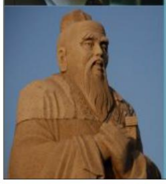



The only way to learn mathematics is to do mathematics.
— Paul Halmos —

MATRICES

I hear and I forget.
I see and I remember.
I do and I understand.
— Confucius —

DEFINITION

An ordered set of mn elements arranged in a rectangular array of m rows and n columns and enclosed by a pair of brackets [ ] or ( ) is called a matrix of order $m \times n$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n} = [a_{ij}]_{m \times n}$$

TYPES OF MATRICES

- Row Matrix or Row Vector
- Column Matrix or Column Vector
- Square Matrix
- Identity or Unit Matrix
- Null or Zero Matrix
- Diagonal Matrix
- Scalar Matrix
- Upper Triangular Matrix
- Lower Triangular Matrix

ROW MATRIX OR ROW VECTOR

A matrix having only one row and n number of columns is called a row matrix or a row vector. The order of a row matrix is $1 \times n$.

Examples:

$$[2 \quad 7]_{1\times2}$$

$$[5 \quad -2 \quad 0]_{1\times3}$$

COLUMN MATRIX OR COLUMN VECTOR

A matrix having only one column and n number of rows is called a column matrix or a column vector. The order of a column matrix is $n \times 1$.

Examples:

$$\begin{bmatrix} -1 \\ 4 \end{bmatrix}_{2\times1} \quad \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}_{3\times1}$$

SQUARE MATRIX

A matrix having same number of rows and column is called a square matrix. The order of a square matrix is $n \times n$. We also call it as square matrix of order n.

Examples:

$$\begin{bmatrix} -1 & 0 \\ 4 & 7 \end{bmatrix}_{2\times2} \quad \begin{bmatrix} 1 & -2 & 3 \\ 4 & 0 & 5 \\ 2 & 6 & 7 \end{bmatrix}_{3\times3}$$

IDENTITY MATRIX OR UNIT MATRIX

A square matrix in which diagonal elements are 1 and non-diagonal elements are zero is called a identity matrix or unit matrix. Identity matrix of order n is denoted by I_n.

Examples:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

NULL MATRIX OR ZERO MATRIX

A matrix with all elements zero is called a null matrix or zero matrix. The null matrix is denoted by $0_{m \times n}$.

Examples:

$$0_{2\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad 0_{2\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

DIAGONAL MATRIX

A square matrix in which all non-diagonal elements are zero is called a diagonal matrix.

Examples:

$$\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} = diag[2 \quad 6]$$

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix} = diag[-3 \quad 7 \quad 0]$$

SCALAR MATRIX

A square matrix in which all non-diagonal elements are zero and diagonal elements are same is called a scalar matrix. It is denoted by kI_n since scalar matrices are scalar multiple of identity matrix.

Examples:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I_2$$

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = 7I_3$$

UPPER TRIANGULAR MATRIX

A square matrix in which all elements below diagonal are zero is called an upper triangular matrix.

Examples:

$$\begin{bmatrix} 1 & 4 \\ 0 & -3 \end{bmatrix} \quad \begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 7 \end{bmatrix}$$

LOWER TRIANGULAR MATRIX

A square matrix in which all elements above diagonal are zero is called an lower triangular matrix.

Examples:

$$\begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & 0 \\ 3 & 4 & 7 \end{bmatrix}$$

EXERCISE

1) What is order of a row matrix?
2) What is order of a column matrix?
3) What is order of a square matrix?
4) Give example of each type of matrix discussed in the session.
5) Give example of a non-square matrix.
6) In a matrix of order 5×7, number of rows = ____, number of columns = __ and number of elements = ___.
7) What are the possible orders of a matrix with 12 elements?
8) Is it possible to have a square matrix with 18 elements? Justify.

EXERCISE

9) In A = $\begin{bmatrix} -2 & 7 & 4 & 6 \\ 0 & 2 & 1 & 5 \\ -1 & 3 & 9 & 8 \end{bmatrix}$

a) $a_{12} =$ ____
b) $a_{31} =$ ____
c) $a_{23} =$ ____
d) $a_{14} =$ ____
e) $a_{31} =$ ____
f) $a_{24} =$ ____
g) $a_{13} =$ ____

EXERCISE

10)Let $A = [1]$. Answer each of the following with justification.

a) Is A a row matrix?
b) Is A a column matrix?
c) Is A a square matrix?
d) Is A an identity matrix?
e) Is A a null matrix?
f) Is A a diagonal matrix?
g) Is A a scalar matrix?
h) Is A an upper triangular matrix?
i) Is A a lower triangular matrix?

EQUALITY OF MATRICES

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if
(i) they have same order (ii) $a_{ij} = b_{ij}$ for all i and j.

Example:

- $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 2 \end{bmatrix}$ $and\ B = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 2 \end{bmatrix}$
  $Here\ A = B.$
- $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 7 & 0 \end{bmatrix}$ $and\ B = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$
  $Here\ A \neq B$, since order is not same.
- $A = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}$ $and\ B = \begin{bmatrix} 4 & 1 \\ 3 & 0 \end{bmatrix}$
  $Here\ A \neq B$, since $a_{22} \neq b_{22}$.

PROBLEM ON EQUALITY OF MATRICES

Find the values of $x, y\ and\ z$ if

$$\begin{bmatrix} x+3 & -1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 6 & y \\ z-3 & 5 \end{bmatrix}$$

Solution:

Since the given two matrices are same,

$$x + 3 = 6$$
$$\Rightarrow x = 3$$
$$y = -1$$
$$z - 3 = 4$$
$$\Rightarrow z = 7$$

PROBLEM ON EQUALITY OF MATRICES

Find the values of a, $b, c\ and\ d$ if

$$\begin{bmatrix} 2a & 5a+b \\ 3d & 5d+c \end{bmatrix} = \begin{bmatrix} 10 & 17 \\ 9 & 15 \end{bmatrix}$$

Solution:

Since the given two matrices are same,

$$2a = 10$$
$$\Rightarrow a = 5$$
$$5a + b = 17$$
$$\Rightarrow b = -8$$
$$3d = 9$$
$$\Rightarrow d = 3$$
$$5d + c = 15$$
$$\Rightarrow c = 0$$

EXERCISE

- Find the values of a, b and c if
$$\begin{bmatrix} a+b+c \\ a+c \\ b+c \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$$
- Find the values of p, q and r if
$$\begin{bmatrix} p+q & 2 \\ r+5 & pq \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$$
- Find the values of w, x, y and z if
$$\begin{bmatrix} w-x & 2w+y \\ 2w-y & 3y+z \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

ADDITION OF MATRICES

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of same order, say $m \times n$ then the sum of the two matrices A and B is obtained by adding corresponding element of both the matrices

i.e. $A + B = [a_{ij} + b_{ij}]_{m \times n}$.

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} & \cdots & a_{2n}+b_{2n} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} & \cdots & a_{3n}+b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & a_{m3}+b_{m3} & \cdots & a_{mn}+b_{mn} \end{bmatrix}_{m \times n}$$

ADDITION OF MATRICES

$$Let\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} and$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} then$$

$$A + B = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

ADDITION OF MATRICES

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 5 & 4 & 7 \end{bmatrix} and\ B = \begin{bmatrix} 5 & 6 & 4 \\ 3 & 2 & 0 \end{bmatrix}$$

$$A + B = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

$$= \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

ADDITION OF MATRICES

$$A = \begin{bmatrix} 5 & 7 & 0 \\ -2 & 1 & -3 \end{bmatrix} and\ B = \begin{bmatrix} 1 & -4 & 6 \\ 3 & 2 & 7 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 5+1 & 7+(-4) & 0+6 \\ (-2)+3 & 1+2 & (-3)+7 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 3 & 6 \\ 1 & 3 & 4 \end{bmatrix}$$

MULTIPLICATION OF A MATRIX BY A SCALAR

If $A = [a_{ij}]_{m \times n}$ and k is a scalar then the scalar multiplication of k by A is obtained by multiplying each element of A by k

i.e. $kA = [ka_{ij}]_{m \times n}$.

$$= \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & ka_{23} & \cdots & ka_{2n} \\ ka_{31} & ka_{32} & ka_{33} & \cdots & ka_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & ka_{m3} & \cdots & ka_{mn} \end{bmatrix}_{m \times n}$$

MULTIPLICATION OF A MATRIX BY A SCALAR

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and

k be any scalar then

$$kA = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

MULTIPLICATION OF A MATRIX BY A SCALAR

Let $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 7 & -6 \end{bmatrix}$ then

$$3A = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

$$= \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

MULTIPLICATION OF A MATRIX BY A SCALAR

Let $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 7 & -6 \end{bmatrix}$ then

$$-2A = \begin{bmatrix} (-2)\times 2 & (-2)\times(-1) & (-2)\times 0 \\ (-2)\times 4 & (-2)\times 7 & (-2)\times(-6) \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 2 & 0 \\ -8 & -14 & 12 \end{bmatrix}$$

MULTIPLICATION OF A MATRIX BY A SCALAR

Let $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 7 & -6 \end{bmatrix}$ then

$$(-1)A = \begin{bmatrix} (-1)\times 2 & (-1)\times(-1) & (-1)\times 0 \\ (-1)\times 4 & (-1)\times 7 & (-1)\times(-6) \end{bmatrix}$$

$$-A = \begin{bmatrix} -2 & 1 & 0 \\ -4 & -7 & 6 \end{bmatrix}$$

$-A$ is called as negative of matrix A.

SUBTRACTION OF MATRICES

If $A = [a_{ij}]$ and B = $[b_{ij}]$ are two matrices of same order, say $m \times n$ then the subtraction of the two matrices A and B is defined as

$A - B = A + (-B)$.

$A - B = [a_{ij} - b_{ij}]_{m\times n}$.

$$= \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} & \cdots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} & \cdots & a_{2n}-b_{2n} \\ a_{31}-b_{31} & a_{32}-b_{32} & a_{33}-b_{33} & \cdots & a_{3n}-b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & a_{m3}-b_{m3} & \cdots & a_{mn}-b_{mn} \end{bmatrix}_{m\times n}$$

SUBTRACTION OF MATRICES

$A = \begin{bmatrix} 5 & 7 & 0 \\ -2 & 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -4 & 6 \\ 3 & 2 & 7 \end{bmatrix}$

$$A - B = \begin{bmatrix} 5-1 & 7-(-4) & 0-6 \\ (-2)-3 & 1-2 & (-3)-7 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 11 & -6 \\ -5 & -1 & -10 \end{bmatrix}$$

PROPERTIES

- *Matrix Addition:*
- All matrices are of order $m \times n$.

1. Commutative Law: $A + B = B + A$.
2. Associative Law: $(A + B) + C = A + (B + C)$.
3. Existence of 0: $A + 0 = 0 + A = A$.
4. Existence of Additive Inverse:
$$A + (-A) = (-A) + A = 0.$$

- *Scalar Multiplication:*

1. $k(A + B) = kA + kB$.
2. $(k + l)A = kA + lA$.

PROBLEMS

- Find the matrix X such that $3A + 2X = 4B$ where
$$A = \begin{bmatrix} 3 & 0 \\ 4 & 1 \\ -2 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 4 \\ 6 & 1 \\ 2 & 7 \end{bmatrix}.$$

Some initial steps:
$$3A + 2X = 4B$$
Adding $-3A$ on both the sides, we get
$$-3A + (3A + 2X) = -3A + 4B$$
$$(-3A + 3A) + 2X = 4B - 3A$$
$$0 + 2X = 4B - 3A$$
$$2X = 4B - 3A$$

PROBLEMS

Solution:

Given that $3A + 2X = 4B \quad \Rightarrow 2X = 4B - 3A$
$$\therefore 2X = 4\begin{bmatrix} -1 & 4 \\ 6 & 1 \\ 2 & 7 \end{bmatrix} - 3\begin{bmatrix} 3 & 0 \\ 4 & 1 \\ -2 & 6 \end{bmatrix}$$
$$= \begin{bmatrix} -4 & 16 \\ 24 & 4 \\ 8 & 28 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 12 & 3 \\ -6 & 18 \end{bmatrix}$$
$$= \begin{bmatrix} -13 & 16 \\ 12 & 1 \\ 14 & 10 \end{bmatrix}$$
$$\therefore X = \begin{bmatrix} -13/2 & 8 \\ 6 & 1/2 \\ 7 & 5 \end{bmatrix}$$

PROBLEMS

Find X and Y, if $X + Y = \begin{bmatrix} 7 & 2 \\ 0 & 3 \end{bmatrix}$ and $X - Y = \begin{bmatrix} -3 & -2 \\ 0 & 5 \end{bmatrix}$.

Solution: Given that

$X + Y = \begin{bmatrix} 7 & 2 \\ 0 & 3 \end{bmatrix}$ ..... (I)

$X - Y = \begin{bmatrix} -3 & -2 \\ 0 & 5 \end{bmatrix}$ ..... (II)

(I) + (II) gives, $2X = \begin{bmatrix} 7 & 2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ 0 & 5 \end{bmatrix}$
$$= \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}$$
$$\Rightarrow X = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

(I) - (II) gives, $2Y = \begin{bmatrix} 7 & 2 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} -3 & -2 \\ 0 & 5 \end{bmatrix}$
$$= \begin{bmatrix} 10 & 4 \\ 0 & -2 \end{bmatrix}$$
$$\Rightarrow Y = \begin{bmatrix} 5 & 2 \\ 0 & -1 \end{bmatrix}$$

PROBLEMS

Find x and y, if $3\begin{bmatrix} x & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ -2 & y+2 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 7 & 4 \end{bmatrix}$.

