Planar Graphs

A graph is **planar** if it can be drawn in the plane with no edge crossings.

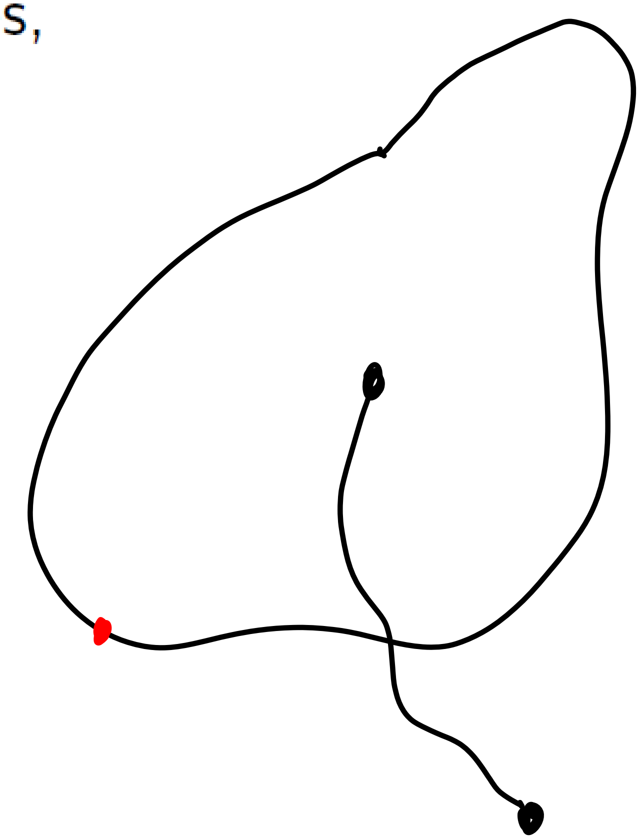
A drawing of G is a **planar embedding** of G if it has no crossings.

Plane graph: a particular planar embedding of a planar graph.

Jordan Curve Theorem: A simple closed curve C partitions the plane $-C$ into exactly two faces, each having C as its boundary.

(exterior of C - unbounded face)

(interior of C - bounded face)



Any curve joining a point in the interior of C with a point in the exterior of C must contain a point in the boundary of C.

Prop. 6.1.2 K_5 is not planar

(proof method: "conflicting chords")

Proof: Let C be a spanning cycle of K_5. Then in any planar embedding of K_5, the drawing of C is a simple, closed, curve.

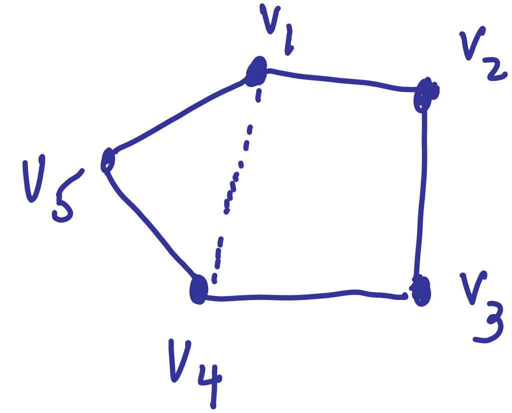
A chord of C is an edge joining 2 non-consecutive vertices on C and must be embedded inside or outside C.

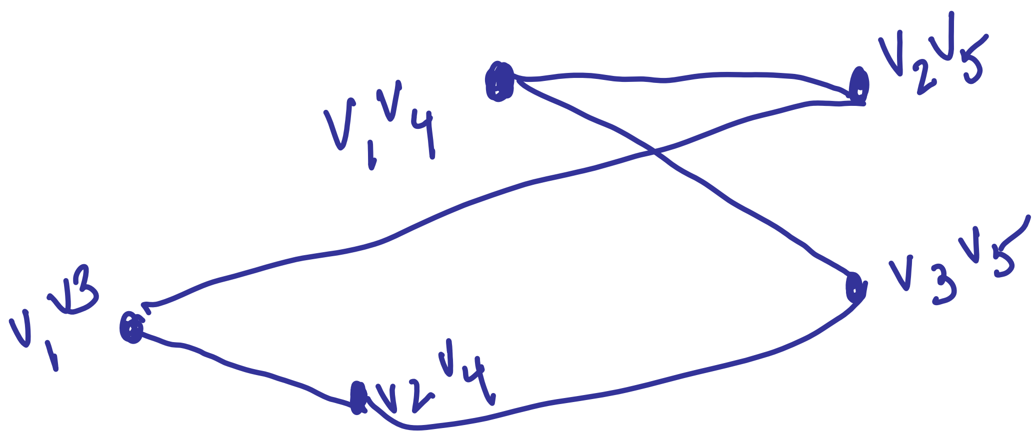
Two chords xy and wv conflict if their end-points appear in the relative order x, w, y, v along C, and conflicting chords must appear on opposite sides of C.

