Orbits of a Regular Polygon

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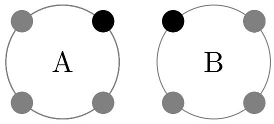
Abstract

In this report we will look at ways to enumerate the number of distinguishable colourings we have when colouring the vertices of a regular polygon. We will look at two methods of obtaining this number. These are Burnside's lemma and Pólya's enumeration theorem.

Contents

1 Introduction

In this report we will be taking a look at methods that may be used in order to solve various counting problems. In particular we will be looking at the colourings of the beads on a necklace. This is different from less complicated counting problems where beads are placed in a line and then coloured. In these types of problems the beads are arranged in a circular shape. Say we have a necklace made of six beads to be coloured with k colours. Initially we might simply think that the number of colourings would be k^6 as there are k colour choices for each bead, however this is not the case. To see why let us consider the following colourings:



if we take A and rotate it anti-clockwise about its centre by 45 degrees we will obtain the colouring B. We can see that these colourings are essentially the same, and so we must find some other way to try and calculate the number of different colourings that are possible given that colourings like these are the same.

Another variation of this problem would be to consider not just the rotation of the necklace but also both sides of the necklace. For example if we look at the previous diagram, colouring B can be obtained by turning over necklace A.

We can relate this problem to the colouring of the n vertices of a regular polygon. This gives us a much easier way to look at the problem. We will look at two different methods that will allow us to obtain a solution for colouring the vertices of a polygon. These are Burnside's lemma and Pólya's enumeration theorem.

First of all we will look at some background knowledge that will be very useful when trying to understand these methods and then we will look at some example problems that were the starting point for this report.

1.1 Background Knowledge

Before looking at the initial problems we will set out some basic definitions. These definitions will relate to the colouring of the vertices of a regular polygon.

Definition 1.1

A **binary operation** takes two elements from the same set and uses them to obtain an element also from this set.

Definition 1.2

A **group** consists of a set of elements say, G, and a binary operation which we will denote $*$. The group must satisfy four conditions:

1. The set G must be closed under $*$. This means if we have $g_1, g_2 \in G$ then $g_1 * g_2 \in G$ as well.
2. The operation must be associative in G. This means that $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$ for all $g_1, g_2, g_3 \in G$.
3. Thirdly there must be an identity element say $I \in G$ such that $I * g = g * I = g$ for all $g \in G$.
4. Finally for each $g \in G$ where g is non-zero there must be an inverse element say g^{-1}, such that $g * g^{-1} = g^{-1} * g = I$.

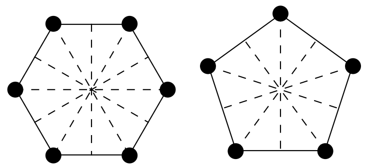
An example of a group would be the set of integers with addition. (Weisstein, E. n.d.)

Definition 1.3

A **transformation** is a function that can translate, rotate, scale or reflect an object with respect to some point or line. For the vertices of a polygon a transformation will map each vertex of a polygon to a unique position. There are two types of transformation that we will look at in this report. These are reflections and rotations. We will denote reflections with M and rotations R.

Definition 1.4

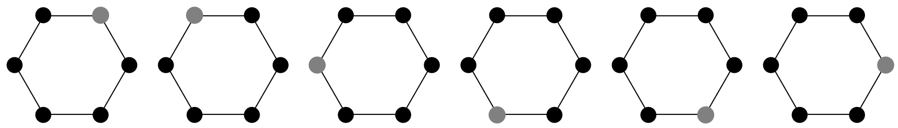
Symmetries are transformations which map a shape to itself. So in the image, vertices will be mapped to one of the locations of the vertices in the pre-image. For example if we reflect the hexagon or pentagon below across any of the dashed lines the image will still look like a regular hexagon or pentagon.



If we have a polygon with n vertices, then in general we see that regular polygons all have $2n$ different symmetries. Of these symmetries n of them are rotational symmetries. The remaining n vary depending on whether n is odd or even. If n is even, there will be $\frac{n}{2}$ reflectional symmetries that pass through the mid points of opposite edges and $\frac{n}{2}$ reflectional symmetries that pass through opposite vertices. The reflectional symmetries of a hexagon are shown above. If n is odd then we will have n reflectional symmetries that pass through a vertex and the mid point of the opposite edge. An example of the reflectional symmetries of a pentagon is shown above as well.

Definition 1.5

An **orbit** is a subset of X, where X is the set of colourings on the vertices of a polygon. If we take the set of symmetries of this polygon G, acting on the set X then for any chosen element $x \in X$ the members of the subset are $\{y \in X : y = g(x),\ \forall g \in G\}$. The orbit contains all the colourings you can achieve by rotating and reflecting a chosen colouring. We will denote the orbit of an element of X by Gx. Below is an example of an orbit on the colouring of the vertices of a hexagon: (Biggs, 2002)



Definition 1.6

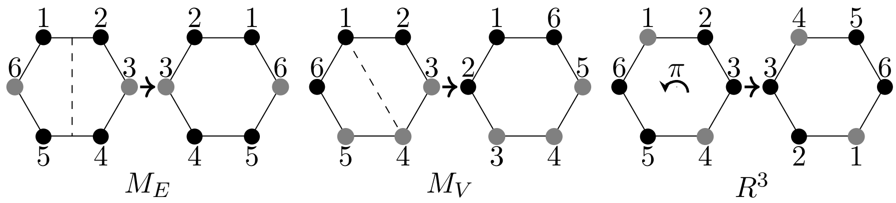
Two colourings of the vertices of a polygon are said to be **indistinguishable** if you can obtain one of the colourings by rotating and reflecting the other. This definition of distinguishable colourings only considers the colours of vertices and not the positions of the vertices themselves. So if we have a polygon with labeled vertices, although we see that these vertex labels are moved around after each transformation, the position of these vertices does not determine whether the colouring is indistinguishable. Indistinguishable colourings all lie in the same orbit.

Definition 1.7

Two colourings are said to be **distinguishable** from each other if you can not obtain one of the colourings from the other through any combination of reflections and rotations. Similarly to before, the position of the vertices of a polygon does not determine whether the colouring is distinguishable, only the colour of the vertices matter. Distinguishable colourings of the vertices of a polygon do not lie in the same orbit as one another.

Definition 1.8

A colouring is said to be **fixed** under a symmetry, say g, if the image is indistinguishable from the initial colouring after going through transformation g. If we consider a group of symmetries G acting on some set of colourings, X, then fixed colourings are the members of the set $\{x \in X : g(x) = x\}$. These are the members of x that will be mapped to themselves. We will denote the set of fixed colourings under g as $F(g)$. Below are some examples of fixed colourings. We can see that although the vertices have changed position, the colourings are visually indistinguishable (Biggs, 2002). The transformation below denoted by M_E is a reflection where the line of reflection passes through the midpoint of opposite edges of the polygon, M_V represents a reflection where the line of reflection passes through opposite vertices. The rotation R is a rotation by $\frac{2\pi}{n}$ where n is the number of vertices in the polygon.



Definition 1.9

A **stabiliser** for some colouring is a transformation that will fix it. This means for some group of symmetries, G acting on the set of colourings, X, for some chosen $x \in X$, the stabilisers will be given by the set $\{g \in G : g(x) = x\}$. We will denote the set of stabilisers G_x. (Biggs, 2002)

Definition 1.10

A **necklace** is a circular arrangement of objects whose colourings are said to be distinguishable under rotation only. (Weisstein, E. n.d.)

Definition 1.11

A **bracelet** is similar to a necklace but its colourings are said to be distinguishable under both rotation and reflection.

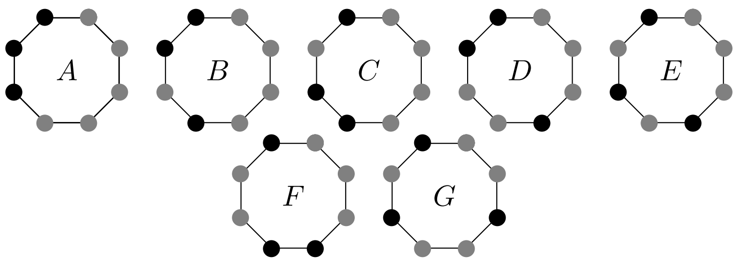
1.2 Initial Problems

Next we will look at the initial problems set before writing this report. The following solutions were found without the use of the methods that we will look at later in the report.