Solution: Given that
$$3\begin{bmatrix} x & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ -2 & y+2 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 7 & 4 \end{bmatrix}$$
$$\begin{bmatrix} 3x & 6 \\ 9 & 3 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ -2 & y+2 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 7 & 4 \end{bmatrix}$$
$$\begin{bmatrix} 3x+4 & 5 \\ 7 & y+5 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 7 & 4 \end{bmatrix}$$
$$\Rightarrow 3x + 4 = 10 \text{ and } y + 5 = 4$$
$$\Rightarrow x = 2 \text{ and } y = -1$$

EXERCISE

- Find the matrix X such that $5A + 2X = 3B$ where
$$A = \begin{bmatrix} -2 & 1 & 0 \\ 6 & -3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -3 & 2 \\ 0 & 1 & 6 \end{bmatrix}.$$
- Find X and Y, if
$$2X + Y = \begin{bmatrix} 8 & 3 \\ 1 & 10 \end{bmatrix} \text{ and } X - Y = \begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}.$$
- Find x and y, if $2\begin{bmatrix} x & 3 \\ 5 & y-2 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 9 \\ 11 & 4 \end{bmatrix}$.

MULTIPLICATION OF MATRICES

The product AB of matrices A and B is defined if number of columns of A is equal to the number of rows of B.

Let $A = [a_{ij}]_{m\times n}$ and B = $[b_{jk}]_{n\times p}$. Then the product AB of matrices A and B is a matrix of order $m \times p$ and is defined as

$$AB = \left[\sum_{j=1}^{n} a_{ij}b_{jk}\right]_{m\times p}$$

MULTIPLICATION OF MATRICES

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2k} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & b_{3k} & \cdots & b_{3p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nk} & \cdots & b_{np} \end{bmatrix}_{n\times p}$$

$$AB = \left[\sum_{j=1}^{n} a_{ij}b_{jk}\right]_{m\times p}$$

MULTIPLICATION OF MATRICES

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & 10 \\ 11 & 5\times 3 + 3\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 10 \\ 11 & 27 \end{bmatrix}$$

MULTIPLICATION OF MATRICES

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 2 & 4 \\ 8 & 1 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} -10 & 3 & 5 \\ 44 & 14 & 34 \end{bmatrix}$$

POWERS OF MATRIX

- $A^2 = A \times A$
- $A^3 = A^2 \times A$
- In general,

$$A^n = A^{n-1} \times A, \quad where\ n \geq 2.$$

POWERS OF MATRIX

Example:

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$$

$$\therefore A^2 = A \times A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$$

$$\therefore A^2 = \begin{bmatrix} 2(2) + 1(-3) & 2(1) + 1(4) \\ (-3)2 + 4(-3) & (-3)1 + 4(4) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 \\ -18 & 13 \end{bmatrix}$$

POWERS OF MATRIX

Example:

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} \text{ and } A^2 = \begin{bmatrix} 1 & 6 \\ -18 & 13 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 1 & 6 \\ -18 & 13 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$$

$$\therefore A^3 = \begin{bmatrix} 1(2) + 6(-3) & 1(1) + 6(4) \\ (-18)2 + 13(-3) & (-18)1 + 13(4) \end{bmatrix}$$

$$= \begin{bmatrix} -16 & 25 \\ -75 & 34 \end{bmatrix}$$

PROPERTIES

- All matrices taken are conformable with respect to the operations.

1. $AB \neq BA$ i.e. matrix multiplication is not commutative.

Example:

$$Let\ A = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 4 \\ 1 & 7 \end{bmatrix}$$

$$Then\ AB = \begin{bmatrix} 1 & 17 \\ 5 & 15 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 6 & 10 \\ 13 & 10 \end{bmatrix}$$

$$\therefore AB \neq BA$$

PROPERTIES

Example:

$$Let\ A = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$Then\ AB = \begin{bmatrix} -2 & 6 \\ 4 & 2 \end{bmatrix} \text{ and } BA = \begin{bmatrix} -2 & 6 \\ 4 & 2 \end{bmatrix}$$

$$\therefore AB = BA$$

PROPERTIES

2. $(AB)C = A(BC)$ i.e. matrix multiplication is associative.

3. Existence of multiplicative identity I: $AI = IA = A$.

4. Distributive law:

i. $A(B + C) = AB + AC$.

ii. $(A + B)C = AC + BC$.

PROPERTIES

5. $AB = 0 \nRightarrow A = 0 \text{ or } B = 0$.

Example:

$$Let\ A = \begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 1 \\ 0 & 0 \end{bmatrix}$$

$$Then\ AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So here we have $AB = 0$ but neither $A = 0$ nor $B = 0$.

So, $AB = 0 \nRightarrow A = 0 \text{ or } B = 0$.

PROPERTIES

6. $AB = 0 \nRightarrow BA = 0$.

Example:

$$Let\ A = \begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 1 \\ 0 & 0 \end{bmatrix}$$

$$Then\ AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 0 & 17 \\ 0 & 0 \end{bmatrix}.$$

So here we have $AB = 0$ but $BA \neq 0$.

So, $AB = 0 \nRightarrow BA = 0$.

$(A+B)^2 = ?$

$$(A+B)^2 = (A+B)(A+B)$$

$$= A(A+B) + B(A+B)$$

$$= A^2 + AB + BA + B^2$$

$$(A+B)^2 = A^2 + AB + BA + B^2$$

EXERCISE

- Note: Examples to be given other than discussed in the session.
- If $A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 1 & 2 \\ 4 & -2 & -1 \end{bmatrix}$ then compute A^2 and A^4.
- If $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ then compute AB and BA. Is $AB = BA$?
- Give example of A and B such that $AB = BA$.
- If $A = \begin{bmatrix} -1 & 3 \\ 2 & 6 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$ then verify

i. $(AB)C = A(BC)$

ii. $(A+B)C = AC + BC$

EXERCISE

- Verify the existence of multiplicative identity property for $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 2 \\ 6 & -3 & 8 \end{bmatrix}$.
- Give example of A and B such that

i. $A \neq 0, B \neq 0$ but $AB = 0$

ii. $AB = 0$ but $BA \neq 0$

- Expand $(A+B)^3$.

TRANSPOSE OF A MATRIX

The matrix obtained by interchanging rows and columns of a matrix is called a transpose of a matrix.

Let $A = [a_{ij}]_{m \times n}$, then its transpose is denoted by A' or A^T, and $A' = [a_{ji}]_{n \times m}$.

Example:

- $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 2 \end{bmatrix}$

$A' = \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ 3 & 2 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 7 & 5 \end{bmatrix}$$

$$A' = \begin{bmatrix} 2 & 4 \\ 1 & 7 \\ 3 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & -1 & 3 \\ 0 & 2 & 4 \\ 6 & -3 & 7 \end{bmatrix}$$

$$A' = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

PROPERTIES

1) $(A')' = A$
2) $(A+B)' = A' + B'$
3) $(kA)' = kA'$
4) $(AB)' = B'A'$

SYMMETRIC MATRIX

A square matrix A is said to be symmetric if $A = A'$.

Example:

$A = \begin{bmatrix} 5 & -1 & 3 \\ -1 & 2 & 4 \\ 3 & 4 & 7 \end{bmatrix}$

$$A = \begin{bmatrix} 5 & -1 & 3 \\ -1 & 2 & 4 \\ 3 & 4 & 7 \end{bmatrix}$$

$$A' = \begin{bmatrix} 5 & -1 & 3 \\ -1 & 2 & 4 \\ 3 & 4 & 7 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & -1 & 3 \\ -1 & 2 & 4 \\ 3 & 4 & 7 \end{bmatrix}$$

$$A' = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

CONSTRUCTION OF EXAMPLE

- Recall
- In a square matrix $A = [a_{ij}]$, diagonal elements are of the form a_{ii} and non-diagonal elements are of the form a_{ij} where $i \neq j$.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 0 & 5 \\ 6 & 5 & -1 \end{bmatrix}$$

- If $A = [a_{ij}]$ then $A' = [a_{ji}]$.
- For a symmetric matrix,

$A = A'$

i.e. $[a_{ij}] = [a_{ji}]$

$\Rightarrow a_{ij} = a_{ji}\ \forall i, j$

On diagonal, $a_{ii} = a_{ii}$

On non-diagonal, $a_{ij} = a_{ji}$ where $i \neq j$.

SKEW–SYMMETRIC MATRIX

A square matrix A is said to be skew-symmetric if $A = -A'$.

Example:

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 6 \\ -3 & 6 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

CONSTRUCTION OF EXAMPLE

- Recall
- In a square matrix $A = [a_{ij}]$, diagonal elements are of the form a_{ii} and non-diagonal elements are of the form a_{ij} where $i \neq j$.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & -2 & 6 \\ 2 & 0 & -5 \\ -6 & 5 & 0 \end{bmatrix}$$

- If $A = [a_{ij}]$ then $A' = [a_{ji}]$.
- For a skew-symmetric matrix,

$A = -A'$

i.e. $[a_{ij}] = [-a_{ji}]$

$\Rightarrow a_{ij} = -a_{ji}\ \forall i, j$

On diagonal, $a_{ii} = -a_{ii}$

$\Rightarrow a_{ii} = 0$

On non-diagonal, $a_{ij} = -a_{ji}$ where $i \neq j$.

EXERCISE

- Give example of symmetric and skew-symmetric matrix of order

1) 3×3
2) 4×4

PROBLEMS

Find a, b, c and d such that X is symmetric where

$$X = \begin{bmatrix} 1 & a & 3 \\ 4 & b & c \\ d & -1 & 5 \end{bmatrix}.$$

Solution:

Since the given matrix X is symmetric, $x_{ij} = x_{ji}\ \forall i, j$

$$a = x_{12} = x_{21} = 4$$
$$c = x_{23} = x_{32} = -1$$
$$d = x_{31} = x_{13} = 3$$

b is on the diagonal and we can have any real number on the diagonal of a symmetric matrix.

Therfore b = any real number.

PROBLEMS

Find w, x, y and z such that A is skew-symmetric where

$$A = \begin{bmatrix} w & 2 & x \\ y & 0 & -1 \\ 4 & z & 0 \end{bmatrix}.$$

Solution:

Since the given matrix A is skew − symmetric, $a_{ij} = -a_{ji}\ \forall i, j$

w is on the diagonal and we can have only 0 on the diagonal of a skew − symmetric matrix.

Therfore w = 0.

$$x = a_{13} = -a_{31} = -4$$
$$y = a_{21} = -a_{12} = -2$$
$$z = a_{32} = -a_{23} = -(-1)$$

IMPORTANT RESULTS

Result 1: Sum of two symmetric matrices is symmetric.

(i.e.) If A and B are symmetric matrices then $A + B$ is symmetric.

Proof:

Let A and B be any two symmetric matrices of same order.

To prove: $A + B$ is symmetric.

We know that, matrix X is symmetric if $X' = X$.

(i.e.) to prove, $(A + B)' = A + B$.

Consider, $(A + B)' = A' + B'$

Given that A and B are symmetric, therefore $A' = A$ and $B' = B$.

$$\therefore (A + B)' = A + B$$

Hence proved.

IMPORTANT RESULTS

Result 2: Scalar multiple of a symmetric matrix is symmetric.

(i.e.) If A is symmetric matrix then kA is symmetric, where k is any scalar.

Proof:

Let A be any symmetric matrix and k be any scalar.

To prove: kA is symmetric.

We know that, matrix X is symmetric if $X' = X$.

(i.e.) to prove, $(kA)' = kA$.

Consider, $(kA)' = kA'$

Given that A is symmetric, therefore $A' = A$.

$$\therefore (kA)' = kA$$

Hence proved.

IMPORTANT RESULTS

Result 3: Sum of two skew-symmetric matrices is skew-symmetric.

(i.e.) If A and B are skew-symmetric matrices then $A + B$ is skew-symmetric.

Proof:

Let A and B be any two skew-symmetric matrices of same order.

To prove: $A + B$ is skew-symmetric.

We know that, matrix X is skew-symmetric if $X' = -X$.

(i.e.) to prove, $(A + B)' = -(A + B)$.

Consider, $(A + B)' = A' + B'$

Given that A and B are skew-symmetric, therefore $A' = -A$ and $B' = -B$.

$$\therefore (A + B)' = (-A) + (-B) = -(A + B)$$

Hence proved.

IMPORTANT RESULTS

Result 4: Scalar multiple of a skew-symmetric matrix is skew-symmetric.

(i.e.) If A is skew-symmetric matrix then kA is skew-symmetric, where k is any scalar.

Proof:

Let A be any skew-symmetric matrix and k be any scalar.

To prove: kA is skew-symmetric.

We know that, matrix X is skew-symmetric if $X' = -X$.

(i.e.) to prove, $(kA)' = -(kA)$.

Consider, $(kA)' = kA'$

Given that A is skew-symmetric, therefore $A' = -A$.

$$\therefore (kA)' = k(-A) = -(kA)$$

Hence proved.

IMPORTANT RESULTS

Result 5: For any square matrix A

i. $A + A'$ is symmetric and

ii. $A - A'$ is skew-symmetric.

Proof :

Let A be any square matrix.

i. To prove: $A + A'$ is symmetric.

We know that, matrix X is symmetric if $X' = X$.

(i.e.) to prove, $(A + A')' = A + A'$.

Consider, $(A + A')' = A' + (A')' = A' + A = A + A'$

Hence $A + A'$ is symmetric.

IMPORTANT RESULTS

Result 5: For any square matrix A

i. $A + A'$ is symmetric and

ii. $A - A'$ is skew-symmetric.

Proof :

Let A be any square matrix.

ii. To prove: $A - A'$ is skew-symmetric.

We know that, matrix X is skew-symmetric if $X' = -X$.

(i.e.) to prove, $(A - A')' = -(A - A')$.

Consider, $(A - A')' = A' - (A')'$

$= A' - A$

$= -(A - A')$

Hence $A - A'$ is skew-symmetric.

