Hamiltonian Cycles

Hamiltonian cycle (path) in G: spanning cycle (path) in G.

G **is Hamiltonian** if G has a Hamiltonian cycle

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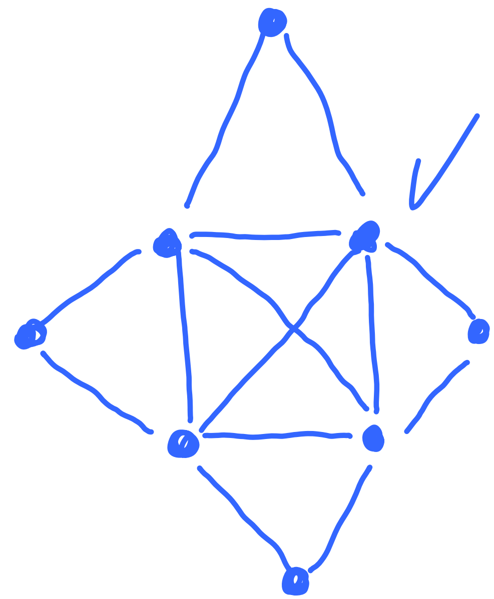
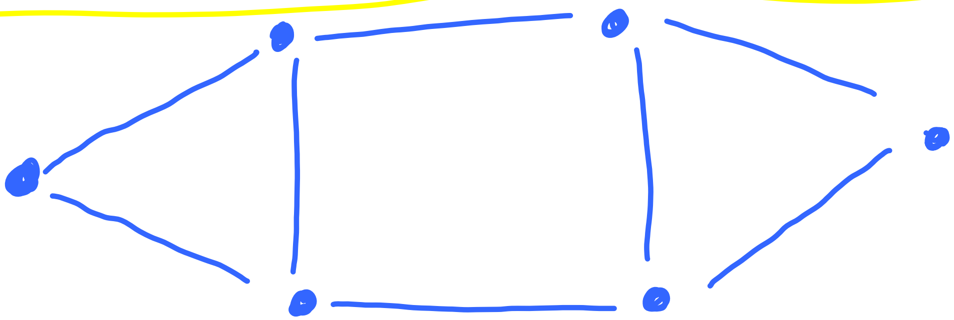
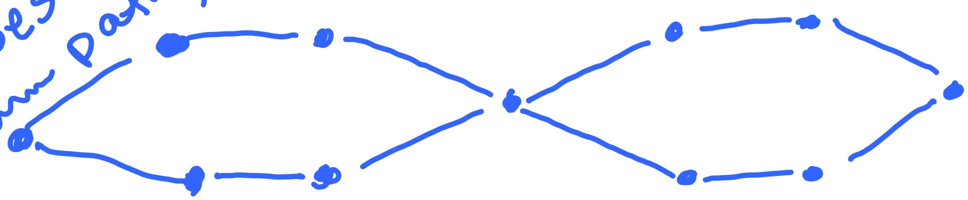




K_n ?

No (but does have Ham path)



$K_{n,m}$?

(Ham iff $n=m$

$Q_k \to$ yes - binary reflected Gray code

Conjecture: Every connected vertex transitive graph has a Hamiltonian path ~~is Hamiltonian~~ except for Petersen, K_2, K_1, + one or two more

G is <u>vertex transitive</u> if for any vertices u and v there is an isomorphism $f: G \to G$ such that $f(u) = v$

<u>Examples</u>

K_n

C_n

Q_k

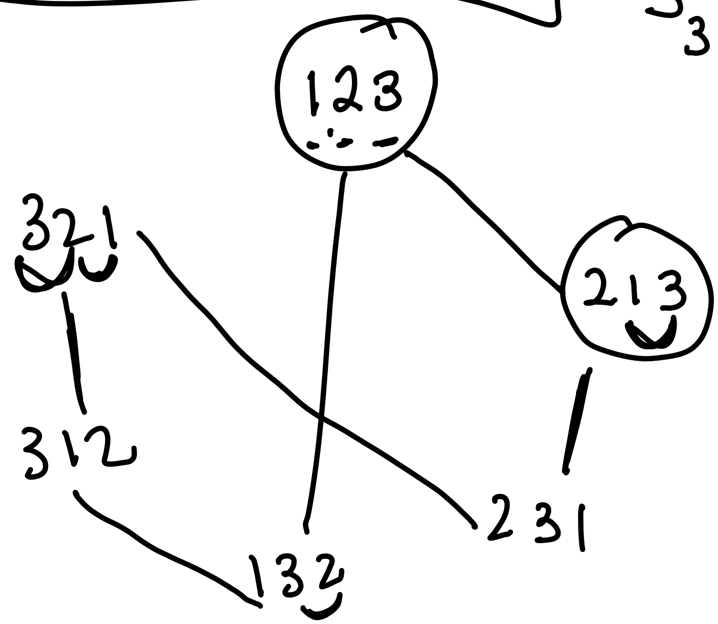
Petersen

$K_{n,n}$

vertex transitive

$\boxed{\text{Cayley graphs of groups}}$

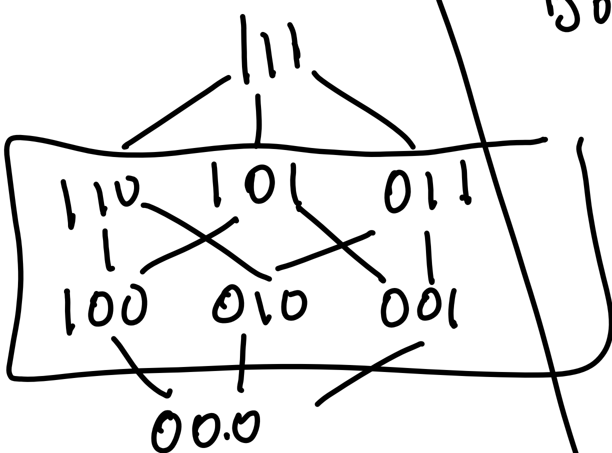
Cayley graph of S_3 with generating set $\{(12), (23)\}$



Middle Levels Problem

Partially ordered set

Boolean Lattice



Q_3

Yes: Torsten Mutze

arxiv

11111 (1)

11110 (5)

11100 (10)

11000 (10)

1000 (5)

00000 (1)

Q_5

regular

bipartite

vertex transitive

Is it Hamiltonian?