1.2.1 Problem One

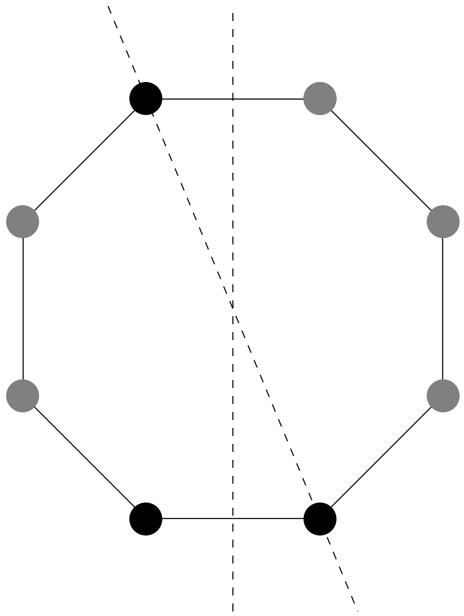
Determine the number of distinguishable designs for a bracelet comprised of five white and three black beads.

First we will attempt to solve this problem by inspection. We will consider the problem of colouring the vertices of an octagon with three black vertices and five white vertices. We will first check how many necklaces that exist with vertices coloured in this way. As there are eight beads in total the number of ways of arranging these is 8!, however there are five white beads and three black beads. Beads of the same colour are indistinguishable from each other and so the order of these beads within the necklace is not important and must be removed from the total number of arrangements giving us $\frac{8!}{5!3!}$. Next, to remove the colourings indistinguishable under rotation we can divide by eight as there are eight different places we can start the necklace from. So the number of necklaces will be $\frac{8!}{5!3!} \cdot \frac{1}{8} = 7$. These colourings are shown below.



These colourings were found by starting with the three black vertices together and then placing the white vertices in between them. These are the seven possible necklaces as they are all distinguishable under rotation. By inspection of these colourings you can see colourings A, E and G are distinguishable. We can also see that B is a reflection of C and that D is a reflection of F. So in total there are five distinguishable bracelets that can be made using this composition of beads. Here we notice that some colourings could be grouped together such as B and C. These were the colourings that were asymmetrical.

In this second attempt we will try to find a way to remove the number of symmetrical colourings from the number of necklaces as they are not repeated due to reflections so we would only be left with the colourings like B and C that could be paired together. Then we would just need to half the number of these colourings and then add back on the number of symmetrical colourings which would allow us to get the number of bracelets. The previous method would not be very useful for much larger problems.



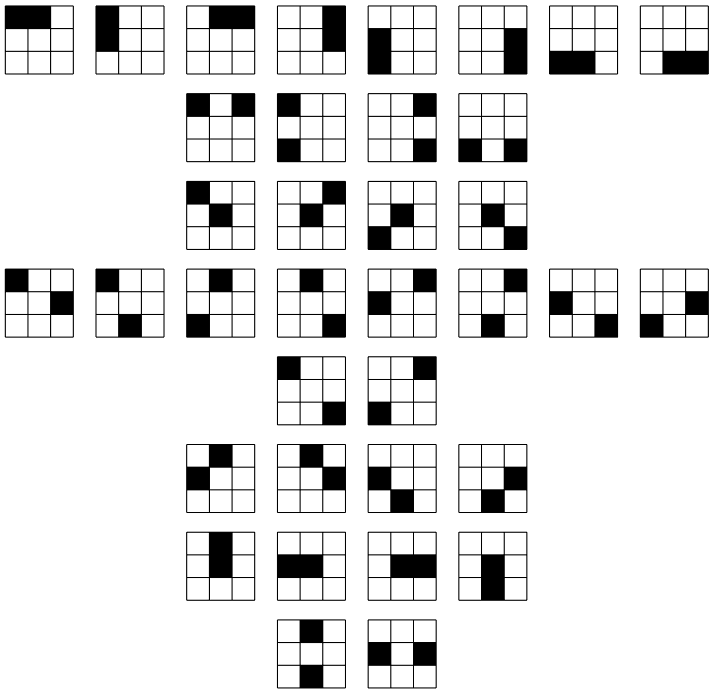
The E and V above describe two different types of reflection that we can have in a regular polygon with an even number of vertices. The E represents a reflection where the line of reflection passes through the midpoints of opposite edges, and V the line that passes through opposite vertices. As the bracelet is made up of odd numbers of black and white beads there can be no arrangements that are symmetrical across the line M_E. For the bracelet to be symmetrical across line M_V, one of the vertices that lies on the line must be coloured white and the other must be coloured black. This leaves six beads left to colour, four white and two black, however these beads must be chosen in pairs in order for the arrangement of the bracelet to be symmetrical so there are $\frac{3!}{2!}$ ways of doing this. We do not need to worry about switching the colours on the vertices that lie on M_V as that is a rotation of the arrangement and they have already been removed from the number of arrangements in the previous attempt. So the number of distinguishable bracelets can be given by the number of necklaces minus the number of symmetrical colourings all over two, after that we just need to add the symmetrical colourings back and we get:

$$\frac{1}{2}\left(\frac{8!}{5!3!8} - 3\right) + 3 = 5.$$

1.2.2 Problem Two

The design for a square table mat is based on a 3x3 grid of nine glass squares. Two of these squares are to be blue; the remainder red. The mat is transparent and can be turned over. Determine the number of different designs.

When trying to solve this problem we can again try by inspection. We find all the possible colourings and try to group the colourings that are indistinguishable together. There are 9! ways to arrange the nine squares but similarly to before some of the panes are the same colour so in total there are $\frac{9!}{7!2!} = 36$ colourings. These are shown below.



From this we can group the indistinguishable colourings together. Above the colourings have been split in to rows where each row contains the colourings that can be grouped together. As there are eight rows there are eight possible arrangements.

1.2.3 Problem Three

Determine the number of distinguishable designs for a bracelet comprised of thirteen white and three black beads.

To find the number of distinguishable bracelets we will take the same approach as in the second attempt of Problem One as this is essentially the same problem but with more white beads. Similarly to Problem One the only symmetrical colourings possible are going to be reflections across the line that goes through two opposite vertices. This leaves us with six white beads and one black bead to colour. This can be done in $\binom{7}{1}$ ways so the number of distinguishable bracelets will be given by the following expression.

$$\frac{1}{2}\left(\frac{16!}{13!3!16} - 7\right) + 7 = 21$$

We can create an expression for this type of problem. This expression is:

$$N = \frac{1}{2}\left(\frac{1}{n}\binom{n}{v_{c_1}} - \binom{\frac{n-2}{2}}{\frac{v_{c_1}-1}{2}}\right) + \binom{\frac{n-2}{2}}{\frac{v_{c_1}-1}{2}},$$

here n is the number of beads to be coloured, v_{c_i} are the number of beads of colour c_i and N just denotes the number of distinguishable colourings. Unfortunately this formula only applies to a small number of cases. To use this formula the problem must fit these conditions: 1) the number of beads n, must be even, 2) only two colours of bead, c_1 and c_2, may be used, 3) the number of beads v_{c_1} and v_{c_2} must both be odd, 4) the number of beads $v_{c_1} \neq v_{c_2}$.

2 Burnside's Lemma

Although the solutions for the problems above were obtained quite easily, such methods could not be applied to much larger or more intricate problems, so we must find another way to solve them. One method of solving these problems is by using Burnside's lemma. Burnside's lemma states that the number of distinguishable patterns can be given by the average of the sum of all fixed colourings under each symmetry acting on the bracelet or necklace. For some group of symmetries, G, acting on a set of colourings X, the number of orbits N, is given by the equation below. Here $F(g)$ is the number of fixed colourings under symmetry g. (Biggs, 2002)

$$N = \frac{1}{|G|} \sum_{g \in G} |F(g)|$$

To see why this is true we need to first understand the Orbit-Stabiliser theorem, which is a key result in combinatorics and is vital to proving Burnside's lemma.

2.1 The Orbit Stabiliser Theorem

The Orbit-Stabiliser theorem says that for some group of symmetries, G, acting on X, the set of colourings on the vertices of a polygon, the number of elements in group G is equal to the number of elements in the orbit of $x \in X$ multiplied by the number of stabilisers for that x. (Biggs, 2002)

$$|G| = |Gx| \times |G_x|$$

Orbit-Stabiliser Theorem Proof

For a group of symmetries, G, acting on set of colourings X, let us first consider the set of (g, y) pairs, A, where:

$$A = \{(g, y) : g(x) = y,\ g \in G,\ x \in X\}.$$

These are the (g, y) pairs given any chosen x such that $g(x)$ is y. Meaning that we can represent these pairs in a grid as shown below.

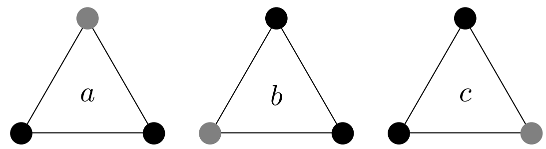
$$\begin{array}{c|ccccccccc}
 & \cdots & \cdots & \cdots & \cdots & y & \cdots & \cdots & \cdots & \cdots \\
\hline
: & & * & & & & & & & \\
: & & * & & & & & & & \\
g & & & & & & * & & & \\
: & & & * & & & & & & \\
: & & & & & & & & & *
\end{array}$$

Along the top of the grid the y's represent all the members of X and the g's are all transformations in G. The asterisks inside the grid indicate that the corresponding pair is a member of A.

