrightarrow{s'} \mathcal{C}' \to \cdots \overset{rt+mn-1\ bendings}{\to \cdots \to} \ldots \to \mathcal{C}'_{**} \overset{\infty}{\to} \mathcal{C}'_*.$$

By Lemma 3.11 $\mathcal{C}_*$ and $\mathcal{C}'_*$ are M-equivalent and thus related by a finite sequence of Ω-moves. This means that the two sequences satisfy the conditions of the lemma. Earlier in the proof we made the assumption that c' was directed from S_1 to S_2. If we were to reverse this orientation we would follow the same steps and find that $\langle \delta, \gamma, \beta, \alpha | +, + \rangle$ becomes $\langle \delta, \gamma, \beta, \alpha | +, - \rangle$. By Lemma 3.11 this does not affect the M-equivalence of the braid and so the claim is still true.

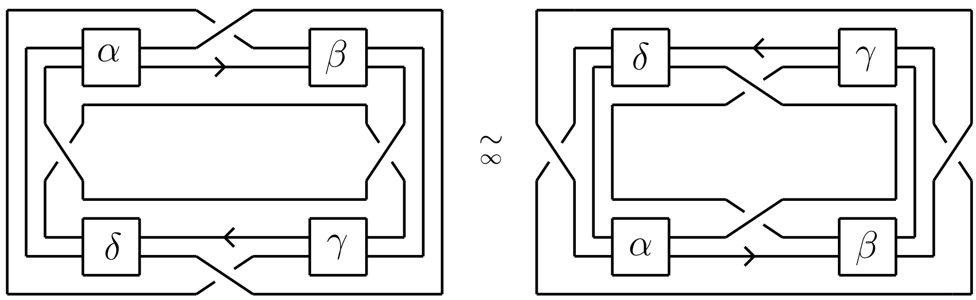

Figure 80. *Diagram of* $\mathcal{C}'_{**} \underset{\infty}{\sim} \mathcal{C}'_*$

$\square$

As mentioned before this allows us to replace a local maximum with a sequence whose diagrams all have height less than D. In particular we have,

$$\mathcal{C} \to \cdots \to \mathcal{C}_* \sim \mathcal{C}'_* \leftarrow \cdots \leftarrow \mathcal{C}'.$$

The equivalence in the middle is to show that these two diagrams could either be equal to one another or that they are 0-diagrams and can be related through a finite number of Ω moves.

Lemma 3.19. *If we can relate two 0-diagrams through a finite sequence of bendings, tightenings and isotopies in the sphere then we can also relate them* *through a finite sequence of* Ω*-moves*

Proof. In this proof we will talk about the height of a sequence. By this we mean the the height of the diagram with the largest height overall in the sequence. To prove the lemma we will perform induction of the height of a sequence, say m. Let us first consider the base case where $m = 0$. This implies that the sequence contains no bending or tightening, only isotopies in S^2. In Lemma 3.16 we consider this exact case and so we may relate the two diagrams by a sequence of Ω-moves.

Let us now examine the case when $m > 0$ and assume that it is true for all values less than this. Consider an isotopy of a diagram followed by a bending or tightening. We see that we could in fact perform the bending or tightening first and then the isotopy. In this way we can push all the isotopies to one side of our sequences in such a way that we are left with a sequence of bendings and tightenings followed by the sequence of isotopies. In these sequences any diagram of height m must be local maxima ($\mathcal{C} \xleftarrow{s} D \xrightarrow{s'} \mathcal{C}'$) as m is the height of the sequence itself which means the diagrams before and after it must have height $m-1$. By Lemma 3.17 we can find alternate sequences that have the same start and end diagrams such that $s \cdot s' = 0$ for all local maxima. The height of such a sequence will also be m. By Lemma 3.18 we can replace every local maximum in our sequence with a sequence like the following:

$$\mathcal{C} \to \cdots \to \mathcal{C}_* \sim \mathcal{C}'_* \leftarrow \cdots \leftarrow \mathcal{C}'.$$

Here we have managed to re-express a sequence which initially had height m with another whose height is less than m. By the inductive hypothesis two 0-diagrams related by a sequence such as this can also be related through a finite series of Ω-moves. $\square$

This lemma as well as Lemma 3.15 implies our reformulated Markov's theorem is true and thus Markov's original theorem is as well.

4 The Alexander-Conway Polynomial

In this section we develop an invariant of oriented links. In particular we will look at the Alexander-Conway polynomial of oriented links introduced by Conway in 1969.

This section is written with reference to sections 3.1-3.4 of Kassel and Turaev's book [2] and a set of notes written by Massuyeau on the Alexander-Conway polynomial [3].

4.1 The Burau and Reduced Burau Representation

Here we will present the Burau representation for braids. This is an alternate presentation for the braid group B_n in the form of matrices. The matrices are defined over the Laurent ring of polynomials $\mathbb{Z}[t^{\pm 1}]$. That is the ring of polynomials with both positive and negative powers of t with integer coefficients. In addition to this we will define its reduced form. We begin by giving the definitions after which we will justify their existence.

Definition 4.1 (Burau representation)**.** *The map $Bu_n : B_n \to GL_n(\mathbb{Z}[t^{\pm 1}])$ defined by,*

$$Bu_n(\sigma_i) = U_i = I_{i-1} \oplus U \oplus I_{n-i-1}, \quad \textit{where } U = \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}$$

such that $n > 1$ and I_j is the unit matrix of size j is known as the ***Burau representation*** *of B_n.*

More simply as i increases incrementally from 1 the block U moves diagonally from the top left to the bottom right corner by one row and one column per increment.

