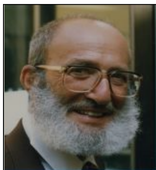


The only way to learn mathematics is to do mathematics.
— Paul Halmos —

SET THEORY

I hear and I forget.
I see and I remember.
I do and I understand.
— Confucious —

SET

Definition: A set is any well defined collection of distinct objects.

Notation: Usually sets are denoted by capital letters and the elements (or members) of the sets are denoted by small letters.

Examples:

$$A = \{a, e, i, o, u\}$$
$$B = \{2,4,6,8\}$$

SET

$\in$ *and* $\notin$ Notation:

- If A is a set, the notation $x \in A$ means that x is a member of set A.
- The notation $x \notin A$ means that x is not a member of set A.

Example: Let $A = \{2,4,6,8\}$,
then members of A are: $2, 4, 6$ *and* 8.
And we can say $2 \in A, 6 \in A$ etc.
But since 3 is not the member of set A, we can say $3 \notin A$.

SET ROSTER NOTATION

A set may be specified using the set-roster notation by writing all of its elements between braces.

Examples:

$$A = \{2,4,6,8\}$$
$$B = \{1,3,5, \dots \dots \dots, 99\}$$
$$C = \{1,2,3, \dots \dots \dots\}$$

BASIC SETS

$$\mathbb{Z} = \textit{the set of integers} = \{\dots \dots \dots, -3, -2, -1, 0, 1, 2, 3, \dots \dots \dots\}$$

$$\mathbb{N} = \textit{the set of natural numbers} = \{1,2,3, \dots \dots \dots\}$$
$$= \textit{the set positive integers} = \mathbb{Z}^+$$

$$\mathbb{W} = \textit{the set of whole numbers} = \{0,1,2,3, \dots \dots \dots\}$$

$$\mathbb{R} = \textit{the set of real numbers}$$

$$\mathbb{R}^+ = \textit{the set of positive real numbers}$$

SET BUILDER NOTATION

The fact that the element x of the set A having a property P is described as $P(x)$ and the set A in set builder notation is written as

$$A = \{x \mid P(x)\}$$

Examples:

$A = \{2,4,6,8\}$

$\therefore\ A = \{x \mid x \textit{ is an even integer and } 2 \le x \le 8\}$

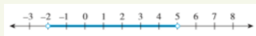
$B = \{1,3,5, \dots \dots \dots, 99\}$

$\therefore\ B = \{x \mid x \textit{ is an odd integer and } 1 \le x \le 99\}$

SET BUILDER NOTATION

Describe each of the following sets:

a) $\{x \in \mathbb{R} \mid -2 < x < 5\}$
b) $\{x \in \mathbb{Z} \mid -2 < x < 5\}$
c) $\{x \in \mathbb{Z}^+ \mid -2 < x < 5\}$
d) $\{x \in \mathbb{R}^+ \mid 0 < x < 1\}$
e) $\{x \in \mathbb{R} \mid 2 \leq x < 7\}$
f) $\{n \in \mathbb{Z} \mid n\ is\ a\ factor\ of\ 6\}$
g) $\{n \in \mathbb{Z}^+ \mid n\ is\ a\ factor\ of\ 6\}$
h) $\{n \in \mathbb{Z} \mid n = (-1)^k, for\ some\ integer\ k\}$
i) $\{m \in \mathbb{Z} \mid m = 1 + (-1)^i, for\ some\ integer\ i\}$
j) $\{r \in \mathbb{Z} \mid 2 \leq r \leq -2\}$
k) $\{s \in \mathbb{Z} \mid s \leq 4\ or\ s \geq 1\}$



EMPTY SET

A set without any element is called an empty set.
It is denoted by ϕ or $\{\ \}$.

Examples:

1. Let $X = \{x \mid x \in \mathbb{Z}\ and\ 2x = 5\}$.
Here $2x = 5 \Longrightarrow x = 2.5$ but 2.5 is not an integer.
$\therefore X = \{\ \}$

2. Let $A = \{x \mid x \in \mathbb{N}\ and\ x^2 + 9x + 18 = 0\}$
Here $x^2 + 9x + 18 = 0 \Longrightarrow x = -3, -6$ but $-3\ and\ -6$ are not the natural numbers.
$\therefore A = \phi$

Note: $\phi \subseteq A$, for any set A.

BASIC TERMS

Singleton Set: A set with only one element is called a singleton set.

Example:

$A = \{5\}$

$B = \{x \mid x \in \mathbb{N}\ and\ x^2 - 7x - 18 = 0\}$
$= \{9\}$

Doubleton Set: A set with only two elements is called a doubleton set.

