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MEM-320 Fluid Dynamics

Normal Shock Equation Derivation

Manual derivation on next page.

MATLAB assisted derivation on pages after.

MATLAB file and LaTeX file available in submission.

Normal Shock Equation: ${M_2}^2 = \frac{\frac{2}{\gamma-1}+{M_1}^2}{\frac{2\,{M_1}^2\,\gamma}{\gamma-1}-1} = f(\gamma, M_1) < 1.0$

Continuity Equation: $\frac{M_1\,p_1}{\sqrt{T_1}} = \frac{M_2\,p_2}{\sqrt{T_2}}$

Energy Equation: $T_1\left(\frac{{M_1}^2\,(\gamma-1)}{2}+1\right) = T_2\left(\frac{{M_2}^2\,(\gamma-1)}{2}+1\right)$

Momentum Equation: $p_1\left(\gamma\,{M_1}^2+1\right) = p_2\left(\gamma\,{M_2}^2+1\right)$

Divide Continuity Equation by Momentum Equation to remove p_1: $\frac{M_1}{\sqrt{T_1}\,(\gamma\,{M_1}^2+1)} = \frac{M_2}{\sqrt{T_2}\,(\gamma\,{M_2}^2+1)}$

Multiply that by the square root of the Energy Equation, $\sqrt{T_1\left(\frac{{M_1}^2\,(\gamma-1)}{2}+1\right)} = \sqrt{T_2\left(\frac{{M_2}^2\,(\gamma-1)}{2}+1\right)}$ to get:

$\frac{M_1\,\sqrt{\frac{{M_1}^2\,(\gamma-1)}{2}+1}}{\gamma\,{M_1}^2+1} = \frac{M_2\,\sqrt{\frac{{M_2}^2\,(\gamma-1)}{2}+1}}{\gamma\,{M_2}^2+1}$

Now, we solve for M_2^2 in terms of M_1^2 and γ as in the orginal Normal Shock Equation:

Step 1: Cross Multiply:

$M_1\,\sqrt{\frac{{M_1}^2\,(\gamma-1)}{2}+1}\left(\gamma\,{M_2}^2+1\right) = M_2\,\sqrt{\frac{{M_2}^2\,(\gamma-1)}{2}+1}\left(\gamma\,{M_1}^2+1\right)$

Step 2: Square both sides:

$M_1^2\left(\frac{{M_1}^2\,(\gamma-1)}{2}+1\right)\left(\gamma\,{M_2}^2+1\right)^2 = M_2^2\left(\frac{{M_2}^2\,(\gamma-1)}{2}+1\right)\left(\gamma\,{M_1}^2+1\right)^2$

Step 3: Expand the squares:

$M_1^2\left(\frac{{M_1}^2\,(\gamma-1)}{2}+1\right)\left(\gamma^2\,{M_2}^4+2\,\gamma\,{M_2}^2+1\right) = M_2^2\left(\frac{{M_2}^2\,(\gamma-1)}{2}+1\right)\left(\gamma^2\,{M_1}^4+2\,\gamma\,{M_1}^2+1\right)$

Step 4: Expand and Collect in terms of M_2:

$M_2^2(1+2\gamma M_1^2+\gamma^2 M_1^4)+M_2^4\left(-\frac{1}{2}+\frac{\gamma}{2}-\gamma M_1^2+\gamma^2 M_1^2-\frac{1}{2}\gamma^2 M_1^4+\frac{1}{2}\gamma^3 M_1^4\right) = M_1^2-\frac{M_1^4}{2}+\frac{\gamma M_1^4}{2}+M_2^2(2\gamma M_1^2-\gamma M_1^4+\gamma^2 M_1^4)+M_2^4(\gamma^2 M_1^2-\frac{1}{2}\gamma^2 M_1^4+\frac{1}{2}\gamma^3 M_1^4)$

Step 5: Move all terms to one side of the Equation:

$-M_1^2+\frac{M_1^4}{2}-\frac{\gamma M_1^4}{2}+M_2^2(1+2\gamma M_1^2+\gamma^2 M_1^4)-M_2^2(2\gamma M_1^2-\gamma M_1^4+\gamma^2 M_1^4)$

$-M_2^4(\gamma^2 M_1^2-\frac{1}{2}\gamma^2 M_1^4+\frac{1}{2}\gamma^3 M_1^4)+M_2^4\left(-\frac{1}{2}+\frac{\gamma}{2}-\gamma M_1^2+\gamma^2 M_1^2-\frac{1}{2}\gamma^2 M_1^4+\frac{1}{2}\gamma^3 M_1^4\right)=0$

Step 6: Expand and Collect in terms of M_2:

$-M_1^2+\frac{M_1^4}{2}-\left(\gamma\frac{M_1^4}{2}\right)+M_2^2(\gamma M_1^4+1)+M_2^4\left(-\frac{1}{2}+\frac{\gamma}{2}-\gamma M_1^2\right)=0$

Step 7: Factor into product with 3 terms:

$\frac{1}{2}\left(M_1^2-M_2^2\right)\left(-2+M_2^2-\gamma M_2^2+M_1^2-\gamma M_1^2+2\gamma M_2^2 M_1^2\right)=0$

Step 8: Multiply both sides by 2 to remove $\frac{1}{2}$:

$\left(M_1^2-M_2^2\right)\left(-2+M_2^2-\gamma M_2^2+M_1^2-\gamma M_1^2+2\gamma M_2^2 M_1^2\right)=0$

Step 9: Now there are 2 equations:

$M_1^2-M_2^2=0 \quad \text{and} \quad -2+M_2^2-\gamma M_2^2+M_1^2-\gamma M_1^2+2\gamma M_2^2 M_1^2=0$

Step 10: The first equation can be easily simplified:

$M_2=M_1 \quad \text{or} \quad M_2=-M_1$

Step 11: In the second equation, collect in terms of M_2^2:

$M_2^2(1-\gamma+2\gamma M_1^2)+M_1^2-\gamma M_1^2-2=0$

Step 12: And then finish solving for M_2^2:

$M_2^2=\frac{2-M_1^2+\gamma M_1^2}{1-\gamma+2\gamma M_1^2}$

Step 13: Factor out M_1^2 in the numerator and -1 in the denominator:

$M_2^2=\frac{2+M_1^2(\gamma-1)}{-1(\gamma-1)+2\gamma M_1^2}$

Step 14: Multiply both the top and bottom by $\frac{1}{(\gamma-1)}$ to get the Normal Shock Equation in its original format:

${M_2}^2 = \frac{\frac{2}{\gamma-1}+{M_1}^2}{\frac{2\,{M_1}^2\,\gamma}{\gamma-1}-1}$

We can also do the derivation faster and easier with MATLAB:

Define Existing Equations:

```
syms M_1 M_2 gamma f p_1 p_2 T_1 T_2

Normal_Shock = M_2^2 == (M_1^2 + 2/(gamma-1))/((2*gamma*M_1^2/(gamma-1))-1)
```

Normal_Shock =

$$M_2^2 = \frac{\frac{2}{\gamma - 1} + M_1^2}{\frac{2\,M_1^2\,\gamma}{\gamma - 1} - 1}$$

```
continuity = p_1*M_1/sqrt(T_1) == p_2*M_2/sqrt(T_2)
```

continuity =

$$\frac{M_1\,p_1}{\sqrt{T_1}} = \frac{M_2\,p_2}{\sqrt{T_2}}$$

```
energy = T_1*((gamma-1)*M_1^2/2+1) == T_2*((gamma-1)*M_2^2/2+1)
```

energy =

$$T_1\left(\frac{M_1^2\,(\gamma - 1)}{2} + 1\right) = T_2\left(\frac{M_2^2\,(\gamma - 1)}{2} + 1\right)$$

```
momentum = (gamma*M_1^2+1)*p_1 == (gamma*M_2^2+1)*p_2
```

momentum = $p_1\,\left(\gamma\,M_1^2 + 1\right) = p_2\,\left(\gamma\,M_2^2 + 1\right)$

