Multivariate Time Series Prediction Using Tensor Decompositions

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March, 2022

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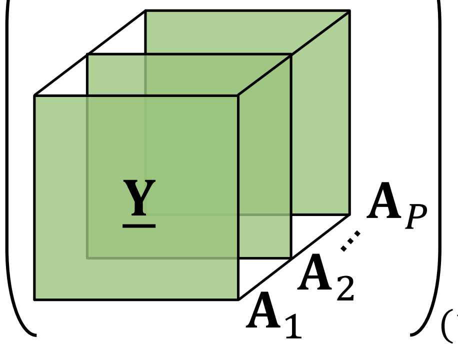
Tensor Vector Auto-Regression (VAR)

$$\mathbf{y}_t = \mathbf{A}_1 \mathbf{y}_{t-1} + \mathbf{A}_2 \mathbf{y}_{t-2} + \cdots + \mathbf{A}_P \mathbf{y}_{t-P} + \boldsymbol{\epsilon}_t$$

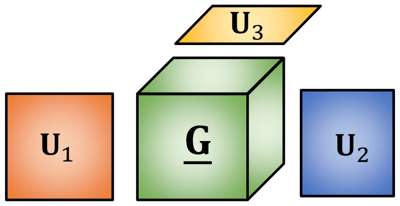
$\mathbf{y}_t = \Big($



$\Big)_{(1)} \mathbf{z}_\mathrm{t} + \boldsymbol{\epsilon}_t,$

$$\mathbf{z}_\mathrm{t} = \left(\mathbf{y}_{t-1}^{\mathrm{T}}, \cdots, \mathbf{y}_{\mathrm{t}-P}^{\mathrm{T}}\right)^{\mathrm{T}}$$

$\mathbf{y}_t = \Big($



$\Big)_{(1)} \mathbf{z}_\mathrm{t} + \boldsymbol{\epsilon}_t,$

$$\mathbf{y}_t = \mathbf{U}_1 \underline{\mathbf{G}}_{(1)} (\mathbf{U}_3 \otimes \mathbf{U}_2)^{\mathrm{T}} \mathbf{z}_t + \boldsymbol{\epsilon}_t = \mathbf{U}_1 \underline{\mathbf{G}}_{(1)} \mathrm{vec}(\mathbf{U}_2^T \mathbf{Z}_t \mathbf{U}_3) + \boldsymbol{\epsilon}_t,$$

where $\mathbf{Z}_t = (\mathbf{y}_{t-1}, \cdots, \mathbf{y}_{t-P})$

Tensor Vector Auto-Regression (VAR)

Multilinear low-rank least squares estimator

$$\underline{\mathbf{A}}_{MLR} \equiv \left[\!\left[\underline{\widehat{\mathbf{G}}}, \widehat{\mathbf{U}}_{\mathbf{1}}, \widehat{\mathbf{U}}_{\mathbf{2}}, \widehat{\mathbf{U}}_{\mathbf{3}} \right]\!\right] = \arg\min L(\underline{\mathbf{G}}, \mathbf{U}_{\mathbf{1}}, \mathbf{U}_{\mathbf{2}}, \mathbf{U}_{\mathbf{3}})$$

$$L(\underline{\mathbf{G}}, \mathbf{U}_{\mathbf{1}}, \mathbf{U}_{\mathbf{2}}, \mathbf{U}_{\mathbf{3}}) = \frac{1}{T} \sum_{t=1}^{T} \left\| \boldsymbol{y}_t - \left(\underline{\mathbf{G}} \times_1 \mathbf{U}_{\mathbf{1}} \times_{\mathbf{2}} \mathbf{U}_{\mathbf{2}} \times_{\mathbf{3}} \mathbf{U}_{\mathbf{3}} \right)_{(1)} \boldsymbol{x}_t \right\|_2^2$$

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Tensor Vector Auto-Regression (VAR)