EXERCISE

- Find a, b, c and d such that X is symmetric where
$$X = \begin{bmatrix} a & -2 & b \\ c & 0 & 4 \\ 3 & d & 0 \end{bmatrix}.$$
- Find w, x, y and z such that A is skew-symmetric where
$$A = \begin{bmatrix} w & x & 4 \\ 2 & 0 & y \\ z & -3 & 0 \end{bmatrix}.$$
- Prove or disprove:

i. AB is symmetric for any symmetric matrices A and B.
ii. AB is skew-symmetric for any skew-symmetric matrices A and B.
iii. A' is symmetric for any symmetric matrix A.
iv. A' is skew-symmetric for any skew-symmetric matrix A.

EXERCISE

- Prove the following:

i. If A is symmetric then A^2 and A^3 is symmetric.
ii. If A is skew-symmetric then A^2 is symmetric and A^3 is skew-symmetric.
iii. For any square matrix A, AA' and $A'A$ are symmetric.

IMPORTANT RESULTS

Result 1: If A and B are symmetric matrices then $A + B$ is symmetric.
Result 2: If A is symmetric matrix then kA is symmetric, where k is any scalar.
Result 3: If A and B are skew-symmetric matrices then $A + B$ is skew-symmetric.
Result 4: If A is skew-symmetric matrix then kA is skew-symmetric, where k is any scalar.
Result 5: For any square matrix A
i. $A + A'$ is symmetric and
ii. $A - A'$ is skew-symmetric.

IMPORTANT RESULTS

Theorem: Any square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix.
Proof:
Let A be any square matrix such that
$A = P + Q$, where P is symmetric and Q is skew-symmetric.
Since P is symmetric, $P' = P$ and
since Q is skew-symmetric, $Q' = -Q$.
Consider, $A' = (P + Q)' = P' + Q' = P - Q$
$A = P + Q$ ..... (1)
$A' = P - Q$ ..... (2)
(1) + (2) gives, $A + A' = 2P \implies P = \frac{1}{2}[A + A']$
(1) − (2) gives, $A - A' = 2Q \implies Q = \frac{1}{2}[A - A']$

$\therefore A = \left\{\frac{1}{2}[A + A']\right\} + \left\{\frac{1}{2}[A - A']\right\}$
where $\frac{1}{2}[A + A']$ is symmetric
and $\frac{1}{2}[A - A']$ is skew − symmetric

IMPORTANT RESULTS

Theorem: Any square matrix A can be uniquely expressed as the sum of a symmetric and skew-symmetric matrix.
Proof of Uniqueness:
Let $A = X + Y$, where X is symmetric and Y is skew-symmetric.
Since X is symmetric, $X' = X$ and
since Y is skew-symmetric, $Y' = -Y$.
Consider, $A' = (X + Y)' = X' + Y' = X - Y$
$A = X + Y$ ..... (1)
$A' = X - Y$ ..... (2)
(1) + (2) gives, $A + A' = 2X \implies X = \frac{1}{2}[A + A'] = P$
(1) − (2) gives, $A - A' = 2Y \implies Y = \frac{1}{2}[A - A'] = Q$
Hence the expression is unique.

STEPS TO EXPRESS

Any square matrix as the sum of a symmetric and skew−symmetric matrix

Step 1: Consider the given matrix A and compute it's transpose A'
Step 2: Compute $A + A'$ and $A - A'$
Step 3: Perform scalar multiplication by $\frac{1}{2}$ with both the matrices (which you got in step 2) to get
$$\frac{1}{2}[A + A'] \text{ and } \frac{1}{2}[A - A']$$
Step 4: Now write A as sum of the two matrices (which you got in step 3 i.e. $A = \left\{\frac{1}{2}[A + A']\right\} + \left\{\frac{1}{2}[A - A']\right\}$
which is the required expression

PROBLEMS

Express the matrix $A = \begin{bmatrix} -1 & 2 & 4 \\ 6 & 3 & 5 \\ -2 & 7 & 1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

Solution:

Consider $A = \begin{bmatrix} -1 & 2 & 4 \\ 6 & 3 & 5 \\ -2 & 7 & 1 \end{bmatrix}$

$\therefore A' = \begin{bmatrix} -1 & 6 & -2 \\ 2 & 3 & 7 \\ 4 & 5 & 1 \end{bmatrix}$

$\therefore A + A' = \begin{bmatrix} -2 & 8 & 2 \\ 8 & 6 & 12 \\ 2 & 12 & 2 \end{bmatrix}$

And $A - A' = \begin{bmatrix} 0 & -4 & 6 \\ 4 & 0 & -2 \\ -6 & 2 & 0 \end{bmatrix}$

$\therefore \frac{1}{2}[A + A'] = \begin{bmatrix} -1 & 4 & 1 \\ 4 & 3 & 6 \\ 1 & 6 & 1 \end{bmatrix}$

And $\frac{1}{2}[A - A'] = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}$

$\therefore A = \begin{bmatrix} -1 & 4 & 1 \\ 4 & 3 & 6 \\ 1 & 6 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}$

Required expression

PROBLEMS

Express the matrix $A = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & -6 \\ 7 & 3 & -1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

Solution:

Consider $A = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & -6 \\ 7 & 3 & -1 \end{bmatrix}$

$\therefore A' = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 4 & 3 \\ 5 & -6 & -1 \end{bmatrix}$

$\therefore A + A' = \begin{bmatrix} 4 & 4 & 12 \\ 4 & 8 & -3 \\ 12 & -3 & -2 \end{bmatrix}$

And $A - A' = \begin{bmatrix} 0 & -2 & -2 \\ 2 & 0 & -9 \\ 2 & 9 & 0 \end{bmatrix}$

$\therefore \frac{1}{2}[A + A'] = \begin{bmatrix} 2 & 2 & 6 \\ 2 & 4 & -3/2 \\ 6 & -3/2 & -1 \end{bmatrix}$

And $\frac{1}{2}[A - A'] = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -9/2 \\ 1 & 9/2 & 0 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 2 & 2 & 6 \\ 2 & 4 & -3/2 \\ 6 & -3/2 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -9/2 \\ 1 & 9/2 & 0 \end{bmatrix}$

Required expression

EXERCISE

- Express the matrix $A = \begin{bmatrix} 4 & 0 & 3 \\ 4 & 2 & -4 \\ 7 & 6 & -1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.
- Express the matrix $B = \begin{bmatrix} -1 & 5 & 3 \\ 7 & 3 & 6 \\ 4 & -2 & 2 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

The only way to learn mathematics is to do mathematics.
— Paul Halmos —

ELEMENTARY OPERATIONS

I hear and I forget.
I see and I remember.
I do and I understand.
— Confucious —

6 Elementary Operations

- 3 On Rows
  - Type – I: $R_i \leftrightarrow R_j$
  - Type – II: $R_i \to kR_i, k \neq 0$
  - Type – III: $R_i \to R_i + kR_j, k \neq 0$
- 3 On Columns
  - Type – I: $C_i \leftrightarrow C_j$
  - Type – II: $C_i \to kC_i, k \neq 0$
  - Type – III: $C_i \to C_i + kC_j, k \neq 0$

TYPE – I $R_i \leftrightarrow R_j$ OR $C_i \leftrightarrow C_j$

Example:

Let $A = \begin{bmatrix} -1 & 2 & 4 \\ 6 & 3 & 5 \\ -2 & 7 & 1 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_3$ to A, we get $\begin{bmatrix} -2 & 7 & 1 \\ 6 & 3 & 5 \\ -1 & 2 & 4 \end{bmatrix}$

Applying $C_2 \leftrightarrow C_3$ to A, we get $\begin{bmatrix} -1 & 4 & 2 \\ 6 & 5 & 3 \\ -2 & 1 & 7 \end{bmatrix}$

TYPE – II $R_i \to kR_i$ OR $C_i \to kC_i$, $k \neq 0$

Example:

Let $A = \begin{bmatrix} -1 & -2 & 4 \\ 6 & -8 & 5 \\ -2 & 4 & 1 \end{bmatrix}$

Applying $R_3 \to 5R_3$ to A, we get $\begin{bmatrix} -1 & -2 & 4 \\ 6 & -8 & 5 \\ -10 & 20 & 5 \end{bmatrix}$

Applying $C_2 \to \left(-\frac{1}{2}\right) C_2$ to A, we get $\begin{bmatrix} -1 & 1 & 4 \\ 6 & 4 & 5 \\ -2 & -2 & 1 \end{bmatrix}$

TYPE – III $R_i \to R_i + kR_j$ OR $C_i \to C_i + kC_j$, $k \neq 0$

Example:

$-2 \quad -8 \quad 4 \quad \leftarrow C_2$

$-2 \quad 12 \quad -4 \quad \leftarrow 2C_1$

$0 \quad -20 \quad 8 \quad \leftarrow C_2 - 2C_1$

Let $A = \begin{bmatrix} -1 & -2 & 4 \\ 6 & -8 & 5 \\ -2 & 4 & 1 \end{bmatrix}$

Applying $R_2 \to R_2 + 6R_1$ to A, we get $\begin{bmatrix} -1 & -2 & 4 \\ 0 & -20 & 29 \\ -2 & 4 & 1 \end{bmatrix}$

Applying $C_2 \to C_2 - 2C_1$ to A, we get $\begin{bmatrix} -1 & 0 & 4 \\ 6 & -20 & 5 \\ -2 & 8 & 1 \end{bmatrix}$

EXERCISE

Perform following elementary operations on $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$

1) $R_2 \leftrightarrow R_3$
2) $C_1 \leftrightarrow C_3$
3) $R_3 \longrightarrow 2R_3$
4) $C_2 \longrightarrow \frac{1}{2} C_2$
5) $R_3 \longrightarrow R_3 - 5R_1$
6) $C_2 \longrightarrow C_2 + 2C_1$

INVERTIBLE MATRIX

A square matrix A is said to be invertible if there exists a square matrix B such that $AB = BA = I$.

Here B is called the (multiplicative) inverse matrix of A and it is denoted by A^{-1}.

Note: Here matrix B is also invertible and $B^{-1} = A$.

Example:

Let $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$.

Here $AB = BA = I_2$.

Therefore A is invertible with $B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = A^{-1}$

and B is also invertible with $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = B^{-1}$.

INVERTIBLE MATRIX

Example:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$.

Here $AB = BA = I_3$.

Therefore A is invertible with $B = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} = A^{-1}$

and B is also invertible with $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} = B^{-1}$.

PROPERTIES

1) Inverse of a square matrix, if it exists, is unique.
2) If A and B are two square matrices of order n such that $AB = I_n$ then $BA = I_n$.
3) If A and B are invertible matrices of the same order then $(AB)^{-1} = B^{-1}A^{-1}$.
4) In general, $(A_1A_2A_3 \ldots\ldots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1} \ldots\ldots A_3^{-1}A_2^{-1}A_1^{-1}$.
5) $\left(A^k\right)^{-1} = (A^{-1})^k$.

EXERCISE

Give example of invertible matrix of order

1) 2×2
2) 3×3

INVERSE OF A MATRIX BY ELEMETARY OPERATIONS

Consider, $AB = I$ ..... (1)

We will apply elementary row (or column) operations on matrix A to transform matrix A to I

and simultaneously same row (or column) operations we will apply on I

and matrix I will be transformed to some matrix X.

So A transforms to I and I transforms to X

and equation (1) becomes, $IB = X$

$$\Rightarrow B = X$$

$$\Rightarrow A^{-1} = X$$

Note: While performing row (or column) operations on matrix A, if any row or column of A becomes zero then it means A^{-1} does not exist.

INVERSE OF A MATRIX BY ELEMETARY OPERATIONS

Find inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ by elementary operations.

INVERSE OF A MATRIX BY ELEMETARY OPERATIONS

Find inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$ by elementary operations.

INVERSE OF A MATRIX BY ELEMETARY OPERATIONS

Find inverse of $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$ by elementary operations.

DETERMINANT

Find determinant of each of the following matrices:

(i) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

(ii) $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 4 \\ 5 & -6 \end{bmatrix}$

(iv) $\begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}$

DETERMINANT

Find determinant of each of the following matrices:

(i) $\begin{bmatrix} 3 & 1 & 2 \\ 7 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & -3 & 1 \\ 0 & 4 & 6 \\ 5 & -1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 2 & 4 \\ 3 & -6 & -12 \\ 2 & -4 & -8 \end{bmatrix}$

SINGULAR AND NON-SINGULAR MATRIX

A square matrix A is said to be *singular* if its determinant is zero.

Example:

$\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 & 3 \\ 1 & 3 & -3 \\ 5 & 3 & 3 \end{bmatrix}$

A square matrix A is said to be *non-singular* if its determinant is non-zero.

Example:

$\begin{bmatrix} 4 & -2 \\ 5 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$

MINORS AND COFACTORS

$\begin{bmatrix} 2 & -3 & 1 \\ 0 & 4 & 6 \\ 5 & -1 & 3 \end{bmatrix}$

ADJOINT

$\begin{bmatrix} 2 & -3 & 1 \\ 0 & 4 & 6 \\ 5 & -1 & 3 \end{bmatrix}$

INVERSE BY ADJOINT FORMULA

Find inverse of $A = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 4 & 6 \\ 5 & -1 & 3 \end{bmatrix}$ by adjoint formula.

INVERSE BY ADJOINT FORMULA

Find inverse of $A = \begin{bmatrix} -2 & 1 & 0 \\ 2 & 2 & 1 \\ 3 & 2 & 5 \end{bmatrix}$ by adjoint formula.

ORTHOGONAL MATRIX

A square matrix A is said to be orthogonal if $AA' = A'A = I$.

Example:

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$C = \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}$$

PROPERTIES

1) All identity matrices are orthogonal matrices.
2) The determinant of an orthogonal matrix is ± 1.
3) The orthogonal matrices are invertible. Moreover, the inverse of an orthogonal matrix is its transpose
   i.e. $A^{-1} = A'$ *for an orthogonal matrix* A.
4) The transpose of an orthogonal matrix is also an orthogonal matrix.
5) The inverse of an orthogonal matrix is also an orthogonal matrix.

ORTHOGONAL MATRIX

Show that $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ is orthogonal and hence find its inverse.

