ASYMPTOTIC METHODS AND STATISTICAL APPLICATIONS: ASSESSMENT

H. S. Battey
Department of Mathematics, Imperial College London

DEADLINE: 20 APRIL 2020.

Please send your solution by email (h.battey@imperial.ac.uk).

Write a brief summary of the asymptotic arguments behind the paper by W. Richter (1957), Local limit theorems for large deviations, *Theory of Probability and its Applications*, 11, 206–219. A copy of the paper is attached.

Note: the deadline is a month away but I do not expect you to do more than what is reasonably achievable within a few hours.

THEORY OF PROBABILITY AND ITS APPLICATIONS

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LOCAL LIMIT THEOREMS FOR LARGE DEVIATIONS

WOLFGANG RICHTER

(*Translated by Allen L. Shields*)

1. Introduction

Suppose a sequence of independent random variables $X_1, X_2, \cdots$ is given. Their distribution functions will be denoted $V_1(x), V_2(x), \cdots$. We assume that the dispersions all exist, $\mathbf{D}X_j = \sigma_j^2$, $s_n^2 = \sum_{j=1}^n \sigma_j^2$. Without loss of generality we may assume that the mathematical expectations of the quantities X_j equal zero, $\mathbf{E}X_j = 0$, $j = 1, 2, \cdots$. Let us introduce into consideration the generating function of the moments of X_j:

$$M_j(z) = \mathbf{E}e^{zX_j} = \int_{-\infty}^{\infty} e^{zy}\, dV_j(y). \tag{1}$$

Put

$$Z_n = \frac{\sum\limits_{j=1}^{n} X_j}{s_n}. \tag{2}$$

One of the important problems in probability theory is to study the limit behavior of the function $\mathbf{P}\{Z_n < x\} = F_n(x)$ or the corresponding local distribution functions of the normalized sums Z_n as $n \to \infty$. The classical formulation of the problem limited itself to the case $x = O(1)$ as $n \to \infty$. But sometimes, especially in applications to mathematical statistics, problems arise connected with the study of the asymptotic behavior of the probabilities of the normalized sums, where x together with n increases without bound. In this case of so-called large deviations there are at present but few results. The local and integral theorems, established under the assumption that $x = O(1)$ as $n \to \infty$, give in this case only a trivial answer. Besides certain results of A. Ya. Khinchin and N. V. Smirnov relating to Bernoulli schemes, there are in the general case certain integral theorems, solving in large measure the problem of large deviations.

H. Cramér [1] proved a theorem giving an estimate for the ratio of the "tails" of the distributions $(1-F_n(x))/(1-\Phi(x))$ (where $\Phi(x)$ is the normal distribution function) for the case of identically distributed quantities under the assumption $x = o(\sqrt{n}/\log n)$. In the work of Feller [3] generalizing Cramér's results a theorem is given for non-identically distributed quantities in case $|X_j|$ has an upper bound of the form $o(s_j)$. The next step was taken in 1953 by V. V. Petrov [5], [6]. He succeeded in obtaining a complete generalization of Cramér's theorem to the case of non-identically distributed quantities, and at the same time he improved the remainder term and replaced the order of growth $x = o(\sqrt{n}/\log n)$ by $x = o(\sqrt{n})$.

In the present work we give a number of local theorems, which in the well-known sense solve the problem of large deviations, and at the same time the results are analogous to those already known in the integral theorems. The method of proof is the saddlepoint method of the theory of functions of a complex variable, and seems to have no relation to the methods previously used. But in reality this is not the case. In all the previously mentioned works it is assumed that $\int_{-\infty}^{\infty} e^{hy} dV_j(y)$ converges for all real numbers h in some interval about the origin, fixed for all $j = 1, 2, \cdots$. This means that they are assuming the analyticity of the generating function of the moments of X_j in some strip $|\operatorname{Re} z| < A$, the same strip for all X_j. Further, in the works cited above a certain transformation of the given probability laws is applied, which, in essence, leads to the introduction of the "conjugate" probability distributions (by A. Ya. Khinchin [8]). But this transformation is really a hidden application of the saddlepoint method. This is clarified in the proofs of our theorems below. Our proofs are in structure analogous to the proof of Cramér's theorem. In them, however, the saddlepoint method is applied consistently.

H. Daniels [2] indicated the possible value of this method for obtaining asymptotic expressions in the theory of probability and in mathematical statistics. He used this method to obtain an asymptotic expression for the density of the arithmetic means of identically distributed independent random variables under the condition $x = O(1)$, $n \to \infty$.

2. Results

On the given sequence $X_1, X_2, \cdots$ we shall impose the following conditions:

Condition A. There exist positive numbers A, K and k such that in the circle $|z| < A$, we have the inequality

$$k \leqq \left| \int_{-\infty}^{\infty} e^{zy} dV_j(y) \right| \leqq K, \qquad j = 1, 2, \cdots.$$

Condition B. For all n,

$$\frac{s_n^2}{n} \geqq \delta > 0.$$

Condition C. The sequence $\{X_j\}$, $j = 1, 2, \cdots$, contains a subsequence $X_{j_1}, X_{j_2}, \cdots$ with a sufficiently large number of terms in each section $X_1, \cdots, X_n$ of the original sequence, and such that for the moment generating function we have the uniform estimate

$$(C) \qquad \left| \int_{-\infty}^{\infty} e^{(v+it)y} dV_{j_k}(y) \right| \leqq \frac{L}{|t|^{\beta}}$$

for $|t| \geqq N$ and $|v| < A$, where $L, N > 0$ and $\beta > 0$ may depend on v. The number n^* of terms X_{j_k} among the first n quantities $X_1, X_2, \cdots, X_n$ satisfies the inequality

$$\varliminf_{n\to\infty} \frac{n^*}{s_n^\gamma} > 0$$

for some $\gamma > 0$.

