Joint Confidence Regions for Rankings based on Correlated Estimates

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University of the Philippines

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- In the problem of estimating ranks of several unknown real-valued parameters $\theta_1, \theta_2, \ldots, \theta_K$, it is desired to rank these K parameters from smallest to largest, $\theta_{(1)} < \theta_{(2)} < \ldots < \theta_{(K)}$.

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- Example 30 day mortality rate in hospitals, mean travel time to work by Klein
- Such tables motivate "implicit" rankings.
- Because rankings based on the observed values of $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_K$ can vary because of sampling variability, widely understood statements of uncertainty should accompany each released ranking.
- While the margin of error gives uncertainty in the estimate $\hat{\theta}_k$ for each unit k separately, A direct assessment of the uncertainty in the estimated overall ranking would jointly involve all units and their relative standing to each other.

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Klein's

Suppose that for each $k \in \{1, 2, \ldots, K\}$ there exists values L_k and U_k ST

$$\theta_k \in (L_k, U_k), k = 1, 2, \ldots, K. \tag{2}$$

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If the condition in (2) holds, the main result from Klein et al. (2020) gives a range for the value of r_k for each $k \in \{1, 2, \ldots, K\}$ as follows:

$$r_k \in \{|\Lambda_{Lk}| + 1, |\Lambda_{Lk}| + 2, |\Lambda_{Lk}| + 3, \ldots, |\Lambda_{Lk}| + |\Lambda_{Ok}| + 1\} \tag{3}$$

where

$$\Lambda_{Lk} = \{j \in I_k \mid U_j \leq L_k\} \tag{4}$$
$$\Lambda_{Ok} = \{j \in I_k \mid U_j > L_k \text{ and } U_k > L_j\} \tag{5}$$
$$I_k = \{1, 2, \ldots, K\} \setminus \{k\} \tag{6}$$

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2025-12-11

Joint Confidence Regions for Rankings

└─Example

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Suppose that for random quantities L_k and U_k the event defined in (2) satisfies the following probability condition:

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- top-ranked candidates is composed of people who can take the most advantage of name recall: All belong to at least one of the following types: media celebrity, member of political dynasty, or had prior experience in the Senate (labeled henceforth Celebrities and Dynasties). Of the eight candidates, three are former movie and television actors, three are offspring of former presidents and senators, and six had prior experience in the Senate. They come from different political parties and different tickets
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Definitions and Assumptions

- Define $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_K)'$ and assume that $\hat{\boldsymbol{\theta}} \sim N(\boldsymbol{\theta}, \boldsymbol{\Sigma})$ where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_K)'$ is unknown and $\boldsymbol{\Sigma}$ is a known $K \times K$ positive definite matrix. The diagonal elements of $\boldsymbol{\Sigma}$ are $\sigma_1^2, \ldots, \sigma_K^2$.

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Procedure

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$$\mathfrak{R}_1 = [\hat{\theta}_1 \pm t \times \sigma_1] \times [\hat{\theta}_2 \pm t \times \sigma_2] \times \cdots \times [\hat{\theta}_K \pm t \times \sigma_K]. \quad (9)$$

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Proposed methodology to compute the joint confidence region for the unordered parameters: Algorithm 1

Let the data be represented by $\hat{\boldsymbol{\theta}} = \left(\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_K\right)'$ and suppose that $\boldsymbol{\Sigma}$ is known

for $b = 1, 2, \dots, B$ **do**

  Generate $\hat{\boldsymbol{\theta}}_b^* \sim N_K\left(\hat{\boldsymbol{\theta}}, \boldsymbol{\Sigma}\right)$ and write $\hat{\boldsymbol{\theta}}_b^* = \left(\hat{\theta}_{b1}^*, \hat{\theta}_{b2}^*, \dots, \hat{\theta}_{bK}^*\right)'$

  Compute

$$t_b^* = \max_{1 \le k \le K} \left| \frac{\hat{\theta}_{bk}^* - \hat{\theta}_k}{\sigma_k} \right|$$

end for

Compute the $(1-\alpha)$-sample quantile of $t_1^*, t_2^*, \dots, t_B^*$, call this $\hat{t}$.

The joint confidence region for $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_K)'$ is given by

$$\mathfrak{R}_1 = \left[\hat{\theta}_1 \pm \hat{t} \times \sigma_1\right] \times \left[\hat{\theta}_2 \pm \hat{t} \times \sigma_2\right] \times \cdots \times \left[\hat{\theta}_K \pm \hat{t} \times \sigma_K\right].$$

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Algorithm 1: Quantile Calculation

We want the joint confidence region in (9) to satisfy the following probability condition:

$$P\left(\hat{\theta}_k - t \cdot \sigma_k \leq \theta_k \leq \hat{\theta}_k + t \cdot \sigma_k,\ \forall\, k = 1, 2, \ldots, K\right) = 1 - \alpha. \tag{10}$$

Equivalently, we require

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Proposed methodology to compute a joint confidence region for the ordered parameters

for $b = 1, 2, \ldots, B$ **do**

  Generate $\hat{\boldsymbol{\theta}}_b^* = \left(\hat{\theta}_{b1}^*, \hat{\theta}_{b2}^*, \ldots, \hat{\theta}_{bK}^*\right)' \sim N_K\left(\hat{\boldsymbol{\theta}}, \boldsymbol{\Sigma}\right)$ and let $\hat{\theta}_{b(1)}^*, \hat{\theta}_{b(2)}^*, \ldots, \hat{\theta}_{b(K)}^*$ be the corresponding ordered values

  Compute $\hat{\sigma}_{b(k)}^*$ using:

  - asymptotic variance definition
  - second-level bootstrap

  Compute

$$t_b^* = \max_{1 \le k \le K} \left| \frac{\hat{\theta}_{b(k)}^* - \hat{\theta}_k^*}{\hat{\sigma}_{b(k)}^*} \right|$$

end for

Compute the $(1 - \alpha)$-sample quantile of $t_1^*, \ldots, t_B^*$, call this $\hat{t}$.