Subgraph of Q_k Induced by middle 2 levels when k is odd

recent result: yes

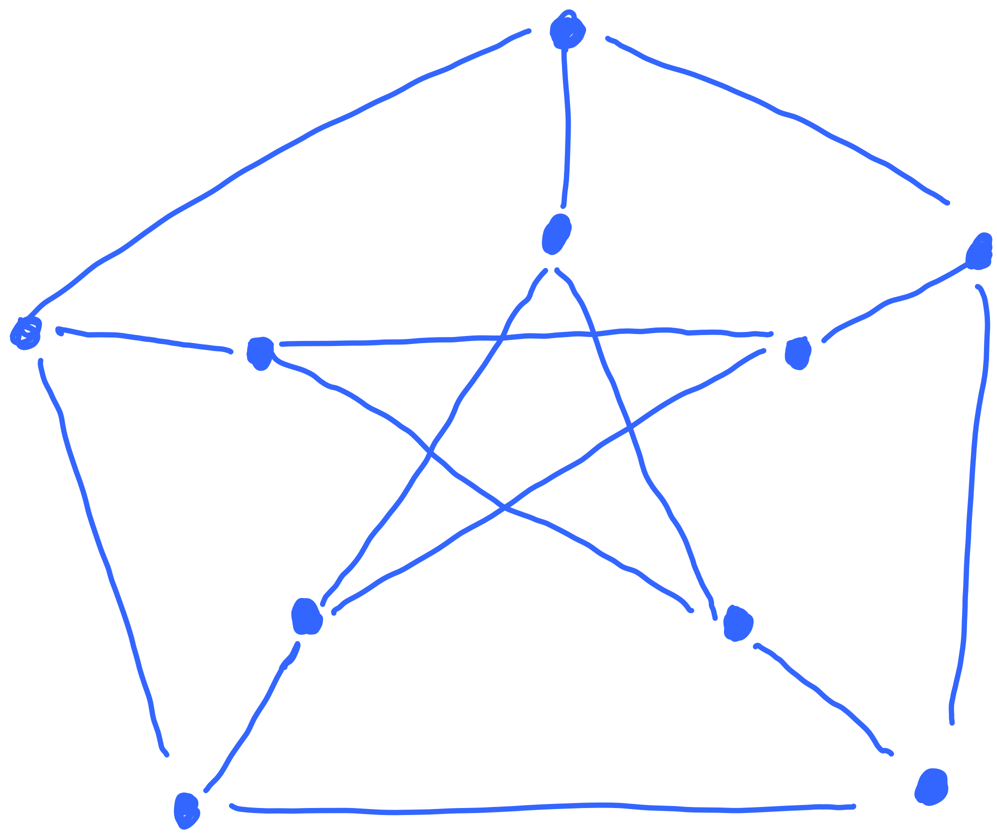
<u>**Circumference of G**</u>: length of a longest cycle.

Example: Petersen graph

Hamiltonian cycle?

Hamiltonian path?

Circumference?



At this time,

no polynomial-time algorithms are Known for Hamiltonian cycle, path, & for circumference.

(NP hard)

No useful necessary + sufficient conditions.

But . . .

Necessary condition

If G is Hamiltonian then for any $S \subseteq V$, $G - S$ has at most $|S|$ components.

Sufficient condition

(Dirac) If G is simple and $n(G) \geq 3$ and $\delta(G) \geq n(G)/2$, then G is Hamiltonian.

Necessary and sufficient but ...

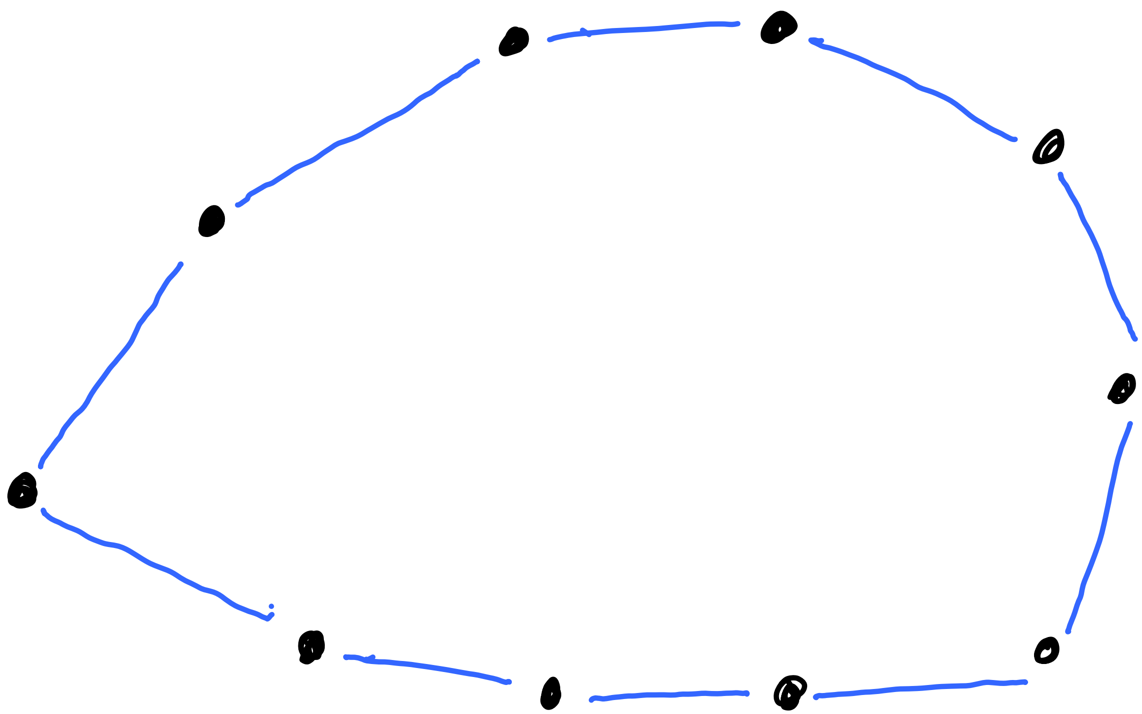
(Bondy-Chvátal) G is Hamiltonian if and only if the closure of G is Hamiltonian.