With these assumptions we obtain the following result:

Theorem 1. *Let conditions* (A)—(C) *be satisfied. Then for all sufficiently large n the density $p_{z_n}(x)$ of the quantity Z_n exists. Further, let x be a real number depending on n, and such that $x > 1$, $x = o(\sqrt{n})$ as $n \to \infty$. Then*

$$\frac{p_{z_n}(x)}{\varphi(x)} = e^{(x^2/\sqrt{n})\lambda_n((x/\sqrt{n}))}\left[1 + O\left(\frac{x}{\sqrt{n}}\right)\right],$$

where $\lambda_n(t)$ is a power series converging for all sufficiently small values of $|t|$, uniformly for all n, and $\varphi(x)$ is the density of the normal distribution.

REMARK. The uniform estimate (C) in condition C is satisfied if, for example, the distribution functions of the quantities X_{j_k} are absolutely continuous functions and their derivatives $p_{j_k}(x)$, together with the functions $e^{vx}p_{j_k}(x)$, $-A < v < A$, have uniformly (for each individual v) bounded total variations, that is, if the sequence $\int_{-\infty}^{\infty} e^{vx}|dp_{j_k}(x)|$ is uniformly bounded for each v, $|v| < A$.

If the quantities $X_1, X_2, \cdots$ are identically distributed then condition C can be considerably relaxed. One has

Theorem 2. *Let the independent quantities $X_1, X_2, \cdots$ with the common distribution function $V(x)$, and $\mathbf{E}X_j = 0$, $\mathbf{D}X_j = \sigma^2 > 0$, $j = 1, 2, \cdots$, be subjected to the following conditions:*

Condition 1. *There exists a positive number A such that for every real number s, $|s| < A$, the integral $\int_{-\infty}^{\infty} e^{sy}dV(y)$ is convergent.*

Condition 2. *There exists an n_0 such that the distribution function of the sum $X_1 + \cdots + X_n$ is absolutely continuous with a bounded derivative.*

Then, if $x > 1$ and $x = o(\sqrt{n})$, we have, as $n \to \infty$,

$$\frac{p_{z_n}(x)}{\varphi(x)} = e^{(x^3/\sqrt{n})\lambda((x/\sqrt{n}))}\left[1 + O\left(\frac{x}{\sqrt{n}}\right)\right],$$

where $\lambda(t)$ is a power series converging for all sufficiently small values of $|t|$.

In a completely analogous manner one has local theorems also for the case of identically distributed lattice terms $X_1, X_2, \cdots$, with the maximal span of the distribution equal to h.

One has

Theorem 3. *Let there be given a sequence $X_1, X_2, \cdots$ of independent random variables, $\mathbf{E}X_j = 0$, $\mathbf{D}X_j = \sigma^2 > 0$, which take values only in a certain arithmetic progression with probabilities $p_k = \mathbf{P}\{X_j = a+kh\}$, h is the maximal span of the distribution, k is an integer, a is a fixed real number. Let us put*

$$P_n(k) = \mathbf{P}\{\sum_1^n X_j = an+kh\}.$$

Let the sequence $X_1, X_2, \cdots$ *be subjected to condition* 1 *of Theorem* 2. *If we put* $x = x_{nk} = (an+kh)/(\sigma\sqrt{n})$, *then for* $x > 1$, $x = o(\sqrt{n})$ *as* $n \to \infty$,

$$\frac{\dfrac{\sigma\sqrt{n}}{h} P_n(k)}{\varphi(x)} = e^{(x^3/\sqrt{n})\,\lambda\,((x/\sqrt{n}))} \left[1 + O\left(\frac{x}{\sqrt{n}}\right)\right],$$

where $\lambda(t)$ *is a power series converging for all sufficiently small values of* $|t|$.

REMARK. For the case of a Bernoulli scheme, that is, if X_k takes the values $-p, q,\ p \geqq 0,\ q \geqq 0,\ p+q = 1$, with probabilities $p_0 = \mathbf{P}\{X_k = -p\} = q$, $p_1 = \mathbf{P}\{X_k = q\} = p$, there is the well known result obtained by A. Ya. Khinchin [7] in 1929 by a simple application of Stirling's formula. The following is valid:

If $n \to \infty$ and k varies in such a way that $1 < x = (k-np)/\sqrt{npq} = o(\sqrt{n})$, then

$$P_n(k) = \frac{1}{\sqrt{2\pi pqn}}\, e^{-f(x/\sqrt{n})+O(x/\sqrt{n})},$$

where

$$f(t) = n\sum_{k=2}^{\infty} \frac{p^{k-1}-(-q)^{k-1}}{k(k-1)(pq)^{k/2-1}}\, t^k = n\left[\frac{t^2}{2} + \frac{p-q}{6\sqrt{pq}}\, t^3 + \frac{p^3+q^3}{12pq}\, t^4 + \cdots\right],$$

which corresponds to our result.

The same asymptotic expression is valid for negative values of the argument x, if one assumes $x < -1$ and $|x| = o(\sqrt{n})$ as $n \to \infty$. It is sufficient to replace x by $|x|$ in the remainder terms of the limit expressions in Theorems 1—3.

Finally it should be observed that the series $\lambda(t), \lambda_n(t)$ respectively, used in Theorems 1—3, is the same as in the works of Cramér [1], Petrov [5].

3. Some Corollaries of Theorems 1—3

A number of simple corollaries follow from Theorems 1—3. We shall formulate the most interesting of them for Theorem 1.

Corollary 1. *Under the assumptions of Theorem* 1 *we have*:

(a) *if* $x > 1$, $x = O(n^{1/6})$ *as* $n \to \infty$, *then*

$$\frac{p_{z_n}(x)}{\varphi(x)} = e^{c_n x^3}\left[1 + O\left(\frac{x}{\sqrt{n}}\right)\right],$$

where $c_n = (1/6s_n^3)\sum_{j=1}^n \alpha_{3j}$ (α_{3j} *denotes the third moment of the quantity* X_j) *is of the order* $c_n = O(1/\sqrt{n})$ *as* $n \to \infty$.

