2

Filters

FIR | IIR

* $h[n]$ is finite in length

* $h[n]$ is infinite in length.

$$x[n] \longrightarrow \boxed{\begin{matrix} h[n] \\ H(z) \end{matrix}} \longrightarrow y[n]$$

$$X(z) \qquad\qquad\qquad Y(z)$$

* Generally,

$$y[n] = \sum_{k=-M_1}^{M_1} a_k x[n-k] - \sum_{k=1}^{N} b_k y[n-k]$$

$$y[n] + b_1 y[n-1] + b_2 y[n-2] + \dots + b_N y[n-N] = a_{-M_1} x[n+M_1] + a_{-M_1+1} x[n+M_1-1]$$
$$+ \dots + a_0 x[n] + \dots + a_M x[n-M]$$

Non-recursive

$b_k = 0 \quad \forall\, k = 1, 2, \dots, N$

$\Downarrow$

$$y[n] = a_{-M_1} x[n+M_1] + \dots + a_{-1} x[n+1]$$
$$+ a_0 x[n] + a_1 x[n-1] + \dots + a_{M_1} x[n-M_1]$$

(FIR Filters)

$\downarrow$ Transfer Function

$$H(z) = a_{M_1} z^{M_1} + \dots + a_0 + a_1 z^{-1} + \dots + a_{M_1} z^{-M_1}$$

For a causal filter

$a_k = 0$ for $k < 0$

$$H(z) = a_0 + a_1 z^{-1} + \dots + a_{M_1} z^{-M_1}$$

Recursive

$b_k \neq 0$ for some $k = 1, 2, \dots, N$

$\Downarrow$

(All IIR Filters, Some FIR Filters)

$\downarrow$

Ex: Moving average filter

$\downarrow$ Transfer Function

$$H(z) = \frac{a_{M_1} z^{M_1} + a_{-M_1+1} z^{M_1-1} + \dots + a_0 + a_1 z^{-1} + \dots + a_{M_1} z^{-M_1}}{1 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_N z^{-N}}$$

For a causal filter

$a_k = 0$ for $k < 0$

$$H(z) = \frac{a_0 + a_1 z^{-1} + \dots + a_{M_1} z^{-M_1}}{1 + b_1 z^{-1} + \dots + b_N z^{-N}}$$

FIR Filters

$$y[n] = \sum_{k=0}^{M} a_k x[n-k]$$

$$y[n] = a_0 x[n] + a_1 x[n-1] + \dots + a_M x[n-M]$$

If $x[n] = \delta[n] \Rightarrow y[n] = h[n]$,

$$h[n] = a_0 \delta[n] + a_1 \delta[n-1] + \dots + a_M \delta[n-M]$$

Then,

$$h[0] = a_0 \delta[0] + a_1 \delta[-1] + \dots + a_M \delta[-M] = a_0$$

$$h[1] = a_0 \delta[1] + a_1 \delta[0] + \dots + a_M \delta[1-M] = a_1$$

$\vdots$

$$h[k] = a_k$$

- Designing Filters
  - Closed Form
    - Direct Method (FIR)
    - Indirect Method (IIR)
  - Optimization Form

Constant Delay in Non-recursive Filters

* A causal non-recursive filter has the transfer function,

$$H(z) = \sum_{n=0}^{N-1} h[n] z^{-n}$$

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h[n] e^{-j\omega n} = M(\omega) e^{j\theta(\omega)}$$

$\theta(\omega) \leftarrow$ Phase

$M(\omega) \uparrow$ Magnitude

$$M(\omega) = |H(e^{j\omega})| \qquad \theta(\omega) = \arg[H(e^{j\omega})]$$

Phase delay,

$$\tau_p = -\frac{\theta(\omega)}{\omega} \qquad \tau_g = -\frac{d\theta(\omega)}{d\omega}$$

$\therefore$ Constant phase and group delays can be achieved by making the impulse response symmetric about its center

i.e.

$$H(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}$$

N = order of the filter Length of $h[n] = N+1$

$$H(e^{j\omega}) = a_0 + a_1 e^{-j\omega} + a_2 e^{-j2\omega} + \dots + a_N e^{-jN\omega}$$

$$= a_0 + a_1 \cos(\omega) + a_2 \cos(2\omega) + \dots + a_N \cos(N\omega)$$

$$-j\left[a_1 \sin(\omega) + a_2 \sin(2\omega) + \dots + a_N \sin(N\omega)\right]$$

$$H(e^{j\omega}) = \sum_{n=0}^{N} a_n \cos(\omega n) - j \sum_{n=0}^{N} a_n \sin(\omega n)$$

$$\therefore \theta(\omega) = \tan^{-1}\left[\frac{-\sum_{n=0}^{N} a_n \sin(\omega n)}{\sum_{n=0}^{N} a_n \cos(\omega n)}\right] = -\tau\omega \quad \text{(Linear phase response)}$$

$$\tan(\tau\omega) = \frac{\sum_{n=0}^{N} a_n \sin(\omega n)}{\sum_{n=0}^{N} a_n \cos(\omega n)}$$

$$\frac{\sin(\tau\omega)}{\cos(\tau\omega)} = \frac{\sum_{n=0}^{N} a_n \sin(\omega n)}{\sum_{n=0}^{N} a_n \cos(\omega n)}$$

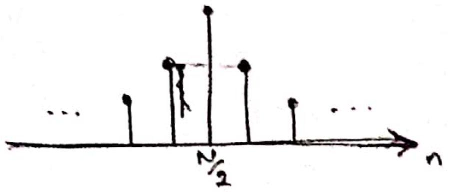
$$\sum_{n=0}^{N} a_n \sin(\tau\omega)\cos(\omega n) - \sum_{n=0}^{N} a_n \cos(\tau\omega)\sin(\omega n) = 0$$

$$\sum_{n=0}^{N} a_n \left[\sin(\tau\omega)\cos(\omega n) - \cos(\tau\omega)\sin(\omega n)\right] = 0$$

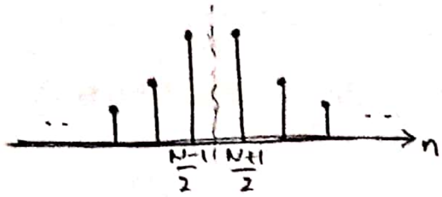
$$\sum_{n=0}^{N} a_n \sin(\omega\tau - \omega n) = 0$$