ORTHOGONAL MATRIX

Show that $A = \frac{1}{7}\begin{bmatrix} 3 & 2 & 6 \\ -6 & 3 & 2 \\ 2 & 6 & -3 \end{bmatrix}$ is orthogonal and hence find its inverse.

EIGEN VALUES AND EIGEN VECTORS

For a square matrix A, the characteristic equation is given by,

$$|A - \lambda I| = 0.$$

The roots of characteristic equation are the eigen values of the matrix.

For an eigen value λ_1, there is a vector X_1 satisfying $[A - \lambda_1 I]X_1 = 0$, is known as eigen vector of corresponding eigen value.

EIGEN VALUES AND EIGEN VECTORS

Find eigen values and eigen vectors of $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

EIGEN VALUES AND EIGEN VECTORS

Find eigen values and eigen vectors of $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$.

EIGEN VALUES AND EIGEN VECTORS

For 2×2 matrix, the characteristic equation is given by,

$$\lambda^2 - (sum\ of\ diag\ elts)\lambda + |A| = 0.$$

For 3×3 matrix, the characteristic equation is given by,

$$\lambda^3 - (sum\ of\ diag\ elts)\lambda^2 + (sum\ of\ minors\ of\ diag\ elts)\lambda - |A| = 0.$$

EIGEN VALUES AND EIGEN VECTORS

Find eigen values and eigen vectors of $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$.

EIGEN VALUES AND EIGEN VECTORS

Find eigen values and eigen vectors of

(i) $\begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$

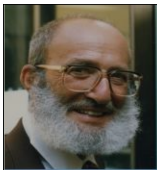
(ii) $\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

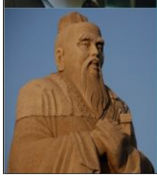
(iii) $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$

(iv) $\begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$

PROPERTIES

1) The sum of eigen values of a matrix is the sum of diagonal elements.
2) The product of eigen values of a matrix is the determinant of the matrix.
3) If λ is an eigen value of an invertible matrix A then $1/\lambda$ is an eigen value of A^{-1}.
4) If λ is an eigen value of a matrix A then λ is also an eigen value of A^T.
5) If λ is an eigen value of an orthogonal matrix A then $1/\lambda$ is also its eigen value.
6) If λ is an eigen value of a matrix A then λ^m is an eigen value of A^m.
7) Eigen vectors corresponding to distinct eigen values are linearly independent.
8) Eigen vectors corresponding to distinct eigen values of a symmetric matrix are orthogonal.





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MEASURES OF DISPERSION

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— Confucious —

INTRODUCTION

The second important characteristic of a distribution is given by dispersion. Two distributions may be identical in respect of its first important characteristic i.e. central tendency and yet they may differ on account of scatterness.

Dispersion for a given set of observations may be defined as the amount of deviation of the observations, usually, from an appropriate measure of central tendency.

Measures of dispersion may be broadly classified into

1. Absolute measures of dispersion
2. Relative measures of dispersion (unit free measure)

INTRODUCTION

Absolute measures of dispersion are classified into

i. Range
ii. Mean Deviation
iii. Quartile Deviation
iv. Standard Deviation

Likewise, we have the following relative measures of dispersion :

i. Coefficient of Range
ii. Coefficient of Mean Deviation
iii. Coefficient of Quartile Deviation
iv. Coefficient of Variation

RANGE

For a given set of observations, range may be defined as the difference between the largest and smallest of observations.

$$Range = Max - Min$$

The corresponding relative measure of dispersion, known as coefficient of range, is given by

$$Coefficient\ of\ Range = \frac{Max - Min}{Max + Min} \times 100$$

RANGE

Example 1: Find range and coefficient of range for the following:

$$12, 6, 7, 3, 15, 10, 18, 5$$

Solution:

$$Range = 18 - 3 = 15$$

$$Coefficient\ of\ Range = \frac{18 - 3}{18 + 3} \times 100 = 71.43$$

RANGE

Example 2: Find range and coefficient of range for the following:

Size	Frequency
8	14
9	21
10	17
11	12
12	6

Ans: 4, 20

Weight (lb)	Frequency
118–126	3
127–135	5
136–144	9
145–153	12
154–162	5
163–171	4
172–180	2

Ans: 63, 21.14

RANGE

Result: Range remains unaffected due to a change of origin but affected in the same ratio due to a change in scale i.e., if for any two constants a and b, two variables x and y are related by y = a + bx,

$$Range(y) = |b| \times Range(x)$$

Example: If the relationship between x and y is given by 2x+3y=10 and the range of x is 15, what would be the range of y?

Solution: Given that $2x + 3y = 10 \Rightarrow y = \frac{10}{3} - \frac{2}{3}x$

$$\therefore Range(y) = \frac{2}{3} Range(x) = 10$$

MEAN DEVIATION

1. MD from Mean:
   - For raw data: $MD\ from\ Mean = \frac{1}{N}\sum|x - \bar{x}|$
   - For frequency data: $MD\ from\ Mean = \frac{1}{N}\sum f|x - \bar{x}|$
2. MD from Median:
   - For raw data: $MD\ from\ Median = \frac{1}{N}\sum|x - M|$
   - For frequency data: $MD\ from\ Median = \frac{1}{N}\sum f|x - M|$

- $Coefficient\ of\ MD = \frac{MD\ about\ A}{A} \times 100$

MEAN DEVIATION

Result 1: Mean deviation takes its minimum value when the deviations are taken from the median.

Result 2: Mean deviation remains unchanged due to a change of origin but changes in the same ratio due to a change in scale i.e. if y = a + bx, a and b being constants, then $MD(y) = |b| \times MD(x)$.

Example: If x and y are related as 4x+3y+11 = 0 and mean deviation of x is 5.40, what is the mean deviation of y?

Answer: $MD(y) = 7.20$.

MEAN DEVIATION

Example 1: Find MD from mean, MD from median and corresponding coefficients of MD for the following:

$$12, 6, 7, 3, 15, 10, 18, 5$$

Solution:

For MD from mean:

$$\bar{x} = \frac{76}{8} = 9.5$$

$$MD\ from\ \bar{x} = \frac{2.5 + 3.5 + 2.5 + 6.5 + 5.5 + 0.5 + 8.5 + 4.5}{8} = 4.25$$

$$Coefficient\ of\ MD = \frac{4.25}{9.5} \times 100 = 44.74$$

MEAN DEVIATION

Example 1: Find MD from mean, MD from median and corresponding coefficients of MD for the following:

$$12, 6, 7, 3, 15, 10, 18, 5$$

Solution:

Cont....

For MD from median:

$$M = 8.5$$

$$MD\ from\ M = \frac{3.5 + 2.5 + 1.5 + 5.5 + 6.5 + 1.5 + 9.5 + 3.5}{8} = 4.25$$

$$Coefficient\ of\ MD = \frac{4.25}{8.5} \times 100 = 51.83$$

MEAN DEVIATION

Example 2: Compute MD from mean and coefficient of MD for the following:

x	f	fx	$\lvert x-\bar{x}\rvert$	$f\lvert x-\bar{x}\rvert$
6	3	18	2.12	6.36
7	5	35	1.12	5.6
8	7	56	0.12	0.84
9	6	54	0.88	5.28
10	4	40	1.88	7.52
	25	203		25.6

$$\bar{x} = \frac{203}{25} = 8.12$$

$$MD\ from\ \bar{x} = \frac{25.6}{25} = 1.024$$

$$CMD = \frac{1.024}{8.12} \times 100 = 12.61$$

MEAN DEVIATION

Example 3: Compute MD from median and coefficient of MD for the following:

CI	Frequency	cf	x	$\lvert x-M\rvert$	$f\lvert x-M\rvert$
0 - 10	5	5	5	26.67	133.35
10 - 20	10	15	15	16.67	166.7
20 - 30	12	27	25	6.67	80.04
30 - 40	18	45	35	3.33	59.94
40 - 50	9	54	45	13.33	119.97
50 - 60	6	60	55	23.33	139.98
	60				699.98

$M = 31.67$

$MD\ from\ M = \frac{699.98}{60}$

$= 11.67$

$\text{CMD} = \frac{11.67}{31.67} \times 100$

$= 36.85$

QUARTILE DEVIATION

- Quartile deviation is given by

$$QD = Q_3 - Q_1$$

- Coefficient of quartile deviation is given by

$$CQD = \frac{Q_3 - Q_1}{Q_3 + Q_1} \times 100$$

- Semi-Interquartile Range:

$$SQR = \frac{Q_3 - Q_1}{2}$$

QUARTILE DEVIATION

Example 1: Find quartile deviation and coefficient of quartile deviation for the following:

CI	Frequency	cf
0 - 10	5	5
10 - 20	10	15
20 - 30	12	27
30 - 40	18	45
40 - 50	9	54
50 - 60	6	60
	60	

QUARTILE DEVIATION

Example 2: Find quartile deviation and coefficient of quartile deviation for the following:

Rainfall in cms	No. of Cities	CF
0 - 10	10	10
10 - 20	15	25
20 - 30	20	45
30 - 40	10	55
40 - 50	5	60
	60	

QUARTILE DEVIATION

- Quartile deviation provides the best measure of dispersion for open-end classification.
- It is also less affected due to sampling fluctuations. Like other measures of dispersion.
- Quartile deviation remains unaffected due to a change of origin but is affected in the same ratio due to change in scale.

Example: If the quartile deviation of x is 6 and 3x + 6y = 20, what is the quartile deviation of y?

Answer: $QD(y) = 3$.

STANDARD DEVIATION

Although mean deviation is an improvement over range so far as a measure of dispersion is concerned, mean deviation is difficult to compute and further more, it cannot be treated mathematically. The best measure of dispersion is, usually, standard deviation which does not possess the demerits of range and mean deviation.

Standard deviation for a given set of observations is defined as the root mean square deviation when the deviations are taken from the AM of the observations.

STANDARD DEVIATION

- For raw data:

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{N}} = \sqrt{\frac{\sum x^2}{N} - \bar{x}^2}$$

- For frequency data:

$$s = \sqrt{\frac{\sum f(x-\bar{x})^2}{N}} = \sqrt{\frac{\sum fx^2}{N} - \bar{x}^2}$$

The square of standard deviation, known as variance, is regarded as a measure of dispersion.

STANDARD DEVIATION

A relative measure of dispersion using standard deviation is given by coefficient of variation (cv) which is defined as the ratio of standard deviation to the corresponding arithmetic mean, expressed as a percentage.

$$Coefficient\ of\ variation\ (CV) = \frac{SD}{AM} \times 100$$

STANDARD DEVIATION

Example 1: Find the standard deviation and the coefficient of variation for the following numbers:

12, 6, 7, 3, 15, 10, 18, 5

Solution:

$$\bar{x} = \frac{76}{8} = 9.5$$

$$s = \sqrt{\frac{912}{8} - 9.5^2} = 4.87$$

$$cv = \frac{4.87}{9.5} \times 100 = 51.26$$

x	x²
3	9
5	25
6	36
7	49
10	100
12	144
15	225
18	324
76	912

STANDARD DEVIATION

Example 2: Find the standard deviation and the coefficient of variation for the following:

Solution:

$$\bar{x} = \frac{1840}{60} = 30.67$$

$$s = \sqrt{\frac{68300}{60} - 30.67^2}$$

$$= 14.06$$

$$cv = \frac{14.06}{30.67} \times 100 = 45.84$$

CI	Frequency	x	fx	fx²
0 - 10	5	5	25	125
10 - 20	10	15	150	2250
20 - 30	12	25	300	7500
30 - 40	18	35	630	22050
40 - 50	9	45	405	18225
50 - 60	6	55	330	18150
	60		1840	68300

PROPERTIES OF STANDARD DEVIATION

1. If all the observations assumed by a variable are constant i.e. equal, then the SD is zero. This means that if all the values taken by a variable x is k, say, then s = 0. This result applies to range as well as mean deviation.
2. SD remains unaffected due to a change of origin but is affected in the same ratio due to a change of scale i.e., if there are two variables x and y related as y = a+bx for any two constants a and b, then SD of y is given by

$$s_y = |b| s_x$$

PROPERTIES OF STANDARD DEVIATION

3. If there are two groups containing n_1 and n_2 observations, $\overline{x_1}$ and $\overline{x_2}$ as respective AMs, s_1 and s_2 are respective SDs, then combined SD is given by

$$s = \sqrt{\frac{n_1({s_1}^2 + {d_1}^2) + n_2({s_2}^2 + {d_2}^2)}{n_1 + n_2}}$$

$$where, d_1 = \bar{x} - \overline{x_1}$$

$$d_2 = \bar{x} - \overline{x_2}$$

$$and\ \bar{x} = \frac{n_1\overline{x_1} + n_2\overline{x_2}}{n_1 + n_2}$$

STANDARD DEVIATION (CODING METHOD EXAMPLE)

Example 3: Find the standard deviation for the following:

Height (in)	Frequency	x	$u = \frac{x-67}{3}$	fu	fu²
60 - 62	5	61	-2	-10	20
63 - 65	18	64	-1	-18	18
66 - 68	42	67	0	0	0
69 - 71	27	70	1	27	27
72 - 74	8	73	2	16	32
	100			15	97

STANDARD DEVIATION (CODING METHOD EXAMPLE)

Example 3: Continue…

$$s_u = \sqrt{\frac{\sum fu^2}{N} - \bar{u}^2}$$

$$s_x = \sqrt{\frac{\sum fu^2}{N} - \bar{u}^2} \times C$$

$$s_x = \sqrt{\frac{97}{100} - \left(\frac{15}{100}\right)^2} \times 3 = 2.92$$

STANDARD DEVIATION

Example 4: If AM and coefficient of variation of x are 10 and 40 respectively, what is the variance of (15–2x)?