(b) *if* $x > 0$, $x = o(n^{1/6})$ *as* $n \to \infty$, *then*

$$\lim_{n\to\infty} p_{z_n}(x) = \varphi(x).$$

Corollary 2. *From the proof of Theorem* 1 *it follows that one also has an analogue of Theorem* 2 *of V. V. Petrov* [5]. *The only requirement is that* $x/\sqrt{n}$ *remain for large* n *in modulus less than some quantity* $\varepsilon_0 > 0$. *Selecting such a constant* ε_0 *we can assert that in the interval* $1 < x < \varepsilon_0\sqrt{n}$ *we have*

$$\frac{p_{z_n}(x)}{\varphi(x)} = e^{(x^3/\sqrt{n})\,\lambda_n((x/\sqrt{n}))}\,[1+r\varepsilon_0],$$

where $|r|$ *is less than some constant quantity* d.

4. The Proof of Theorem 2

The method of proof is almost the same for all three theorems; however, for each theorem there are certain peculiarities, connected with the different nature of the given random variables. We shall use the proof of Theorem 2 to explain the substance of the method.

Let us pass to the proof of Theorem 2. As is known, condition 1 is equivalent to the analyticity of the moment generating function $M(z) = \int_{-\infty}^{\infty} e^{zx}dV(x)$ in some strip $|\mathrm{Re}\, z| < A$.

Since $M(0) = 1$ one can find positive numbers k and $A_1 \leqq A$ such that $|M(z)| \geqq k > 0$ in the circle $|z| \leqq A_1$. Therefore we can form $K(z) = \log M(z)$ and take for the logarithm its principal branch, tending to 0 as $z \to 0$.

By the analyticity of $K(z)$ in the circle $|z| \leqq A_1$, $K(z)$ may be expanded in a power series converging uniformly on that circle. Clearly

$$K(z) = \sum_{k=2}^{\infty} \frac{\gamma_k z^k}{k!} = \frac{\sigma^2 z^2}{2} + \frac{\gamma_3 z^3}{3!} + \cdots \tag{1}$$

and

$$K'(z) = \sum_{k=1}^{\infty} \frac{\gamma_{k-1} z^k}{k!} = \sigma^2 z + \gamma_3 \frac{z^2}{2} = \cdots, \tag{2}$$

where γ_k denotes the semi-invariant of k-th order of the quantities X_j, $j = 1, 2, \cdots$.

The moment generating function for the sum of the first n quantities $X_1, \cdots, X_n$ is equal, as is known, to the product of the moment generating functions of the individual terms: $M_n(z) = [M(z)]^n$. But we shall have to establish conditions for the validity of an inversion formula for $M_n(z)$.

For this we need the following theorem on Fourier transforms (cf. [4], p. 20, or B. V. Gnedenko, *Course in Probability Theory,* 1955, p. 238 (Russian edition)):

If the function $g(x)$ is absolutely integrable $(g \in L_1(-\infty, +\infty))$, continuous and bounded, and if its Fourier transform $h(t) = \int_{-\infty}^{\infty} e^{itx}g(x)dx$ is non-negative, then $h \in L_1(-\infty, +\infty)$.

From this theorem with the aid of some simple considerations one can obtain the following lemma:

Lemma. *If* $g \in L_1(-\infty, +\infty)$ *and* $e^{cx}g \in L_1(-\infty, +\infty)$ *for all* c, $|c| < A$, *and if* $g(x)$ *and* $e^{cx}g(x)$ *are bounded, then the square of the generating function*

$$M_g(z) = \int_{-\infty}^{\infty} e^{zx} g(x)dx, \qquad |\mathrm{Re}\, z| < A,$$

is absolutely integrable along each vertical line $z = c+iy$, $|c| < A$:

$$\int_{-\infty}^{\infty} |M^2(c+iy)|dy < +\infty.$$

For the proof we form the auxiliary function

$$\Psi_c(x) = \int_{-\infty}^{\infty} g(x+y)e^{c(x+y)} g(y)e^{cy}\, dy,$$

where c is some fixed number from the interval $(-A, A)$. The function $\Psi_c(x)$ is absolutely integrable, bounded and continuous on $(-\infty, +\infty)$. Let us apply the theorem formulated above to the function $\Psi_c(x)$. This is possible since the Fourier transform of the function $\Psi_c(x)$ is really non-negative:

$$\int_{-\infty}^{\infty} e^{itx}\Psi_c(x)dx = \int_{-\infty}^{\infty} g(y)e^{cy}\, dy \int_{-\infty}^{\infty} e^{itx} e^{c(x+y)} g(x+y)dx$$
$$= M(c-it)M(c+it) = |M(c+it)|^2 \geqq 0.$$

By the theorem we have

$$\int_{-\infty}^{\infty} |M(c+it)|^2 dt < +\infty.$$

At the same time c, $|c| < A$, is arbitrary, which completes the proof.

By condition 2 of Theorem 2 there exists a natural number n_0 such that the distribution function of the sum $\sum_{j=1}^{n_0} X_j$ is absolutely continuous. Its derivative—let us denote it by $g(x)$—is bounded. By condition 1, $\int_{-\infty}^{\infty} e^{cx} g(x)dx < +\infty$ for all c from $(-A, A)$ and $e^{cx}g(x)$ is bounded. An application of the lemma gives us an inversion formula for all $n \geqq 2n_0$:

$$p_{z_n}(x) = \frac{\sigma\sqrt{n}}{2\pi i} \int_{c-i\infty}^{c+i\infty} M^n(z)e^{-\sigma\sqrt{n}\, zx}\, dz \tag{3}$$

and

$$\int_{c-i\infty}^{c+i\infty} |M^{2n_0}(z)|dz < +\infty$$

for all c from $(-A, A)$. Here we take into account that $p_{z_n}(x) = \sigma\sqrt{n}g(\sigma\sqrt{n}x)$.