Definition 4.2 (Reduced Burau representation)**.** *The map $\overline{Bu}_n : B_n \to GL_{n-1}(\mathbb{Z}[t^{\pm 1}])$ defined by the following,*

$$\overline{Bu}_n(\sigma_1) = V_1 = \begin{pmatrix} -t & 0 \\ 1 & 1 \end{pmatrix} \oplus I_{n-3}, \quad \overline{Bu}_n(\sigma_{n-1}) = V_{n-1} = I_{n-3} \oplus \begin{pmatrix} 1 & t \\ 0 & -t \end{pmatrix},$$

$$\overline{Bu}_n(\sigma_i) = V_i = I_{i-2} \oplus \begin{pmatrix} 1 & t & 0 \\ 0 & -t & 0 \\ 0 & 1 & 1 \end{pmatrix} \oplus I_{n-i-2}, \quad \overline{Bu}_2(\sigma_1) = (-t),$$

such that $n > 1$ and I_j is the unit matrix of size j is known as the ***reduced Burau representation*** *of B_n.*

As with the U_i we can see that the non trivial matrix in our expression for $\overline{Bu}_n(\sigma_i)$ travels from the top left to the bottom right of V_i as i increases. However at $i = 1$ the first row and column of the block extends outside of the matrix and when $i = n - 1$ the last row and column extend outside of the matrix. From these definitions we can form a short exact sequence.

Proposition 4.3. *There exists a short exact sequence of B_n-modules:*

$$1 \longrightarrow \mathbb{Z}[t^{\pm 1}] \longrightarrow Bu_n \longrightarrow \overline{Bu}_n \longrightarrow 1$$

Proof. We claim that the map from $Bu_n \to \overline{Bu}_n$ is defined by the expression,

$$C_n^{-1} U_u C_n = \begin{pmatrix} V_i & 0 \\ {}^*i & 1 \end{pmatrix} = V_i', \quad \text{where } C_n = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

where ${}^*i$ is the row of length $n-1$ $(0\ 0\ \dots\ 0\ \delta_{i,n-1})$, $\delta_{i,j}$ being the Kronecker delta. To prove this claim is true we simply need to verify that $U_i C_n = V_i' C_n$ for all values of i. This comes down to a fair amount of matrix multiplication so here we will just write the results obtained in each of the three cases, the reader can verify these by direct computation. Firstly if $i = n - 1$ we obtain the matrix below:

$$U_i C = C V_i^{'} = \begin{pmatrix} C_{n-2} & 1 & 1 \\ 0 & 1-t & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

If $n-1>i>1$:

$$U_iC = CV_i^{'} = \begin{pmatrix} C_{i-1} & 1 & 1 & 1 & 1 \\ 0 & 1-t & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & C_{n-i-2} \end{pmatrix}.$$

Finally if $i=1$:

$$U_iC = CV_i^{'} = \begin{pmatrix} 1-t & 1 \\ {}^*l & C_{n-1} \end{pmatrix}, \text{ where } {}^*l = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T.$$

The second map from $\mathbb{Z}[t^{\pm 1}]$ to Bu is given by the action of Bu on $\mathbb{Z}[t^{\pm 1}]^{\oplus n}$. Notice that in the expression $C_n^{-1}Bu(\beta)C_n$ the final column is fixed. This means that the nth copy of $\mathbb{Z}[t^{\pm 1}]$ remains unchanged. We will map the first $n-1$ copies of $\mathbb{Z}[t^{\pm 1}]$ on to the final one. As this is the map to $\overline{Bu}$ we must reverse the conjugation. This leaves us with the inclusion,

$$C_n \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & 0 & \vdots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix} C_n^{-1} = \begin{pmatrix} 0 & \dots & 0 & 1 \\ \vdots & 0 & \vdots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

□

Let us now look at whether these definitions are justified or not. In making these definitions we have assumed that U_i and V_i are both invertible and follow the two braid relations. We will show that these properties are indeed met.

First consider the invertability of U_i. To do this we can directly compute the determinant:

$$det\left(I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}\right).$$

Notice that the first $i-1$ rows and columns of the matrix can be ignored as they will only add a factor of 1 to the determinant. Similarly we can neglect the final $n-i-1$ rows and columns of the matrix and so we see that the determinant is given by $(1-t)0 - 1(t) = -t \neq 0$. This means that U_i is invertible for all i. If we take the product of two invertible matrices it is clear due to the associativity of matrix multiplication that it will also be invertible. In other words if X and Y are invertible then $XYY^{-1}X^{-1} = I$ and so any combination of U_i is invertible. To see that the V_i are also invertible we once again (knowing that we can eliminate the I_j components of the matrices) calculate the determinants:

$$det\begin{pmatrix} -t & 0 \\ 1 & 1 \end{pmatrix}, \quad det\begin{pmatrix} 1 & t \\ 0 & -t \end{pmatrix}, \quad det\begin{pmatrix} 1 & t & 0 \\ 0 & -t & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

This gives us the result $-t$ for all three cases which implies that V_i is invertible for all i and in turn that any combination of V_i is also invertible. This means that the maps Bu and $\overline{Bu}$ do end in the general linear group.

We now verify that the braid relations hold. If we look at U_iU_j we see that if $|i-j| \leqslant 1$ then U occupies at least one of the same entries in their respective matrices and so we see that $U_iU_j \neq U_jU_i$. However if $|i-j| > 1$ the Us do not overlap and so they are multiplied by some I_j which gives the same result whether it is multiplied to the right or the left and so $U_iU_j = U_jU_i$.

Now let us also consider the relation $U_IU_{i_1}U_i = U_{i+1}U_iU_{i+1}$ and check if it also holds. In order to do this it will suffice to show that the following equality holds. This is because the unit parts of the matrix are trivial under matrix multiplication.

$$\begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{pmatrix}$$

To see that the relations also hold for the V_i recall that they are obtained by conjugating the U_i. So the product of two V_i is given by the expression below:

$$C_n^{-1}U_iC_nC_n^{-1}U_jC_n = C_n^{-1}U_iU_jC_n.$$

We see that the C_n on the inside of the expression cancel to leave us with all the U_i together. The relation holds for the U_i and so taking a conjugate of them will also preserve the braid relation.

From the expression above it is clear that we can obtain an expression for $\overline{Bu}_n(\beta)$ by finding $Bu_n(\beta)$ and then conjugating it by C_n as follows,

$$C_n^{-1}Bu_n(\beta)C_n = \begin{pmatrix} \overline{Bu}_n(\beta) & 0 \\ {}^*\beta & 1 \end{pmatrix}.$$

Unlike ${}^*i$ the term ${}^*\beta$ is not so easily found. Let us now consider it in more detail.

