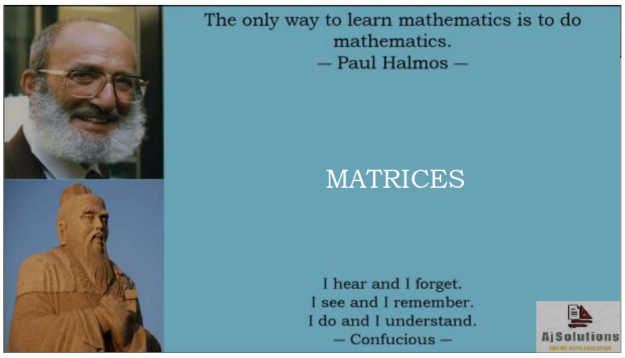


DEFINITION

An ordered set of mn elements arranged in a rectangular array of m rows and n columns and enclosed by a pair of brackets [ ] or ( ) is called a matrix of order $m \times n$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n} = [a_{ij}]_{m\times n}$$

TYPES OF MATRICES

- Row Matrix or Row Vector
- Column Matrix or Column Vector
- Square Matrix
- Identity or Unit Matrix
- Null or Zero Matrix
- Diagonal Matrix
- Scalar Matrix
- Upper Triangular Matrix
- Lower Triangular Matrix

ROW MATRIX OR ROW VECTOR

A matrix having only one row and n number of columns is called a row matrix or a row vector. The order of a row matrix is $1 \times n$.

Examples:

$$\begin{bmatrix} 2 & 7 \end{bmatrix}_{1\times 2}$$

$$\begin{bmatrix} 5 & -2 & 0 \end{bmatrix}_{1\times 3}$$

COLUMN MATRIX OR COLUMN VECTOR

A matrix having only one column and n number of rows is called a column matrix or a column vector. The order of a column matrix is $n \times 1$.

Examples:

$$\begin{bmatrix} -1 \\ 4 \end{bmatrix}_{2\times 1} \qquad \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}_{3\times 1}$$

SQUARE MATRIX

A matrix having same number of rows and column is called a square matrix. The order of a square matrix is $n \times n$. We also call it as square matrix of order n.

Examples:

$$\begin{bmatrix} -1 & 0 \\ 4 & 7 \end{bmatrix}_{2\times 2} \qquad \begin{bmatrix} 1 & -2 & 3 \\ 4 & 0 & 5 \\ 2 & 6 & 7 \end{bmatrix}_{3\times 3}$$

IDENTITY MATRIX OR UNIT MATRIX

A square matrix in which diagonal elements are 1 and non-diagonal elements are zero is called a identity matrix or unit matrix. Identity matrix of order n is denoted by I_n.

Examples:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

NULL MATRIX OR ZERO MATRIX

A matrix with all elements zero is called a null matrix or zero matrix. The null matrix is denoted by $0_{m\times n}$.

Examples:

$$0_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad 0_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

DIAGONAL MATRIX

A square matrix in which all non-diagonal elements are zero is called a diagonal matrix.

Examples:

$$\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} = diag[2 \quad 6]$$

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix} = diag[-3 \quad 7 \quad 0]$$

SCALAR MATRIX

A square matrix in which all non-diagonal elements are zero and diagonal elements are same is called a scalar matrix. It is denoted by kI_n since scalar matrices are scalar multiple of identity matrix.

Examples:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I_2$$

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = 7I_3$$

UPPER TRIANGULAR MATRIX

A square matrix in which all elements below diagonal are zero is called an upper triangular matrix.

Examples:

$$\begin{bmatrix} 1 & 4 \\ 0 & -3 \end{bmatrix} \quad \begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 7 \end{bmatrix}$$

LOWER TRIANGULAR MATRIX

A square matrix in which all elements above diagonal are zero is called an lower triangular matrix.

Examples:

$$\begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & 0 \\ 3 & 4 & 7 \end{bmatrix}$$

EXERCISE

1) What is order of a row matrix?
2) What is order of a column matrix?
3) What is order of a square matrix?
4) Give example of each type of matrix discussed in the session.
5) Give example of a non-square matrix.
6) In a matrix of order 5×7, number of rows = ____, number of columns = __ and number of elements = ___.
7) What are the possible orders of a matrix with 12 elements?
8) Is it possible to have a square matrix with 18 elements? Justify.

EXERCISE

9) In A = $\begin{bmatrix} -2 & 7 & 4 & 6 \\ 0 & 2 & 1 & 5 \\ -1 & 3 & 9 & 8 \end{bmatrix}$

a) a_{12} = ____
b) a_{31} = ____
c) a_{23} = ____
d) a_{14} = ____
e) a_{31} = ____
f) a_{24} = ____
g) a_{13} = ____

EXERCISE

10)Let $A = [1]$. Answer each of the following with justification.

a) Is A a row matrix?
b) Is A a column matrix?
c) Is A a square matrix?
d) Is A an identity matrix?
e) Is A a null matrix?
f) Is A a diagonal matrix?
g) Is A a scalar matrix?
h) Is A an upper triangular matrix?
i) Is A a lower triangular matrix?

EQUALITY OF MATRICES

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if
(i) they have same order (ii) $a_{ij} = b_{ij}$ for all i and j.

Example:

- $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 2 \end{bmatrix}$ $and\ B = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 2 \end{bmatrix}$
  $Here\ A = B.$
- $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 7 & 0 \end{bmatrix}$ $and\ B = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$
  $Here\ A \neq B,$ since order is not same.
- $A = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}$ $and\ B = \begin{bmatrix} 4 & 1 \\ 3 & 0 \end{bmatrix}$
  $Here\ A \neq B,$ since $a_{22} \neq b_{22}$.