Let us pass to the calculation of the integral (3) under the conditions of Theorem 2. The number $c \in (-A, A)$ is for the time being arbitrary. Let us try to make the best possible choice of it.

For this purpose we consider the integral

$$I = \int_{c-i\infty}^{c+i\infty} M^n(z)e^{-n\sigma\tau z}\, dz.$$

We can apply the saddlepoint method, at least for $|\mathrm{Re}\, z| < A_1$. To find the saddlepoint z_0 for the given problem it is necessary to find the solution z_0 to the following equation:

$$\frac{d}{dz}\,[K(z)-\sigma\tau z] = K'(z)-\sigma\tau = 0$$

or

(4) $$\tau = \sigma z+\frac{\gamma_3 z^2}{2\sigma}+\frac{\gamma_4 z^3}{3!\,\sigma}+\cdots.$$

Since $\sigma > 0$ this power series may be inverted for sufficiently small $|\tau|$, and in a unique manner. The series obtained, expressing z as a function of τ, converges absolutely for small τ and tends to zero as $\tau \to 0$. The solution z_0 of equation (4) has the same sign as τ. In our case τ and z_0 are positive. We obtain

(5) $$z_0 = \frac{\tau}{\sigma}-\frac{\gamma_3}{2\sigma^4}\tau^2+\frac{3\gamma_3^2-\gamma_4\sigma^2}{6\sigma^7}\tau^3+\cdots.$$

For the saddlepoint contour one may take the vertical line $z = z_0+it$, $-\infty < t < +\infty$. Along the line $z = z_0+it$ one may expand $K(z)-z\tau\sigma$ in powers of t in a neighborhood of $t = 0$:

(6) $$K(z)-z\tau\sigma = K(z_0)-z_0\tau\sigma+\sum_{j=2}^{\infty}\frac{K^{(j)}(z_0)\,(it)^j}{j!}.$$

We calculate immediately

$$K(z_0)-z_0\tau\sigma = K(z_0)-z_0K'(z_0) = -\sum_{k=2}^{\infty}\frac{k-1}{k}\,\gamma_k z_0^k = -\frac{\tau^2}{2}+\tau^3\lambda(\tau),$$

where

(7) $$\lambda(\tau) = \frac{\gamma_3}{6\sigma^3}+\frac{\gamma_4\sigma^2-3\gamma_3^2}{24\sigma^6}\tau+\cdots$$

converges absolutely in some neighborhood of $\tau = 0$.

Let $\varepsilon > 0$ be a number smaller than the radius of convergence of the series (6). By the saddlepoint method the principal part of the integral I is obtained by integrating along an arbitrarily small segment of the saddlepoint contour near the saddlepoint z_0. Therefore one may write

(8) $$I = \int_{z_0-i\varepsilon}^{z_0+i\varepsilon} M^n(z)e^{-n\sigma\tau z}\,dz+R.$$

First let us find an estimate for the modulus of the quantity

$$R = \int_{z_0-i\infty}^{z_0-i\varepsilon} M^n(z)e^{-n\sigma\tau z}\,dz+\int_{z_0+i\varepsilon}^{z_0+i\infty} M^n(z)e^{-n\sigma\tau z}\,dz.$$

$M^n(z)$, for $n \geqq n_0$, is the moment generating function of the absolutely continuous distribution function. This means that on each line $z = c+it$, $-\infty<t<\infty$, $c \in (-A, A)$, the modulus $|M^n(z)|$ tends to zero as $|t| \to \infty$. There exists, consequently, a positive number $\alpha = \alpha(z_0)$, such that for $|t| \geqq \varepsilon$

$$|M^{2n_0}(z_0+it)| < e^{-\alpha}M^{2n_0}(z_0) = e^{-\alpha+2n_0K(z_0)}.$$

For $n \geqq 2n_0$ we obtain

$$\left|\frac{\sigma\sqrt{n}}{2\pi i} R\right| < \exp\{-n\sigma\tau z_0+nK(z_0)\} \frac{\sigma\sqrt{n}}{2\pi} \exp\left\{-\alpha\left(\left[\frac{n}{2n_0}\right]-1\right)\right\} \cdot L_1,$$

where

$$L_1 = e^{-2n_0K(z_0)} \int_{z_0-i\infty}^{z_0+i\infty} |M(z)|^{2n_0} dz < +\infty$$

is bounded above by a number independent of n and z_0. Then

$$\left|\frac{\sigma\sqrt{n}}{2\pi} R\right| = \exp\{-n\sigma\tau z_0+nK(z_0)\} o(n^{-m}), \tag{9}$$

where m is arbitrarily large as $n \to \infty$.

Now we begin to estimate the principal term of the integral I. We have

$$I_1 = \int_{z_0-i\varepsilon}^{z_0+i\varepsilon} \exp\{n[K(z)-\sigma\tau z]\} dz \tag{10}$$

and by (6)

$$I_1 = i \exp\{n[K(z_0)-z_0\tau\sigma]\} \int_{-\varepsilon}^{+\varepsilon} \exp\left\{n \sum_{k=2}^{\infty} \frac{K^{(k)}(z_0)(it)^k}{k!}\right\} dt.$$

For all (real) z_0, $|z_0| < A_1$, the function $K''(z)$ is positive:

$$\begin{aligned} 0 &< \frac{1}{M(z_0)} \int_{-\infty}^{\infty} \left(y - \frac{M'(z_0)}{M(z_0)}\right)^2 e^{yz} dV(y) \\ &= \frac{1}{M(z_0)} \left[\int_{-\infty}^{\infty} y^2 e^{yz} dV(y) - \frac{M'(z_0)}{M(z_0)} \int_{-\infty}^{\infty} y e^{yz} dV(y)\right] \\ &= \frac{M''(z_0)M(z_0) - M'^2(z_0)}{M^2(z_0)} = K''(z_0). \end{aligned}$$

