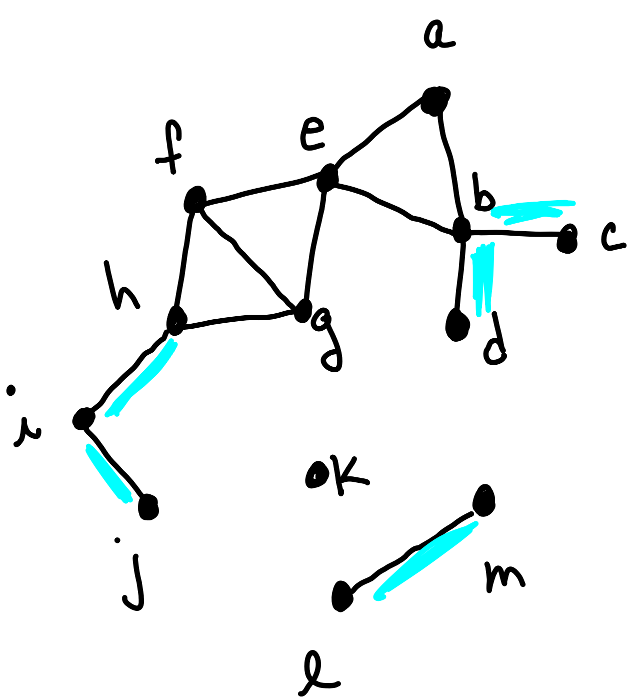
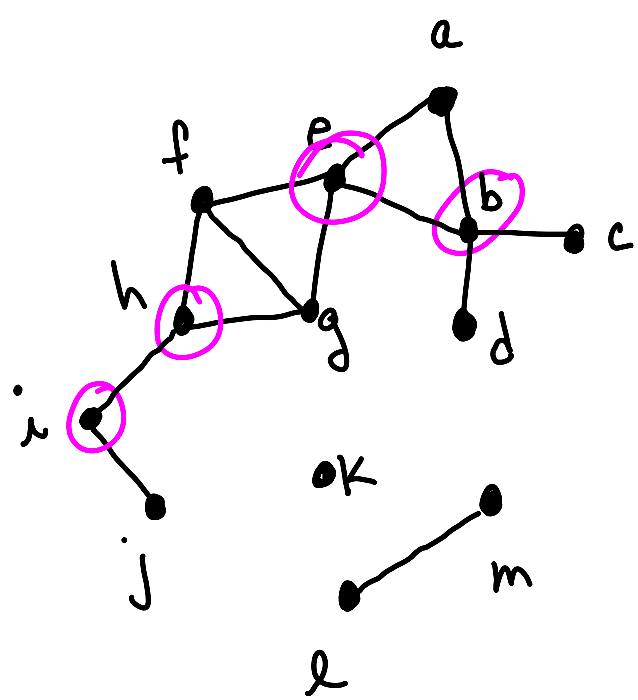
Cut edge - edge whose deletion increases the number of components

Cut vertex - vertex whose deletion increases the number of components





Theorem 1.2.14. An edge e is a cut-edge of a graph G if and only if it belongs to no cycle.

Proof. Let H be the component of e. It suffices to prove that $H - e$ is connected if and only if e belongs to a cycle.

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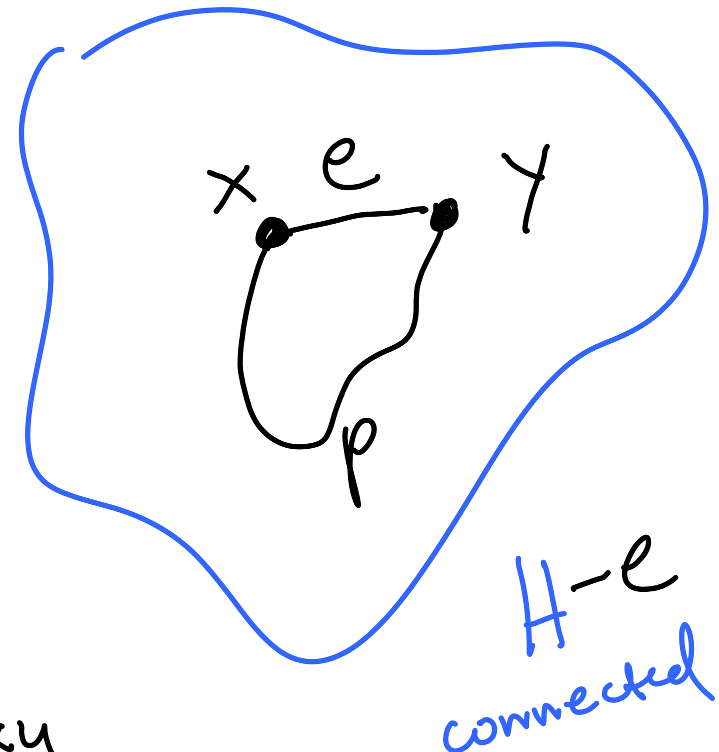