Example:

$A = \{x \mid x^2 - 9x + 18 = 0\} = \{3,6\}$

BASIC TERMS

Finite Set: A set with n distinct elements is called a finite set.

Example:

$A = \{100, 2000, 30000, 400000\}$

$B = \{x \mid x \in \mathbb{N}\ and\ 16 \leq x < 100\}$
$= \{16, 17, 18, \dots\dots\dots, 99\}$

Infinite Set: A set which is not finite is called an infinite set.

Example:

$A = \mathbb{N}$
$B = \{x \mid x \in \mathbb{R}\ and\ 0 < x < 1\}$

SUBSET

If A and B are sets, then A is called a subset of B, written $A \subseteq B$, if, and only if, every element of A is also an element of B.

Examples:

1. Let $X = \{2,4\}$ and $Y = \{1,2,3,4\}$.
Then $X \subseteq Y$, since every element of X is also an element of Y.
2. Let $A = \{x \mid x^2 - 9x + 18 = 0\}$ and $B = \{0,3,6,9,12\}$.
A in set roster notation, $A = \{3,6\}$
Then $A \subseteq B$, since every element of A is also an element of B.

SUBSET

$A \nsubseteq B$

means that there is at least one element x such that $x \in A$ but $x \notin B$.

Examples:

1. Let $X = \{2,4,6\}$ and $Y = \{1,2,3,4\}$.
Here $X \nsubseteq Y$, since $6 \in X$ but $6 \notin Y$.
2. Let $A = \{x \mid x^2 - 3x - 18 = 0\}$ and $B = \{0,3,6,9,12\}$.
A in set roster notation, $A = \{-3,6\}$
Then $A \nsubseteq B$, since $-3 \in A$ but $-3 \notin B$.

PROPER SUBSET

Let A and B be sets. A is proper subset of B if, and only if, every element of A is in B but there is at least one element of B that is not in A.

A subset which is not proper is called improper subset.

Examples:

1. Let $X = \{2,4\}$ and $Y = \{1,2,3,4\}$.
Here X is proper subset of Y, i.e. $X \subsetneq Y$.
2. Let $A = \{x \mid x^2 - 9x + 18 = 0\}$ and $B = \{0,3,6,9,12\}$.
A in set roster notation, $A = \{3,6\}$.
Here A is proper subset of B, i.e. $A \subsetneq B$.
3. Let $C = \{x \mid x^2 - 9 = 0\}$ and $D = \{-3,3\}$.
C in set roster notation, $C = \{-3,3\}$.
Here C is improper subset of D.

EQUAL SETS

The two sets A and B are said to be equal if they have same elements and we write $A = B$.

Example:

Let $A = \{2,5,3\}$ and $B = \{2,3,2,5\}$.

Here we can rewrite B as $\{2,3,5\}$ i.e. $B = \{2,3,5\}$.

$\therefore A = B$

Note: Equality of two sets A and B can also be defined as,

$$A = B \; iff \; A \subseteq B \; and \; B \subseteq A$$

PROBLEMS

Q.1 How many elements are there in the set $A = \{1,1,2,3,3,3\}$?

Q.2 Is $\{0\} = 0$?

Q.3 How many elements are there in the set $\{1,\{1\}\}$? List them.

PROBLEMS

Q.4 How many elements are there in the set $\{1,\{1\},\{1,2\}\}$? List them.

Q.5 Let $A = \{1,2,3\}, B = \{3,1,2\}$ and $C = \{1,1,2,3,3,3\}$. How A, B and C are related?

Q.6 For each non-negative n, let $U_n = \{-n, n\}$. Find U_1, U_2 and U_0.

PROBLEMS

Q.7 Answer each of the following:

a) Is $3 \in \{1,2,3\}$?
b) Is $1 \subseteq \{1\}$?
c) Is $\{2\} \in \{1,2\}$?
d) Is $\{3\} \in \{1,\{2\},\{3\}\}$?
e) Is $1 \in \{1\}$?
f) Is $\{3\} \subseteq \{1,\{2\},\{3\}\}$?
g) Is $\{1\} \subseteq \{1,2\}$?
h) Is $1 \in \{\{1\},2\}$?
i) Is $\{1\} \subseteq \{1,\{2\}\}$?

PROBLEMS

Q.8 Let $A = \{c,d,f,g\}, B = \{f,j\}$ and $C = \{d,g\}$.