The joint confidence region of $\theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(K)}$ is given by

$$\mathfrak{R}_2 = \left[\hat{\theta}_{(1)} \pm \hat{t} \times \hat{\sigma}_{(1)}\right] \times \left[\hat{\theta}_{(2)} \pm \hat{t} \times \hat{\sigma}_{(2)}\right] \times \cdots \times \left[\hat{\theta}_{(K)} \pm \hat{t} \times \hat{\sigma}_{(K)}\right]$$

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Algorithm 2: Asymptotic Definition of Variance

$$\hat{\sigma}^*_{b(k)} = \sqrt{\left[\text{kth ordered value among } \left\{\hat{\theta}^{*2}_{b1} + \sigma^2_1, \ldots, \hat{\theta}^{*2}_{bK} + \sigma^2_K\right\}\right] - \hat{\theta}^{*2}_{(k)}}$$

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Algorithm 3: Variance from Second-Level Bootstrap

for $c = 1, 2, \ldots, C$ **do**

Generate $\hat{\boldsymbol{\theta}}^{**}_{bc} = \left(\hat{\theta}^{**}_{bc1}, \hat{\theta}^{**}_{bc2}, \ldots, \hat{\theta}^{**}_{bcK}\right) \sim N_K\left(\hat{\boldsymbol{\theta}}^{*}_{b}, \boldsymbol{\Sigma}\right)$ and let $\hat{\theta}^{**}_{bc(1)}, \hat{\theta}^{**}_{bc(2)}, \ldots, \hat{\theta}^{**}_{bc(K)}$ be the corresponding ordered values of $\hat{\theta}^{**}_{bc1}, \hat{\theta}^{**}_{bc2}, \ldots, \hat{\theta}^{**}_{bcK}$

Compute $\hat{\sigma}^{*}_{b(k)} = \dfrac{\sum_{c=1}^{C}\left(\hat{\theta}^{**}_{bc(k)} - \bar{\hat{\theta}}^{**}_{b\cdot(k)}\right)^2}{C-1}, \quad \bar{\hat{\theta}}^{**}_{b\cdot(k)} = \dfrac{1}{C}\sum_{c=1}^{C}\hat{\theta}^{**}_{bc(k)}$

end for

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Evaluation Algorithm

For given values of $\boldsymbol{\Sigma}$ and $\theta_1, \theta_2, \ldots, \theta_K$ (with corresponding $\theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(K)}$ for rank-based methods)

1: **for** replications $= 1, 2, \ldots, 5000$ **do**
2: Generate $\hat{\boldsymbol{\theta}} \sim N_K(\boldsymbol{\theta}, \boldsymbol{\Sigma})$
3: Compute the confidence region $\mathfrak{R}_1$ for the unordered parameters using Algorithm 1 and the confidence region for the ordered parameters $\mathfrak{R}_2$ using Algorithms 2 and 3.
4: For the unordered parameters, check if $(\theta_1, \theta_2, \ldots, \theta_K) \in \mathfrak{R}_1$ and compute T_1, T_2, and T_3. For the ordered parameters, check if $\left(\theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(K)}\right) \in \mathfrak{R}_2$
5: **end for**
6: Compute the proportion of times that the condition in line 4 is satisfied and the average of T_1, T_2, and T_3.

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Covariance Matrix $\boldsymbol{\Sigma}$

- The covariance matrix $\boldsymbol{\Sigma}$ need not be a diagonal matrix.
- We assume that $V(\hat{\boldsymbol{\theta}}) = \boldsymbol{\Sigma}$ is known and express $\boldsymbol{\Sigma}$ as in (12), where $\mathbf{R}$ is the population correlation matrix.

$$\boldsymbol{\Sigma} = \boldsymbol{\Delta}^{1/2}\mathbf{R}\boldsymbol{\Delta}^{1/2}; \quad \boldsymbol{\Delta} = \text{diag}\left\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_K^2\right\}. \tag{12}$$

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$$\mathbf{R}_{\text{eq}} = (1-\rho)\,\mathbf{I}_K + \rho \mathbf{1}_K \mathbf{1}_K' = \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix}_{K \times K} \tag{13}$$

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where $\parallel \cdot \parallel$ denotes the Euclidean distance and K_ν is the second kind of the modified Bessel function. It has a scale parameter $\kappa > 0$ and a smoothness parameter $\nu > 0$. ρ_{matern} reduces to the exponential correlation when $\nu = 0.5$ and to Gaussian correlation function when $\nu = \infty$.

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Simulation settings

The following settings are considered for $K = \{10, 20, 30, 40, 50\}$. Each K has a set of corresponding population variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_K^2$. A nominal level of $1 - \alpha = 0.95$ will be used.

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low variability	$\mathbf{R}_{\text{eq}}$, $\mathbf{R}_{\text{block}}$
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