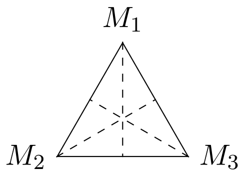
Now let us pick any $x \in X$ and consider the value of the row sums given the asterisks have a value of one. We can see that the row sums will all be equal to one for the selected x, as each transformation of this x can only yield one result. In total there are $|G|$ rows, so the sum of all the rows over all $g \in G$ will be $1 \times |G|$.

Next we will consider the column totals of the table. To make it easier let us first consider the sum of the $g(x) = x$ column. If this is the case, by the definition of a stabiliser the total of this column is just the number of stabilisers that x has, this is $|G_x|$. Next if $g(x) = y$ then this implies y is in the orbit of x meaning y will have the same number of stabilisers as x. This means that every column where y is a member of the orbit of x will total to $|G_x|$. As there are $|Gx|$ elements in the orbit of x, the total contribution of these terms to the sum of the columns will be $|Gx| \times |G_x|$. For the rest of the columns the corresponding y does not belong to the orbit of x and so the column will sum to zero as there is no permutation of x that could result in such a y and so the total over all columns will be $|Gx| \times |G_x| + 0$. As the sum over all columns will be the same as the sum over all rows we can equate these results to get $|G| = |Gx| \times |G_x|$ (Biggs, 2002).

Let us look at an example of the theorem in order to better understand what is going on. Say we wish to colour the vertices of a triangle so that there are two black vertices and one that is white. As we only have one vertex that is white and the rest are black then we can see that the number of colourings will just be the number of places which the white vertex can be placed. So there are three colourings in this example which are shown below.



Now lets consider the set $A = \{(g, y) : g(x) = y,\ g \in G,\ x \in X\}$. In our example the possible y's will be a, b and c. The members of G will be anti-clockwise rotations by $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, and 2π radians and three reflection which are shown below.



If we now choose $x = a$ we can construct the table that we used to prove the theorem. The table for this example is shown below.

	a	b	c
M_1	*	-	-
M_2	-	-	*
M_3	-	*	-
$\frac{2\pi}{3}$	-	*	-
$\frac{4\pi}{3}$	-	-	*
2π	*	-	-

As expected the sum of each row is one as each transformation can only yield one result, so the sum over all rows is $1 \times 6 = 6$. Next if we consider our chosen x we can see that it has two stabilisers. These are a rotation by 2π radians and the reflection M_1. So we would expect every column where $y \in Gx$ to sum to two which is the case. So the sum over all columns is $3 \times 2 = 6$. Our two sums are equal.

2.2 Proof of Burnside's Lemma

Now we can use the previous result to obtain Burnside's lemma. We will use a very similar method as we did for the orbit-stabiliser theorem. Let G be a group of symmetries acting on a set X. Let us first consider the set of (g, x) pairs, B, where:

$$B = \{(g, x) : g(x) = x,\ g \in G,\ x \in X\}$$

These are the pairs such that $g(x) = x$. Meaning that every x will be in a pair with each of its stabilisers. We can represent this set of (g, x) pairs as a grid in the same way as before, shown below.

$$
\begin{array}{c|ccccccccc}
 & \dots & \dots & \dots & \dots & x & \dots & \dots & \dots & \dots \\
\hline
\vdots & & * & & & & & & & * \\
\vdots & & & & * & & & & & \\
g & & & * & & & & * & & \\
\vdots & & & & & & & & & * \\
\vdots & * & & & * & & & & & *
\end{array}
$$

Along the x line are all members of set X and down the g line are all members of G. Here the asterisks indicate that the pair is a member of set B. Now let us consider the row totals of the grid where each asterisk again has a value of one. As $g(x) = x$ then the row totals must equal the number of fixed colourings that exist under g denoted $|F(g)|$. We can then take the sum of the row totals to give us:

$$\sum_{g \in G} |F(g)|.$$

The same approach can now be taken for the column totals. The column totals are the $g \in G$ that fix x so these are the stabilisers of x and so the total of a column will be $|G_x|$. From this we can see that:

$$\sum_{g \in G} |F(G)| = \sum_{x \in X} |G_x|.$$

Let us choose some other element, say $z \in X$, then if x is in the orbit of z then the number of stabilisers of both x and z will be the same. This means that for this particular orbit in X it will contribute $|Gx| \times |G_x|$ to the sum. So lets say there are N orbits, this means that there are N different z's we can choose in the same way as before and so we will have N lots of $|Gx| \times |G_x|$. From the orbit-stabiliser theorem we know that $|Gx| \times |G_x| = |G|$ so we may now express the equation above as:

$$\sum_{g \in G} |F(g)| = N|G|,$$

here N represents the number of orbits in X. This can now be rearranged to give us the desired result (Biggs, 2002):

$$N = \frac{1}{|G|} \sum_{g \in G} |F(g)|$$

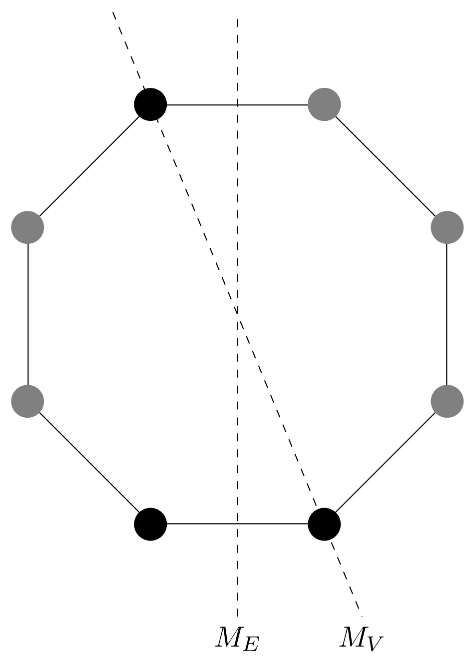
2.3 Examples Using Burnside's Lemma

We will now use the initial problems as examples to demonstrate how Burnside's lemma may be used.

Problem One

Determine the number of distinguishable designs for a bracelet comprised of five white and three black beads.

Solution/ First we will start by finding how many symmetries there are to consider. We can see that an octagon has sixteen symmetries. These are made up of eight reflections and eight rotations. There are two types of reflections for an octagon as shown below. Type M_E is a reflection over a line that passes through the mid-point of opposite edges and type M_V is a reflection over a line that passes through two opposite vertices, there are four of both of these. We will refer to type M_E as an edge to edge reflection and type M_V as a vertex to vertex reflection from now on. The eight rotations are rotations about the centre of the polygon by multiples of $\frac{2\pi}{8}$ radians and we will denote the rotations R^s where this is a rotation by $s\frac{2\pi}{8}$.



For this particular example the set of symmetries, G, is shown below.

$$G = \{I, M_{E_i}, M_{V_i}, R^s\}, \text{ where } i = 1, 2, 3, 4 \text{ and } s = 1, 2, ...7$$

We can now examine each transformation individually, finding out the number of fixed patterns each will yield. First if we look at the identity (rotation by 2π) the number of fixed patterns will just be the number of distinguishable necklaces that there are. The number of fixed colourings under the identity is $\frac{8!}{5!3!}$. Next we can look at the edge to edge reflections. As there is an odd number of black beads and an odd number of white beads there is no way to arrange them symmetrically across the line of reflection as the beads must be chosen in twos. For vertex to vertex reflections we must have one black bead and one white bead on the line of reflection as again, beads must be chosen in twos and placing a black and white bead on these vertices will leave them unchanged after the transformation and will also leave an even number of both black and white beads. So there will be $\binom{3}{1}$ (ways of arranging three beads) $\times$ $\binom{2}{1}$ (ways of choosing the vertex colours) fixed patterns. Finally if we look at the remaining rotations, as sixteen is not divisible by three there will be no fixed patterns under rotation. Now we can use Burnside's Lemma to obtain the answer:

$$N = \frac{1}{2n}\left[\left(|F(I)|\right) + \left(\frac{n}{2}|F(M_E)|\right) + \left(\frac{n}{2}|F(M_V)|\right) + \left(|F(R)| + |F(R^2)| + ... + |F(R^7)|\right)\right]$$

$$= \frac{1}{16}\left[\left(\frac{8!}{5!3!}\right) + (4 \cdot 0) + \left(4 \cdot \binom{3}{1}\binom{2}{1}\right) + (7 \cdot 0)\right]$$

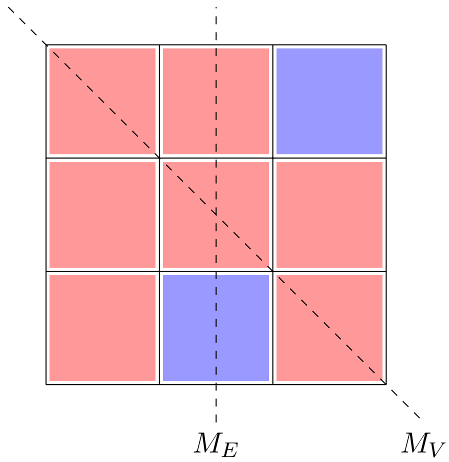
$$= 5.$$

This is the same result as we obtained before.