Solutions to this equation, $\tau = N/2$

$$a_n = a_{N-n} \quad ; \quad n = 0, 1, 2, \dots, N/2$$



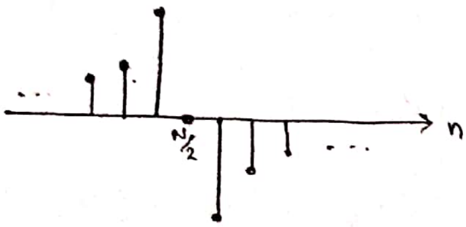
N is even
$\Rightarrow$ odd length filter



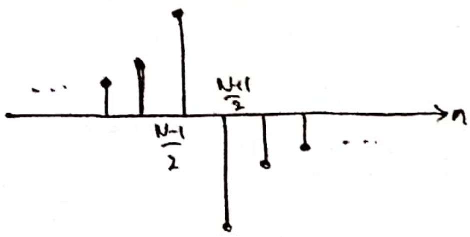
N is odd
$\Rightarrow$ even length filter

$\rightarrow$ Constant phase and group delays

$\boxed{\theta = -\tau\omega}$



N is even
$\Rightarrow$ Odd length filter



N is odd
$\Rightarrow$ even length filter

$\rightarrow$ $\theta = \theta_0 - \tau\omega$
where
$\theta_0 = \pm \pi/2$

Group delay is a constant

Locations of the Poles

$$H(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}$$

$$H(z) = \frac{a_0 z^N + a_1 z^{N-1} + \dots + a_N}{z^N}$$

N poles at the origin $\Rightarrow$ within the unit ~~cycle~~ circle.
$\downarrow$
Causal FIR filters are always stable

$\therefore$ All the poles are at the origin

Locations of the Zeros

* Consider the case of linear phase response. $\Rightarrow$ $a_n = a_{N-n}$

Suppose N is even,

$$H(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}$$

$$H(z) = a_0 + a_1 z^{-1} + \dots + a_{N/2} z^{-N/2} + a_{N/2+1} z^{-N/2+1} + \dots + a_{N-1} z^{-N+1} + a_N z^{-N}$$

$$= a_0 + a_1 z^{-1} + \dots + a_{N/2-1} z^{-N/2+1} + a_{N/2} z^{-N/2} + a_{N/2-1} z^{-N/2-1} + \dots + a_1 z^{-N+1} + a_0 z^{-N}$$

$$= \frac{1}{z^{N/2}} \left[a_0 z^{+N/2} + a_1 z^{N/2-1} + \dots + a_{N/2-1} z^{1} + a_{N/2} + a_{N/2-1} z^{-1} + \dots + a_1 z^{-N/2+1} + a_0 z^{-N/2} \right]$$

$$H(z) = \frac{a_{N/2} + \sum_{n=1}^{N/2} a_{N/2-n}(z^n + z^{-n})}{z^{N/2}} = \frac{N(z)}{D(z)}$$

where

$$N(z) = a_{N/2} + \sum_{n=1}^{N/2} a_{N/2-n}(z^n + z^{-n})$$

Suppose $\hat{z}$ is a root of $N(z)$ i.e. $N(\hat{z}) = 0$

$$\Rightarrow a_{N/2} + \sum_{n=1}^{N/2} a_{N/2-n}(\hat{z}^{-n} + \hat{z}^{n}) = 0 \quad -①$$

Now, consider $N(\hat{z}^{-1}) = a_{N/2} + \sum_{n=1}^{N/2} a_{N/2-n}(\hat{z}^{-n} + \hat{z}^{n}) = 0$ (By ①)

$\therefore$ $\hat{z}$ is a root of ~~z~~ $N(z)$ $\Rightarrow$ $\hat{z}^{-1}$ is a root of $N(z)$

1. An arbitary even number of zeros can be located at $z = \pm 1$

$z = 1 \Rightarrow z^{-1} = 1$

$z = -1 \Rightarrow z^{-1} = -1$

~~$N(1) = 0 \Rightarrow$ Two zeros at $z = 1$~~

2. An arbitary number of complex-conjugate pairs of zeros can be located on the unit circle

~~$z = |z| e^{j\theta}$~~ $z = e^{j\theta}$ (on the unit circle)

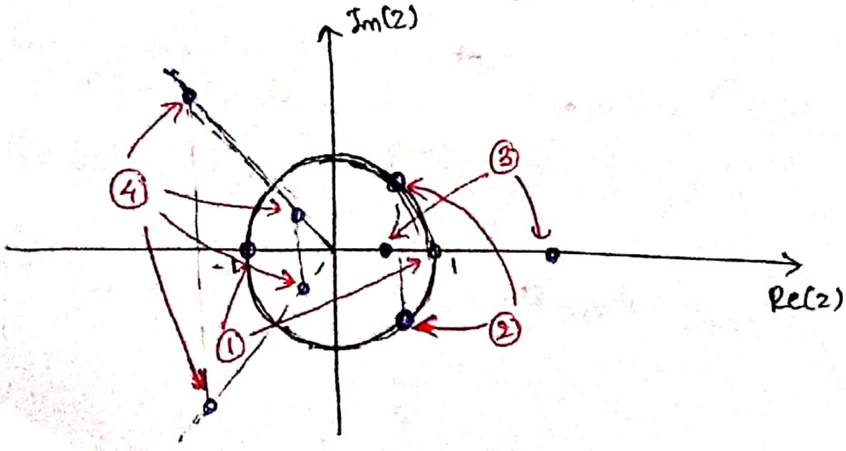
$\Downarrow$

$z^{-1} = e^{-j\theta}$ (also on the unit circle)

$= z^*$

3. Off the unit circle, zeros on the real axis occur in reciprocal pairs.

$z = \hat{z} \Rightarrow z^{-1} = \frac{1}{\hat{z}}$



4. Complex zeros off the unit circle must occur as groups of four.

$z = |\hat{z}| e^{j\theta} \Rightarrow z^* = |\hat{z}| e^{-j\theta}$

$\Downarrow \qquad\qquad \Downarrow$

$z^{-1} = \frac{1}{|\hat{z}|} e^{-j\theta} \qquad \frac{1}{z^*} = \frac{1}{|\hat{z}|} e^{j\theta}$