Multilinear low-rank least squares estimation

$F_{t-1} = \sigma(\boldsymbol{\epsilon}_t, \boldsymbol{\epsilon}_{\boldsymbol{t-1}}, \cdots)$) be the σ-field generated by $\{\boldsymbol{\epsilon}_s, s < t\}$, and
$\mathbf{X}_t = (\mathbf{y}_{t-1}, \cdots, \mathbf{y}_{t-P})$

$$\mathbb{E}(\mathbf{y}_\mathbf{t} \,|F_{t-1}) = \left(\left(\mathbf{x}_t^T(\mathbf{U}_3 \otimes \mathbf{U}_2)\underline{\mathbf{G}}_{(1)}^T\right) \otimes \mathbf{I}_N\right) vec(\mathbf{U}_1)$$

$$= \mathbf{U}_1\underline{\mathrm{G}}_{(1)}\left((\mathbf{U}_3^T\mathbf{X}_t^T) \otimes \mathbf{I}_{r2}\right) vec(\mathbf{U}_2^T) = \mathbf{U}_1\underline{\mathbf{G}}_{(1)}(\mathbf{I}_{r3} \otimes (\mathbf{U}_2^T\mathbf{X}_t))vec(\mathbf{U}_3)$$

$$= \left(\left((\mathbf{U}_3 \otimes \mathbf{U}_2)^T\mathbf{x}_t\right)^T \otimes \mathbf{U}_1\right) vec(\underline{\mathbf{G}}_{(1)})$$

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Tensor Vector Auto-Regression (VAR) ALS

Algorithm Alternating least squares algorithm for $\widehat{\mathcal{A}}_{\mathrm{MLR}}$

Initialize: $\mathcal{A}^{(0)}$

HOSVD: $\mathcal{A}^{(0)} \approx \mathcal{G}^{(0)} \times_1 \boldsymbol{U}_1^{(0)} \times_2 \boldsymbol{U}_2^{(0)} \times_3 \boldsymbol{U}_3^{(0)}$ with multilinear ranks (r_1, r_2, r_3)

repeat $k = 0, 1, 2, \ldots$

$$\boldsymbol{U}_1^{(k+1)} \leftarrow \arg\min_{\boldsymbol{U}_1} \sum_{t=1}^{T} \|\boldsymbol{y}_t - ((\boldsymbol{x}_t'(\boldsymbol{U}_3^{(k)} \otimes \boldsymbol{U}_2^{(k)})\mathcal{G}_{(1)}^{(k)\prime}) \otimes \boldsymbol{I}_N)\mathrm{vec}(\boldsymbol{U}_1)\|_2^2$$

$$\boldsymbol{U}_2^{(k+1)} \leftarrow \arg\min_{\boldsymbol{U}_2} \sum_{t=1}^{T} \|\boldsymbol{y}_t - \boldsymbol{U}_1^{(k+1)}\mathcal{G}_{(1)}^{(k)}((\boldsymbol{X}_t\boldsymbol{U}_3^{(k)})' \otimes \boldsymbol{I}_{r_2})\mathrm{vec}(\boldsymbol{U}_2')\|_2^2$$

$$\boldsymbol{U}_3^{(k+1)} \leftarrow \arg\min_{\boldsymbol{U}_3} \sum_{t=1}^{T} \|\boldsymbol{y}_t - \boldsymbol{U}_1^{(k+1)}\mathcal{G}_{(1)}^{(k)}(\boldsymbol{I}_{r_3} \otimes (\boldsymbol{U}_2^{(k+1)\prime}\boldsymbol{X}_t))\mathrm{vec}(\boldsymbol{U}_3)\|_2^2$$

$$\mathcal{G}^{(k+1)} \leftarrow \arg\min_{\mathcal{G}} \sum_{t=1}^{T} \|\boldsymbol{y}_t - (((\boldsymbol{U}_3^{(k+1)} \otimes \boldsymbol{U}_2^{(k+1)})'\boldsymbol{x}_t)' \otimes \boldsymbol{U}_1^{(k+1)})\mathrm{vec}(\mathcal{G}_{(1)})\|_2^2$$

$$\mathcal{A}^{(k+1)} \leftarrow \mathcal{G}^{(k+1)} \times_1 \boldsymbol{U}_1^{(k+1)} \times_2 \boldsymbol{U}_2^{(k+1)} \times_3 \boldsymbol{U}_3^{(k+1)}$$

until convergence

Finalize: $\widehat{\boldsymbol{U}}_i \leftarrow$ top r_i left singular vectors of $\widehat{\mathcal{A}}_{(i)}$ with positive first elements, $1 \leq i \leq 3$

$$\widehat{\mathcal{G}} \leftarrow [\![\widehat{\mathcal{A}}; \widehat{\boldsymbol{U}}_1', \widehat{\boldsymbol{U}}_2', \widehat{\boldsymbol{U}}_3']\!]$$