Sufficient condition

(Chvátal) Suppose G has degree sequence $(d_1, \ldots, d_n)$ where $d_1 \leq d_2 \leq \ldots \leq d_n$. If $i < n/2$ implies that either $d_i > i$ or $d_{n-i} \geq n - i$, then G is Hamiltonian.

Cycle C $\qquad$ $S \subseteq V(C)$

How many components in $G-S$? (compare to $|S|$)



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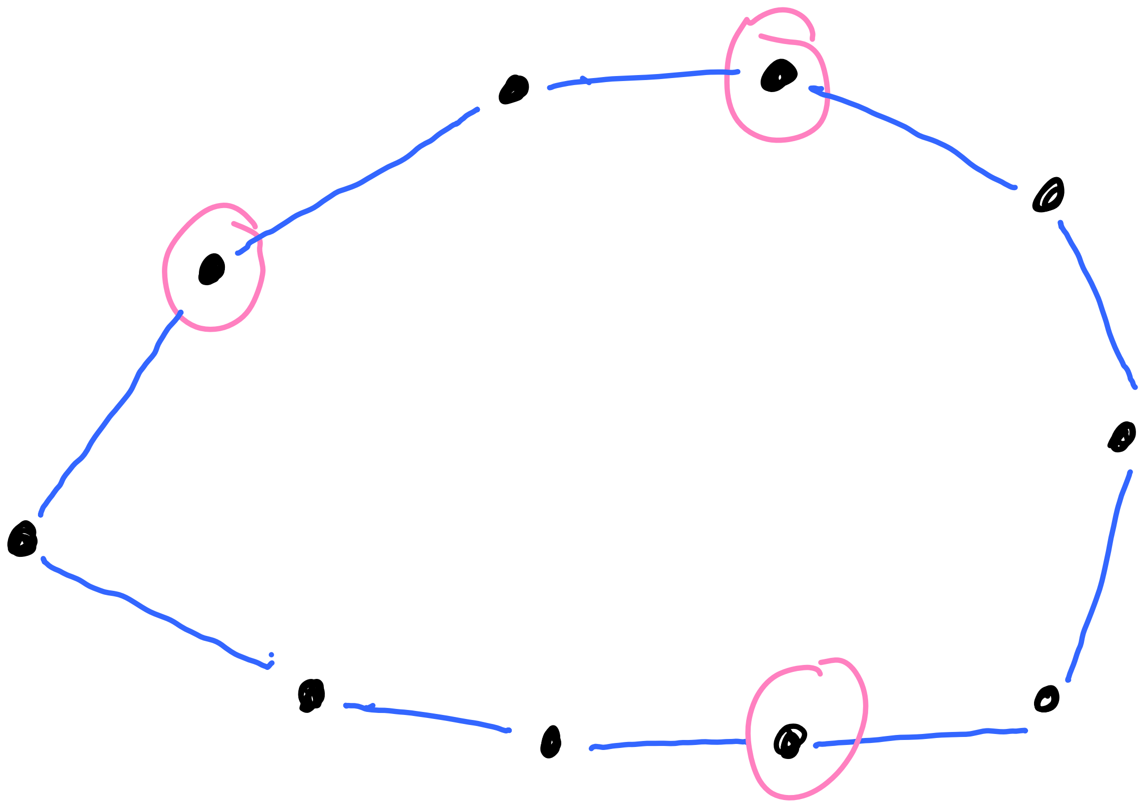

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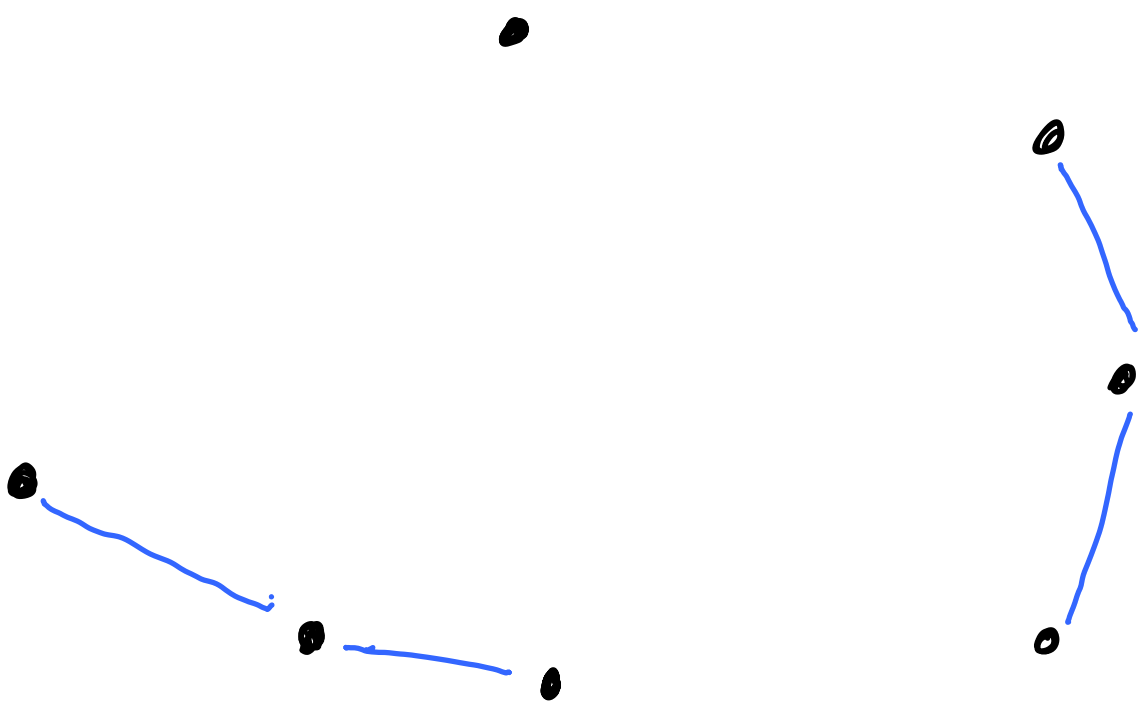

Necessary condition: If G is Hamiltonian then for any $S \subseteq V$, graph $G - S$ has at most $|S|$ components.

Proof.