$h[n]$ is a real valued sequence (i.e $a_n \in \mathbb{R}$)

* These are mirror-image polynomials

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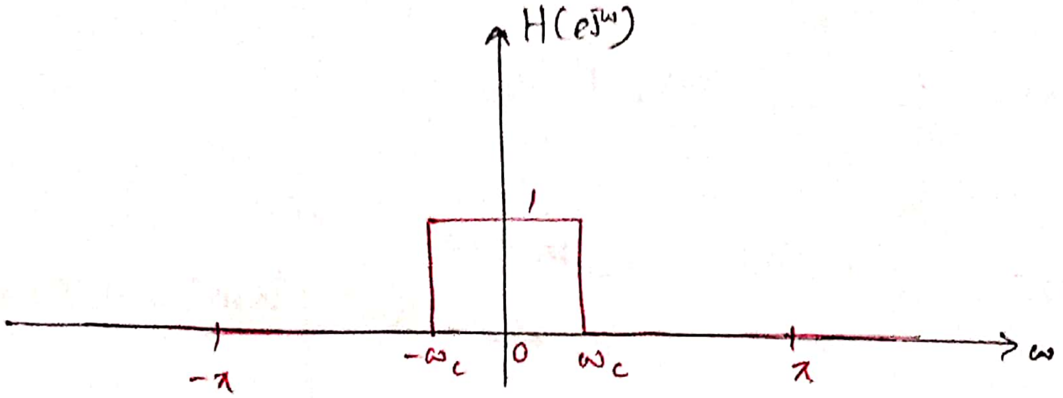
$$H(e^{j\omega}) = \begin{cases} 1 & ; \ |\omega| \le \omega_c \\ 0 & ; \ \cancel{|\omega| > \omega_c} \ \omega_c < |\omega| < \pi \end{cases}$$



$$h_I[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\, \frac{\left[e^{j\omega n}\right]_{-\omega_c}^{\omega_c}}{jn} \quad ; \ n \neq 0$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right]$$

$$h_I[n] = \frac{\sin(\omega_c n)}{\pi n} \quad ; \ n \neq 0$$

$n=0$,

$$h_I[0] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} \quad ; \ n = 0$$

$$h_I[n] = \begin{cases} \omega_c/\pi & ; \ n = 0 \\ \dfrac{\sin(\omega_c n)}{n\pi} & ; \ n \neq 0 \end{cases}$$

$$h_I[-n] = \frac{\sin(-\omega_c n)}{-n\pi} = \frac{\sin(\omega_c n)}{n\pi} = h_I[n] \Rightarrow h_I[n] \text{ is symmetric.}$$

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$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n]\, e^{-j\omega n}$$

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$\boxed{\text{Causal}}$

$$h[n] = \begin{cases} h_d[n] & : 0 \le n \le M \\ 0 & : \text{otherwise} \end{cases} \Leftarrow \quad h[n] = h_d[n]\,w[n]$$

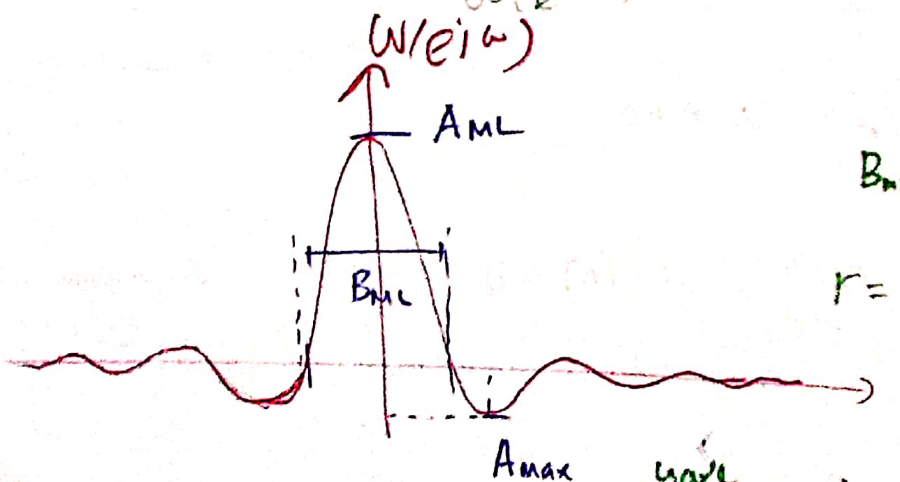
$$w[n] = \begin{cases} 1; & 0 \le n \le M \\ 0; & \text{otherwise} \end{cases} \leftarrow \text{Rectangular window}$$

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})\, W(e^{j(\theta-\omega)})\, d\theta$$

$$W(e^{j\omega}) = \sum_{n=0}^{M} e^{-j\omega n} = \frac{1 - e^{-j\omega(M+1)}}{1 - e^{-j\omega}}$$

$$= \frac{e^{-j\omega\left(\frac{M+1}{2}\right)}}{e^{-j\omega/2}} \frac{\left[e^{j\omega\left(\frac{M+1}{2}\right)} - e^{-j\omega\left(\frac{M+1}{2}\right)}\right]}{\left[e^{j\omega/2} - e^{-j\omega/2}\right]}$$

$$W(e^{j\omega}) = e^{-j\omega M/2}\, \frac{\sin\left[\omega\left(\frac{M+1}{2}\right)\right]}{\sin\left(\frac{\omega}{2}\right)}$$



B_{ML} = Main lobe width

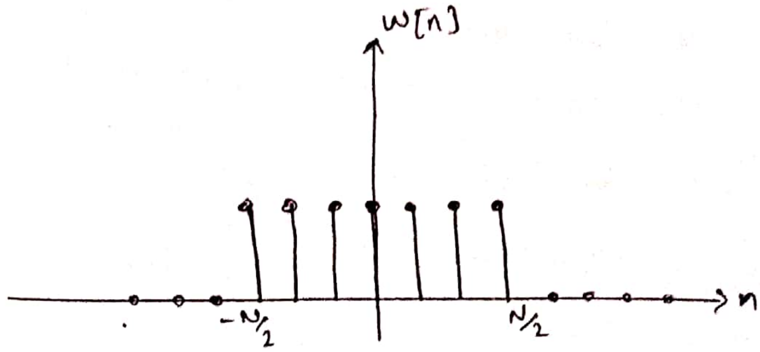
$r = \frac{A_{max}}{A_{ML}} \times 100\%$ or