PROBLEM ON EQUALITY OF MATRICES

Find the values of $x, y\ and\ z$ if

$$\begin{bmatrix} x+3 & -1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 6 & y \\ z-3 & 5 \end{bmatrix}$$

Solution:

Since the given two matrices are same,

$$x + 3 = 6$$
$$\Rightarrow x = 3$$
$$y = -1$$
$$z - 3 = 4$$
$$\Rightarrow z = 7$$

PROBLEM ON EQUALITY OF MATRICES

Find the values of a, $b, c\ and\ d$ if

$$\begin{bmatrix} 2a & 5a+b \\ 3d & 5d+c \end{bmatrix} = \begin{bmatrix} 10 & 17 \\ 9 & 15 \end{bmatrix}$$

Solution:

Since the given two matrices are same,

$$2a = 10$$
$$\Rightarrow a = 5$$
$$5a + b = 17$$
$$\Rightarrow b = -8$$
$$3d = 9$$
$$\Rightarrow d = 3$$
$$5d + c = 15$$
$$\Rightarrow c = 0$$

EXERCISE

- Find the values of a, b *and* c if
$$\begin{bmatrix} a+b+c \\ a+c \\ b+c \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$$
- Find the values of p, q *and* r if
$$\begin{bmatrix} p+q & 2 \\ r+5 & pq \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$$
- Find the values of w, x, y *and* z if
$$\begin{bmatrix} w-x & 2w+y \\ 2w-y & 3y+z \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

ADDITION OF MATRICES

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of same order, say $m \times n$ then the sum of the two matrices A and B is obtained by adding corresponding element of both the matrices

i.e. $A + B = [a_{ij} + b_{ij}]_{m\times n}$.

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} & \cdots & a_{2n}+b_{2n} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} & \cdots & a_{3n}+b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & a_{m3}+b_{m3} & \cdots & a_{mn}+b_{mn} \end{bmatrix}_{m\times n}$$

ADDITION OF MATRICES

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and}$$
$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \text{ then}$$
$$A + B = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

ADDITION OF MATRICES

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 5 & 4 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 6 & 4 \\ 3 & 2 & 0 \end{bmatrix}$$
$$A + B = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$
$$= \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

ADDITION OF MATRICES

$$A = \begin{bmatrix} 5 & 7 & 0 \\ -2 & 1 & -3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -4 & 6 \\ 3 & 2 & 7 \end{bmatrix}$$
$$A + B = \begin{bmatrix} 5+1 & 7+(-4) & 0+6 \\ (-2)+3 & 1+2 & (-3)+7 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & 3 & 6 \\ 1 & 3 & 4 \end{bmatrix}$$

MULTIPLICATION OF A MATRIX BY A SCALAR

If $A = [a_{ij}]_{m\times n}$ and k is a scalar then the scalar multiplication of k by A is obtained by multiplying each element of A by k

i.e. $kA = [ka_{ij}]_{m\times n}$.

$$= \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & ka_{23} & \cdots & ka_{2n} \\ ka_{31} & ka_{32} & ka_{33} & \cdots & ka_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & ka_{m3} & \cdots & ka_{mn} \end{bmatrix}_{m\times n}$$

MULTIPLICATION OF A MATRIX BY A SCALAR

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and}$$

k be any scalar then

$$kA = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

MULTIPLICATION OF A MATRIX BY A SCALAR

$$\text{Let } A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 7 & -6 \end{bmatrix} \text{ then}$$

$$3A = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

$$= \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

MULTIPLICATION OF A MATRIX BY A SCALAR

$$\text{Let } A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 7 & -6 \end{bmatrix} \text{ then}$$

$$-2A = \begin{bmatrix} (-2)\times 2 & (-2)\times(-1) & (-2)\times 0 \\ (-2)\times 4 & (-2)\times 7 & (-2)\times(-6) \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 2 & 0 \\ -8 & -14 & 12 \end{bmatrix}$$

MULTIPLICATION OF A MATRIX BY A SCALAR

$$\text{Let } A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 7 & -6 \end{bmatrix} \text{ then}$$

$$(-1)A = \begin{bmatrix} (-1)\times 2 & (-1)\times(-1) & (-1)\times 0 \\ (-1)\times 4 & (-1)\times 7 & (-1)\times(-6) \end{bmatrix}$$

$$-A = \begin{bmatrix} -2 & 1 & 0 \\ -4 & -7 & 6 \end{bmatrix}$$

$-A$ is called as negative of matrix A.

SUBTRACTION OF MATRICES

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of same order, say $m \times n$ then the subtraction of the two matrices A and B is defined as

$A - B = A + (-B)$.

$A - B = [a_{ij} - b_{ij}]_{m\times n}$.

$$= \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} & \cdots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} & \cdots & a_{2n}-b_{2n} \\ a_{31}-b_{31} & a_{32}-b_{32} & a_{33}-b_{33} & \cdots & a_{3n}-b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & a_{m3}-b_{m3} & \cdots & a_{mn}-b_{mn} \end{bmatrix}_{m\times n}$$

SUBTRACTION OF MATRICES

$$A = \begin{bmatrix} 5 & 7 & 0 \\ -2 & 1 & -3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -4 & 6 \\ 3 & 2 & 7 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 5-1 & 7-(-4) & 0-6 \\ (-2)-3 & 1-2 & (-3)-7 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 11 & -6 \\ -5 & -1 & -10 \end{bmatrix}$$

PROPERTIES

- *Matrix Addition:*
- All matrices are of order $m \times n$.

1. Commutative Law: $A + B = B + A$.
2. Associative Law: $(A + B) + C = A + (B + C)$.
3. Existence of 0: $A + 0 = 0 + A = A$.
4. Existence of Additive Inverse:

$$A + (-A) = (-A) + A = 0.$$

- *Scalar Multiplication:*

1. $k(A + B) = kA + kB$.
2. $(k + l)A = kA + lA$.