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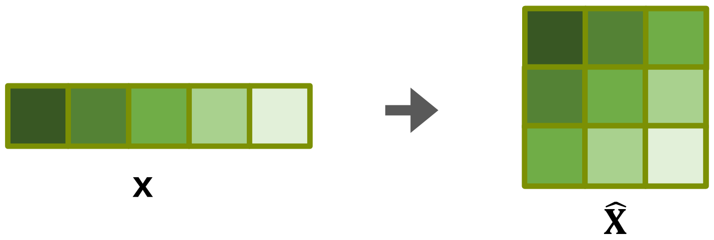
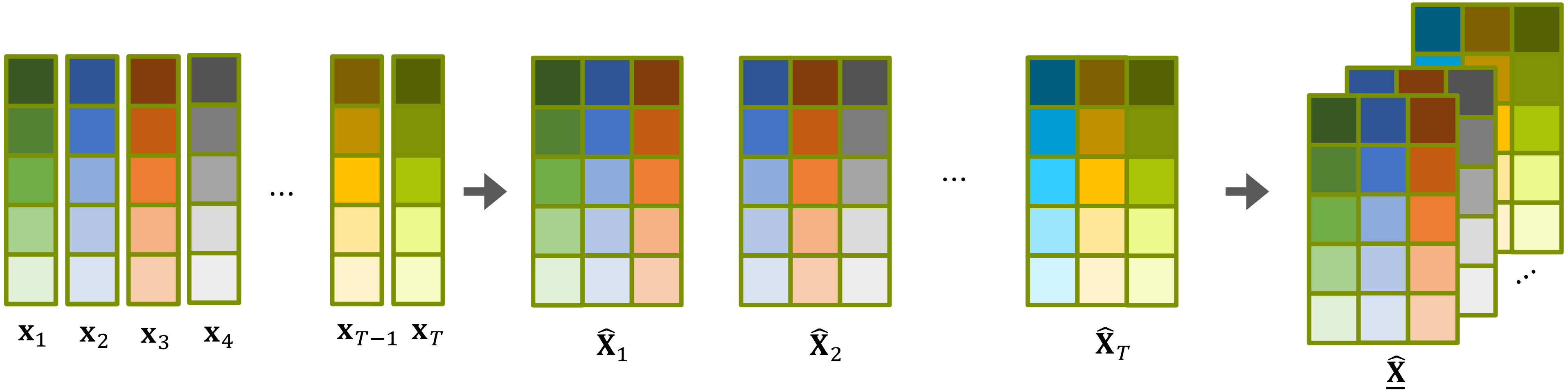
Block Hankel Tensor-ARIMA



Hankelization of vector (Tensorization)



Multi-way delay embedding (extension of Hankelization)

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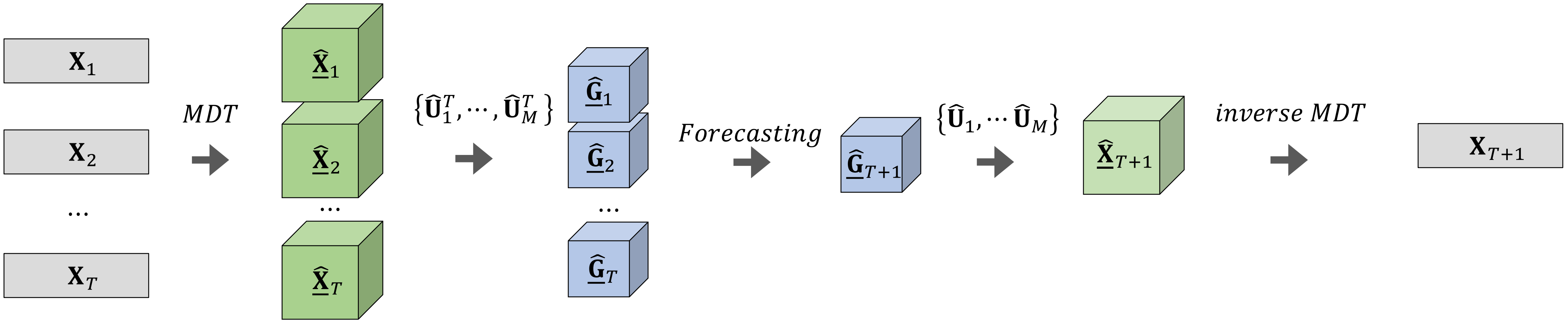
Block Hankel Tensor-ARIMA



