L12 Spectral Partitioning of Bipartite Graphs

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May 8, 2018

1 Spectral Partitioning of Bipartite Graphs

Typical example of bipartite graph is a graph obtained from a collection of documents presented as a *term* $\times$ *document* matrix.

1.1 Prerequisites

The reader should be familiar with k-means algorithm and spectral graph partitioning theory and algorithms.

1.2 Competences

The reader should be able to apply spectral partitioning of bipartite graphs to data clustering problems.

Credits: The notebook is based on [Mir05].

References

[Mir05] I. Mirošević, 'Spectral Graph Partitioning and Application to Knowledge Extraction', M.Sc. Thesis, University of Zagreb, 2005 (in Croatian).

1.3 Definitions

Undirected bipartite graph G is a triplet $G = (T, D, E)$, where $T = \{t_1, \cdots, t_m\}$ and $D = \{d_1, ..., d_n\}$ are two sets of vertices and $E = \{(t_i, d_j) : t_i \in R, d_j \in D\}$, is a set of edges.

G is **weighted** if there is weight $\omega(e)$ associated with each edge $e \in E$.

For example, D is a set of documents, T is a set of terms (words) and edge $e = (t_i, d_j)$ exists if document d_j contains term t_i. Weight $\omega(e)$ can be number of appearances of the term t_i in the document d_j.

A **term-by-document-matrix** is a matrix $A \in \mathbb{R}^{m\times n}$ with $A_{ij} = \omega((t_i, d_j))$.

1.4 Facts

1. The weight matrix of G is $W = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$.

2. The Laplacian matrix of G is
$$L = \begin{bmatrix} \Delta_1 & -A \\ -A^T & \Delta_2 \end{bmatrix},$$
where Δ_1 and Δ_2 are diagonal matrices with elements $\Delta_{1,ii} = \sum_{j=1}^{n} A_{ij}$ for $i = 1, \ldots, m$, and $\Delta_{1,jj} = \sum_{i=1}^{m} A_{ij}$ for $j = 1, \ldots, n$.

3. The normalized Laplacian matrix of G is
$$L_n = \begin{bmatrix} I & -\Delta_1^{-\frac{1}{2}} A \Delta_2^{-\frac{1}{2}} \\ -\Delta_2^{-\frac{1}{2}} A^T \Delta_1^{-\frac{1}{2}} & I \end{bmatrix} \equiv \begin{bmatrix} I & -A_n \\ -A_n^T & I \end{bmatrix}.$$

4. Let λ be an eigenvalue of L_n with an eigenvector $w = \begin{bmatrix} u \\ v \end{bmatrix}$, where $u \in \mathbb{R}^m$ $v \in \mathbb{R}^n$. Then $L_n w = \lambda w$ implies $A_n v = (1-\lambda)u$ and $A_n^T u = (1-\lambda)v$. Vice versa, if (u, σ, v) is a singular triplet of A_n, then $1 - \sigma$ is an eigenvalue of L_n with (non-unit) eigenvector $w = \begin{bmatrix} u \\ v \end{bmatrix}$.

5. The second largest singular value of A_n corresponds to the second smallest eigenvalue of L_n, and computing the former is numerically more stable.

6. **Bipartitioning algorithm** is the following:

    1. For given A compute A_n.
    2. Compute singular vectors of A_n, $u^{[2]}$ and $v^{[2]}$, which correspond to the second largest singular value, $\sigma_2(A_n)$.
    3. Assign the partitions $T = \{T_1, T_2\}$ and $D = \{D_1, D_2\}$ according to the signs of $u^{[2]}$ and $v^{[2]}$. The pair (T, D) is now partitioned as $\{(T_1, D_1), (T_2, D_2)\}$.