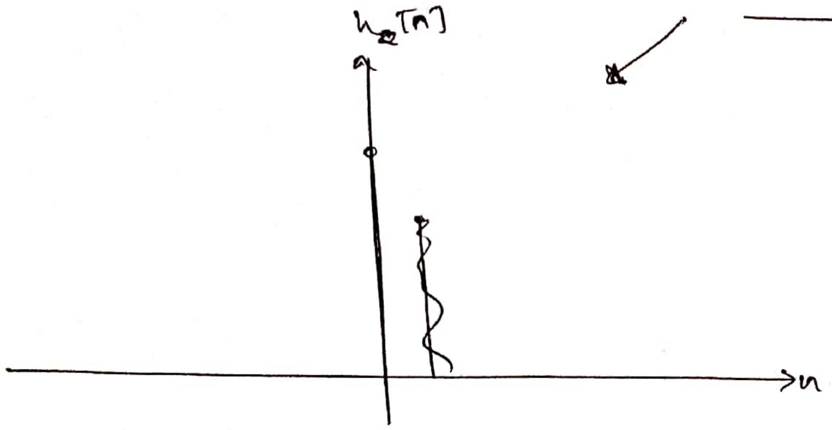
$R = 20 \log\left(\frac{A_{max}}{A_{ML}}\right)$ dB

Must have $B_{ML} \downarrow$, $r \downarrow$

$$H(z) = \cdots + h_I[-1]\, z + h_I[0] + h_I[1]\, z^{-1} + \cdots$$

$$h[n] = h_I[n]\, w[n]$$





$h_c[n] = h[n - N/2]$ ← To make $h[n]$ causal

$$H_c(e^{j\omega}) = e^{-j\omega N/2}\, \underbrace{H(e^{j\omega})}_{\text{Has zero phase (Real)}}$$

$$\Downarrow$$

$H_c(e^{j\omega})$ has a linear phase response

$$H_c(z) = h_I[0] + h_I[1]\, z^{-1} + h_I[2]\, z^{-2} + \cdots + h_I[N]\, z^{-N}$$

Main lobe width →	Steepness of the transition
Ripple ratio →	Amplitudes of the passband and stopband ripples

Rectangular window → * Ripples at the edges of the passband cannot be eliminated.

* Maximum stopband attenuation is −45 dB

Other windows

01. Von Hann
02. Hamming
03. Blackman

$\}$ One parameter to change.

04. Dolph – Chebyshev
05. Kaiser

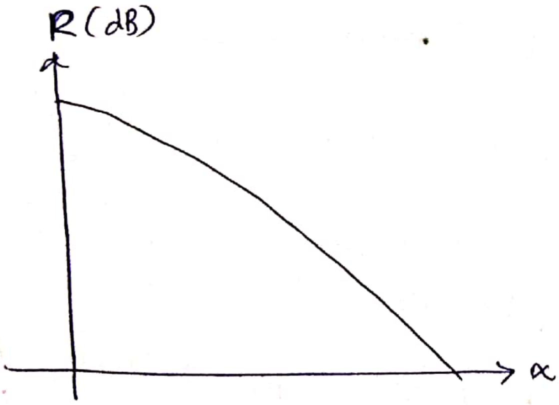
$\}$ Two parameters to change

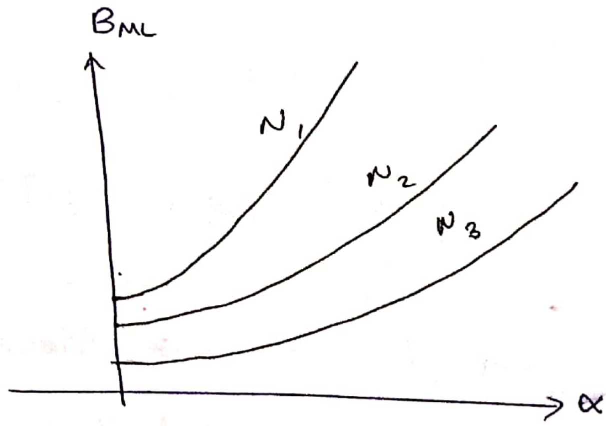
06. Ultraspherical window – Three parameters to change

Kaiser Window

$$w_K[n] = \begin{cases} \dfrac{I_0(\beta)}{I_0(\alpha)} & ; \quad |n| \leq \left(\dfrac{N-1}{2}\right) \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$I_0(x) = 1 + \sum_{k=1}^{\infty}\left[\frac{1}{k!}\left(\frac{x}{2}\right)^k\right]^2 \quad \text{and} \quad \beta = \alpha\sqrt{1-\left(\frac{2n}{N-1}\right)^2}$$



