Application of Kronecker Product Operation in Interpolation Via Least Squares Method

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الخلاصة

في هذا البحث نصف تطبيق عملية ضرب كرونكر بصياغة بناء الدالة المجدولة . شروط البناء حسبت على شكل ضرب كرونكر مع أستعمال طريقة المربعات الصغرى .

سنعطي نبذة مختصرة لبعض خواص ضرب كرونكر وعمليات vec– للمصفوفات . لهذه الطريقة تم استخدم ماتلاب (ver.6.5) في الاحتسابات وايضا اعطيت بعض الامثلة .

ABSTRACT

In this paper, we describe the application of the kronecker product operation to formulate the interpolation of tabulated function. Interpolation conditions are computed as kronecker product using least squares method.

We shall briefly review some properties of the kronecker product and the Vec–operator of matrices. For this method, the MATLAB (ver.6.5) is used in computations and some examples are given.

INTRODUCTION

We give a short introduction of the kronecker product operation and the method of least squares which are two of the important tools in statistics, for more details see [1] , [2] [3].We need these two tools for the interpolation of tabulated function values.

The kronecker product operation (also known as outer product or tensor product) has been successfully used as a framework for understanding different variants of the fast Fourier transform and has proved to be very useful in various branches of mathematics such as approximation theory, combinatorial theory and linear Algebra [4].

The least squares method (LSM) is one of the oldest techniques of modern statistics, and even though ancestors of LSM can be traced up to Greek mathematics, the first modern precursor is probably Galileo [5]. The modern approach was first exposed in 1805 by the French mathematician Legendre in a now classic memoir, but this method is somewhat older because it turned out that, after the publication of Legendre's memoir, Gauss (the famous German mathematician) contested Legendre's priority .

Now a day, the least squares method is widely used to find or estimate the numerical values of the unknown parameters to fit a function to a set of data points and to characterize the statistical properties of estimates. It exists with several variations: It's simpler version called ordinary least squares (OLS).

The Least Squares Method

In the standard linear statistical model [1],[3],[6]

$$\underline{Y} = X\,\underline{\beta} + \underline{\varepsilon} \qquad \text{... (1)}$$

where $\underline{Y}$ is the ($n\times1$) response vector, X is an ($n\times p$) model design matrix, $\underline{\beta}$ is a ($p\times1$) vector of unknown parameters to be estimated; and $\underline{\varepsilon}$ is an ($n\times1$) vector of random errors.

Assuming that $\underline{\varepsilon} \sim N(0, \sigma^2 I_n)$ leads to the familiar ordinary–least squares (OLS) estimator of $\underline{\beta}$ which is given in the following form :

$$\underline{b}_{OLS} = \hat{\beta} = (X'X)^{-1}X'Y$$

... (2)

$$V(b_{OLS}) = \sigma^2 (X'X)^{-1}$$

The kronecker product Operation

The kronecker product operation is a special operator used in statistics and matrix algebra for multiplication of two matrices[1],[3],[4]. This product operation, written as ⊗, gives the possibility to obtain a composite matrix of the elements of any pair of matrices. " any " stresses here that the kronecker product operation works without the assumption on the size of composing matrices, as it is the case with ordinary matrix multiplication [1].

The kronecker product of two matrices $A = (a_{ij})$ and $B = (b_{st})$ is defined as[1],[3],[4]:

$$A_{(m\times n)} \otimes B_{(p\times q)} = (a_{ij}B)\ ,\ i=1,...,m;\ j=1,.....,n$$

The result of this product is a new matrix of order (mp) × (nq) composed of all possible $a_{ij}\ b_{st}$, with s=1,……,p; t=1,…..q.

Some properties of the kronecker product operation

1- $(A\otimes B) \bullet (E\otimes F) = (A\cdot E) \otimes (B\cdot F)$
2- $(A+B) \otimes E = A\otimes E + B\otimes E$
3- $(A\otimes B) \otimes E = A\otimes (B\otimes E)$
4- $(A\otimes B)^{-1} = (A^{-1}\otimes B^{-1})$
5- $(A\otimes B)^{t} = (A^{t}\otimes B^{t})$
6- $(\alpha A)\otimes(\beta B) = (\alpha\beta)(A\otimes B), \alpha,\beta \in R$

Vec – Operator

Vec–Operator, written as Vec, is another statistical and linear algebra tool which is important in the multidimensional regression matrix representation. The mechanism of Vec–Operator is simple and can be applied to a matrix of any order. This operator transforms the matrix into a column vector, by stacking all the columns of the matrix one underneath the other [7].