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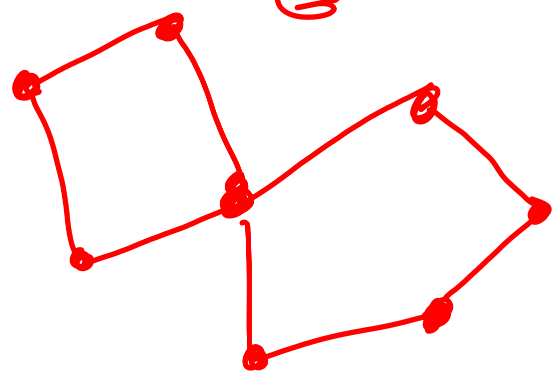
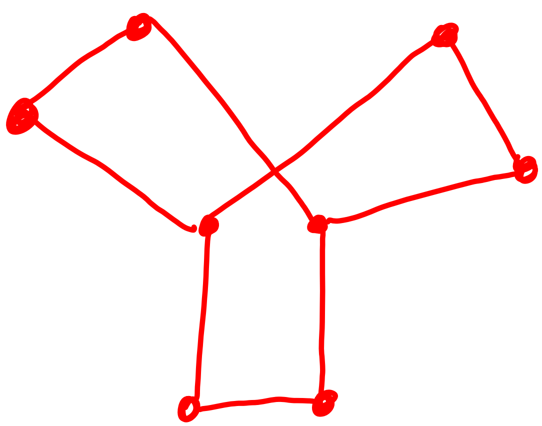
Proof.

Examples

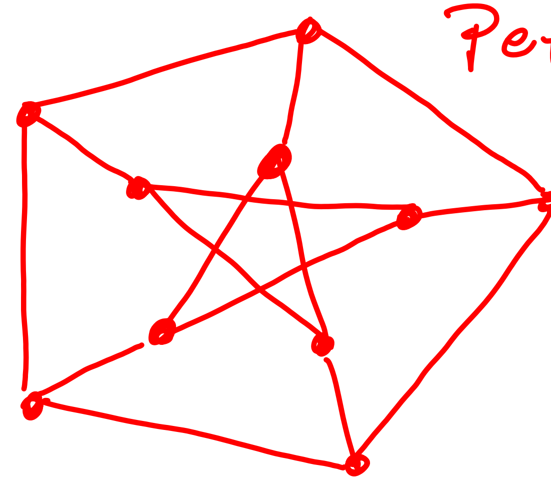
G





H

Petersen



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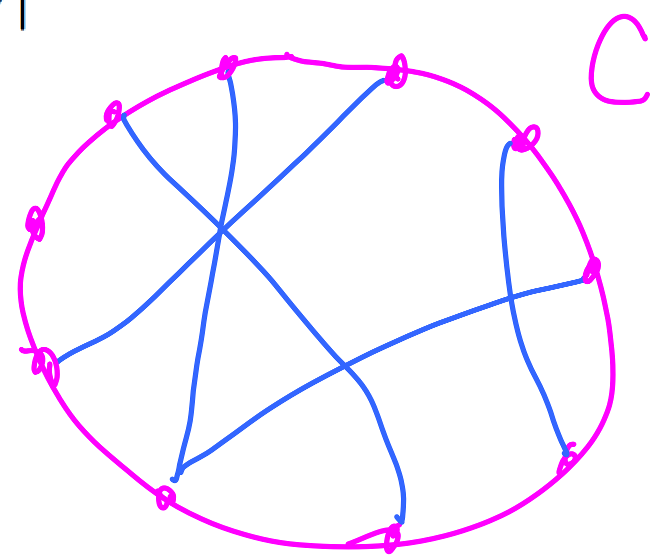

Proof.

Suppose C is a Hamiltonian cycle in G.

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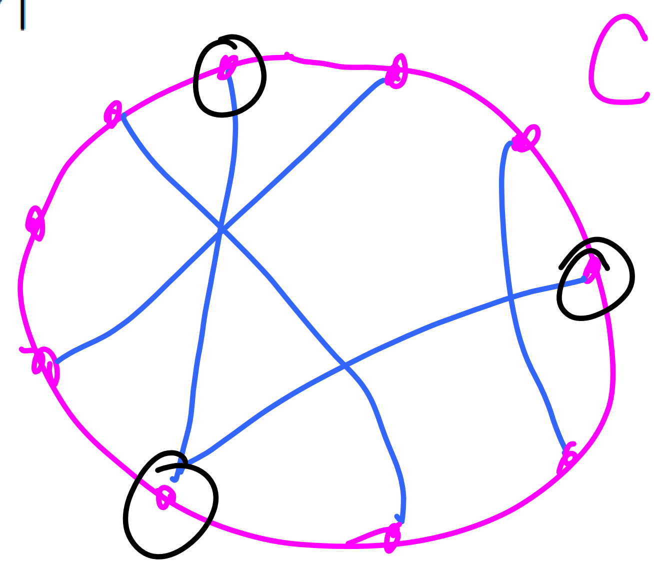

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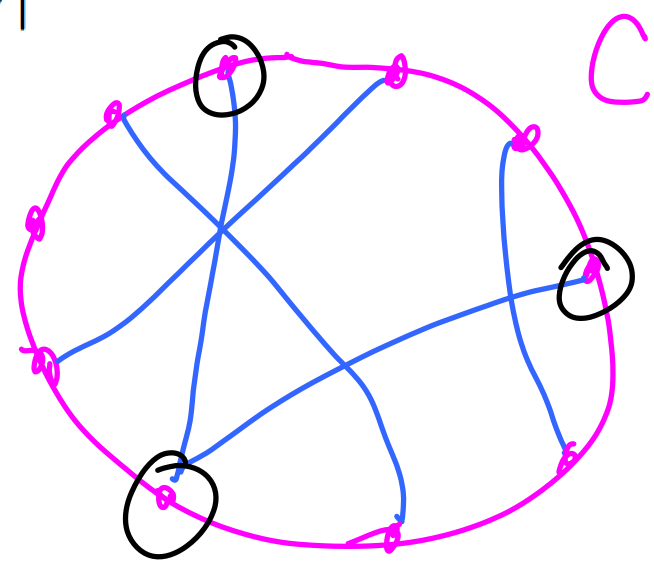