Solution: Let $y = 15 - 2x \quad \Longrightarrow s_y = 2s_x$

$$CV(x) = \frac{s_x}{\bar{x}} \times 100$$

$$40 = \frac{s_x}{10} \times 100$$

$$\therefore\ s_x = 4 \qquad \Longrightarrow s_y = 2(4) = 8$$

$$\therefore var(y) = 8^2 = 64$$

STANDARD DEVIATION

Example 5: For a group of 50 male workers the mean and standard deviation of their daily wages are 63 dollars and 9 dollars respectively. For a group of 40 female workers these values are 54 dollars and 6 dollars respectively. Find the mean and variance of the combined group of 90 workers.

Solution:

Here $n_1 = 50, \overline{x_1} = 63, s_1 = 9$ and $n_2 = 40, \overline{x_2} = 54, s_2 = 6$

$$\bar{x} = \frac{50 \times 63 + 40 \times 54}{50 + 40} = 59$$

$$d_1 = -4, d_2 = 5$$

$$s = \sqrt{\frac{50(81+16) + 40(36+25)}{50+40}} = 9 \qquad \Longrightarrow s^2 = variance = \$81$$

STANDARD DEVIATION

Example 6: For a group of 50 male workers the mean and standard deviation of their daily wages are 63 dollars and 9 dollars respectively. For a group of 40 female workers these values are 54 dollars and 6 dollars respectively. Examine consistency of both the group.

Solution:

Here $n_1 = 50, \overline{x_1} = 63, s_1 = 9$ and $n_2 = 40, \overline{x_2} = 54, s_2 = 6$

$$CV(M) = \frac{9}{63} \times 100 = 14.29$$

$$CV(F) = \frac{6}{54} \times 100 = 11.11$$

Therefore, female group has more consistent wage.

BASIC QUESTIONS

Q.1 Which measure of dispersion is most useful?

Ans: Standard deviation

Q.2 Which measure of dispersion is the easiest to calculate?

Ans: Range

Q.3 If $3x + 2y = 40$ and range(x) = 12 then range(y) = ?

Ans: $y = \frac{40}{2} - \frac{3x}{2} \quad \Longrightarrow range(y) = \frac{3}{2} range(x) = \frac{3}{2} \times 12 = 18.$

BASIC QUESTIONS

Q.4 If $3x - 4y = 20$ and MD(x) = 8 then MD(y) = ?

Ans: $y = \frac{3x}{4} - \frac{20}{4} \quad \Rightarrow MD(y) = \frac{3}{4} MD(x) = \frac{3}{4} \times 8 = 6.$

Q.5 If $3x + y = 12$ and QD(x) = 15 then QD(y) = ?

Ans: $y = 12 - 3x \quad \Rightarrow QD(y) = 3\, QD(x) = 3 \times 15 = 45.$

Q.6 If $2x + y = 10$ and SD(x) = 4 then SD(y) = ?

Ans: $y = 10 - 2x \quad \Rightarrow SD(y) = 2\, SD(x) = 2 \times 4 = 8.$

BASIC QUESTIONS

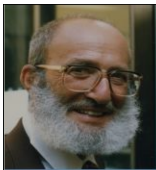
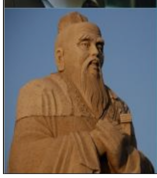
Q.7 If $4x - 3y = 15$ and Var(x) = 3 then Var(y) = ?

Ans: $y = \frac{4x}{3} - \frac{15}{3} \quad \Rightarrow SD(y) = \frac{4}{3} SD(x)$

$$\Rightarrow Var(y) = \left(\frac{4}{3}\right)^2 \times Var(x) = \frac{16}{9} \times 3 = 5.33.$$

Q.8 If mean = 20 and variance = 4 then coefficient of variation = ?

Ans: Coefficient of Variation $= \frac{SD}{Mean} \times 100 = \frac{2}{20} \times 100 = 10\%.$

The only way to learn mathematics is to do mathematics.
— Paul Halmos —

MOMENTS, SKEWNESS and KURTOSIS

I hear and I forget.
I see and I remember.
I do and I understand.
— Confucious —

MOMENTS

For Raw Data:

r^{th} moments about A is defined as,

$$m'_r = \overline{(X-A)^r} = \frac{\sum (X-A)^r}{N}$$

MOMENTS

For Raw Data:

r^{th} moments about $A = 0$ is defined as,

$$m'_r = \overline{X^r} = \frac{\sum X^r}{N}$$

First four moments about $A = 0$,

$$m'_1 = \bar{X} = \frac{\sum X}{N}$$
$$m'_2 = \overline{X^2} = \frac{\sum X^2}{N}$$
$$m'_3 = \overline{X^3} = \frac{\sum X^3}{N}$$
$$m'_4 = \overline{X^4} = \frac{\sum X^4}{N}$$

MOMENTS

For Raw Data:

r^{th} moments about $A = \bar{X}$ is defined as,

$$m_r = \overline{(X-\bar{X})^r} = \frac{\sum (X-\bar{X})^r}{N}$$

First four moments about $A = \bar{X}$,

$$m_1 = \overline{(X-\bar{X})} = \frac{\sum (X-\bar{X})}{N}$$
$$m_2 = \overline{(X-\bar{X})^2} = \frac{\sum (X-\bar{X})^2}{N}$$
$$m_3 = \overline{(X-\bar{X})^3} = \frac{\sum (X-\bar{X})^3}{N}$$
$$m_4 = \overline{(X-\bar{X})^4} = \frac{\sum (X-\bar{X})^4}{N}$$

MOMENTS

For Frequency Data:

r^{th} moments about A is defined as,

$$m'_r = \overline{(X-A)^r} = \frac{\sum f(X-A)^r}{N}$$

MOMENTS

For Frequency Data:

r^{th} moments about $A = 0$ is defined as,

$$m'_r = \overline{X^r} = \frac{\sum fX^r}{N}$$

First four moments about $A = 0$,

$$m'_1 = \bar{X} = \frac{\sum fX}{N}$$
$$m'_2 = \overline{X^2} = \frac{\sum fX^2}{N}$$
$$m'_3 = \overline{X^3} = \frac{\sum fX^3}{N}$$
$$m'_4 = \overline{X^4} = \frac{\sum fX^4}{N}$$

MOMENTS

For Frequency Data:

r^{th} moments about $A = \bar{X}$ is defined as,

$$m_r = \overline{(X-\bar{X})^r} = \frac{\sum f(X-\bar{X})^r}{N}$$

First four moments about $A = \bar{X}$,

$$m_1 = \overline{(X-\bar{X})} = \frac{\sum f(X-\bar{X})}{N}$$

$$m_2 = \overline{(X-\bar{X})^2} = \frac{\sum f(X-\bar{X})^2}{N}$$

$$m_3 = \overline{(X-\bar{X})^3} = \frac{\sum f(X-\bar{X})^3}{N}$$

$$m_4 = \overline{(X-\bar{X})^4} = \frac{\sum f(X-\bar{X})^4}{N}$$

MOMENTS

Note:

1. First moment about $A = 0$, $m_1' = \bar{X} = AM$.
2. First moment about $A = \bar{X}$, $m_1 = \overline{(X-\bar{X})} = 0$.
3. Second moment about $A = \bar{X}$, $m_2 = \overline{(X-\bar{X})^2} = Variance$.

RELATION BETWEEN MOMENTS

Let m_r denotes moments about the mean and m_r' denotes moments about an arbitrary origin.

$$m_2 = m_2' - m_1'^2$$

$$m_3 = m_3' - 3m_1'm_2' + 2m_1'^3$$

$$m_4 = m_4' - 4m_1'm_3' + 6m_1'^2 m_2' - 3m_1'^4$$

Note that $m_1' = \bar{X} - A$.

MOMENTS

Find the first four moments for the set 4, 7, 5, 9, 8, 3, 6.

X	X²	X³	X⁴
4	16	64	256
7	49	343	2401
5	25	125	625
9	81	729	6561
8	64	512	4096
3	9	27	81
6	36	216	1296
42	280	2016	15316

$$First\ Moment = m_1' = \frac{\sum X}{N} = \frac{42}{7} = 6$$

$$Second\ Moment = m_2' = \frac{\sum X^2}{N} = \frac{280}{7} = 40$$

$$Third\ Moment = m_3' = \frac{\sum X^3}{N} = \frac{2016}{7} = 288$$

$$Fourth\ Moment = m_4' = \frac{\sum X^4}{N} = \frac{15316}{7} = 2188$$

MOMENTS

Find the first four moments about mean for the set 4, 7, 5, 9, 8, 3, 6.

X	$(X-\bar{X})$	$(X-\bar{X})^2$	$(X-\bar{X})^3$	$(X-\bar{X})^4$
4	-2	4	-8	16
7	1	1	1	1
5	-1	1	-1	1
9	3	9	27	81
8	2	4	8	16
3	-3	9	-27	81
6	0	0	0	0
42	0	28	0	196

$\bar{X} = 6$

$$First\ Moment = m_1 = \frac{\sum(X-\bar{X})}{N} = 0$$

$$Second\ Moment = m_2 = \frac{\sum(X-\bar{X})^2}{N} = \frac{28}{7} = 4$$

$$Third\ Moment = m_3 = \frac{\sum(X-\bar{X})^3}{N} = 0$$

$$Fourth\ Moment = m_4 = \frac{\sum(X-\bar{X})^4}{N} = \frac{196}{7} = 28$$

MOMENTS

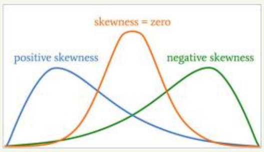
Find the first four moments about 7 for the set 4, 7, 5, 9, 8, 3, 6.

X	X - 7	(X - 7)²	(X - 7)³	(X - 7)⁴
4	-3	9	-27	81
7	0	0	0	0
5	-2	4	-8	16
9	2	4	8	16
8	1	1	1	1
3	-4	16	-64	256
6	-1	1	-1	1
	-7	35	-91	371

SKEWNESS

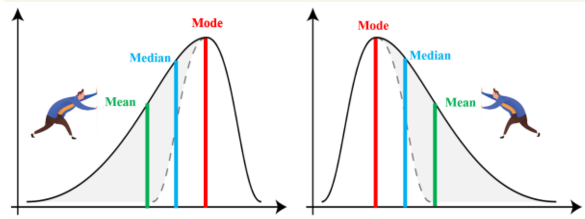
Skewness is the degree of asymmetry, or departure from asymmetry, of a distribution.

If the frequency curve of a distribution has a longer tail to the right of the central maximum than to the left, the distribution is said to be skewed to the right, or to have positive skewness. If the reverse is true, it is said to be skewed to the left, or to have negative skewness.



SKEWNESS

For skewed distributions, the mean tends to lie on the same side of the mode as the longer tail.



SKEWNESS

Pearson's First and Second Coefficient of Skewness:

$$Skewness = \frac{mean - mode}{Standard\ Deviation}$$

$$Skewness = \frac{3(mean - median)}{Standard\ Deviation}$$

Other measures of Skewness,

$$Quartile\ Coefficient\ of\ Skewness = \frac{Q_3 - 2Q_2 + Q_1}{Q_3 - Q_1}$$

$$10 - 90\ Percentile\ Coefficient\ of\ Skewness = \frac{P_{90} - 2P_{50} + P_{10}}{P_{90} - P_{10}}$$

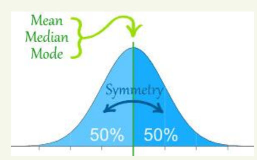
$$Moment\ Coefficient\ of\ Skewness = a_3 = \frac{m_3}{\left(\sqrt{m_2}\right)^3}$$

SKEWNESS

Another measure of skewness is given by

$$b_1 = \beta_1 = a_3^2 = \frac{m_3^2}{m_2^3}$$

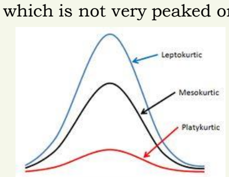
For perfectly symmetric curves, such as normal curve, a_3 and β_1 are zero.



KURTOSIS

Kurtosis is the degree of peakedness of a distribution, usually taken relative to a normal distribution.

A distribution having a relatively high peak is called leptokurtic, while one which is flat-topped is called platykurtic. A normal distribution, which is not very peaked or very flat-topped, is called mesokurtic.



KURTOSIS

Measures of Kurtosis,

$$Moment\ Coefficient\ of\ Kurtosis = a_4 = \beta_2 = \frac{m_4}{m_2^2}$$

For the normal distribution, $Moment\ Coefficient\ of\ Kurtosis = 3$.

So $Moment\ Coefficient\ of\ Kurtosis > 3$, for leptokurtic distributions and $Moment\ Coefficient\ of\ Kurtosis < 3$, for platykurtic distributions.

Another measure of kurtosis is given by,

$$\kappa = \frac{Semi - Interquartile\ Range}{10 - 90\ Percentile\ Range} = \frac{(Q_3 - Q_1)/2}{P_{90} - P_{10}}$$

For the normal distribution, $\kappa = 0.263$.

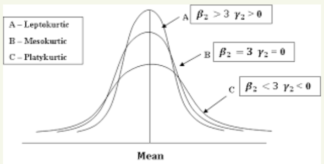
So $\kappa > 0.263$, for leptokurtic distributions and $\kappa < 0.263$, for platykurtic distributions.

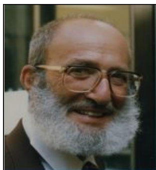
KURTOSIS

The Kurtosis is sometimes defined by

$$\gamma_2 = \beta_2 - 3$$

Which is positive for leptokurtic distribution, negative for platykurtic distribution and zero for normal distribution.





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SET THEORY

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I see and I remember.
I do and I understand.
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SET

Definition: A set is any well defined collection of distinct objects.

Notation: Usually sets are denoted by capital letters and the elements (or members) of the sets are denoted by small letters.

Examples:

$$A = \{a, e, i, o, u\}$$
$$B = \{2,4,6,8\}$$

SET

$\in$ *and* $\notin$ Notation:

- If A is a set, the notation $x \in A$ means that x is a member of set A.
- The notation $x \notin A$ means that x is not a member of set A.