Lemma 4.4. *For all $i \in \{1, \ldots, n-1\}$ given the matrix $\overline{Bu}_n(\beta) - I_{n-1}$ we see that,*

$$\sum_{i=1}^{n-1}(1 + t + \cdots + t^{i-1})a_i + (1 + t + \cdots + t^{n-1})^*\beta = 0,$$

where a_i is the respective row of $\overline{Bu}_n(\beta) - I_{n-1}$.

Proof. Let us define the following row matrices of length n. The first we shall call E. This is the matrix whose ith entry is t^{i-1}.

$$E = (1, t, t^2, \ldots, t^{n-1})$$

The second of these row matrices we will call F. This is given by EC_n. So we see that the ith entry of F is the sum of the first i entries in E.

$$F = EC_n = (1, 1 + t, 1 + t + t^2, \ldots, 1 + t + \cdots + t^{n-1})$$

Let us now consider EU_i:

$$EU_i = \begin{pmatrix} 1 \\ t \\ \vdots \\ t^{n-1} \end{pmatrix}^T \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix}$$

It is clear that the first $i-1$ terms and the last $n-i-1$ terms are going to be the same as that of E since we are multiplying by the unit matrix. The ith entry will be given by $(1-t)t^{i-1} + t^i = t^{i-1}$ and the $i+1$th entry is given by $t(t^{i-1}) = t^i$. These two entries are also the same as that of E so we see that $EU_i = E$. This implies also that $EBu_n(\beta) = E$. We see from this realisation that,

$$F \begin{pmatrix} \overline{Bu}_n & 0 \\ {}^*\beta & 1 \end{pmatrix} = EC_nC_n^{-1}Bu_n(\beta)C_n = EC_n = F.$$

Subtracting FI_n from both sides:

$$F \begin{pmatrix} \overline{Bu}_n & 0 \\ {}^*\beta & 1 \end{pmatrix} - FI_n = F \begin{pmatrix} \overline{Bu}_n(\beta) - I_{n-1} & 0 \\ {}^*\beta & 0 \end{pmatrix} = \mathbf{0}.$$

We have almost proven our claim. Let us expand the final expression out slightly ignoring the final column as it will just give us a zero in the final matrix:

$$\begin{pmatrix} 1 \\ 1+t \\ 1+t+t^2 \\ \vdots \\ 1+t+\cdots+t^{n-2} \\ 1+t+\cdots+t^{n-1} \end{pmatrix}^T \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1(n-1)} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2(n-1)} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & a_{(n-1)3} & \cdots & a_{(n-1)(n-1)} \\ \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_{n-1} \end{pmatrix}.$$

As this expression is equal to zero it is clear that the sum of all entries in the matrix will also equal zero. Here we see that all the elements in a_i are multiplied by the ith entry of F and so we have $\sum_{i=1}^{n-1}(1+t+\cdots+t^{i-1})a_i$ as the sum of all entries in the first $n-2$ rows of the matrix. The final row of the matrix is multiplied by $1 + t + \cdots + t^{n-1}$ and so the sum of these entries is equal to $(1 + t + \cdots + t^{n-1})^* \beta$. This proves the claim.

$\square$

An Alternate Form of the Burau Representation

An interesting thing to note about the Burau representation is that there are two forms that can be used. These forms are equivalent to one another. To obtain the second form Bu'_n we transpose the centre block U in the matrix, that is:

$$\sigma_i \longmapsto I_{i-1} \oplus \begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix} \oplus I_{n-i-1}.$$

As a consequence we also have an alternate version of the reduced Burau representation defined by conjugation of Bu' by C_n^T in the following way:

$$C_n^T Bu'_n(\sigma_i)(C_n^T)^{-1} = \begin{pmatrix} \overline{Bu}'_n & {*i}^T \\ 0 & 1 \end{pmatrix}.$$

To see that these representations are equivalent consider the matrix,

$$R_n = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & t & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & t^{n-2} & 0 \\ 0 & 0 & \dots & 0 & t^{n-1} \end{pmatrix}.$$

To obtain the inverse of this matrix we take the elements along the diagonal $r_{i,i}$ and replace them with $1/r_{i,i}$. If we change basis using this matrix we see that,

$$R_n^{-1} Bu'(\sigma_i) R_n = R_n^{-1} I_{i-1} \oplus \begin{pmatrix} (1-t)t^{i-1} + (1)0 & (1-t)0 + (1)t^i \\ (t)t^{i-1} + (0)0 & (t)0 + (0)0 \end{pmatrix} \oplus I_{n-i-1}$$

$$= I_{i-1} \oplus \begin{pmatrix} (t^{-(i-1)})t^{i-1}(1-t) + (0)t^i & (t^{-(i-1)})t^i + (0)0 \\ (0)t^{i-1}(1-t) + (t^{-i})t^i & (0)t^i + (0)0 \end{pmatrix} \oplus I_{n-i-1}$$

$$= I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1} = Bu(\sigma_i).$$

4.2 Markov Functions

We will now introduce the idea of a Markov function.

Definition 4.5 (Markov function)**.** *A **Markov function** is a set of functions f_n which act on the elements of B_n taking them to some set E. These functions must also satisfy the conditions for M-equivalence. That is:*

1) $f_n(\alpha\beta) = f_n(\beta\alpha)$ for all α and $\beta \in B_n$ where $n \geqslant 1$,

2) $f_n(\beta) = f_{n+1}(\sigma_n\beta) = f_{n+1}(\sigma_n^{-1}\beta)$ for all β where $n \geqslant 1$.

This definition is not particularly informative on its own so we will now look at an example of a Markov function in some detail.

4.2.1 An Example of a Markov Function

Let us now look at an example of such a function. Consider the following function over $\mathbb{Z}[s^{\pm 1}]$:

$$f_n(\beta) = (-1)^{n+1}\frac{s^{-\langle\beta\rangle}(s-s^{-1})}{s^n - s^{-n}}g(det(\overline{Bu}_n(\beta) - I_{n-1})), \quad f_1(\beta) = 1$$

where g is a mapping from $\mathbb{Z}[t^{\pm 1}] \to \mathbb{Z}[s^{\pm 1}]$ such that t is mapped to s^2. The term $\langle\beta\rangle$ is obtained by taking the sum of all powers of σ in β. For example consider the braid $\sigma_1\sigma_2 \in B_3$. We obtain the following:

$$\overline{Bu}_3(\sigma_1\sigma_2) - I_2 = \begin{pmatrix} -t & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & -t \end{pmatrix} - I_2 = \begin{pmatrix} -t-1 & -t^2 \\ 1 & -1 \end{pmatrix} = V$$

$$f_3(\sigma_1\sigma_2) = (-1)^4\frac{s^{-2}(s-s^{-1})}{s^3 - s^{-3}}g(t^2+t+1)$$

$$= \frac{s^{-2}(s-s^{-1})(s^4+s^2+1)}{s^3-s^{-3}} = 1$$

To prove that this set of f_n are Markov functions we must show that both of the conditions of a Markov function hold.