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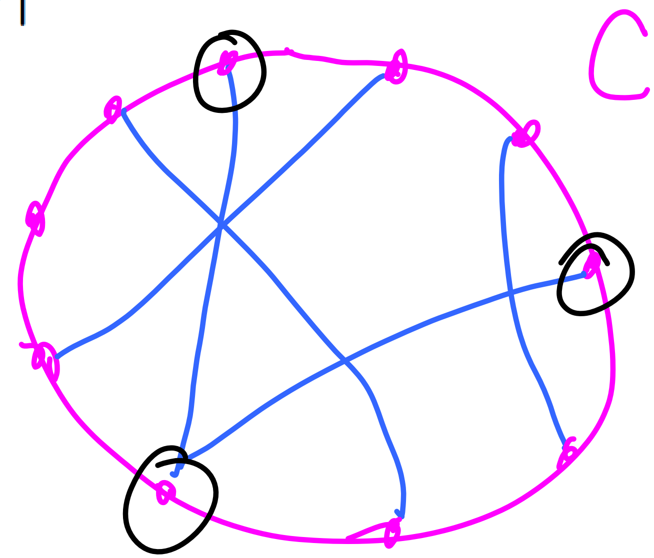

Suppose C is a Hamiltonian cycle in G.

For any $S \subseteq V$, $C - S$ has at most $|S|$ components.

So, $G - S$ has at most $|S|$ components,

(since $C - S$ is a spanning subgraph of $G - S$)

<u>Sufficient condition</u>: (Dirac) If G is simple and $n(G) \geq 3$ and $\delta(G) \geq n(G)/2$, then G is Hamiltonian.

Proof.

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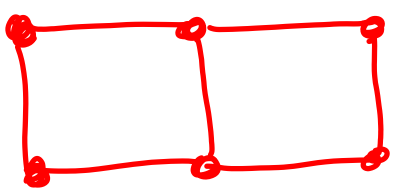
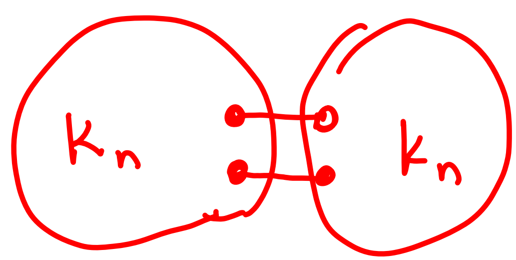
(1952)

Proof.

Examples

$K_{n,n}$ $H_{6,3}$





$T_{n,k}$?

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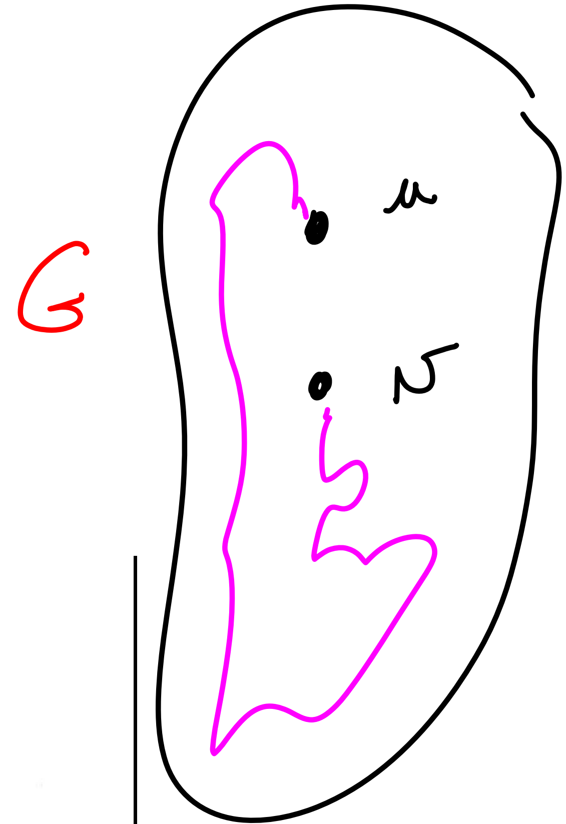
<u>Proof</u>. Suppose Dirac's Thm is false.

Then there must be a <u>counterexample</u>.

Let G be a maximal counterexample, i.e.

G is a counterexample, but

$G + uv$ is <u>not</u> for any nonadj. u, v in $V(G)$



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G is a counterexample, but

G + uv is not for any nonadj. u, v in V(G)

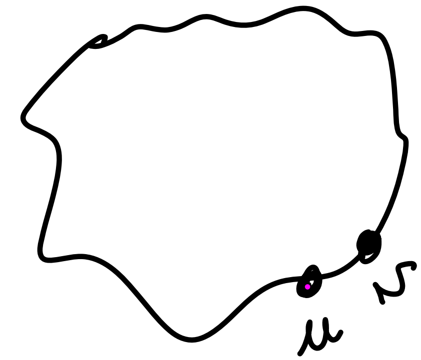
Then: $n(G) \geqslant 3$, $\delta(G) \geqslant n/2$, G not Hamilt.

G not K_n, G + uv is Hamilt for $uv \notin E(G)$

Thm: $n(G) \geq 3$, $\delta(G) \geq n/2$, G not Hamilt.

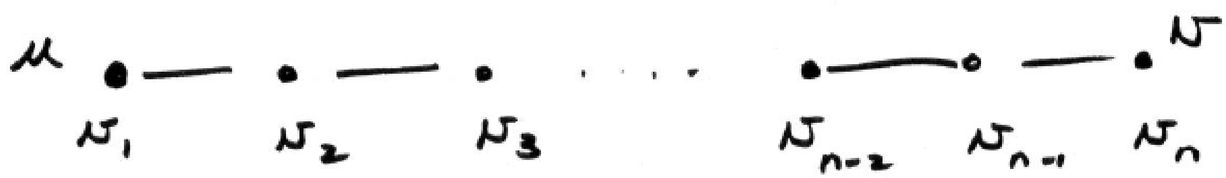
G not K_n, $G + uv$ is Hamilt for $uv \notin E(G)$

C



Let u, v be nonadj in G.