Problem Two

The design for a square table mat is based on a 3x3 grid of nine glass squares. Two of these squares are to be blue, the remainder red. The mat is transparent and can be turned over. Determine the number of different designs.

Solution/ Again we will start by finding the number of symmetries to consider. As the tiles are arranged in a square there will be eight symmetries. Four reflections and four rotations.



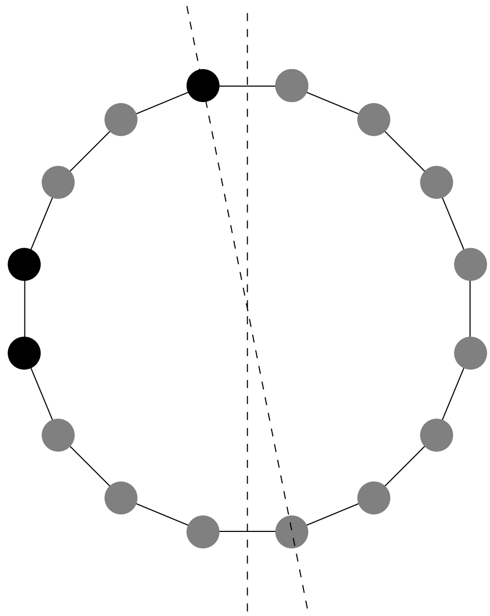
We can now look at each of these transformations individually similarly to Question One. Looking at the identity there will be $\frac{9!}{7!2!}$ fixed patterns. For the edge to edge reflections, the fixed patterns will occur when the two blue squares are on symmetrically opposite sides of the line of reflection or when they both lie on the line. So there will be six fixed patterns. The same will apply for vertex to vertex reflections and so there will be six fixed patterns again. The only fixed patterns after rotation are after a rotation of π radians. These patterns will be when the blue tiles lie on opposite corners or the center of opposite edges, there are four of these. So the number of distinguishable tilings will be:

$$\frac{1}{8}\left(\left(\frac{9!}{7!2!}\right) + (2 \cdot 6) + (2 \cdot 6) + 4\right) = 8$$

Problem Three

Determine the number of distinguishable designs for a bracelet comprised of thirteen white and three black beads.

Solution/ Again we will find the number of symmetries we need to consider. A hexadecagon has 32 symmetries, 16 reflections and 16 rotations.



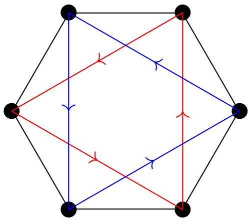
First if we look at the identity there will be $\frac{16!}{13!3!}$ fixed patterns. Next looking at the edge to edge reflections there will be no fixed patterns as there are an odd number of black and white beads and they must be chosen in pairs for a fixed pattern to exist across this line. For the vertex to vertex reflections one vertex on the line must be coloured black and the other must be coloured white and then the remaining colours must be picked in pairs so similarly to the first question there are $\binom{2}{1}\binom{7}{1}$ fixed patterns. There will be no fixed patterns after any of the remaining rotations as 16 is not divisible by three. So the number of distinguishable bracelets will be given by:

$$\frac{1}{32}\left(\left(\frac{16!}{13!3!}\right) + (8 \cdot 0) + (8 \cdot 2 \cdot 7) + (15 \cdot 0)\right) = 21$$

2.4 Colouring the Vertices of a Regular Polygon With k Colours

In this section we will find an expression to find the number of distinguishable colourings on the vertices of a polygon that has n vertices with k different colours. Recall that a polygon with n vertices will always have $2n$ symmetries no matter how many vertices there are. Of these symmetries n will always be rotational symmetries. The remaining n symmetries will vary whether n is even or odd. If even there will be $\frac{n}{2}$ edge to edge reflections and $\frac{n}{2}$ vertex to vertex reflections. If odd then there will be n instances of a reflection where the line of reflection will pass through a vertex and the midpoint of the edge opposite it. We will call this an edge to vertex reflection. To find a formula for the number of distinguishable colourings we need to find expressions that will give us the number of fixed patterns under each symmetry. First we will start by looking at the rotational symmetries of the polygon.

The n rotations of a polygon will each be a multiple of $s(\frac{2\pi}{n})$, where $1 \leqslant s \leqslant n$. If s is a divisor of n then the vertices will be split into s lots of n/s vertices. In order to obtain a fixed colouring all vertices in a lot must be assigned the same colour. For example if we take a hexagon and rotate it by $2(\frac{2\pi}{6})$ the vertices will be grouped in to two lots of three. In the diagram below the coloured lines show where each vertex is mapped to after the rotation.



If we choose one of the vertices in the hexagon and follow the line, eventually we will reach the vertex we started at. This means that every vertex in this lot must be the same colour for us to have a fixed pattern under this symmetry.

Any multiples of s that are not divisors of n will also divide n in to s lots of $\frac{n}{s}$. For example if we take the hexagon, $2|6$ but $2 \cdot 2 \not| 6$ yet it still splits the vertices in to two lots of three. Finally if s is coprime to n then every vertex must be the same colour for us to get a fixed pattern.

In order to find an expression that shows this we will use Euler's totient function. Euler's totient function, written $\varphi(n)$, will give us the number of positive integers that are smaller than n and also coprime to it. Two numbers are said to be coprime if the only positive integer that can divide them both is one (doc.ic.ac.uk, n.d.). For example $\varphi(6) = 2$ as the only numbers coprime to it are five and one. Numbers two and three both divide six so they are not coprime to it and the greatest common divisor (gcd) of four and six is two, so four is also not coprime to six.

We can use Euler's totient function to help us find the number of fixed patterns under the rotational symmetries. We can see for each $d \in D(n)$ where $D(n)$ is the set of divisors of n, there are $\varphi(d)$ terms of the power n/d in the number of fixed patterns under rotation (Weisstein, E. n.d.).

Proof/ In order to prove this let us consider the values say, g, that divide n in turn. For any particular g we see that $n = ag$. As we previously mentioned if we have a rotation R^s where s is a divisor of n then the vertices get split into s lots of $\frac{n}{2}$ vertices in which every vertex in a lot must be assigned the same colour in order to obtain a fixed colouring. We also mentioned that any multiple of s that was not a divisor of n as well would have the same effect. This property of these s values may be represented in the following way:

$$gcd(n, s) = g.$$

So to obtain our result we want to find the number of these particular s. As s is a multiple of g it may be expressed as $s = bg$. Now we already know that:

$$gcd(s, g) = gcd(n, g) = g,$$

so for this to be true a and b must be coprime or else $gcd(s, g)$ and $gcd(n, g)$ would be greater than g as we would be able to take another common factor other than one out of a and b. So the number of s will simply be given by $\varphi(a)$. If we re-arrange $n = ag$ we see that $\varphi(a) = \varphi\left(\frac{n}{g}\right)$. If we take $\frac{n}{g} = d$ we will obtain our result. (Royle, G. 2004)

Knowing this we can now form an expression that will give the number of fixed patterns under rotation. This is:

$$\frac{1}{2n} \sum_{d \in D(n)} \varphi(d) k^{\frac{n}{d}}.$$

This expression will apply to polygons with an odd or even number of vertices. Next we need to find the number of fixed patterns under each reflection.