Claim 4.6. *The set of f_n defined above form a Markov function.*

Proof. First of all we will show that $f_n(\alpha\beta) = f_n(\beta\alpha)$. To see this equivalence consider the values of both $\langle\beta\alpha\rangle$ and $\langle\alpha\beta\rangle$. It is clear that the sums of the powers in both expressions are equal. Next we will consider the determinant portion of the expression. Let us consider $det(\overline{Bu}_n(\alpha\beta) - I_{n-1})$, Let us calculate the determinant in the usual way expanding across the main diagonal but ignore the -1 terms until the end. It is clear that this is equal to $det(\overline{Bu}_n(\alpha\beta)) - k_1$ where k_1 is introduced by the -1 terms. Similarly we can say that $det(\overline{Bu}_n(\beta\alpha) - I_{n-1}) = det(\overline{Bu}_n(\beta\alpha)) - k_2$. Recall that if we have two square matrices X and Y then $det(XY) = det(X)det(Y) = det(Y)det(X) = det(YX)$ and that $tr(XY) = tr(YX)$. These two equalities imply that when calculating the determinant of $\overline{Bu}_n(\alpha\beta)$ the additive and subtractive parts of the expression will be equal to those of $\overline{Bu}_n(\beta\alpha)$ and that the trace of each expression will contain the same terms just appearing in a different order. This means that as the -1 terms are added along the diagonal any terms they add will be the same for both expressions ans so k_1 must be equal to k_2. This means both determinants are the same and the first condition holds.

We will now show that the second condition also holds. Let us set $\beta_+ = \iota(\beta)\sigma_n \in B_{n+1}$. We want to show that $f_{n+1}(\beta_+) = f_n(\beta)$. Let us first consider the case where $n > 1$. Notice that,

$$\frac{s^{-\langle\beta\rangle}(s - s^{-1})}{s^n - s^{-n}} = \frac{s^{n-1-\langle\beta\rangle}}{1 + s^2 + s^4 + \cdots + s^{2(n-1)}}.$$

In this expression if we replace n and β with $n+1$ and β_+ we see that $s^{n-1-\langle\beta\rangle} = s^{(n+1)-1-\langle\beta_+\rangle}$ as $\langle\beta_+\rangle$ is one larger than $\langle\beta\rangle$. We will now replace the s^2 terms with t. With these observations we can simplify the expression

$f_{n+1}(\beta_+) = f_n(\beta)$ to the following:

$$\begin{aligned}
\rightarrow &(-1)^{n+2}\frac{s^{n+1-1-\langle\beta_+\rangle}}{1+t+t^2+\cdots+t^n} det(\overline{Bu}_{n+1}(\beta_+) - I_n) \\
&= (-1)^{n+1}\frac{s^{n-1-\langle\beta\rangle}}{1+t+t^2+\cdots+t^{n-1}} \det(\overline{Bu}_n(\beta) - I_{n-1})
\end{aligned}$$

$$\begin{aligned}
\rightarrow &(1+t+t^2+\cdots+t^{n-1}) det(\overline{Bu}_{n+1}(\beta_+) - I_n) \\
&= -(1+t+t^2+\cdots+t^n) det(\overline{Bu}_n(\beta) - I_{n-1}).
\end{aligned}$$

Next we would like to expand the $\overline{Bu}_{n+1}(\beta_+)$ term of the expression above. To do this let us consider the Burau representation of the inclusion $B_n \hookrightarrow B_{n+1}$. The bottom right block in the Burau matrices is the unit matrix I and the entries above this and to the left are all zero. So to represent the inclusion we need only add a 1 to the bottom right and zeroes everywhere else,

$$Bu_{n+1}(\iota(\beta)) = \begin{pmatrix} Bu_n(\beta) & 0 \\ 0 & 1 \end{pmatrix}.$$

Recall the following,

$$C_n^{-1} Bu_n(\beta) C_n = \begin{pmatrix} \overline{Bu}_n(\beta) & 0 \\ *\beta & 1 \end{pmatrix}.$$

Substituting this expression into the expression for the inclusion of β in to B_{n+1} we obtain:

$$Bu_{n+1}(\iota(\beta)) = \begin{pmatrix} Bu_n(\beta) & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} C_n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{Bu}_n(\beta) & 0 & 0 \\ *\beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_n^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

Let us now use the expression we recalled above for $\overline{Bu}_{n+1}$ using the fact that

$\beta_+ = \iota(\beta)\sigma_n$:

$$\begin{aligned}
\begin{pmatrix} \overline{Bu}_{n+1}(\beta_+) & 0 \\ *\beta & 1 \end{pmatrix} &= C_{n+1}^{-1} Bu_{n+1}(\beta_+) C_{n+1} \\
&= C_{n+1}^{-1} Bu_{n+1}(\iota(\beta)) Bu_{n+1}(\sigma_n) C_{n+1} \\
&= C_{n+1}^{-1} \begin{pmatrix} C_n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{Bu}_n(\beta) & 0 & 0 \\ *\beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_n^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{pmatrix} C_{n+1}.
\end{aligned}$$

We will now compute the product of the first three and the last three matrices. This gives us the following result,

$$\begin{pmatrix} \overline{Bu}_{n+1}(\beta_+) & 0 \\ *\beta & 1 \end{pmatrix} = \begin{pmatrix} \overline{Bu}_n(\beta) & 0 & 0 \\ *\beta & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_{n-2} & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1-t & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