7. **Recursive bipartitioning algorithm** is the following:

    1. Compute the bipartition $\pi = \{(T_1, D_1), (T_2, D_2)\}$ of (T, D). Set the counter $c = 2$.
    2. While $c < k$ repeat
        1. compute bipartitions of each of the subpartitions of (T, D),
        2. among all $(c+1)$-subpartitions, choose the one with the smallest $pcut(\pi_{c+1})$ or $ncut(\pi_{c+1})$, respectively.
    3. Set $c = c + 1$
    4. Stop

8. **Multipartitioning algorithm** is the following:

    1. For given A compute A_n.

2. Compute k left and right singular vectors, $u^{[1]},\ldots,u^{[k]}$ and $v^{[1]},\ldots,v^{[k]}$, which correspond to k largest singular values $\sigma_1 \geq \cdots \geq \sigma_k$ of A_n.
3. Partition the rows of matrices $\Delta_1^{-\frac{1}{2}} \begin{bmatrix} u^{[1]} & \ldots & u^{[k]} \end{bmatrix}$ and $\Delta_2^{-\frac{1}{2}} \begin{bmatrix} v^{[1]} & \ldots & v^{[k]} \end{bmatrix}$ with the k-means algorithm.

1.4.1 Example - Small term-by- document matrix

In [1]:
```
# Some packages
using LightGraphs
using GraphPlot
using Clustering
```

In [2]:
```
# Some functions
function my_weight_matrix(src::Array,dst::Array,weights::Array)
    n=nv(G)
    sparse([src;dst],[dst;src],[weights;weights],n,n)
end

my_laplacian(W::AbstractMatrix)=sparse(diagm(vec(sum(W,2))))-W

function my_normalized_laplacian(L::AbstractMatrix)
    D=1.0./sqrt.(diag(L))
    n=length(D)
    [L[i,j]*(D[i]*D[j]) for i=1:n, j=1:n]
end
```

Out[2]:
```
my_normalized_laplacian (generic function with 1 method)
```

In [3]:
```
# Sources, targets, and weights
n=7
dn=[6,6,7,6,7,7]
tn=[1,2,2,3,4,5]
wn=[3,1,3,2,2,3]
[dn tn wn]
```

Out[3]:
```
6×3 Array{Int64,2}:
 6  1  3
 6  2  1
 7  2  3
 6  3  2
 7  4  2
 7  5  3
```

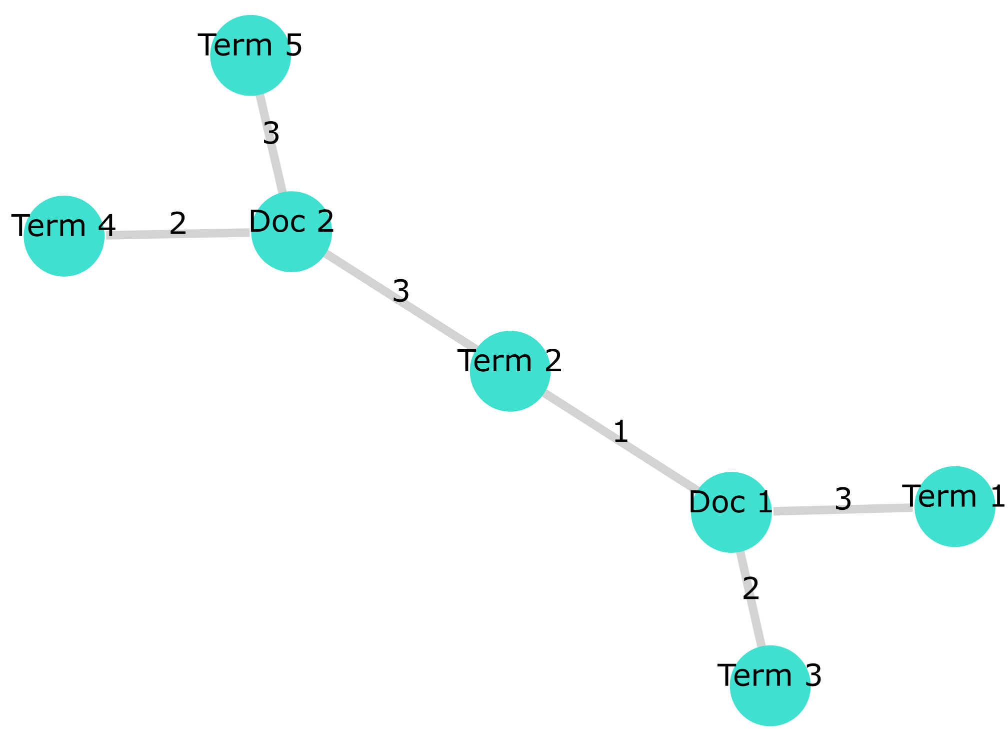
In [4]:
```
mynames=["Term 1";"Term 2";"Term 3";"Term 4";"Term 5";"Doc 1";"Doc 2"]
```

