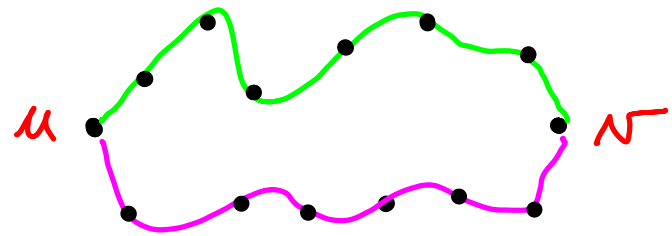
Characterization of 2-connected Graphs

<u>**Theorem**</u> [illegible]:
Graph G with $n(G) \geq 3$ is 2-connected if and only if for each pair u, v of distinct vertices, G has at least 2 internally disjoint u, v paths.



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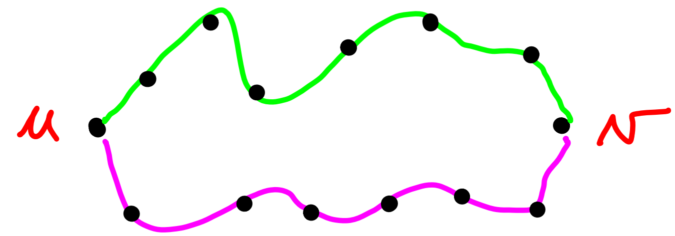

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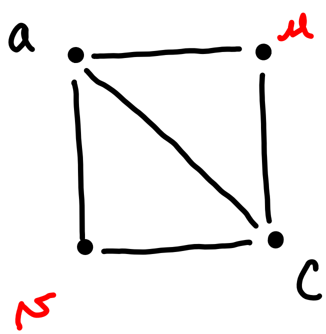
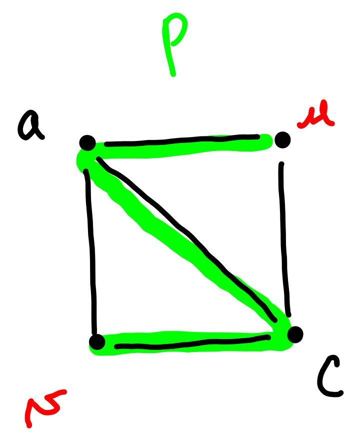
But . . .



this graph is 2- connected

and here is a u, v path P



yet ???

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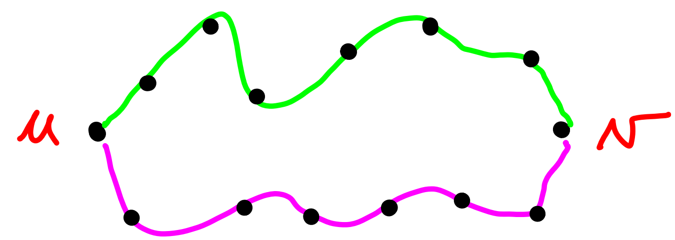
Theorem [Whitney 1932]:
Graph G with $n(G) \geq 3$ is 2-connected if and only if for each pair u, v of distinct vertices, G has at least 2 internally disjoint u, v paths.

Proof ($\Leftarrow$) In this case, no pair of vertices u, v can be disconnected by the removal of a single vertex. Thus $\kappa(G) \geq 2$.



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($\Rightarrow$) Assume G is 2-connected. Proof by induction on $d(u, v)$.

(Basis) If $d(u, v) = 1$, uv is not a cut edge since $\kappa'(G) \geq \kappa(G) \geq 2$. Therfore $G - uv$ contains a (u, v)-path. The edge uv is the other path in G.

(Inductive step:)

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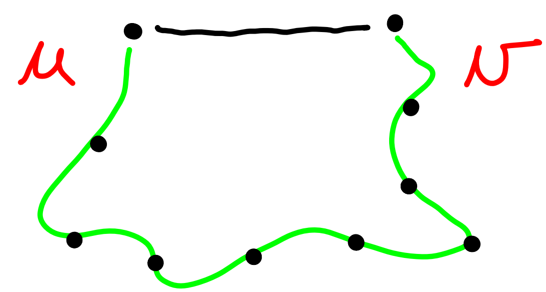
(Inductive step:)

Recall
$\kappa(G) \leq \kappa'(G) \leq \delta(G)$

$\Rightarrow \kappa(G) \geq 2$

$\therefore \kappa'(G) \geq 2$

$\therefore$ no cut edge



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assume true for vertices a distance $< t$ apart.