PROBLEMS

- Find the matrix X such that $3A + 2X = 4B$ where

$$A = \begin{bmatrix} 3 & 0 \\ 4 & 1 \\ -2 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 4 \\ 6 & 1 \\ 2 & 7 \end{bmatrix}.$$

Some initial steps:

$$3A + 2X = 4B$$

Adding $-3A$ on both the sides, we get

$$-3A + (3A + 2X) = -3A + 4B$$

$$(-3A + 3A) + 2X = 4B - 3A$$

$$0 + 2X = 4B - 3A$$

$$2X = 4B - 3A$$

PROBLEMS

Solution:

Given that $3A + 2X = 4B \quad \Rightarrow 2X = 4B - 3A$

$$\therefore 2X = 4\begin{bmatrix} -1 & 4 \\ 6 & 1 \\ 2 & 7 \end{bmatrix} - 3\begin{bmatrix} 3 & 0 \\ 4 & 1 \\ -2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 16 \\ 24 & 4 \\ 8 & 28 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 12 & 3 \\ -6 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} -13 & 16 \\ 12 & 1 \\ 14 & 10 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} -13/2 & 8 \\ 6 & 1/2 \\ 7 & 5 \end{bmatrix}$$

PROBLEMS

Find X and Y, if $X + Y = \begin{bmatrix} 7 & 2 \\ 0 & 3 \end{bmatrix}$ and $X - Y = \begin{bmatrix} -3 & -2 \\ 0 & 5 \end{bmatrix}$.

Solution: Given that

$X + Y = \begin{bmatrix} 7 & 2 \\ 0 & 3 \end{bmatrix}$ ..... (I)

$X - Y = \begin{bmatrix} -3 & -2 \\ 0 & 5 \end{bmatrix}$ ..... (II)

(I) + (II) gives, $2X = \begin{bmatrix} 7 & 2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ 0 & 5 \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}$

$\Rightarrow X = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

(I) - (II) gives, $2Y = \begin{bmatrix} 7 & 2 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} -3 & -2 \\ 0 & 5 \end{bmatrix}$

$= \begin{bmatrix} 10 & 4 \\ 0 & -2 \end{bmatrix}$

$\Rightarrow Y = \begin{bmatrix} 5 & 2 \\ 0 & -1 \end{bmatrix}$

PROBLEMS

Find x and y, if $3\begin{bmatrix} x & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ -2 & y+2 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 7 & 4 \end{bmatrix}$.

Solution: Given that

$$3\begin{bmatrix} x & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ -2 & y+2 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 7 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3x & 6 \\ 9 & 3 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ -2 & y+2 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 7 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3x+4 & 5 \\ 7 & y+5 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 7 & 4 \end{bmatrix}$$

$$\Rightarrow 3x + 4 = 10 \text{ and } y + 5 = 4$$

$$\Rightarrow x = 2 \text{ and } y = -1$$

EXERCISE

- Find the matrix X such that $5A + 2X = 3B$ where

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 6 & -3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -3 & 2 \\ 0 & 1 & 6 \end{bmatrix}.$$

- Find X and Y, if

$$2X + Y = \begin{bmatrix} 8 & 3 \\ 1 & 10 \end{bmatrix} \text{ and } X - Y = \begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}.$$

- Find x and y, if $2\begin{bmatrix} x & 3 \\ 5 & y-2 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 9 \\ 11 & 4 \end{bmatrix}$.

MULTIPLICATION OF MATRICES

The product AB of matrices A and B is defined if number of columns of A is equal to the number of rows of B.

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$. Then the product AB of matrices A and B is a matrix of order $m \times p$ and is defined as

$$AB = \left[\sum_{j=1}^{n} a_{ij}b_{jk}\right]_{m\times p}$$

MULTIPLICATION OF MATRICES

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2k} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & b_{3k} & \cdots & b_{3p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nk} & \cdots & b_{np} \end{bmatrix}_{n\times p}$$

$$AB = \left[\sum_{j=1}^{n} a_{ij}b_{jk}\right]_{m\times p}$$

MULTIPLICATION OF MATRICES

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & 10 \\ 11 & 5\times 3 + 3\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 10 \\ 11 & 27 \end{bmatrix}$$

MULTIPLICATION OF MATRICES

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 2 & 4 \\ 8 & 1 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} -10 & 3 & 5 \\ 44 & 14 & 34 \end{bmatrix}$$

POWERS OF MATRIX

- $A^2 = A \times A$
- $A^3 = A^2 \times A$
- In general,

$$A^n = A^{n-1} \times A, \qquad \text{where } n \geq 2.$$

POWERS OF MATRIX

Example:

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$$

$$\therefore A^2 = A \times A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$$

$$\therefore A^2 = \begin{bmatrix} 2(2) + 1(-3) & 2(1) + 1(4) \\ (-3)2 + 4(-3) & (-3)1 + 4(4) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 \\ -18 & 13 \end{bmatrix}$$

POWERS OF MATRIX

Example:

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} \text{ and } A^2 = \begin{bmatrix} 1 & 6 \\ -18 & 13 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 1 & 6 \\ -18 & 13 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$$

$$\therefore A^3 = \begin{bmatrix} 1(2) + 6(-3) & 1(1) + 6(4) \\ (-18)2 + 13(-3) & (-18)1 + 13(4) \end{bmatrix}$$

$$= \begin{bmatrix} -16 & 25 \\ -75 & 34 \end{bmatrix}$$

PROPERTIES

- All matrices taken are conformable with respect to the operations.

1. $AB \neq BA$ i.e. matrix multiplication is not commutative.