Out[4]:
```
7-element Array{String,1}:
 "Term 1"
```

```
 "Term 2"
 "Term 3"
 "Term 4"
 "Term 5"
 "Doc 1"
 "Doc 2"
```

In [5]:
```
G=Graph(n)
for i=1:length(dn)
    add_edge!(G,tn[i],dn[i])
end
gplot(G, nodelabel=mynames, edgelabel=wn)
```

Out[5]:



In [6]:
```
W=my_weight_matrix(tn,dn,wn)
```

Out[6]:
```
7×7 SparseMatrixCSC{Int64,Int64} with 12 stored entries:
  [6, 1]  =  3
  [6, 2]  =  1
  [7, 2]  =  3
  [6, 3]  =  2
```

```
  [7, 4]  =  2
  [7, 5]  =  3
  [1, 6]  =  3
  [2, 6]  =  1
  [3, 6]  =  2
  [2, 7]  =  3
  [4, 7]  =  2
  [5, 7]  =  3
```

In [7]:
```
full(W)
```

Out[7]:
```
7×7 Array{Int64,2}:
 0  0  0  0  0  3  0
 0  0  0  0  0  1  3
 0  0  0  0  0  2  0
 0  0  0  0  0  0  2
 0  0  0  0  0  0  3
 3  1  2  0  0  0  0
 0  3  0  2  3  0  0
```

In [8]:
```
L=my_laplacian(W)
full(L)
```

Out[8]:
```
7×7 Array{Int64,2}:
  3   0   0   0   0  -3   0
  0   4   0   0   0  -1  -3
  0   0   2   0   0  -2   0
  0   0   0   2   0   0  -2
  0   0   0   0   3   0  -3
 -3  -1  -2   0   0   6   0
  0  -3   0  -2  -3   0   8
```

In [9]:
```
Ln=my_normalized_laplacian(L)
```

Out[9]:
```
7×7 Array{Float64,2}:
  1.0        0.0        0.0        0.0   0.0        -0.707107   0.0
  0.0        1.0        0.0        0.0   0.0        -0.204124  -0.53033
  0.0        0.0        1.0        0.0   0.0        -0.57735    0.0
  0.0        0.0        0.0        1.0   0.0         0.0       -0.5
  0.0        0.0        0.0        0.0   1.0         0.0       -0.612372
 -0.707107  -0.204124  -0.57735    0.0   0.0         1.0        0.0
  0.0       -0.53033    0.0       -0.5  -0.612372    0.0        1.0
```

In [10]:
```
A=W[1:5,6:7]
Δ1=sqrt.(sum(A,2))
Δ2=sqrt.(sum(A,1))
An=[A[i,j]/(Δ1[i]*Δ2[j]) for i=1:size(A,1), j=1:size(A,2)]
```

Out[10]: 5×2 Array{Float64,2}:
```
 0.707107  0.0
 0.204124  0.53033
 0.57735   0.0
 0.0       0.5
 0.0       0.612372
```

In [11]:
```
# The partitioning - explain the results!
U,σ,V=svd(An)
```

Out[11]: ([-0.46291 0.604743; -0.534522 -0.218218; ... ; -0.377964 -0.370328; -0.46291 -0.453557

In [12]:
```
U[:,2]
```

Out[12]: 5-element Array{Float64,1}:
```
  0.604743
 -0.218218
  0.493771
 -0.370328
 -0.453557
```