(i) Assume $H - e$ is connected. Let $e = xy$

Let P be an x,y path in $H-e$

Adding e to P creates a cycle in H containing e

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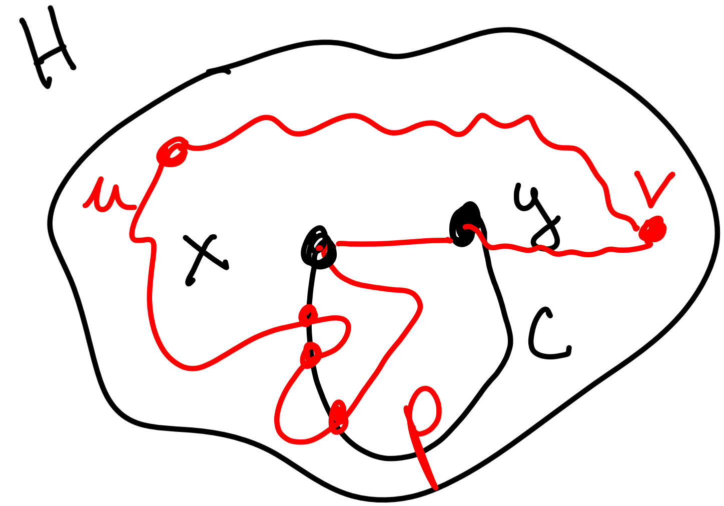

Eulerian Graphs

Circuit - closed trail

Graph G is **Eulerian** if there is a circuit which contains every edge of G.

Such a circuit is called an **Eulerian circuit**.

A trail which contains every edge of graph G is called an **Eulerian trail**.

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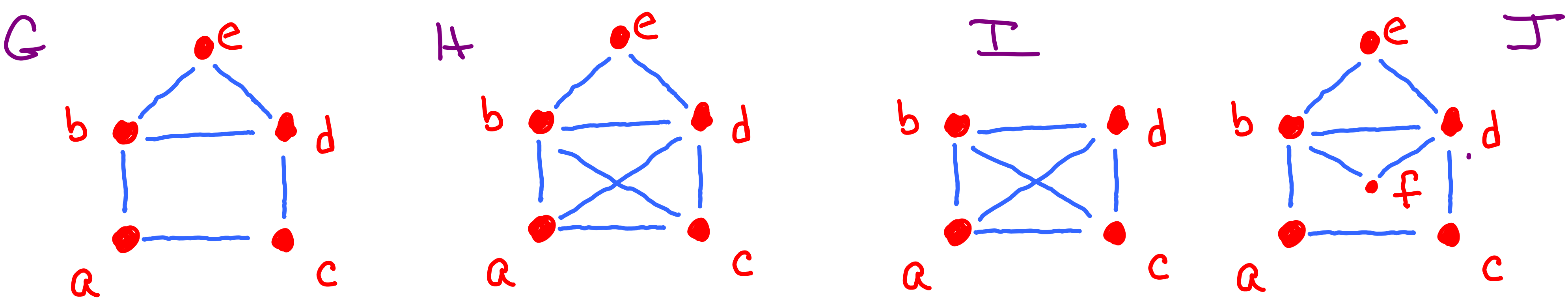

Necessary condition for Euler circuit in a connected graph?

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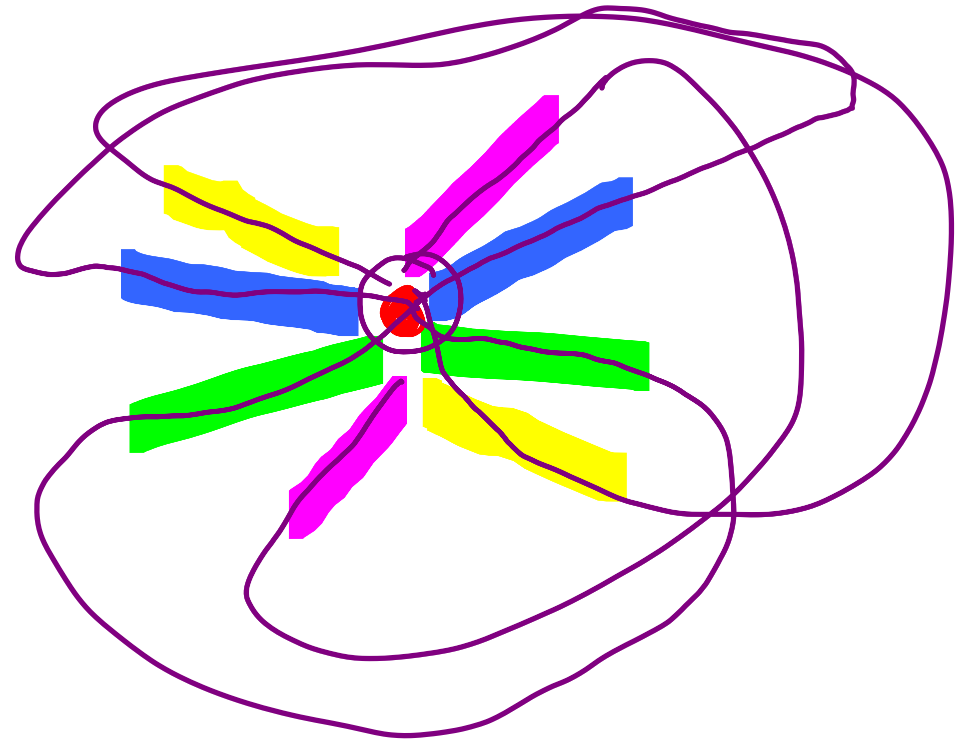