For any matrix $A_{(m\times n)}$ with the i – th column defined as a_i

$$\mathrm{Vec}\ (A) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Some properties of theVec–Operator

1- Vec(A+B) = Vec(A) + Vec(B)

2- Vec(αA) = α Vec A, $\alpha \in R$

3- Vec a′ = Vec a = a, for any a .

Some relationships between theVec–operator and the kronecker product operation

1- Vec ab′ = $b\otimes a$ for any vectors a and b

2- Vec (ABC) = ($C'\otimes A$) Vec B , whenever ABC is well defined.

The Interpolation

The kronecker product also arise form interpolation of tabulated function. Let F be a matrix , $F = F_{(ij)}$ represent tabulated function values for $F_{ij} = F(x_i , y_j)$. The function F(x,y) can be approximated as [8]:

$$F(x,\ y) = \sum_{k,l} C_{kl}\ \varphi_k(x)\ \varphi_l(y) \qquad ...(3)$$

In this paper the basis functions φ are chosen as:

$$\varphi_k(x) = x^{k-1} \quad \text{and} \quad \varphi_l(y) = y^{l-1} \qquad ...(4)$$

The coefficients C_{kl} can be computed to ordinary–least squares (OLS) satisfy the interpolation conditions.

$$F_{ij} = \sum_{k,l} C_{kl}\ \varphi_k(x_i)\ \varphi_l(y_j) \qquad ...(5)$$

The interpolation condition can be expressed as a kronecker product, $F = (T_y \otimes T_x)\bullet C + \varepsilon$

where

$$T_x = \begin{pmatrix} \varphi_1(x_1)\cdots\varphi_n(x_1) \\ \vdots \\ \varphi_1(x_n)\cdots\varphi_n(x_n) \end{pmatrix}, T_y = \begin{pmatrix} \varphi_1(y_1)\cdots\varphi_n(y_1) \\ \vdots \\ \varphi_1(y_n)\cdots\varphi_n(y_n) \end{pmatrix} \qquad ...(6)$$

The columns of T_x and T_y contain the values of the basis function evaluated as the interpolation knots. The coefficients C_{kl} can be efficiently computed using the properties of the kronecker product and equation(1) as:

$$F = (T_y \otimes T_x) \bullet C_{OLS} + \varepsilon$$

$$[F - (T_y \otimes T_x) \bullet C_{OLS}]'[F - (T_y \otimes T_x) \bullet C_{oLs}] = \varepsilon'\varepsilon$$

$$F'F - [(T_y \otimes T_x) \bullet C_{OLS}]'F - F'[(T_y \otimes T_x) \bullet C_{OLS}] +$$
$$[(T_y \otimes T_x) \bullet C_{OLS}]'[(T_y \otimes T_x) \bullet C_{OLS}] = S$$

$$\frac{\partial S}{\partial C_{OLS}} = 0 - 2(T_y \otimes T_x)'F + 2(T_y \otimes T_x)'[(T_y \otimes T_x) \bullet C_{OLS}] = 0$$

$$(T_y \otimes T_x)'F = (T_y \otimes T_x)'[(T_y \otimes T_x) \bullet C_{OLS}]$$

$$\therefore C_{OLS} = [(T_y \otimes T_x)' \bullet (T_y \otimes T_x)]^{-1}(T_y \otimes T_x)'F$$
$$= [(T_y \otimes T_x)^{-1} \bullet (T_y \otimes T_x)'^{-1}](T_y \otimes T_x)'F$$

$$C_{OLS} = (T_y \otimes T_x)^{-1}[(T_y \otimes T_x)^{-1'} \bullet (T_y \otimes T_x)']F$$
$$= (T_y \otimes T_x)^{-1}[(T_y \otimes T_x) \bullet (T_y \otimes T_x)^{-1}]'F$$
$$= (T_y \otimes T_x)^{-1} \bullet I \bullet F$$
$$\therefore C_{OLS} = (T_y \otimes T_x)^{-1} \bullet F \qquad ... (7)$$

where

$$C_{OLS} = \mathrm{Vec}\ (C_{OLS}) = \begin{pmatrix} C_{11} \\ C_{21} \\ \vdots \\ \underline{C_{2n}} \\ C_{12} \\ C_{22} \\ \vdots \\ \underline{C_{n2}} \\ C_{1n} \\ C_{2n} \\ \vdots \\ C_{nn} \end{pmatrix}$$

$$F = \mathrm{Vec}\ (F) = \begin{pmatrix} F_{11} \\ F_{21} \\ \vdots \\ \underline{F_{2n}} \\ F_{12} \\ F_{22} \\ \vdots \\ \underline{F_{n2}} \\ F_{1n} \\ F_{2n} \\ \vdots \\ F_{nn} \end{pmatrix}$$

Examples

The performance of the proposed method described earlier in this paper, will be tested using two assumed numerical examples.