In [13]:
```
V[:,2]
```

Out[13]: 2-element Array{Float64,1}:
```
  0.755929
 -0.654654
```

1.4.2 Example - Sets of points

In [14]:
```
using Gadfly
using Images
```

In [15]:
```
?sprand;
```

search: sprand sprandn StepRange StepRangeLen spectral_distance

In [16]:
```
# Define sizes
m=[200,100,100]
n=[100,200,100]
density=[0.5,0.7,0.4]
A=Array{Any}(3)
s=srand(421)
for i=1:3
    A[i]=sprand(m[i],n[i],density[i])
end
B=blkdiag(A[1],A[2],A[3])
```

Out[16]: 400×400 SparseMatrixCSC{Float64,Int64} with 28016 stored entries:

```
  [4  ,   1]  =  0.0793511
  [6  ,   1]  =  0.140307
  [9  ,   1]  =  0.298078
  [13 ,   1]  =  0.982495
  [20 ,   1]  =  0.0537056
  [23 ,   1]  =  0.977629
  [25 ,   1]  =  0.432631
  [27 ,   1]  =  0.954235
  [29 ,   1]  =  0.330135
  [31 ,   1]  =  0.246655
  ⋮
  [382, 400]  =  0.439011
  [385, 400]  =  0.112492
  [387, 400]  =  0.287446
  [388, 400]  =  0.300438
  [390, 400]  =  0.907918
  [392, 400]  =  0.782529
  [393, 400]  =  0.587851
  [394, 400]  =  0.568105
  [395, 400]  =  0.568407
  [398, 400]  =  0.923563
  [399, 400]  =  0.276277
```

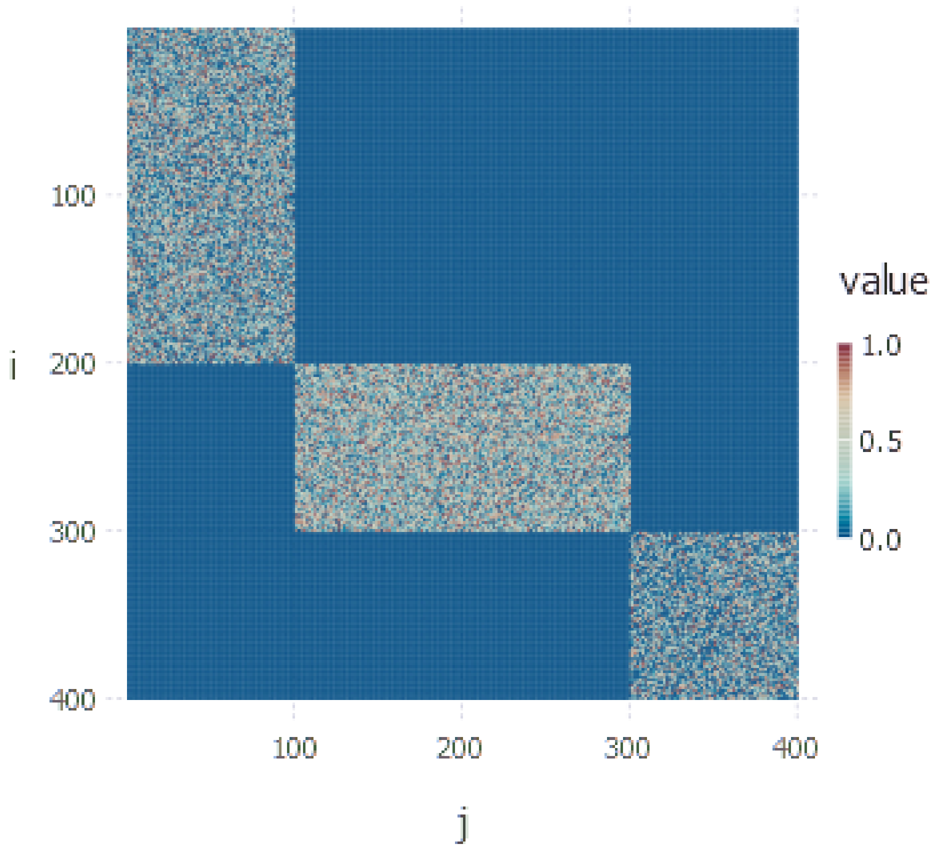
In [17]:

```
pB=spy(B)
draw(Gadfly.PNG("files/pB.png", 4inch, 4inch), pB)
```

