1 Survival time

As a small example, the survival times of 9 rats were 10, 27, 30, 40, 46, 51, 52, 104, and 146 days. Because of the skewness in the data, consider estimating the population median survival time θ through the sample median.

```
CODE FILENAME: ../R/01_01_load_data.R*****

  survival_times <- c(10, 27, 30, 40, 46, 51, 52, 104, 146)
  sample_median <- median(survival_times)
  seed <- 7
  B1 <- 1000
  B2 <- 50
  n <- 9

  bootstrap_fn <- function(estimate = "median", meth = "percentile") {
    # taking 1st level boot
    survival_boot <- sample(survival_times, n, replace = TRUE)

    if (estimate == "median") {
      est_boot <- median(survival_boot)
      if (meth == "percentile") {
        return(est_boot)
      } else if (meth == "bootstrap_t") {
        sample_est <- median(survival_times)
        # taking 2nd level boot
        est_boot2 <- replicate(B2, {
          survival_boot2 <- sample(survival_boot, n, replace = TRUE)
          median(survival_boot2)
        })
      }
    } else if (estimate == "mean") {
      est_boot <- mean(survival_boot)
      if (meth == "percentile") {
        return(est_boot)
      } else if (meth == "bootstrap_t") {
        sample_est <- mean(survival_times)
        #taking 2nd level boot
        est_boot2 <- replicate(B2, {
          survival_boot2 <- sample(survival_boot, n, replace = TRUE)
          mean(survival_boot2)
        })
      }
    }

    se_boot <- sd(est_boot2)
    t_boot <- (est_boot - sample_est) / se_boot
    result_list <- list(r = est_boot, t = t_boot)
    return(result_list)
  }
end---------------------------------------------------------------------------
```

1.1 Bootstrap-t method: median

Compute a 95% CI for θ using the bootstrap-t method. Use $B_1 = 1000$ first-level bootstrap samples and $B_2 = 50$ second level bootstrap samples (to estimate the standard error).

We are 95% confident that the true value of the median is between 20.46346 and 78.24754.

```
CODE FILENAME: ../R/01_02_bst_median.R*****

  set.seed(seed)
  res = sapply(1:B1, function(.){bootstrap_fn(estimate = "median",
                                              meth = "bootstrap_t")})
  ses <- unlist(res[1,])
  tbs <- unlist(res[2,])

  se_median <- sd(ses) #bootstrap estimate of the SE
  lower <- sample_median - quantile(tbs,.975)*se_median
  upper <- sample_median - quantile(tbs,.025)*se_median
  #c(lower,upper)

end-------------------------------------------------------------------------
```

1.2 Bootstrap percentile CI: median

Compute a 95% CI for θ using the bootstrap percentile CI with B = 1000 bootstrap samples.

We are 95% confident that the true value of the median is between 27 and 53.3.

```
CODE FILENAME: ../R/01_03_percentile_median.R*****

  set.seed(seed)
  res_median = sapply(1:B1, function(.){bootstrap_fn(estimate = "median",
                                                     meth = "percentile")})
  #c(quantile(res_median,.025), quantile(res_median,.975))

end-------------------------------------------------------------------------
```

1.3 Bootstrap percentile CI: mean

Compute a 95% confidence interval for the mean time between failures θ using the basic bootstrap method with B = 1000 bootstrap samples.

