# Joint Confidence Regions for Rankings based on Correlated Estimates

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- In the problem of estimating ranks of several unknown real-valued parameters $\theta_1, \theta_2, \ldots, \theta_K$, it is desired to rank these $K$ parameters from smallest to largest, $\theta_{(1)} < \theta_{(2)} < \ldots < \theta_{(K)}$.





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- Example 30 day mortality rate in hospitals, mean travel time to work by Klein
- Such tables motivate "implicit" rankings.
- Because rankings based on the observed values of $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_K$ can vary because of sampling variability, widely understood statements of uncertainty should accompany each released ranking.
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$$r_k = \sum_{j=1}^{K} I(\theta_j \leq \theta_k) = 1 + \sum_{j:j\neq k} I(\theta_j \leq \theta_k), \qquad \text{for } k = 1, 2, \ldots, K.$$





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- Example

| $k$ | $\hat{\theta}_k$ | $\hat{r}_k$ |
|---|---|---|
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |





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$k$ $\hat{\theta}_k$ $\hat{r}_k$
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2025-12-14

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Klein et al. (2020) assumes that $\hat{\theta}_k \sim N(\theta_k, \sigma_k^2), k = 1, 2, \ldots, K$ where $\theta_k$ is unknown but $\sigma_k^2$ is known.





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- Note that the number of elements in the range given in (2) is $|\Lambda_{Ok}| + 1$. Since the smaller difference between $U_k$ and $L_k$ leads to a smaller $|\Lambda_{Ok}|$, narrower confidence intervals for $\theta_1, \theta_2, \ldots, \theta_K$ are desirable.

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Suppose that for each $k \in \{1, 2, \ldots, K\}$ there exists values $L_k$ and $U_k$ ST

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$$\theta_k \in (L_k, U_k), k = 1, 2, \ldots, K, \tag{1}$$

their main result gives a range for $r_k$ for each $k \in \{1, 2, \ldots, K\}$ as follows:

$$r_k \in \{|\Lambda_{Lk}| + 1, |\Lambda_{Lk}| + 2, |\Lambda_{Lk}| + 3, \ldots, |\Lambda_{Lk}| + |\Lambda_{Ok}| + 1\} \tag{2}$$

$$\begin{aligned} \text{where } I_k &= \{1, 2, \ldots, K\} \setminus \{k\} \\ \Lambda_{Lk} &= \{j \in I_k \mid U_j \leq L_k\} \\ \Lambda_{Ok} &= \{j \in I_k \mid U_j > L_k \text{ and } U_k > L_j\} \end{aligned}$$





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| $k$ | $\hat{\theta}_k$ | $\hat{r}_k$ | $(L_k, U_k)$ | $\lvert\Lambda_{Lk}\rvert$ | $\lvert\Lambda_{Ok}\rvert$ | range |
|---|---|---|---|---|---|---|
| 1 | 12.3 | 1 | | | | |
| 2 | 17.2 | 2 | | | | |
| 3 | 19.1 | 5 | | | | |
| 4 | 18.0 | 3 | | | | |
| 5 | 19.0 | 4 | | | | |





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|---|---|---|---|---|---|---|
| 1 | 12.3 | 1 | (11.0, 14.3) | | | |
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| 5 | 19.0 | 4 | (18.4, 21.0) | | | |



Joint Confidence Regions for Rankings
2025-12-14
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2025-12-14
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| 2 | 17.2 | 2 | (14.1, 21.2) | 0 | 4 | |
| 3 | 19.1 | 5 | (18.2, 20.1) | 2 | | |
| 4 | 18.0 | 3 | (17.3, 18.0) | 1 | | |
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Joint Confidence Regions for Rankings
2025-12-14
└Klein's Approach
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- One may use the Bonferroni approach to choose $t$. If such an approach is used, the choice of $t$ that would satisfy (3) is $t = z_{\alpha/2K}$.
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**Political setting — David & Legara (2015)**

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- Define $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_K)'$ and assume that $\hat{\boldsymbol{\theta}} \sim N(\boldsymbol{\theta}, \boldsymbol{\Sigma})$ where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_K)'$ is unknown and $\boldsymbol{\Sigma}$ is a known $K \times K$ positive definite matrix. The diagonal elements of $\boldsymbol{\Sigma}$ are $\sigma_1^2, \ldots, \sigma_K^2$.





