Selecting a Set

Peter Stuckey

Choosing from a set of objects

- Many problems require us to select a subset from a set of objects that
  - Meets some criteria; and
  - Optimizes some objective function
- Example 0-1 knapsack (at most one copy of each object)
  - Limit choices of x variable
  - Make the x an array of Booleans
  - Use a set variable

0-1Knapsack Model

```
int: n; % number of objects
set of int: OBJ = 1..n;
int: capacity;
array[OBJ] of int: profit;
array[OBJ] of int: size;

array[OBJ] of var 0..1: x;

constraint forall(i in OBJ)(x[i] >= 0);
constraint sum(i in OBJ)(size[i] * x[i])
           <= capacity;
solve maximize sum(i in OBJ)(profit[i] * x[i]);


output ["x = ", show(x), "\n"];
```

knapsack01.mzn

0-1Knapsack Model

```
int: n; % number of objects
set of int: OBJ = 1..n;
int: capacity;
array[OBJ] of int: profit;
array[OBJ] of int: size;

array[OBJ] of var bool: x;


constraint sum(i in OBJ)(size[i] *
                   bool2int(x[i])) <= capacity;
solve maximize sum(i in OBJ)
                  (profit[i] * bool2int(x[i]));

output ["x = ", show(x), "\n"];
```

knapsack01bool.mzn

0-1Knapsack Model

```
int: n; % number of objects
set of int: OBJ = 1..n;
int: capacity;
array[OBJ] of int: profit;
array[OBJ] of int: size;

array[OBJ] of var bool: x;


constraint sum(i in OBJ)(size[i] *
                bool2int(x[i])) <= capacity;
solve maximize sum(i in OBJ)
                  (profit[i] * bool2int(x[i]));

output [”x = “, show(x), “\n”];
```

knapsack01bool.mzn

0-1Knapsack Model

```
int: n; % number of objects
set of int: OBJ = 1..n;
int: capacity;
array[OBJ] of int: profit;
array[OBJ] of int: size;

var set of OBJ: x;


constraint sum(i in OBJ)(size[i] *
           bool2int(i in x)) <= capacity;
solve maximize sum(i in OBJ)(profit[i] *
                             bool2int(i in x));

output [”x = “, show(x), “\n”];
```

knapsack01set.mzn

0-1Knapsack Model

```
int: n; % number of objects
set of int: OBJ = 1..n;
int: capacity;
array[OBJ] of int: profit;
array[OBJ] of int: size;

var set of OBJ: x;


constraint sum(i in OBJ)(size[i] *
           bool2int(i in x)) <= capacity;
solve maximize sum(i in OBJ)(profit[i] *
                             bool2int(i in x));

output [”x = “, show(x), “\n”];
```

knapsack01set.mzn

0-1Knapsack Model

```
int: n; % number of objects
set of int: OBJ = 1..n;
int: capacity;
array[OBJ] of int: profit;
array[OBJ] of int: size;

var set of OBJ: x;


constraint sum(i in x)(size[i]) <= capacity;

solve maximize sum(i in x)(profit[i]);


output [”x = “, show(x), “\n”];
```

knapsack01set_concise.mzn

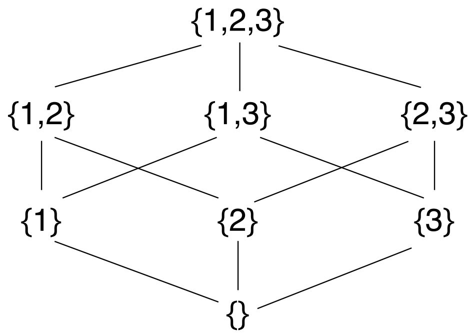
bool2int

- Coerce a Boolean to an integer
  - `bool2int(false) = 0`
  - `bool2int(true) = 1`
- When you use a Boolean where MiniZinc expects an integer, MiniZinc will automatically “add” the `bool2int` coercion
- Many solvers will use the same internal representation for `x` and `b` where
  - `x = bool2int(b)`