Therefore it is legitimate to make the following change of variables: $t' = \sqrt{nK''(z)}t$. We have

$$I_1 = \frac{ie^{n\{K(z_0)-z_0\tau\sigma\}}}{\sqrt{nK''(z_0)}} \int_{-\varepsilon\sqrt{nK''(z_0)}}^{+\varepsilon\sqrt{nK''(z_0)}} e^{-\frac{1}{2}t^2} \exp\left\{\sum_{j=3}^{\infty} \frac{\rho_j(z_0)(it)^j}{j!\, n^{j/2-1}}\right\} dt,$$

where

$$\rho_j(z) = \frac{K^{(j)}(z)}{[K''(z)]^{j/2}}.$$

We calculate

$$\exp\left\{\sum_{j=3}^{\infty} \frac{\rho_j(z_0)(it)^j}{j!\, n^{j/2-1}}\right\} = 1 - \frac{i\rho_3 t^3}{3!\sqrt{n}} - \left\{\frac{\rho_3^2 t^6}{(3!)^2} - \frac{\rho_4 t^4}{4!}\right\} \frac{1}{n} + o\left(\frac{1}{n}\right). \tag{11}$$

Then we obtain

$$I_1 = \frac{ie^{n\{K(z_0)-z_0\tau\sigma\}}}{\sqrt{nK''(z_0)}} \left[\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} \left\{1 - \frac{i\rho_3 t^3}{3!\sqrt{n}} - \left[\frac{\rho_3^2 t^6}{(3!)^2} - \frac{\rho_4 t^4}{4!}\right] \frac{1}{n} + \cdots\right\} dt + R_1\right].$$

The error R_1 which we make when we replace the finite limits of integration by the infinite limits is of the order $o(1/n)$ as $n \to \infty$.

Integrating, we obtain the following expression for I_1:

$$I_1 = \frac{i\sqrt{2\pi}e^{n\{K(z_0)-z_0\tau\sigma\}}}{\sqrt{nK''(z_0)}}\left[1+\left\{\frac{\rho_4(z_0)}{9}-\tfrac{5}{12}\rho_3^2(z_0)\right\}\frac{1}{n}+o\left(\frac{1}{n}\right)\right].$$

Let us rewrite this expression, replacing z_0 everywhere by its expression (5) in terms of τ. For this we first estimate

$$\frac{\sigma}{\sqrt{K''(z_0)}} = 1-\frac{\gamma_3}{2\sigma^3}\tau+\frac{5\gamma_3^2-2\gamma_4\sigma^2}{8\sigma^6}\tau^2+\cdots, \tag{12}$$

$$\rho_4(z_0) = \frac{\gamma_4}{\sigma^4}+\frac{\gamma_5\sigma^2-2\gamma_3\gamma_4}{\sigma^7}\tau+\cdots, \tag{13}$$

$$\rho_3^2(z_0) = \frac{\gamma_3^2}{\sigma^6}+\frac{2\gamma_3\gamma_4\sigma^2-3\gamma_3^2}{\sigma^9}\tau+\cdots. \tag{14}$$

Thus, by (7), (12)—(14), as $\tau\to 0$, $n\to\infty$,

$$I_1 = \frac{i\sqrt{2\pi}}{\sigma\sqrt{n}}e^{-(n\tau^2/2)+n\tau^3\lambda(\tau)}\left[1-\frac{\gamma_3}{2\sigma^3}\tau+o(\tau)+O\left(\frac{1}{n}\right)\right], \tag{15}$$

and finally, by (8), (9), (11) and (15)

$$\frac{\sigma\sqrt{n}}{2\pi i}I = \frac{e^{-(n\tau^2/2)+n\tau^3\lambda(\tau)}}{\sqrt{2\pi}}\left[1-\frac{\gamma_3}{2\sigma^3}\tau+o(\tau)+O\left(\frac{1}{n}\right)\right].$$

From this relation it is now easy to obtain the assertion of the theorem. Put $\tau = x/\sqrt{n}$ and subject x to the condition $x = o(\sqrt{n})$ as $n\to\infty$, $x>1$. Then $\tau\to 0$ as $n\to\infty$, and by (3) we obtain

$$p_{z_n}(x) = \frac{1}{\sqrt{2\pi}}e^{-(x^2/2)+(x^3/\sqrt{n})\lambda((x/\sqrt{n}))}\left[1+O\left(\frac{x}{\sqrt{n}}\right)\right],$$

where $\lambda(t)$ is defined by (7). Theorem 2 is proven.

5. The Proof of Theorem 3

The proof of Theorem 3 is in many ways analogous to the proof of Theorem 2. But there are a number of essential differences with which we shall deal at once.

As in the preceding proof, there exist numbers $k>0$ and $A_1 \leqq A$ such that $|M(z)| \geqq k > 0$ in the circle $|z| \leqq A_1$. In this circle one has the expansions (1) and (2) for $K(z)$ and $K'(z)$.

The quantities $X_1, X_2, \cdots$ take the values $a+kh$ with probabilities p_k. Then

$$M(z) = \mathbf{E}e^{zX_j} = \sum_{k=-\infty}^{+\infty} p_k e^{(a+kh)/z}.$$

$M(z)$ is analytic in the strip $|\mathrm{Re}\, z| < A$ and is periodic with period $2\pi i/h$, where h is the maximal step of the distribution. Therefore $|M(z)|$ takes its maximal value $M(v)$ on each vertical line $z = v+it$, $|v| < A$, only at the points $z = v+i2\pi k/h$. This means that for a given v_0, $|v_0| < A$, for each ε, $0 < \varepsilon < \pi/h$, there exists a positive number α such that

$$|M(v_0+it)| < M(s_0)e^{-\alpha} \tag{16}$$

for $\varepsilon \leqq |t| \leqq \pi/h$.