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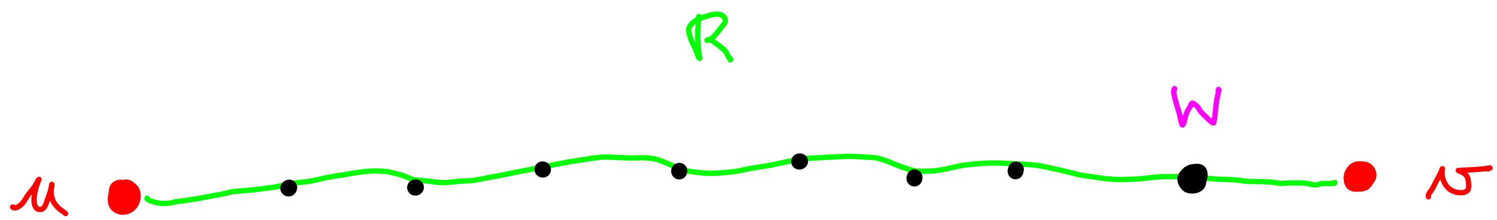
There is a u, v path R of length t. Let w be the predecessor of v on R



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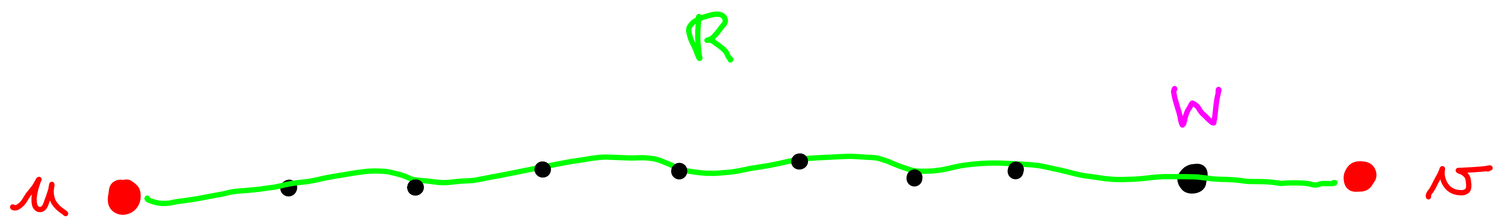

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$\exists$ internally disjoint u, w paths P, Q

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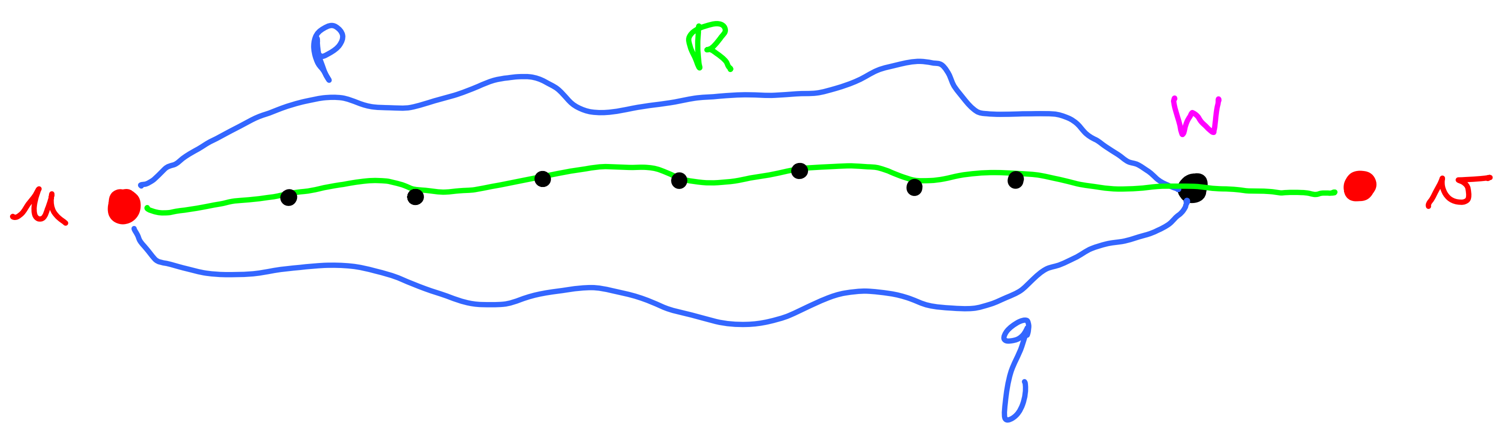

