Graph Theory Problem Set 1

Zadid Hasan

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1 Mathematical Challenges

- **Easy**

1. By counting the number of edges of a complete graph with n vertices, prove that,
   (a) $\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$.
   (b) if $\sum_{i=1}^{k} n_i = n$, then $\sum_{i=1}^{k} \binom{n_i}{2} < \binom{n}{2}$.
2. Show that for any graph G = (V, E) the vertex set V can be partitioned into two sets V_1 and V_2 such that, $e(V_1) + e(V_2) \leq \frac{|E|}{2}$ where $e(V_i)$ means the number of edges in E with both end points in V_i.
3. Prove that a regular bipartite graph of degree at least 2 does not contain a bridge.
4. There are 2011 mathematicians in a party. It is known that, Mahbub, the host of the party (who is also a mathematician) knows all other mathematicians. Two mutually unacquainted mathematicians will become friend of each other eventually after the party if they have a common friend/acquaintance (who will introduce them to each other of course). After the end of the party how many pairs of mathematicians will be left who are not yet introduced to each other?
5. If in a directed graph number of incoming edges is equal to number of outgoing edges then the graph has an Eulerian Path.
6. There are n cities in the country. Between any two cities there is at most one road. Suppose that the total number of roads is n. Prove that there is a city such that starting from there it is possible to come back to it without ever travelling the same road twice.
7. Suppose there are $n \geq 2$ people at a party. And people are shaking hands with each other. Everyone is taking a count of how many people they have shaken hands with. Each pair shakes hands at most once. Prove that, there are at least 2 persons who shook hands with the same number of people.
8. There are n players in a chess tournament.Every player plays every other player exactly once and there are no draws.Prove that,the players can be labeled $1, 2, \cdots, n$ so that i beats $i + 1$ for each $i \in 1, 2, 3, \cdots, n-1$.
9. There are 36 participants at a competition. Some of the participants shook hands with each other. But no two participants shook hands with each other more than once. Each participants recorder the number of handshakes they made. It was found that no two participants with the same number of handshakes made, had shaken hands with each other. Find the maximum number of handshakes at the party with proof.(when two participants shook hands with each other, this will be counted as one handshake).
10. Let G be a simple graph with n vertices. Let $t(G)$ be the total number of triangles in G and $\bar{G}$ altogether.
    (a) Prove that $t(G) = \binom{n}{3} - (n-2)e(G) + \sum_{v \in V(G)} \binom{d(v)}{2}$

(b) Prove that $t(G) \geq \frac{n(n-1)(n-5)}{24}$

11. Prove that 101 x 999 grid does not have a Hamiltonian Cycle.
12. Prove that a graph with n vertices and n + 2 edges in planar.
13. How many 4-cycles are there in $K_{n,n}$?
14. Prove that either G or $\bar{G}$ is connected.
15. In a classroom there are A girls and B boys. Each girl is friends with r boys. And each boy is friends with r girls. Friendship is mutual. Prove that A = B.
16. In Bangladesh, there are n airports. From any airport, there is no way to return to that airport after visiting exactly 2 other airports. Determine the highest number of direct flights in Bangladesh.
17. How many edges must a simple graph with n vertices have so that it is connected?
18. Show that for graphs with $n \geq 2$, there are at least 2 vertices that are not cut-vertices.

