Joint Confidence Regions for Rankings based on Correlated Estimates

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- In the problem of estimating ranks of several unknown real-valued parameters $\theta_1, \theta_2, \ldots, \theta_K$, it is desired to rank these K parameters from smallest to largest, $\theta_{(1)} < \theta_{(2)} < \ldots < \theta_{(K)}$.

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- Example 30 day mortality rate in hospitals, mean travel time to work by Klein
- Such tables motivate "implicit" rankings.
- Because rankings based on the observed values of $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_K$ can vary because of sampling variability, widely understood statements of uncertainty should accompany each released ranking.
- While the margin of error gives uncertainty in the estimate $\hat{\theta}_k$ for each unit k separately, A direct assessment of the uncertainty in the estimated overall ranking would jointly involve all units and their relative standing to each other.
- If we repeat the entire process of generating all K intervals many times, at least 95 percent of the time, all K of the resulting intervals will simultaneously contain their respective true parameters $\theta_1, \ldots, \theta_K$

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$$r_k = \sum_{j=1}^{K} I(\theta_j \leq \theta_k) = 1 + \sum_{j:j\neq k} I(\theta_j \leq \theta_k), \qquad \text{for } k = 1, 2, \ldots, K.$$

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k	$\hat{\theta}_k$	$\hat{r}_k$
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Klein et al. (2020) assumes that $\hat{\theta}_k \sim N(\theta_k, \sigma_k^2), k = 1, 2, \ldots, K$ where θ_k is unknown but σ_k^2 is known.

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Suppose that for each $k \in \{1, 2, \ldots, K\}$ there exists values L_k and U_k ST

$$\theta_k \in (L_k, U_k), k = 1, 2, \ldots, K, \tag{1}$$

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the range for r_k for each $k \in \{1, 2, \ldots, K\}$ is as follows:

$$r_k \in \{|\Lambda_{Lk}| + 1, |\Lambda_{Lk}| + 2, |\Lambda_{Lk}| + 3, \ldots, |\Lambda_{Lk}| + |\Lambda_{Ok}| + 1\} \tag{2}$$

$$\begin{aligned} \text{where } I_k &= \{1, 2, \ldots, K\} \setminus \{k\} \\ \Lambda_{Lk} &= \{j \in I_k \mid U_j \leq L_k\} \\ \Lambda_{Ok} &= \{j \in I_k \mid U_j > L_k \text{ and } U_k > L_j\} \end{aligned}$$

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Proposed methodology to compute the joint confidence region for the unordered parameters: Algorithm 1

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Remarks about bootstrap (Hesterberg,2021):

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  Compute

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We want our joint confidence region to satisfy the probability condition:

$$P\left(\hat{\theta}_k - t \cdot \sigma_k \leq \theta_k \leq \hat{\theta}_k + t \cdot \sigma_k,\ \forall\ k = 1, 2, \ldots, K\right) = 1 - \alpha$$

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1: **for** $b = 1, 2, \ldots, B$ **do**

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- We can say we are 95 percent confident that the smallest parameter up to the largest parameter falls between the calculated bounds.

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The joint confidence region of $\theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(K)}$ is given by

$$\mathfrak{R}_2 = \left[\hat{\theta}_{(1)} \pm \hat{t} \times \hat{\sigma}_{(1)}\right] \times \left[\hat{\theta}_{(2)} \pm \hat{t} \times \hat{\sigma}_{(2)}\right] \times \cdots \times \left[\hat{\theta}_{(K)} \pm \hat{t} \times \hat{\sigma}_{(K)}\right]$$

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Algorithm 2: Asymptotic Definition of Variance

- This uses results from Chen (1976) and Dudewicz (1972) to obtain an expression of the asymptotic variance of $\hat{\theta}_{(k)}$

$$\hat{\sigma}^*_{b(k)} = \sqrt{\left[\text{kth ordered value among } \left\{\hat{\theta}^{*2}_{b1} + \sigma^2_1, \ldots, \hat{\theta}^{*2}_{bK} + \sigma^2_K\right\}\right] - \hat{\theta}^{*2}_{(k)}}$$

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Joint Confidence Regions for Rankings
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Algorithm 3: Variance from Second-Level Bootstrap

- Second-level bootstrap algorithm

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Compute $\hat{\sigma}^{*}_{b(k)} = \sqrt{\dfrac{\sum_{c=1}^{C}\left(\hat{\theta}^{**}_{bc(k)} - \bar{\hat{\theta}}^{**}_{b\cdot(k)}\right)^2}{C-1}}$, $\bar{\hat{\theta}}^{**}_{b\cdot(k)} = \dfrac{1}{C}\sum_{c=1}^{C}\hat{\theta}^{**}_{bc(k)}$