Assume K_5 is planar.

It has a Hamilton cycle

$v_1, v_2, v_3, v_4, v_5, v_1$ which must be a simple closed curve in any planar embedding

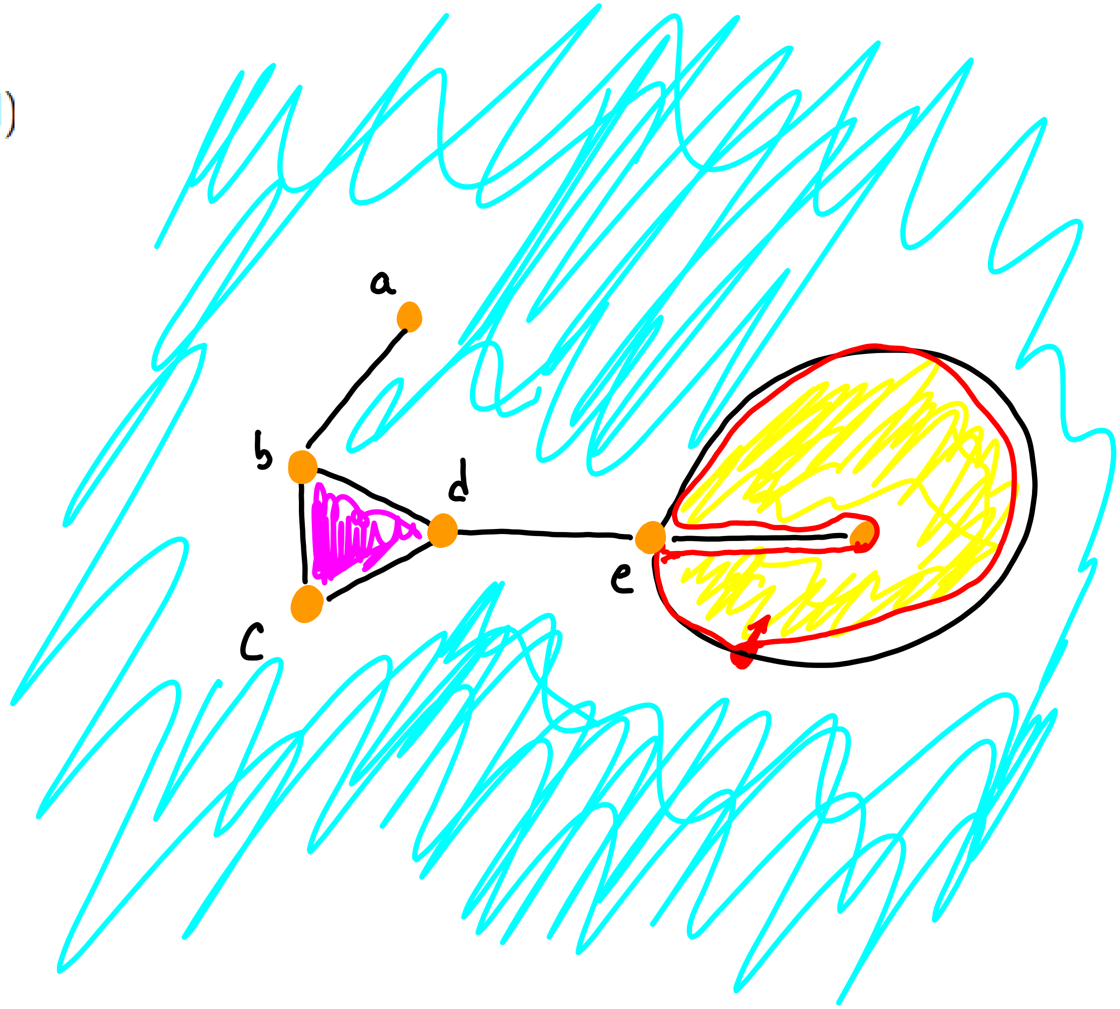




Exercise : Try to use the same method to show that $\boxed{K_{3,3}}$ is not planar

