

Joint Confidence Regions for Rankings based on Correlated Estimates

Matala, Shaine Rosewel

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- In the problem of estimating ranks of several unknown real-valued parameters $\theta_1, \theta_2, \dots, \theta_K$, it is desired to rank these K parameters from smallest to largest, $\theta_{(1)} < \theta_{(2)} < \dots < \theta_{(K)}$.

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- Example 30 day mortality rate in hospitals, mean travel time to work by Klein
- Such tables motivate "implicit" rankings.
- Because rankings based on the observed values of $\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_K$ can vary because of sampling variability, widely understood statements of uncertainty should accompany each released ranking.
- While the margin of error gives uncertainty in the estimate $\hat{\theta}_k$ for each unit k separately, A direct assessment of the uncertainty in the estimated overall ranking would jointly involve all units and their relative standing to each other.

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$$r_k = \sum_{j=1}^K I(\theta_j \leq \theta_k) = 1 + \sum_{j:j \neq k} I(\theta_j \leq \theta_k), \quad \text{for } k = 1, 2, \dots, K. \tag{1}$$

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- Example

k	$\hat{\theta}_k$	\hat{r}_k
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2		
3		
4		
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$$\longrightarrow \left(\hat{\theta}_{(1)}, \hat{\theta}_{(2)}, \hat{\theta}_{(3)}, \hat{\theta}_{(4)}, \hat{\theta}_{(5)} \right)' = (12, 17, 18, 19, 19)'$$

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Suppose that for each $k \in \{1, 2, \dots, K\}$ there exists values L_k and U_k ST

$$\theta_k \in (L_k, U_k), k = 1, 2, \dots, K. \tag{2}$$

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If the condition in (2) holds, the main result from Klein et al. (2020) gives a range for the value of r_k for each $k \in \{1, 2, \dots, K\}$ as follows:

$$r_k \in \{|\Lambda_{Lk}| + 1, |\Lambda_{Lk}| + 2, |\Lambda_{Lk}| + 3, \dots, |\Lambda_{Lk}| + |\Lambda_{Ok}| + 1\} \tag{3}$$

where

$$\Lambda_{Lk} = \{j \in I_k \mid U_j \leq L_k\} \tag{4}$$

$$\Lambda_{Ok} = \{j \in I_k \mid U_j > L_k \text{ and } U_k > L_j\} \tag{5}$$

$$I_k = \{1, 2, \dots, K\} \setminus \{k\} \tag{6}$$

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Example

k	$\hat{\theta}_k$	\hat{r}_k	(L_k, U_k)	$ \Lambda_{L_k} $	$ \Lambda_{O_k} $	range
1	12	1				
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k	$\hat{\theta}_k$	\hat{r}_k	(L_k, U_k)	$ \Lambda_{L_k} $	$ \Lambda_{O_k} $	range
1	12	1	(11, 14)			
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Joint Confidence Regions for Rankings

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Political setting — David & Legara (2015)

- Name recall is a powerful predictor of likely victory in elections.

- Candidates with a name-recall advantage, such as media celebrities, incumbents, and members of dynastic families, received majority of the votes in the 2010 senatorial elections
- top-ranked candidates is composed of people who can take the most advantage of name recall: All belong to at least one of the following types: media celebrity, member of political dynasty, or had prior experience in the Senate (labeled henceforth Celebrities and Dynasties). Of the eight candidates, three are former movie and television actors, three are offspring of former presidents and senators, and six had prior experience in the Senate. They come from different political parties and different tickets
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Definitions and Assumptions

- Define $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_K)'$ and assume that $\hat{\theta} \sim N(\theta, \Sigma)$ where $\theta = (\theta_1, \theta_2, \dots, \theta_K)'$ is unknown and Σ is a known $K \times K$ positive definite matrix. The diagonal elements of Σ are $\sigma_1^2, \dots, \sigma_K^2$.

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- Note that in the literature on inferences on the ranks, it is customary to assume that the variances are known.

- 1 Derive simultaneous confidence intervals for $\theta_1, \theta_2, \dots, \theta_K$ of the form

$$\mathfrak{R}_1 = [\hat{\theta}_1 \pm t \times \sigma_1] \times [\hat{\theta}_2 \pm t \times \sigma_2] \times \dots \times [\hat{\theta}_K \pm t \times \sigma_K]. \quad (9)$$

- bullet 2: That is, we also get a joint confidence region for r_1, r_2, \dots, r_K .

