HW 5 & 6 – Motion Tracking and Lucas–Kanade with Affine Motion

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Fall 2025

(a) Derivation of the Motion Tracking Equation (Optical Flow Constraint)

Image formation and motion

A grayscale video is represented by an intensity function:

$$I(x, y, t)$$

A point in the scene appears at pixel coordinates $(x(t), y(t))$.
After a small time interval Δt, it moves to:

$$(x(t + \Delta t), y(t + \Delta t)) = (x + u\Delta t,\ y + v\Delta t)$$

where:

$$u = \frac{dx}{dt}, \qquad v = \frac{dy}{dt}$$

are the optical flow components.

Brightness Constancy

Brightness constancy states:

$$I(x, y, t) = I(x + u\Delta t,\ y + v\Delta t,\ t + \Delta t)$$

Taylor Expansion

Applying a first-order Taylor expansion:

$$I(x + u\Delta t, y + v\Delta t, t + \Delta t) \approx I + I_x(u\Delta t) + I_y(v\Delta t) + I_t\Delta t$$

Substitute into the brightness constancy equation and simplify:

$$0 \approx I_x u + I_y v + I_t$$

Optical Flow Constraint Equation

Thus:

$$I_x u + I_y v + I_t = 0$$

Vector form:

$$\nabla I^\top \begin{bmatrix} u \\ v \end{bmatrix} + I_t = 0$$

Because this is a single equation with two unknowns, additional assumptions are needed to estimate flow.

(b) Lucas–Kanade for Affine Motion

Affine Motion Model

Motion in a small window is modeled as affine:

$$u(x, y) = a_1 x + b_1 y + c_1, \qquad v(x, y) = a_2 x + b_2 y + c_2$$

Parameter vector:

$$p = [a_1,\ b_1,\ c_1,\ a_2,\ b_2,\ c_2]^\top$$

Optical Flow Constraint with Affine Motion

Substituting the affine model into the optical flow equation:

$$I_x(a_1 x + b_1 y + c_1) + I_y(a_2 x + b_2 y + c_2) + I_t = 0$$

This can be written as:

$$A(x, y)p + I_t = 0$$

where:

$$A(x, y) = [I_x x,\ I_x y,\ I_x,\ I_y x,\ I_y y,\ I_y]$$

Lucas–Kanade Least Squares Formulation

Over all pixels (x, y) in a window W, minimize:

$$E(p) = \sum_{(x,y)\in W} (A(x, y)p + I_t)^2$$

Normal equations:

$$Hp = b$$

where:

$$H = \sum A^\top A, \qquad b = -\sum A^\top I_t$$

Solution:

$$p = H^{-1} b$$

Iterative Lucas–Kanade with Affine Warp

Warp Function

Coordinates are warped using:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + u(x,y) \\ y + v(x,y) \end{bmatrix}$$

Affine matrix form:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + a_1 & b_1 & c_1 \\ a_2 & 1 + b_2 & c_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Error Function

Lucas–Kanade minimizes:

$$E(p) = \sum_{(x,y) \in W} \left(I_2(W(x,y;p)) - I_1(x,y) \right)^2$$

Linearization

Define residual:

$$e(x,y) = I_2(W(x,y;p)) - I_1(x,y)$$

Steepest-descent row:

$$J = \nabla I_2 \cdot \frac{\partial W}{\partial p}$$

Solve for parameter increment:

$$H \Delta p = -g$$

Update:

$$p \leftarrow p + \Delta p$$

Stop when $\|\Delta p\|$ is sufficiently small.

Final Summary

- The optical flow equation comes from brightness constancy and first-order Taylor expansion.
- Lucas–Kanade with affine motion fits 6 affine parameters over a window using least squares.
- The iterative version repeatedly warps the second image and refines the parameters until convergence.