Set Variables

- Set variables in MiniZinc choose a set from a given fixed superset

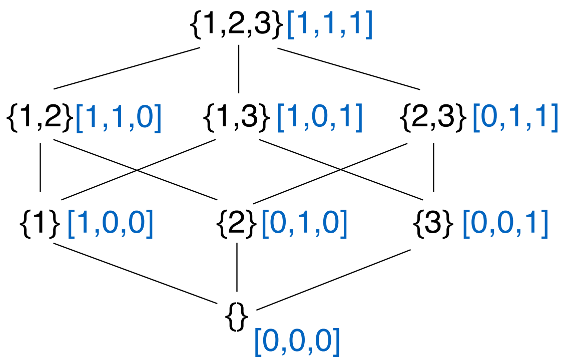
```
var set of {1,2,3}: x;
```



Set Representations

- Other set representations encode or model the possible values of the set

```
array[1..3] of var 0..1: x;
```



Set Constraints

- MiniZinc provides the (infix) set operations
  - in, (membership e.g. x in s)
  - subset, superset
  - intersect (intersection)
  - union
  - card (cardinality)
  - diff (set difference e.g x diff y = x \ y)
  - symdiff (symmetic difference)
    - e.g. {1, 2, 5, 6} symdiff {2, 3, 4, 5} = {1, 3, 4, 6}

Which model is best?

- Most solvers will treat each model the same
  - CP solvers may treat the last model better since they can combine cardinality reasoning with other set reasoning
- Model whichever makes it easier to express the constraints
  - for knapsack01 the first version
- Model using the highest level model
  - the last version

Overview

- Modeling with sets is common for combinatorial problems
- There are at least three representations
  - Indicator variables: 0-1 or bool variables
  - Native set variables

EOF

Choosing a Set Representation: Example

Peter Stuckey

SetSelect Question

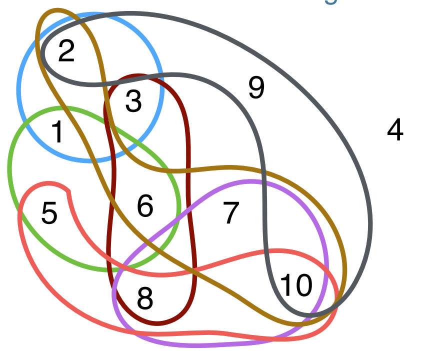
- Write a MiniZinc model that given an array of `k` subsets of numbers `1..n`, chooses a subset of `1..n` which includes at most one from each subset and maximizes the sum of the chosen set.

```
n = 10;
k = 7;
s = [{1,5,6}, {2,6,7,10}, {3,6,8}, {1,2,3},
     {2,9,10}, {5,8,10}, {7,8,10}];
```

setselect.dzn

Simple Set Select Algorithm

- Greedy algorithm
  - choose the largest available element
  - eliminate choices that are no longer valid




```
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s = [{1,5,6}, {2,6,7,10}, {3,6,8}, {1,2,3},
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```


Simple Set Select Algorithm

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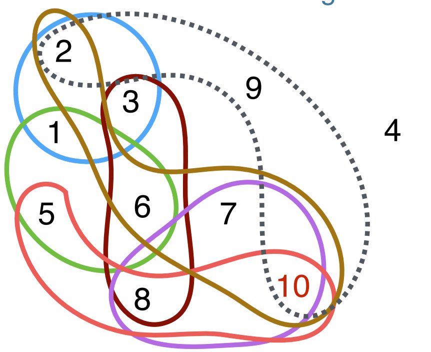


```
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s = [{1,5,6}, {2,6,7,10}, {3,6,8}, {1,2,3},
     {2,9,10}, {5,8,10}, {7,8,10}];
```


Simple Set Select Algorithm

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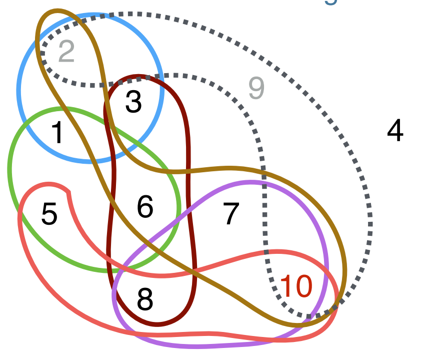