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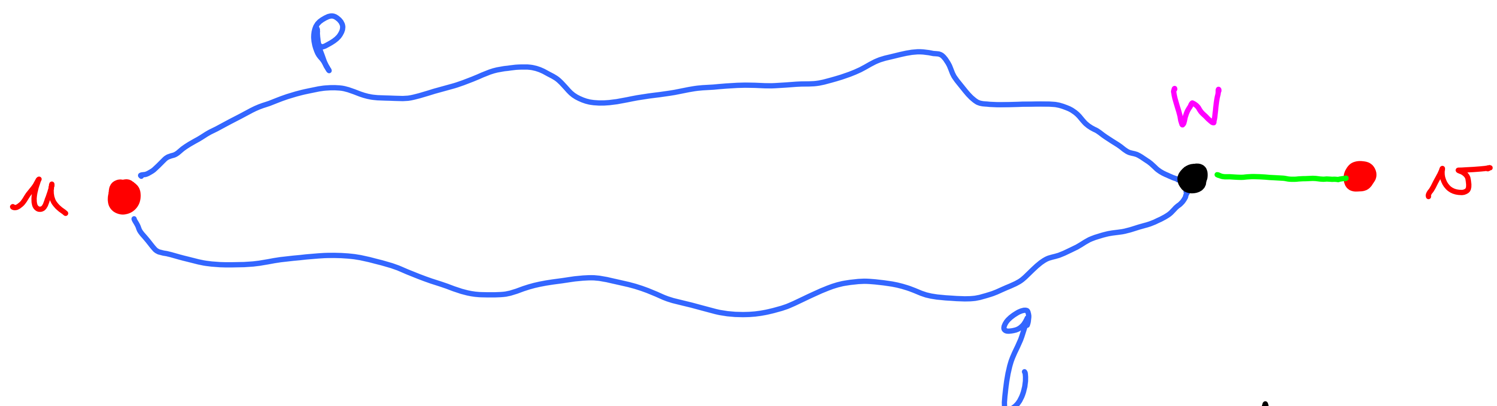

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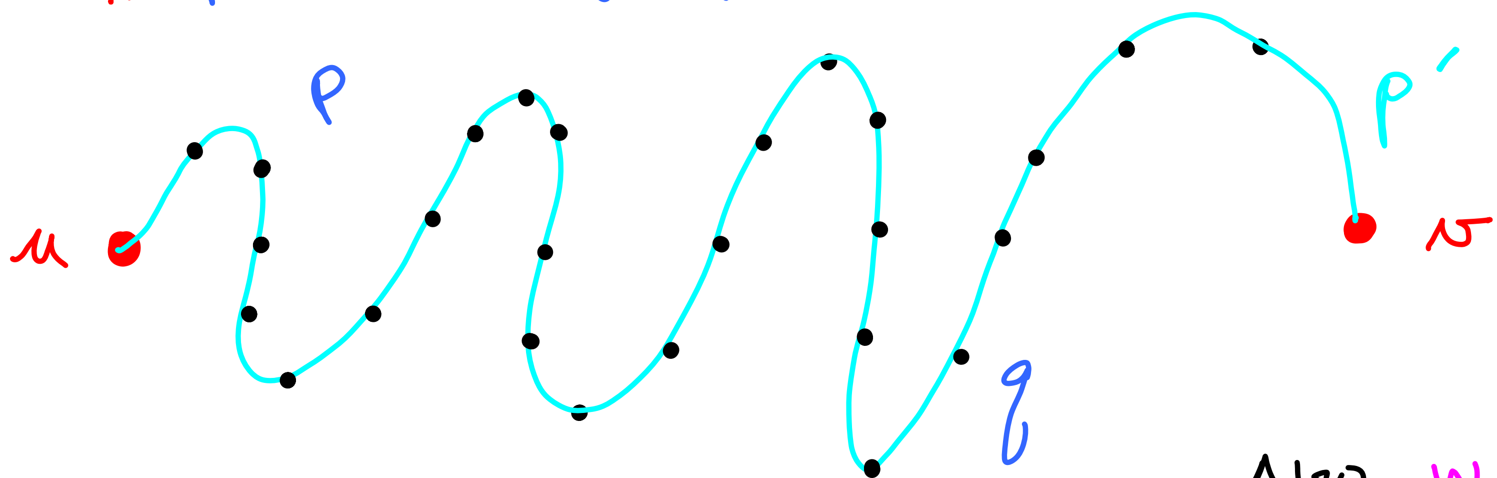

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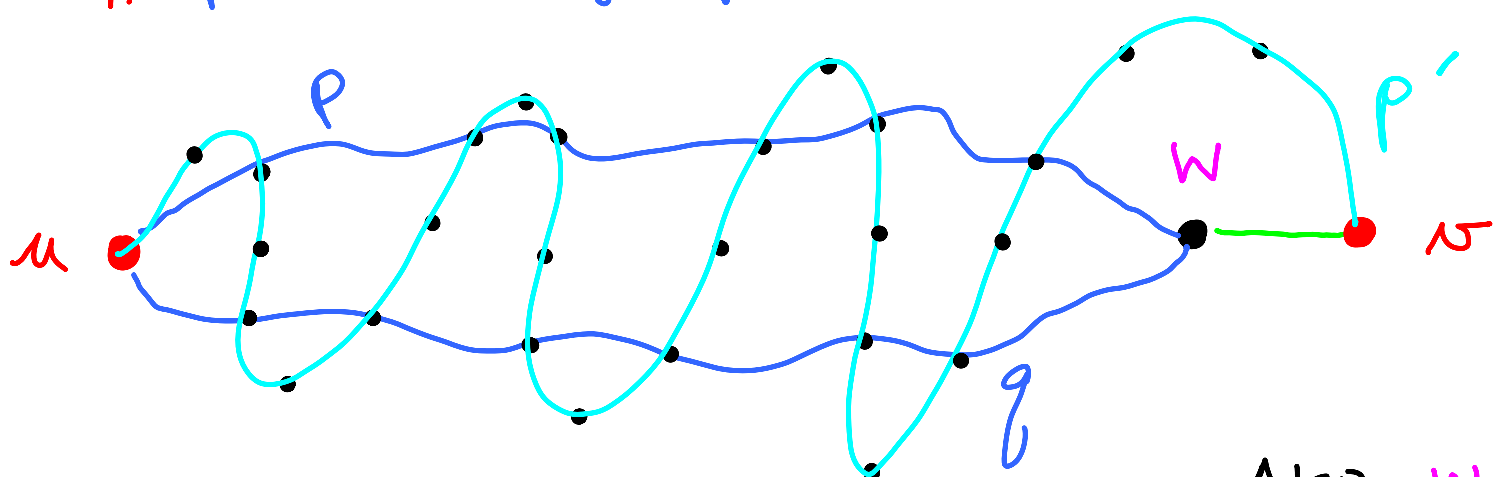

