Shakil Ali

Statistical Methods - Quiz 7

Question 1 (a)

Question 1
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Consider a collection of RVs, X_i (for $i = 1, \ldots, 5$), where each X_i can take integer values between 1 and 6 inclusive with equal probability. That is $\Pr(X_i = 1) = \Pr(X_i = 2) = \ldots = \Pr(X_i = 6) = \frac{1}{6}$ (a discrete uniform distribution). Give your answers to the following questions, up to 3 significant figures.

- (a) What is the expected value and SD of each X_i?

$E[X_i] =$ 3.50 [2 marks] and $SD(X_i) =$ 1.71 [3 marks]

E[Xi] and SD(Xi) was calculated as follows:

$$Pr(Xi = 1) = Pr(Xi = 2) = \ldots = Pr(Xi = 6) = \frac{1}{6}$$

$$E[X_i] = (1 * \frac{1}{6}) + (2 * \frac{1}{6}) + (3 * \frac{1}{6}) + (4 * \frac{1}{6}) + (5 * \frac{1}{6}) + (6 * \frac{1}{6}) = 3.500 = 3.50 \text{ (3 sig figs)}$$

$$SD(X_i) = \sqrt{Var(X_i)}$$

$$Var(X_i) = E[X_i^2] - \mu^2 = \frac{91}{6} - (3.5^2) = \frac{35}{12}$$

$$SD(X_i) = \sqrt{\frac{35}{12}}$$

$$SD(X_i) = 1.7078... = 1.71 \text{ (3 sf)}$$

Assistance was from:

Example UCL

A museum has a donation point where every family that enters either donates \$0, \$5, \$10 and \$20 with respective probabilities 0.8, 0.16, 0.03 and 0.01. Assuming $1000 = 10^3$ families visit in a day:

a) What is the expected total donated that day?

$$E[X_i] = \mu = 0.8 \cdot 0 + 0.16 \cdot 5 + 0.03 \cdot 10 + 0.01 \cdot 20 = \$1.30$$

$$E[\sum_i X_i] = 1000\mu = \$1300$$

b) What is the standard deviation of this total?

$$var(X_i) = \sigma^2 = E[X_i^2] - \mu^2 = 9.31$$

$$SD(\sum_i X_i) = \sqrt{n}\sigma = \$96.5$$

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Question 1 (b)

- (b) What is the expected value and SD of the mean, $\bar{X} = \frac{\sum_i X_i}{5}$?

$E[\bar{X}] =$ 3.50 [2 marks] and $SD(X) =$ 1.47 [3 marks]

E[bar X] and SD(bar X) was calculated as follows:

$E[\bar{X}]$ = ?

P$\{\bar{X} = 1\}$ = P$\{(1, 1)\}$ = $\frac{1}{36}$

P$\{\bar{X} = 1.5\}$ = P$\{(1, 2) or (2, 1)\}$ = $\frac{1}{6}$

P$\{\bar{X} = 2\}$ = P$\{(2, 2)\}$ = $\frac{1}{36}$

P$\{\bar{X} = 2.5\}$ = P$\{(2, 3) or (3, 2)\}$ = $\frac{1}{6}$

P$\{\bar{X} = 3\}$ = P$\{(3, 3)\}$ = $\frac{1}{36}$

P$\{\bar{X} = 3.5\}$ = P$\{(3, 4) or (4, 3)\}$ = $\frac{1}{6}$

P$\{\bar{X} = 4\}$ = P$\{(4, 4)\}$ = $\frac{1}{36}$

P$\{\bar{X} = 4.5\}$ = P$\{(4, 5) or (5, 4)\}$ = $\frac{1}{6}$

P$\{\bar{X} = 5\}$ = P$\{(5, 5)\}$ = $\frac{1}{36}$

P$\{\bar{X} = 5.5\}$ = P$\{(5, 6) or (6, 5)\}$ = $\frac{1}{6}$

P$\{\bar{X} = 6\}$ = P$\{(6, 6)\}$ = $\frac{1}{36}$

$E[\bar{X}] = 1(\frac{1}{36}) + 1.5(\frac{1}{6}) + 2(\frac{1}{36}) + 2.5(\frac{1}{6}) + 3(\frac{1}{36}) + 3.5(\frac{1}{6}) + 4(\frac{1}{36}) + 4.5(\frac{1}{6}) + 5(\frac{1}{36}) + 5.5(\frac{1}{6}) + 6(\frac{1}{36})$

$E[\bar{X}] = \frac{7}{2}$ = 3.5 = 3.50 (2 s.f.)

$SD(\bar{X})$ = ?