$G + uv$ is Hamit $\therefore$ G has a u, v Hamilt. path



← all in G

Thm: $n(G) \geq 3$, $\delta(G) \geq n/2$, G not Hamilt.

G not K_n, $G + uv$ is Hamilt for $uv \notin E(G)$

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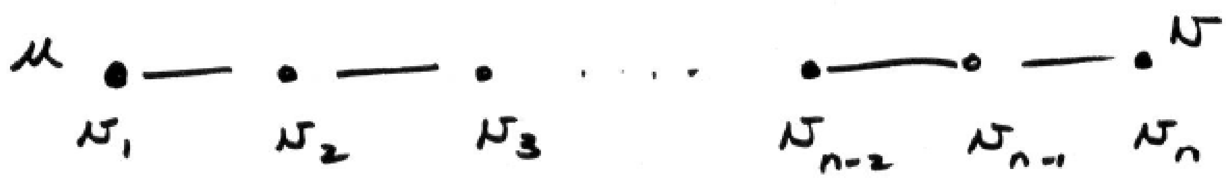

Let $S = \{ j-1 \mid uv_j \in E(G) \}$

$T = \{ i \mid v_i v \in E(G) \}$

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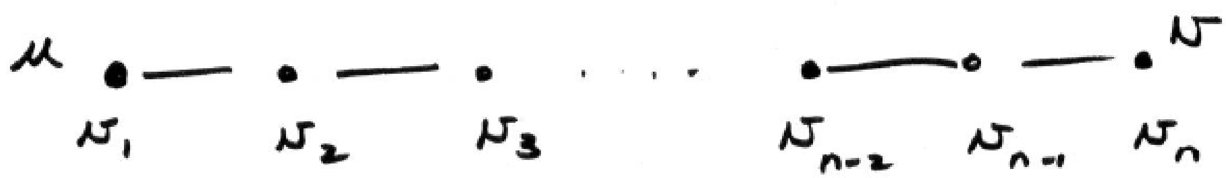

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<u>Note</u>: $n \notin S$ (why) $\quad n \notin T$ (why?)

so $|S \cup T| \leq n-1$ $\quad (*)$

Also, $\underbrace{|S|}_{d(u)} \geq n/2 \quad \overbrace{|T|}^{d(v)} \geq n/2$ $\quad$ (why?)

$|S| + |T| \geq n$

Putting together with $(*)$ means there must be

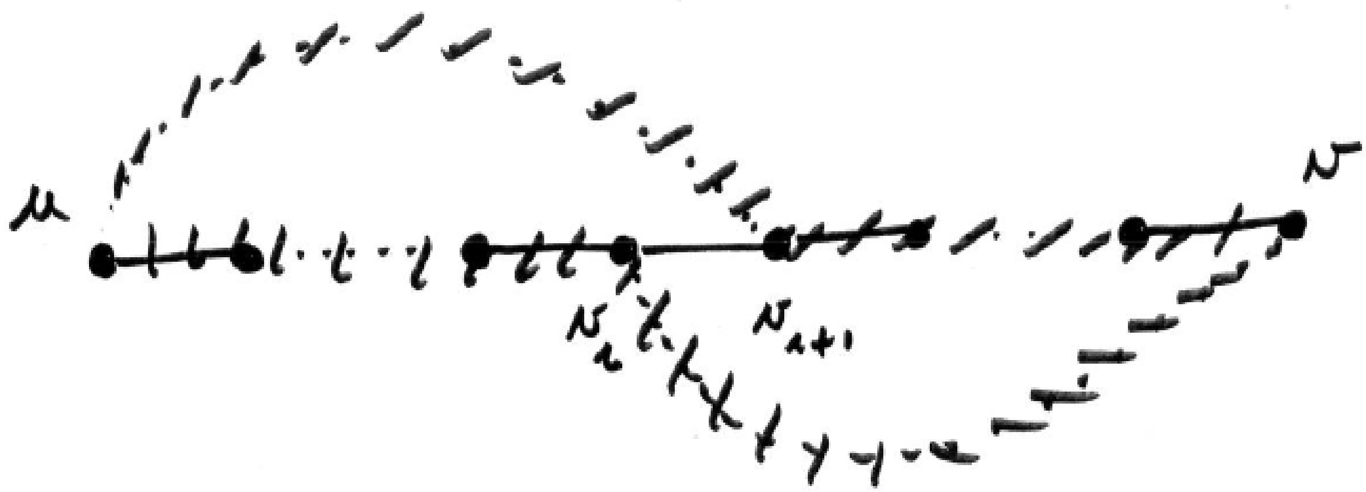
<u>some</u> $i \in S \cap T$

<u>But</u> $i \in S \cap T$ means

$u v_{i+1} \in E \qquad (i \in S)$

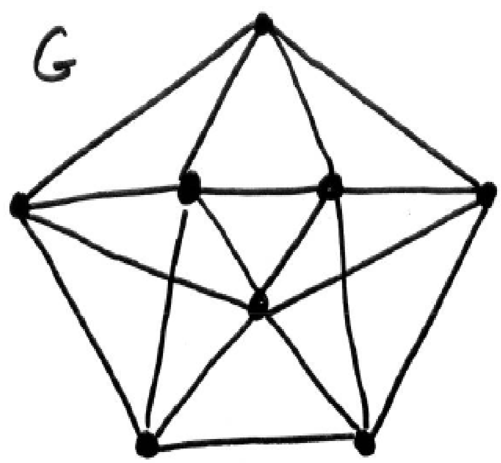
$v_i v \in E \qquad (i \in T)$

So, we have this picture in G:



giving a Hamiltonian cycle in G, a contradiction

Dirac's condition is sufficient,



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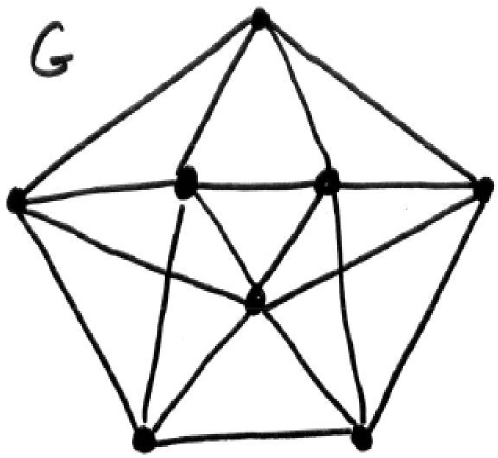
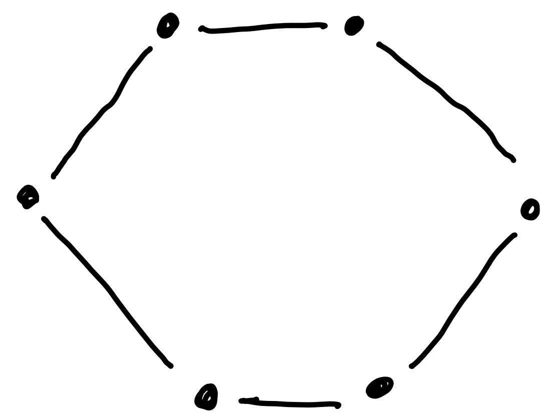


But not necessary



Re-examine proof of Dirac.

