A $\boxed{\text{walk}}$ of length k in G is a sequence of vertices + edges:

$$v_0, e_1, v_1, e_2, \ldots, e_k, v_k$$

where $e_i = v_{i-1} v_i$ for all i

(Can omit edges if G is simple)

---

u, v - walk if first vertex u and last is v

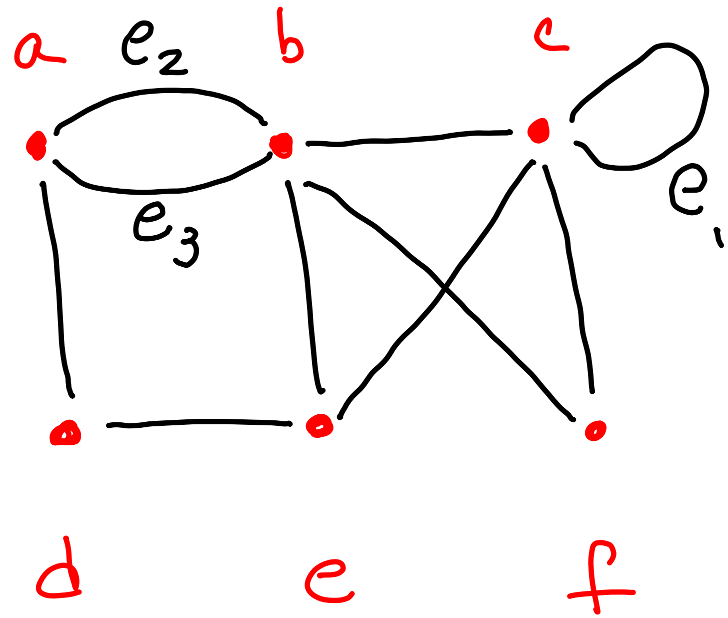
$\boxed{\text{trail}}$ if no repeated edge;

$\boxed{\text{closed}}$ if $u = v$;

$\boxed{\text{cycle}}$ - closed trail of length ≥ 1 with no repeated vertex, except $u = v$

$\boxed{\text{path}}$ if no repeated vertex

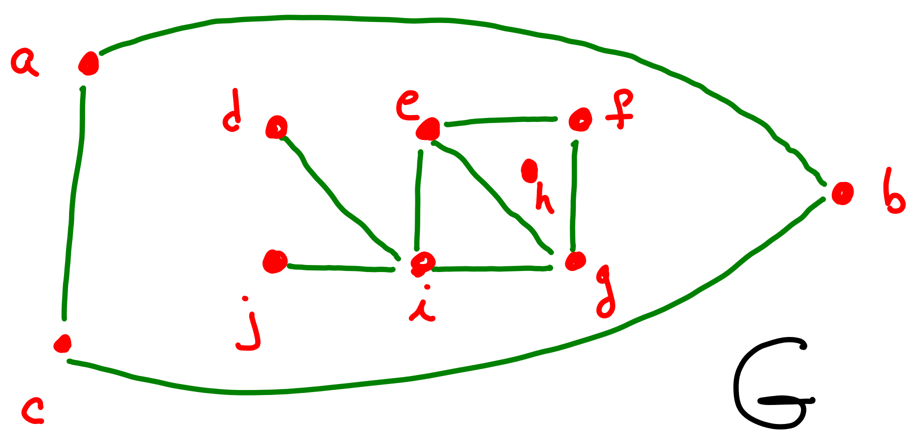
sequence	w?	tr?	pa?	cl?	cy?
d, b, d, e, c, f, b, d					
$a, e_2, b, c, f, b, e_3, a$					
a, d, e, c, f					
c, e_1, c					
a, e_2, b, e_3, a					
a, e_2, b, e_2, a					
a					



Vertex u is connected to vertex v if G has a u,v-path

G is connected if u is connected to v for every $u, v \in V(G)$; otherwise disconnected.