Let us consider the sums $\bar{Z}_n = \sum_{j=1}^n X_j$. The quantity $\bar{Z}_n$ takes the value $an+kh$ with probability $P_n(k)$. Therefore the moment generating function for the quantity $\bar{Z}_n$ is equal to

$$M^n(z) = \sum_{k=-\infty}^{+\infty} P_n(k)e^{z(an+kh)}.$$

Multiply this expression by $e^{-z(an+k_0h)}$ and integrate from $c-i\pi/h$ to $c+i\pi/h$, $|c| < A$, along the line. We obtain

$$P_n(k) = \frac{h}{2\pi i}\int_{c-i(\pi/h)}^{c+i(\pi/h)} M^n(z)e^{-z(an+k_0h)}\,dz.$$

$\sum_{k=1}^n X_k/\sigma\sqrt{n}$ takes the value $x = x_{kn} = (an+kh)/\sigma\sqrt{n}$, that is, $kh = \sigma\sqrt{n}x - an$. Therefore we may write

$$P_n(k) = \frac{h}{2\pi i}\int_{c-i(\pi/h)}^{c+i(\pi/h)} M^n(z)e^{-z\sigma\sqrt{n}x}\,dz. \tag{17}$$

For the calculation of this integral under the assumptions of Theorem 3 we put $x/\sqrt{n} = \tau$ and we use the saddlepoint method, assuming $|c| < A_1$.

We obtain the saddlepoint (5), $z_0 = \tau/\sigma+\gamma_3\tau^2/2\sigma^4+\cdots$. The contour of integration will be $z = z_0+it$, $-i\pi/h \leqq t \leqq i\pi/h$. The principal part of the integral

$$\int_{z_0-i\varepsilon}^{z_0+i\varepsilon} M^n(z)e^{-z\sigma\tau n}\,dz$$

is obtained just as in the preceding proof ((10), (15)):

$$\int_{z_0-i\varepsilon}^{z_0+i\varepsilon} M^n(z)e^{-z\sigma\tau n}\,dz = \frac{i\sqrt{2\pi}}{\sigma\sqrt{n}}e^{-(n\tau^2/2)+n\tau^3\lambda(\tau)}\left[1-\frac{\gamma_3}{2\sigma^3}\tau+o(\tau)+O\left(\frac{1}{n}\right)\right].$$

The estimation of the remaining integrals is not difficult (16):

$$R = \left[\int_{z_0-i(\pi/h)}^{z_0-i\varepsilon} + \int_{z_0+i\varepsilon}^{z_0+i(\pi/h)}\right] M^n(z)e^{-n\sigma\tau z}\,dz,$$

$$|R| < \frac{2\pi}{h}e^{-n\sigma\tau z_0+nK(z_0)}\cdot e^{-n\alpha} = e^{-n\sigma\tau z_0+nK(z_0)}\cdot o(n^{-m}),$$

where m is any large positive number. Therefore in the estimation of our integral we may neglect R. Returning to the original problem, put $\tau = x/\sqrt{n}$, $x = o(\sqrt{n})$ as $n \to \infty$, $x > 1$. Then by (17)

$$P_n(k) = \frac{h}{\sigma\sqrt{n}}\cdot\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\cdot e^{(x^3/\sqrt{n})\lambda((x/\sqrt{n}))}\left[1+O\left(\frac{x}{\sqrt{n}}\right)\right],$$

which was to be proven.

6. The Proof of Theorem 1

Now that the method has been clarified in the proof of Theorem 2, the proof of Theorem 1 is not difficult. Condition A is equivalent to the following, that the functions

$$K_j(z) = \mathrm{Log} \int_{-\infty}^{\infty} e^{zx} dV_j(x)$$

(Log denotes the principal branch of the logarithm) are analytic in one and the same circle $|z| < A$ and that the functions $K_j'(z)$ are uniformly bounded in modulus, for example, in the circle $|z| < A_1 = A/2$ (cf. [6]).

We have

$$K_j(z) = \sum_{k=2}^{\infty} \frac{\gamma_{kj}}{k!} z^k,$$

where γ_{kj} is the semi-invariant of order k of the random variable X_j. Since $|K_j(z)| \leqq B$ on the circumference $|z| = A_2 < A_1$, by the Cauchy inequality we have

$$|\gamma_{kj}| \leqq \frac{B(k-1)!}{A_2^{k-1}}.$$

Then the quantities $\Gamma_{kn} = (1/n) \sum_{j=1}^{n} \gamma_{kj}$ are also bounded by numbers not depending on n. Consequently,

$$\sum_{j=1}^{n} K_j'(z) = n \sum_{k=2}^{\infty} \Gamma_{kn} \frac{z^{k-1}}{(k-1)!}$$

is majorized by means of a series converging for $|z| < A_2$ with coefficients not depending on n.

By condition C for some positive μ and $n \geqq n_1$ we have

$$n^* > \mu s_n^{\gamma}.$$

But $n^*\beta > \mu\beta s_n^{\gamma} \to \infty$, that is, there is an $n_0 \geqq n_1$ such that $n^*\beta \geqq 2$.

For $|t| \geqq N$, we have $|M_{j_k}(c+it)| \leqq L(c)/|t|^{\beta}$ and, consequently,

$$\Big| \prod_{j=1}^{n_0} M_j(c+it) \Big| \leqq \frac{L_1}{|t|^{n_0^*\beta}} \leqq \frac{L_1}{t^2},$$

and the integral

$$\int_{-\infty}^{\infty} \Big| \prod_{j=1}^{n} M_j(c+it) \Big| dt$$

converges for $n \geqq n_0$, $|c| < A_2$.

Therefore we have the inversion formula

$$p_{z_n}(x) = \frac{s_n}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{-zxs_n} \prod_{j=1}^{n} M_j(z) dz \tag{18}$$

for all $n \geqq n_0$ independently of the special choice of the number c from the interval $(-A_2, A_2)$.

Let us put $x/\sqrt{n} = \tau$ and calculate the integral

$$I = \frac{s_n}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-z\tau_n\sqrt{n}} \prod_{j=1}^{n} M_j(z)dz \tag{19}$$

by the saddlepoint method.