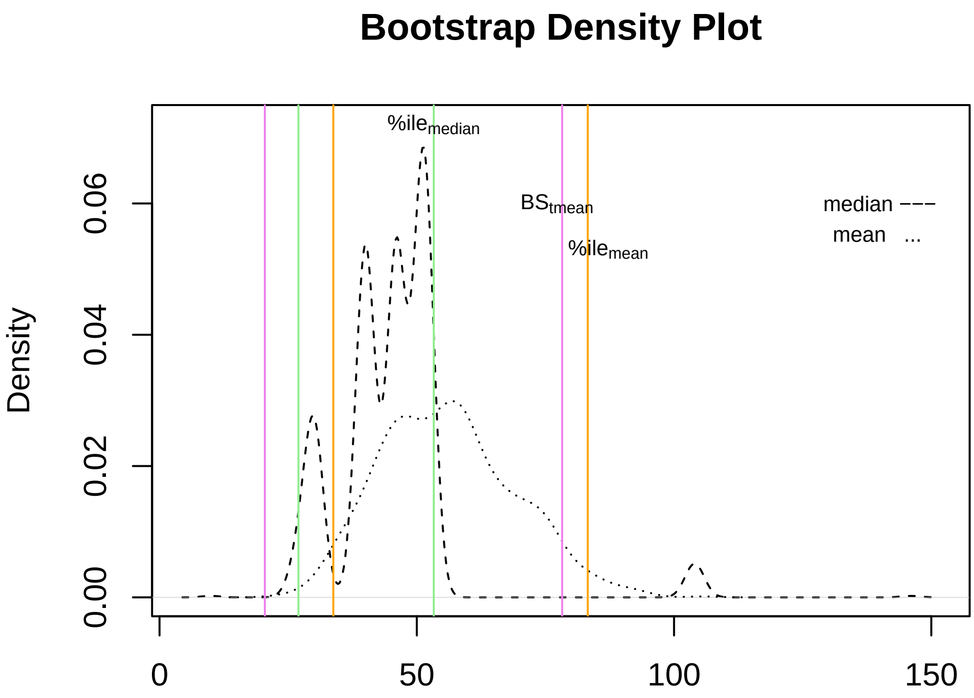
We are 95% confident that the true value of the mean is between 33.775 and 83.22778.

```
CODE FILENAME: ../R/01_04_percentile_mean.R*****

  set.seed(seed)
  res_mean = sapply(1:B1, function(.){bootstrap_fn(estimate = "mean",
                                                   meth = "percentile")})
  #c(quantile(res_mean,.025),quantile(res_mean,.975))

end-------------------------------------------------------------------------
```

1.4 Density estimate

Plot a density estimate of the data. In R, you can do this through the density function. Compare the results in parts 1.1, 1.2, and 1.3.

First, the percentile CI's covers areas in which the the bulk of statistics are seen. Because the data is skewed to the right and median is robust to outliers, percentile CI based on it can be seen to the left of the percentile CI based on the mean. Next, the mean is pulled to the right due to some extremely high values (104, 146). Finally, it could be observed that the percentile CI's are narrower than that of the bootstrap-t, and that the latter appears to cover areas of large concentration from both the mean and the median percentile CI's.




```
CODE FILENAME: ../R/01_05_density_plot.R*****

  plot(density(res_median),
       ylim = range(density(res_median)$y, density(ses)$y),
       lty = 'dashed', main = "Bootstrap Density Plot", xlab = '')
  lines(density(res_mean), lty = 'dotted')

  abline(v=quantile(res_mean,.025),col="orange")
  abline(v=quantile(res_mean,.975),col="orange")
  text(quantile(res_mean,.975)+4, 0.053,
       expression(paste("%", ile[mean])), cex=0.7)

  abline(v=quantile(res_median,.025),col="lightgreen")
  abline(v=quantile(res_median,.975),col="lightgreen")
  text(quantile(res_median,.975), 0.072,
       expression(paste("%", ile[median])), cex=0.7)

  abline(v=quantile(lower,.025),col="violet")
  abline(v=quantile(upper,.975),col="violet")
  text(quantile(upper,.975)-1, 0.06, expression(BS[tmean]), cex=0.7)
```

```
text(140, 0.06, "median ---", cex=0.7)
text(139.5, 0.055, "mean   ...", cex=0.7)
```

end--

2 Spatial test

Consider the spatial test data from Table 14.1 of Efron and Tibshirani (1993) shown below. From the table's description, it is clear that the measurements A and B are paired. Suppose the data consist of a random sample from an unknown joint distribution of A and B. Whenever ratios are scientifically or statistically preferred to differences, we gain stability by considering the logarithm of the ratios. Let $\theta_1 = \log E\left(\frac{A_i}{B_i}\right)$, $\theta_2 = E\left(\log \frac{A_i}{B_i}\right)$ for all i. Exclude observation #14 because the logarithm of its ratio is undefined. Use 2000 bootstrap samples.

2.1 Bootstrap percentile CI for θ_1

Compute a bootstrap percentile confidence interval for θ_1.

We are 95% confident that the true value of θ_1 is between -0.06867 and 0.16487.