Polygons with an even number of vertices have two different types of reflection that need to be considered. These are the edge to edge reflections and the vertex to vertex reflections. Finding the number of fixed patterns each of these reflections contributes to the formula is quite simple. For the edge-edge reflection, the vertices are evenly split in two so for every one colour chosen two vertices must be given this colour. So the number of fixed colourings under an edge to edge reflection will be $k^{\frac{n}{2}}$. For the vertex-vertex reflection, the two vertices on the line can be coloured with no restriction as they get mapped to the same location after the transformation. This leaves $n-2$ vertices left to colour in the same way as the edge-edge type. So these reflections contribute $k^{\frac{n-2}{2}+2}$ fixed patterns, the number of patterns can be rearranged to $k^{\frac{n}{2}+1}$. There are $\frac{n}{2}$ instances of each type of reflection so in total the number of fixed patterns is:

$$\frac{n}{2}k^{\frac{n}{2}} + \frac{n}{2}k^{\frac{n}{2}+1} = \frac{n}{2}(k+1)k^{\frac{n}{2}}.$$

Next we will look at polygons with an odd number of vertices. These are slightly less complicated than those with an even number of vertices. Unlike before, for an odd number of vertices we only need to consider an edge to vertex reflection. Similarly to before the vertex on the line can be coloured without restriction and then the remaining $n-1$ vertices must be coloured in pairs, so this reflection will contribute $k^{\frac{n-1}{2}+1}$ fixed patterns. There are n instances of this reflection so the total number of fixed patterns is:

$$nk^{\frac{n-1}{2}+1}.$$

We can now collate this information to give us a general expression for the number of distinguishable bracelets possible of length n, coloured with k colours. This expression is as follows:

$$N = \frac{1}{2n}\begin{cases} \left(\sum_{d\in D(n)} \varphi(d)k^{\frac{n}{d}} + \frac{n}{2}(k+1)k^{\frac{n}{2}}\right), & \text{if } n \text{ even} \\ \\ \left(\sum_{d\in D(n)} \varphi(d)k^{\frac{n}{d}} + nk^{\frac{n-1}{2}+1}\right), & \text{if } n \text{ odd.} \end{cases}$$

There is one special case that occurs when n is prime which allows us to create an expression for the number of colourings without the use of Euler's totient function. First let us think about the number of colourings contributed by the vertex-edge reflections. This is exactly the same as above so we get $nk^{\frac{n-1}{2}+1}$ colourings. The identity also provides the same number k^n. Finally for the remaining rotations, as there is no $d \in D(n)$ other than one, every remaining rotation will contribute k fixed patterns. So to find the number of orbits for a regular polygon with a prime number of vertices we have the expression:

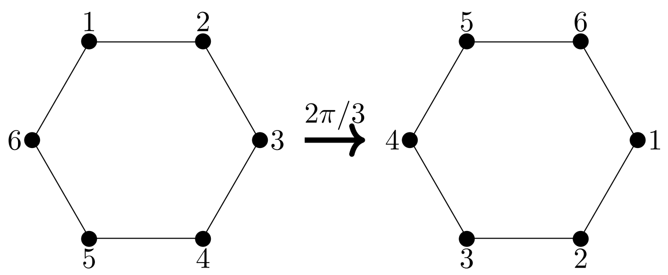
$$N = \frac{1}{2n}\left(k^n + nk^{\frac{n-1}{2}+1} + (n-1)k\right).$$

3 Pólya Enumeration Theorem

Pólya's enumeration theorem is a great improvement on Burnside's lemma. It is a generating function that allows us to calculate the number of orbits made up of specified numbers of coloured beads. Before looking at this we must know about the cycle index.

3.1 Cycle Index

We can obtain a cycle when looking at a transformation acting on the vertices of a polygon. To do this we can pick one of the vertices of the polygon and see what vertex it gets mapped to after the transformation. We can then do the same thing for this vertex and so on until we reach the initial vertex again. This will be one of the cycles given by the transformation. For example let us look at a clockwise rotation by $\frac{2\pi}{3}$ in a hexagon:

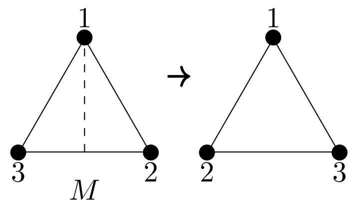


We can see that vertex 1 gets mapped to vertex 3, 3 to 5 and 5 to 1. This forms a cycle of length three. The same is also true for vertices 2, 4, and 6 in this case. This gives us useful information when trying to colour the vertices of the polygon. For example if you wanted to find the number of fixed colourings under a rotation of $\frac{2\pi}{3}$ of a hexagon that existed, we now know that all vertices in the three cycle that we found must be of the same colour. We can also see that a fixed colouring can contain at most two different colours at one time in this case.

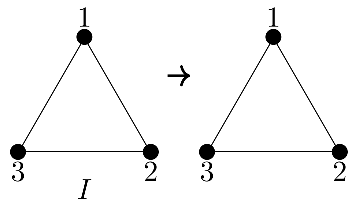
The cycle index is a way of collating all this information and representing it as a polynomial. The polynomial gives us information about how many, and of what length, are the cycles present after each transformation. If we consider a group of permutations, G, acting on the set of vertices of a polygon, X, then we will denote the cycle index as:

$$C(z_1, z_2, ..., z_n) = \frac{1}{|G|}\sum_{g\in G} z_1^{\alpha_1} z_2^{\alpha_2} ... z_n^{\alpha_n} = \frac{1}{|G|}\sum_{g\in G}\prod_{i=1}^{n} z_i^{\alpha_i},$$

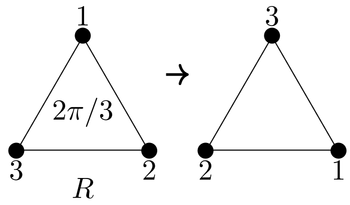
where z_i is represents a cycle of length i and α_i represents the number of cycles of length i (Jang Soo Kim, 2016). It will be best to demonstrate how to find the cycle index through the use of an example. An example of finding the cycle index of the vertices of a triangle is shown below.



Above we can see that vertex 1 gets mapped to itself (we will say this is a cycle of length one) and also a two cycle is formed between vertices 2 and 3. With this information we can write the cycle monomial for this transformation. A monomial is just a polynomial consisting of one term. This is $z_1^1 z_2^1 z_3^0 = z_1 z_2$. Now if we consider the identity:



We can see that there are three vertices that will get mapped to themselves. So we get the cycle monomial $z_1 z_1 z_1 = z_1^3$. Finally we can consider the two remaining rotations. We only need to look at one of these as they will be the same:



This forms a three cycle so the final cycle monomial is $z_1^0 z_2^0 z_3^1 = z_3$. Like Burnside's lemma, the cycle index is obtained by taking the average of all the cycle monomials. So the cycle index of the vertices on a triangle is:

$$C(z_1, z_2, z_3) = \frac{1}{6}\bigg(z_1^3 + 3z_1 z_2 + 2z_3\bigg).$$

3.2 Pólya's Enumeration Theorem

Now that we know about the cycle index we can look at what Pólya's theorem states. The theorem takes the cycle index and for each z_i in the polynomial we substitute the expression $(c_1^i + c_2^i + ... + c_k^i)$. Here each c_k represents a unique colour. This substitution gives us a generating function to find the number of orbits made up of a specified number of each colour concerned. We will write Pólya's enumeration theorem as follows (Jang Soo Kim, 2016):

$$Z(c_1, c_2, ..., c_k) = C(z_1, z_2, ..., z_n), \text{ where } z_i = c_1^i + c_2^i + ... + c_k^i.$$

If we look at the same example of a triangle it will be easier to explain how we can use the generating function. After the substitution we obtain:

$$\begin{aligned} Z(c_1, c_2, ..., c_k) = \frac{1}{6}\bigg(&(c_1^1 + c_2^1 + ... + c_k^1)^3 + 3(c_1^1 + c_2^1 + ... + c_k^1) \\ &\times(c_1^2 + c_2^2 + ... + c_k^2) + 2(c_1^3 + c_2^3 + ... + c_k^3)\bigg). \end{aligned}$$

Now say you wanted to find all possible distinguishable colourings possible with all k colours, then you would simply sum all the coefficients of every term in expression. In addition to allowing us to find all the colourings possible Pólya's theorem also allows us to look at specific cases. Say you wanted to find the number of distinguishable colourings using exactly two of c_2 and one c_5, then we would just need to find the coefficient of the $c_2^2 c_5^1$ term in the expression.

3.3 Proof of Pólya's Enumeration Theorem

Before proving Pólya's enumeration theorem there are a few preliminary definitions and results we must know about first.

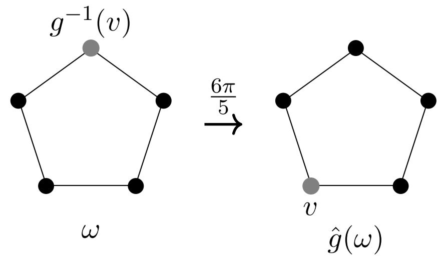
Say that we have a group of permutations, G, acting on a set V. In our case V will be the vertices of a regular polygon. Each of these vertices is to be coloured with one of k unique colours. We will denote this set of colours as K. The colouring of the vertices is a function from V to K that we will call ω. Finally the set of all possible colourings of V will be denoted X.