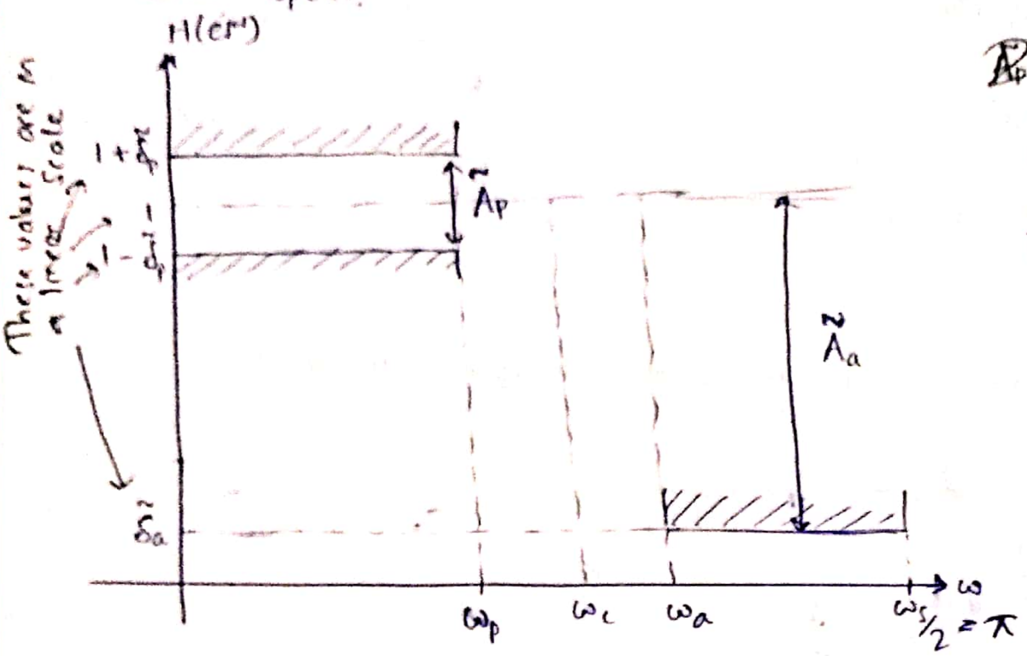


$N_1 < N_2 < N_3$

* First set α for the desired ripple ratio /

Designing a non-recursive LPF using Kaiser window

* Given specs



(Specified)
$\tilde{A}_p$ = Maximum passband ripple (dB)
(Specified)
$\tilde{A}_a$ = Minimum stopband attenuation (dB)

ω_p = Upper passband edge

ω_c = Cutoff frequency

ω_a = Lower stopband edge

$$\omega_c = \frac{1}{2}(\omega_p + \omega_a)$$

Step 1 : Determine the impulse response assuming ideal frequency response.

$$H(e^{j\omega}) = \begin{cases} 1 ; & |\omega| \leq \omega_c \\ 0 ; & \omega_c \leq |\omega| \leq \pi \end{cases}$$

$$h[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{jn}\right] = \frac{\sin(\omega_c n)}{\pi n}$$

; $n \neq 0$

$$h[n] = \begin{cases} \frac{\omega_c}{\pi} ; & n = 0 \\ \frac{\sin(\omega_c n)}{\pi n} ; & n \neq 0 \end{cases}$$

Step 2 : Find δ

* When we find δ, we use $\tilde{\delta}_p = \tilde{\delta}_a = \delta$.

$\therefore$ δ should be choosen such that,

$A_p \leq \tilde{A}_p$ and $A_a \geq \tilde{A}_a$

$\therefore$ $\delta = \min(\tilde{\delta}_p, \tilde{\delta}_a)$

$$20 \log\left(\frac{1+\tilde{\delta}_p}{1-\tilde{\delta}_p}\right) = \tilde{A}_p \qquad\qquad 20 \log\left(\frac{1}{\tilde{\delta}_a}\right) = \tilde{A}_a$$

$$\frac{1+\tilde{\delta}_p}{1-\tilde{\delta}_p} = 10^{0.05\tilde{A}_p} \qquad\qquad \frac{1}{\tilde{\delta}_a} = 10^{0.05\tilde{A}_a}$$

$$\boxed{\tilde{\delta}_p = \frac{10^{0.05\tilde{A}_p} - 1}{10^{0.05\tilde{A}_p} + 1}} \qquad\qquad \boxed{\tilde{\delta}_a = 10^{-0.05\tilde{A}_a}}$$

Step 3 : Calculate the actual stopband attenuation using δ.

$$A_a = -20 \log(\delta)$$

Step 4 : Choose α using A_a as,

$$\alpha = \begin{cases} 0 & ; \quad A_a \leq 21 \\ 0.5842\,(A_a - 21)^{0.4} + 0.07886\,(A_a - 21) & ; \quad 21 \leq A_a \leq 50 \\ 0.1102\,(A_a - 8.7) & ; \quad A_a > 50 \end{cases}$$

Step 5 : Choose N ~~Cont~~ (length of the filter) as follows.

* Find D as,

$$D = \begin{cases} 0.9222 & ; \quad A_a \leq 21 \\ \dfrac{A_a - 7.95}{14.36} & ; \quad A_a > 21 \end{cases}$$

* Choose the lowest odd value ~~to~~ N, which satisfies the following inequality as N,

$$N \geq \frac{2\pi D}{\omega_a - \omega_p} + 1 \qquad\qquad B_t = \omega_a - \omega_p$$

Step 6 : Form the Kaiser window as follows.

$$w_K[n] = \begin{cases} \dfrac{I_0(\beta)}{I_0(\alpha)} & ; \quad |n| \leq \left(\frac{N-1}{2}\right) \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$\beta = \alpha\sqrt{1 - \left(\frac{2n}{N-1}\right)^2}, \qquad I_0(x) = 1 + \sum_{k=1}^{\infty}\left[\frac{1}{k!}\left(\frac{x}{2}\right)^k\right]^2$$

Step 7 : Obtain the transfer function $H_w'(z)$,

$$H_w(z) = \mathcal{Z}\{w_k[n]\,h[n]\}$$

$$H_w'(z) = z^{-(N-1)/2} H_w(z)$$

$$H_w'(z) = z^{-\left(\frac{N-1}{2}\right)} H_w(z) \leftarrow \text{Causal filter}$$

