

# Joint Confidence Regions for Rankings based on Correlated Estimates

Matala, Shaine Rosewel

University of the Philippines

December 11, 2025

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- Example 30 day mortality rate in hospitals, mean travel time to work by Klein
- Such tables motivate "implicit" rankings.
- Because rankings based on the observed values of  $\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_K$  can vary because of sampling variability, widely understood statements of uncertainty should accompany each released ranking.
- While the margin of error gives uncertainty in the estimate  $\hat{\theta}_k$  for each unit  $k$  separately, A direct assessment of the uncertainty in the estimated overall ranking would jointly involve all units and their relative standing to each other.

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Let  $r_1, r_2, \dots, r_K$  be the true unknown ranks of  $\theta_1, \theta_2, \dots, \theta_K$ . A mathematical definition of  $r_k$  is as follows:

$$r_k = \sum_{j=1}^K I(\theta_j \leq \theta_k) = 1 + \sum_{j:j \neq k} I(\theta_j \leq \theta_k), \quad \text{for } k = 1, 2, \dots, K. \quad (1)$$

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- Example

$k$	$\hat{\theta}_k$	$\hat{r}_k$
1		
2		
3		
4		
5		

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k	$\hat{\theta}_k$	$\hat{r}_k$
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2	17	
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$k$	$\hat{\theta}_k$	$\hat{r}_k$
1	12	1
2	17	2
3	19	5
4	18	3
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$$\rightarrow (\hat{\theta}_{(1)}, \hat{\theta}_{(2)}, \hat{\theta}_{(3)}, \hat{\theta}_{(4)}, \hat{\theta}_{(5)})' = (12, 17, 18, 19, 19)'$$

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Example

1	$\hat{\theta}_1$	$\hat{r}_1$
1	12	1
2	17	2
3	19	5
4	18	3
5	19	5

$$\rightarrow (\hat{\theta}_{(1)}, \hat{\theta}_{(2)}, \hat{\theta}_{(3)}, \hat{\theta}_{(4)}, \hat{\theta}_{(5)})' = (12, 17, 18, 19, 19)'$$

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└ Klein's

Suppose that for each  $k \in \{1, 2, \dots, K\}$  there exists values  $L_k$  and  $U_k$  ST

$$\theta_k \in (L_k, U_k), k = 1, 2, \dots, K. \quad (2)$$

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$$\theta_k \in (L_k, U_k), k = 1, 2, \dots, K. \quad (2)$$

If the condition in (2) holds, the main result from Klein et al. (2020) gives a range for the value of  $r_k$  for each  $k \in \{1, 2, \dots, K\}$  as follows:

$$r_k \in \{|\Lambda_{Lk}| + 1, |\Lambda_{Lk}| + 2, |\Lambda_{Lk}| + 3, \dots, |\Lambda_{Lk}| + |\Lambda_{Ok}| + 1\} \quad (3)$$

where

$$\Lambda_{Lk} = \{j \in I_k \mid U_j \leq L_k\} \quad (4)$$

$$\Lambda_{Ok} = \{j \in I_k \mid U_j > L_k \text{ and } U_k > L_j\} \quad (5)$$

$$I_k = \{1, 2, \dots, K\} \setminus \{k\} \quad (6)$$

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$k$	$\hat{\theta}_k$	$\hat{r}_k$	$(L_k, U_k)$	$ \Lambda_{L_k} $	$ \Lambda_{O_k} $	range
1	12	1				
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$k$	$\hat{\theta}_k$	$\hat{r}_k$	$(L_k, U_k)$	$ \Lambda_{L_k} $	$ \Lambda_{O_k} $	range
1	12	1	(11, 14)			
2	17	2				
3	19	5				
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1	12	1	(11, 14)	0		
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2	17	2	(14, 21)	1	3	
3	19	5	(18, 20)	2		
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Suppose that for random quantities  $L_k$  and  $U_k$  the event defined in (2) satisfies the following probability condition:

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- Name recall is a powerful predictor of likely victory in elections.

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- top-ranked candidates is composed of people who can take the most advantage of name recall: All belong to at least one of the following types: media celebrity, member of political dynasty, or had prior experience in the Senate (labeled henceforth Celebrities and Dynasties). Of the eight candidates, three are former movie and television actors, three are offspring of former presidents and senators, and six had prior experience in the Senate. They come from different political parties and different tickets
- This set of candidates was aggressively campaigned alongside candidate for President Aquino, who was popular throughout the election season and eventually won by a 12

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Travel  
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Time

Common Locations

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# Proposed methodology to compute the joint confidence region for the unordered parameters:

## Algorithm 1

Let the data be represented by  $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_K)'$  and suppose that  $\Sigma$  is known

- 1: **for**  $b = 1, 2, \dots, B$  **do**
- 2:     Generate  $\hat{\theta}_b^* \sim N_K(\hat{\theta}, \Sigma)$  and write  $\hat{\theta}_b^* = (\hat{\theta}_{b1}^*, \hat{\theta}_{b2}^*, \dots, \hat{\theta}_{bK}^*)'$
- 3:     Compute

$$t_b^* = \max_{1 \leq k \leq K} \left| \frac{\hat{\theta}_{bk}^* - \hat{\theta}_k}{\sigma_k} \right|$$