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- 2 Once the confidence intervals in (9) have been obtained, we can then use the result of Klein et al. (2020) in (8) to get the lower and upper bounds on the ranks $r_k, k = 1, 2, \dots, K$.

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Proposed methodology to compute the joint confidence region for the unordered parameters: Algorithm 1

Let the data be represented by $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_K)'$ and suppose that Σ is known

- 1: **for** $b = 1, 2, \dots, B$ **do**
- 2: Generate $\hat{\theta}_b^* \sim N_K(\hat{\theta}, \Sigma)$ and write $\hat{\theta}_b^* = (\hat{\theta}_{b1}^*, \hat{\theta}_{b2}^*, \dots, \hat{\theta}_{bK}^*)'$
- 3: Compute

$$t_b^* = \max_{1 \leq k \leq K} \left| \frac{\hat{\theta}_{bk}^* - \hat{\theta}_k}{\sigma_k} \right|$$

- 4: **end for**
- 5: Compute the $(1 - \alpha)$ -sample quantile of $t_1^*, t_2^*, \dots, t_B^*$, call this \hat{t} .
- 6: The joint confidence region for $\theta = (\theta_1, \theta_2, \dots, \theta_K)'$ is given by

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Joint Confidence Regions for Rankings

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Algorithm 1: Quantile Calculation

We want the joint confidence region in (9) to satisfy the following probability condition:

$$P\left(\hat{\theta}_k - t \cdot \sigma_k \leq \theta_k \leq \hat{\theta}_k + t \cdot \sigma_k, \forall k = 1, 2, \dots, K\right) = 1 - \alpha. \tag{10}$$

Equivalently, we require

$$P\left(\max_{k=1,2,\dots,K} \left| \frac{\hat{\theta}_k - \theta_k}{\sigma_k} \right| \leq t\right) = 1 - \alpha. \tag{11}$$

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Proposed methodology to compute a joint confidence region for the ordered parameters

- 1: **for** $b = 1, 2, \dots, B$ **do**
- 2: Generate $\hat{\boldsymbol{\theta}}_b^* = (\hat{\theta}_{b1}^*, \hat{\theta}_{b2}^*, \dots, \hat{\theta}_{bK}^*)' \sim N_K(\hat{\boldsymbol{\theta}}, \boldsymbol{\Sigma})$ and let $\hat{\theta}_{b(1)}^*, \hat{\theta}_{b(2)}^*, \dots, \hat{\theta}_{b(K)}^*$ be the corresponding ordered values
- 3: Compute $\hat{\sigma}_{b(k)}^*$ using:
 - asymptotic variance definition
 - second-level bootstrap
- 4: Compute

$$t_b^* = \max_{1 \leq k \leq K} \left| \frac{\hat{\theta}_{b(k)}^* - \hat{\theta}_{(k)}}{\hat{\sigma}_{b(k)}^*} \right|$$

- 5: **end for**
- 6: Compute the $(1 - \alpha)$ -sample quantile of t_1^*, \dots, t_B^* , call this \hat{t} .
- 7: The joint confidence region of $\theta_{(1)}, \theta_{(2)}, \dots, \theta_{(K)}$ is given by

$$\mathfrak{R}_2 = [\hat{\theta}_{(1)} \pm \hat{t} \times \hat{\sigma}_{(1)}] \times [\hat{\theta}_{(2)} \pm \hat{t} \times \hat{\sigma}_{(2)}] \times \dots \times [\hat{\theta}_{(K)} \pm \hat{t} \times \hat{\sigma}_{(K)}]$$

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Joint Confidence Regions for Rankings

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Algorithm 2: Asymptotic Definition of Variance

$$\hat{\sigma}_{b(k)}^* = \sqrt{\left[\text{kth ordered value among } \left\{ \hat{\theta}_{b1}^{*2} + \sigma_1^2, \dots, \hat{\theta}_{bK}^{*2} + \sigma_K^2 \right\} \right] - \hat{\theta}_{(k)}^{*2}}$$

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Algorithm 3: Variance from Second-Level Bootstrap