$$\Delta^d \hat{\underline{\mathbf{G}}}_t = \Delta^d \hat{\underline{\mathbf{X}}}_t \times_1 \hat{\mathbf{U}}_1^T \times_2 \hat{\mathbf{U}}_2^T \cdots \times_M \hat{\mathbf{U}}_M^T$$

$$\Delta^d \hat{\underline{\mathbf{G}}}_{T+1} = \sum_{i=1}^{p} \alpha_i \Delta^d \hat{\underline{\mathbf{G}}}_{T-i} - \sum_{i=1}^{q} \beta_i \hat{\epsilon}_{T-i}$$

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Results of the Experiment. Yahoo.finance

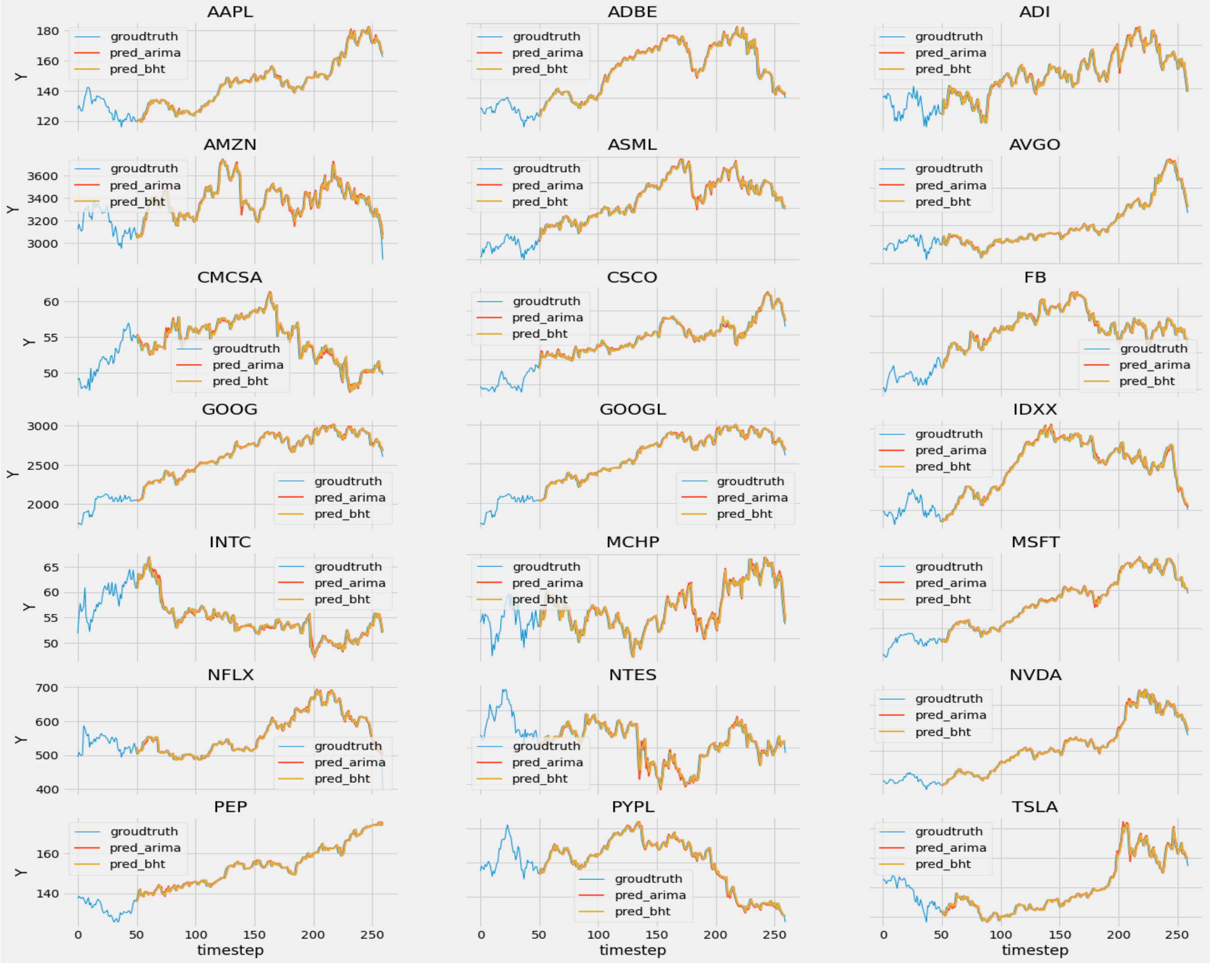

Fig. 1 BHT-ARIMA(1, 1, 0) with MDT-rank [21, 4], Tucker rank [4, 4] and window size - 50