- 4: **end for**
- 5: Compute the  $(1 - \alpha)$ -sample quantile of  $t_1^*, t_2^*, \dots, t_B^*$ , call this  $\hat{t}$ .
- 6: The joint confidence region for  $\theta = (\theta_1, \theta_2, \dots, \theta_K)'$  is given by  
$$\mathfrak{R}_1 = [\hat{\theta}_1 \pm \hat{t} \times \sigma_1] \times [\hat{\theta}_2 \pm \hat{t} \times \sigma_2] \times \dots \times [\hat{\theta}_K \pm \hat{t} \times \sigma_K].$$

## Joint Confidence Regions for Rankings

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- └ Proposed methodology to compute the joint confidence region for the unordered parameters:

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## Joint Confidence Regions for Rankings

We want the joint confidence region in (9) to satisfy the following probability condition:

$$P(\hat{\theta}_k - t \cdot \sigma_k \leq \theta_k \leq \hat{\theta}_k + t \cdot \sigma_k, \forall k = 1, 2, \dots, K) = 1 - \alpha. \quad (10)$$

Equivalently, we require

$$P\left(\max_{k=1,2,\dots,K} \left|\frac{\hat{\theta}_k - \theta_k}{\sigma_k}\right| \leq t\right) = 1 - \alpha. \quad (11)$$

## Algorithm 1: Quantile Calculation

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Algorithm 1: Quantile Calculation

# Proposed methodology to compute a joint confidence region for the ordered parameters

1: **for**  $b = 1, 2, \dots, B$  **do**  
2:   Generate  $\hat{\theta}_b^* = (\hat{\theta}_{b1}^*, \hat{\theta}_{b2}^*, \dots, \hat{\theta}_{bK}^*)'$   $\sim N_K(\hat{\theta}, \Sigma)$  and let  
    $\hat{\theta}_{b(1)}^*, \hat{\theta}_{b(2)}^*, \dots, \hat{\theta}_{b(K)}^*$  be the corresponding ordered values  
3:   Compute  $\hat{\sigma}_{b(k)}^*$  using:

- asymptotic variance definition
- second-level bootstrap

4:   Compute

$$t_b^* = \max_{1 \leq k \leq K} \left| \frac{\hat{\theta}_{b(k)}^* - \hat{\theta}_{(k)}}{\hat{\sigma}_{b(k)}^*} \right|$$

5: **end for**

6: Compute the  $(1 - \alpha)$ -sample quantile of  $t_1^*, \dots, t_B^*$ , call this  $\hat{t}$ .

7: The joint confidence region of  $\theta_{(1)}, \theta_{(2)}, \dots, \theta_{(K)}$  is given by

$$\mathfrak{R}_2 = [\hat{\theta}_{(1)} \pm \hat{t} \times \hat{\sigma}_{(1)}] \times [\hat{\theta}_{(2)} \pm \hat{t} \times \hat{\sigma}_{(2)}] \times \cdots \times [\hat{\theta}_{(K)} \pm \hat{t} \times \hat{\sigma}_{(K)}]$$

## Joint Confidence Regions for Rankings

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└ Proposed methodology to compute a joint confidence region for the ordered parameters

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```
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```

## Algorithm 2: Asymptotic Definition of Variance

$$\hat{\sigma}_{b(k)}^* = \sqrt{\left[ \text{kth ordered value among } \{ \hat{\theta}_{b1}^{*2} + \sigma_1^2, \dots, \hat{\theta}_{bK}^{*2} + \sigma_K^2 \} \right] - \hat{\theta}_{(k)}^{*2}}$$

### Joint Confidence Regions for Rankings

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## Joint Confidence Regions for Rankings

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└ Algorithm 3: Variance from Second-Level Bootstrap

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```

1: for  $c = 1, 2, \dots, C$  do
2:   Generate  $\hat{\theta}_{bc}^{**} = (\hat{\theta}_{bc1}^{**}, \hat{\theta}_{bc2}^{**}, \dots, \hat{\theta}_{bcK}^{**}) \sim N_K(\hat{\theta}_b^*, \Sigma)$  and let
    $\hat{\theta}_{bc(1)}^{**}, \hat{\theta}_{bc(2)}^{**}, \dots, \hat{\theta}_{bc(K)}^{**}$  be the corresponding ordered values of
    $\hat{\theta}_{bc1}^{**}, \hat{\theta}_{bc2}^{**}, \dots, \hat{\theta}_{bcK}^{**}$ 
3:   Compute  $\hat{\sigma}_{b(k)}^* = \frac{\sum_{c=1}^C (\hat{\theta}_{bc(k)}^{**} - \bar{\hat{\theta}}_{b\cdot(k)}^{**})^2}{C - 1}$ ,  $\bar{\hat{\theta}}_{b\cdot(k)}^{**} = \frac{1}{C} \sum_{c=1}^C \hat{\theta}_{bc(k)}^{**}$ 
4: end for

```