In [18]:

```
load("files/pB.png")
```

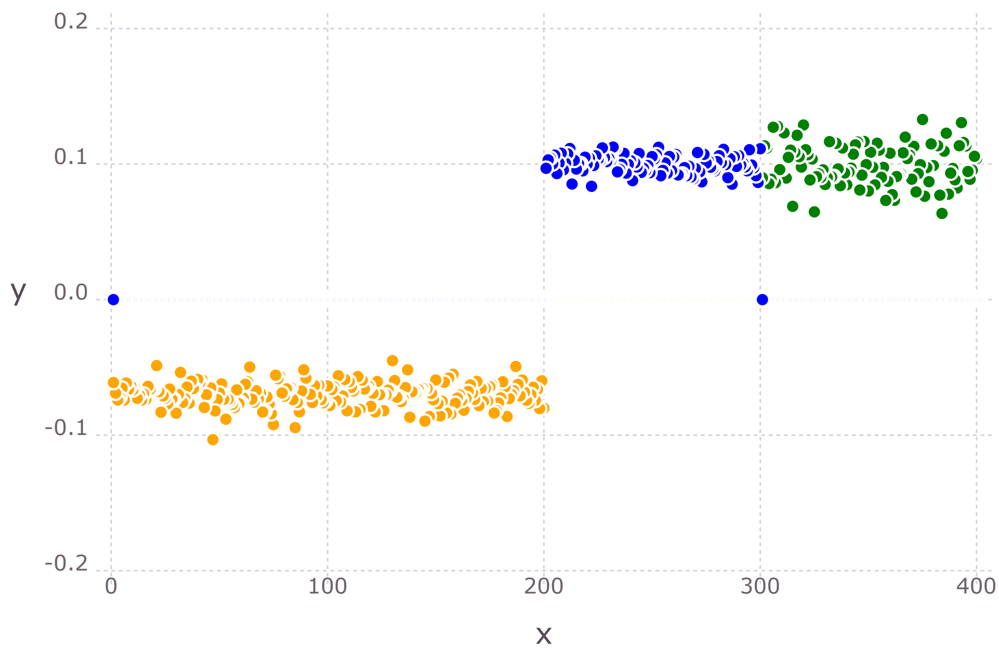
Out[18]:




```
In [19]: # The structure of singular vectors reflects the blocks
         S,rest=svds(B,nsv=3)

Out[19]: (Base.LinAlg.SVD{Float64,Float64,Array{Float64,2}}([-3.78886e-17 -0.0612099 1.7273e-17;

In [20]: # S is a structure
         fieldnames(S)

Out[20]: 3-element Array{Symbol,1}:
          :U
          :S
          :Vt

In [21]: # Plot the first three left singular vectors
         k=size(B,1)
```