- 1: **for** $c = 1, 2, \dots, C$ **do**
- 2: Generate $\hat{\theta}_{bc}^{**} = (\hat{\theta}_{bc1}^{**}, \hat{\theta}_{bc2}^{**}, \dots, \hat{\theta}_{bcK}^{**}) \sim N_K(\hat{\theta}_b^*, \Sigma)$ and let $\hat{\theta}_{bc(1)}^{**}, \hat{\theta}_{bc(2)}^{**}, \dots, \hat{\theta}_{bc(K)}^{**}$ be the corresponding ordered values of $\hat{\theta}_{bc1}^{**}, \hat{\theta}_{bc2}^{**}, \dots, \hat{\theta}_{bcK}^{**}$
- 3: Compute $\hat{\sigma}_{b(k)}^* = \frac{\sum_{c=1}^C (\hat{\theta}_{bc(k)}^{**} - \bar{\hat{\theta}}_{b \cdot (k)}^{**})^2}{C - 1}$, $\bar{\hat{\theta}}_{b \cdot (k)}^{**} = \frac{1}{C} \sum_{c=1}^C \hat{\theta}_{bc(k)}^{**}$
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Joint Confidence Regions for Rankings

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Evaluation Algorithm

For given values of Σ and $\theta_1, \theta_2, \dots, \theta_K$ (with corresponding $\theta_{(1)}, \theta_{(2)}, \dots, \theta_{(K)}$ for rank-based methods)

- 1: **for** replications = 1, 2, ..., 5000 **do**
- 2: Generate $\hat{\theta} \sim N_K(\theta, \Sigma)$
- 3: Compute the confidence region \mathfrak{R}_1 for the unordered parameters using Algorithm 1 and the confidence region for the ordered parameters \mathfrak{R}_2 using Algorithms 2 and 3.
- 4: For the unordered parameters, check if $(\theta_1, \theta_2, \dots, \theta_K) \in \mathfrak{R}_1$ and compute T_1, T_2 , and T_3 . For the ordered parameters, check if $(\theta_{(1)}, \theta_{(2)}, \dots, \theta_{(K)}) \in \mathfrak{R}_2$
- 5: **end for**
- 6: Compute the proportion of times that the condition in line 4 is satisfied and the average of T_1, T_2 , and T_3 .

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Covariance Matrix Σ

- The covariance matrix Σ need not be a diagonal matrix.
- We assume that $V(\hat{\theta}) = \Sigma$ is known and express Σ as in (12), where \mathbf{R} is the population correlation matrix.

$$\Sigma = \Delta^{1/2} \mathbf{R} \Delta^{1/2}; \quad \Delta = \text{diag} \left\{ \sigma_1^2, \sigma_2^2, \dots, \sigma_K^2 \right\}. \quad (12)$$

- The diagonal elements of Σ , which are $\sigma_k^2 = V(\hat{\theta}_k)$ for $k = 1, 2, \dots, K$, are treated as known quantities in practice.

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Correlation Structures: Equicorrelation

- We assume certain correlation structures among the $\hat{\theta}$ s.

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$$\mathbf{R}_{\text{eq}} = (1 - \rho) \mathbf{I}_K + \rho \mathbf{1}_K \mathbf{1}_K' = \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix}_{K \times K} \quad (13)$$

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- In a block correlation matrix \mathbf{R}_{block} with G blocks, each diagonal block represents an equicorrelation structure within group g , denoted by

$$\mathbf{R}_{eq,g} = (1 - \rho_g) \mathbf{I}_{n_g} + \rho_g \mathbf{1}_{n_g} \mathbf{1}_{n_g}'$$

where ρ_g is the within-block correlation and n_g is the number of variables in block g such that $\sum_{g=1}^G n_g = K$.

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$$\rho_{\text{matern}} = \frac{2^{1-\nu}}{\Gamma(\nu)} (\kappa \|\mathbf{s}_i - \mathbf{s}_j\|)^\nu K_\nu(\kappa \|\mathbf{s}_i - \mathbf{s}_j\|)$$

where $\|\cdot\|$ denotes the Euclidean distance and K_ν is the second kind of the modified Bessel function. It has a scale parameter $\kappa > 0$ and a smoothness parameter $\nu > 0$. ρ_{matern} reduces to the exponential correlation when $\nu = 0.5$ and to Gaussian correlation function when $\nu = \infty$.

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Simulation settings

The following settings are considered for $K = \{10, 20, 30, 40, 50\}$. Each K has a set of corresponding population variances $\sigma_1^2, \sigma_2^2, \dots, \sigma_K^2$. A nominal level of $1 - \alpha = 0.95$ will be used.

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