The components of G are the maximal connected subgraphs of G



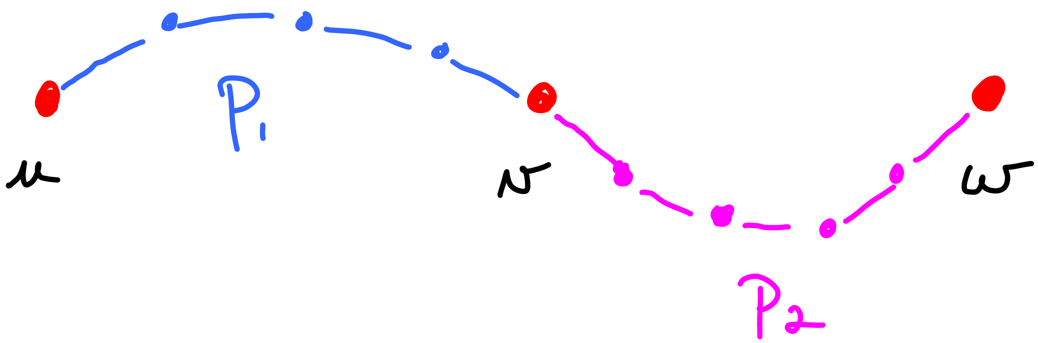
Components?

If u is connected to v and v is connected to w, is u connected to w?

P_1

P_2

P_3?



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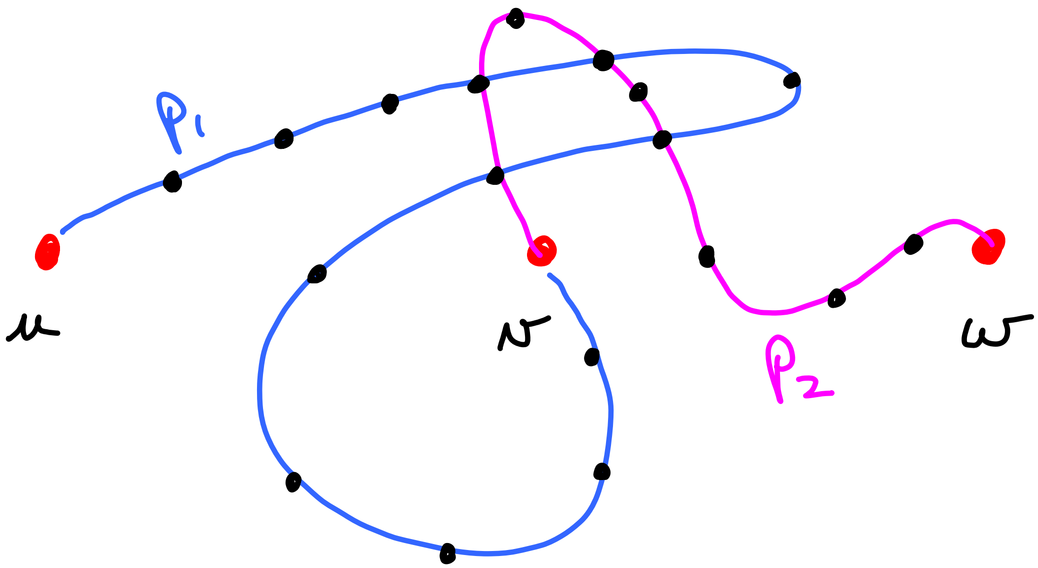

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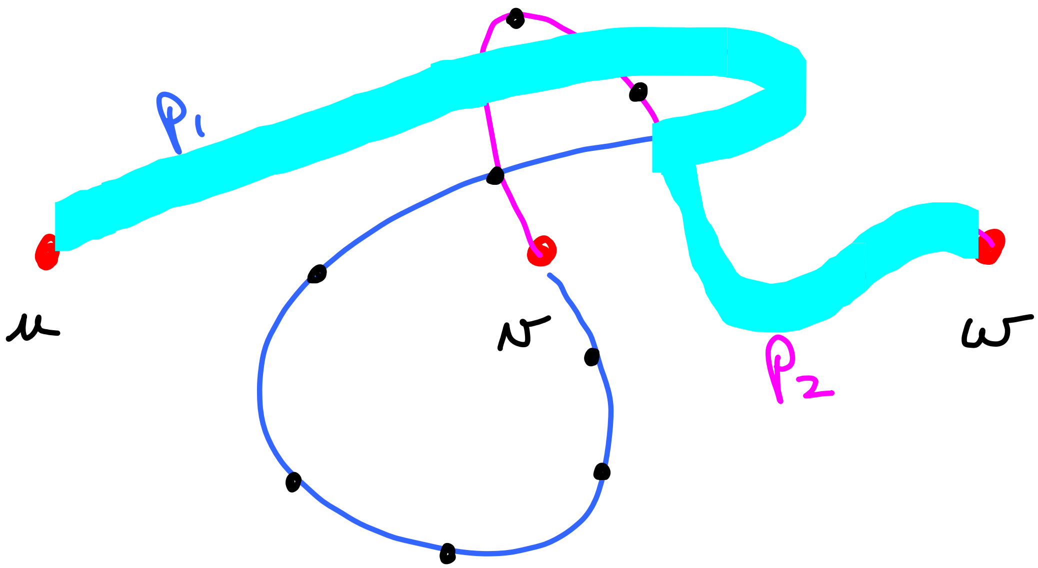

Strong Induction Principle

Assume $P(n)$ is a statement with integer parameter n.