To multiply these matrices together we will make the following expansion:

$$\begin{pmatrix} \overline{Bu}_n(\beta) & 0 & 0 \\ *\beta & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} X & Y & 0 & 0 \\ Z & T & 0 & 0 \\ P & Q & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this matrix X is an $(n-2) \times (n-2)$ matrix, Y is a column matrix with $n-2$ entries, Z and P are row matrices with $n-2$ entries and T and Q are just polynomials in $\mathbb{Z}[t^{\pm 1}]$. These particular substitutions allow us to multiply the matrices together in the usual way. Giving us,

$$\begin{pmatrix} \overline{Bu}_{n+1}(\beta_+) & 0 \\ *\beta & 1 \end{pmatrix} = \begin{pmatrix} X & Y & tY & 0 \\ Z & T & tT & 0 \\ P & Q & tQ-t & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

and so we see that,

$$\overline{Bu}_{n+1}(\beta_+) - I_n = \begin{pmatrix} X - I_{n-2} & Y & tY \\ Z & T-1 & tT \\ P & Q & tQ - t - 1 \end{pmatrix}.$$

To calculate the determinant of this matrix we must recall that if we take a multiple of a row or column in a matrix and subtract it from another the determinant will be unchanged. To this effect we will multiply the $(n-1)$th column by $(-t)$ and add it to the nth column. This gives us the following equivalence,

$$det(\overline{Bu}_{n+1}(\beta_+) - I_n) = det \begin{pmatrix} X - I_{n-2} & Y & 0 \\ Z & T-1 & t \\ P & Q & -t-1 \end{pmatrix}.$$

Notice that in this matrix we also have the terms $\overline{Bu}_n(\beta) - I_{n-1}$ and $*\beta$. The term $*\beta = (P \quad Q)$ and the four blocks above this form $\overline{Bu}_n(\beta) - I_{n-1}$. If we use the same determinant trick as before we will multiply the first row of $\overline{Bu}_{n+1}(\beta_+) - I_n$ by 1 the second by $1+t$ the third by $1+t+t^2$ and so on until we reach the second last row. For the last row we return to the full expression $(1+t+t^2+\cdots+t^{n-1})det(\overline{Bu}_{n+1}(\beta_+)-I_n)$. Notice that multiplying the determinant of a matrix by some factor will give the same result as if we multiplied every entry of a row or column in the matrix by that factor and then calculated the determinant. This means that we can take this factor inside multiplying the entries of the last row of the matrix. Now if we add the rest of these rows to the final one we obtain the following matrix by Lemma 4.4,

$$\begin{aligned} &(1+t+t^2+\cdots+t^{n-1})det(\overline{Bu}_{n+1}(\beta_+) - I_n) \\ &\qquad = det \begin{pmatrix} X - I_{n-2} & Y & 0 \\ Z & T-1 & t \\ 0 & 0 & -(1+t+\cdots+t^n) \end{pmatrix}. \end{aligned}$$

With a simple calculation this shows us that,

$$(1+t+t^2+\cdots+t^{n-1})det(\overline{Bu}_{n+1}(\beta_+)-I_n)$$
$$= -(1+t+\cdots+t^n)det\begin{pmatrix} X-I_{n-2} & Y \\ Z & T-1 \end{pmatrix}.$$

This is exactly what we were trying to show. Now to show that the condition holds for the case where $n=1$. We see that,

$$\begin{aligned} f_2(\sigma_1) &= -\frac{s^{-1}(s-s^{-1})}{s^2-s^{-2}}(-s^2-1) \\ &= -s^{-1}(s+s^{-1})^{-1}(-s^2-1) = 1 = f_1(\beta), \end{aligned}$$

so the second condition holds for σ^n. To show that it is true for σ^{-n} we form an argument in a similar way to this one. This means that both conditions hold and the set of f_n form a Markov function.

□

4.2.2 An Invariant of Equivalent Links

Consider some braid β whose closure is the link L. We define $\hat{f}(L) = f_n(\beta)$ where f_n is the function defined in the previous section. Consider now some link $L' = \widehat{\beta'}$ which is equivalent to L. By Markov's theorem we know that the braids β and β' are M-equivalent. Now by our definition of a Markov function we see that $f_n(\beta)$ and $f_n(\beta')$ must be equal. As we chose β arbitrarily this must be the case for all braids so the function $\hat{f}(L)$ is an invariant under the equivalence of links. In the following section we will discuss this invariant of links further.

4.3 The Alexander-Conway Polynomial

Consider some oriented link L, in $\mathbb{R}^3$. If we examine each of the crossings in L we notice that locally they look like a 2-braid in $\mathbb{R}^3$. The three possible 2-braids we could see are called the Conway triple. The Conway triple is shown diagrammatically in Figure 81.

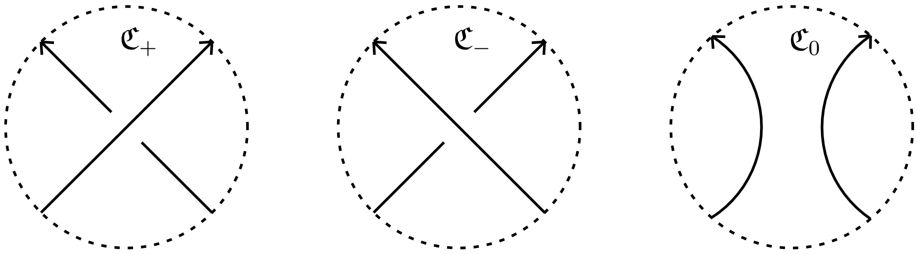

Figure 81. *The Conway triple*

Notice that we have oriented these in such a way that the orientation of both strands points up. This is the angle from which we will view all crossings. We will label the triple $\mathfrak{C}_+$, $\mathfrak{C}_-$ and $\mathfrak{C}_0$ as shown in Figure 81. If we replace one of the crossings in our link with one of the $\mathfrak{C}_i$ we will write the new link L_i. With this we can now introduce the Alexander-Conway polynomial of a link L denoted $\nabla(L) \in \mathbb{Z}[s^{\pm 1}]$. This polynomial follows three rules:

$$
\begin{aligned}
&1)\nabla(L) \text{ is invariant if we take an isotopy of } L,\\
&2)\nabla(\circ) = 1, \text{ where} \circ \text{is the unknot},\\
&3)\nabla(L_+) - \nabla(L_-) = x\nabla(L_0), \text{where } x = (s - s^{-1}).
\end{aligned}
$$

The third rule is known as the Alexander-Conway skein relation. Although these rules gives us all we need to compute the polynomial for any given link they are rather unhelpful for building intuition, so let us work through an example. We will compute $\nabla(K)$ where K is the trefoil. To begin consider Figure 82.