Definition 5.1 (Definition taken from Biggs, Discrete Mathematics for completeness)

Every permutation in G will induce a permutation say, $\hat{g}$ in X. This means that for every permutation of V there is a corresponding permutation in X. We will define $\hat{g}$:

$$(\hat{g}(\omega))(v) = \omega(g^{-1}(v)).$$

This is much easier to visualise pictorially. Say we choose g to be a clockwise rotation of a pentagon by $\frac{6\pi}{5}$ radians then the definition can be represented:



we can see that $(\hat{g}(\omega))(v)$ is gray and that $\omega(g^{-1}(v))$ is also gray. We can show that this mapping of $g \to \hat{g}$ is a bijection. In order to do this let us assume that we have $\hat{g_1} = \hat{g_2}$. This means that we have $(\hat{g_1}(\omega))(v) = (\hat{g_2}(\omega))(v)$ and also $\omega(g_1^{-1}(v)) = \omega(g_2^{-1}(v))$. As these expressions are true for all $v \in V$ and all $\omega \in X$, it would imply that g_1 is equal to g_2 and so $g \to \hat{g}$ is a bijection. This is very useful as now we know G is isomorphic to $\hat{G}$ and so we see that (Biggs, 2002) :

$$\frac{1}{|\hat{G}|}\sum_{\hat{g}\in\hat{G}} F(\hat{g}) = \frac{1}{|G|}\sum_{g\in G} F(\hat{g})$$

Definition 5.2

Say we have a colouring of a set of vertices, ω, then the **indicator** of ω will be given by the following expression:

$$ind(\omega) = c_1^{v_{c_1}} c_2^{v_{c_2}} \times ... \times c_k^{v_{c_k}}.$$

Here v_{c_i} is the number of vertices coloured with c_i. (Biggs, 2002)

Definition 5.3

Next we will define a **generating function** for a set of colourings, X. This expression will tell us exactly how many colourings exist using a specified number of colours. We will denote a generating function of this set of colourings as U_X. The expression can be obtained in the following way:

$$U_X(c_1, c_2, ..., c_k) = \sum_{\omega \in X} ind(\omega).$$

Here, ω is a colouring in X and c_i represents one of the k colours. (Biggs, 2002)

Finally we will look at a result that we will need to know before proving Pólya's theorem. Say we have a set of vertices V partitioned in the following way:

$$V = V_1 \cup V_2 \cup ... \cup V_h.$$

The size of a subset V_i is equal to m_i and the sum of all m_i will be n.

Now consider a set of colourings such that every vertex in V_i must be assigned the same colour. If this is the case then we can express the generating function for this set of colourings as:

$$U_X(c_1, c_2, ..., c_k) = (c_1^{m_1} + c_2^{m_1} + ... + c_k^{m_1})(c_1^{m_2} + c_2^{m_2} + ... + c_k^{m_2}) \times ...$$
$$... \times (c_1^{m_h} + c_2^{m_h} + ... + c_k^{m_h}).$$

To prove this we will simply consider the expansion of the right hand side of the expression. If we were to multiply out these brackets we would get all combinations of the products of one term from each bracket. For example say we took the first term from each bracket. This means that the members of V_1 would be coloured with c_1, V_2 with c_1 and so on. This will give us $c_1^{m_1} c_1^{m_2} \times ... \times c_k^{m_k}$. This is just one of the possible colourings in X, the rest of the combinations will give us all the remaining colourings in X. This means the right side equals $\sum_{\omega \in X} ind(\omega)$ which by definition is U_X. (Biggs ,2002)

We are now ready to look at Pólya's theorem. Consider a set of colourings, D, that contains of representative from each orbit of the set of colourings X. The generating function for this is :

$$U_D(c_1, c_2, ..., c_k) = \sum_{\omega \in D} ind(\omega).$$

We can re-express $\sum_{\omega \in D} ind(\omega)$ using Burnside's lemma. This will give us:

$$\sum_{\omega \in D} ind(\omega) = \frac{1}{|\hat{G}|} \sum_{\hat{g} \in \hat{G}} \left(\sum_{\omega \in F(\hat{g})} ind(\omega) \right).$$

By definition we can express $\sum_{\omega \in F(\hat{g})} ind(\omega)$ as $U_{F(\hat{g})}$. As we saw before, a permutation on the vertices of a polygon will split them in to cycles in which all of the vertices in the cycle must be the same colour in order to obtain a fixed pattern. This forms a partition as in the previous result so this means we now have the following expression:

$$\begin{aligned} U_{F(\hat{g})}(c_1, c_2, ..., c_k) = (c_1^{m_1} + c_2^{m_1} + ... + c_k^{m_1})(c_1^{m_2} + c_2^{m_2} + ... + c_k^{m_2}) \times ... \\ ... \times (c_1^{m_h} + c_2^{m_h} + ... + c_k^{m_h}). \end{aligned}$$

If we allow $(c_1^i + c_2^i + ... + c_k^i)$ to equal z_i, the expression above becomes:

$$U_{F(\hat{g})}(c_1, c_2, ..., c_k) = z_{m_1} z_{m_2} \times ... \times z_{m_h}.$$

As the m_i's are the sizes of each V_i in the partition this means that in this case they will be the sizes of the cycles present. As the cycles cannot exceed the length of n the previous expression may also be expressed:

$$U_{F(\hat{g})}(c_1, c_2, ..., c_k) = z_1^{\alpha_1} z_2^{\alpha_2} \times ... \times z_n^{\alpha_n},$$

here the α_i's are the number of cycles of length i. If we substitute this back in to the formula we obtain:

$$U_D(c_1, c_2, ..., c_k) = \frac{1}{|\hat{G}|} \sum_{\hat{g} \in \hat{G}} z_1^{\alpha_1} z_2^{\alpha_2} \times ... \times z_n^{\alpha_n}.$$

As the group G is isomorphic to $\hat{G}$, then the right hand side of the above expression is equal to $C(z_1, z_2, ..., z_n)$.

$$\begin{aligned} U_D(c_1, c_2, ..., c_k) &= \frac{1}{|\hat{G}|} \sum_{\hat{g} \in \hat{G}} z_1^{\alpha_1} z_2^{\alpha_2} \times ... \times z_n^{\alpha_n} \\ &= \frac{1}{|G|} \sum_{g \in G} z_1^{\alpha_1} z_2^{\alpha_2} \times ... \times z_n^{\alpha_n} \\ &= C(z_1, z_2, ..., z_n) \\ &= Z(c_1, c_2, ..., c_k) \end{aligned}$$

This is Pólya's enumeration theorem. (Biggs, 2002)

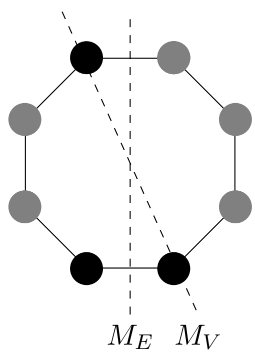
3.4 Examples Using Pólya's Enumeration Theorem

We will take a look at the initial problems this time making use of Pólya's enumeration theorem.

Problem One

Determine the number of distinguishable designs for a bracelet comprised of five white and three black beads.

Solution/ To start this question we will want to find the cycle monomials for each transformation of an octagon.



We will only need to consider four of the rotations seeing as a rotation by R radians is essentially the same as a rotation by $2\pi - R$ radians. First for a clockwise rotation by $\frac{\pi}{4}$, each vertex gets mapped to the vertex next to it in a clockwise direction and so we have an eight cycle. For a rotation of $\frac{\pi}{2}$ the vertices get split in to two lots of four and so we have two four cycles. A rotation by $\frac{3\pi}{4}$ will leave us with a six cycle as eight is not divisible by three. For a rotation by π we will obtain four two cycles as each vertex gets mapped to the vertex directly opposite to it. An edge to edge reflection will give us four two cycles. Finally for a vertex to vertex reflection there are two vertices which get mapped to themselves so these will give us two one cycles and the remaining vertices form three two cycles. Below are the cycle monomials for each transformation where R^s is a rotation by $s\left(\frac{2\pi}{n}\right)$ and the M are reflections:

R	R^2	R^3	R^4	R^5	R^6	R^7	R^8	M_E	M_V
z_8	z_4^2	z_8	z_2^4	z_8	z_4^2	z_8	z_1^8	z_2^4	$z_1^2 z_2^3$

So the cycle index for the vertices of a regular octagon is,

$$C = \frac{1}{16}\left(z_1^8 + 4z_1^2 z_2^3 + 5z_2^4 + 2z_4^2 + 4z_8\right).$$

Now if we substitute $(B^i + W^i)$ for all z_i's in the cycle index where B represents the colour black and W white. Then we will obtain the expression:

$$Z = \frac{1}{16}\left((B+W)^8 + 4(B+W)^2(B^2+W^2)^3 + 5(B^2W^2)^4 + 2(B^4+W^4)^2 + 4(B^8+W^8)\right).$$

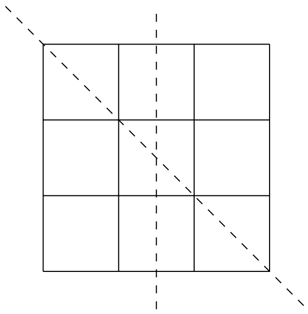
To find the number of orbits containing three black beads and five white beads we want to find the coefficient of the B^3W^5 term in the expression. For the first term of this expression there is $\binom{8}{3}$ ways to obtain a B^3W^5 term. For the second term there are $4\binom{2}{1}\binom{3}{1}$ ways. For the remainder of the terms it is not possible to obtain B^3W^5, as it is not possible to obtain B^3 in any way. Now to obtain the number of orbits we divide the sum of the coefficients by the number of elements in G. So the number of orbits is given by:

$$\frac{1}{16}\left(\frac{8!}{5!3!} + 4 \cdot 2 \cdot 3\right) = 5$$

Problem Two

The design for a square table mat is based on a 3x3 grid of nine glass squares. Two of these squares are to be blue; the remainder red. The mat is transparent and can be turned over. Determine the number of different designs.