```
n = 10;
k = 7;
s = [{1,5,6}, {2,6,7,10}, {3,6,8}, {1,2,3},
     {2,9,10}, {5,8,10}, {7,8,10}];
```

Simple Set Select Algorithm

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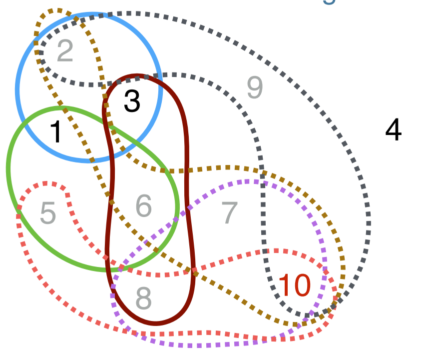
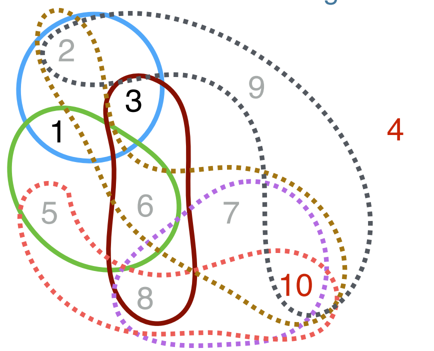



```
n = 10;
k = 7;
s = [{1,5,6}, {2,6,7,10}, {3,6,8}, {1,2,3},
     {2,9,10}, {5,8,10}, {7,8,10}];
```


Simple Set Select Algorithm

- Greedy algorithm
  - choose the largest available element
  - eliminate choices that are no longer valid

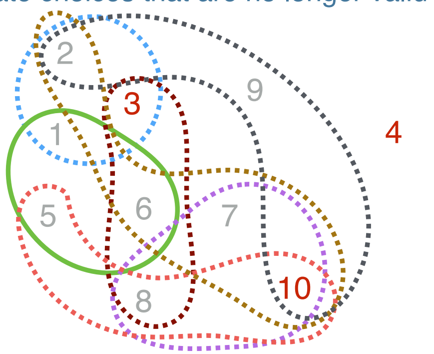



```
n = 10;
k = 7;
s = [{1,5,6}, {2,6,7,10}, {3,6,8}, {1,2,3},
     {2,9,10}, {5,8,10}, {7,8,10}];
```


Simple Set Select Algorithm

- Greedy algorithm
  - choose the largest available element
  - eliminate choices that are no longer valid

{3,4,10}




```
n = 10;
k = 7;
s = [{1,5,6}, {2,6,7,10}, {3,6,8}, {1,2,3},
     {2,9,10}, {5,8,10}, {7,8,10}];
```


SetSelect Data + Decisions

- Data

```
int: n;
set of int: OBJ = 1..n;
int: k;
set of int: SET = 1..k;
array[SET] of set of OBJ: s;
```

- Decisions

```
var set of OBJ: x;
```

SetSelect Constraints + Objective

- At most one intersection

```
forall(i in SET)
        (card(x intersect s[i]) <= 1);
```

- Objective

```
solve maximize sum(i in x)(i);
```


Solving the model

- Executing the model

```
$ minizinc setselect.mzn setselect.dzn

x = {3,4,5,7,9};
----------
==========
```

- Much better than the greedy algorithm

SetSelect Revised Question

- Write a MiniZinc model that given an array of `k` subsets of numbers `1..n`, chooses a subset of `1..n` of size `u` which includes at most one from each subset and maximizes the sum of the chosen set.

```
n = 10;
k = 7;
u = 3;
s = [{1,5,6}, {2,6,7,10}, {3,6,8}, {1,2,3},
     {2,9,10}, {5,8,10}, {7,8,10}];
```

setselectr.dzn

SetSelect Revised Addition

- Additional constraint: `card(x) = u;`

- Executing the model

```
$ minizinc setselectr.mzn setselectr.dzn

x = {4,8,9};
----------
==========
% failures = 237
```

- But we can model a set of known cardinality differently!

EOF

Choosing a Fixed Cardinality Set

Peter Stuckey

SetSelect Revised Question

- Write a MiniZinc model that given an array of `k` subsets of numbers `1..n`, chooses a subset of `1..n` of size `u` which includes at most one from each subset and maximizes the sum of the chosen set.