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Also, w is not a cut vertex (why?) so $G - w$ has a u,v path p'

Now we can get 2 internally disjoint u, v paths:

Let x be the last vertex of p' which is also on p or q.

then do this:

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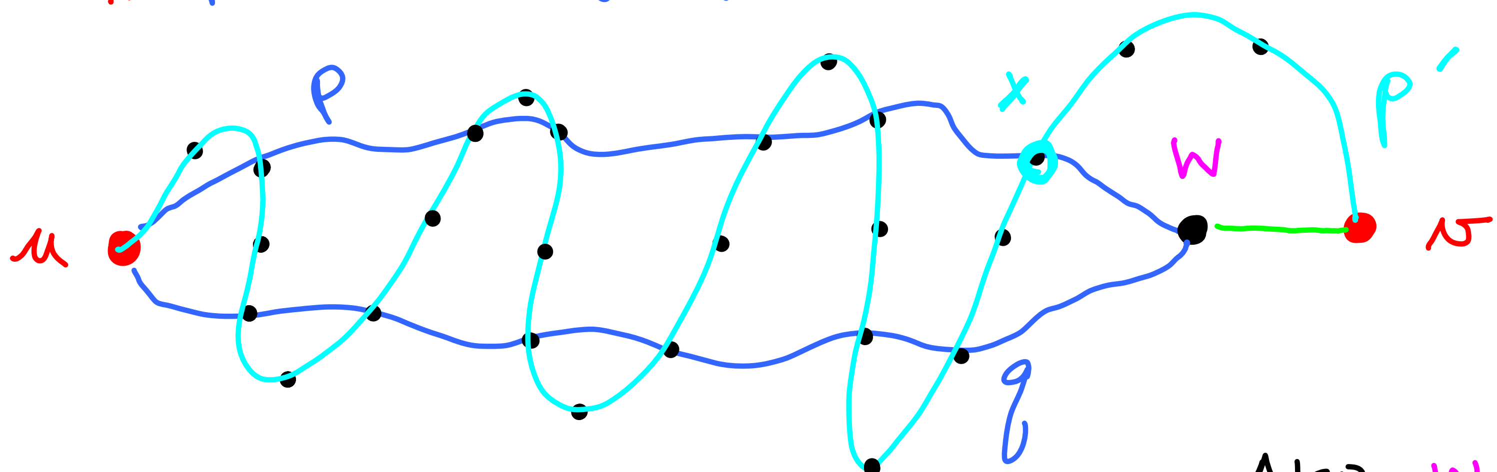

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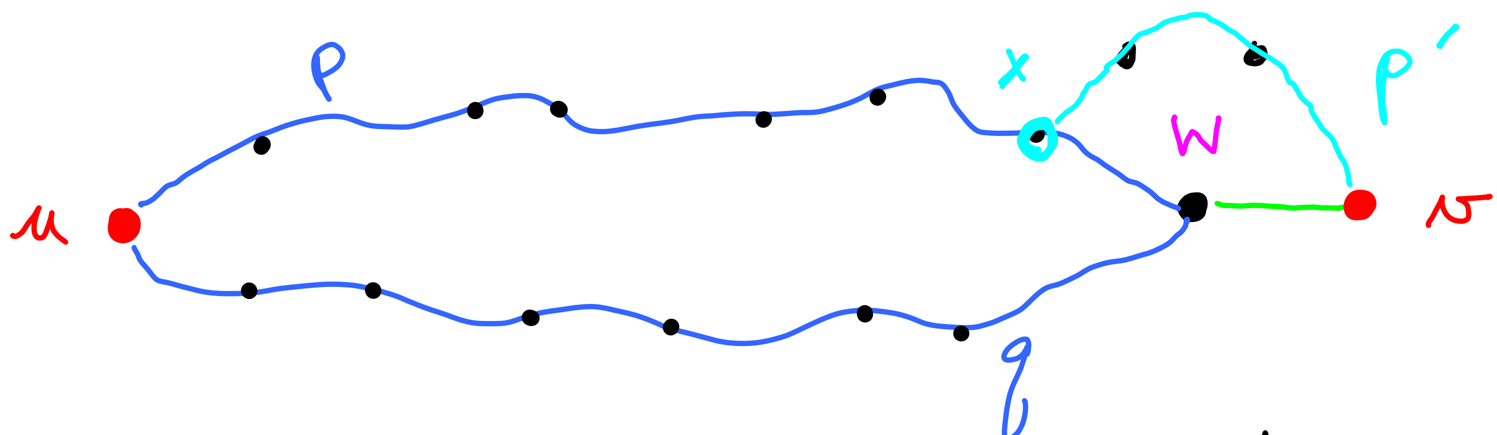