Complete Step 1: divide contunity equaiton by momentum equation to remove pressure

```
step1 = continuity/momentum
```

step1 =

$$\frac{M_1}{\sqrt{T_1}\,\left(\gamma\,M_1^2 + 1\right)} = \frac{M_2}{\sqrt{T_2}\,\left(\gamma\,M_2^2 + 1\right)}$$

Complete Step 2: multiply step1 by sqare root of energy equation to remove temperature

```
% redefining energy equation with 2 square roots (MATLAB was not removing
% T_1 and T_2 from the equation in step2 unless this was done)
energy = sqrt(T_1)*sqrt(((gamma-1)*M_1^2/2+1)) == sqrt(T_2)*sqrt(((gamma-1)*M_2^2/2+1));

step2 = step1*energy % sqrt(energy) already performed above
```

step2 =

$$\frac{M_1 \sqrt{\frac{{M_1}^2\ (\gamma-1)}{2}+1}}{\gamma\, {M_1}^2+1} = \frac{M_2 \sqrt{\frac{{M_2}^2\ (\gamma-1)}{2}+1}}{\gamma\, {M_2}^2+1}$$

Solve for M_2^2 **(using MATLAB solve command on our symbolic equaiton skips almost all of the manual derivation)**:

```
M2 = solve(step2,M_2);
```

```
Warning: Solutions are only valid under certain conditions. To include parameters and conditions in the solution, specify the 'ReturnConditions' value as 'true'.
```

```
M2_squared_option1 = M2(1,1)^2
```

M2_squared_option1 = ${M_1}^2$

```
M2_squared_option2 = M2(2,1)^2
```

M2_squared_option2 =

$$\frac{{M_1}^2\,\gamma - {M_1}^2 + 2}{2\,\gamma\, {M_1}^2 - \gamma + 1}$$

These two answers line up with what was seen in manual derivation.

From here we can do as shown in the manual derivation:

`Factor out M1^2 in the numerator and -1 in the denominator:`

$$M_2^2 = \frac{2+M_1^2(\gamma-1)}{-1(\gamma-1)+2\gamma M_1^2}$$

`Multiply both the top and bottom by 1/(gamma-1) to get the Normal Shock Equation in its original format:`

```
Normal_Shock
```

Normal_Shock =

$${M_2}^2 = \frac{\frac{2}{\gamma-1}+{M_1}^2}{\frac{2\,{M_1}^2\,\gamma}{\gamma-1}-1}$$

We can also check with MATLAB to check if our derived equation and orignal equaiton are equal:

```
% Simplify the difference between the two expressions
difference = simplify(M2_squared_option2 - rhs(Normal_Shock));

% Check if the simplified difference is always zero
```

```
if isAlways(difference == 0)
    disp('The expressions are equivalent');
else
    disp('The expressions are not equivalent');
end
```

```
The expressions are equivalent
```

We can also verify Numerically via a manual version of MATLAB isAlways() function

by substituting any random value for M1 and gamma, many times, into our derived equation and orignal equaiton

```
difference_list = [];
for i = 1:50
    random_M1 = randomInRange(0,10);
    random_gamma = randomInRange(0,10);

    check_derivation = random_check(M2_squared_option2,random_M1,random_gamma);

    check_original = random_check(rhs(Normal_Shock),random_M1,random_gamma);

    difference = check_derivation - check_original;

    difference_list = [difference_list difference];
end

if all(difference == 0)
    disp('The expressions are equivalent');
else
    disp('The expressions are not equivalent');
end
```

```
The expressions are equivalent
```

```
function random_value = randomInRange(min_value, max_value)
    random_value = min_value + (max_value - min_value) * rand;
end

function check = random_check(equation,M1_value,gamma_value)
    check = subs(equation, 'M_1', M1_value);
    check = subs(check, 'gamma', gamma_value);
    check = vpa(check, 4); %precise to 4 decimal places
    check = double(check);
end
```