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# Procedure

1. Derive simultaneous confidence intervals for $\theta_1, \theta_2, \ldots, \theta_K$ of the form

$$\mathfrak{R}_1 = [\hat{\theta}_1 \pm t \times \sigma_1] \times [\hat{\theta}_2 \pm t \times \sigma_2] \times \cdots \times [\hat{\theta}_K \pm t \times \sigma_K]. \quad (5)$$





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- bullet 2: Once the confidence intervals in (5) have been obtained, we can then use the result of Klein et al. (2020) in (4) to get the lower and upper bounds on the ranks $r_k, k = 1, 2, \ldots, K$. That is, we also get a joint confidence region for $r_1, r_2, ..., r_K$.

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# Proposed methodology to compute the joint confidence region for the unordered parameters: Algorithm 1

Let the data be represented by $\hat{\boldsymbol{\theta}} = \left(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_K\right)'$ and suppose that $\boldsymbol{\Sigma}$ is known

**for** $b = 1, 2, \ldots, B$ **do**

**end for**





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Remarks about bootstrap (Hesterberg,2021):

- Plug-in principle: We can plugin the estimator of an unknown parameter to the normal distribution
- Bootstrap is implemented by Monte Carlo sampling. Estimator of SE improves as $B$ increases, as long as sampling is *iid*.
- When sampling from estimated distribution, the resulting bootstrap distribution provides useful information about the sampling distribution of the statistic.
- Bootstrap distribution is centered at the observed statistic, not the parameter. That is, the mean of the bootstrap estimates is not $\mu$ but $\bar{x}$. As a result, we do not use the mean of the bootstrap statistic as a replacement for the original estimator (e.g., bootstrap cannot improve upon $\bar{x}$) as an estimator for $\mu$.
- Bootstrap is for estimating only the SE (it is only use to estimate parameter for some, e.g. quantiles)

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2: Generate $\hat{\boldsymbol{\theta}}^*_b \sim N_K\left(\hat{\boldsymbol{\theta}}, \boldsymbol{\Sigma}\right)$ and write $\hat{\boldsymbol{\theta}}^*_b = \left(\hat{\theta}^*_{b1}, \hat{\theta}^*_{b2}, \ldots, \hat{\theta}^*_{bK}\right)'$

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Remarks about bootstrap (Hesterberg,2021):

- Plug-in principle: We can plugin the estimator of an unknown parameter to the normal distribution
- Bootstrap is implemented by Monte Carlo sampling. Estimator of SE improves as $B$ increases, as long as sampling is *iid*.
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6: The joint confidence region for $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_K)'$ is given by

$$\mathfrak{R}_1 = \left[\hat{\theta}_1 \pm \hat{t} \times \sigma_1\right] \times \left[\hat{\theta}_2 \pm \hat{t} \times \sigma_2\right] \times \cdots \times \left[\hat{\theta}_K \pm \hat{t} \times \sigma_K\right].$$





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# Algorithm 1: Quantile Calculation

We want our joint confidence region to satisfy the probability condition:

$$P\left(\hat{\theta}_k - t \cdot \sigma_k \leq \theta_k \leq \hat{\theta}_k + t \cdot \sigma_k,\ \forall\ k = 1, 2, \ldots, K\right) = 1 - \alpha$$





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## Proposed methodology to compute a joint confidence region for the ordered parameters

**for** $b = 1, 2, \ldots, B$ **do**

**end for**





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# Algorithm 2: Asymptotic Definition of Variance

- This uses results from Chen (1976) and Dudewicz (1972) to obtain an expression of the asymptotic variance of $\hat{\theta}_{(k)}$

$$\hat{\sigma}^*_{b(k)} = \sqrt{\left[\text{kth ordered value among } \left\{\hat{\theta}^{*2}_{b1} + \sigma^2_1, \ldots, \hat{\theta}^{*2}_{bK} + \sigma^2_K\right\}\right] - \hat{\theta}^{*2}_{(k)}}$$



Joint Confidence Regions for Rankings
2025-12-14
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# Algorithm 3: Variance from Second-Level Bootstrap

- Second-level bootstrap algorithm

**for** $c = 1, 2, \ldots, C$ **do**

**end for**





Algorithm 3: Variance from Second-Level Bootstrap
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for $c = 1, 2, \ldots, C$ do
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Compute $\hat{\sigma}^{*}_{b(k)} = \sqrt{\dfrac{\sum_{c=1}^{C}\left(\hat{\theta}^{**}_{bc(k)} - \bar{\hat{\theta}}^{**}_{b\cdot(k)}\right)^2}{C-1}}$, $\bar{\hat{\theta}}^{**}_{b\cdot(k)} = \dfrac{1}{C}\sum_{c=1}^{C}\hat{\theta}^{**}_{bc(k)}$