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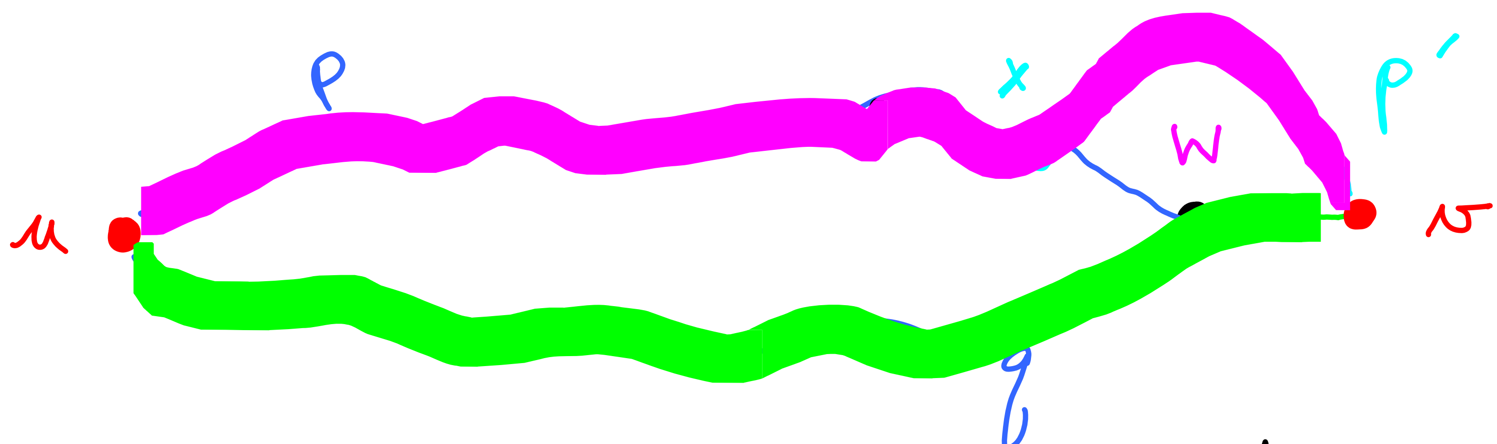

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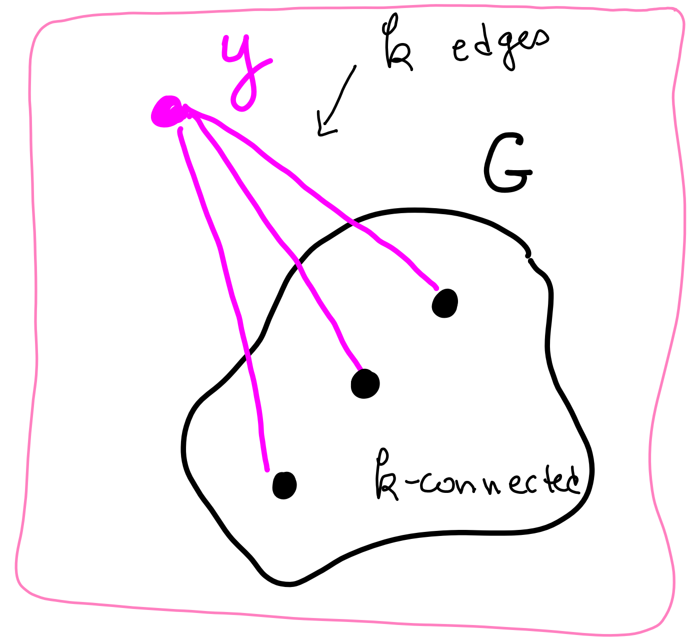
<u>Expansion Lemma</u> 4.2.3. If G is k-connected and G' is obtained from G by adding a new vertex y adjacent to at least k vertices of G, then G' is k-connected.