	RMSE	Execute time
BHT-ARIMA	**18.24**	**192.15s**
ARIMA	20.26	442.71s

Table 1. Methods performance

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Results of the Experiment. Air Quality Data Set

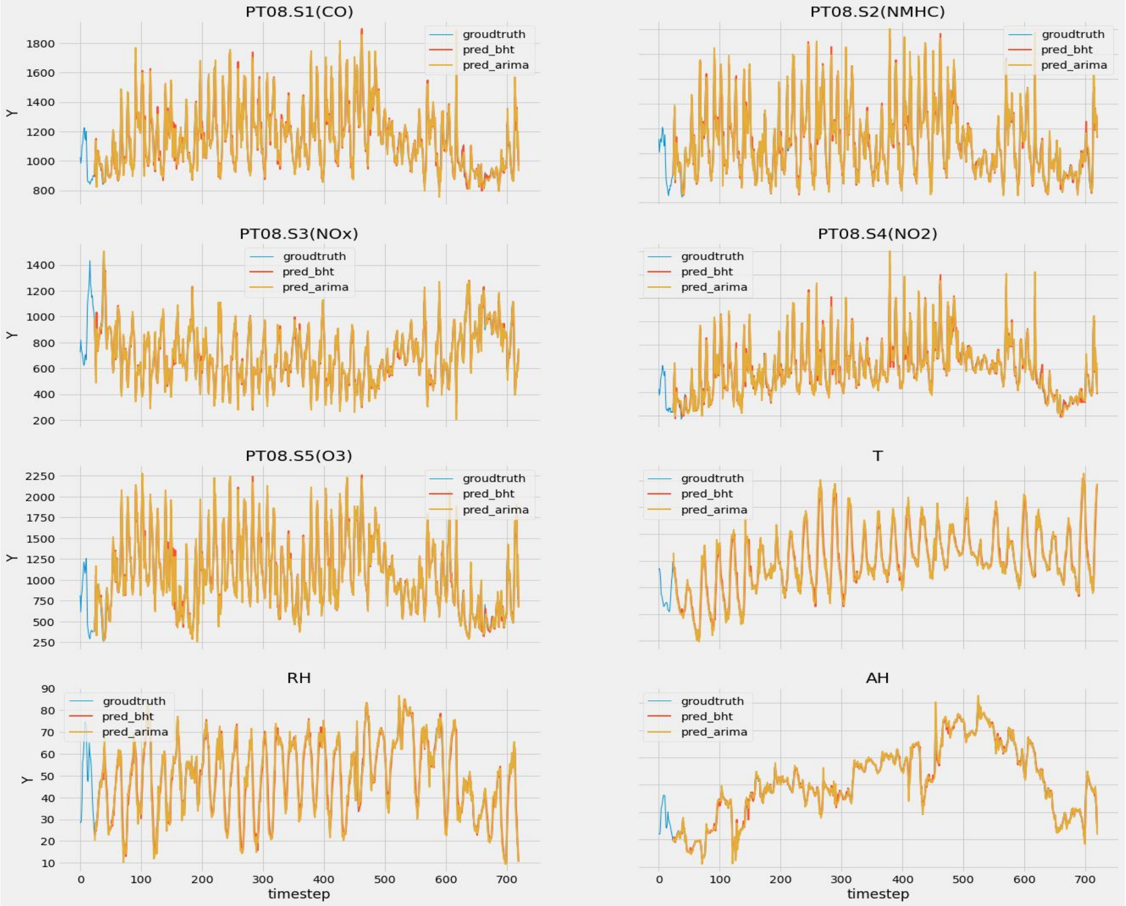

Fig. 2 BHT-ARIMA(2, 1,1) with MDT-rank [8, 4], Tucker rank [4, 4] and window size - 24