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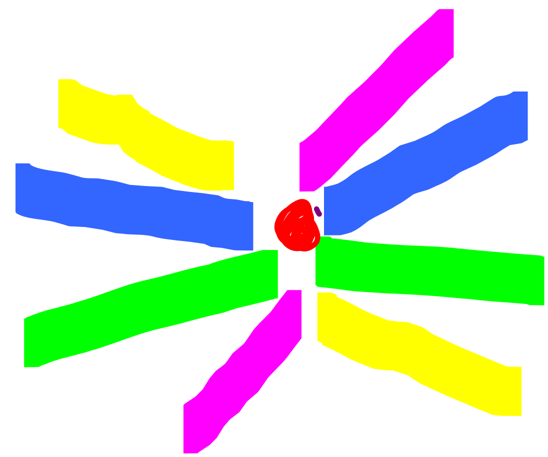
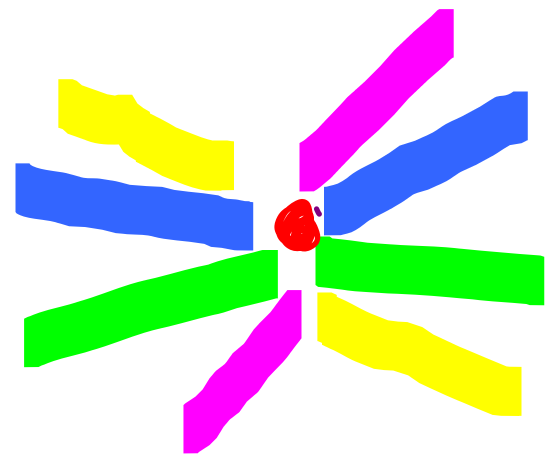


For Euler trail?

Necessary condition for Euler circuit in a connected graph?



Sufficient?

Theorem 1.2.26.
A connected graph G is Eulerian if and only if every vertex has even degree.

($\Rightarrow$): easy

($\Leftarrow$): Induction on number of edges.

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necessity

($\Rightarrow$): easy

sufficiency

($\Leftarrow$): Induction on number of edges.

Theorem 1.2.26.
A connected graph G is Eulerian if and only if every vertex has even degree.

necessity ($\Rightarrow$): easy

sufficiency ($\Leftarrow$): Induction on number of edges.

We will use: Lemma 1.2.25. If every vertex of a graph G has degree ≥ 2, then G contains a cycle

(its proof is by extremality, like Prop 1.2.28, but note that Prop 1.2.28 required G to be simple.)

Theorem 1.2.26.
A connected graph G is Eulerian if and only if every vertex has even degree.

<u>Sufficiency</u>: Let G be a connected even graph. Prove G Eulerian.

Induction on # of edges of G.