- **Medium**

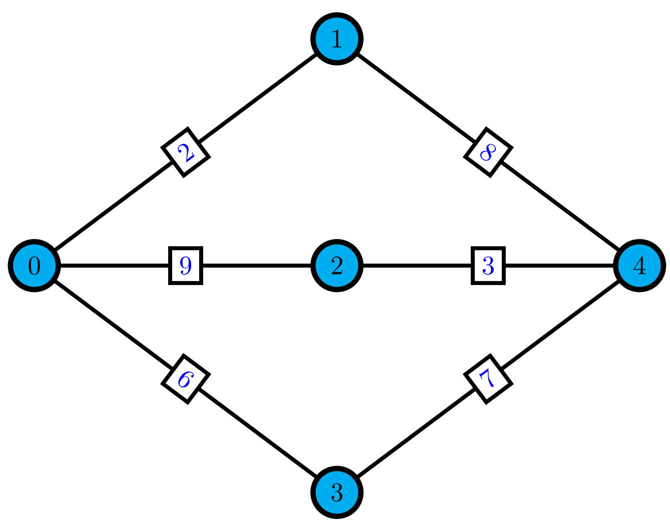
1. Bhaskaracharya has set up a strange study group. Any member of that group has exactly one immediate teacher (the teacher who teaches him) except for Bhaskaracharya himself, although teacher of teacher is also respected as a teacher. As the chancellor of the group, Bhaskaracharya is not taught by anybody. No two members of that group can be teachers of each other. The study group operates in a pairs where each pair consists of one member and his immediate teacher.If such a pairing is possible, is it unique? Justify your answer.
2. If $\delta(G) \leq d$ and at least one vertex has degree strictly less than d then prove that the graph is d-colorable.
3. Prove that a planar bipartite graph has at most $2n - 4$ edges.
4. If G is a bipartite graph with n nodes in each partition and $\delta(G) \geq \frac{n}{2}$ then prove that it has a perfect matching.
5. A chord of a cycle is an edge that connects two non-adjacent vertices in the cycle. Prove that if $\delta(G) \geq 3$ then G contains a cycle with a chord.
6. Let $T_1, \cdots, T_k$ be subtrees of a tree T such that for all i, j the trees T_i and T_j have a vertex in common. Show that T has a vertex that is in all T_i.
7. If G = (V, E) is a graph on n vertices such that all the vertices have even degree. Show that the edge set E can be partitioned into pairwise disjoint sets $C_1, C_2, \cdots, C_k$ such that for all $1 \leq i \leq k$ the subgraphs (V, C_i) is a cycle and a collection of isolated vertices.
8. Given a group of 6 people, show that 3 of them are mutual friends or complete strangers.
9. Let G be a planar graph, with edges colored red and blue. Show that there is a vertex v such that going round the vertex in a clockwise direction we encountered no more than two change of colors.
10. Given a group of 17 people.Any 2 of them are either strangers or enemies or good friends. Show that we can find a trio all whom are mutual good friends or mutual enemies or mutual strangers.
11. In a classroom of n students, every student has at least $\frac{n-1}{2}$ first cousins. Prove that, every two students are either first cousins or second cousins.
12. In a group of n people, every two persons is either friends or enemies. Friend of a friend is also friend. Enemy of an enemy is friend. Determine the values of n.
13. 20 football teams take part in a tournament. On the first day all the teams play one match. On the second day all the teams play a further match. Prove that after the second day it is possible to select 10 teams, so that no two of them have yet played each other.

14. There are 1000 cities in the country of Euleria, and some pairs of cities are linked by dirt roads. It is possible to get from any city to any other city by traveling along these roads. Prove that the government of Euleria may pave some of the roads so that every city will have an odd number of paved roads leading out of it.
15. If every face of a planar graph has even number of edges then prove that the graph is bipartite.

- **Hard**
  1. Prove that there is a tournament T with n players and at least $\frac{n!}{2^{(n-1)}}$ Hamiltonian paths.
  2. If a graph has $\Delta(G) \leq k$ then it is (k + 1)-colorable.
  3. Every pair of communities in a country are linked by exactly one mode of transportation: bus, train or automobile. All three modes of transportation are used in the country with no community being served by all three modes and no three communities being linked pairwise by the same mode. Determine the maximum number of communities in the country.
  4. The 20 members of a local tennis club have scheduled exactly 14 two-person games among themselves with each member playing in at least one game. Prove that, within this schedule, there must be a set of 6 games with 12 distinct players.

2 Programming Challenges

1. I am going to my home. There are many cities and many bi-directional roads between them. The cities are numbered from 0 to n-1 and each road has a cost. There are m roads. You are given the number of my city t where I belong. Now from each city you have to find the minimum cost to go to my city. The cost is defined by the cost of the maximum road you have used to go to my city.