```

1: for  $c = 1, 2, \dots, C$  do
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# Evaluation Algorithm

For given values of  $\Sigma$  and  $\theta_1, \theta_2, \dots, \theta_K$  (with corresponding  $\theta_{(1)}, \theta_{(2)}, \dots, \theta_{(K)}$  for rank-based methods)

- 1: **for** replications = 1, 2, ..., 5000 **do**
- 2:     Generate  $\hat{\theta} \sim N_K(\theta, \Sigma)$
- 3:     Compute the confidence region  $\mathfrak{R}_1$  for the unordered parameters using Algorithm 1 and the confidence region for the ordered parameters  $\mathfrak{R}_2$  using Algorithms 2 and 3.
- 4:     For the unordered parameters, check if  $(\theta_1, \theta_2, \dots, \theta_K) \in \mathfrak{R}_1$  and compute  $T_1$ ,  $T_2$ , and  $T_3$ . For the ordered parameters, check if  $(\theta_{(1)}, \theta_{(2)}, \dots, \theta_{(K)}) \in \mathfrak{R}_2$
- 5: **end for**
- 6: Compute the proportion of times that the condition in line 4 is satisfied and the average of  $T_1$ ,  $T_2$ , and  $T_3$ .

## Joint Confidence Regions for Rankings

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### └ Evaluation Algorithm

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```

# Covariance Matrix $\Sigma$

- The covariance matrix  $\Sigma$  need not be a diagonal matrix.
- We assume that  $V(\hat{\theta}) = \Sigma$  is known and express  $\Sigma$  as in (12), where  $\mathbf{R}$  is the population correlation matrix.

$$\Sigma = \Delta^{1/2} \mathbf{R} \Delta^{1/2}; \quad \Delta = \text{diag} \left\{ \sigma_1^2, \sigma_2^2, \dots, \sigma_K^2 \right\}. \quad (12)$$

- The diagonal elements of  $\Sigma$ , which are  $\sigma_k^2 = V(\hat{\theta}_k)$  for  $k = 1, 2, \dots, K$ , are treated as known quantities in practice.

## Joint Confidence Regions for Rankings

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## Joint Confidence Regions for Rankings

We assume certain correlation structures among the  $\hat{\theta}$ s.

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## └ Correlation Structures: Equicorrelation

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- This assumes that the  $k$  variables are equally correlated, i.e., that  $\rho_{jk} = \rho$  where  $\rho \in [-1, 1]$  for  $j \neq k \in \{1, \dots, K\}$ .

$$\mathbf{R}_{\text{eq}} = (1 - \rho) \mathbf{I}_K + \rho \mathbf{1}_K \mathbf{1}'_K = \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix}_{K \times K} \quad (13)$$

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- Useful in the context of pre-election surveys

## Joint Confidence Regions for Rankings

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- In a block correlation matrix  $\mathbf{R}_{block}$  with  $G$  blocks, each diagonal block represents an equicorrelation structure within group  $g$ , denoted by

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where  $\rho_g$  is the within-block correlation and  $n_g$  is the number of variables in block  $g$  such that  $\sum_{g=1}^G n_g = K$ .

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- The full block correlation matrix can be expressed as in (14).

$$\mathbf{R}_{block} = \begin{bmatrix} \mathbf{R}_{eq,1} & \mathbf{C}_{12} & \cdots & \mathbf{C}_{1G} \\ \mathbf{C}_{21} & \mathbf{R}_{eq,2} & \cdots & \mathbf{C}_{2G} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_{G1} & \mathbf{C}_{G2} & \cdots & \mathbf{R}_{eq,G} \end{bmatrix}_{K \times K} \quad (14)$$

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# Correlation Structures: Distance-based correlation

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## Joint Confidence Regions for Rankings

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$$\rho_{\text{matern}} = \frac{2^{1-\nu}}{\Gamma(\nu)} (\kappa \| s_i - s_j \|)^\nu K_\nu(\kappa \| s_i - s_j \|)$$

where  $\| \cdot \|$  denotes the Euclidean distance and  $K_\nu$  is the second kind of the modified Bessel function. It has a scale parameter  $\kappa > 0$  and a smoothness parameter  $\nu > 0$ .  $\rho_{\text{matern}}$  reduces to the exponential correlation when  $\nu = 0.5$  and to Gaussian correlation function when  $\nu = \infty$ .

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## Simulation settings

The following settings are considered for  $K = \{10, 20, 30, 40, 50\}$ . Each  $K$  has a set of corresponding population variances  $\sigma_1^2, \sigma_2^2, \dots, \sigma_K^2$ . A nominal level of  $1 - \alpha = 0.95$  will be used.

$\theta$	Correlation matrix
low variability	$\mathbf{R}_{\text{eq}}, \mathbf{R}_{\text{block}}$
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