If

(1) $\boxed{P(1) \text{ is true}}$

and

(2) For all $n > 1$,

$\boxed{P(k) \text{ true for } 1 \le k < n}$ implies $\boxed{P(n) \text{ is true}}$

then

$\boxed{P(n) \text{ is true for every positive integer } n.}$

Lemma 1.2.5 Every u,v-walk contains a u,v-path

Proof: Induction on the length ℓ of the walk W

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Proof: Induction on the length l of the walk W

basis $l = 0$ W: •

ind. $l > 1$, assume lemma true for walks of length $< l$

Case 1: If W has no repeated vertex, then ... path!

Case 2: W has a repeated vertex, w

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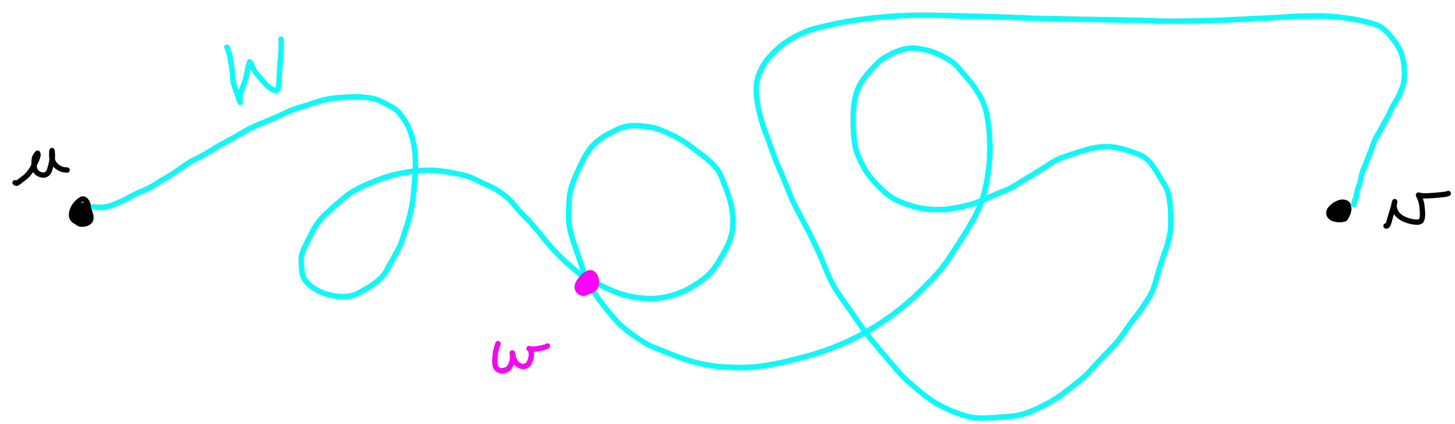