CODE FILENAME: ../R/02_02_percentile_logmean.R*****

```
B <- 2000

plug_in_estimator <- function(A,B, est = "theta_1"){
  if (est == "theta_1") {
    return(log(mean(A/B)))
  } else if (est == "theta_2") {
    return(mean(log(A/B)))
  }
}

theta_1_hat_star <- c()
theta_2_hat_star <- c()
set.seed(7)
for (b in 1:B) {
  bs_sample <- spatial_test_data[sample(spatial_test_data[,1],
                                        n1,
                                        replace = TRUE),]
  theta_1_hat_star[b] <- plug_in_estimator(bs_sample$A,
                                           bs_sample$B,
                                           est = "theta_1")
  theta_2_hat_star[b] <- plug_in_estimator(bs_sample$A,
                                           bs_sample$B,
                                           est = "theta_2")
}

theta_1_pci <- c(quantile(theta_1_hat_star,.025),
                 quantile(theta_1_hat_star,.975))
#theta_1_pci
```

end--

2.2 BC_a CI for θ_1

Compute a BC_a confidence interval for θ_1. Interpret the CI.

We are 95% confident that the true value of θ_1 is between -0.07823 and 0.12771.

CODE FILENAME: ../R/02_03_bca_logmean.R*****

```
# helper functions==================================================

bias_correction <- function(bootstrap_estimates,
                            plug_in_estimate,
                            B){
  return(qnorm(sum(bootstrap_estimates < plug_in_estimate)/B))
}

acceleration_parameter <- function(data = spatial_test_data,
                                   est = "theta_1"){
  for (i in n1) {
    summ <- ((sum(
      plug_in_estimator(data$A[-i],
                        data$B[-i],
                        est = est))/n1) - plug_in_estimator(data$A[-i],
                                                            data$B[-i],
                                                            est = est))
    return(sum(summ^3) / (6*((sum(summ)^2))^(3/2)))
  }
}

alpha <- function(confid = 0.95,
                  est = "theta_1",
                  bootstrap_estimates = theta_1_hat_star,
                  plug_in_estimate = plug_in_estimate_theta_1){

  bc <- bias_correction(bootstrap_estimates = bootstrap_estimates,
                        plug_in_estimate = plug_in_estimate,
                        B = B)

  ap <- acceleration_parameter(data = spatial_test_data,
                               est = est)

  return(pnorm(bc + (bc + qnorm(confid))/(1-(ap*(bc + qnorm(confid))))))
}

BC_a <- function(confid = .95,
                 est = "theta_1",
                 bootstrap_estimates = theta_1_hat_star,
                 plug_in_estimate = plug_in_estimate_theta_1
){
  return(c(quantile(bootstrap_estimates,
                    alpha(1-confid, est = est,
                          plug_in_estimate = plug_in_estimate)),
```

```
            quantile(bootstrap_estimates,
                     alpha(confid, est = est,
                           plug_in_estimate = plug_in_estimate))))
}

# BCa for logmean computation ============================================

plug_in_estimate_theta_1 <- plug_in_estimator(spatial_test_data$A,
                                              spatial_test_data$B,
                                              est = "theta_1")

theta_1_BCa <- BC_a(confid = .95,
                    est = "theta_1",
                    bootstrap_estimates = theta_1_hat_star,
                    plug_in_estimate = plug_in_estimate_theta_1)

#theta_1_BCa
```

end---

2.3 BC_a CI for θ_2

Compute a BC_a confidence interval for θ_2.

We are 95% confident that the true value of θ_2 is between -0.26296 and 0.00192.