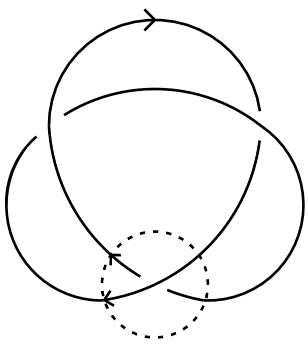

Figure 82. *The trefoil K*

We would like to express it in terms of rule 3 in such a way that we can deduce the polynomial. To do this we consider one of the crossings in K. We will focus on the one circled in Figure 82. Imagine removing the crossing and replacing it with $\mathfrak{C}_+$ and $\mathfrak{C}_0$. This allows us to form the expression $\nabla(K_+) - \nabla(K) = x\nabla(K_0)$.

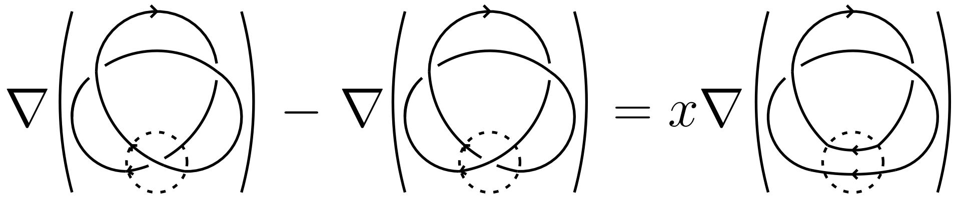

Figure 83. *Diagrams of* $\nabla(K_+) - \nabla(K_-) = x\nabla(K_0)$

Notice that K_+ is in fact equivalent to the unknot. By definition the Alexander-Conway polynomial of this is 1. All that is left for us to compute is $\nabla(K_0)$. In the same way as before we will replace one of the crossings of $K_0 = H_-$ with the other $\mathfrak{C}_i$. We obtain the expression $\nabla(H_+) - \nabla(H_-) = x\nabla(H_0)$:

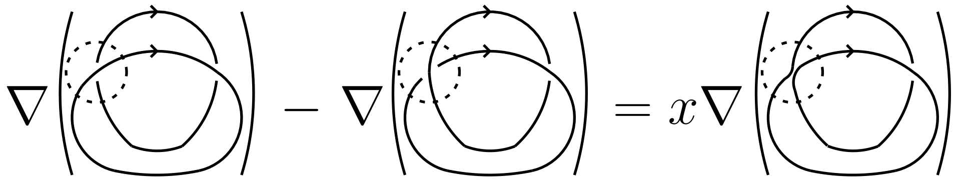

Figure 84. *Diagrams of* $\nabla(H_+) - \nabla(H_-) = x\nabla(H_0)$

Here we see that H_0 is equivalent to the unknot so its polynomial is 1. There is just one link left that we do not know the value for, $\nabla(H_+)$. This final link is the trivial link of two disjoint unknots T_0. To find the polynomial for this we see that if we shrink the upper circle so that it rests inside the other the space between them resembles $\mathfrak{C}_0$ so we can once again use rule 3 to find its value. The expression we obtain is $\nabla(T_+) - \nabla(T_-) = x\nabla(T_0)$ shown in Figure 85.

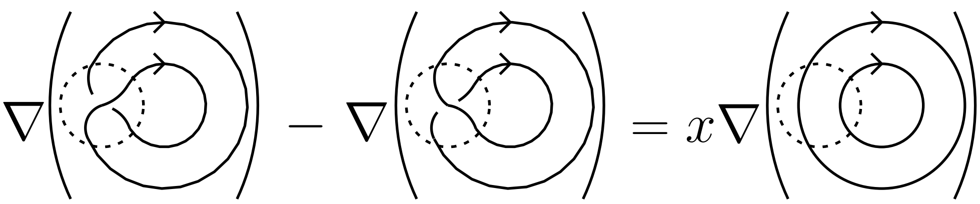

Figure 85. *Diagrams of* $\nabla(T_+) - \nabla(T_-) = x\nabla(T_0)$

Both T_+ and T_- are equivalent to the unknot which means we are left with the expression $1 - 1 = x\nabla(T_0)$. This implies that $T_0 = 0$. We can now substitute this back in to our expression for H_-, giving us $-\nabla(H_-) = x$. Finally we substitute our expression for H_- back in to the expression for the trefoil, $1 - \nabla(K_-) = -x^2$. This means that the Alexander-Conway for the trefoil K is $\nabla(K) = \nabla(K_-) = 1 + x^2$. In this way we can break any given knot or link in to a series of unknots in order to compute its polynomial.

One interesting thing thing to note about $\nabla(L)$ is that if the link contains an unknot completely disjoint from the rest of the knots the polynomial is 0. To see this consider taking some link and performing Ω_1 anywhere on one of its components. Now take the same link and perform Ω_1^{-1} at the same point as before. These two new links are clearly equivalent. In addition to this the crossings created by these moves resemble $\mathfrak{C}_+$ and $\mathfrak{C}_-$ so we can describe these two links as L_+ and L_-. This implies that $x\nabla(L_0) = 0$. The link L_0 is the original link L plus a disjoint unknot. A nicer way to phrase this is to say that $\nabla(L_0) = 0$ implies L_+ and L_- are equivalent. From this phrasing it is clear that $\nabla(L)$ is equal to 0 for all cases where L is a union some link (or knot) with another link (or knot) such that a plane can be drawn between the two components that does not intersect with either one. In this case L_+ and L_- are clearly equivalent.

It is rather a large claim to make so we will now look at the existence of the polynomial in more detail.

Theorem 4.7. *The Alexander-Conway polynomial exists and is unique.*

Proof. We begin by proving the uniqueness of ∇. Let us assume that there exists two mappings ∇^* and ∇' from the set of oriented links to the ring of Laurent polynomials that satisfy the three rules from before. If these two functions are the same then their difference say ∇ will be zero. To prove that this difference is zero for all links we form an inductive argument on the number of crossings N present in the link L.