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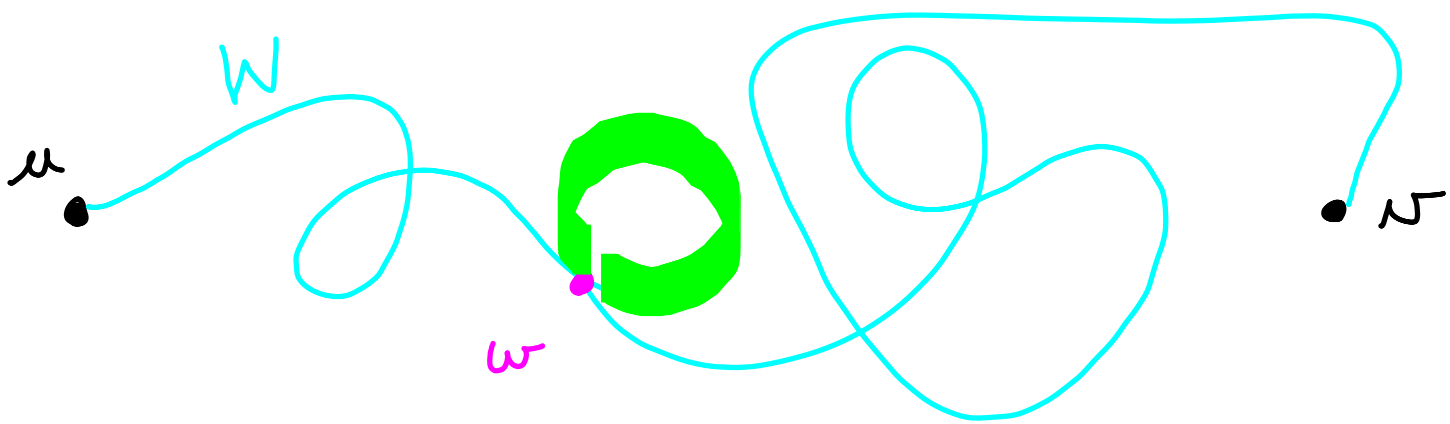

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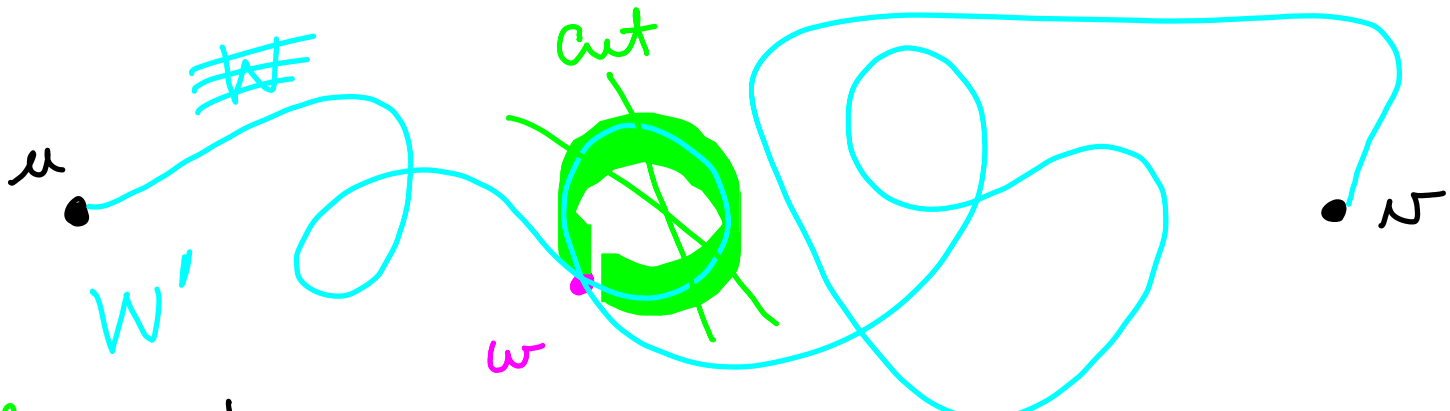

length $< \ell$, so by induction W' contains a u,v-path & therefore so does W.

We just proved:

Lemma 1.2.5 Every u,v-walk contains a u,v-path

Is it similarly true that

every closed walk contains a cycle?

Lemma 1.2.15 Every closed odd walk contains an odd cycle.

Proof. Induction on the length ℓ of closed odd walk W.

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Proof. Induction on the length ℓ of closed odd walk W.

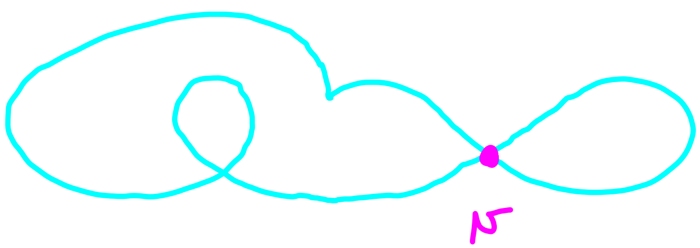
Basis If $\ell = 1$ then W :

Ind. $\ell > 1$, assume lemma true for closed odd walks of length $< \ell$.

Case 1: W has no repeated vertex (except first = last): then done!