Plane graph has vertices, edges, **faces**

Faces of a plane graph G: maximal (connected) regions of the plane which are disjoint from the drawing of G.



Every planar graph has exactly one unbounded face.

Every edge is on the boundary of two (not necessarily distinct) faces.

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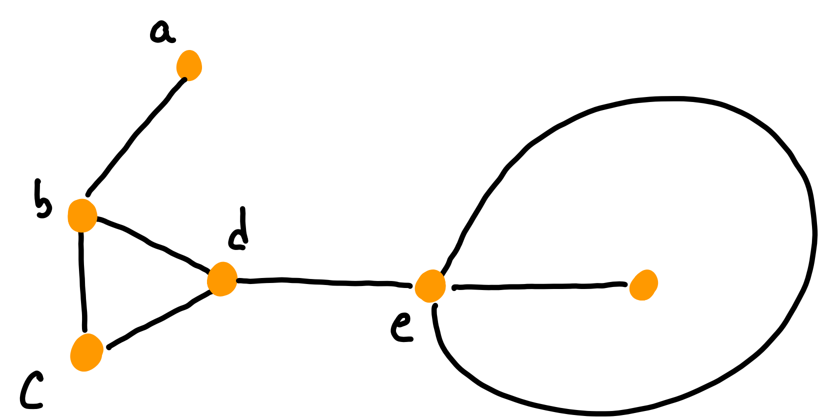

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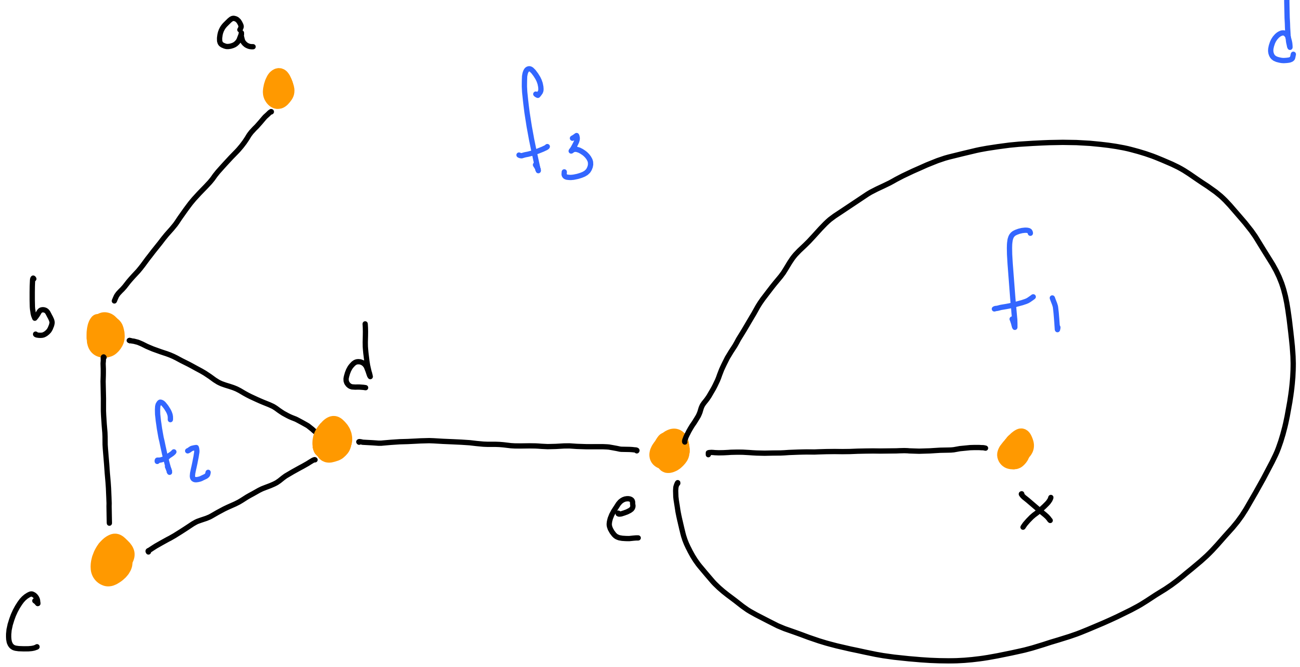
e is a cut edge $\iff$ it is on the boundary of only 1 face