For example, in the above picture, if we want to go from 0 to 4, then we can choose,

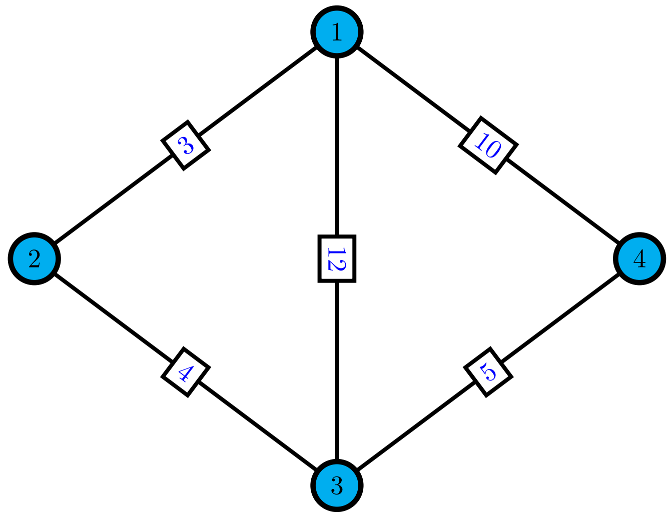
(a) 0 - 1 - 4 which costs 8, as 8 (1 - 4) is the maximum road we used

(b) 0 - 2 - 4 which costs 9, as 9 (0 - 2) is the maximum road we used

(c) 0 - 3 - 4 which costs 7, as 7 (3 - 4) is the maximum road we used

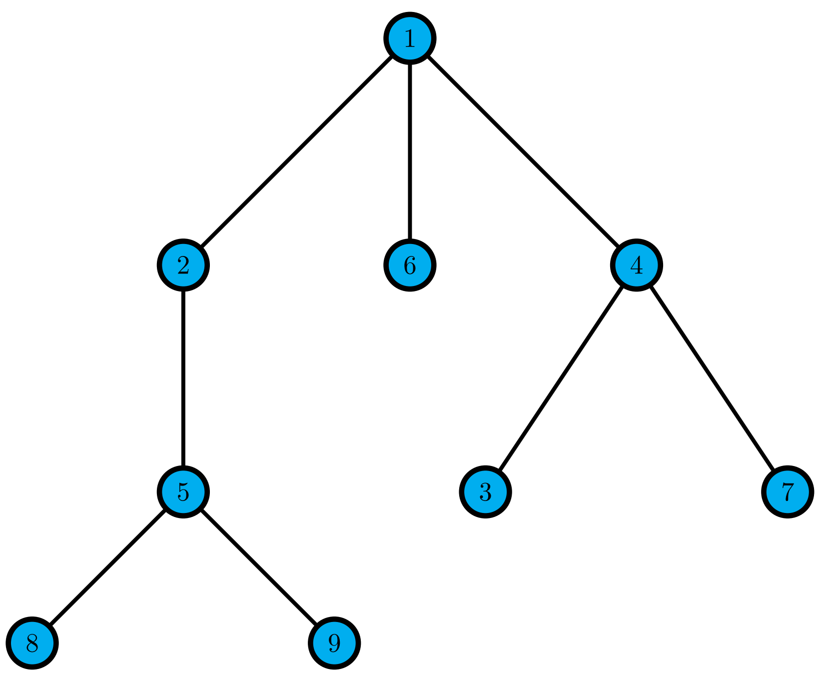
So, our result is 7, as we can use 0 - 3 - 4. Find an algorithm and justify your approach.

2. Robin is training for a marathon. Behind his house is a park with a large network of jogging trails connecting water stations. Robin wants to find the shortest jogging route that travels along every trail at least once.



For example, if he follows the path 2 - 1 - 4 - 3 - 1 - 2 - 3 - 2, then the total distance he has covered is 41. And there is no other way he could achieve a shorter route.

3. Given a tree (a connected graph with no cycles), you have to find the distance between farthest nodes in the tree. The edges of the tree are un-weighted and undirected. That means you have to find two nodes in the tree whose distance is maximum amongst all nodes.



For example, 5 is the diameter of this tree. The pairs (3,8), (3,9), (7,8), (7,9) all realize this distance.

4. Suppose you have a to-do list before you go to school where you have a number of tasks. You have ordered pair (a, b) of tasks which means you have to finish task a before task b. Find an ordering of all the tasks so that you do not perform any task in the wrong order.