Example: Let $A = \{2,4,6,8\}$,
then members of A are: $2, 4, 6$ *and* 8.
And we can say $2 \in A, 6 \in A$ etc.
But since 3 is not the member of set A, we can say $3 \notin A$.

SET ROSTER NOTATION

A set may be specified using the set-roster notation by writing all of its elements between braces.

Examples:

$$A = \{2,4,6,8\}$$

$$B = \{1,3,5, \dots \dots \dots, 99\}$$

$$C = \{1,2,3, \dots \dots \dots\}$$

BASIC SETS

$\mathbb{Z} = \textit{the set of integers} = \{\dots \dots \dots, -3, -2, -1, 0, 1, 2, 3, \dots \dots \dots\}$

$\mathbb{N} = \textit{the set of natural numbers} = \{1,2,3, \dots \dots \dots\}$
$= \textit{the set positive integers} = \mathbb{Z}^+$

$\mathbb{W} = \textit{the set of whole numbers} = \{0,1,2,3, \dots \dots \dots\}$

$\mathbb{R} = \textit{the set of real numbers}$

$\mathbb{R}^+ = \textit{the set of positive real numbers}$

SET BUILDER NOTATION

The fact that the element x of the set A having a property P is described as $P(x)$ and the set A in set builder notation is written as

$$A = \{x \mid P(x)\}$$

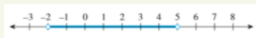
Examples:

$A = \{2,4,6,8\}$
$\therefore\ A = \{x \mid x \textit{ is an even integer and } 2 \le x \le 8\}$
$B = \{1,3,5, \dots \dots \dots, 99\}$
$\therefore\ B = \{x \mid x \textit{ is an odd integer and } 1 \le x \le 99\}$

SET BUILDER NOTATION

Describe each of the following sets:

a) $\{x \in \mathbb{R} \mid -2 < x < 5\}$



b) $\{x \in \mathbb{Z} \mid -2 < x < 5\}$
c) $\{x \in \mathbb{Z}^+ \mid -2 < x < 5\}$
d) $\{x \in \mathbb{R}^+ \mid 0 < x < 1\}$
e) $\{x \in \mathbb{R} \mid 2 \leq x < 7\}$
f) $\{n \in \mathbb{Z} \mid n\ is\ a\ factor\ of\ 6\}$
g) $\{n \in \mathbb{Z}^+ \mid n\ is\ a\ factor\ of\ 6\}$
h) $\{n \in \mathbb{Z} \mid n = (-1)^k, for\ some\ integer\ k\}$
i) $\{m \in \mathbb{Z} \mid m = 1 + (-1)^i, for\ some\ integer\ i\}$
j) $\{r \in \mathbb{Z} \mid 2 \leq r \leq -2\}$
k) $\{s \in \mathbb{Z} \mid s \leq 4\ or\ s \geq 1\}$

EMPTY SET

A set without any element is called an empty set.
It is denoted by ϕ or $\{\ \}$.

Examples:

1. Let $X = \{x \mid x \in \mathbb{Z}\ and\ 2x = 5\}$.
Here $2x = 5 \Longrightarrow x = 2.5$ but 2.5 is not an integer.
$\therefore X = \{\ \}$

2. Let $A = \{x \mid x \in \mathbb{N}\ and\ x^2 + 9x + 18 = 0\}$
Here $x^2 + 9x + 18 = 0 \Longrightarrow x = -3, -6$ but $-3\ and\ -6$ are not the natural numbers.
$\therefore A = \phi$

Note: $\phi \subseteq A$, for any set A.

BASIC TERMS

Singleton Set: A set with only one element is called a singleton set.

Example:

$A = \{5\}$

$B = \{x \mid x \in \mathbb{N}\ and\ x^2 - 7x - 18 = 0\}$
$= \{9\}$

Doubleton Set: A set with only two elements is called a doubleton set.

Example:

$A = \{x \mid x^2 - 9x + 18 = 0\} = \{3,6\}$

BASIC TERMS

Finite Set: A set with n distinct elements is called a finite set.

Example:

$A = \{100, 2000, 30000, 400000\}$

$B = \{x \mid x \in \mathbb{N}\ and\ 16 \leq x < 100\}$
$= \{16, 17, 18, \ldots \ldots \ldots, 99\}$

Infinite Set: A set which is not finite is called an infinite set.

Example:

$A = \mathbb{N}$
$B = \{x \mid x \in \mathbb{R}\ and\ 0 < x < 1\}$

SUBSET

If A and B are sets, then A is called a subset of B, written $A \subseteq B$, if, and only if, every element of A is also an element of B.

Examples:

1. Let $X = \{2,4\}$ and $Y = \{1,2,3,4\}$.
Then $X \subseteq Y$, since every element of X is also an element of Y.

2. Let $A = \{x \mid x^2 - 9x + 18 = 0\}$ and $B = \{0,3,6,9,12\}$.
A in set roster notation, $A = \{3,6\}$
Then $A \subseteq B$, since every element of A is also an element of B.

SUBSET

$A \nsubseteq B$

means that there is at least one element x such that $x \in A$ but $x \notin B$.

Examples:

1. Let $X = \{2,4,6\}$ and $Y = \{1,2,3,4\}$.
Here $X \nsubseteq Y$, since $6 \in X$ but $6 \notin Y$.

2. Let $A = \{x \mid x^2 - 3x - 18 = 0\}$ and $B = \{0,3,6,9,12\}$.
A in set roster notation, $A = \{-3,6\}$
Then $A \nsubseteq B$, since $-3 \in A$ but $-3 \notin B$.

PROPER SUBSET

Let A and B be sets. A is proper subset of B if, and only if, every element of A is in B but there is at least one element of B that is not in A.

A subset which is not proper is called improper subset.

Examples:

1. Let $X = \{2,4\}$ and $Y = \{1,2,3,4\}$.
Here X is proper subset of Y, i.e. $X \subsetneq Y$.
2. Let $A = \{x \mid x^2 - 9x + 18 = 0\}$ and $B = \{0,3,6,9,12\}$.
A in set roster notation, $A = \{3,6\}$.
Here A is proper subset of B, i.e. $A \subsetneq B$.
3. Let $C = \{x \mid x^2 - 9 = 0\}$ and $D = \{-3,3\}$.
C in set roster notation, $C = \{-3,3\}$.
Here C is improper subset of D.

EQUAL SETS

The two sets A and B are said to be equal if they have same elements and we write $A = B$.

Example:

Let $A = \{2,5,3\}$ and $B = \{2,3,2,5\}$.

Here we can rewrite B as $\{2,3,5\}$ i.e. $B = \{2,3,5\}$.

$\therefore A = B$

Note: Equality of two sets A and B can also be defined as,

$$A = B \; iff \; A \subseteq B \text{ and } B \subseteq A$$

PROBLEMS

Q.1 How many elements are there in the set $A = \{1, 1, 2, 3, 3, 3\}$?

Q.2 Is $\{0\} = 0$?

Q.3 How many elements are there in the set $\{1, \{1\}\}$? List them.

PROBLEMS

Q.4 How many elements are there in the set$\{1, \{1\}, \{1,2\}\}$? List them.

Q.5 Let $A = \{1, 2, 3\}, B = \{3, 1, 2\}$ and $C = \{1, 1, 2, 3, 3, 3\}$. How A, B and C are related?

Q.6 For each non-negative n, let $U_n = \{-n, n\}$. Find U_1, U_2 and U_0.

PROBLEMS

Q.7 Answer each of the following:

a) Is $3 \in \{1,2,3\}$?
b) Is $1 \subseteq \{1\}$?
c) Is $\{2\} \in \{1,2\}$?
d) Is $\{3\} \in \{1, \{2\}, \{3\}\}$?
e) Is $1 \in \{1\}$?
f) Is $\{3\} \subseteq \{1, \{2\}, \{3\}\}$?
g) Is $\{1\} \subseteq \{1,2\}$?
h) Is $1 \in \{\{1\}, 2\}$?
i) Is $\{1\} \subseteq \{1, \{2\}\}$?

PROBLEMS

Q.8 Let $A = \{c, d, f, g\}, B = \{f, j\}$ and $C = \{d, g\}$.

Answer each of the following with justification:

a) Is $B \subseteq A$?
b) Is $C \subseteq A$?
c) Is $C \subseteq C$?
d) Is C a proper subset of A?

UNIVERSAL SET

Any larger set which contains all other sets is called an universal set.

It is denoted by U.

Examples:

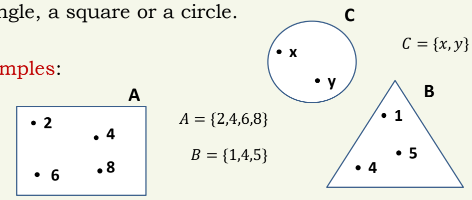
1. For $\mathbb{N}, \mathbb{W}, \mathbb{Z}$,
   $\mathbb{R}$ *or* $\mathbb{C}$ can be universal set.
2. For $A = \{1, 3, 5, 7\}, B = \{2, 4, 6\}$ and $C = \{1, 2, 5, 6, 9, 10\}$
   $U = \{1,2,3, \dots \dots \dots, 10\}$ can be universal set.

VENN DIAGRAM

A pictorial representation of a set is called as Venn diagram.

Elements of a set are denoted by dots enclosed in a triangle, a square or a circle.

Examples:

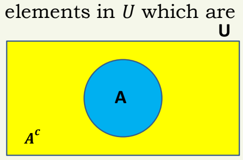


COMPLEMENT OF A SET

The complement of a set A has elements in U which are not in A.

It is denoted by A' *or* A^c *or* $\bar{A}$.

i.e. $A^c = \{x \mid x \in U \text{ and } x \notin A\}$



Example:

Let $U = \{1, 2, 3, \dots \dots \dots, 10\}$ and $A = \{1, 3, 5, 7\}$.

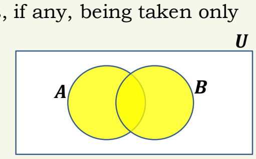
Then $A^c = \{2, 4, 6, 8, 9, 10\}$.

UNION OF SETS

The union of two sets A and B is the set of all elements of A and B, the common elements, if any, being taken only once.

It is denoted by $A \cup B$.

i.e. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



Example:

Let $A = \{1,2,4,5\}$ and $B = \{3,4,5,8,9\}$.

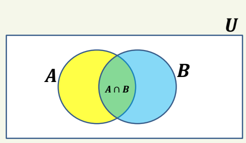
Then $A \cup B = \{1,2,3,4,5,8,9\}$.

INTERSECTION OF SETS

The intersection of two sets A and B is the set of all common elements of A and B.

It is denoted by $A \cap B$.

i.e. $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



Example:

Let $A = \{1,2,4,5\}$ and $B = \{3,4,5,8,9\}$.

Then $A \cap B = \{4,5\}$.

DISJOINT SETS

The two sets A and B are disjoint if they don't have any common element.

i.e. if $A \cap B = \phi$

Example:

Let $A = \{0,2,4,6\}$ and $B = \{1,3,5\}$.

Then $A \cap B = \{\}$.

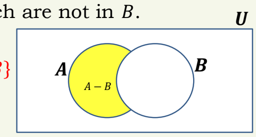
SET DIFFERENCE

The relative complement of A with respect to B is the set of all those elements of A which are not in B.

It is denoted by $A - B$ or A/B.

i.e. $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

$B - A = \{x \mid x \in B \text{ and } x \notin A\}$



Example:

Let $A = \{1,2,4,5\}$ and $B = \{3,4,5,8,9\}$.

Then $A - B = \{1,2\}$

and $B - A = \{3,8,9\}$.

PROBLEM

Ex.1 For $U = \{1,2,3, \ldots \ldots ,10\}$, let $A = \{1,2,3,4,5\}$, $B = \{1,2,4,8\}$, $C = \{1,2,3,5,7\}$ and $D = \{2,4,6,8\}$. Determine each of the following :

(i) $C' \cup D'$ (ii) $(C \cap D)'$ (iii) $A \cup (B/C)$

(iv) $(A \cup B)/(C \cap D)$

INCLUSION EXCLUSION PRINCIPLE

For two sets:

Let A and B be any two finite sets, then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

For three sets:

Let A, B and C be any three finite sets, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

INCLUSION EXCLUSION PRINCIPLE

Q.1 In a class of 50 students, 25 like Maths and 15 like Programming, 10 like Maths and Programming. Find

a) How many like at least one of them?

b) How many do not any of the subjects?

INCLUSION EXCLUSION PRINCIPLE

Q.2 Out of 150 residents of a building, 105 speak Marathi, 75 speak Gujarati and 45 speak both the languages. Find the number of residents who do not speak either of the languages also find the number of residents who speak only Marathi.

INCLUSION EXCLUSION PRINCIPLE

Q.3 In a survey of people it was found that 80 people watch football, 60 watch cricket, 50 watch hockey, 30 watch football and cricket, 20 watch football and hockey, 15 watch cricket and hockey and 10 watch all three games.

a) How many people watch at least one game?

b) How many people watch only cricket?

c) How many people watch football and cricket but not hockey?

INCLUSION EXCLUSION PRINCIPLE

Q.4 In a survey of 120 people, it was found that:
65 read Newsweek magazine, 45 read Time, 42 read Fortune,
20 read both Newsweek and Time, 25 read both Newsweek and Fortune,
15 read both Time and Fortune, 8 read all the three magazines.

(a) Find the number of people who read at least one of the three magazines.
(b) Find the number of people who read exactly one magazine.

PROPERTIES OF SETS

Theorem 6.2.1 Some Subset Relations

1. *Inclusion of Intersection:* For all sets A and B,

   (a) $A \cap B \subseteq A$ and (b) $A \cap B \subseteq B$.

2. *Inclusion in Union:* For all sets A and B,

   (a) $A \subseteq A \cup B$ and (b) $B \subseteq A \cup B$.