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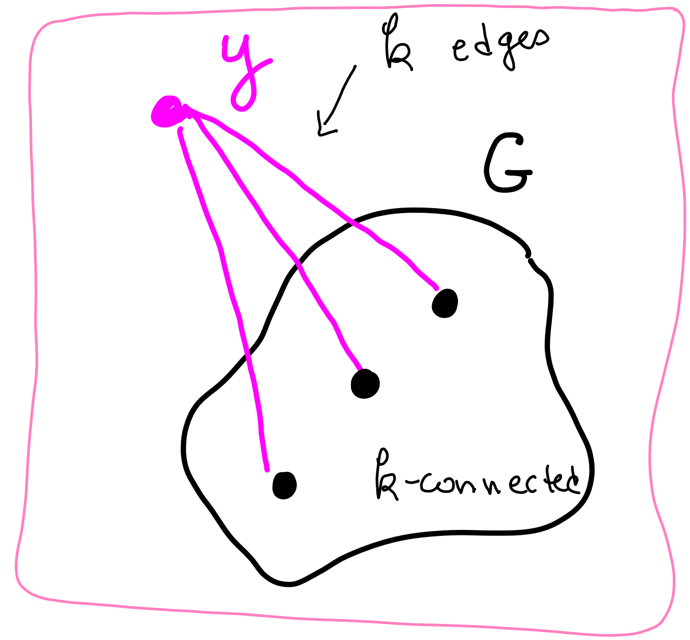


G' is k-connected

(Show $|S| \geq k$)

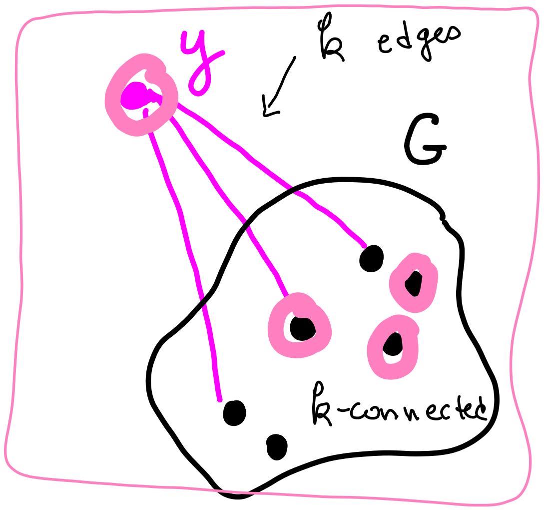
Proof. Suppose S separates G'.

If $y \in S$, then $S - y$ separates G. Thus, $|S - y| \geq k$, i.e., $|S| \geq k + 1$.

If $y \notin S$, then if $N(y) \subseteq S$ then $|S| \geq |N(y)| = k$.

Otherwise, S separates 2 vertices of G and therefore $|S| \geq k$.

If S disconnects G' then one of these:

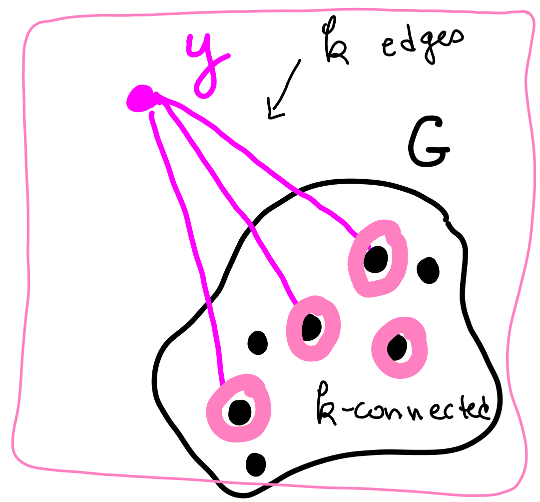


$y \in S$

Then $S - y$ disconnects G

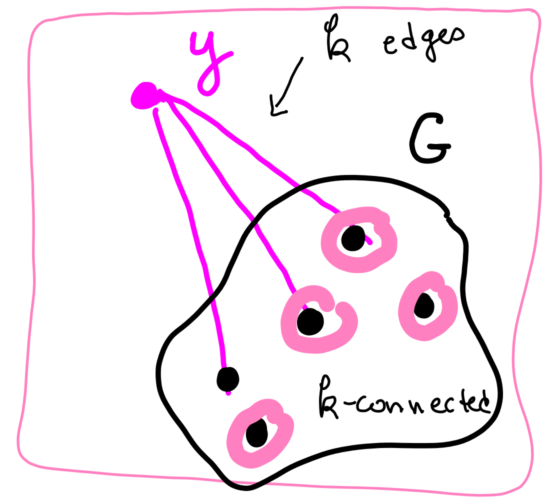
So $|S - y| \geq k$

$\therefore |S| \geq k+1$



$y \notin S$ but $N(y) \subseteq S$

Then $|S| \geq |N(y)| = k$



otherwise

Removing S disconnects G

$\therefore |S| \geq k$

Theorem 4.2.4. If $n(G) \geq 3$, the f.a.e.

(A) G is 2-connected

(B) For all $x, y \in V(G)$, there are internally disjoint x, y-paths.

(C) For all $x, y \in V(G)$, there is a cycle containing x and y.

