Homework 3
Graph Theory CSC/MA/OR 565
Due Tuesday, February 9, 2016
Sketch of Solutions

1. In the graph shown on the last page, find the eccentricity of each vertex. Find the diameter and radius of the graph.

Eccentricities of the vertices:

$$\epsilon(A) = 4 \quad \epsilon(B) = 5 \quad \epsilon(C) = 4 \quad \epsilon(D) = 5 \quad \epsilon(E) = 6$$

$$\epsilon(F) = 7 \quad \epsilon(G) = 7 \quad \epsilon(H) = 6 \quad \epsilon(I) = 7 \quad \epsilon(J) = 7$$

$$\epsilon(K) = 5 \quad \epsilon(L) = 6 \quad \epsilon(M) = 7 \quad \epsilon(N) = 5 \quad \epsilon(O) = 6$$

$$\epsilon(P) = 6 \quad \epsilon(Q) = 6 \quad \epsilon(R) = 6 \quad \epsilon(S) = 7 \quad \epsilon(T) = 7$$

Radius of the graph: 4;

Diameter of the graph: 7.

2. In the graph shown on the next page, find the girth and the circumference of the graph. What is the length of the longest path? Find the center of the graph.

Girth of the graph: 3;

Circumference of the graph: 7;

Length of longest path: 14;

Center: subgraph induced by vertices A and C.

3. Problem 1.4.15 in the text. There is a typo in the first line: "$0 \leq n$" should be "$0 \leq j \leq n$".

A path from $(0, 0)$ to (m, n) is obtained by (in some intermixed order) increasing the first coordinate by 1 m times and increasing the second coordinate by 1 n times. Encode the order as a binary string with m zeroes and n ones: a 0 means increase the first coordinate, a 1 means increase the second coordinate. This encoding is a bijection between the paths from $(0, 0)$ to (m, n) and the binary strings of length $n + m$ with n ones, of which there are $\binom{n+m}{n}$.

4. Problem 1.4.36 in the text.

a. Actually, I will prove that if the indegree of a vertex v is positive, then (at least) one of its in-neighbors is a king. It would follow that this is also true if v is a king.

Let v be a vertex with $\delta^-(v) > 0$ and let y be the vertex in $N^-(v)$ with maximum outdegree. We show y is a king. If not, there is a vertex w which is not in $N^+(y)$ and cannot be reached from y by a path of length 2. Then since T is a tournament, all the vertices of $N^+(y)$ are in $N^+(w)$. In particular, (w, v) is an edge, so $w \in N^-(v)$. Since (y, w) is not an edge, (w, y) must be an edge, so also $y \in N^+(w)$. But then this means w has larger outdegree than y, contradicting choice of y.

b. We know that every tournament has a king, x. By part (a), there is another king y with $(y, x) \in E(T)$. Also by (a), there is a king z with $(z, y) \in E(T)$ and z cannot be x, since the edge joining x and y is oriented (y, x). Thus $x, y,$ and z are three different kings.

c. Select 3 vertices and call them x, y, z (these will be the three kings). Orient edges $(z, y), (y, x), (x, z)$. For $w \notin \{x, y, z\}$, orient these edges: $(x, w), (y, x), (z, w)$. Orient remaning edges arbitrarily. Then $x, y,$and z will be vertices of maximum out degree and therefore kings. But none of x, y, z are even reachable from the other vertices, so there are no other kings.

5. Problem 2.1.27 in the text.

If , for $n \geq 2$, $d_1, \ldots, d_n$ is a list of the vertex degrees of a tree T, then since T has $n-1$ edges, by the degree-sum formula, $\sum_{i=1}^{n} d_i = 2(n-1)$.

Conversely, assume $n \geq 2$ and $d_1, \ldots, d_n$ is a sequence of positive integers that sum to $2(n-1)$. We prove by induction that there is a tree with these numbers as its vertex degrees. If $n = 2$, it must be that $d_1 = d_2 = 1$ and T consists of two vertices joined by an edge. So assume $n > 2$ and that the result holds for smaller values. At least one of the numbers $d_1, \ldots, d_n$ must be 1 (otherwise $\sum_{i=1}^{n} d_i >= 2n$, which is too big) and at least one must be 2 or more (otherwise $\sum_{i=1}^{n} d_i = n$, which would be too small). Without loss of generality, assume $d_n = 1$ and $d_j > 1$. The the following is a list of positive numbers that sum to $2(n-2)$: $d_1, \ldots, d_{j-1}, d_j - 1, d_{j+1}, \ldots, d_{n-1}$. By induction there is a tree T' with this sequence of degrees. Add a new vertex v_n to T' and an edge from v_n to v_j to get a tree with degrees $d_1, \ldots, d_n$.