Let us first consider the base step where $N = 0$. By definition both ∇^* and ∇' map this to 1 and so ∇ is trivially 0. Let us now assume that the claim holds for a link with N crossings and consider the case when we have $N+1$ crossings. Choose some crossing in this link and consider the relation $\nabla(L_+) - \nabla(L_-) = x\nabla(L_0)$. By the inductive hypothesis $\nabla(L_0) = 0$ as L_0 contains N crossings. With this information we see that $\nabla(L_+) = \nabla(L_-)$. As this crossing was chosen arbitrarily this is the case for any crossing chosen within the link. Thus we can transform any $(-)$ crossing into a $(+)$ one without changing the value of $\nabla(L)$ or vice versa. Notice that with this we can transform any link so that it is of the form shown in Figure 86. In this diagram we have two parts of a link connected by two twisted strands. The diagram obtained need not look exactly like the one in Figure 86, it is possible that L' and L'' overlap, the diagram is merely to indicate that these two sections are connected by two strands only. As ∇^* and ∇' are isotopy invariant we can 'untwist' these two connecting strands. We now have a link of N crossings obtained from L by exchanging crossing types which we noted had no affect on the value of ∇.

Another way to show that this is true in a link would be to notice that exchanging crossings as before allows us to 'free' the components of the link so that they become completely disjoint from the rest of the link. As we noted before if we have two disjoint components $\nabla = 0$. Thus by induction $\nabla(L) = 0$ for all links and knots and so the Alexander-Conway polynomial is unique.

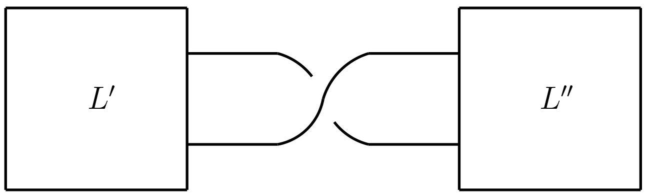

Figure 86. *Transformation of L by exchanging crossing types*

Now we will tackle the first claim of the lemma. Recall the example Markov function $\hat{f}(\widehat{\beta}) = f_n(\beta)$ from the previous section. We claim that this function corresponds exactly to ∇. We already know that $\hat{f}$ is invariant under the equivalence of links so we just need to show that it also meets the other two conditions of the Alexander-Conway polynomial. If $\widehat{\beta}$ is the unknot then β must be equivalent to the trivial braid on one strand and so $\hat{f}(\widehat{\beta}) = 1$ by definition. All that is left to do is verify that the function also satisfies the

Alexander-Conway skein relation. To do this we will need to construct three braids whose closure depicts the Conway triple so that we may calculate f_n for each. We see that the braids $\alpha\sigma_i\beta$ and $\alpha\sigma_i^{-1}\beta$ depict $\mathfrak{C}_-$ and $\mathfrak{C}_+$ respectively. To construct $\mathfrak{C}_0$ we simply remove the σ component leaving us with $\alpha\beta$. So we must show that,

$$f_n(\alpha\sigma_i^{-1}\beta) - f_n(\alpha\sigma_i\beta) = (s - s^{-1})f_n(\alpha\beta).$$

Fortunately we can simplify this expression considerably. The first thing to notice is that σ_i is a conjugate of σ_1. To see this imagine taking the trivial braid in B_{m+n} and performing $\sigma^+_{m,n}$ on it. Notice that now the $(m+1)$th strand of the of the braid is now in position 1 and so we can now perform σ_1 on the braid and then perform $\sigma^-_{n,m} = (\sigma^+_{m,n})^{-1}$. In this way all strands start and end in the same position and the m strands that were moved are not entwined with one another or the other n strands so the braid is equivalent to σ_{m+1} which means σ_{m+1} is a conjugate of σ_1. An example of this is shown in Figure 87.

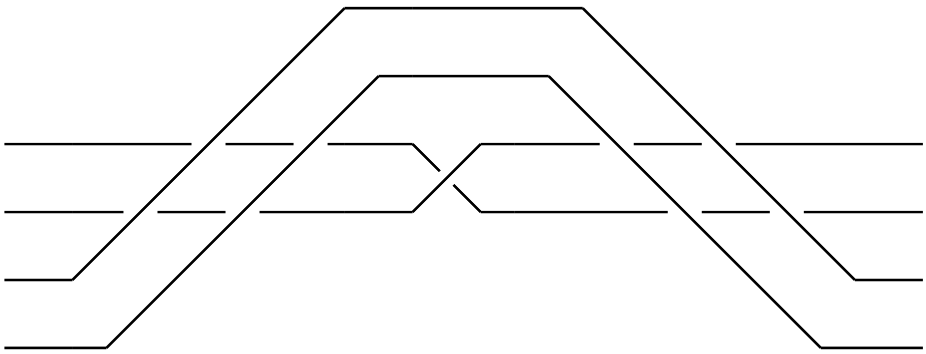

Figure 87. *Example showing σ_3 is a conjugate of σ_1*

With this fact it is enough for us to simply consider σ_1. We can make a second simplification to our formula by conjugating the braids on the left by α. This brings α to be on the right of β so we can consider them to be one. Making these simplifications we are left with the expression,

$$f_n(\sigma_1^{-1}\beta) - f_n(\sigma_1\beta) = (s - s^{-1})f_n(\beta)$$

Expanding this more we obtain the following:

$$\begin{aligned}&(-1)^{n+1}\frac{s^{-\langle\beta\rangle+1}(s-s^{1})}{s^{n}-s^{-n}}g(det(\overline{Bu}_n(\sigma_1^{-1}\beta)-I_{n-1}))\dots\\-&(-1)^{n+1}\frac{s^{-\langle\beta\rangle-1}(s-s^{-1})}{s^{n}-s^{-1}}g(det(\overline{Bu}_n(\sigma_1\beta)-I_{n-1}))\\=&(s-s^{-1})(-1)^{n+1}\frac{s^{-\langle\beta\rangle}(s-s^{-1})}{s^{n}-s^{-n}}g(det(\overline{Bu}_n(\beta)-I_{n-1})).\end{aligned}$$