Differentiating the expression

$$-\frac{s_n}{\sqrt{n}} z\tau + \frac{1}{n}\sum_{j=1}^{n} K_j(z)$$

with respect to z and equating the result to zero, we find the saddlepoint of our problem as the solution of this equation, if a solution exists.

We obtain as the equation

$$\sqrt{\frac{s_n^2}{n}}\,\tau = \sum_{k=2}^{\infty} \Gamma_{kn} \frac{z^{k-1}}{(k-1)!} = \frac{s_n^2}{n} z + \Gamma_{3n}\frac{z^2}{2} + \cdots, \tag{20}$$

$|z| < A_2$. By condition B we have $s_n^2/n \geqq \delta > 0$. Then one can invert the series in a unique manner and express z as a function of τ. One can find a common neighborhood of $\tau = 0$ for all n, within which the new series is majorized by a series with coefficients not depending on n, which converges there.

Inverting the series (20) we obtain

$$z = \frac{\tau}{\sqrt{\Gamma_{2n}}} - \frac{\Gamma_{3n}}{2\Gamma_{2n}^2}\tau^2 + \frac{3\Gamma_{3n}^2 - \Gamma_{4n}\Gamma_{2n}}{6\Gamma_{2n}^{7/2}}\tau^3 + \cdots. \tag{21}$$

One can find an $\varepsilon_0 > 0$ such that this series converges absolutely for all $\tau \in (-\varepsilon_0, \varepsilon_0)$ and the corresponding sum z of the series (21) lies in the interval $(-A_2, A_2)$. Positive z correspond to positive τ. Fix a sufficiently small τ. Let z_0 be the corresponding saddlepoint, the solution to equation (20). Then in (19) also we take $c = z_0$ and we split the integral into two parts.

Choose $\varepsilon > 0$ so small that the circle $|z - z_0| \leqq \varepsilon$ still lies inside the circle $|z| < A_2$. One can write

$$I = \frac{s_n}{2\pi i}\int_{z_0-i\varepsilon}^{z_0+i\varepsilon} \exp\left\{n\left[-\sqrt{\frac{s_n^2}{n}}\,z\tau + \sum_{k=2}^{\infty}\Gamma_{kn}\frac{z^k}{k!}\right]\right\} dz + R.$$

The modulus of R is easy to estimate. For every $\varepsilon > 0$ by estimate (C) there exists an $\alpha = \alpha(z_0) > 0$, such that for each term of the subsequence (X_{j_k}), $k = 1, 2, \cdots$,

$$|M_{j_k}(z_0 + it)| < e^{-\alpha} M_{j_k}(z_0)$$

for all t, $|t| \geqq \varepsilon$. For all j, naturally, one has the estimate $|M_j(z_0 + it)| \leqq M_j(z_0)$. Consequently,

$$|R| < \frac{s_n}{2\pi} e^{-\alpha n^*} \exp\left\{n\left[-\sqrt{\frac{s_n^2}{n}}\,z_0\tau + \frac{1}{n}\sum_{j=1}^{n} K_j(z_0)\right]\right\} \cdot L,$$

where L is a certain constant number (not depending on n), $n \geqq n_0$, and for $n \geqq n_0$ we have $n^* > \mu s_n^\gamma$, that is $e^{-\alpha n^*} < e^{-\mu\alpha s_n^\gamma}$ and, further

$$s_n e^{-\alpha n^*} < s_n e^{-\alpha\mu s_n^\gamma} \leqq c_1\sqrt{n}e^{-c_2 n^{\gamma/2}} = o(n^{-2}).$$

On the interval $z = z_0+it$, $|t| < \varepsilon$, one has the series expansions

$$\text{(22)} \qquad -\sqrt{\frac{s_n^2}{n}}z\tau+\frac{1}{n}\sum_{j=1}^{n}K_j(z) = -\tau z_0\sqrt{\frac{s_n^2}{n}}+\frac{1}{n}\sum_{j=1}^{n}K_j(z_0) -\frac{1}{2n}\sum_{j=1}^{n}K_j''(z_0)t^2-\frac{1}{6n}\sum_{j=1}^{n}K_j'''(z_0)it^3+\cdots.$$

At the same time,

$$\text{(23)} \qquad \begin{aligned} -\frac{\tau z_0 s_n}{\sqrt{n}}+\frac{1}{n}\sum_{j=1}^{n}K_j(z_0) &= -z_0\sum_{k=2}^{\infty}\Gamma_{kn}\frac{z_0^{k-1}}{(k-1)!}+\sum_{k=2}^{\infty}\Gamma_{kn}\frac{z_0^k}{k!} \\ &= \sum_{k=2}^{\infty}\Gamma_{kn}\frac{1-k}{k!}z_0^k = -\frac{\tau^2}{2}+\tau^3\lambda_n(\tau), \end{aligned}$$

where

$$\text{(24)} \qquad \lambda_n(t) = \frac{\Gamma_{3n}}{6\Gamma_{2n}^{3/2}}-\frac{3\Gamma_{3n}^2-\Gamma_{4n}\Gamma_{2n}}{24\Gamma_{2n}^3}t+\cdots$$

is a power series converging for sufficiently small $|t|$ uniformly for all n.