A connected plane graph is a tree $\iff$ it has only 1 face



$d(f_1) = 3$

$d(f_2) = 3$

$d(f_3) = 8$

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(Length)

Degree of a face $d(f)$ is the number of edges on its boundary, cut edges count twice

$$\sum_{f \text{ a face}} d(f) = 2e$$

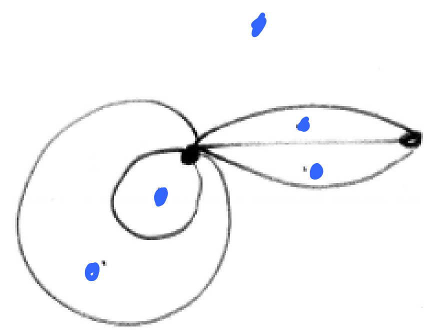
in a plane graph

Recommended

Can two planar embeddings of the same planar graph have different "face length" sequences?

Euler's Formula (1758). If a connected plane graph G has n vertices, e edges, and f faces, then

$$n - e + f = 2.$$



$n=2$

$e=5$

$f=5$

$2-5+5=2 \checkmark$

Proof. (Induction on f.)

Basis $f=1$: G is a tree

$e = n-1$

$n-(n-1)+1=2 \checkmark$

Assume $f>1$ and E.F. true for graphs with fewer faces.

There is some edge e^* which lies on boundary of 2 faces. Those faces become one in $G-e^*$. So $G-e^*$ has $f-1$ faces so E.F. holds in $G-e^*$: $n-(e-1)+(f-1)=2$

$n-e+f=2 \checkmark$

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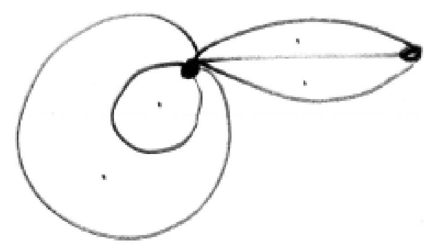


proof: induction on f

Basis: $f=1 \Rightarrow G$ is a tree, so $e=n-1$

$$n-e+f = n-(n-1)+1=2$$

Induction. Let $f \geq 2$, assume true for $< f$ faces. Then G is not a tree, so G has a cycle C, let e be an edge of C. Then e' is on the boundary of 2 faces. Removing e' merges these faces. The new graph has $f-1$ faces, $e-1$ edges & since Euler holds for it (by ind), Euler holds for G 5

Rec. ex. If G is a plane graph with k components, find an appropriate "Euler's formula"

Theorem A simple, connected, planar graph G, with $n(G) \geq 3$ has at most $3\,n(G) - 6$ edges

Proof Let H be a planar embedding of G.

$$2e = \sum_{f \text{ face of } H} d(f) \geq 3f$$

$$f \leq \frac{2}{3} e$$

$$n - e + f = 2$$

$$e = n + f - 2$$

$$\leq n + \frac{2}{3}e - 2$$

$$\frac{e}{3} \leq n - 2$$

$$e \leq 3n - 6$$

K_5: $n = 5$

$e = 10$

Not planar

$3n - 6 = 9$

$e = 10 > 9$

Use this to prove K_5 not planar? $K_{3,3}$?