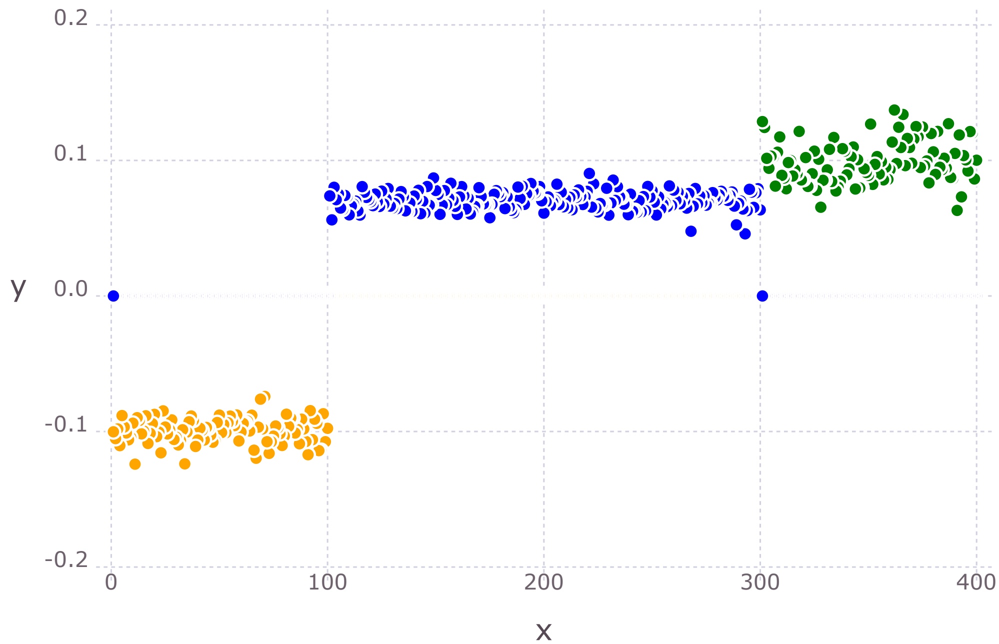
```
x=collect(1:k)
Gadfly.plot(layer(x=x,y=S.U[:,1],Geom.point,Theme(default_color=colorant"blue")),
layer(x=x,y=S.U[:,2], Geom.point,Theme(default_color=colorant"orange")),
layer(x=x,y=S.U[:,3], Geom.point,Theme(default_color=colorant"green")))
```

Out[21]:



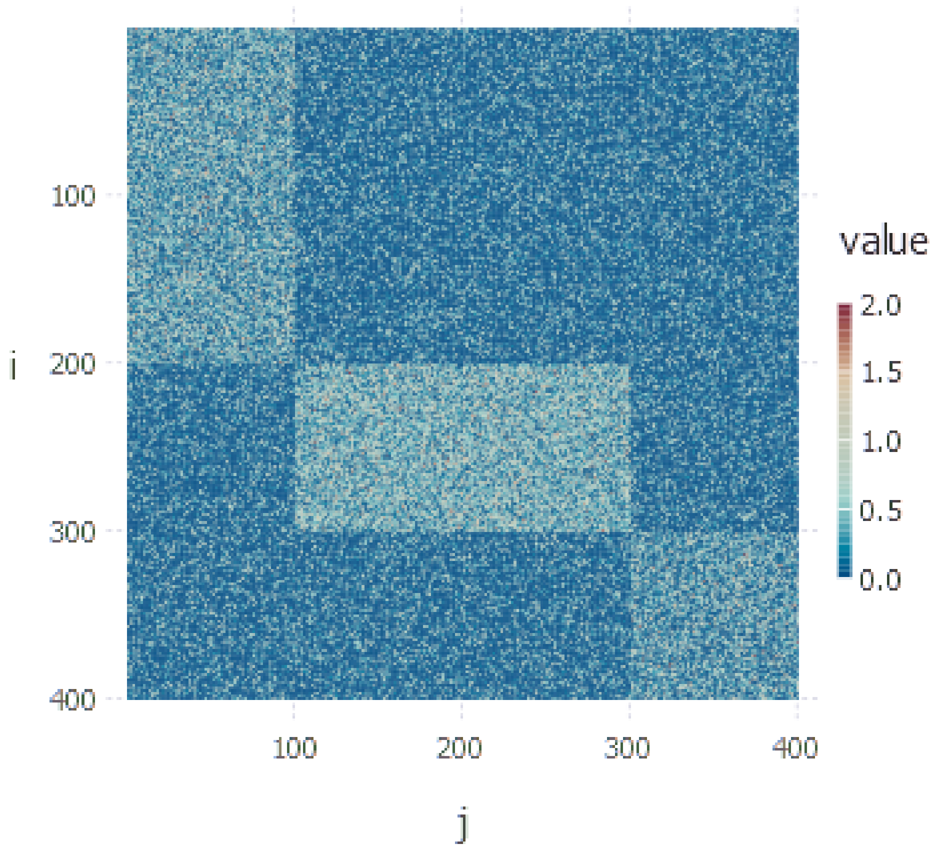
In [22]:
```
# Plot the first three right singular vectors
Gadfly.plot(layer(x=x,y=S.Vt[1,:],Geom.point,Theme(default_color=colorant"blue")),
layer(x=x,y=S.Vt[2,:], Geom.point,Theme(default_color=colorant"orange")),
layer(x=x,y=S.Vt[3,:], Geom.point,Theme(default_color=colorant"green")))
```