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**end for**

In $\mathfrak{R}_2$, $\hat{\sigma}_{(k)} = \sqrt{\dfrac{\sum_{b=1}^{B}\left(\hat{\theta}^{*}_{b(k)} - \bar{\hat{\theta}}^{*}_{\cdot(k)}\right)^2}{B-1}}$, $\bar{\hat{\theta}}^{*}_{\cdot(k)} = \dfrac{1}{B}\sum_{b=1}^{B}\hat{\theta}^{*}_{b(k)}$





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# Evaluation Algorithm

For given values of $\boldsymbol{\Sigma}$ and $\theta_1, \theta_2, \ldots, \theta_K$ (with corresponding $\theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(K)}$ for ordered parameters)

**for** replications $= 1, 2, \ldots, 5000$ **do**

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For given values of $\boldsymbol{\Sigma}$ and $\theta_1, \theta_2, \ldots, \theta_K$ (with corresponding $\theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(K)}$ for ordered parameters)

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5: **end for**

6: Compute the proportion of times that the condition in line 4 is satisfied and the average of $T_1$, $T_2$, and $T_3$.





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# Measures of tightness

$$T_1 = \frac{1}{K}\sum_{k=1}^{K} \left|\Lambda_{Ok}\right|$$

$$T_2 = \prod_{k=1}^{K} \left|\Lambda_{Ok}\right|$$

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- Moreover, the tightness of the joint confidence region that results from Algorithm 1 is assessed using three summary measures: the arithmetic mean ($T_1$), geometric mean ($T_2$), and the metric $T_3$ introduced by Wright (2025)
- $OP$ denotes the total number of occupied positions in a joint confidence region out of the total number of positions $K^2$; or the sum of the differences between the upper and lower bound of the simultaneous rank intervals added by 1, for each population $k$. Higher values of $T_1$ and $T_2$ indicate wider confidence intervals and are therefore less desirable, whereas higher values of $T_3$ are preferable. $T_3$ can range from 0, indicating no tightness, to $\frac{K-1}{K}$, implying the confidence region only contains the estimated ranking which is likely the true ranking.

# Simulation settings

The following settings are considered for $K = \{10, 20, 30, 40, 50\}$. Each $K$ has a set of corresponding variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_K^2$. A nominal level of $1 - \alpha = 0.95$ will be used.

| $\boldsymbol{\theta}$ | Correlation matrix |
|---|---|
| low variability | $\mathbf{R}_{\text{eq}}$, $\mathbf{R}_{\text{block}}$ |
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- We assume certain correlation structures among the $\hat{\theta}$s.
- Equicorrelation is considered for simplicity.

# Correlation Structures: Equicorrelation

- This assumes that the $k$ variables are equally correlated, i.e., that $\rho_{jk} = \rho$ where $\rho \in [-1, 1]$ for $j \neq k \in \{1, \ldots, K\}$.

$$\mathbf{R}_{\text{eq}} = (1-\rho)\,\mathbf{I}_K + \rho \mathbf{1}_K \mathbf{1}_K' = \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix}_{K \times K}$$





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- The full block correlation matrix can be expressed as

$$\mathbf{R}_{\text{block}} = \begin{bmatrix} \mathbf{R}_{eq,1} & \mathbf{C}_{12} & \cdots & \mathbf{C}_{1G} \\ \mathbf{C}_{21} & \mathbf{R}_{eq,2} & \cdots & \mathbf{C}_{2G} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_{G1} & \mathbf{C}_{G2} & \cdots & \mathbf{R}_{eq,G} \end{bmatrix}_{K\times K}$$





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- Useful in the context of pre-election surveys
- In a block correlation matrix $\mathbf{R}_{block}$ with $G$ blocks, each diagonal block represents an equicorrelation structure within group $g$, denoted by

$$\mathbf{R}_{\text{eq,g}} = (1-\rho_g)\,\mathbf{I}_{n_g} + \rho_g \mathbf{1}_{n_g}\mathbf{1}'_{n_g}$$

  where $\rho_g$ is the within-block correlation and $n_g$ is the number of variables in block $g$ such that $\sum_{g=1}^{G} n_g = K$.
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# Correlation Structures: Distance-based correlation

- Spatial dependence can be modeled using a stationary Matérn correlation function, which for two locations $\mathbf{s}_i$ and $\mathbf{s}_j$ is expressed as

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- Useful in the mean travel time from the study of Klein et al. (2020).

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- Proposed contribution: Develop a procedure to construct joint confidence intervals for the ranks and the ranked parameters when the estimates to be ranked may be correlated.





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