Example:

$$\text{Let } A = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 4 \\ 1 & 7 \end{bmatrix}$$

$$\text{Then } AB = \begin{bmatrix} 1 & 17 \\ 5 & 15 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 6 & 10 \\ 13 & 10 \end{bmatrix}$$

$$\therefore AB \neq BA$$

PROPERTIES

Example:

$$\text{Let } A = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\text{Then } AB = \begin{bmatrix} -2 & 6 \\ 4 & 2 \end{bmatrix} \text{ and } BA = \begin{bmatrix} -2 & 6 \\ 4 & 2 \end{bmatrix}$$

$$\therefore AB = BA$$

PROPERTIES

2. $(AB)C = A(BC)$ i.e. matrix multiplication is associative.

3. Existence of multiplicative identity I: $AI = IA = A$.

4. Distributive law:

i. $A(B + C) = AB + AC$.

ii. $(A + B)C = AC + BC$.

PROPERTIES

5. $AB = 0 \nRightarrow A = 0 \text{ or } B = 0$.

Example:

$$\text{Let } A = \begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\text{Then } AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So here we have $AB = 0$ but neither $A = 0$ nor $B = 0$.

So, $AB = 0 \nRightarrow A = 0 \text{ or } B = 0$.

PROPERTIES

6. $AB = 0 \nRightarrow BA = 0$.

Example:

$$\text{Let } A = \begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\text{Then } AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 0 & 17 \\ 0 & 0 \end{bmatrix}.$$

So here we have $AB = 0 \text{ but } BA \neq 0$.

So, $AB = 0 \nRightarrow BA = 0$.

$(A+B)^2 = ?$

$$(A+B)^2 = (A+B)(A+B)$$

$$= A(A+B) + B(A+B)$$

$$= A^2 + AB + BA + B^2$$

$$(A+B)^2 = A^2 + AB + BA + B^2$$

EXERCISE

- Note: Examples to be given other than discussed in the session.
- If $A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 1 & 2 \\ 4 & -2 & -1 \end{bmatrix}$ then compute A^2 *and* A^4.
- If $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ then compute AB *and* BA. Is $AB = BA$?
- Give example of A *and* B such that $AB = BA$.
- If $A = \begin{bmatrix} -1 & 3 \\ 2 & 6 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix}$ *and* $C = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$ then verify

i. $(AB)C = A(BC)$

ii. $(A+B)C = AC + BC$

EXERCISE

- Verify the existence of multiplicative identity property for $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 2 \\ 6 & -3 & 8 \end{bmatrix}$.
- Give example of A and B such that

i. $A \neq 0, B \neq 0$ *but* $AB = 0$

ii. $AB = 0$ *but* $BA \neq 0$

- Expand $(A+B)^3$.

TRANSPOSE OF A MATRIX

The matrix obtained by interchanging rows and columns of a matrix is called a transpose of a matrix.

Let $A = [a_{ij}]_{m\times n}$, then its transpose is denoted by A' *or* A^T, and $A' = [a_{ji}]_{n\times m}$.

Example:

- $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 2 \end{bmatrix}$

$A' = \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ 3 & 2 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 7 & 5 \end{bmatrix}$$

$$A' = \begin{bmatrix} 2 & 4 \\ 1 & 7 \\ 3 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & -1 & 3 \\ 0 & 2 & 4 \\ 6 & -3 & 7 \end{bmatrix}$$

$$A' = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

PROPERTIES

1) $(A')' = A$
2) $(A+B)' = A' + B'$
3) $(kA)' = kA'$
4) $(AB)' = B'A'$

SYMMETRIC MATRIX

A square matrix A is said to be symmetric if $A = A'$.

Example:

$$A = \begin{bmatrix} 5 & -1 & 3 \\ -1 & 2 & 4 \\ 3 & 4 & 7 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & -1 & 3 \\ -1 & 2 & 4 \\ 3 & 4 & 7 \end{bmatrix}$$

$$A' = \begin{bmatrix} 5 & -1 & 3 \\ -1 & 2 & 4 \\ 3 & 4 & 7 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & -1 & 3 \\ -1 & 2 & 4 \\ 3 & 4 & 7 \end{bmatrix}$$

$$A' = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

CONSTRUCTION OF EXAMPLE

- Recall
- In a square matrix $A = [a_{ij}]$, diagonal elements are of the form a_{ii} and non-diagonal elements are of the form a_{ij} where $i \neq j$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- If $A = [a_{ij}]$ then $A' = [a_{ji}]$.
- For a symmetric matrix,

$$A = A'$$

i.e. $[a_{ij}] = [a_{ji}]$

$$\Rightarrow a_{ij} = a_{ji} \ \forall i, j$$

On diagonal, $a_{ii} = a_{ii}$

On non-diagonal, $a_{ij} = a_{ji}$ where $i \neq j$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 0 & 5 \\ 6 & 5 & -1 \end{bmatrix}$$

SKEW–SYMMETRIC MATRIX

A square matrix A is said to be skew-symmetric if $A = -A'$.

Example:

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 6 \\ -3 & 6 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

CONSTRUCTION OF EXAMPLE

- Recall
- In a square matrix $A = [a_{ij}]$, diagonal elements are of the form a_{ii} and non-diagonal elements are of the form a_{ij} where $i \neq j$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- If $A = [a_{ij}]$ then $A' = [a_{ji}]$.
- For a skew-symmetric matrix,

$$A = -A'$$

i.e. $[a_{ij}] = [-a_{ji}]$

$$\Rightarrow a_{ij} = -a_{ji} \ \forall i, j$$

On diagonal, $a_{ii} = -a_{ii}$

$$\Rightarrow a_{ii} = 0$$

On non-diagonal, $a_{ij} = -a_{ji}$ where $i \neq j$.