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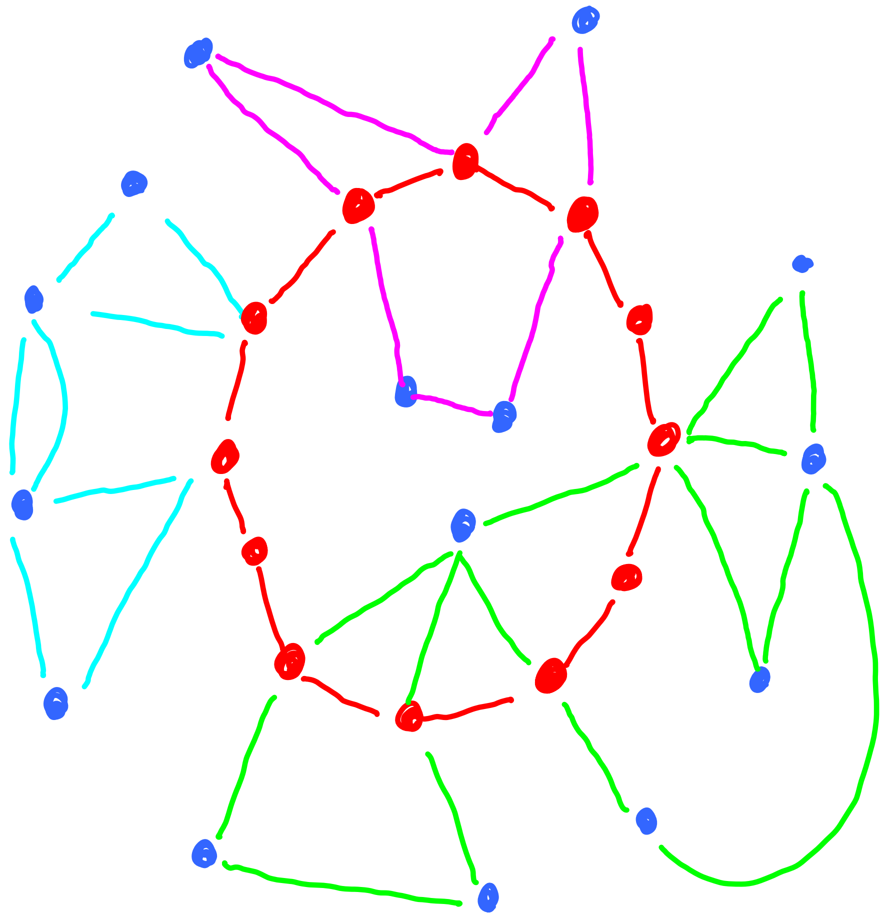
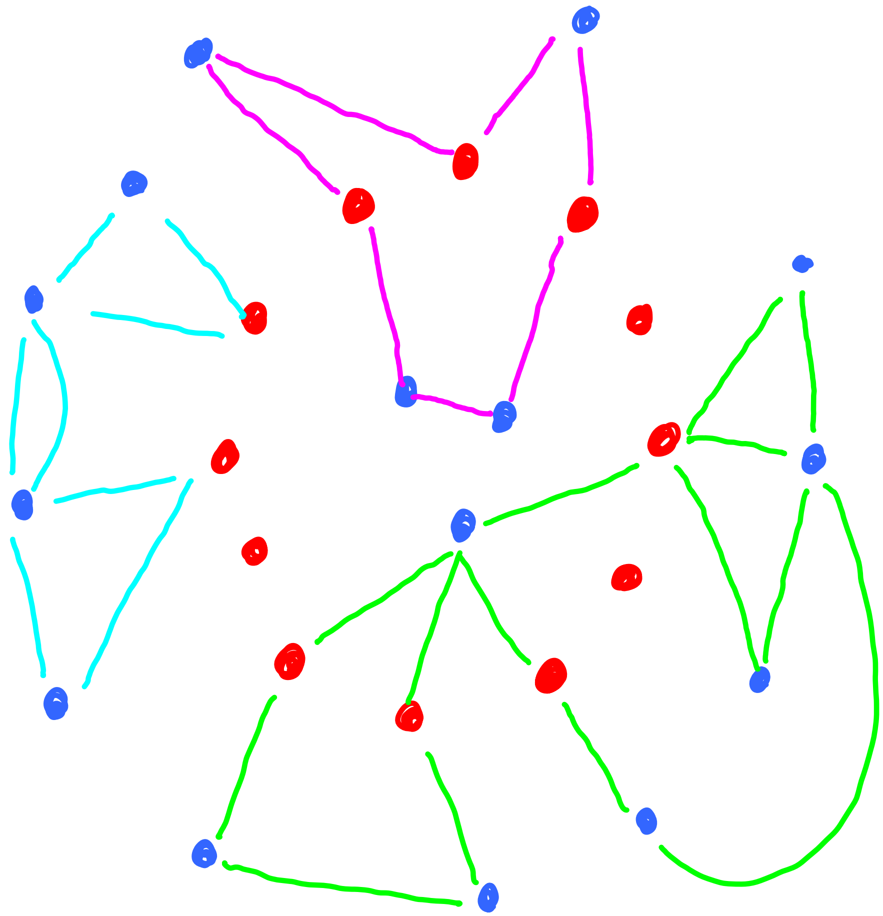
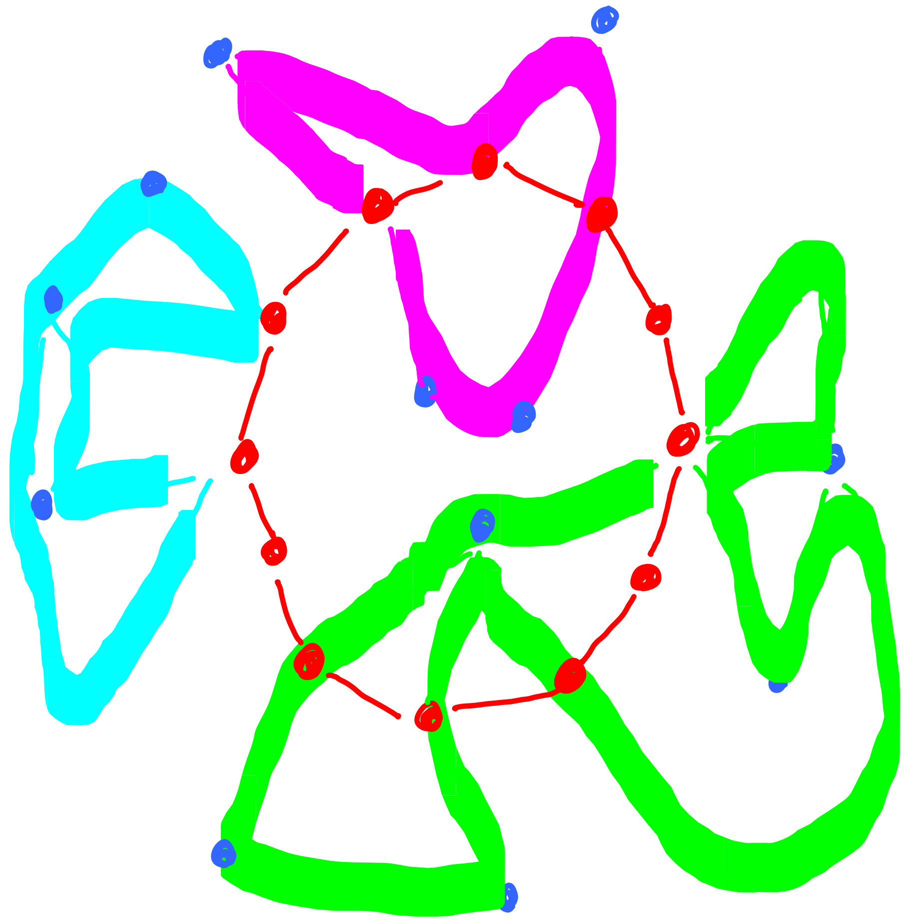
Splice each of these into C. (see text)