(D) $\delta(G) \geq 1$ and every two edges of G lie on a common cycle.

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$A \Longleftrightarrow B$ is Whitney's Theorem

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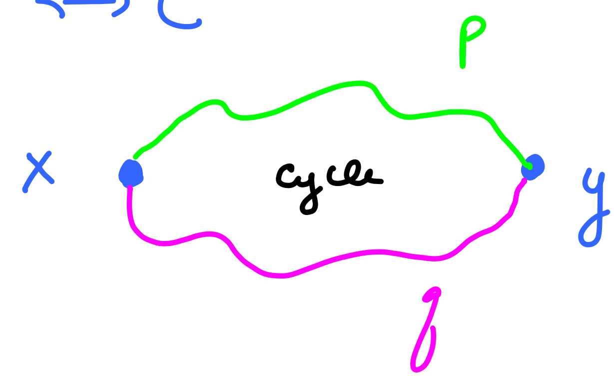
B $\Longleftrightarrow$ C



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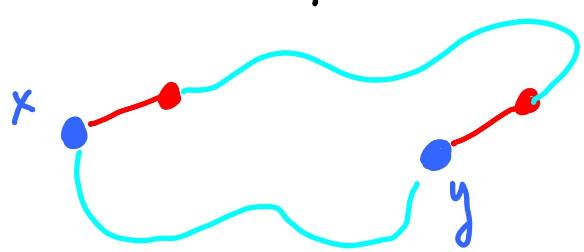
Every 2 vertices lie on a cycle

$\Updownarrow$

Every 2 edges lie on a cycle

D → C is easy



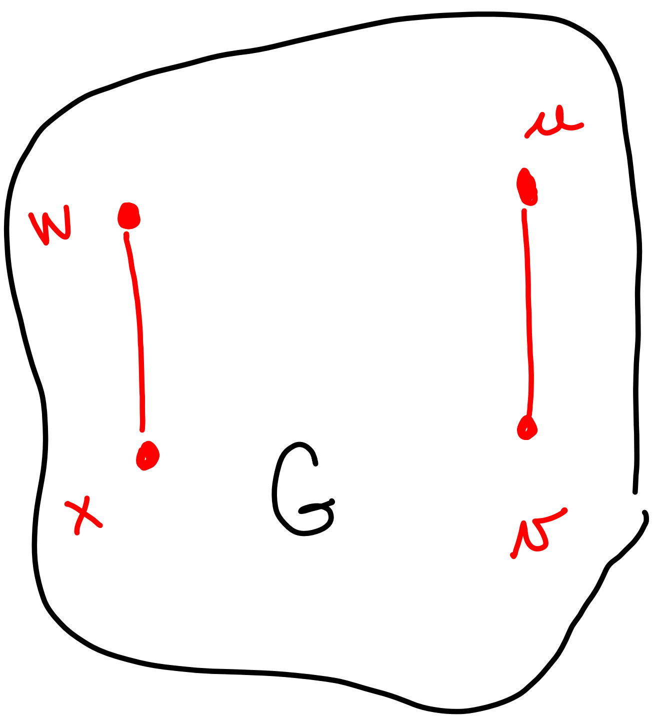
For C → D need "expansion lemma"

Given $A \Leftrightarrow B \Longleftrightarrow C$, show $C \Rightarrow D$.

Assume any 2 vertices of G lie on a common cycle.

Then G is 2 connected (since $A \Leftrightarrow C$).

Let uv and wx be any 2 edges of G.



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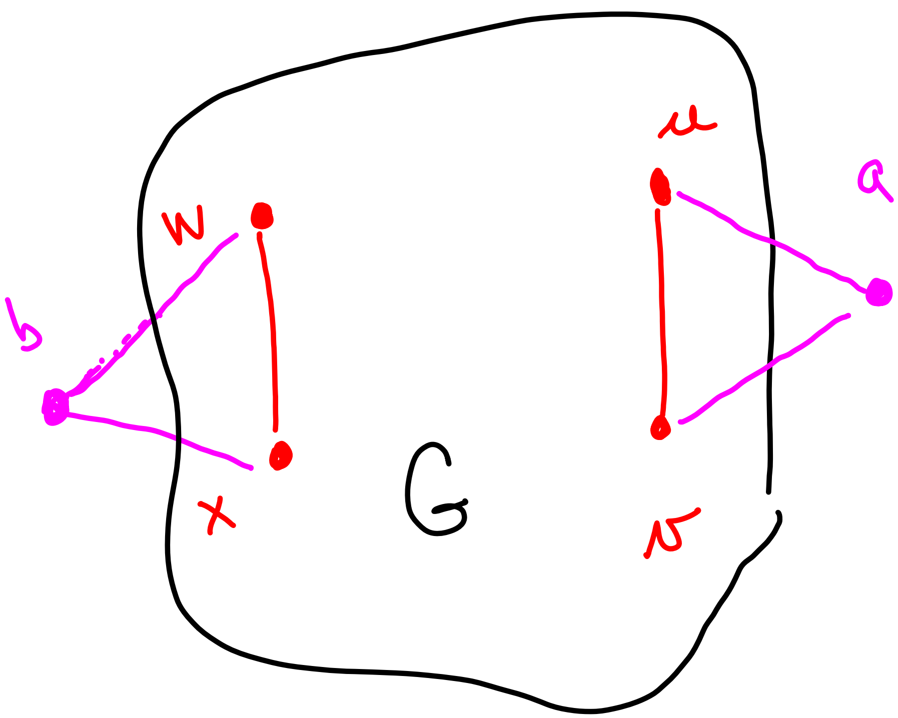
Use expansion lemma like this:



Result is 2-connected, so a & b lie on a common cycle, which must contain (consecutively) w, x and u, v. Now

$w, b, x \to wx$

$u, a, v \to uv$

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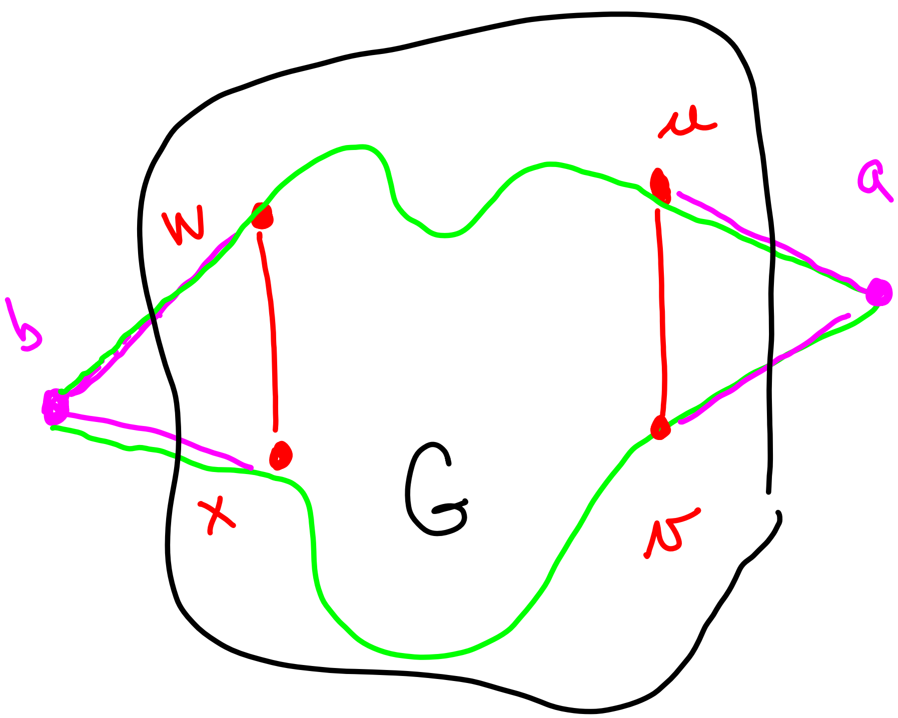

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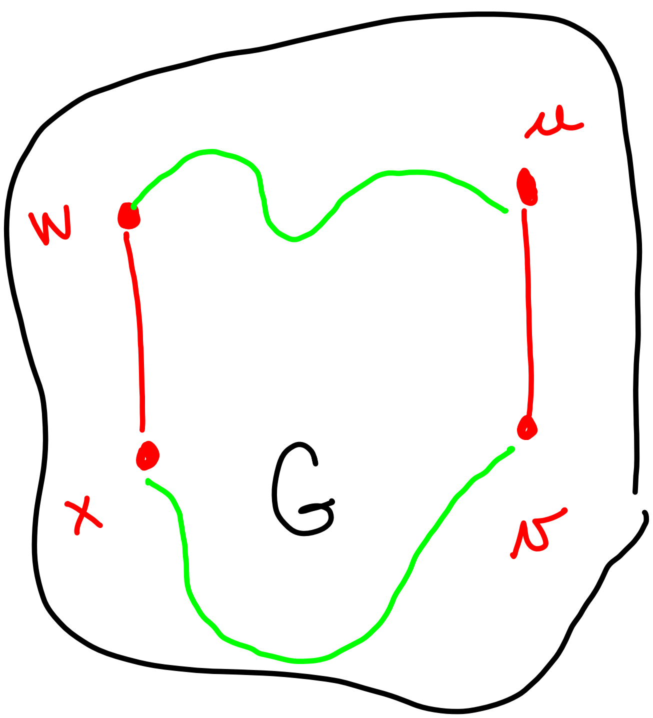

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Can generalize:

Menger's Theorem: Graph G with at least $k+1$ vertices is k-connected iff for every pair of vertices u, v of G, there are at least k pairwise internally disjoint (u, v)-paths.

Also: k-edge-connected iff at least k pairwise edge-disjoint u, v-paths.