Case 2: W has a repeated vertex v

W



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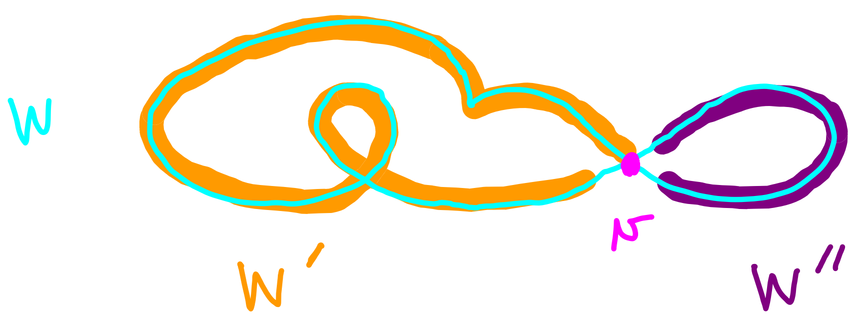

Both W' and W'' are closed walks of length $< \ell$.

At least one must have odd length (why?)

By induction it contains a closed odd cycle, $\therefore$ so does W

Implication

$A \rightarrow B$

A is the hypothesis

B is the conclusion

Equivalence

A if and only if B

means $A \rightarrow B$ (B is necessary for A)

and $B \rightarrow A$ (B is sufficient for A)

both are true

"iff"

Theorem 1.2.18 A graph is bipartite iff it has no odd cycle.

Proof

Necessity (yellow) → (magenta): If G is bipartite any closed walk must be even since it alternates between the partite sets.

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Let $u \in V(G)$.

For each $v \in V(G)$, let $d_G(u,v)$ be length of shortest u,v-path.

Let $X = \{v \mid d_G(u,v) \text{ is even}\}$

Let $Y = \{v \mid d_G(u,v) \text{ is odd}\}$

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Sufficiency $\rightarrow$: If G has no odd cycle, construct a bipartition of G as follows (first assuming G is connected)

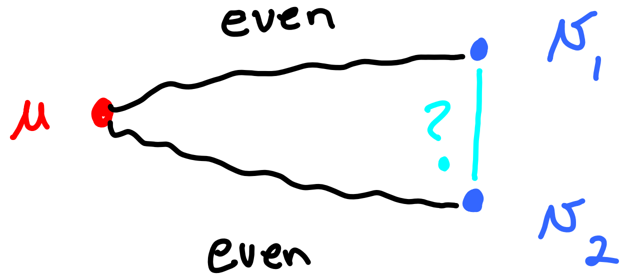
Let $u \in V(G)$.

For each $v \in V(G)$, let $d_G(u,v)$ be length of shortest u,v-path.

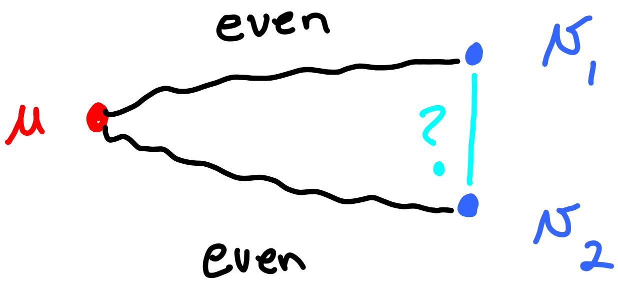
$$\text{Let } X = \{v \mid d_G(u,v) \text{ is even}\}$$

$$\text{Let } Y = \{v \mid d_G(u,v) \text{ is odd}\}$$

independent sets! (why?)



$\nu_1, \nu_2 \in X$, can't have:



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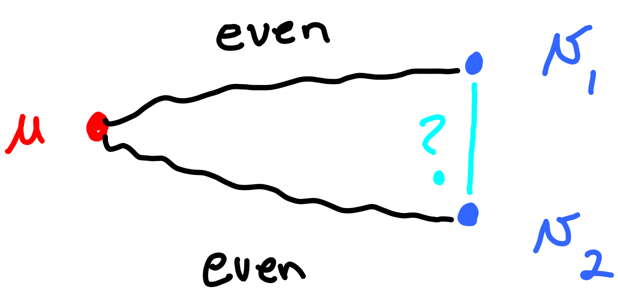
Closed odd walk, by Lemma 1.2.15 contains an odd cycle