Directed Graph (Digraph):

vertex set $V(G)$, **edge set** $E(G)$

function: edges $\longrightarrow$ endpoints

$e \in E(G) \longrightarrow$ ordered pairs of vertices in $V(G)$

Example:

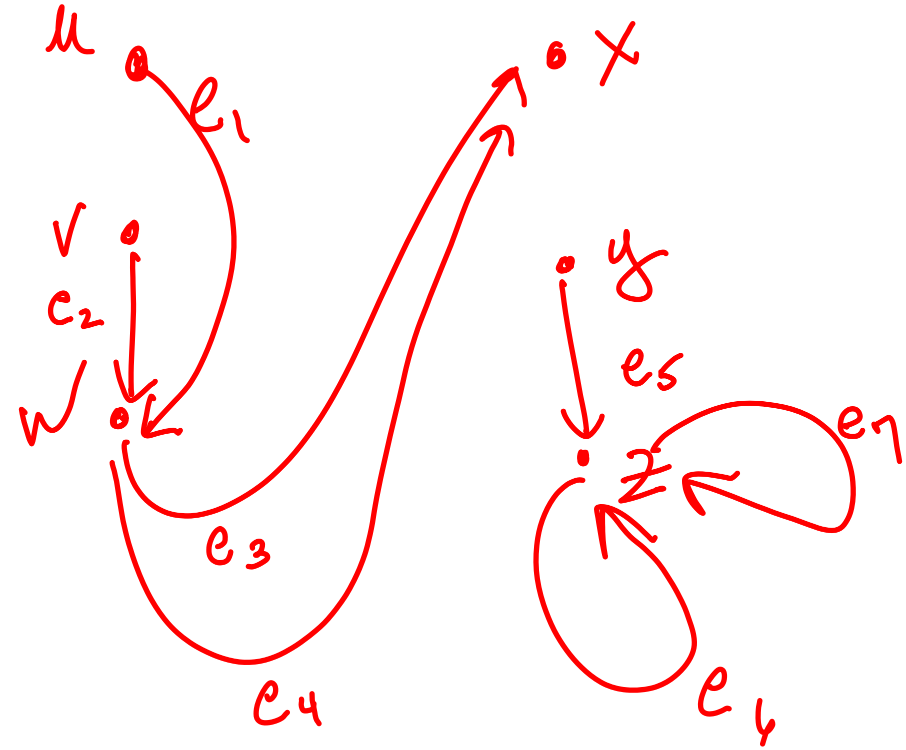
$$V(G) = \{u, v, w, x, y, z\}$$

$$E(G) = \{e_1, e_2, \ldots, e_7\}$$

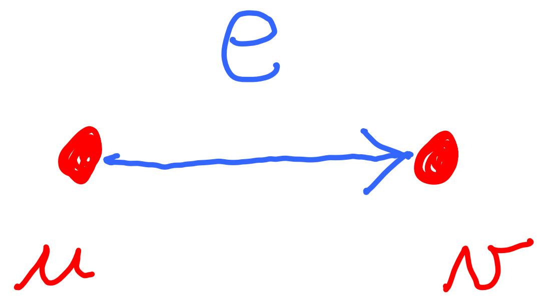
$e_1 \longrightarrow uw, \quad e_2 \longrightarrow vw, \quad e_3 \longrightarrow wx,$