Can also be used to show that if

u & v are nonadj & $\underline{\underline{d(u)+d(v) \geq n}}$

then if $G + uv$ Hamilt., so is G

Proof of Dirac's Theorem also gives:

Lemma 7.2.9 (Ore): If G is simple and u and v are nonadjacent vertices with (1960)

$$d(u) + d(v) \geq n(G),$$

then G is Hamiltonian iff $G+uv$ is Hamiltonian.

Define the **closure** of G to be the graph obtained from G as follows: until no longer possible, join non-adjacent vertices whose degrees sum to at least $n(G)$.

Then it follows from Ore's lemma:

Theorem 7.2.11 (Bondy-Chvátal): G is Hamiltonian iff the closure of G is Hamiltonian.

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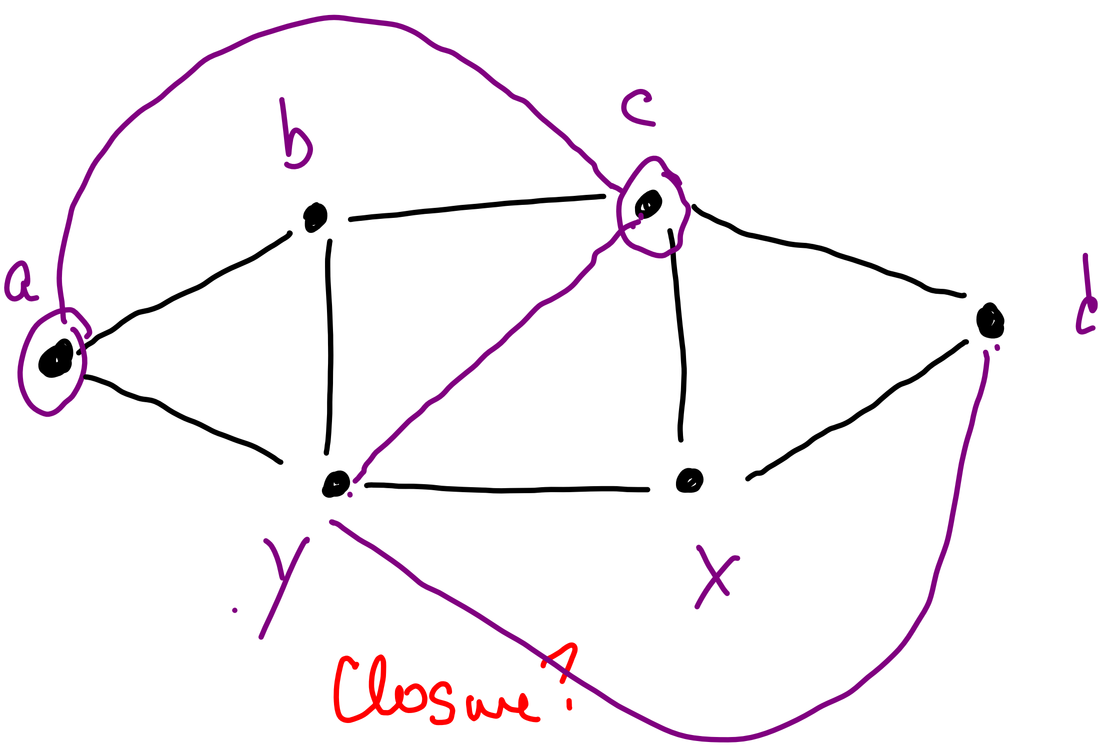
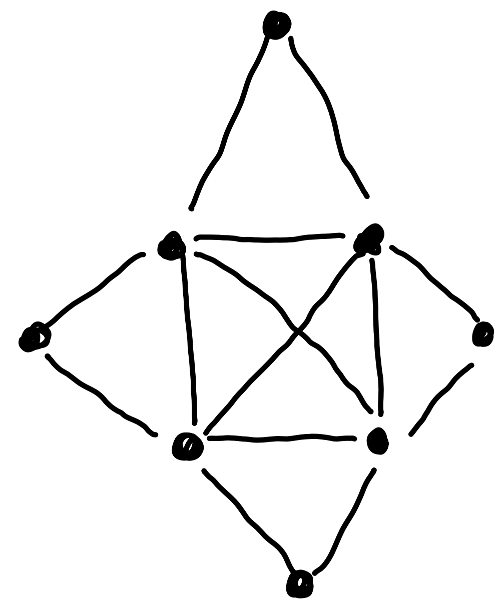
Lemma 7.2.12

Well- defined



Closure?