Ex : $\tilde{A}_p = 0.05\ dB$, $\tilde{A}_a = 53\ dB$,

$\Omega_{ps} = 1100\ rad/s$, $\Omega_a = 1200\ rad/s$

$\Omega_s = 4000\ rad/s$

$$\omega = \Omega T$$

$$\omega = \frac{2\pi\Omega}{\Omega_s}$$

$$\omega_p = \frac{2\pi \times 1100}{4000} = \frac{11\pi}{20}\ rad/sample$$

$$\omega_a = \frac{2\pi \times 1200}{4000} = \frac{3\pi}{5}\ rad/sample$$

$$\omega_c = \frac{(\omega_p + \omega_a)}{2} = \frac{\left(\frac{11\pi}{20} + \frac{3\pi}{5}\right)}{2}$$

$$\omega_c = \frac{23\pi}{40}\ rad/sample$$

$$h[n] = \begin{cases} \frac{23}{40} & ; \quad n = 0 \\ \frac{\sin\left(\frac{23\pi n}{40}\right)}{\pi n} & ; \quad n \neq 0 \end{cases}$$

$$\tilde{\delta}_p = \frac{10^{0.05\tilde{A}_p} - 1}{10^{0.05\tilde{A}_p} + 1}$$

$$\tilde{\delta}_p = 2.878 \times 10^{-3}$$

$$\tilde{\delta}_a = 10^{-0.05\tilde{A}_a}$$

$$\tilde{\delta}_a = 2.239 \times 10^{-3}$$

$$\delta = \min(\tilde{\delta}_p, \tilde{\delta}_a)$$

$$\therefore \delta = \tilde{\delta}_a = 2.239 \times 10^{-3}$$

$$A_a = -20\log(\delta_a) = -20\log(2.239 \times 10^{-3})$$

$$A_a = \cancel{52.99}\ 53\ dB > 50$$

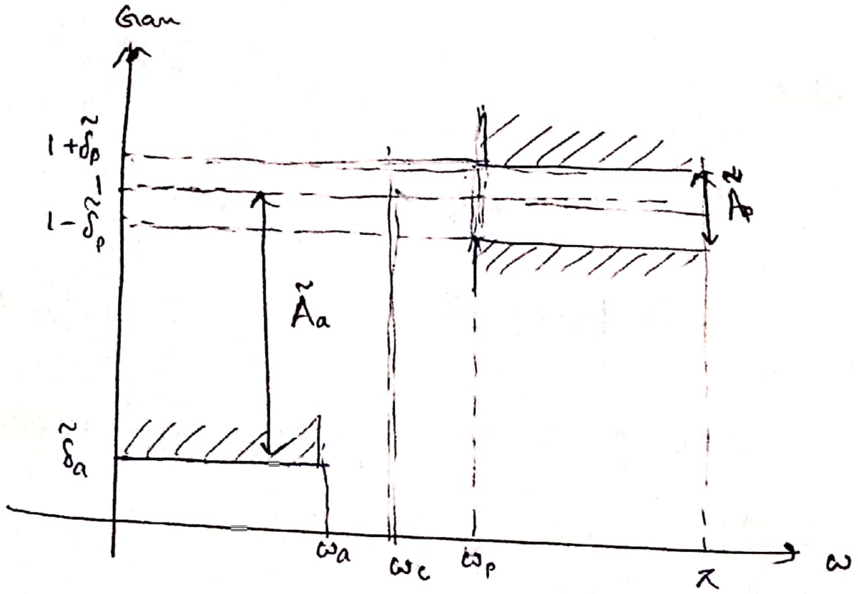
$\therefore \quad \alpha = 0.1102\,(53 - 8.7) = 4.8819$

$D = 3.137$

$$\frac{2\pi D}{\omega_a - \omega_p} + 1 = \frac{2\pi \times 3.137}{\left(\frac{3\pi}{5} - \frac{11\pi}{20}\right)} + 1 = 126.48$$

$$\Rightarrow N = 127$$

Design of a non-recursive HPF



Only difference is in step 1.

$$H(e^{j\omega}) = \begin{cases} 1 ; & \omega_c \leq |\omega| \leq \pi \\ 0 ; & \text{otherwise} \end{cases} = \begin{cases} 1 ; & -\pi \leq \omega \leq -\omega_c \\ 1 ; & \omega_c \leq \omega \leq \pi \\ 0 ; & \text{otherwise} \end{cases}$$

$$h[n] = \frac{1}{2\pi}\left\{ \int_{-\pi}^{-\omega_c} e^{+j\omega n}\, d\omega + \int_{\omega_c}^{\pi} e^{j\omega n}\, d\omega \right\}$$

$$= \frac{1}{2\pi}\left\{ \frac{\left[e^{j\omega n}\right]_{-\pi}^{-\omega_c}}{jn} + \frac{\left[e^{j\omega n}\right]_{\omega_c}^{\pi}}{jn} \right\} \quad ; n \neq 0$$

$$= \frac{1}{\pi n}\left\{ \frac{e^{-j\omega_c n} - e^{-jn\pi} + e^{j\pi n} - e^{j\omega_c n}}{2j} \right\}$$

$$h[n] = \frac{1}{\pi n}\left\{\frac{\cos(n\pi) - \cos(-n\pi) - e^{j\omega_c n} + e^{-j\omega_c n}}{2j}\right\}$$