Out[22]:



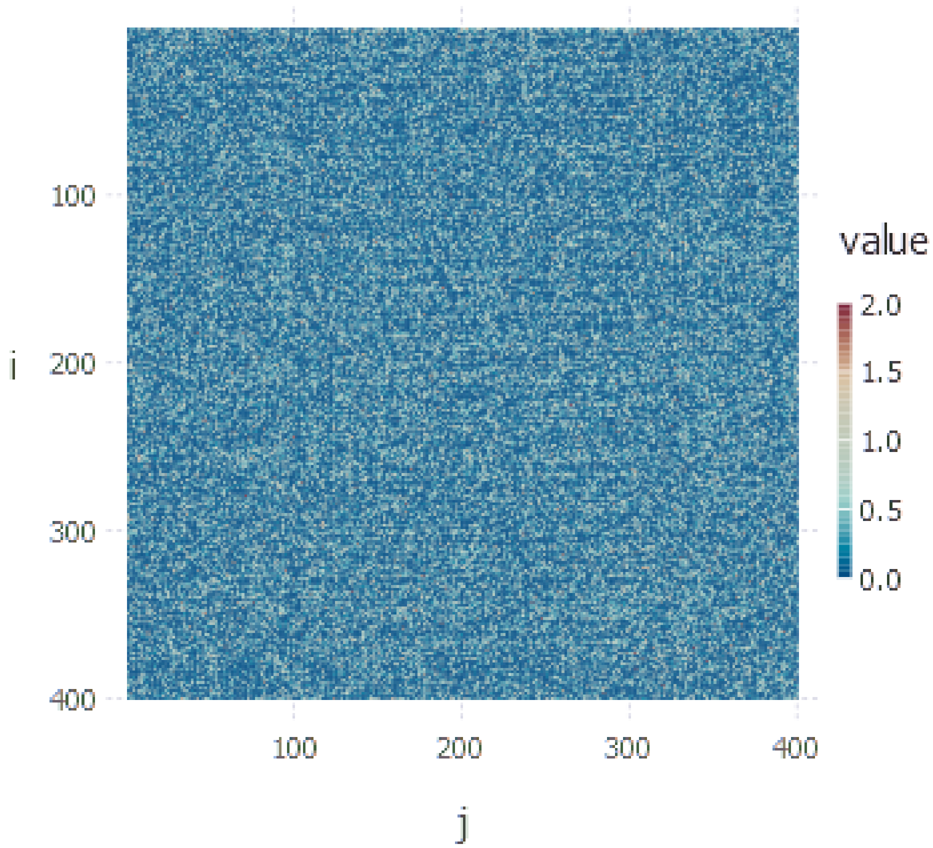

```
In [23]: # Add random noise
         noise=sprand(k,k,0.3)
         C=B+noise
         pC=spy(C)
         draw(Gadfly.PNG("files/pC.png", 4inch, 4inch), pC)
         load("files/pC.png")
```

Out[23]:




```
In [24]: # Apply random permutation to rows and columns of C
         D=C[randperm(k),randperm(k)]
         pD=spy(D)
         draw(Gadfly.PNG("files/pD.png", 4inch, 4inch), pD)
         load("files/pD.png")
```