CODE FILENAME: ../R/02_05_bca_meanlog.R*****

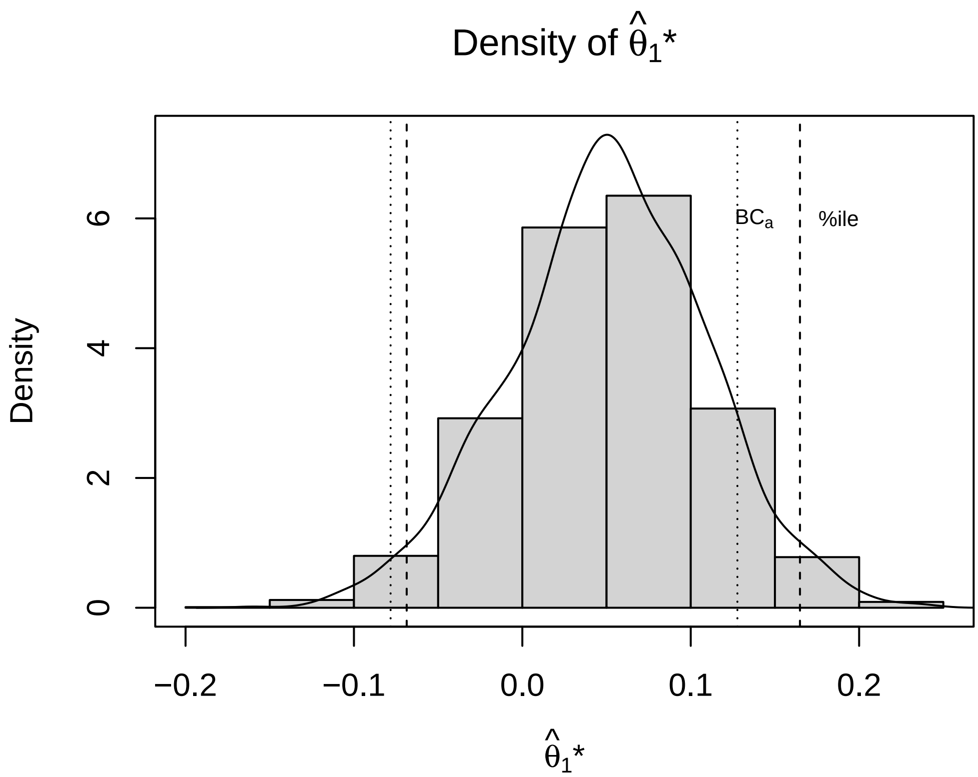
```
plug_in_estimate_theta_2 <- plug_in_estimator(spatial_test_data$A,
                                              spatial_test_data$B,
                                              est = "theta_2")

theta_2_BCa <- BC_a(confid = .95,
                    est = "theta_2",
                    bootstrap_estimates = theta_2_hat_star,
                    plug_in_estimate = plug_in_estimate_theta_2)

#theta_2_BCa
```

end---

2.4 Bootstrap percentile vs BC_a CI for θ_1

Compare your CIs in 2.1 and 2.2. How different are the two CIs?

The CI based on the BC_a is narrower than that of the bootstrap percentile CI. Also, the endpoints of the former are found to the left of the latter. The percentile CI captures the bulk of the statistics.



CODE FILENAME: ../R/02_06_compare_ci.R*****

```
hist(theta_1_hat_star,
     prob = TRUE,
     main = expression(paste("Density of ", hat(theta)[1], "*")),
     xlab = expression(paste(hat(theta)[1], "*")),
     ylim = range(density(theta_1_hat_star)$y, density(theta_1_hat_star)$y))

lines(density(theta_1_hat_star))

box(col = "black")

abline(v=theta_1_pci[1],lty="dashed")
abline(v=theta_1_pci[2],lty="dashed")
text(theta_1_pci[2]+0.023, 6, expression(paste("%", ile)), cex=0.7)

abline(v=theta_1_BCa[1],lty="dotted")
abline(v=theta_1_BCa[2],lty="dotted")
text(theta_1_BCa[2]+0.01, 6, expression(BC[a]), cex=0.7)
```

end---

3 References

Efron, B., & Tibshirani, R. (1993). *An introduction to the bootstrap.* Chapman & Hall/CRC.

4 Appendix

4.1 Code to read data for item 2

CODE FILENAME: ../R/02_01_load_data.R

```
data <- "../../../problems/ps_02/datasets/spatial_test_data.RData"

if (file.exists(data)) {
  print(paste(c("The file exists; loading", data), collapse = ' '))
  load(data)
} else {
  paste(c("The file does not exist; creating, loading and saving", data),
        collapse = ' ')
  spatial_test_data <- data.frame(
    'i' = 1:25,
    'A' = c(48, 36, 20, 29, 42, 42, 20, 42, 22, 41, 45, 14, 6,
            33, 28, 34, 4, 32, 24, 47, 41, 24, 26, 30, 41),
    'B' = c(42, 33, 16, 39, 38, 36, 15, 33, 20, 43, 34, 22, 7,
            34, 29, 41, 13, 38, 25, 27, 41, 28, 14, 28, 40)
  )
  n1 <- dim(spatial_test_data)[1]
  seed <- 7

  save(spatial_test_data, seed, n1, file=data)
}

rm(data)
```