$$h[n] = -\frac{\sin(\omega_c n)}{n\pi} \quad ; n \neq 0$$

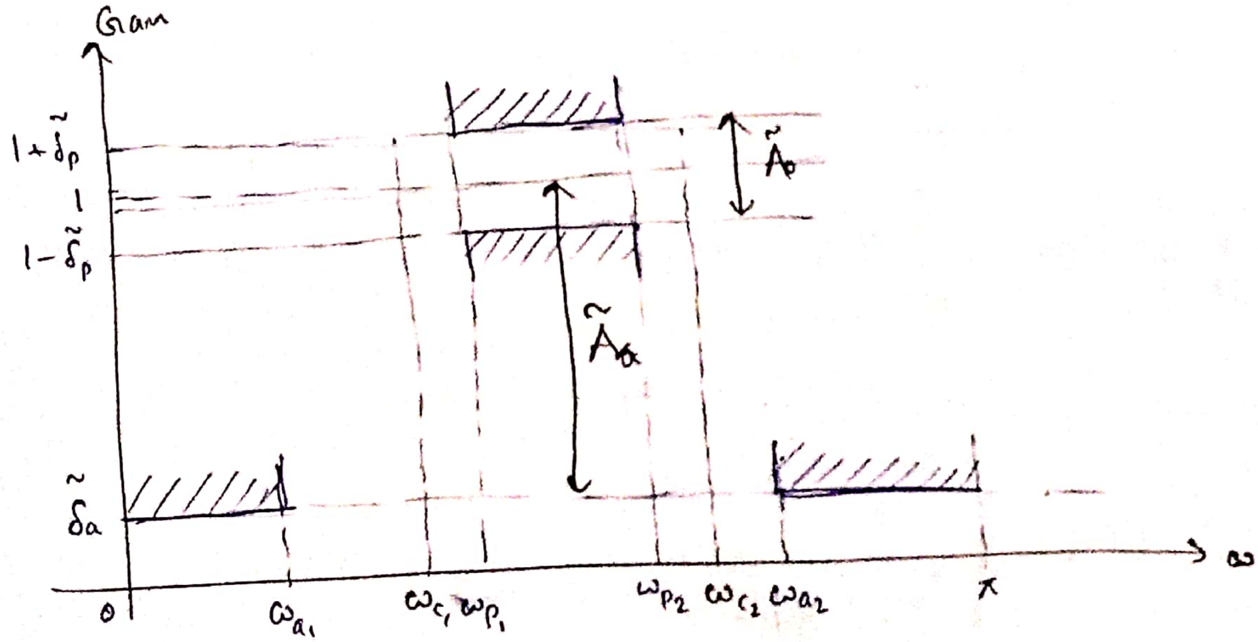
$n = 0$,

$$h[0] = \frac{1}{2\pi}\left\{(\omega_c + \pi) + (\pi - \omega_c)\right\}$$

$$= 1 - \frac{\omega_c}{\pi}$$

$$\therefore h[n] = \begin{cases} 1 - \frac{\omega_c}{\pi} & ; n = 0 \\ -\frac{\sin(\omega_c n)}{n\pi} & ; n \neq 0 \end{cases}$$

Design of a non-recursive Bandpass Filter



* Only difference is to use the most critical transition out of the two transitions.

i.e $\quad B_t = \min\left\{(\omega_{p_1} - \omega_{a_1}),\ (\omega_{a_2} - \omega_{p_2})\right\}$

Then, $\quad \omega_{c_1} = \omega_{p_1} - \frac{B_t}{2} \qquad \omega_{c_2} = \omega_{p_2} + \frac{B_t}{2}$

* Then the ideal frequency response of the BPF,

$$H(e^{j\omega}) = \begin{cases} 1 \; ; & -\omega_{c_2} \leq \omega \leq -\omega_{c_1} \\ 1 \; ; & \omega_{c_1} \leq \omega \leq \omega_{c_2} \\ 0 \; ; & \text{otherwise} \end{cases}$$

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) \, e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi} \left\{ \int_{-\omega_{c_2}}^{-\omega_{c_1}} e^{j\omega n} \, d\omega + \int_{\omega_{c_1}}^{\omega_{c_2}} e^{j\omega n} \, d\omega \right\}$$

$$= \frac{1}{2\pi} \left\{ \frac{\left[e^{j\omega n} \right]_{-\omega_{c_2}}^{-\omega_{c_1}}}{jn} + \frac{\left[e^{j\omega n} \right]_{\omega_{c_1}}^{\omega_{c_2}}}{jn} \right\} \; ; \; n \neq 0$$

$$= \frac{1}{\pi} \left\{ \frac{e^{-j\omega_{c_1} n} - e^{-j\omega_{c_2} n} + e^{j\omega_{c_2} n} - e^{j\omega_{c_1} n}}{2j} \right\}$$

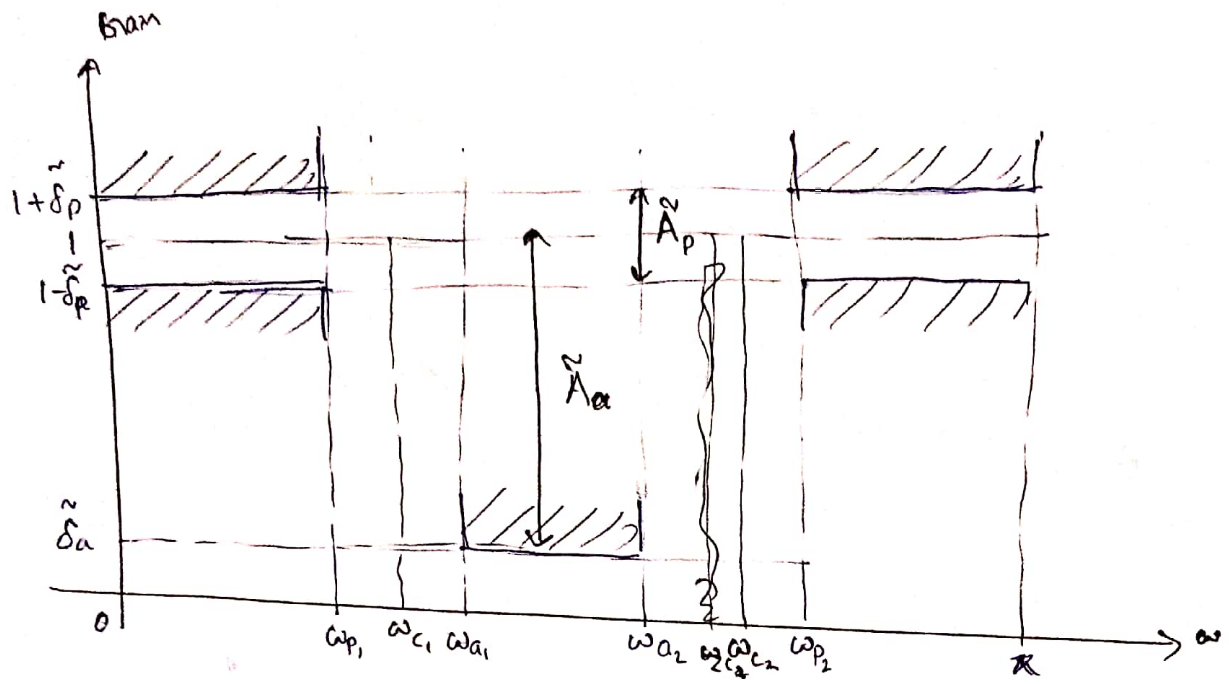
$$h[n] = \frac{\sin(\omega_{c_2} n) - \sin(\omega_{c_1} n)}{n\pi} \; ; \; n \neq 0$$

$$h[0] = \frac{1}{2\pi} \left\{ \int_{-\omega_{c_2}}^{-\omega_{c_1}} d\omega + \int_{\omega_{c_1}}^{\omega_{c_2}} d\omega \right\} = \frac{1}{2\pi} \left\{ -\omega_{c_1} + \omega_{c_2} + \omega_{c_2} - \omega_{c_1} \right\}$$

$$h[0] = \frac{\omega_{c_2} - \omega_{c_1}}{\pi}$$

$$\therefore \; h[n] = \begin{cases} \dfrac{\omega_{c_2} - \omega_{c_1}}{\pi} \; ; & n = 0. \\ \dfrac{\sin(\omega_{c_2} n) - \sin(\omega_{c_1} n)}{n\pi} \; ; & n \neq 0 \end{cases}$$