Out[24]:



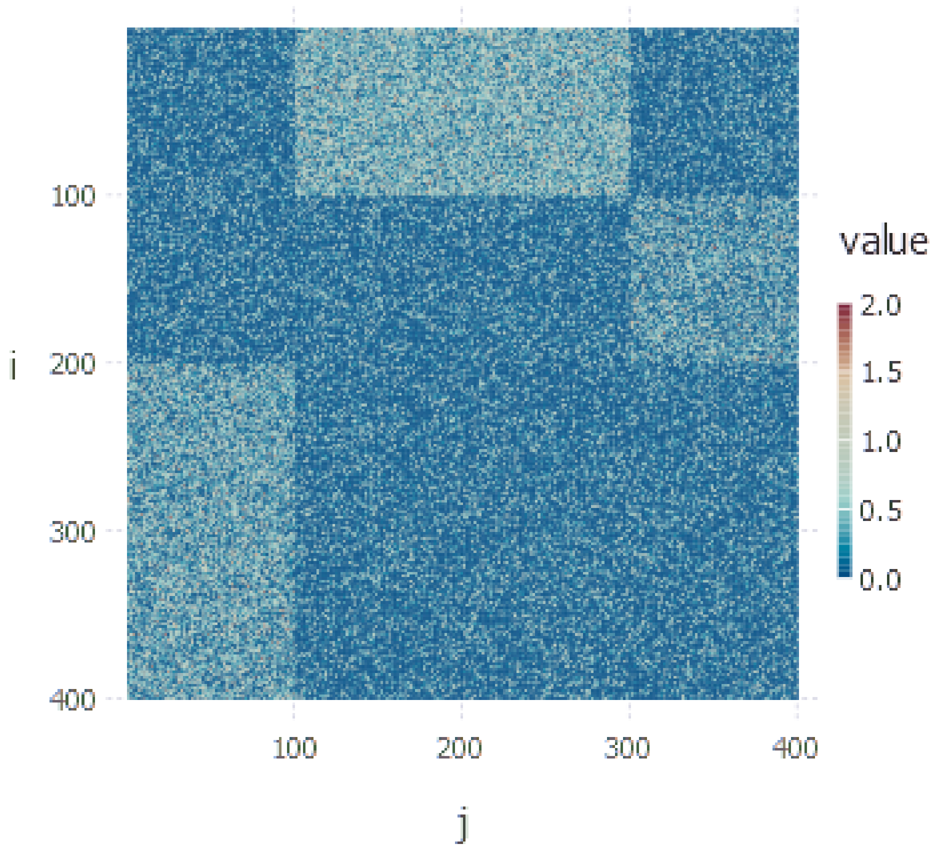

```
In [25]: # Given D, can we recover C (with spectral partitioning)?
         S,rest=svds(D,nsv=3)

Out[25]: (Base.LinAlg.SVD{Float64,Float64,Array{Float64,2}}([-0.0651511 -0.0643788 -0.00822275;

In [26]: # K-means on rows of U
         outU=kmeans((S.U)',3)

Out[26]: Clustering.KmeansResult{Float64}([-0.0688622 -0.0377666 -0.0435021; -0.0694819 0.015123

In [27]: # K-means on Vt
         outV=kmeans(S.Vt,3)

Out[27]: Clustering.KmeansResult{Float64}([-0.0518106 -0.0541609 -0.0374349; 0.0740591 -0.043118
```

In [28]:
```
sortperm(outV.assignments)
```

Out[28]:
```
400-element Array{Int64,1}:
   9
  13
  28
  29
  36
  39
  43
  45
  49
  50
  51
  52
  53
   ⋮
 339
 345
 361
 363
 365
 374
 381
 385
 386
 388
 390
 400
```

In [29]:
```
E=D[sortperm(outU.assignments),sortperm(outV.assignments)]
pE=spy(E)
draw(Gadfly.PNG("files/pE.png", 4inch, 4inch), pE)
load("files/pE.png")
```

Out[29]:




```
In [ ]:
```