$e_4 \longrightarrow wx, \quad e_5 \longrightarrow yz, \quad e_6 \longrightarrow zz,$

$e_7 \longrightarrow zz$

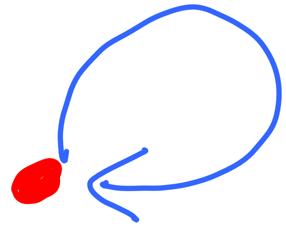


Edge $e \longrightarrow uv$ in digraph

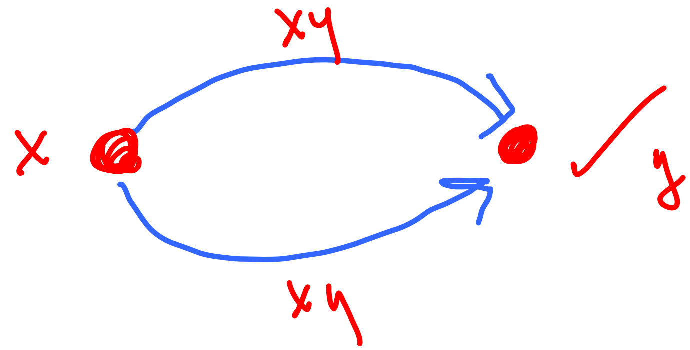


- e is an edge **from** u to v.
- u and v are the **endpoints** of e.
- u is the **tail** of e.
- v is the **head** of e.
- u is a **predecessor** of v.
- v is a **successor** of u.
- "$u \longrightarrow v$"

loop:



multiple edges:



loopless digraph: no loops allowed

outdegree: $d^+(v)$ # of successors of v

indegree: $d^-(v)$ # of predecessors of v

successor set: $N^+(v)$

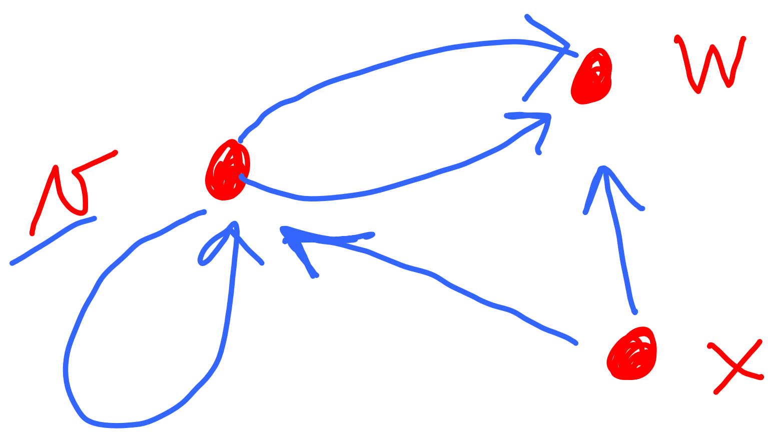
predecessor set: $N^-(v)$

$N^+(v) = \{w, v\}$

$N^-(v) = \{x, v\}$

$d^+(v) = 3$

$d^-(v) = 2$



Proposition 1.4.18.