Theorem (a) A simple, connected, planar graph G, with $n(G) \geq 3$ has at most $3n(G)-6$ edges

(b) If, in addition, G is triangle free then $e(G) \leq 2n(G)-4$

Pf: $2e = \sum d(f) \geq 4f$

etc. like part a

Use (b) to prove $K_{3,3}$ not planar

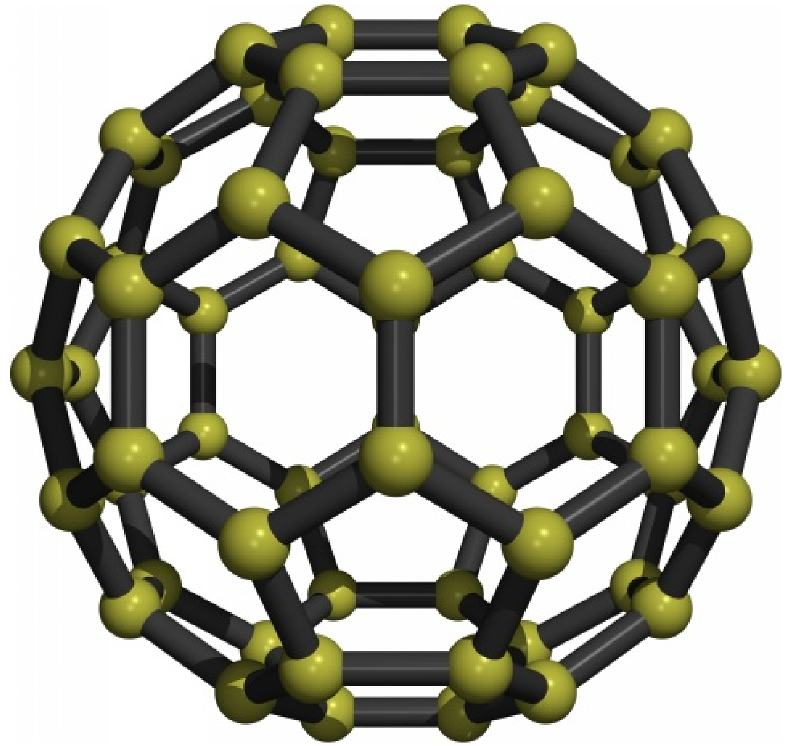
Apply Euler's formula to "fullerenes" to count number of pentagons.

C_{60} molecule: **Buckminsterfullerene**

- synthesized in 1985 by Curl and Smalley
- third known pure form of carbon (other two: diamond, graphite)
- trivalent polyhedron; each face a hexagon or pentagon (like soccer ball)
- 12 pentagons, 20 hexagons

What other forms satisfying third condition are theoretically possible?

Show: Euler's formula restricts number p of pentagons, but not the number h of hexagons.



http://www.ornl.gov/~pk7/pictures/c60.html

[Coxeter, < 1960] Does one exist for every value of h?

[Grunbaum, Motzkin 1963] Yes, except for $h = 1$.

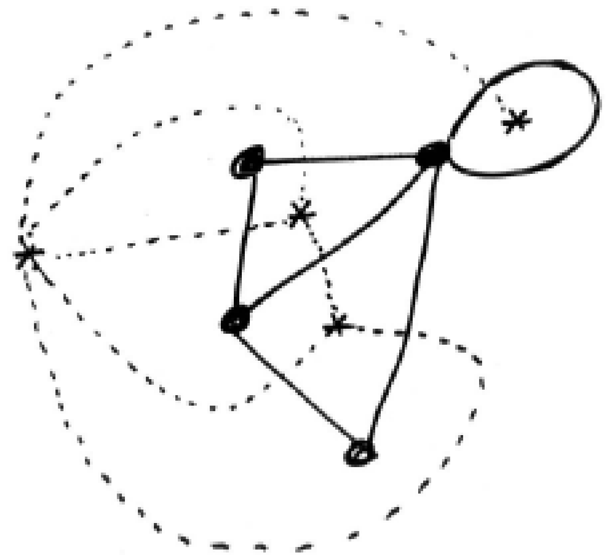
G - plane graph

G^* - **dual** of G (plane graph)

$V(G^*) =$ faces of G

$E(G^*) \sim E(G)$

If e is on the boundary of faces F_i and F_j in G then e^* joins F_i and F_j in G^*.



Note: If H and H' are different planar embeddings of G, the duals of H and H' may not be isomorphic.

Exercise: Find an example to illustrate this