3. *Transitive Property of Subsets:* For all sets A, B, and C,

   if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

PROPERTIES OF SETS

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

   (a) $A \cup B = B \cup A$ and (b) $A \cap B = B \cap A$.

2. *Associative Laws:* For all sets A, B, and C,

   (a) $(A \cup B) \cup C = A \cup (B \cup C)$ and
   (b) $(A \cap B) \cap C = A \cap (B \cap C)$.

3. *Distributive Laws:* For all sets, A, B, and C,

   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

4. *Identity Laws:* For all sets A,

   (a) $A \cup \emptyset = A$ and (b) $A \cap U = A$.

PROPERTIES OF SETS

5. *Complement Laws:*

   (a) $A \cup A^c = U$ and (b) $A \cap A^c = \emptyset$.

6. *Double Complement Law:* For all sets A,

   $(A^c)^c = A$.

7. *Idempotent Laws:* For all sets A,

   (a) $A \cup A = A$ and (b) $A \cap A = A$.

8. *Universal Bound Laws:* For all sets A,

   (a) $A \cup U = U$ and (b) $A \cap \emptyset = \emptyset$.

9. *De Morgan's Laws:* For all sets A and B,

   (a) $(A \cup B)^c = A^c \cap B^c$ and (b) $(A \cap B)^c = A^c \cup B^c$.

PROPERTIES OF SETS

10. *Absorption Laws:* For all sets A and B,

    (a) $A \cup (A \cap B) = A$ and (b) $A \cap (A \cup B) = A$.

11. *Complements of U and $\emptyset$:*

    (a) $U^c = \emptyset$ and (b) $\emptyset^c = U$.

12. *Set Difference Law:* For all sets A and B,

    $A - B = A \cap B^c$.

PROPERTIES OF SETS

Q.1 Let $U = \{x \mid x \in \mathbb{N} \text{ and } x \leq 15\}, A = \{1,3,4,5,9\}$ and $B = \{3,5,7,9,12\}$.
State and verify De Morgan's laws.

Solution: Here $U = \{1,2,3, \dots \dots \dots, 15\}, A = \{1,3,4,5,9\}$ and $B = \{3,5,7,9,12\}$.

De Morgan's laws: i) $(A \cup B)^c = A^c \cap B^c$
ii) $(A \cap B)^c = A^c \cup B^c$

Verification of $(A \cup B)^c = A^c \cap B^c$:

$A \cup B = \{1,3,4,5,7,9,12\}$

$\therefore (A \cup B)^c = \{2,6,8,10,11,13,14,15\}$ ……… (I)

$A^c = \{2,6,7,8,10,11,12,13,14,15\}$ and $B^c = \{1,2,4,6,8,10,11,13,14,15\}$

$\therefore A^c \cap B^c = \{2,6,8,10,11,13,14,15\}$ ……… (II)

By (I) and (II), $(A \cup B)^c = A^c \cap B^c$

Hence verified

PROPERTIES OF SETS

Solution Cont: Here $U = \{1,2,3, \dots\dots\dots, 15\}, A = \{1,3,4,5,9\}$ and $B = \{3,5,7,9,12\}$.

Verification of $(A \cap B)^c = A^c \cup B^c$:

$A \cap B = \{3,5,9\}$

$\therefore (A \cap B)^c = \{1,2,4,6,7,8,10,11,12,13,14,15\}$ … … … (III)

$A^c = \{2,6,7,8,10,11,12,13,14,15\}$ and $B^c = \{1,2,4,6,8,10,11,13,14,15\}$

$\therefore A^c \cup B^c = \{1,2,4,6,7,8,10,11,12,13,14,15\}$ … … … (IV)

By (III) and (IV), $(A \cap B)^c = A^c \cup B^c$

Hence verified

PROPERTIES OF SETS

Q.2 Let $A = \{1,3,4,5,9\}, B = \{3,5,7,9,12\}$ and $C = \{2,3,5,6\}$.

State and verify Distributive laws.

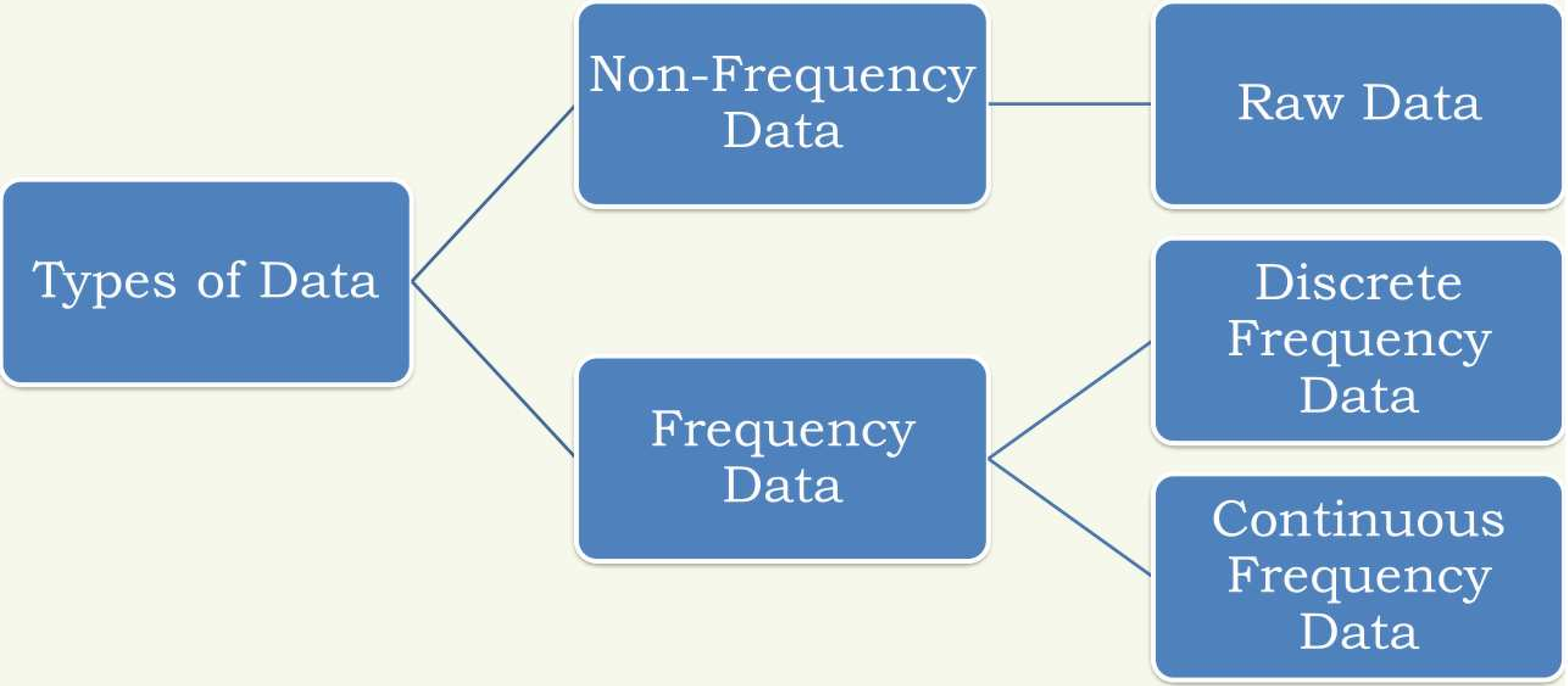
The only way to learn mathematics is to do mathematics.
— Paul Halmos —

MEASURES OF CENTRAL TENDANCY

I hear and I forget.
I see and I remember.
I do and I understand.
— Confucious —

AjSolutions

BASICS


Types of Data
Non-Frequency Data
Frequency Data
Raw Data
Discrete Frequency Data
Continuous Frequency Data

BASICS

1. Non-Frequency Data or Raw Data:

$$x_1, x_2, x_3, \dots \dots \dots, x_n$$

Example:

1. $2, 5, 26, 13, 11, 15, 10$

2. The grades of a student on six examinations were 84, 91, 72, 68, 87, and 78.

3. A review of the first 30 pages of a book reveals the following printing mistakes :

0	1	3	3	2	5	6	0	1	0
4	1	1	0	2	3	2	5	0	4
2	3	2	2	3	3	4	6	1	4

BASICS

2. Frequency Data:

i. Discrete Frequency Data:

OR

Ungrouped Frequency Data:

Class	Frequency
x_1	f_1
x_2	f_2
x_3	f_3
.	.
.	.
.	.
x_n	f_n

Example:

1.

Size	Frequency
8	14
9	21
10	17
11	12
12	6

BASICS

AjSolutions

2. Frequency Data:

i. Discrete Frequency Data:

OR

Ungrouped Frequency Data:

Class	Frequency
x_1	f_1
x_2	f_2
x_3	f_3
.	.
.	.
.	.
x_n	f_n

Example:

2. A review of the first 30 pages of a book reveals the following printing mistakes :

0	1	3	3	2	5	6	0	1	0
4	1	1	0	2	3	2	5	0	4
2	3	2	2	3	3	4	6	1	4

Printing Mistake	Tally marks	Frequency (No. of Pages)
0	~~IIII~~	5
1	~~IIII~~	5
2	~~IIII~~ I	6
3	~~IIII~~ I	6
4	IIII	4
5	II	2
6	II	2
Total	–	30

BASICS

2. Frequency Data:

ii. Continuous Frequency Data:

OR

Grouped Frequency Data:

a. Inclusive Type Class Interval Grouped Frequency Data:

b. Exclusive Type Class Interval Grouped Frequency Data:

Height (in)	Frequency
60 - 62	5
63 - 65	18
66 - 68	42
69 - 71	27
72 - 74	8
	100

CI	Frequency
0 - 10	5
10 - 20	10
20 - 30	12
30 - 40	18
40 - 50	9
50 - 60	6

BASICS

Continuous Frequency Data:

Following are the weights in kgs. of 36 students of a class:

70	73	49	61	61	47	57	50	59
59	68	45	55	65	68	56	68	55
70	70	57	44	69	73	64	49	63
65	70	65	62	64	73	67	60	50

Solution 1 with Inclusive type Class Interval

Weight in kg (Class Interval)	Tally marks	No. of Students (Frequency)
44-48	III	3
49-53	IIII	4
54-58	卌	5
59-63	卌 II	7
64-68	卌 IIII	9
69-73	卌 III	8
Total	–	36

Solution 2 with Exclusive type Class Interval

Weight in kg (Class interval)	No. of Students (Frequency)
44 – 49	3
49 – 54	4
54 – 59	5
59 – 64	7
64 – 69	9
69 – 74	8
Total	36

BASICS

Continuous Frequency Data:

Some Important Terms:

i. Class Limits (CL)

ii. Class Boundaries (CB)

iii. Mid-Point or Class Mark (CM)

iv. Class Width or Size of a Class Interval

v. Cumulative Frequency

Weight in kg (Class interval)	No. of Students (Frequency)
43.5 – 48.5	3
48.5 – 53.5	4

Weight in kg (Class Interval)	No. of Students (Frequency)
44-48	3
49-53	4
54-58	5
59-63	7
64-68	9
69-73	8
Total	36

BASICS

Continuous Frequency Data:

Conversion of Class Intervals From Inclusive Type to Exclusive Type:

Step 1: Compute D = LLCI – ULPCI

Step 2: Compute D/2

Step 3: Subtract D/2 from each LL
and add D/2 to each UL.

Resultant class intervals will be exclusive type.

Weight in kg (Class Interval)	No. of Students (Frequency)
44-48	3
49-53	4
54-58	5
59-63	7
64-68	9
69-73	8
Total	36

INTRODUCTION

In many a case, like the distributions of height, weight, marks, profit, wage and so on, it has been noted that starting with rather low frequency, the class frequency gradually increases till it reaches its maximum somewhere near the central part of the distribution and after which the class frequency steadily falls to its minimum value towards the end.

Thus, central tendency may be defined as the tendency of a given set of observations to cluster around a single central or middle value and the single value that represents the given set of observations is described as a measure of central tendency or, location, or average.

INTRODUCTION

Hence, it is possible to condense a vast mass of data by a single representative value. The computation of a measure of central tendency plays a very important part in many a sphere.

Furthermore, the central tendency also facilitates us in providing a basis for comparison between different distribution. Following are the different measures of central tendency:

i. Arithmetic Mean (AM)

ii. Median (Me)

iii. Mode (Mo)

ARITHMETIC MEAN

- For Raw Data: $\bar{x} = \frac{\sum x_i}{n}$

- For Frequency Data: $\bar{x} = \frac{\sum f_i x_i}{N}$ *where* $N = \sum f_i$

  In case of grouped data x_i is mid point of each class interval.

- Coding Method: $\bar{x} = A + \frac{\sum f_i u_i}{N} \times C$

  where, $u_i = \frac{x_i - A}{C}$

ARITHMETIC MEAN

Example 1: The grades of a student on six examinations were 84, 91, 72, 68, 87, and 78. Find the arithmetic mean of the grades.

Solution:

Formula: $\bar{x} = \frac{\sum x_i}{n}$

$$\therefore AM\ of\ the\ grades = \frac{84 + 91 + 72 + 68 + 87 + 78}{6}$$

$$= \frac{480}{6} = 80$$

ARITHMETIC MEAN

Example 2: Find the arithmetic mean of the numbers

5, 3, 6, 5, 4, 5, 2, 8, 6, 5, 4, 8, 3, 4, 5, 4, 8, 2, 5, and 4.