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & -2 & 6 \\ 2 & 0 & -5 \\ -6 & 5 & 0 \end{bmatrix}$$

EXERCISE

- Give example of symmetric and skew-symmetric matrix of order

1) 3×3
2) 4×4

PROBLEMS

Find a, b, c and d such that X is symmetric where

$$X = \begin{bmatrix} 1 & a & 3 \\ 4 & b & c \\ d & -1 & 5 \end{bmatrix}.$$

Solution:

Since the given matrix X is symmetric, $x_{ij} = x_{ji} \ \forall \ i, j$

$$a = x_{12} = x_{21} = 4$$

$$c = x_{23} = x_{32} = -1$$

$$d = x_{31} = x_{13} = 3$$

b is on the diagonal and we can have any real number on the diagonal of a symmetric matrix.

Therfore $b =$ *any real number.*

PROBLEMS

Find w, x, y and z such that A is skew-symmetric where

$$A = \begin{bmatrix} w & 2 & x \\ y & 0 & -1 \\ 4 & z & 0 \end{bmatrix}.$$

Solution:

Since the given matrix A is skew − symmetric, $a_{ij} = -a_{ji} \ \forall \ i, j$

w is on the diagonal and we can have only 0 *on the diagonal of a skew − symmetric matrix.*

Therfore $w = 0$.

$$x = a_{13} = -a_{31} = -4$$

$$y = a_{21} = -a_{12} = -2$$

$$z = a_{32} = -a_{23} = -(-1)$$

IMPORTANT RESULTS

Result 1: Sum of two symmetric matrices is symmetric.

(i.e.) If A *and* B are symmetric matrices then $A + B$ is symmetric.

Proof:

Let A *and* B be any two symmetric matrices of same order.

To prove: $A + B$ is symmetric.

We know that, matrix X is symmetric if $X' = X$.

(i.e.) to prove, $(A + B)' = A + B$.

Consider, $(A + B)' = A' + B'$

Given that A *and* B are symmetric, therefore $A' = A$ *and* $B' = B$.

$$\therefore (A + B)' = A + B$$

Hence proved.

IMPORTANT RESULTS

Result 2: Scalar multiple of a symmetric matrix is symmetric.

(i.e.) If A is symmetric matrix then kA is symmetric, where k is any scalar.

Proof:

Let A be any symmetric matrix and k be any scalar.

To prove: kA is symmetric.

We know that, matrix X is symmetric if $X' = X$.

(i.e.) to prove, $(kA)' = kA$.

Consider, $(kA)' = kA'$

Given that A is symmetric, therefore $A' = A$.

$$\therefore (kA)' = kA$$

Hence proved.

IMPORTANT RESULTS

Result 3: Sum of two skew-symmetric matrices is skew-symmetric.

(i.e.) If A *and* B are skew-symmetric matrices then $A + B$ is skew-symmetric.

Proof:

Let A *and* B be any two skew-symmetric matrices of same order.

To prove: $A + B$ is skew-symmetric.

We know that, matrix X is skew-symmetric if $X' = -X$.

(i.e.) to prove, $(A + B)' = -(A + B)$.

Consider, $(A + B)' = A' + B'$

Given that A *and* B are skew-symmetric, therefore $A' = -A$ *and* $B' = -B$.

$$\therefore (A + B)' = (-A) + (-B) = -(A + B)$$

Hence proved.

IMPORTANT RESULTS

Result 4: Scalar multiple of a skew-symmetric matrix is skew-symmetric.

(i.e.) If A is skew-symmetric matrix then kA is skew-symmetric, where k is any scalar.

Proof:

Let A be any skew-symmetric matrix and k be any scalar.

To prove: kA is skew-symmetric.

We know that, matrix X is skew-symmetric if $X' = -X$.

(i.e.) to prove, $(kA)' = -(kA)$.

Consider, $(kA)' = kA'$

Given that A is skew-symmetric, therefore $A' = -A$.

$$\therefore (kA)' = k(-A) = -(kA)$$

Hence proved.

IMPORTANT RESULTS

Result 5: For any square matrix A

i. $A + A'$ is symmetric and

ii. $A - A'$ is skew-symmetric.

Proof :

Let A be any square matrix.

i. To prove: $A + A'$ is symmetric.

We know that, matrix X is symmetric if $X' = X$.

(i.e.) to prove, $(A + A')' = A + A'$.

Consider, $(A + A')' = A' + (A')' = A' + A = A + A'$

Hence $A + A'$ is symmetric.

IMPORTANT RESULTS

Result 5: For any square matrix A

i. $A + A'$ is symmetric and

ii. $A - A'$ is skew-symmetric.

Proof :

Let A be any square matrix.

ii. To prove: $A - A'$ is skew-symmetric.

We know that, matrix X is skew-symmetric if $X' = -X$.

(i.e.) to prove, $(A - A')' = -(A - A')$.

Consider, $(A - A')' = A' - (A')'$

$$= A' - A$$

$$= -(A - A')$$

Hence $A - A'$ is skew-symmetric.

EXERCISE

- Find a, b, c *and* d such that X is symmetric where

$$X = \begin{bmatrix} a & -2 & b \\ c & 0 & 4 \\ 3 & d & 0 \end{bmatrix}.$$

- Find w, x, y *and* z such that A is skew-symmetric where

$$A = \begin{bmatrix} w & x & 4 \\ 2 & 0 & y \\ z & -3 & 0 \end{bmatrix}.$$

- Prove or disprove:

i. AB is symmetric for any symmetric matrices A and B.

ii. AB is skew-symmetric for any skew-symmetric matrices A and B.

iii. A' is symmetric for any symmetric matrix A.

iv. A' is skew-symmetric for any skew-symmetric matrix A.