Answer each of the following with justification:

a) Is $B \subseteq A$?
b) Is $C \subseteq A$?
c) Is $C \subseteq C$?
d) Is C a proper subset of A?

UNIVERSAL SET

Any larger set which contains all other sets is called an universal set.

It is denoted by U.

Examples:

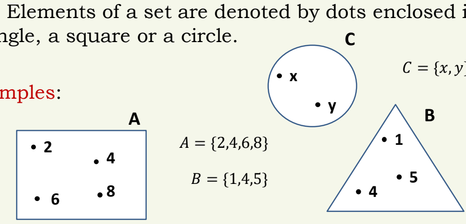
1. For $\mathbb{N}, \mathbb{W}, \mathbb{Z}$,
   $\mathbb{R}$ or $\mathbb{C}$ can be universal set.
2. For $A = \{1,3,5,7\}, B = \{2,4,6\}$ and $C = \{1,2,5,6,9,10\}$
   $U = \{1,2,3, \dots \dots \dots, 10\}$ can be universal set.

VENN DIAGRAM

A pictorial representation of a set is called as Venn diagram.

Elements of a set are denoted by dots enclosed in a triangle, a square or a circle.

Examples:

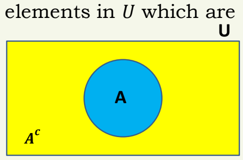


COMPLEMENT OF A SET

The complement of a set A has elements in U which are not in A.

It is denoted by A' or A^c or $\bar{A}$.

i.e. $A^c = \{x \mid x \in U \text{ and } x \notin A\}$



Example:

Let $U = \{1,2,3, \dots \dots \dots, 10\}$ and $A = \{1,3,5,7\}$.

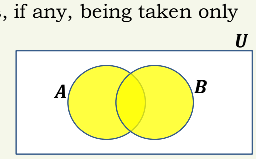
Then $A^c = \{2,4,6,8,9,10\}$.

UNION OF SETS

The union of two sets A and B is the set of all elements of A and B, the common elements, if any, being taken only once.

It is denoted by $A \cup B$.

i.e. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



Example:

Let $A = \{1,2,4,5\}$ and $B = \{3,4,5,8,9\}$.

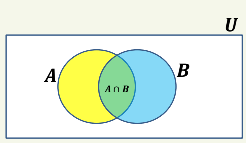
Then $A \cup B = \{1,2,3,4,5,8,9\}$.

INTERSECTION OF SETS

The intersection of two sets A and B is the set of all common elements of A and B.

It is denoted by $A \cap B$.

i.e. $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



Example:

Let $A = \{1,2,4,5\}$ and $B = \{3,4,5,8,9\}$.

Then $A \cap B = \{4,5\}$.

DISJOINT SETS

The two sets A and B are disjoint if they don't have any common element.

i.e. if $A \cap B = \phi$

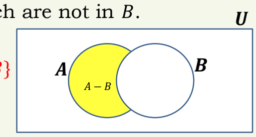
Example:

Let $A = \{0,2,4,6\}$ and $B = \{1,3,5\}$.

Then $A \cap B = \{\}$.

SET DIFFERENCE

The relative complement of A with respect to B is the set of all those elements of A which are not in B.



It is denoted by $A - B$ or A/B.

i.e. $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

$B - A = \{x \mid x \in B \text{ and } x \notin A\}$

Example:

Let $A = \{1,2,4,5\}$ and $B = \{3,4,5,8,9\}$.

Then $A - B = \{1,2\}$

and $B - A = \{3,8,9\}$.

PROBLEM

Ex.1 For $U = \{1,2,3, \ldots \ldots, 10\}$, let $A = \{1,2,3,4,5\}$, $B = \{1,2,4,8\}$, $C = \{1,2,3,5,7\}$ and $D = \{2,4,6,8\}$. Determine each of the following :

(i) $C' \cup D'$ (ii) $(C \cap D)'$ (iii) $A \cup (B/C)$

(iv) $(A \cup B)/(C \cap D)$

INCLUSION EXCLUSION PRINCIPLE

For two sets:

Let A and B be any two finite sets, then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

For three sets:

Let A, B and C be any three finite sets, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

INCLUSION EXCLUSION PRINCIPLE

Q.1 In a class of 50 students, 25 like Maths and 15 like Programming, 10 like Maths and Programming. Find

a) How many like at least one of them?

b) How many do not any of the subjects?

INCLUSION EXCLUSION PRINCIPLE

Q.2 Out of 150 residents of a building, 105 speak Marathi, 75 speak Gujarati and 45 speak both the languages. Find the number of residents who do not speak either of the languages also find the number of residents who speak only Marathi.