$Var(\bar{X}) = E[(\bar{X} - 1.5)^2]$

$Var(\bar{X}) = (1 - 3.5)^2\frac{1}{36} + (1.5 - 3.5)^2\frac{1}{6} + (2 - 3.5)^2\frac{1}{36} + ... + (5 - 3.5)^2\frac{1}{36} + (5.5 - 3.5)^2\frac{1}{6} + (6 - 3.5)^2\frac{1}{36}$

$Var(\bar{X}) = \frac{25}{144} + \frac{2}{3} + \frac{1}{16} + \frac{1}{6} + \frac{1}{144} + 0 + \frac{1}{144} + \frac{1}{6} + \frac{1}{16} + \frac{2}{3} + \frac{25}{144}$

$Var(\bar{X}) = \frac{155}{72}$

$SD(\bar{X}) = \sqrt{\frac{155}{72}}$

$SD(\bar{X})$ = 1.46723... = 1.47 (3 s.f.)

Assistance from (introductory statistics page 301):

with their respective probabilities are as follows:

$$P\{\overline{X} = 1\} = P\{(1, 1)\} = \frac{1}{4}$$
$$P\{\overline{X} = 1.5\} = P\{(1, 2) \text{ or } (2, 1)\} = \frac{2}{4} = \frac{1}{2}$$
$$P\{\overline{X} = 2\} = P\{(2, 2)\} = \frac{1}{4}$$

Therefore,

$$E[\overline{X}] = 1\left(\frac{1}{4}\right) + 1.5\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) = \frac{6}{4} = 1.5$$

Also

$$\begin{aligned}\text{Var}(\overline{X}) &= E\left[(\overline{X} - 1.5)^2\right] \\ &= (1 - 1.5)^2\left(\frac{1}{4}\right) + (1.5 - 1.5)^2\left(\frac{1}{2}\right) + (2 - 1.5)^2\left(\frac{1}{4}\right) \\ &= \frac{1}{16} + 0 + \frac{1}{16} = \frac{1}{8}\end{aligned}$$

which, since $\mu = 1.5$ and $\sigma^2 = 1/4$, verifies that $E[\overline{X}] = \mu$ and $\text{Var}(\overline{X}) = \sigma^2/2$.

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Question 1(c)

Consider instead the finite population of values $\{1, 2, 3, 4, 5, 6\}$. An experiment is performed where 5 values in sequence are taken out of this population without replacement. The ith value drawn in this experiment is RV Y_i.

- (c) What is the expected value and SD of Y_3?

  $E[Y_3] =$ 5.25 [2 marks] and $SD(Y_3) =$ 11.0 [3 marks]

I found that question 1(c) is similar to question 1(a)

6 * 5 * 4 * 3 * 2 = 720 possible sets of 5 numbers from 6 digits

E[Y3] indicates that it is the third value in the sequence (2 numbers have been selected)

Therefore:

4 * 3 * 2 = 24 possible sets of 3 numbers from 4 digits

Not sure what to do from this point on, although I know how to calculate E[X] and SD[X]

I guess the question here is how to calculate E[X] and SD[X] when you don't know the X values in advanced? I don't think calculating 24 E[X]'s then averaging them is the right way to go either ☹

Due to the failure of doing **Question 1(c),** I cannot do **Question 1(d)** and **Question 1(e)**

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Question 2(a)

> I go out for a meal with 4 friends each month, to a restaurant that costs £20 a head. We are each given a number between 1 and 5 (I am number 3) and each time we go out we roll a die. If the die face shows one of our numbers that person pays the bill for everyone. If a 6 is rolled, then we each pay for our meal.
>
> a. What is the expected value of the cost of a single meal to me?
>
> Expected cost of a single meal is 1.00 [2 marks]

Expected single cost means a 6 has been rolled

The characteristic (k) is 6 being rolled

Therefore Pr(X = 0) for rolls 1,2,3,4,5 (except 6)

Pr(X_6 = 1) = k/N = 1/6 (rolling 6)

p = 1/6

E[X] = *np*

E[X] = 6 * (1/6)

E[X] = 1.00

NOTE: I think answer can also be 20.0 because they say value (£20)

Assistance from:

Samples from finite populations

Definition: A sample of size n selected from a population of N elements is **a random sample** if it is selected so that it is equally likely to be any subset of size n.

Suppose such a random sample is taken, a characteristic of interest is given, and for $i = 1, \dots, n$

$$X_i = \begin{cases} 1 & \text{if the } i\text{th member of the sample has the characteristic} \\ 0 & \text{otherwise} \end{cases}$$

Consider the sum

$$X = X_1 + X_2 + \dots X_n$$

This counts the number in the sample with the characteristic.

Approximate independence

As we have a finite population, N the number of elements with our characteristic, say K, means that a proportion $p = \frac{K}{N}$ have that characteristic.

For our sample, $\Pr(X_i = 1) = \frac{K}{N} = p$ for any i.