EXERCISE

- Prove the following:

i. If A is symmetric then A^2 *and* A^3 is symmetric.

ii. If A is skew-symmetric then A^2 is symmetric *and* A^3 is skew-symmetric.

iii. For any square matrix A, AA' *and* $A'A$ are symmetric.

IMPORTANT RESULTS

Result 1: If A *and* B are symmetric matrices then $A + B$ is symmetric.

Result 2: If A is symmetric matrix then kA is symmetric, where k is any scalar.

Result 3: If A *and* B are skew-symmetric matrices then $A + B$ is skew-symmetric.

Result 4: If A is skew-symmetric matrix then kA is skew-symmetric, where k is any scalar.

Result 5: For any square matrix A

i. $A + A'$ is symmetric and

ii. $A - A'$ is skew-symmetric.

IMPORTANT RESULTS

Theorem: Any square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix.

Proof:

Let A be any square matrix such that

$A = P + Q$, where P is symmetric and Q is skew-symmetric.

Since P is symmetric, $P' = P$ and

since Q is skew-symmetric, $Q' = -Q$.

Consider, $A' = (P + Q)' = P' + Q' = P - Q$

$A = P + Q$ ..... (1)

$A' = P - Q$ ..... (2)

(1) + (2) gives, $A + A' = 2P \quad \Rightarrow P = \frac{1}{2}[A + A']$

(1) – (2) gives, $A - A' = 2Q \quad \Rightarrow Q = \frac{1}{2}[A - A']$

$\therefore A = \left\{\frac{1}{2}[A + A']\right\} + \left\{\frac{1}{2}[A - A']\right\}$

where $\frac{1}{2}[A + A']$ *is symmetric*

and $\frac{1}{2}[A - A']$ *is skew – symmetric*

IMPORTANT RESULTS

Theorem: Any square matrix A can be uniquely expressed as the sum of a symmetric and skew-symmetric matrix.

Proof of Uniqueness:

Let $A = X + Y$, where X is symmetric and Y is skew-symmetric.

Since X is symmetric, $X' = X$ and

since Y is skew-symmetric, $Y' = -Y$.

Consider, $A' = (X + Y)' = X' + Y' = X - Y$

$A = X + Y$ ..... (1)

$A' = X - Y$ ..... (2)

(1) + (2) gives, $A + A' = 2X \quad \Rightarrow X = \frac{1}{2}[A + A'] = P$

(1) – (2) gives, $A - A' = 2Y \quad \Rightarrow Y = \frac{1}{2}[A - A'] = Q$

Hence the expression is unique.

STEPS TO EXPRESS

Any square matrix as the sum of a symmetric and skew–symmetric matrix

Step 1: Consider the given matrix A and compute it's transpose A'

Step 2: Compute $A + A'$ and $A - A'$

Step 3: Perform scalar multiplication by $\frac{1}{2}$ with both the matrices (which you got in step 2) to get

$$\frac{1}{2}[A + A'] \text{ and } \frac{1}{2}[A - A']$$

Step 4: Now write A as sum of the two matrices (which you got in step 3 i.e. $A = \left\{\frac{1}{2}[A + A']\right\} + \left\{\frac{1}{2}[A - A']\right\}$

which is the required expression

PROBLEMS

Express the matrix $A = \begin{bmatrix} -1 & 2 & 4 \\ 6 & 3 & 5 \\ -2 & 7 & 1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

Solution:

Consider $A = \begin{bmatrix} -1 & 2 & 4 \\ 6 & 3 & 5 \\ -2 & 7 & 1 \end{bmatrix}$

$\therefore A' = \begin{bmatrix} -1 & 6 & -2 \\ 2 & 3 & 7 \\ 4 & 5 & 1 \end{bmatrix}$

$\therefore A + A' = \begin{bmatrix} -2 & 8 & 2 \\ 8 & 6 & 12 \\ 2 & 12 & 2 \end{bmatrix}$

And $A - A' = \begin{bmatrix} 0 & -4 & 6 \\ 4 & 0 & -2 \\ -6 & 2 & 0 \end{bmatrix}$

$\therefore \frac{1}{2}[A + A'] = \begin{bmatrix} -1 & 4 & 1 \\ 4 & 3 & 6 \\ 1 & 6 & 1 \end{bmatrix}$

And $\frac{1}{2}[A - A'] = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}$

$\therefore A = \begin{bmatrix} -1 & 4 & 1 \\ 4 & 3 & 6 \\ 1 & 6 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}$

Required expression

PROBLEMS

Express the matrix $A = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & -6 \\ 7 & 3 & -1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

Solution:

Consider $A = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & -6 \\ 7 & 3 & -1 \end{bmatrix}$

$\therefore A' = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 4 & 3 \\ 5 & -6 & -1 \end{bmatrix}$

$\therefore A + A' = \begin{bmatrix} 4 & 4 & 12 \\ 4 & 8 & -3 \\ 12 & -3 & -2 \end{bmatrix}$

And $A - A' = \begin{bmatrix} 0 & -2 & -2 \\ 2 & 0 & -9 \\ 2 & 9 & 0 \end{bmatrix}$

$\therefore \frac{1}{2}[A + A'] = \begin{bmatrix} 2 & 2 & 6 \\ 2 & 4 & -3/2 \\ 6 & -3/2 & -1 \end{bmatrix}$