INCLUSION EXCLUSION PRINCIPLE

Q.3 In a survey of people it was found that 80 people watch football, 60 watch cricket, 50 watch hockey, 30 watch football and cricket, 20 watch football and hockey, 15 watch cricket and hockey and 10 watch all three games.

a) How many people watch at least one game?

b) How many people watch only cricket?

c) How many people watch football and cricket but not hockey?

INCLUSION EXCLUSION PRINCIPLE

Q.4 In a survey of 120 people, it was found that:
65 read Newsweek magazine, 45 read Time, 42 read Fortune,
20 read both Newsweek and Time, 25 read both Newsweek and Fortune,
15 read both Time and Fortune, 8 read all the three magazines.
(a) Find the number of people who read at least one of the three magazines.
(b) Find the number of people who read exactly one magazine.

PROPERTIES OF SETS

Theorem 6.2.1 Some Subset Relations

1. *Inclusion of Intersection:* For all sets A and B,
$$\text{(a) } A \cap B \subseteq A \quad \text{and} \quad \text{(b) } A \cap B \subseteq B.$$
2. *Inclusion in Union:* For all sets A and B,
$$\text{(a) } A \subseteq A \cup B \quad \text{and} \quad \text{(b) } B \subseteq A \cup B.$$
3. *Transitive Property of Subsets:* For all sets A, B, and C,
$$\text{if } A \subseteq B \text{ and } B \subseteq C, \text{ then } A \subseteq C.$$

PROPERTIES OF SETS

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,
$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$
2. *Associative Laws:* For all sets A, B, and C,
$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$
3. *Distributive Laws:* For all sets, A, B, and C,
$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$
4. *Identity Laws:* For all sets A,
$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

PROPERTIES OF SETS

5. *Complement Laws:*
$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$
6. *Double Complement Law:* For all sets A,
$$(A^c)^c = A.$$
7. *Idempotent Laws:* For all sets A,
$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$
8. *Universal Bound Laws:* For all sets A,
$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$
9. *De Morgan's Laws:* For all sets A and B,
$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

PROPERTIES OF SETS

10. *Absorption Laws:* For all sets A and B,
$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$
11. *Complements of U and $\emptyset$:*
$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$
12. *Set Difference Law:* For all sets A and B,
$$A - B = A \cap B^c.$$

PROPERTIES OF SETS

Q.1 Let $U = \{x \mid x \in \mathbb{N} \; and \; x \le 15\}, A = \{1,3,4,5,9\}$ and $B = \{3,5,7,9,12\}$.
State and verify De Morgan's laws.

Solution: Here $U = \{1,2,3, \dots \dots \dots, 15\}, A = \{1,3,4,5,9\}$ and $B = \{3,5,7,9,12\}$.

De Morgan's laws: i) $(A \cup B)^c = A^c \cap B^c$

ii) $(A \cap B)^c = A^c \cup B^c$

Verification of $(A \cup B)^c = A^c \cap B^c$:

$A \cup B = \{1,3,4,5,7,9,12\}$

$\therefore (A \cup B)^c = \{2,6,8,10,11,13,14,15\}$ … … … (I)

$A^c = \{2,6,7,8,10,11,12,13,14,15\}$ and $B^c = \{1,2,4,6,8,10,11,13,14,15\}$

$\therefore A^c \cap B^c = \{2,6,8,10,11,13,14,15\}$ … … … (II)

By (I) and (II), $(A \cup B)^c = A^c \cap B^c$

Hence verified

PROPERTIES OF SETS

Solution Cont: Here $U = \{1,2,3, \dots \dots \dots, 15\}$, $A = \{1,3,4,5,9\}$ and $B = \{3,5,7,9,12\}$.

Verification of $(A \cap B)^c = A^c \cup B^c$:

$A \cap B = \{3,5,9\}$

$\therefore (A \cap B)^c = \{1,2,4,6,7,8,10,11,12,13,14,15\}$ … … … (III)

$A^c = \{2,6,7,8,10,11,12,13,14,15\}$ and $B^c = \{1,2,4,6,8,10,11,13,14,15\}$

$\therefore A^c \cup B^c = \{1,2,4,6,7,8,10,11,12,13,14,15\}$ … … … (IV)

By (III) and (IV), $(A \cap B)^c = A^c \cup B^c$

Hence verified

PROPERTIES OF SETS

Q.2 Let $A = \{1,3,4,5,9\}$, $B = \{3,5,7,9,12\}$ and $C = \{2,3,5,6\}$.

State and verify Distributive laws.