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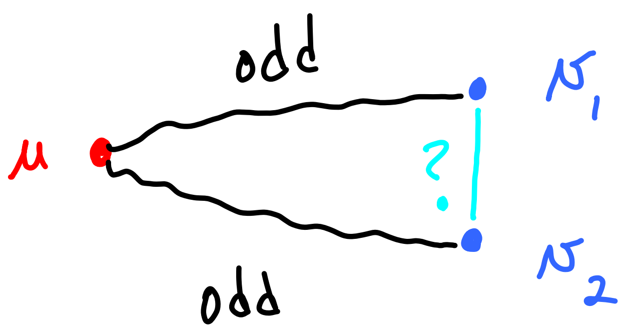
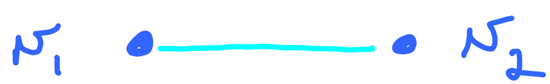


$v_1, v_2 \in Y$, can't have:



Conclude: G is bipartite

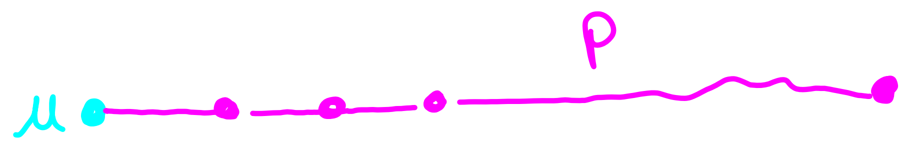
(What if G is not connected?)

Proof by extremality:

Select extreme example of a structure and use the lack of a more extreme example to gain leverage.

Proposition 1.2.28 If G is a simple graph in which every vertex has degree at least k, then G contains a path of length at least k.

Proof Let P be a maximal path in G and let u be one of its endpoints



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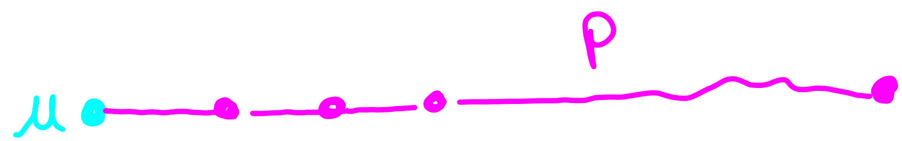

Every neighbor of u must be on P, (why?)
and there are at least k of them (why?)

So, P has length at least k.

Proposition 1.2.28 If G is a simple graph in which every vertex has degree at least k, then G contains a path of length at least k. If $k \geq 2$, then G also contains a cycle of length at least $k+1$.

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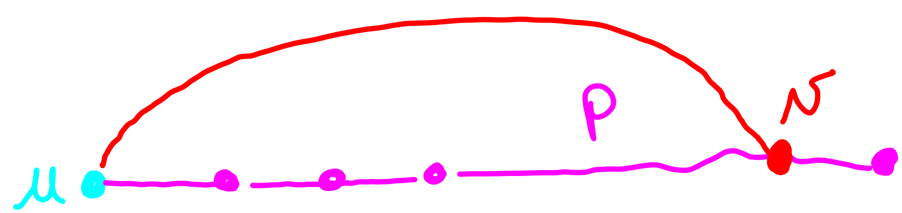


Every neighbor of u must be on P, (why?)

and there are at least k of them (why?)

Let v be the last one on P

This creates a cycle of length at least $k+1$.

Proof by Contradiction

Prove **implication**:

$$A \text{ implies } B$$

by proving:

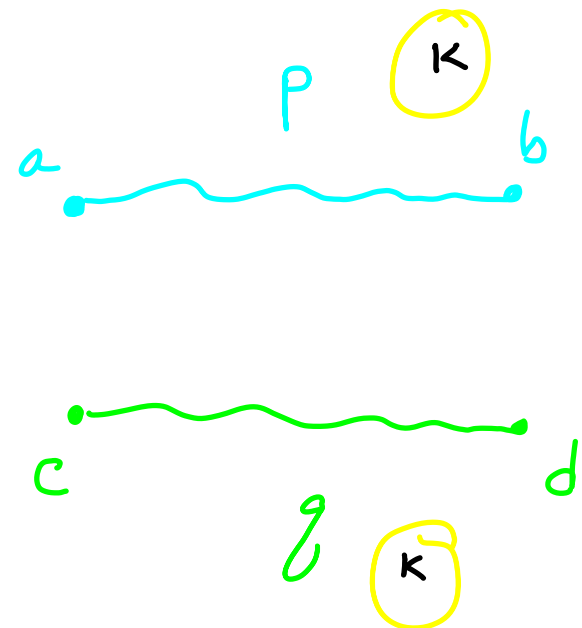
$$(A \text{ true}) \quad \text{and} \quad (B \text{ false}) \quad \text{is impossible.}$$

Example

Prove: In a connected graph, any two longest paths have a vertex in common.