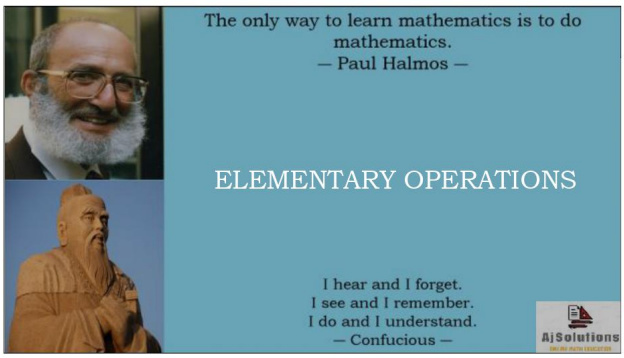
And $\frac{1}{2}[A - A'] = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -9/2 \\ 1 & 9/2 & 0 \end{bmatrix}$

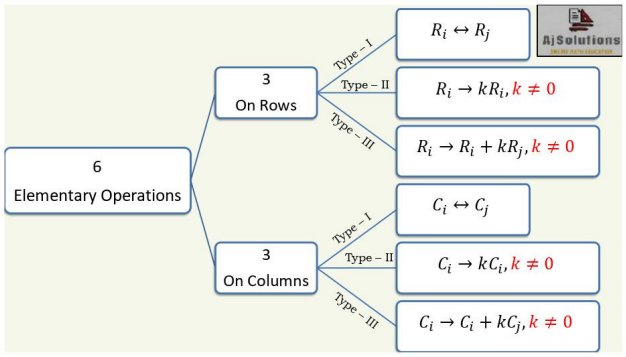
$\therefore A = \begin{bmatrix} 2 & 2 & 6 \\ 2 & 4 & -3/2 \\ 6 & -3/2 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -9/2 \\ 1 & 9/2 & 0 \end{bmatrix}$

Required expression

EXERCISE

- Express the matrix $A = \begin{bmatrix} 4 & 0 & 3 \\ 4 & 2 & -4 \\ 7 & 6 & -1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.
- Express the matrix $B = \begin{bmatrix} -1 & 5 & 3 \\ 7 & 3 & 6 \\ 4 & -2 & 2 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.





TYPE – I $R_i \leftrightarrow R_j$ OR $C_i \leftrightarrow C_j$

Example:

Let $A = \begin{bmatrix} -1 & 2 & 4 \\ 6 & 3 & 5 \\ -2 & 7 & 1 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_3$ to A, we get $\begin{bmatrix} -2 & 7 & 1 \\ 6 & 3 & 5 \\ -1 & 2 & 4 \end{bmatrix}$

Applying $C_2 \leftrightarrow C_3$ to A, we get $\begin{bmatrix} -1 & 4 & 2 \\ 6 & 5 & 3 \\ -2 & 1 & 7 \end{bmatrix}$

TYPE – II $R_i \to kR_i$ OR $C_i \to kC_i$, $k \neq 0$

Example:

$$Let\ A = \begin{bmatrix} -1 & -2 & 4 \\ 6 & -8 & 5 \\ -2 & 4 & 1 \end{bmatrix}$$

Applying $R_3 \to 5R_3$ to A, we get $\begin{bmatrix} -1 & -2 & 4 \\ 6 & -8 & 5 \\ -10 & 20 & 5 \end{bmatrix}$

Applying $C_2 \to \left(-\frac{1}{2}\right)C_2$ to A, we get $\begin{bmatrix} -1 & 1 & 4 \\ 6 & 4 & 5 \\ -2 & -2 & 1 \end{bmatrix}$

TYPE – III $R_i \to R_i + kR_j$ OR $C_i \to C_i + kC_j$, $k \neq 0$

Example:

$$Let\ A = \begin{bmatrix} -1 & -2 & 4 \\ 6 & -8 & 5 \\ -2 & 4 & 1 \end{bmatrix} \qquad \begin{array}{cccl} -2 & -8 & 4 & \leftarrow C_2 \\ -2 & 12 & -4 & \leftarrow 2C_1 \\ 0 & -20 & 8 & \leftarrow C_2 - 2C_1 \end{array}$$

Applying $R_2 \to R_2 + 6R_1$ to A, we get $\begin{bmatrix} -1 & -2 & 4 \\ 0 & -20 & 29 \\ -2 & 4 & 1 \end{bmatrix}$

Applying $C_2 \to C_2 - 2C_1$ to A, we get $\begin{bmatrix} -1 & 0 & 4 \\ 6 & -20 & 5 \\ -2 & 8 & 1 \end{bmatrix}$

EXERCISE

Perform following elementary operations on $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$

1) $R_2 \leftrightarrow R_3$
2) $C_1 \leftrightarrow C_3$
3) $R_3 \longrightarrow 2R_3$
4) $C_2 \longrightarrow \frac{1}{2}C_2$
5) $R_3 \longrightarrow R_3 - 5R_1$
6) $C_2 \longrightarrow C_2 + 2C_1$

INVERTIBLE MATRIX

A square matrix A is said to be invertible if there exists a square matrix B such that $AB = BA = I$.

Here B is called the (multiplicative) inverse matrix of A and it is denoted by A^{-1}.

Note: Here matrix B is also invertible and $B^{-1} = A$.

Example:

$Let\ A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ $and\ B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$.

Here $AB = BA = I_2$.

Therefore A is invertible with $B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = A^{-1}$

and B is also invertible with $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = B^{-1}$.

INVERTIBLE MATRIX

Example:

$Let\ A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$ $and\ B = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$.

Here $AB = BA = I_3$.

Therefore A is invertible with $B = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} = A^{-1}$

and B is also invertible with $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} = B^{-1}$.

PROPERTIES

1) Inverse of a square matrix, if it exists, is unique.
2) If A and B are two square matrices of order n such that $AB = I_n$ then $BA = I_n$.
3) If A and B are invertible matrices of the same order then $(AB)^{-1} = B^{-1}A^{-1}$.
4) In general, $(A_1A_2A_3 \ldots\ldots\ldots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1} \ldots\ldots\ldots A_3^{-1}A_2^{-1}A_1^{-1}$.
5) $(A^k)^{-1} = (A^{-1})^k$.