Example (1):

Consider the following data of function

x_i \ y_j	0	1	2
0	0	1	2
1	1	2	3
2	2	3	4

From equation (3) Let k=l =3 and

$$F(x,\ y) = \sum_{l=1}^{3} \ \sum_{k=1}^{3} C_{kl}\ \varphi_k(x)\ \varphi_l(y) \qquad \text{... (8)}$$

At first, we find T_y and T_x by using equation(4) and equation (6) to get:

$$T_y = \begin{pmatrix} y_1^0 & y_1 & y_2^2 \\ y_2^0 & y_2 & y_2^2 \\ y_3^0 & y_3 & y_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}_{3\times 3}$$

$$T_x = \begin{pmatrix} x_1^0 & x_1 & x_2^2 \\ x_2^0 & x_2 & x_2^2 \\ x_3^0 & x_3 & x_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}_{3\times 3}$$

Now , using MATLAB program to compute T_y^{-1} , T_x^{-1} and $T_y^{-1} \otimes T_x^{-1}$ and then using equation (7) to obtain :

$$\mathrm{Vec}C_{OLS} = \begin{pmatrix} 1 & 0 & 0 \\ -1.5 & 2 & -0.5 \\ 0.5 & -1 & 0.5 \end{pmatrix}_{3\times 3} \otimes \begin{pmatrix} 1 & 0 & 0 \\ -1.5 & 2 & -0.5 \\ 0.5 & -1 & 0.5 \end{pmatrix}_{3\times 3} \bullet \mathrm{Vec}(F)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.5 & 2 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & -1 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.5 & 0 & 0 & 2 & 0 & 0 & -0.5 & 0 & 0 \\ 2.25 & -3 & 0.75 & -3 & 4 & -1 & 0.75 & -1 & 0.23 \\ -0.75 & 1.5 & -0.75 & 1 & -2 & 1 & -0.25 & 0.5 & -0.25 \\ 0.5 & 0 & 0 & -1 & 0 & 0 & 0.5 & 0 & 0 \\ -0.75 & 1 & -0.25 & 1.5 & -2 & 0.5 & -0.75 & 1 & -0.25 \\ 0.25 & -0.5 & 0.25 & -0.5 & 1 & -o.5 & 0.25 & -0.5 & 0.25 \end{pmatrix}_{9\times 9} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 2 \\ 3 \\ 2 \\ 3 \\ 4 \end{pmatrix}_{9\times 1}$$

Then we obtain the coefficient C_{kl}

$$\begin{pmatrix} C_{11} \\ C_{21} \\ C_{31} \\ C_{12} \\ C_{22} \\ C_{32} \\ C_{13} \\ C_{23} \\ C_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Finally, substituting these values in equation (8), then we get F(x,y) = x + y .

Example(2):

Consider the following table lists of data for the unknown function F(x,y):

x_i \ y_j	0	0.5	1	2
0	0	0.5	1	2
1	0.5	2	3.5	6.5
2	1	9.5	18	35
3	1.5	29	56.5	111.5

Let $$F(x,\ y) = \sum_{l=1}^{4} \sum_{k=1}^{4} C_{kl}\ \varphi_k(x)\ \varphi_l(y) \qquad \text{... (9)}$$

To compute T_y and T_x , we use equation (4) and equation (6), we get :

$$T_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0.5 & 0.25 & 0.125 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{pmatrix}_{4\times4} \quad \text{and} \quad T_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix}_{4\times4}$$

Then , by using equation (7) and MATLAB program we find C_{kl} as :

$$\text{Vec}\, C_{OLS} = \begin{pmatrix} 0 & 0.5 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}'_{16\times1}$$

Substituting C_{kl} in equation (9) we get the unknown function

$$F(x, y) = \frac{1}{2} x + y + 2x^3 y$$

Conclusion

In this paper, an interpolation with kronecker product with least squares method have been used to find an approximate function for some given data. Computer program coded in MATLAB software have also been adapted to compute the inverse of a matrix as well as the kronecker product of suitable matrices. the efficiency of the proposed method has discussed and proved, as well.

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