Solution/ Again we will start by finding the cycle monomials for each transformation on the faces of the following shape.



Below are the cycle monomials for each transformation where R^s is a rotation by $s\frac{2\pi}{n}$ and the M are reflections:

R	R^2	R^3	R^4	M_E	M_V
$z_1z_4^2$	$z_1z_2^4$	$z_1z_4^2$	z_1^9	$z_1^3z_2^3$	$z_1^3z_2^3$

So the cycle index is:

$$C = \frac{1}{8}\left(z_1^9 + z_1z_2^4 + 2z_1z_4^2 + 4z_1^3z_2^3\right).$$

Now we can substitute $(R^i + B^i)$ for all z_i's in the cycle index to get:

$$Z = \frac{1}{8}\left((R+B)^9 + (R+B)(R^2+B^2)^4 + 2(R+B)(R^4+B^4)^2 + 4(R+B)^3(R^2+B^2)^3\right).$$

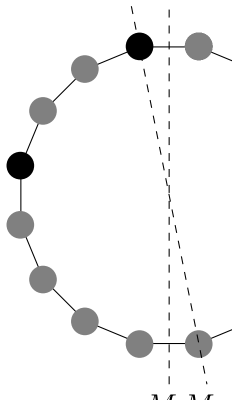
From this expression we want to find the coefficient of the R^7B^2 term. From the first term in Z we can see that there are $\binom{9}{7}$ ways of obtaining R^7B^2, $\binom{4}{1}$ from the second, none from the third, and finally $4\left(\binom{3}{1}+\binom{3}{1}\right)$ for the last. So the number of orbits with seven red tiles and two blue is:

$$\frac{1}{8}\left(\frac{9!}{7!2!} + \frac{4!}{3!1!} + 2(0) + 4\left(\frac{3!}{2!1!} + \frac{3!}{2!1!}\right)\right) = 8$$

Problem Three

Determine the number of distinguishable designs for a bracelet comprised of thirteen white and three black beads.

Solution/ Starting in the same way as before we will find the cycle monomials for each transformation on the vertices of the polygon.



Below the cycle monomials for each transformation are shown. Where R^s is a rotation by $s\frac{2\pi}{n}$ and the M are reflections.

R	R^2	R^3	R^4	R^5	R^6	R^7	R^8	R^9	R^{10}	R^{11}	R^{12}	R^{13}
z_{16}	z_8^2	z_{16}	z_4^4	z_{16}	z_8^2	z_{16}	z_2^8	z_{16}	z_8^2	z_{16}	z_4^4	z_{16}

R^{14}	R^{15}	R^{16}	M_E	M_V
z_8^2	z_{16}	z_1^{16}	z_2^8	$z_1^2 z_2^7$

So the cycle index is:

$$C = \frac{1}{32}\Big(z_1^{16} + 8z_1^2 z_2^7 + 9z_2^8 + 2z_4^4 + 4z_8^2 + 8z_{16}\Big).$$

Now we can substitute $(B^i + W^i)$ for each z_i in this expression where B represents the colour black and W, white. Doing this we obtain:

$$Z = \frac{1}{32}\Big((B+W)^{16} + 8(B+W)^2(B^2+W^2)^7 + 9(B^2+W^2)^8 + 2(B^4+W^4)^4 + 4(B^8+W^8)^2 + 8(B^{16}+W^{16})\Big).$$

Now we just need to find the coefficient of the B^3W^{13} term in this expression. From the first term we can see there is $\binom{16}{3}$ ways of obtaining a B^3W^{13} term. For the second term the only way to get a B^3 is by choosing one of the $(B+W)$ terms and one of the (B^2+W^2) terms. There are $8\binom{2}{1}\binom{7}{1}$ ways to do this. For the remaining terms it is impossible to get a B^3 term as they are all of even power. So now we can obtain the answer:

$$\frac{1}{32}\left(\frac{16!}{13!3!}+8\cdot 2\cdot 7+0\right)=21$$

3.5 Colouring the Vertices of a Regular Polygon

Finding an expression to colour the vertices of a regular polygon using Pólya's enumeration theorem is quite easy. We can simply modify the expression we found for Burnside's lemma. In our previous expression we had $\varphi(d)k^{\frac{n}{d}}$. This means that we had $\varphi(d)$ cases in which the rotations split the vertices in to $\frac{n}{d}$ cycles of d. So with this we can easily modify our expressions to give us the cycle index instead. So $\varphi(d)k^{\frac{n}{d}}$ becomes $\varphi(d)z_d^{\frac{n}{d}}$. The expressions corresponding to the reflections can also be modified in the same way. So in general we can find the number of orbits with the following formulae:

$$Z(c_1,c_2,...,c_k)=\frac{1}{2n}\begin{cases}\left(\sum_{d\in D(n)}\varphi(d)z_d^{\frac{n}{d}}+\frac{n}{2}z_2^{\frac{n}{2}}+\frac{n}{2}z_1^2z_2^{\frac{n-2}{2}}\right), & \text{if } n \text{ even}\\ \left(\sum_{d\in D(n)}\varphi(d)z_d^{\frac{n}{d}}+nz_1z_2^{\frac{n-1}{2}}\right), & \text{if } n \text{ odd}\\ \left(z_1^n+(n-1)z_n+nz_1z_2^{\frac{n-1}{2}}\right), & \text{if } n \text{ prime}\end{cases}$$

Where $z_i=(c_1^i+c_2^i+...+c_k^i)$

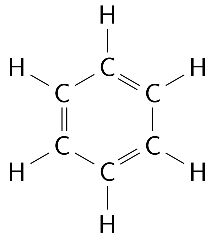
4 Further Application of Pólya Enumeration Theorem

4.1 Chemical Enumeration

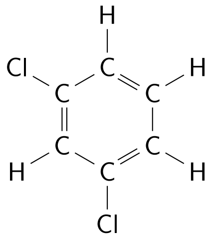
One application of Pólya's enumeration theorem is found in chemistry in the form of chemical and isomer enumeration. An isomer is a molecule that shares the same chemical formula as another molecule but does not share the same molecular structure. Pólyas theorem can be used in order to find the number of isomers or to find a number of compounds with a certain composition that exist.

Example - k-chlorobenzene

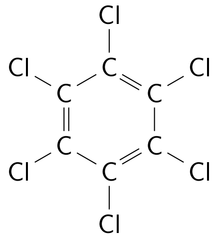
One simple example that we could take a look at is k-chlorobenzene. Benzene has the molecular structure shown below on the left:



Benzene



Dichlorobenzene



Hexachlorobenzene

The compound k-chlorobenzene has the same structure as benzene except k of the hydrogen molecules (H) are replaced with chlorine molecules (Cl). Examples of some k-chlorobenzenes are shown above, one where k is two and the other six. If we look at the shape of the compound we can see that it can be converted in to a simple bracelet problem which can easily be solved using Pólya's enumeration theorem. To find the number of k-chlorobenzenes the problem will now be to find the colourings of the vertices of a hexagon with two colours. The cycle index for this bracelet problem is:

$$C(z_1, z_2, ..., z_6) = \frac{1}{12}\Bigg(z_1^6 + 3z_1^2 z_2^2 + 4z_2^3 + 2z_3^2 + 2z_6\Bigg).$$

Now we just need to substitute an expression like $(H^i + Cl^i)$ for each z_i in the cycle index. This will give us the expression:

$$\begin{aligned} Z(H, Cl) = \frac{1}{12}\Bigg(&(H + Cl)^6 + 3(H + Cl)^2(H^2 + Cl^2)^2 + 4(H^2 + Cl^2)^3 \\ &+ 2(H^3 + Cl^3)^2 + 2(H^6 + Cl^6)\Bigg) \\ = {} & H^6 + H^5Cl + 3H^4Cl^2 + 3H^3Cl^3 + 3H^2Cl^4 + HCl^5 + Cl^6. \end{aligned}$$

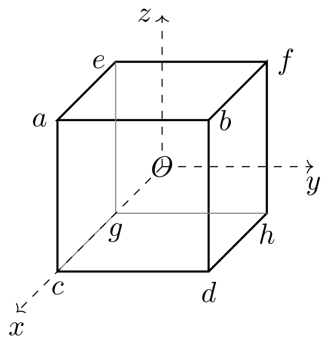
With this expression we will be able to find the number of any type of k-chlorobenzene we like. In total however there are 13 possible k-chlorobenzene compounds.