$$\sum_{v \in V(G)} d^{+}(v) = e(G)$$

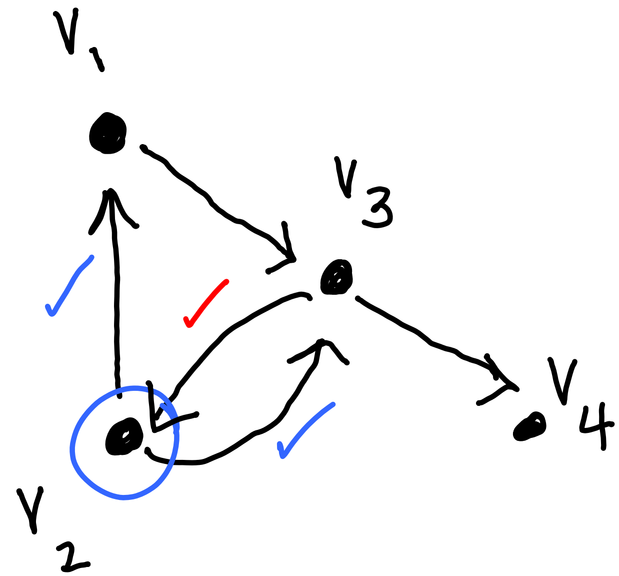
$$\sum_{v \in V(G)} d^{-}(v) = e(G)$$

Given digraph G with

$$V(G) = \{v_1, \dots v_n\}$$

Adjacency matrix $A(G)$ $\quad (n \times n)$

$$A(G)[i,j] = \text{number of edges from } v_i \text{ to } v_j$$



$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

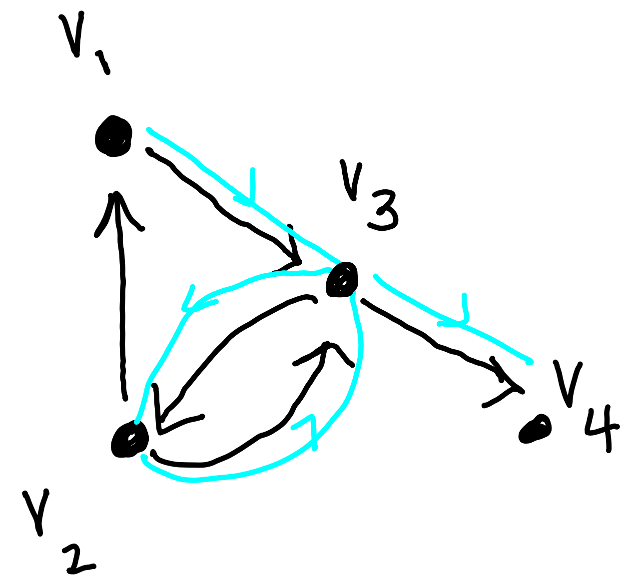
Walk of length k in digraph G:

sequence of vertices and edges of G:

$$v_0, e_1, v_1, e_2, \ldots e_k, v_k$$

where $e_i = v_{i-1}v_i$ for all i.

(Can omit edges if simple)



u, v - walk if first vertex is u and last is v

trail if no repeated edges

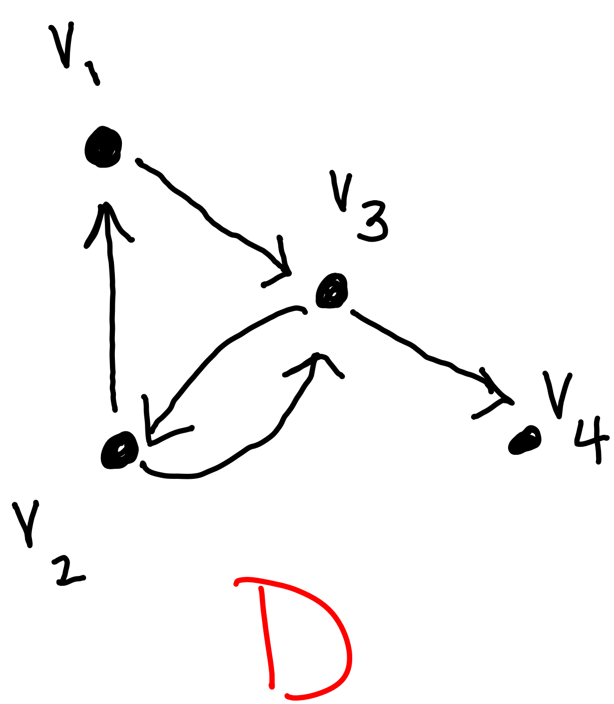
path if no repeated vertex

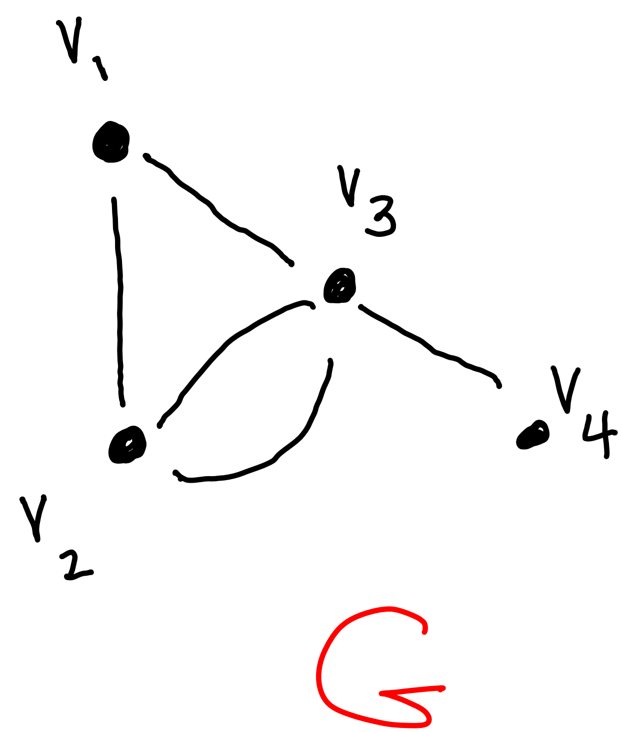
closed if $v_0 = v_k$

cycle closed trail of length at least 1, with no repeated vertex, except first = last

circuit closed trail

The **underlying graph** of digraph D is the graph G obtained from D by regarding the edges of D as unordered pairs.