Design of a non-recursive Bandstop Filter



Step 1: Consider the more critical transition to find cut-off frequencies

$$B_t = \min\{(\omega_{a_1}-\omega_{p_1}),\ (\omega_{p_2}-\omega_{a_2})\}$$

$$\omega_{c_1} = \omega_{p_1} + \frac{B_t}{2} \qquad \omega_{c_2} = \omega_{p_1} - \frac{B_t}{2}$$

Step 2: Use the ideal frequency response to find the ideal impulse response.

$$H(e^{j\omega}) = \begin{cases} 1 ; & 0 \le |\omega| \le \omega_{c_1} \\ 0 ; & \omega_{c_1} < |\omega| < \omega_{c_2} \\ 1 ; & \omega_{c_2} \le |\omega| \le \pi \end{cases}$$

$$\Rightarrow \quad H(e^{j\omega}) = \begin{cases} 1 ; & -\pi \le \omega \le -\omega_{c_2} \\ 0 ; & -\omega_{c_2} < \omega < -\omega_{c_1} \\ 1 ; & -\omega_{c_1} \le \omega \le \omega_{c_1} \\ 0 ; & \omega_{c_1} < \omega < \omega_{c_2} \\ 1 ; & \omega_{c_2} \le \omega \le \pi \end{cases}$$

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi} \left\{ \int_{-\pi}^{-\omega_{c_2}} e^{j\omega n} d\omega + \int_{-\omega_{c_1}}^{\omega_{c_1}} e^{j\omega n} d\omega + \int_{\omega_{c_2}}^{\pi} e^{j\omega n} d\omega \right\}$$

$$= \frac{1}{2\pi} \left\{ \frac{\left[e^{j\omega n}\right]_{-\pi}^{-\omega_{c_2}}}{jn} + \frac{\left[e^{j\omega n}\right]_{-\omega_{c_1}}^{\omega_{c_1}}}{jn} + \frac{\left[e^{j\omega n}\right]_{\omega_{c_2}}^{\pi}}{jn} \right\} \quad ; n \neq 0$$

$$= \frac{1}{2\pi} \left\{ \frac{e^{-j\omega_{c_2} n} - e^{-j\pi n} + e^{j\omega_{c_1} n} - e^{-j\omega_{c_1} n} + e^{j\pi n} - e^{j\omega_{c_2} n}}{2j} \right\}$$

$$h[n] = \frac{\sin(\omega_{c_1} n) - \sin(\omega_{c_2} n)}{n\pi} \quad ; \quad n \neq 0$$

$$h[0] = \frac{1}{2\pi} \left\{ \int_{-\pi}^{-\omega_{c_2}} d\omega + \int_{-\omega_{c_1}}^{\omega_{c_1}} d\omega + \int_{\omega_{c_2}}^{\pi} d\omega \right\}$$

$$= \frac{1}{2\pi} \cdot \left\{ -\omega_{c_2} + \pi + \omega_{c_1} + \omega_{c_1} + \pi - \omega_{c_2} \right\}$$

$$h[0] = \frac{\omega_{c_1} - \omega_{c_2}}{\pi} + 1$$

$$\therefore h[n] = \begin{cases} 1 + \dfrac{(\omega_{c_1} - \omega_{c_2})}{\pi} & ; \quad n = 0 \\[2ex] \dfrac{\sin(\omega_{c_1} n) - \sin(\omega_{c_2} n)}{n\pi} & ; \quad n \neq 0 \end{cases}$$

Step 3 : Find δ

$$\delta = \min(\tilde{\delta}_p, \tilde{\delta}_a)$$

where,

$$\tilde{\delta}_p = \frac{10^{0.05\tilde{A}_p} - 1}{10^{0.05\tilde{A}_p} + 1} \quad \text{and} \quad \tilde{\delta}_a = 10^{-0.05\tilde{A}_a}$$

Step 4 : Calculate the actual stopband attenuation.

$$A_a = -20 \log_{10} \delta$$

Step 5 : Choose α.

$$\alpha = \begin{cases} 0 & ; \quad A_a \leq 21 \\ 0.5842 (A_a - 21)^{0.4} + 0.07886 (A_a - 21) & ; \quad 21 \leq A_a \leq 50 \\ 0.1102 (A_a - 8.7) & ; \quad A_a > 50 \end{cases}$$

Step 6 : Choose N (Length of the filter)

- First find D,

$$D = \begin{cases} 0.9222 & ; \quad A_a \leq 21 \\ \dfrac{A_a - 7.95}{14.36} & ; \quad A_a > 21 \end{cases}$$

- Choose the lowest odd value satisfying the following inequality as N,

$$N \geq \frac{2\pi D}{B_t} + 1$$

Step 7 : Form the Kaiser window.

$$\omega_k[n] = \begin{cases} \dfrac{I_0(\beta)}{I_0(\alpha)} & ; \quad |n| \leq \left(\dfrac{N-1}{2}\right) \\ 0 & ; \quad \text{otherwise} \end{cases}$$

where

$$\beta = \alpha \sqrt{1 - \left(\frac{2n}{N-1}\right)^2} \quad \text{and} \quad I_0(x) = 1 + \sum_{k=1}^{\infty} \left[\frac{1}{k!}\left(\frac{x}{2}\right)^k\right]^2$$

Step 8 : Obtain the transfer function $H_{w'}(z)$

$$H_w(z) = Z\{w_k[n]\, h[n]\}$$

$$H_{w'}(z) = z^{-\left(\frac{N-1}{2}\right)} H_w(z)$$

$$H_{w'}(e^{j\omega}) = H_{w'}(z)\Big|_{z=e^{j\omega}}$$