```
n = 10;
k = 7;
u = 3;
s = [{1,5,6}, {2,6,7,10}, {3,6,8}, {1,2,3},
     {2,9,10}, {5,8,10}, {7,8,10}];
```

setselectrev.dzn

Deciding a set of fixed cardinality

- Instead of a set variable `var set of 1..n: x` with
  - cardinality constraint `card(x) = u`
- An array of `u` values

  `array[1..u] of var 1..n: x`

  - and some other constraints …
- Why: suppose n = 1000, u = 4
- First representation
  - 1000 Boolean variables
- Second representation
  - 4 integer variables

Deciding a set of fixed cardinality

- Consider `array[1..3] of var 1..10: x;`
- How many possible values
  - 1000
- And `var set of 1..10: x` where `card(x) = 3;`
  - 10 * 9 * 8 / 3 * 2 * 1 = 120
- First issue: some array solutions are not sets of cardinality 3
  - e.g. [1,1,1] = {1}, [1,2,1] = {1,2}
- Solution: ensure all different
  - `for(i,j in 1..u where i < j)`
  - `    (x[i] != x[j]);`

Deciding a set of fixed cardinality

- Consider `array[1..3] of var 1..10: x;`
  - `for(i,j in 1..u where i < j)`
  - `    (x[i] != x[j]);`
- How many possible values
  - 10 * 9 * 8 = 720
- Second issue: multiple representations of the same set
  - e.g. {1,6,10} = [1,6,10], [10,1,6], [10,6,1], [1,10,6], [6,10,1], [6,1,10]
- Solution: ensure ordered
  - `for(i in 1..u-1)(x[i] < x[i+1]);`

Critical Issues in Modeling Decisions

- Decisions in the problem
  - are not necessarily the decisions in the model
- If possible make then identical but
  - what if you are deciding
    - a path in a graph,
    - a tree structure
- Critical modeling requirement (1)
  - decisions in the model (satisfying constraints)
  - are valid decisions in the problem
- Add constraints to make this so
  - e.g. add constraints forcing array elements ≠

Critical Issues in Modeling Decisions

- Multiple decisions in the model
  - reflect the same decision in the problem
  - e.g. x = [1,2,7], x = [7,2,1] for x = {1,2,7}
- Critical modeling issue (2)
  - try to have only one set of decisions in the model (satisfying constraints)
  - reflect valid decisions in the problem
- Add constraints to remove all but one set
  - e.g. add constraints forcing array elements <

SetSelect Revised Question

- Write a MiniZinc model that given an array of `k` subsets of numbers `1..n`, chooses a subset of `1..n` of size `u` which includes at most one from each subset and maximizes the sum of the chosen set.

```
n = 10;
k = 7;
u = 3;
s = [{1,5,6}, {2,6,7,10}, {3,6,8}, {1,2,3},
     {2,9,10}, {5,8,10}, {7,8,10}];
```

setselectrev.dzn

SetSelect Revised Model

- Decisions

```
array[1..u] of var 1..n: x;
for(i in 1..u-1)(x[i] < x[i+1]);
```

- At most one intersection

```
forall(i in SET)
       (sum(j in 1..u)
           (x[j] in s[i]) % coercion
        <= 1);
```

- Objective

```
solve maximize sum(x);
```


Solving the model

- Executing the model

```
$ minizinc setselectr2.mzn setselectr.dzn

x = [5,7,9];
----------
==========
% failures = 22
```

- This representation makes search easier

Overview

- There are multiple ways to represent fixed cardinality sets
  - var set of OBJ + cardinality constraint
    - good if the solver natively supports sets
    - good when OBJ is not too big
  - array[1..u] of var OBJ
    - good when u is small
- Two critical issues in modelling decisions
  - ensure each solution to the model is a solution of the problem
  - try to ensure each solution of the problem only has one solution in the model (symmetry)

EOF

Choosing a Bounded Cardinality Set

Peter Stuckey

SetSelect Revised Question

- Write a MiniZinc model that given an array of `k` subsets of numbers `1..n`, chooses a subset of `1..n` of size at most `u` which includes at most one from each subset and maximizes the sum of the chosen set.