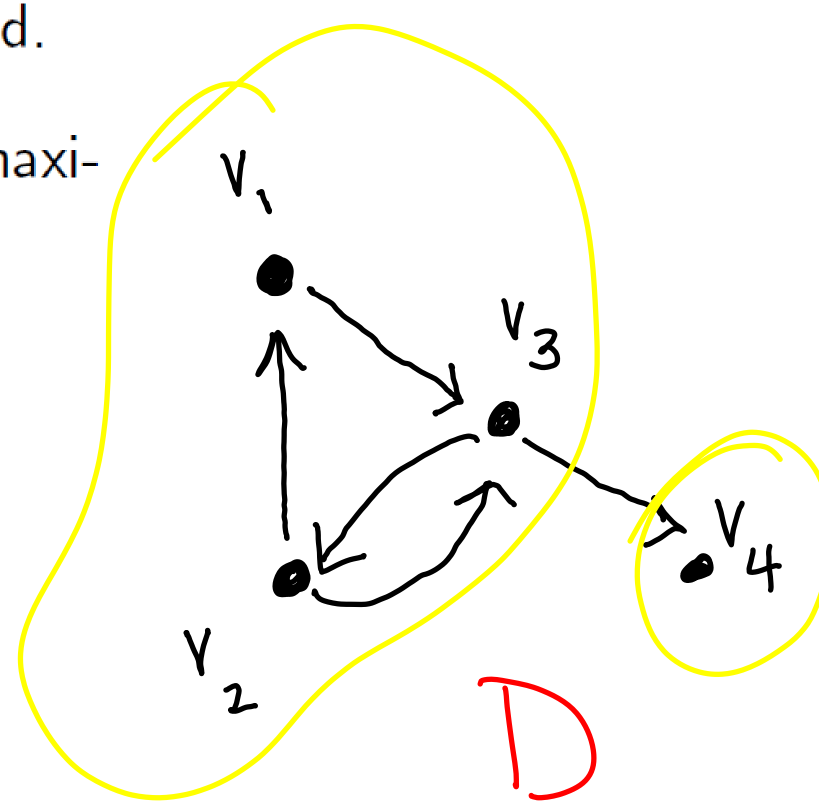


Vertex u is **connected** to vertex v if there is a u, v-path in G.

Digraph D is **strongly connected** if there is a path from u to v for every pair (u, v) of vertices.

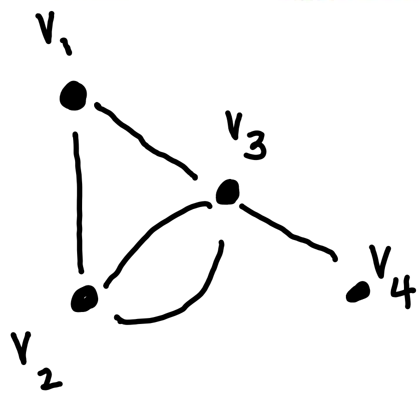
D is **weakly connected** if G is connected.

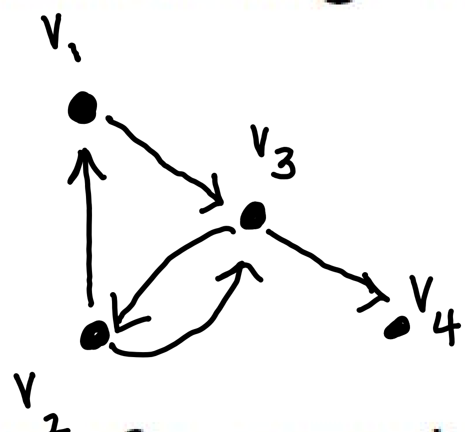
The **strong components** of D are the maximal strongly connected subgraphs of D.



Orientation D of simple graph G:

digraph obtained by assigning orientation $x \longrightarrow y$ or $y \longrightarrow x$ to each edge xy of G.





Tournament: orientation of a complete (simple) graph.

Note: Tournament need not have a “winner”.

However:

Proposition 1.4.~~13~~ 30. Every tournament has a vertex from which every other vertex can be reached by a path of length at most 2.

Proof. Let z be a vertex of maximum outdegree....

- strongly rec. ex.
- note different proof from book

Example: Every tournament has a directed path which includes every vertex.

Proof.

- another strongly recommended ex.
- use this to practice a proof by extremality

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A trail which contains every edge of digraph G is called an **Eulerian trail**.

Theorem 1.4.24.
A weakly connected digraph G is Eulerian if and only if ...