Homework 5
Graph Theory CSC/MA/OR 565
Sketch of Solutions

1. Problem 6.1.8, text. (Prove that every simple planar graph G has a vertex of degree at most 5.)

If $n = n(G) < 3$, this is clearly true. If $n \geq 3$, then by Theorem 6.1.23, $e = e(G) \leq 3n - 6$. Now if $\delta(v) \geq 6$ for every vertex v of G, then by the degree sum formula, $2e \geq 6n$. Combining these inequalities means that $3n \leq e \leq 3n - 6$, which is impossible, and therefore a contradiction.

2. Using only the result of 6.1.8, prove that if G is a simple planar graph, then $\chi(G) \leq 6$.

Prove by induction on $n(G)$. If $n(G) \leq 6$, assign each vertex a different color to get a proper coloring with at most 6 colors.

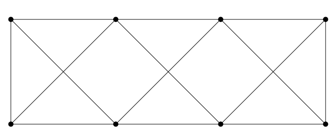
Let G be a simple planar graph with $n(G) \geq 7$ and assume the claim is true for simple planar graphs with fewer than $n(G)$ vertices.

By 6.1.8, G has a vertex v with at most 5 neighbors. By induction, $\chi(G - v) \leq 6$. At most 5 of those 6 colors are used on the neighbors of v, leaving a color available for v to give a proper coloring of G with at most 6 colors.

3. For the first part, we can use Theorem 6.1.23 which says $e = e(G) \leq 3n - 6$ for a simple planar graph with at least 3 vertices.

If G is a planar graph with 11 vertices, then, by Theorem 6.1.23, $e(G) \leq 27$. The number of edges in the complement of G is $11(10)/2 - e(G) \ \geq 55 - 27 = 28$. This is more than 27, so the complement of G cannot be planar.

For the second part, you can check that the graph below is planar and is isomorphic to its complement (so its complement is also planar.



4. Using the fact that $\sum n_i = n$ and the degree sum formula we have

$$\sum (6-i)n_i = \sum 6n_i - \sum in_i = 6n - 2e.$$

From Euler's formula $n - e + f = 2$, since every face is a triangle, $3f = 2e$,

$$2 = n - e + f = n - e + 2e/3 = (3n - e)/3.$$

Multiply through by 6 and get $12 = 6n - 2e$.

5. What is the maximum number of edges (as a function of the number of vertices) in a simple planar graph of girth 5? Use this to prove that the Petersen graph is not planar.

Let G be a planar embedding of a planar graph of girth 5. The $\ell(f) \geq 5$ for every face f of G. Thus $2e = \sum \ell(f) \geq 5f$. Combining this with Euler's formula,

$$2 = n - e + f \leq n - e + 2e/5 = n - 3e/5.$$

So, $e \leq (5n - 10)/3$.

The Petersen graph violates this - it has 15 edges, but $(5 \times 10 - 10)/3 = 40/3 < 15$.

6. Give an example of a simple planar graph with minimum degree 5.

The graph defined by the vertices and edges of the regular icosohedron is one example. It is regular of degree 5. Since it can be embedded on the surface of a sphere with no crossings, it is planar.

7. Find, if possible, a subdivision of K_5 in Q_4.

This example makes use of the following 11 vertices of Q_4, listed in sorted order. The vertices marked (*) will play the role of the vertices in the subdivision of K_5. The other vertices will be part of edge subdivisions.

0000 (*), 0001 (*), 0010 (*), 0011, 0100 (*), 0101, 0110, 1000 (*), 1001, 1010, 1100.

0000 is adjacent to 0001, 0010, 0100, and 1000.

0001 is joined to 0010, 0100, and 1000 by the edge subdivisions 0001-0011-0010, 0001-0101-0100, and 0001-1001-1000.

0010 is joined to 0100 and 1000 by the edge subdivisions 0010-0110-0100, and 0010-1010-1000.

0100 is joined to 1000 by the edge subdivision 0100-1100-1000.

8. Find, if possible, a subdivision of $K_{3,3}$ in the Petersen graph.

Yes, this is easy to do.

9. For which pairs (n, r) is $T_{n,r}$ planar?

For $r \geq 5$, $T_{n,r}$ contains K_5, so cannot be planar.

(I'm going to assume that $n \geq r$. If $n < r$, then $T_{n,r} = T_{n,n}$.)

$T_{n,r}$ is planar if and only it

- $r = 1$ (no edges) or

- $r = 2$ and $n \leq 5$ (contains $K_{3,3}$ if $n \geq 6$) or

- $r = 3$ and $n \leq 6$ (too many edges if $n > 6$, but $T_{6,3}$ is planar) or

- $r = 4$ and $n \leq 5$ (too many edges if $n \geq 6$, but $T_{5,4}$ is planar)

10. Look up the definition of thickness in the text. Find (and prove) the thickness of Q_4. Find (and prove) the thickness of the Petersen graph.

Neither Q_4 nor the Petersen graph are planar (see exercises 7 and 8), so their thickness must be bigger than 1. It is straightforward to show that they each have thickness 2 - there are many ways to decompose these into two planar graphs.