	RMSE	Execute time
BHT-ARIMA	**109.4**	**419.33s**
ARIMA	116.4	553.15s

Table 2. Methods performance

1. PT08.S1 (tin oxide) hourly averaged sensor response
2. PT08.S2 (titania) hourly averaged sensor response
3. PT08.S3 (tungsten oxide) hourly averaged sensor response
4. PT08.S4 (tungsten oxide) hourly averaged sensor response
5. PT08.S5 (indium oxide) hourly averaged sensor response
6. T Temperature
7. RH Relative Humidity (%)
8. AH Absolute Humidity

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Results of the Experiment

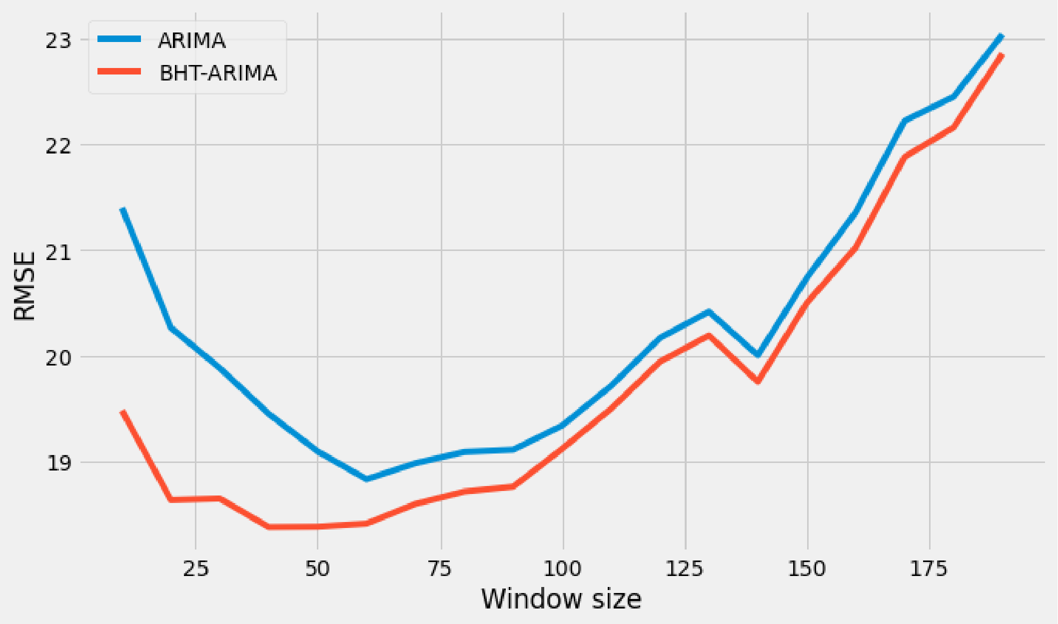

Fig. 3. Dependence of RMSE on the window size

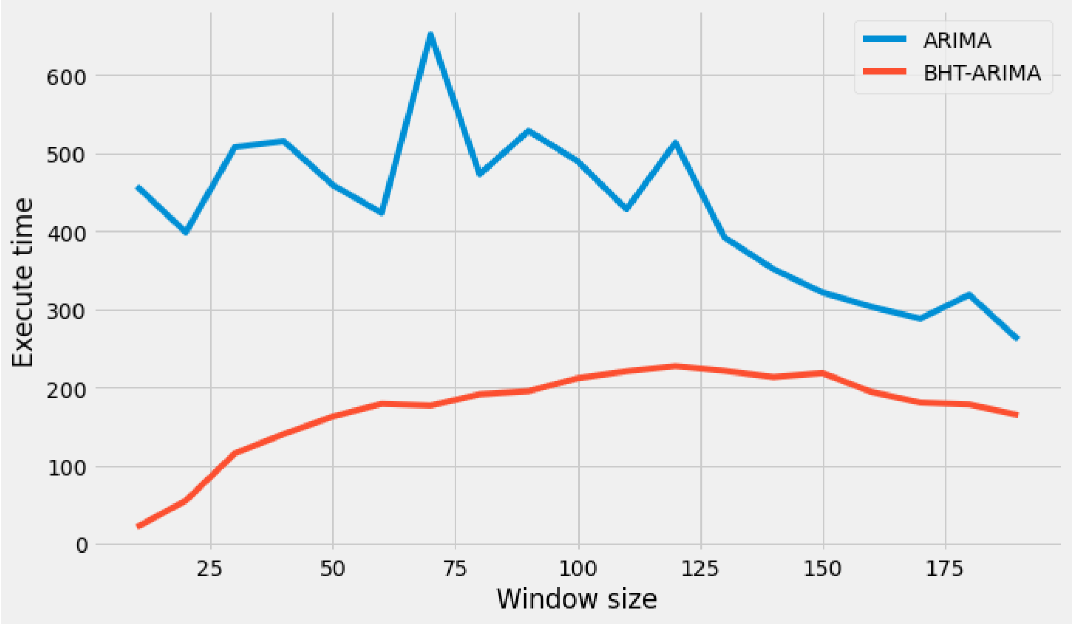

Fig. 4. Dependence of execute time on the window size

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Results of the Experiment

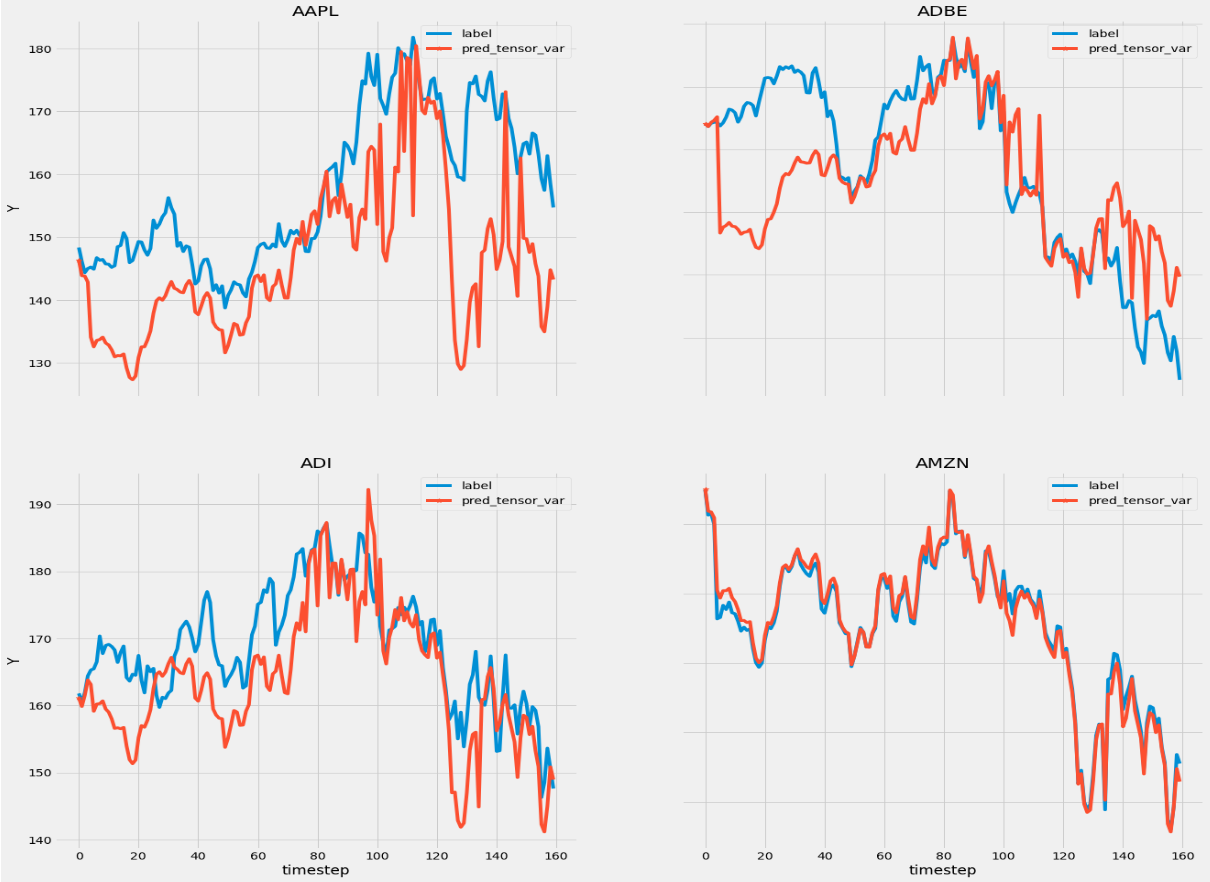

Fig. 5 Tensor-VAR(1) with train size - 100

	RMSE
Tensor-VAR	29.86

Table 3. Method performance

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Thank you for attention!

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