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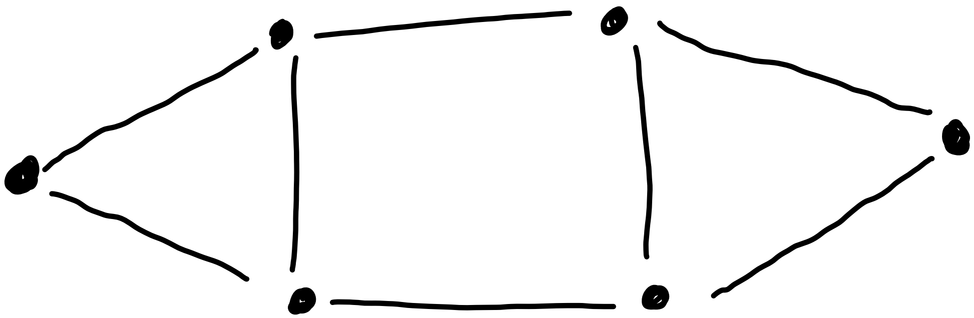

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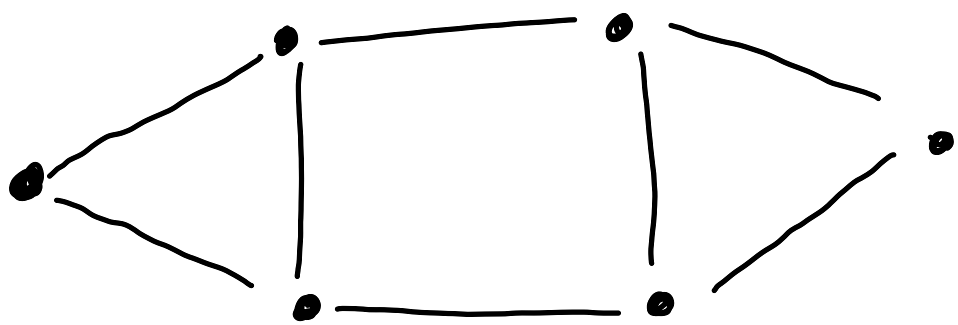
Then it follows from Ore's lemma:

Theorem 7.2.11 (Bondy-Chvátal): G is Hamiltonian iff the closure of G is Hamiltonian.

It follows that if $n(G) \geq 3$ and the closure of G is complete, then G is Hamiltonian.

Example:



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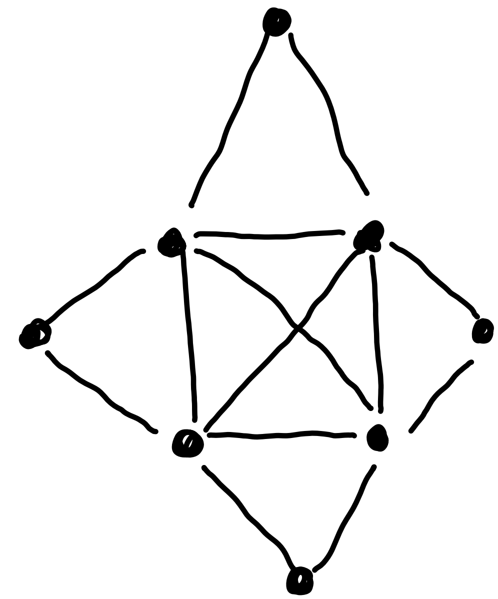
Example:

However, ...



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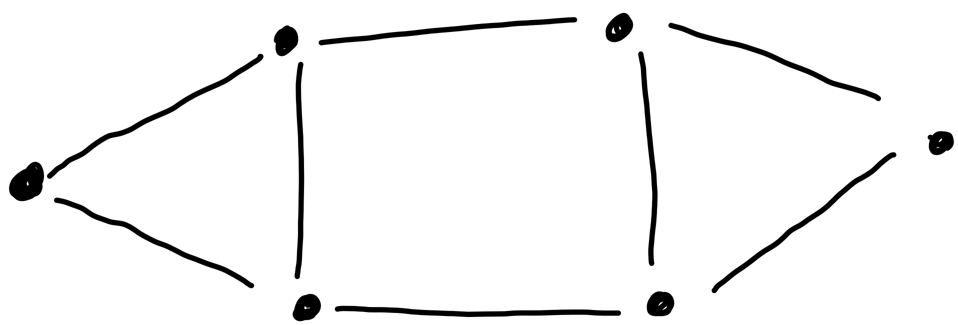

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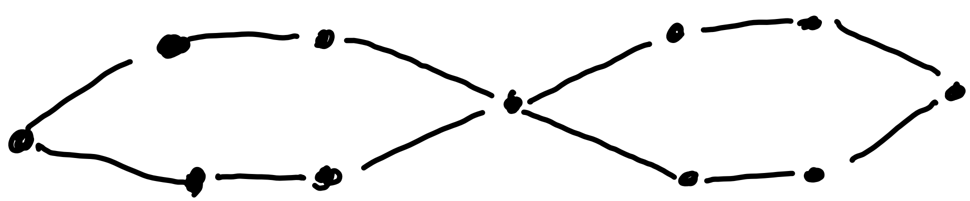


If G satisfies Dirac, what can you say about closure?

If G does not satisfy Dirac, could closure still be complete?

Sufficient condition **(Chvátal):** Suppose G has degree sequence $(d_1, d_2, \ldots, d_n)$ where $d_1 \leq d_2 \leq \ldots \leq d_n$. If $i < n/2$ implies that either

$$d_i > i \text{ or } d_{n-i} \geq n - i,$$

then G is Hamiltonian.

Proof. (next time)

Traveling Salesman Problem

Given weighted K_n (non-negative weights), find <u>**minimum weight Hamiltonian cycle.**</u>

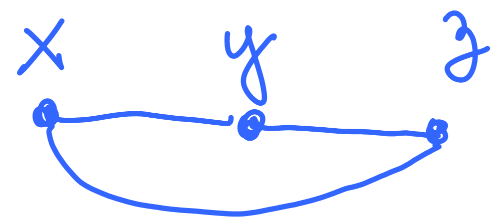
(No polynomial-time solution known.)

Special case:

If the triangle inequality holds for all vertices $x, y, z \in V$:

$$w(xy) + w(yz) \geq w(xz)$$

then it is possible to find a "good approximation" to the TSP.



TSP with Triangle Inequalty

Finding a Spanning Cycle
Whose Weight is at most Twice Optimal

1. Let C^* be the minimum weight Hamiltonian cycle and let T be a minimum spanning tree.

   $w(T) \le w(C^*)$

2. Duplicate every edge of T to get T', which is Eulerian.

   $w(T') \le 2\,w(T)$

3. Find an Eulerian tour W in T'.

   $w(W) = w(T')$

4. Iteratively, until only a spanning cycle C remains, eliminate first repeated vertex remaining on W.

   $w(C) \le w(W)$
   (triangle ineq.)

5. Show: Resulting cycle C is a Hamiltonian cycle of weight at most $2w(C^*)$.

   $w(C) \le 2\,w(C^*)$