Let us set the following expressions, $D_\pm = det(\overline{Bu}_n(\sigma_1^{\mp}\beta)-I_{n-1})$, and $D_0 = det(\overline{Bu}_n(\beta)-I_{n-1})$. It is clear that we can simplify the expression above. After we simplify the expression we will multiply both sides by s. We then re-write the expression in terms of t. This gives us,

$$\begin{aligned}&sg(D_+)-s^{-1}g(D_-)=(s-s^{-1})g(D_0)\\\rightarrow&s^2g(D_+)-g(D_-)=(s^2-1)g(D_0)\\\rightarrow&tD_+-D_-=(t-1)D_0.\end{aligned}$$

Let us now consider the terms D_+, D_- and D_0. We will take the $\overline{Bu}_n(\beta)$ and expand it as follows:

$$\overline{Bu}_n(\beta)=\begin{pmatrix}a&b&x\\c&d&y\\p&q&M\end{pmatrix}.$$

In this matrix the terms a, b, c, and d are Laurent polynomials. The entries p and q are column matrices of height $n-3$ and the entries x and y are row matrices of length $n-3$. Finally M is a square matrix of size $n-3$. Using this matrix we see that,

$$D_0=det(\overline{Bu}_n(\beta)-I_{n-1})=det\begin{pmatrix}a-1&b&x\\c&d-1&y\\p&q&M-I_{n-3}\end{pmatrix}=det(A_0).$$

For D_+ and D_- we will need to calculate $\overline{Bu}_n$ for the braids $\sigma_1^{\pm1}\beta$. We obtain the following,

$$\overline{Bu}_n(\sigma_1\beta) = \begin{pmatrix} -t & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix} \begin{pmatrix} a & b & x \\ c & d & y \\ p & q & M \end{pmatrix} = \begin{pmatrix} -ta & -tb & -tx \\ a+c & b+d & x+y \\ p & q & M \end{pmatrix}$$

$$\overline{Bu}_n(\sigma_1^{-1}\beta) = \begin{pmatrix} -t^{-1} & 0 & 0 \\ t^{-1} & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix} \begin{pmatrix} a & b & x \\ c & d & y \\ p & q & M \end{pmatrix} = \begin{pmatrix} -t^{-1}a & -t^{-1}b & -t^{-1}x \\ t^{-1}a+c & t^{-1}b+d & t^{-1}x+y \\ p & q & M-I_{n-3} \end{pmatrix}$$

Let us now consider D_-. We see that,

$$D_- = det \begin{pmatrix} -ta-1 & -tb & -tx \\ a+c & b+d-1 & x+y \\ p & q & M-I_{n-3} \end{pmatrix}$$

Let us construct a new matrix A_- by multiplying the first row of this matrix by $-t^{-1}$. We will then take the new row one and subtract it from row two. Notice that the determinant of this new matrix differs by a factor of $-t^{-1}$ from that of the original so we will add a factor of $-t$ outside of the matrix. This leaves us with the expression,

$$D_- = -t\ det \begin{pmatrix} a+t^{-1} & b & x \\ c-t^{-1} & d-1 & y \\ p & q & M-I_{n-3} \end{pmatrix} = -t\ det(A_-).$$

Finally we will consider D_+. We have,

$$D_+ = det \begin{pmatrix} -t^{-1}a-1 & -t^{-1}b & -t^{-1}x \\ t^{-1}a+c & t^{-1}b+d-1 & t^{-1}x+y \\ p & q & M-I_{n-3} \end{pmatrix}.$$

Again we will construct a new matrix A_+. To do this we will add the first row of the matrix above to the second. We will then multiply the first row by $-t$. Notice again that multiplying the first row changes the determinant of the matrix by a factor of $-t$ so we add a $-t^{-1}$ term in front of the determinant.

This leaves us with the expression,

$$D_+ = -t^{-1}\ det \begin{pmatrix} a+t & b & x \\ c-1 & d-1 & y \\ p & q & M - I_{n-3} \end{pmatrix} = -t^{-1}\ det(A_+).$$

Now that we have constructed these new matrices notice that the second and third columns of each is identical. This means that we can ignore these columns when calculating the result as the first, second and third entries in the first column of each matrix will be multiplied by the same factor when calculating the determinant. We are now in a position to verify if the statement is true. We have,

$$\begin{aligned} &tD_+ - D_- = (t-1)D_0 \\ \rightarrow &t(-t^{-1})det(A_+) - (-t)det(A_-) = (t-1)det(A_0) \\ \rightarrow &-\begin{pmatrix} a+t \\ c-1 \\ p \end{pmatrix} + t\begin{pmatrix} a+t^{-1} \\ c-t^{-1} \\ p \end{pmatrix} = (t-1)\begin{pmatrix} a-1 \\ c \\ p \end{pmatrix} \end{aligned}$$

We see this statement is true meaning the skein relation holds for f_n and so it corresponds to the Alexander-Conway polynomial. □

References

[1] Joan S. Birman and Tara E. Brendle. “Braids: a survey”. In: *Handbook of knot theory* (2005), pp. 19–103.

[2] Christian Kassel and Vladimir Turaev. *Braid groups.* Vol. 247. Graduate Texts in Mathematics. With the graphical assistance of Olivier Dodane. Springer, New York, 2008.

[3] Gwénaël Massuyeau. *A Short Introduction to the Alexander Polynomial.* 2008. URL: `http://massuyea.perso.math.cnrs.fr/notes/Alex.pdf`.

[4] Kunio Murasugi. *Knot Theory and its Applications*. Translated by Bohdan Kurpita. Birkhäuser, Boston, 1996.

[5] Justin Roberts. *Knot Knotes*. 2015. URL: http://www.math.ucsd.edu/~justin/Roberts-Knotes-Jan2015.pdf.

[6] Dale Rolfsen. *KNOTS AND LINKS*. American Mathematical Society, 2003.

[7] Seifert Surfaces produced with SeifertView. Jarke J. van Wijk, Technische Universiteit Eindhoven. URL: https://www.win.tue.nl/~vanwijk/seifertview/seifertview.exe.

[8] Eric W. Weisstein. *Braid Group*. From MathWorld–A Wolfram Web Resource. URL: https://mathworld.wolfram.com/BraidGroup.html.