Proof. Assume G is connected and let k be the length of a longest path in G. Suppose a, b-path p and c, d-path q are different paths of length k in G. Show by contradiction that p and q have a common vertex.

Assume p and q do not intersect. (Try to make a path longer than k to reach a contradiction.) Let r be an a, c-path (since G is connected.)



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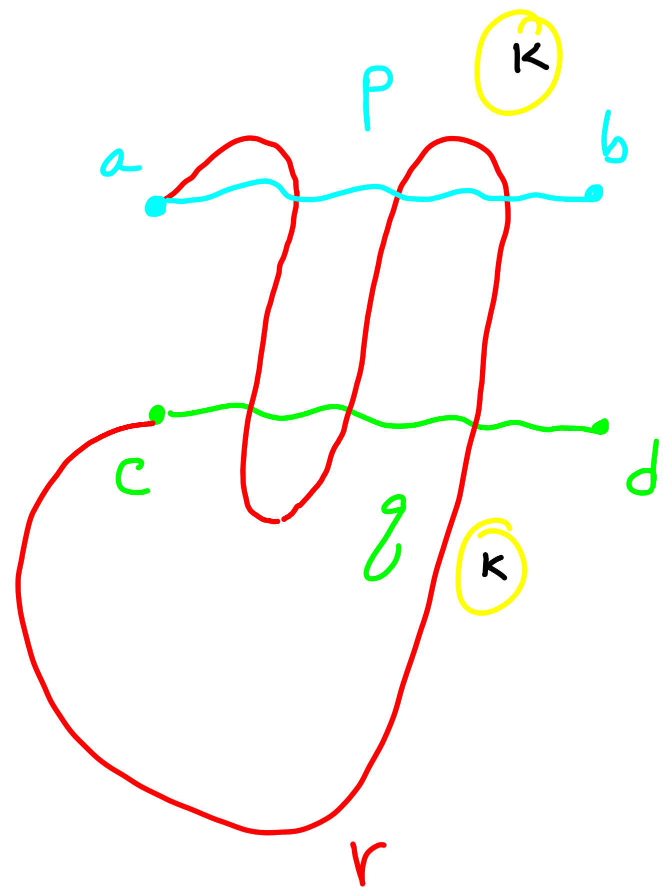

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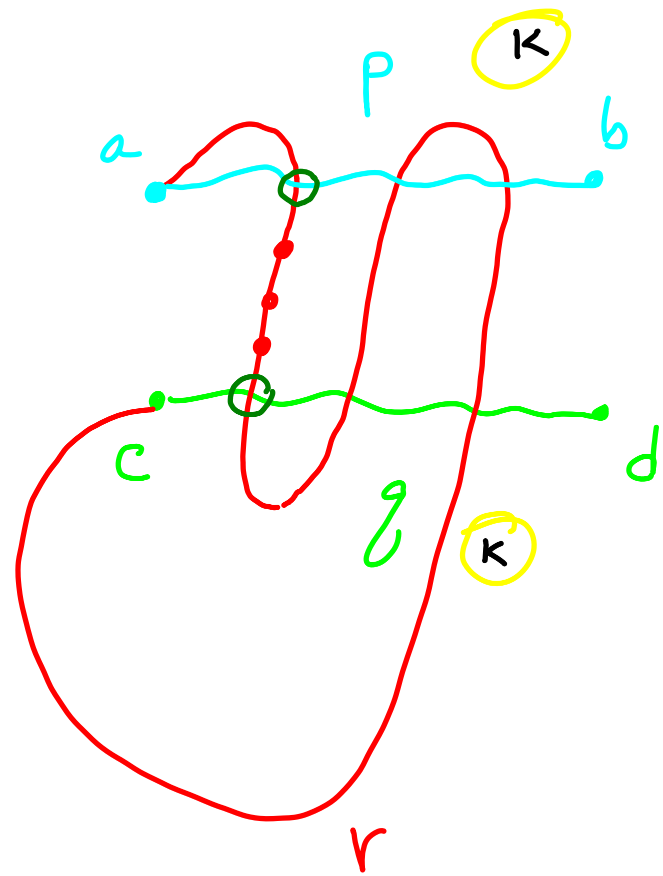

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