Solution:

Formula: $\bar{x} = \frac{\sum f_i x_i}{N}$

$\therefore AM = \frac{96}{20} = 4.8$

x	f	fx
2	2	4
3	2	6
4	5	20
5	6	30
6	2	12
8	3	24
	20	96

ARITHMETIC MEAN

Example 3: Find the mean wage for the following:

Daily Wage	No. of Employees	fx
255	8	2040
265	10	2650
275	16	4400
285	15	4275
295	10	2950
305	8	2440
315	3	945
	70	19700

x ↓ (Daily Wage); f ↓ (No. of Employees); ↑ N (70); ↑ $\sum fx$ (19700)

$$\bar{x} = \frac{\sum f_i x_i}{N} = \frac{19700}{70}$$

$$= 281.43$$

ARITHMETIC MEAN

Solution by Coding Method:

Daily Wage	u = (x - 285)/10	No. of Employees	fu
255	-3	8	-24
265	-2	10	-20
275	-1	16	-16
285	0	15	0
295	1	10	10
305	2	8	16
315	3	3	9
		70	-25

$$\bar{x} = A + \frac{\sum f_i u_i}{N} \times C$$

$$= 285 + \frac{(-25)}{70} \times 10$$

$$= 285 - 3.57$$

$$= 281.43$$

ARITHMETIC MEAN

Example 4: Find the mean height for the following:

Height (in)	Frequency	Class Mark (x)	fx
60 - 62	5	61	305
63 - 65	18	64	1152
66 - 68	42	67	2814
69 - 71	27	70	1890
72 - 74	8	73	584
	100		6745

$$\bar{x} = \frac{\sum f_i x_i}{N} = \frac{6745}{100}$$

$$= 67.45 \text{ in}$$

ARITHMETIC MEAN

Example 5: Find the mean weight for the following:

Weight	No. of Children
0 – 20	3
20 – 40	10
40 – 60	15
60 – 80	12
80 – 100	8
100 – 120	12

PROPERTIES OF ARITHMETIC MEAN

i. If all the observations assumed by a variable are constants, say k, then the AM is also k.

Example: If the height of every student in a group of 10 students is 170 cm, then the mean height is, of course, 170 cm.

ii. The algebraic sum of deviations of a set of observations from their arithmetic-mean is zero.

Example: If a variable "x" assumes five observations, say 5, 7, 12, 15, 26, then $\bar{x} = 13$.

The deviations of the observations from arithmetic mean $(x - \bar{x})$ are -8, -6, -1, 2, 13.

Now, $\sum(x - \bar{x}) = (-8) + (-6) + (-1) + 2 + 13 = 0$.

PROPERTIES OF ARITHMETIC MEAN

iii. Arithmetic-mean is affected due to a change of origin and/or scale which implies that if the original variable "x" is changed to another variable "y" effecting a change of origin, say "a" and scale, say "b", of "x".

i.e. if $y = a + bx$

Then we have,

$$Arithmetic - mean\ of\ y = \bar{y} = a + b\bar{x}$$

COMBINED ARITHMETIC MEAN

iv. If there are two groups containing n_1 and n_2 observations $\overline{x_1}$ and $\overline{x_2}$ are the respective arithmetic means, then the combined arithmetic-mean is given by

$$\bar{x} = \frac{n_1\overline{x_1} + n_2\overline{x_2}}{n_1 + n_2}$$

This property could be extended to more than two groups.

Example: Four groups of students, consisting of 15, 20, 10, and 18 individuals, reported mean weights of 162, 148, 153, and 140 pounds (lb), respectively. Find the mean weight of all the students.

$$\bar{x} = \frac{15 \times 162 + 20 \times 148 + 10 \times 153 + 18 \times 140}{15 + 20 + 10 + 18} = 150$$

WEIGHTED ARITHMETIC MEAN

If $x_1, x_2, x_3, \ldots \ldots \ldots, x_n$ have weights $w_1, w_2, w_3, \ldots \ldots \ldots, w_n$ then $weighted\ average = \bar{x} = \frac{\sum wx}{\sum w}$

ARITHMETIC MEAN

Q1. A student received grades of 85, 76, 93, 82, and 96 in five subjects. Determine the arithmetic mean of the grades.

Q.2 The reaction times of an individual to certain stimuli were measured by a psychologist to be 0.53, 0.46, 0.50, 0.49, 0.52, 0.53, 0.44, and 0.55 seconds respectively. Determine the mean reaction time of the individual to the stimuli.

Q.3 A student's grades in the laboratory, lecture, and recitation parts of a physics course were 71, 78, and 89, respectively. If the weights accorded these grades are 2, 4, and 5, respectively, what is an appropriate average grade?

ARITHMETIC MEAN

Q.4 What is arithmetic mean of $11.8, 12.6, 10.9, 8.8$ *and* 9.6?

Q.5 If there are two groups containing 60 and 40 members and having mean values 30 and 25 respectively, then what is the combined arithmetic mean?

Q.6 If the two variables are given by $2y = 3x - 10$ and mean of x is 6, then what is mean of y?

Q.7 If a group of 5 students have same height 153 cm then what is the mean height?

Q.8 What is the sum of deviations of $10, 18, 12, 15$ *and* 9 from arithmetic mean?

ARITHMETIC MEAN

Q.9 Find AM for the following:

Maximum Load (short tons)	Number of Cables
9.3–9.7	2
9.8–10.2	5
10.3–10.7	12
10.8–11.2	17
11.3–11.7	14
11.8–12.2	6
12.3–12.7	3
12.8–13.2	1
	Total 60

Weight (lb)	Frequency
118–126	3
127–135	5
136–144	9
145–153	12
154–162	5
163–171	4
172–180	2

MEDIAN

- The median of a set of numbers arranged in order of magnitude (i.e., in an array) is either the middle value or the arithmetic mean of the two middle values.

  Let $x_1, x_2, x_3, \dots\dots\dots, x_n$ be the data arranged in order.

  - If n is odd, $Median = \left(\frac{n+1}{2}\right)^{th}$ $observation.$
  - If n is even, $Median = AM\ of\ \frac{n}{2}^{th}\ and \left(\frac{n}{2}+1\right)^{th}\ observation$

MEDIAN

Example:

1. If there are 13 data points in order then median = 7^{th} Observation.
2. If there are 18 data points in order then median = AM of 9^{th} Observation and 10^{th} Observation.

$$= \frac{9^{th} Observation + 10^{th} Observation}{2}$$

MEDIAN

Example 1: Find median of $5, 4, 8, 3, 7, 2, 9$.

Solution: Data in ascending order: $2, 3, 4, 5, 7, 8, 9$

Here we have 7 values (odd),

therefore $Median = 4^{th}\ observation = 5$.

Example 2: Find median of $18.3, 20.6, 19.3, 22.4, 20.2, 18.8, 19.7, 20.0$.

Solution:

Data in descending order: $22.4, 20.6, 20.2, 20.0, 19.7, 19.3, 18.8, 18.3$

Here we have 8 values (even),

therefore $Median = \frac{4^{th}\ obs + 5^{th}\ obs}{2} = \frac{20.0+19.7}{2} = 19.85$

FIND MEDIAN

Size	Frequency	cf	
8	14	14	1-14
9	21	35	15-35
10	17	52	36-52
11	12	64	53-64
12	6	70	65-70

MEDIAN

- For grouped data, the median, obtained by interpolation, is given by

$$Median = l_1 + \left[\frac{(l_2 - l_1)\left(\frac{N}{2} - pcf\right)}{f} \right]$$

where, $l_1 = LL\ of\ median\ class$

$l_2 = UL\ of\ median\ class$

$f = frequency\ of\ median\ class$

$pcf = cf\ of\ pre - median\ class$

$N = total\ number\ of\ observations$

Note: Class intervals has to be exclusive type.

MEDIAN

Example 1: Find median rainfall for the following:

Rainfall in cms	No. of Cities	CF
0 - 10	10	10
10 - 20	15	25
20 - 30	20	45
30 - 40	10	55
40 - 50	5	60
	60	

$$M = l_1 + \frac{(l_2 - l_1)\left(\frac{N}{2} - pcf\right)}{f}$$

$$= 20 + \frac{10 \times 5}{20}$$

$$= 22.5\ cms$$

MEDIAN

Example 2: Find median height for the following:

Height (in)	Frequency	cf
59.5 - 62.5	5	5
62.5 - 65.5	18	23
65.5 - 68.5	42	65
68.5 - 71.5	27	92
71.5 - 74.5	8	100
	100	

$$M = l_1 + \frac{(l_2 - l_1)\left(\frac{N}{2} - pcf\right)}{f}$$

$$= 65.5 + \frac{3 \times 27}{42}$$

$$= 67.43\, in$$

MEDIAN

Example 3: Find median for the following

Maximum Load (short tons)	Number of Cables
9.3–9.7	2
9.8–10.2	5
10.3–10.7	12
10.8–11.2	17
11.3–11.7	14
11.8–12.2	6
12.3–12.7	3
12.8–13.2	1
	Total 60

MEDIAN

Example 4: Find median for the following

Weight (lb)	Frequency
118–126	3
127–135	5
136–144	9
145–153	12
154–162	5
163–171	4
172–180	2

MEDIAN

Example 5: If $2y - 3x = 6$ and median of x = 6 then what is median of y?

QUARTILE

Example: Find first, second and third quartile for the following:

$$5, 15, 12, 7, 30, 25, 22, 14, 36, 53, 42$$

Solution:

Data in ascending order:

$$5, 7, 12, 14, 15, 22, 25, 30, 36, 42, 53$$

$$5, 7, 12, 14, 15, 22, 25, 30, 36, 42, 53$$

$$Q_1 = 12$$

$$Q_2 = 22 = M$$

$$Q_3 = 36$$

QUARTILE

- For grouped data,
  - $Q_1 = l_1 + \frac{(l_2 - l_1)\left(\frac{N}{4} - pcf\right)}{f}$
  - $Q_2 = l_1 + \frac{(l_2 - l_1)\left(\frac{2N}{4} - pcf\right)}{f}$
  - $Q_3 = l_1 + \frac{(l_2 - l_1)\left(\frac{3N}{4} - pcf\right)}{f}$

QUARTILE

Example 1: Find first and third quartile for the following

CI	Frequency	cf
0 - 10	5	5
10 - 20	10	15
20 - 30	12	27
30 - 40	18	45
40 - 50	9	54
50 - 60	6	60
	60	

QUARTILE

Example 2: Find first and third quartile for the following

Rainfall in cms	No. of Cities	CF
0 - 10	10	10
10 - 20	15	25
20 - 30	20	45
30 - 40	10	55
40 - 50	5	60
	60	

DECILE AND PERCENTILE

- For grouped data,
  - $D_k = l_1 + \frac{(l_2 - l_1)\left(\frac{kN}{10} - pcf\right)}{f}, \qquad k = 1, 2, 3, \dots\dots\dots, 9$
  - $P_k = l_1 + \frac{(l_2 - l_1)\left(\frac{kN}{100} - pcf\right)}{f}, \qquad k = 1, 2, 3, \dots\dots\dots, 99$

DECILE AND PERCENTILE

Example: Find D_4 and P_{65} for the following

Rainfall in cms	No. of Cities	CF
0 - 10	10	10
10 - 20	15	25
20 - 30	20	45
30 - 40	10	55
40 - 50	5	60
	60	

QUARTILES, DECILE AND PERCENTILE

Example: Find Q_3, D_4 and P_{65} for the following:

$$20, 15, 18, 5, 10, 17, 21, 19, 25, 28$$

Solution: Data in ascending order:

$$5, 10, 15, 17, 18, 19, 20, 21, 25, 28$$

$$Q_3 = \left(\frac{3(n+1)}{4}\right)^{th} term = 8.25^{th} term$$

$$\therefore Q_3 = 8^{th} term + 0.25 \times \left(9^{th} term\ - 8^{th} term\right)$$

$$\therefore Q_3 = 21 + 0.25 \times 4 = 22$$

$$D_4 = \left(\frac{4(n+1)}{10}\right)^{th} term = 4.4^{th} term$$

$$\therefore D_4 = 4^{th} term + 0.4 \times \left(5^{th} term\ - 4^{th} term\right) = 17.4$$

$$P_{65} = \left(\frac{65(n+1)}{100}\right)^{th} term = 7.15^{th} term$$

$$\therefore P_{65} = 20.15$$

MODE

The mode of a set of numbers is that value which occurs with the greatest frequency; that is, it is the most common value.

The mode may not exist, and even if it does exist it may not be unique. A distribution having only one mode is called unimodal.

Example:

1. 2, 2, 5, 7, 9, 9, 9, 10, 10, 11, 12, 18
   is unimodal with mode = 9.
2. 2, 3, 4, 4, 4, 5, 5, 7, 7, 7, 9
   is bimodal with two modes, 4 and 7.
3. 3, 5, 8, 10, 12, 15, 16
   has no mode.
4. 2, 2, 2, 3, 3, 3
   has no mode.

MODE

Example: Find modal size.

Size	Frequency
8	14
9	21
10	17
11	12
12	6

MODE

For grouped data:

$$Mode = Z = l_1 + \frac{(l_2 - l_1)(f_1 - f_0)}{(2f_1 - f_0 - f_2)}$$

$$where, f_1 = frequency\ of\ modal\ class$$

$$f_0 = frequency\ of\ pre - modal\ class$$

$$f_2 = frequency\ of\ post - modal\ class$$

MODE

Example 1: Find modal rainfall.

Rainfall in cms	No. of Cities
0 - 10	10
10 - 20	15
20 - 30	20
30 - 40	10
40 - 50	5
	60

MODE

Example 2: Find modal weight.

Weight (lb)	Frequency
118–126	3
127–135	5
136–144	9
145–153	12
154–162	5
163–171	4
172–180	2

MODE

Example 3: If $3x - y = 6$ and mode of x = 9 then what is mode of y?

RELATION between Mean, Median and Mode

For moderately skewed distribution, we may consider the following empirical relationship between mean, median and mode,

$$Mean - Mode = 3(Mean - Median)$$

Example: For a moderately skewed distribution of marks in statistics for a group of 200 students, the mean and median marks were found to be 55.60 and 52.40. What is the modal marks?

Solution: WKT, $Mean - Mode = 3(Mean - Median)$

$$\Longrightarrow Mode = 3\,Median\ - 2\,Mean$$

$$= 3\,(52.40) - 2\,(55.60) = 46$$


Thank you