EXERCISE

Give example of invertible matrix of order

1) 2×2
2) 3×3

INVERSE OF A MATRIX BY ELEMETARY OPERATIONS

Consider, $AB = I$ ..... (1)

We will apply elementary row (or column) operations on matrix A to transform matrix A to I

and simultaneously same row (or column) operations we will apply on I

and matrix I will be transformed to some matrix X.

So A transforms to I and I transforms to X

and equation (1) becomes, $IB = X$

$$\Rightarrow B = X$$
$$\Rightarrow A^{-1} = X$$

Note: While performing row (or column) operations on matrix A, if any row or column of A becomes zero then it means A^{-1} does not exist.

INVERSE OF A MATRIX BY ELEMETARY OPERATIONS

Find inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ by elementary operations.

INVERSE OF A MATRIX BY ELEMETARY OPERATIONS

Find inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$ by elementary operations.

INVERSE OF A MATRIX BY ELEMETARY OPERATIONS

Find inverse of $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$ by elementary operations.

DETERMINANT

Find determinant of each of the following matrices:

(i) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

(ii) $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 4 \\ 5 & -6 \end{bmatrix}$

(iv) $\begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}$

DETERMINANT

Find determinant of each of the following matrices:

(i) $\begin{bmatrix} 3 & 1 & 2 \\ 7 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & -3 & 1 \\ 0 & 4 & 6 \\ 5 & -1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 2 & 4 \\ 3 & -6 & -12 \\ 2 & -4 & -8 \end{bmatrix}$

SINGULAR AND NON-SINGULAR MATRIX

A square matrix A is said to be *singular* if its determinant is zero.

Example:

$\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 & 3 \\ 1 & 3 & -3 \\ 5 & 3 & 3 \end{bmatrix}$

A square matrix A is said to be *non-singular* if its determinant is non-zero.

Example:

$\begin{bmatrix} 4 & -2 \\ 5 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$

MINORS AND COFACTORS

$\begin{bmatrix} 2 & -3 & 1 \\ 0 & 4 & 6 \\ 5 & -1 & 3 \end{bmatrix}$

ADJOINT

$\begin{bmatrix} 2 & -3 & 1 \\ 0 & 4 & 6 \\ 5 & -1 & 3 \end{bmatrix}$

INVERSE BY ADJOINT FORMULA

Find inverse of $A = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 4 & 6 \\ 5 & -1 & 3 \end{bmatrix}$ by adjoint formula.

INVERSE BY ADJOINT FORMULA

Find inverse of $A = \begin{bmatrix} -2 & 1 & 0 \\ 2 & 2 & 1 \\ 3 & 2 & 5 \end{bmatrix}$ by adjoint formula.

ORTHOGONAL MATRIX

A square matrix A is said to be orthogonal if $AA' = A'A = I$.

Example:

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$C = \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}$$

PROPERTIES

1) All identity matrices are orthogonal matrices.
2) The determinant of an orthogonal matrix is ± 1.
3) The orthogonal matrices are invertible. Moreover, the inverse of an orthogonal matrix is its transpose
   i.e. $A^{-1} = A'$ *for an orthogonal matrix* A.
4) The transpose of an orthogonal matrix is also an orthogonal matrix.
5) The inverse of an orthogonal matrix is also an orthogonal matrix.

ORTHOGONAL MATRIX

Show that $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ is orthogonal and hence find its inverse.

ORTHOGONAL MATRIX

Show that $A = \frac{1}{7}\begin{bmatrix} 3 & 2 & 6 \\ -6 & 3 & 2 \\ 2 & 6 & -3 \end{bmatrix}$ is orthogonal and hence find its inverse.

EIGEN VALUES AND EIGEN VECTORS

For a square matrix A, the characteristic equation is given by,

$$|A - \lambda I| = 0.$$

The roots of characteristic equation are the eigen values of the matrix.

For an eigen value λ_1, there is a vector X_1 satisfying $[A - \lambda_1 I]X_1 = 0$, is known as eigen vector of corresponding eigen value.

EIGEN VALUES AND EIGEN VECTORS

Find eigen values and eigen vectors of $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

EIGEN VALUES AND EIGEN VECTORS

Find eigen values and eigen vectors of $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$.

EIGEN VALUES AND EIGEN VECTORS

For 2×2 matrix, the characteristic equation is given by,

$$\lambda^2 - (sum\ of\ diag\ elts)\lambda + |A| = 0.$$

For 3×3 matrix, the characteristic equation is given by,

$$\lambda^3 - (sum\ of\ diag\ elts)\lambda^2 + (sum\ of\ minors\ of\ diag\ elts)\lambda - |A| = 0.$$

EIGEN VALUES AND EIGEN VECTORS

Find eigen values and eigen vectors of $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$.

EIGEN VALUES AND EIGEN VECTORS

Find eigen values and eigen vectors of

(i) $\begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$

(iv) $\begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$

PROPERTIES

1) The sum of eigen values of a matrix is the sum of diagonal elements.
2) The product of eigen values of a matrix is the determinant of the matrix.
3) If λ is an eigen value of an invertible matrix A then $1/\lambda$ is an eigen value of A^{-1}.
4) If λ is an eigen value of a matrix A then λ is also an eigen value of A^T.
5) If λ is an eigen value of an orthogonal matrix A then $1/\lambda$ is also its eigen value.
6) If λ is an eigen value of a matrix A then λ^m is an eigen value of A^m.
7) Eigen vectors corresponding to distinct eigen values are linearly independent.
8) Eigen vectors corresponding to distinct eigen values of a symmetric matrix are orthogonal.