ESE 650, Spring 2020
Quaternion UKF Filter

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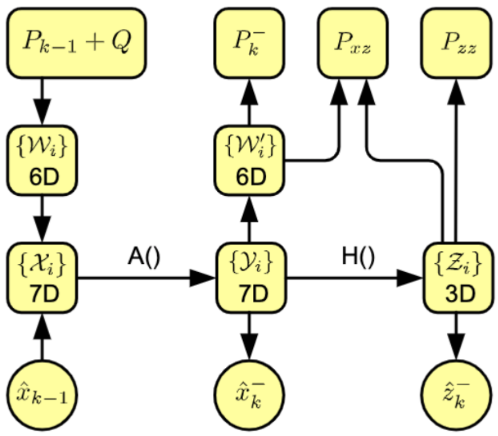
1 Schematic View of Filter



2 State Vector x

$x \in \mathcal{R}^7$ combines orientation q and angular velocity $\vec{\omega}$, $x = \begin{pmatrix} q \\ \vec{\omega} \end{pmatrix}$ where q is a unit quaternion, i.e. $q = (q_0, q_1, q_2, q_3)$. $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$.

3 Process Model A

$x_{k+1} = A(x_k, w_k)$ where $x_k = \begin{pmatrix} q_k \\ \omega_k \end{pmatrix}$, w_k is a random variable (process noise).
Suppose time interval is Δt, We have $q_{k+1} = q_k q_\Delta$ in **quaternion multiplication** form. For 6-dimensional noise w_k, we can write as $w_k = \begin{pmatrix} w_q \\ w_\omega \end{pmatrix}$ where $w_q \in \mathcal{R}^3$ affects orientation and $w_\omega \in \mathcal{R}^3$ affects angular velocity.
Suppose quaternion representation of w_q is q_w.
A disturbed quaternion is $\tilde{q}_k = q_k q_w$. Likewise, a disturbed angular velocity is $\vec{\tilde{\omega}}_k = \vec{\omega}_k + w_\omega$.

Finally, process model A is given by

$$x_{k+1} = \begin{pmatrix} \tilde{q}_k q_\Delta \\ \tilde{\vec{\omega}}_k \end{pmatrix} = \begin{pmatrix} q_k q_w q_\Delta \\ \vec{\omega}_k + w_\omega \end{pmatrix} = \begin{pmatrix} q_{k+1} \\ \vec{\omega}_{k+1} \end{pmatrix}$$

4 Measurement Model H

Generally, $z_k = H(x_k, v_k)$ where v_k is a random variable (measurement noise).
Define $\vec{g}$ be the vector of gravitational field (down), $\vec{b}$ be the vector of magnetic field (north).
Their quaternion vector can be written as $g = (0, [g_x, g_y, g_z]) = (0, \vec{g})$.
Transformation of $\vec{g}$ from **global** coordinate system to **tracker** coordinate system $\vec{g'}$ is $g' = q_k g q_k{}^{-1}$. Similar for b, $b' = q_k b q_k{}^{-1}$ where g, g', b, b' are quaternion vectors.

5 Unscented Kalman Filter (UKF)

5.1 Sigma Points

For a $n \times n$ matrix P_{k-1}, we can derive a set of $2n$ sigma points by Cholesky Decomposition $P_{k-1} = S^T S$. The n **column** vectors of S are multiplied by $\pm\sqrt{2n}$ form the set $\{W_i\}$(with 0 mean and P_{k-1} covariance). By shifting mean, we get

$$\chi_i = \hat{x}_{k-1} + W_i$$

In quaternion representation,

$$\chi_i = \begin{pmatrix} q_{k-1} q_W \\ \vec{\omega}_{k-1} + \vec{\omega}_W \end{pmatrix}$$

where q_W is the quaternion w.r.t the first three components of W_i. $\vec{\omega}_W$ is the angular velocity vector built from the remaining three components.

5.2 Transformation of Sigma Points

We transfer $\{\chi_i\}$ into a different set of state vectors $\{Y_i\}$: $Y_i = A(\chi_i, 0)$. Let

$$\overline{\hat{x}_k} = Mean(\{Y_i\})$$

$$\overline{P_k} = Covariance(\{Y_i\})$$

Further Transfer $\{Y_i\}$ into $\{Z_i\}$: $Z_i = H(Y_i, 0)$. Similarly, Let

$$\overline{z_k} = Mean(\{Z_i\})$$

$$P_{zz} = Covariance(\{Z_i\})$$

Define *innovation* $v_k = z_k - \overline{\hat{z}_k}$. Its covariance $P_{vv} = P_{zz} + R$ where R is the measurement noise covariance. Kalman gain $K_k = P_{xz} P_{vv}^{-1}$.

5.3 Computation of Mean

Unit quaternion are members of a homogenous Riemannian manifold (four dimensional unit sphere). Quaternion representation of error vector $\vec{e}_i$ is $e_i = q_i \overline{q}_t^{-1}$. The barycentric mean $\vec{e}$ of all error vectors is

$$\vec{e} = \frac{1}{2n} \sum_{i=1}^{2n} \vec{e}_i$$

Then we can better estimate $\overline{q}_{t+1}$ by

$$\overline{q}_{t+1} = e\overline{q}_t$$

Keep doing until

$$\left\|\vec{e}\right\|_2 < \varepsilon$$

5.4 Computation of Covariance

Measurement covariance matrix is given by

$$P_{zz} = \frac{1}{2n} \sum_{i=1}^{2n} [Z_i - \overline{z_k}][Z_i - \overline{z_k}]^T$$

Similarly, cross correlation matrix can be computed according to

$$P_{xz} = \frac{1}{2n} \sum_{i=1}^{2n} [W_i'][Z_i - \overline{z_k}]^T = \frac{1}{2n} \sum_{i=1}^{2n} [Y_i - \overline{\hat{x}_k}][Z_i - \overline{z_k}]^T$$