```
n = 10;
k = 7;
u = 3;
s = [{1,5,6}, {2,6,7,10}, {3,6,8}, {1,2,3},
     {2,9,10}, {5,8,10}, {7,8,10}];
```

setselectrev.dzn

Deciding a set of bounded cardinality

- Instead of a set variable `var set of OBJ: x` with
  - cardinality constraint `card(x) <= u`
- An array of `u` values

  `array[1..u] of var EOBJ: x`

  - extended OBJ: EOBJ = OBJ ∪ { extra-value }
  - extra value represents: no element
- For example: OBJ = 1..n
  - EOBJ = 0..n

Two critical issues

- Each solution in the model represents a solution in the problem
  - [3,0,3] ❌ no repeated values
  - [0,2,0] ✅ repeated extra values are OK
- Each solution in the problem has just one solution representative in the model
  - [0,2,0], [0,0,2], [2,0,0] = {2} ❌
  - [0,1,2], [0,2,1], [1,0,2], [1,2,0], [2,0,1], [2,1,0] ❌
- Add constraints to fix these issues

Bounded cardinality constraints

- Constraints to fix

```
array[1..u] of var 0..n: x
```

- Just order the values (decreasing)

```
for(i in 1..u-1)(x[i] > x[i+1]);
```

  - ❌ No representative left for {2} e.g. [2,0,0]
- No strict ordering

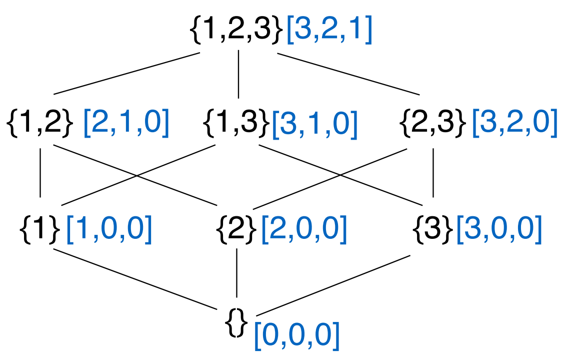
```
for(i in 1..u-1)(x[i] >= x[i+1]);
```

  - ❌ solutions with repeats [3,2,2]
- Combine the two: repeats of 0 allowed

```
for(i in 1..u-1)
   (x[i] >= bool2int(x[i]!=0)+ x[i+1]);
```


Bounded Cardinality

- Representation of `var set of {1,2,3}: x;`
- `array[1..3] of var 0..3: x;`



SetSelect Revised Question

- Write a MiniZinc model that given an array of `k` subsets of numbers `1..n`, chooses a subset of `1..n` of size at most `u` which includes at most one from each subset and maximizes the sum of the chosen set.

```
n = 10;
k = 7;
u = 3;
s = [{1,5,6}, {2,6,7,10}, {3,6,8}, {1,2,3},
     {2,9,10}, {5,8,10}, {7,8,10}];
```

setselectrev.dzn

SetSelect Revised Question

- Decisions

```
array[1..u] of var 0..n: x;
for(i in 1..u-1)
   (x[i] >= bool2int(x[i]=0)+ x[i+1]);
```

- At most one intersection

```
forall(i in SET)
      (sum(j in 1..u)
          (x[j] in s[i]) % 0 is safe
       <= 1);
```

- Objective

```
solve maximize sum(x); % 0 is safe
```


Overview

- There are multiple ways to represent sets
- var set of OBJ
  - good if the solver natively supports sets
  - good when OBJ is not too big
- array[OBJ] of var bool / 0..1
  - good when OBJ is not too big
- array[1..u] of var OBJ
  - only for fixed cardinality u
  - good when u is small
- array[1..u] of var EOBJ
  - need to represent “null” object

EOF

Modeling with MultiSets

Peter Stuckey

Multisets

- A multiset is a set with duplicates allowed
  - set {3,4,5,4,5,7,4,5,3,3,3} = {3,4,5,7}, but
  - multiset {{3,4,5,4,5,7,4,5,3,3,3}} ≠ {{3,4,5,7}}
  - order is not important = {{3,3,3,3,4,4,4,5,5,5,7}}
- Some problems require choosing a multiset
- Multisets are not built in to MiniZinc

Choosing a Multiset

- Typically modeled analogous to sets

```
array[OBJ] of var int: x;
forall(i in OBJ)(x[i] >= 0);
```

- Usually the multiplicity is bounded (by `m`)

```
array[OBJ] of var 0..m: x;
```

- If cardinality `u` is tight we can treat differently

```
array[1..u] of var OBJ0: x;
forall(i in 1..u-1)
      (x[i] >= x[i+1]);
```


Choosing a Multiset

- Knapsack (not 0-1) is an example
- Production planning is an example
- Its often not very interesting to think about choosing a multiset
  - Usually just think about simultaneous choices of how many of each object

Overview

- Multisets also have multiple representations
- In many discrete optimization problems, the multisite viewpoint is not helpful