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4: **end for**

In $\mathfrak{R}_2$, $\hat{\sigma}_{(k)} = \sqrt{\dfrac{\sum_{b=1}^{B}\left(\hat{\theta}^{*}_{b(k)} - \bar{\hat{\theta}}^{*}_{\cdot(k)}\right)^2}{B-1}}$, $\bar{\hat{\theta}}^{*}_{\cdot(k)} = \dfrac{1}{B}\sum_{b=1}^{B}\hat{\theta}^{*}_{b(k)}$

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Evaluation Algorithm

For given values of $\boldsymbol{\Sigma}$ and $\theta_1, \theta_2, \ldots, \theta_K$ (with corresponding $\theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(K)}$ for ordered parameters)

for replications $= 1, 2, \ldots, 5000$ **do**

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Measures of tightness

$$T_1 = \frac{1}{K}\sum_{k=1}^{K}\left|\Lambda_{Ok}\right|$$

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- Moreover, the tightness of the joint confidence region that results from Algorithm 1 is assessed using three summary measures: the arithmetic mean (T_1), geometric mean (T_2), and the metric T_3 introduced by Wright (2025)
- OP denotes the total number of occupied positions in a joint confidence region out of the total number of positions K^2; or the sum of the differences between the upper and lower bound of the simultaneous rank intervals added by 1, for each population k.
- Higher values of T_1 and T_2 indicate wider confidence intervals and are therefore less desirable, whereas higher values of T_3 are preferable. T_3 can range from 0, indicating no tightness, to $\frac{K-1}{K}$, implying the confidence region only contains the estimated ranking which is likely the true ranking.

Simulation settings

The following settings are considered for $K = \{10, 20, 30, 40, 50\}$. Each K has a set of corresponding variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_K^2$. A nominal level of $1 - \alpha = 0.95$ will be used.

$\boldsymbol{\theta}$	Correlation matrix
low variability	$\mathbf{R}_{\text{eq}}$, $\mathbf{R}_{\text{block}}$
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- We assume certain correlation structures among the $\hat{\theta}$s.
- Equicorrelation is considered for simplicity.

Correlation Structures: Equicorrelation

- This assumes that the k variables are equally correlated, i.e., that $\rho_{jk} = \rho$ where $\rho \in [-1, 1]$ for $j \neq k \in \{1, \ldots, K\}$.

$$\mathbf{R}_{\text{eq}} = (1-\rho)\,\mathbf{I}_K + \rho \mathbf{1}_K \mathbf{1}_K' = \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix}_{K \times K}$$

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Correlation Structures: Block correlation

- The full block correlation matrix can be expressed as

$$\mathbf{R}_{\text{block}} = \begin{bmatrix} \mathbf{R}_{eq,1} & \mathbf{C}_{12} & \cdots & \mathbf{C}_{1G} \\ \mathbf{C}_{21} & \mathbf{R}_{eq,2} & \cdots & \mathbf{C}_{2G} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_{G1} & \mathbf{C}_{G2} & \cdots & \mathbf{R}_{eq,G} \end{bmatrix}_{K\times K}$$

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- Useful in the context of pre-election surveys
- In a block correlation matrix $\mathbf{R}_{block}$ with G blocks, each diagonal block represents an equicorrelation structure within group g, denoted by

$$\mathbf{R}_{\text{eq,g}} = (1-\rho_g)\,\mathbf{I}_{n_g} + \rho_g \mathbf{1}_{n_g}\mathbf{1}'_{n_g}$$

  where ρ_g is the within-block correlation and n_g is the number of variables in block g such that $\sum_{g=1}^{G} n_g = K$.
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Correlation Structures: Distance-based correlation

- Spatial dependence can be modeled using a stationary Matérn correlation function, which for two locations $\mathbf{s}_i$ and $\mathbf{s}_j$ is expressed as

$$\rho_{\text{matern}} = \frac{2^{1-\nu}}{\Gamma(\nu)}(\kappa \parallel \mathbf{s}_i - \mathbf{s}_j \parallel)^{\nu} K_{\nu}(\kappa \parallel \mathbf{s}_i - \mathbf{s}_j \parallel)$$

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- Useful in the mean travel time from the study of Klein et al. (2020).

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- Proposed contribution: Develop a procedure to construct joint confidence intervals for the ranks and the ranked parameters when the estimates to be ranked may be correlated.

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