As in the proof of Theorem 2, $K_j''(z_0) > 0$ for $|z_0| < A_2$. Then the same can be asserted for the sum $\sum_{j=1}^n K_j''(z_0)$. By condition B we even have

$$\sum_{j=1}^{n}K_j''(z_0) = s_n^2\left(1+\frac{\Gamma_{3n}}{\Gamma_{2n}^{3/2}}\tau+o(\tau)\right) > n\delta\left(1-\frac{2|\Gamma_{3n}|}{\Gamma_{2n}^{3/2}}|\tau|\right) > n\delta\left(1-\frac{2S|\tau|}{\delta^{3/2}}\right),$$

where $|\Gamma_{3n}| < S$. This means that for sufficiently small $|\tau|$, $|\tau| < \delta^{3/2}/2S$, $(\sum_1^n K_j''(z_0))^{1/2}$ tends to zero at least as fast as $\eta\sqrt{n}$, where $\eta > 0$. Let us make the change of variables $t' = (\sum_1^n K_j''(z_0))^{1/2}t$ in the integral

$$I_1 = \frac{s_n}{2\pi i}\int_{z_0-i\varepsilon}^{z_0+i\varepsilon}\exp\left\{n\left[-\frac{z\tau s_n}{\sqrt{n}}+\sum_{k=2}^{\infty}\Gamma_{kn}\frac{z^k}{k!}\right]\right\}dz.$$

We obtain by (22), (23),

$$I_1 = \frac{s_n}{2\pi}e^{-(n\tau^2/2)+n\tau^3\lambda_n(\tau)}\frac{1}{\sqrt{\sum_{j=1}^{n}K_j''(z_0)}}\int_{-\varepsilon\sqrt{\sum_1^n K_j''(z_0)}}^{+\varepsilon\sqrt{\sum_1^n K_j''(z_0)}}e^{-(t'^2/2)} \times\exp\left\{\sum_{m=3}^{\infty}\frac{\rho_{m,n}(z_0)}{n^{m/2-1}}\cdot\frac{(it')^m}{m!}\right\}dt'.$$

The finite limits of integration may be replaced by infinite limits, making an error of the order $o(1/n)$ as $n\to\infty$. Here we have used the notation

$$\rho_{m,n}(z_0) = \frac{n^{m/2-1}\sum_{j=1}^{n}K_j^{(m)}(z_0)}{\left[\sum_{j=1}^{n}K_j''(z_0)\right]^{m/2}}.$$

We calculate

$$\frac{s_n}{\sqrt{\sum_1^n K_j''(z_0)}} = 1 - \frac{\Gamma_{3n}}{2\Gamma_{2n}^{3/2}}\tau + o(\tau), \qquad \tau \to 0,$$

and

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(t^2/2)}\left\{\sum_{m=3}^{\infty}\frac{\rho_{m,n}(z_0)}{n^{m/2-1}}\cdot\frac{(it)^m}{m!}\right\}dt = 1 + \left[\frac{\rho_{4,n}}{8} - \frac{5\rho_{3,n}^2}{12}\right]\frac{1}{n} + o\left(\frac{1}{n}\right)$$

$$= 1 + \left\{\frac{\Gamma_{4n}}{8\Gamma_{2n}^2} - \frac{5}{12}\cdot\frac{\Gamma_{3n}}{\Gamma_{2n}^3}\right\}\frac{1}{n} + o\left(\frac{1}{n}\right), \qquad n \to \infty.$$

Finally we obtain

$$I = I_n + R = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{n\tau^2}{2} + n\tau^3\lambda_n(\tau)\right\}\left[1 + O(\tau) + O\left(\frac{1}{n}\right)\right]$$

for $\tau \to 0$, $n \to \infty$.

From this it is now easy to obtain the assertion of the theorem. Put $\tau = x/\sqrt{n}$ and subject x to the condition $x = o(\sqrt{n})$ as $n \to \infty$, $x > 1$. Then $x/\sqrt{n} \to 0$ as $n \to \infty$ and for sufficiently large n we have

$$p_{z_n}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2+(x^3\sqrt{n})\,\lambda_n(x/\sqrt{n})}\left[1 + O\left(\frac{x}{\sqrt{n}}\right)\right],$$

where $\lambda_n(t)$ is defined by formula (24). The theorem is proven.

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LOCAL LIMIT THEOREMS FOR LARGE DEVIATIONS

WOLFGANG RICHTER (*LENINGRAD*)

(*Summary*)

Let (X_j), $j = 1, 2, \cdots$, be a sequence of independent random variables with the distribution functions $V_j(x)$. We assume the existence of $\mathbf{D}X_j = \sigma_j^2$, $s_n^2 = \sum_{j=1}^n \sigma_j^2$, $\mathbf{E}X_j = 0$, $j = 1, 2, \cdots$. We put

$$Z_n = \sum_{j=1}^{n} X_j/s_n.$$

With the aid of the saddlepoint method of function theory several local limit theorems are derived, in complete analogy to the previously known integral limit theorems for large deviations of H. Cramér [1] and V. Petrov [5]. These authors considered the behavior of the function $\mathbf{P}\{Z_n < x\} = F_n(x)$ for $n \to \infty$, where x together with n becomes infinite ("large deviations"). V. Petrov generalized Cramér's theorem from the case of identically distributed X_j to the general case and at the same time improved the remainder term and the growth of x. The present work shows that their method of proof, namely the introduction of a definite transformation of the distribution laws of the X_j, was very natural. The present work makes consistent use of the function theoretic possibilities that are given by the assumption that the functions

$$M_j(z) = \mathbf{E}e^{zX_j} = \int_{-\infty}^{\infty} e^{zy}\,dV_j(y)$$

are analytic in a strip $|\mathrm{Re}\, z| < A$.

Theorem 1. *Let conditions* A—C *be fulfilled. Then for sufficiently large n each Z_n possesses a distribution density $p_{z_n}(x)$. Assume further that $x > 1$ and $x = o(\sqrt{n})$ for $n \to \infty$. Then one has*

$$\frac{p_{z_n}(x)}{\varphi(x)} = e^{(x/\sqrt{n})\lambda_n(x/\sqrt{n})}\left[1+O\left(\frac{x}{\sqrt{n}}\right)\right],$$

where $\lambda_n(t)$ is a power series converging, uniformly in n, for sufficiently small values $|t|$, and $\varphi(x)$ is the density of the normal distribution.

For negative x there is a similar relation.

For identically distributed X_j the condition C can be considerably weakened. In this case Theorem 2 holds.

Also in the case of a lattice-like distribution of the random variables X_j an analogous limit relation holds (Theorem 3).