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Question 2(b)

b. What is the standard deviation of the cost of a single meal to me?

Standard deviation of cost of single meal is 0.913 [3 marks]

SD[X] calculated as follows:

Question 2(b)

$SD[X] = \sqrt{np(1-p)}$

n = 6

p = $\frac{1}{6}$

$SD[X] = \sqrt{6 * \frac{1}{6}(1 - \frac{1}{6})}$

$SD[X] = \sqrt{1(\frac{5}{6})}$

$SD[X] = \sqrt{\frac{5}{6}}$

$SD[X]$ = 0.91287... = 0.913 (3sf)

Assistance from:

Binomial Approximation

For large enough N, the sum

$$X = X_1 + X_2 + \ldots X_n$$

is a RV approximately distributed as Binomial(n, p) with

$$\mathbb{E}[X] = np \qquad \text{and} \qquad \mathrm{SD}[X] = \sqrt{np(1-p)}$$

The mean $\overline{X} = X/\,n$ is the proportion of those in the sample with the characteristic and

$$\mathbb{E}[\overline{X}] = p \qquad \text{and} \qquad \mathrm{SD}[\overline{X}] = \sqrt{p(1-p)/\,n}$$

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Question 2(c)

c. What is the expected value of the average cost of a meal over a year'?

Expected value of average yearly cost is | 12.00 | [2 marks]

My answer:

Question 2(c)

Because we have $E[X]$ = 1.00 for cost of meal per month

For a year = $E[X]$ * 12 = 12.00

NOTE: I think answer can also be 240 because they say value (£20 a month for a year)

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Exercise 2(d)

d. What is the standard deviation of the average cost of a meal over a year?

Standard deviation of average yearly cost is 3.16 [3 marks]

My answer:

Question 2(d)

$E[X]$ for a year = 12.00 (part c)

$SD[X]$ for a year = ?

$SD[X] = \sqrt{np(1-p)}$

n = 72 (n = 6 for a month, so for a year = 12 * 6 = 72)

p = $\frac{12}{72}$ = $\frac{1}{6}$

$SD[X]$ for a year = $\sqrt{np(1-p)}$

$SD[X]$ for a year = $\sqrt{72 * \frac{1}{6}(1-\frac{1}{6})}$

$SD[X]$ for a year = $\sqrt{12(\frac{5}{6})}$

$SD[X]$ for a year = $\sqrt{\frac{60}{6}}$

$SD[X]$ for a year = $\sqrt{10}$

$SD[X]$ for a year = 3.16227... = 3.16 (3sf)

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Exercise 2(e)

e. If I go out in this way every month for 3 years, with what probability will I pay in total between £600 and £840 for my meals out?

Pr(I pay in total between £600 and £840) = 0.620 [5 marks]

My answer:

Question 2(e)

Mean E[X] = 12 * 36 (for three years value)

μ = 720

Standardisation formula (Z) = $\frac{X-\mu}{\sigma}$

Pr(600 <= X <= 840)

n = 36 (3 years)

$SD(X) = \sqrt{n}\sigma = 36\sqrt{5}$

Pr(X <= 840) = $\frac{720-840}{36\sqrt{5}}$

Pr(X <= 840) = -1.4907... = -1.49 (3sf)

Z (-1.49) = 0.93189

Pr(X >= 600) = 1 - Pr(X <= 600)

Pr(X >= 600) = 1 - 1.4907..

Pr(X >= 600) = -0.4907..

Pr(X >= 600) = -0.490 (3sf)

Z (-0.490) = 0.31207

Pr(600 <= X <= 840) = 0.93189 - 0.31207

Pr(600 <= X <= 840) = 0.61982 = 0.620 (3sf)

Assistance from:

Example UCL

A museum has a donation point where every family that enters either donates \$0, \$5, \$10 and \$20 with respective probabilities 0.8, 0.16, 0.03 and 0.01. Assuming $1000 = 10^3$ families visit in a day:

a) What is the expected total donated that day?

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$$E[\sum_i X_i] = 1000\mu = \$1300$$

b) What is the standard deviation of this total?

$$var(X_i) = \sigma^2 = E[X_i^2] - \mu^2 = 9.31$$

$$SD(\sum_i X_i) = \sqrt{n}\sigma = \$96.5$$

c) With what probability will this exceed \$1500?

$$\Pr\left(\sum_i X_i > \$1500\right) = \Pr\left(Z > \frac{1500-1300}{96.5}\right) = 0.192$$

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Exercise 2(f)

f. Approximate the amount C such that the following statement is true: There is a roughly 90% chance that I will spend less than £C over 3 years on my meals out, and a roughly 10% chance that I will spend more than £C.

$C =$ 720 [5 marks]

For this question I am completely confused, I suspect it has something to do with lecture 8, but I am not sure.