4.2 Pólya's Enumeration Theorem in Higher Dimensions

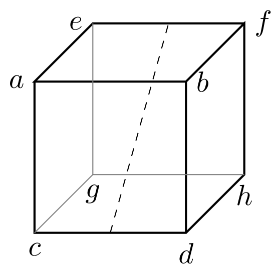
Although we have only been looking at bracelet and necklace problems in this report, Pólya's theorem can be applied in higher dimensions to enumerate the colourings of the faces or vertices of shapes. Here we will only take a look at a simple example of colouring the faces of a cube.

Example - Colouring the Faces of a Cube

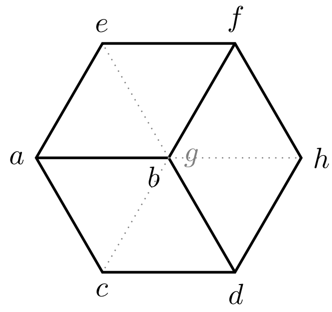
We will start this problem in the same way as the bracelet problems, by considering what the symmetries of a cube are. A cube has 24 symmetries in total. Looking at the diagram below, the vertices of the cube have been labeled a to h and the dashed lines represent axes whose origin are at the center point of the cube. First of all there are four rotations possible around each axes (2π, $\frac{3\pi}{2}$, π, $\frac{\pi}{2}$). However the identity is always going to be the same around each axis so there are ten total rotations of this type (Type A). Next we can rotate the cube at each vertex by multiples of $\frac{2\pi}{3}$, so in total there are eight rotations of this type (Type B). Finally there is a rotation around a line that passes through the midpoints of opposite edges. As there are 12 edges in a cube there will be just six of this type of symmetry (Type C). Now that we know what the symmetries are we can find an expression for the cycle index.



We can obtain the cycle index for a cube in the same way as we did for a polygon except we will keep track of where the faces are mapped to instead. First let us look at the Type C rotation, the line of rotation is represented below by the dashed line.



We can see if we rotate it around this line the face $bdhf$ takes face $acge$'s place and vice versa, so this forms a cycle of length two. Similarly faces $abdc$ and $gcdh$ as well as faces $aefb$ and $eghf$ also form cycles of length two. So the cycle monomial for this type of transformation is just $6z_2^3$ as there are six instances of this type of symmetry. Now let us consider the Type B rotations.



The diagram above shows a cube where vertex b is directly above vertex g. In this orientation it will be easier to visualise the rotation. If we rotate the cube about the line bg by $\frac{2\pi}{3}$ radians we can see that the top three faces of the cube facing us will form a three cycle and do not interact with the hidden faces of the cube. The hidden faces will also give us a three cycle and so the cycle monomial for this symmetry is $8z_3^2$ as there are eight instances of this type of rotation.

Finally we will find the cycle monomials for a Type A rotation. These are slightly more complicated to obtain as there are three cases which we need to consider. We will first look at a rotation about the axes by $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ radians. We can see that the two faces the axes pass through will always be mapped to themselves. Next the four remaining faces will get mapped to one of the faces adjacent to them and so they form a four cycle. This means that the cycle monomial is $6z_1^2z_4$ as there are six of these rotations. Next we will look at a rotation by π radians. In this case the two faces that the axes pass through will be mapped to themselves again leaving four faces. The remaining faces will get mapped to the faces on the opposite side of the cube to them and so they form a two cycle giving us $3z_1^2z_2^2$. Finally the identity will of course give us z_1^6. The cycle monomials are listed below.

Type A	Type B	Type C
$z_1^6,\ 3z_1^2z_2^2,\ 6z_1^2z_4$	$8z_3^2$	$6z_2^3$

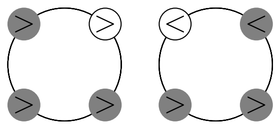
We can now obtain the cycle index:

$$C(c_1, c_2, ..., c_k) = \frac{1}{24}\left(z_1^6 + 3z_1^2z_2^2 + 6z_1^2z_4 + 6z_2^3 + 8z_3^2\right)$$

5 Other Bracelet Problems

5.1 Directed Beads

Here we will consider a variation of the bracelet problem. In this problem as well as assigning a vertex with a colour we will also assign it a direction so we will be able to tell if the bead is pointing clockwise or anti-clockwise. An example of this is shown below:



Example One

In this example we will consider the colourings of a bracelet with six beads. Three of the beads are white and the rest are black. All of these beads display a direction, so they either point clockwise or anti-clockwise.

To solve this we can use Burnside's lemma. First we can consider the rotational symmetries. As we are colouring six vertices we need only consider rotations by $\frac{\pi}{3}$, $\frac{2\pi}{3}$, and π. For a rotation by $\frac{\pi}{3}$ there are no fixed patterns possible as all six beads must be the same colour. For a rotation of $\frac{2\pi}{3}$ we can see that the beads are split in to two cycles of three, so there are two different colourings however we also need to consider the direction of the beads so there is an extra factor of two for each cycle as all beads in a cycle must face in the same direction as well. For a rotation by π there are again no fixed patterns as the beads are split in to three cycles of two and so there are not enough beads of one colour to do this. Next for the identity there are $\binom{6}{3}$ ways of colouring the beads and an added factor of two for each bead, so there are $\binom{6}{3}\times 2^6$ fixed patterns. Finally for the reflections we see that there is never going to be any fixed patterns after a vertex to vertex reflection in any case as this will change the direction of the beads on the vertices. In this case there will be no fixed patterns after an edge to edge reflection either as the beads are split in to three cycles of two and so there are not enough beads of a single colour to create a fixed pattern. So the number of orbits is:

$$N = \frac{1}{12}\left(\frac{6!}{3!3!} \cdot 2^6 + 2 \cdot 2 \cdot 2^2\right) = 108$$

Example Two

In this example we will consider a bracelet composed of eight white and eight black bead that all display a direction.

We will solve this in exactly the same way as the previous example. As we are looking at a polygon with 16 vertices there are 32 symmetries to consider. Below are the number of fixed patterns that exist under each symmetry that has any:

I	R	R^2	R^3	R^4	R^5	R^6	R^7	R^8	M_E	M_V
$\frac{16!}{8!8!} \cdot 2^{16}$	0	$2 \cdot 2^2$	0	$\frac{4!}{2!2!} \cdot 2^4$	0	$2 \cdot 2^2$	0	$\frac{8!}{4!4!} \cdot 2^8$	$\frac{8!}{4!4!} \cdot 2^8$	0

So the number of orbits will be given by:

$$N = \frac{1}{32}\left(\frac{16!}{8!8!} \cdot 2^{16} + 2 \cdot 2 \cdot 2^2 + 2 \cdot \frac{4!}{2!2!} \cdot 2^4 + 2 \cdot 2 \cdot 2^2 + \frac{8!}{4!4!} \cdot 2^8 + 8 \cdot \frac{8!}{4!4!} \cdot 2^8\right)$$

$$= 26\ 362\ 807$$

6 Conclusion

Pólya's enumeration theorem is a very useful tool in trying to enumerate the distinguishable colourings on the vertices of a polygon. It allows us to easily find how many colourings there are whilst also allowing us to select specific colourings to examine if we wish to. We have also seen that Pólya's theorem may be used for more than just the enumeration of bracelets with it being used largely in chemistry to enumerate chemical compounds. Another strength of Pólya's theorem is that it is not bound to one type of question such as bracelet enumeration but is able to be used in many other situations such as in the cube example we took a look at. Although Pólya's enumeration theorem has many strengths it is not a suitable method to solve all types of bracelet problem. For example if we were to impose any restrictions on the colouring of the bracelet the theorem would likely be unusable. One restriction that would not allow us to use the theorem is if we were to say that two of the same colour may not be adjacent to one another in the colouring. In this case we would likely have to return to Burnside's lemma or even use another method entirely. Despite this, Pólya's enumeration theorem is still incredibly useful in enumerating bracelets.

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