6. Problem 2.1.37 in the text. (How does this compare to Propositions 2.1.6 and 2.1.7?)

Given $e \in E(T) - E(T')$, let $e = xy$. Since T is a tree, e is a cut edge. Let U and V be the components of $T - e$ with x in U and y in V. $T' + e$ contains a unique cycle C. $C - e$ is an x, y path in T'. Thus there is an edge $e' = x'y'$ on $C - e$ such that $x' \in U$ and $y' \in V$. Then $T - e + e'$ is connected, with the same number of edges as T and thus is also a spanning tree. Also, since e' lies on a cycle in $T' + e$, it is not a cut edge of $T' + e$ and therefore $T' + e - e'$ is connected. It has the same number of edges as T and is therefore also a spanning tree.

(Problem 2.1.37 shows that by choosing e' carefully, it can satisfy the conditions of both propositions simultaneously.)

7. Problem 2.2.10 in the text. For the first part, use Proposition 2.2.8 to get a recurrence for $\tau(K_{2.m})$. Then solve it.

Applying Proposition 2.2.8, you can get the recurrence

$$\tau(K_{2,m}) = 2\tau(K_{2,m-1}) + 2^{m-1},$$

with initial condition $\tau(K_{2,1}) = 1$. Solving the recurrence gives $\tau(K_{2,m}) = m2^{m-1}$.

For the second part, let X, Y be the bipartition of $K_{2,m}$ into two independent sets with $|X| = 2$ and $|Y| = m$. Let $X = \{a, b\}$. If T is a spanning tree of $K_{2,m}$, then every edge of T is incident with exactly one of a, b. Since T is a tree, this means $d_T(a) + d_T(b) = e(T) = m + 1$. But since no vertex in Y can be isolated in T, one vertex in Y is adjacent to both a and b and all of others are adjacent to exactly one of a, b. So the isomorphism class of a spanning tree is completely determined by the the unordered pair of positive numbers $d_T(a), d_T(b)$ which sum to $m + 1$. There are $\lceil m/2 \rceil$ such pairs.

8. In Lecture 6, we showed that for the n-ribbed fan graph G_n,

$$\tau(G_n) = \tau(G_{n-1}) + \tau(H_{n-1})$$

(this corrects the typo in the slides) and

$$\tau(H_n) = \tau(G_n) + \tau(H_{n-1}),$$

where H_n is the graph obtained from G_n by "doubling" the first rib. Show how to use this to complete the work of finding a formula for $\tau(G_n)$. Can you also find $\tau(H_n)$?

You can show that $\tau(G_n) = 3\tau(G_{n-1}) - \tau(G_{n-2})$ with initial conditions $\tau(G_1) = 1$ and $\tau(G_2) = 3$. This is the same recurrence that you solved in Homewor 1.

You can similarly show that $\tau(H_n) = 3\tau(H_{n-1}) - \tau(H_{n-2})$ with initial conditions $\tau(H_1) = 2$ and $\tau(H_2) = 5$. The solution has the same form, but with different constants due to the different initial conditions.

9. Show how to use one of our tools to compute the number of spanning trees of the Petersen graph.

I think everyone will know to apply the matrix tree theorem. Construct the Laplacian matrix $D - A$ of the Petersen graph. Delete any row or column, and take the determinant. You get the surprising answer (2000).

10. a. Show that Dijkstra's shortest path algorithm can fail in the presence of negative weight edges, even if there are no negative cycles. Specifically, find a (small) weighted graph G that has some negative weight edges, but no negative cycles, and vertices $u, v \in V(G)$ such that Dijkstra's algorithm incorrectly computes $d(u, v)$.

Let $V(G) = \{u, v, x, y, z\}$ and $E(G) = \{uz, zx, xv, uy, yv\}$ with

$$w(uz) = 6, \quad w(zx) = 2, \quad w(xv) = -4, \quad w(uy) = 2, \quad w(yv) = 3.$$

Then Dijkstra's algorithm will say that u, y, v is the shortest path from u to v, but u, z, x, v is the true shortest path.

b. The following is sometimes suggested to fix the problem in part (a): Let $-m$ be the most negative weight in the graph. Add m to the weight of every edge. Now all weights are nonnegative, so run Dijkstra's algorithm. When shortest paths are found, restore edge weights to their original values.

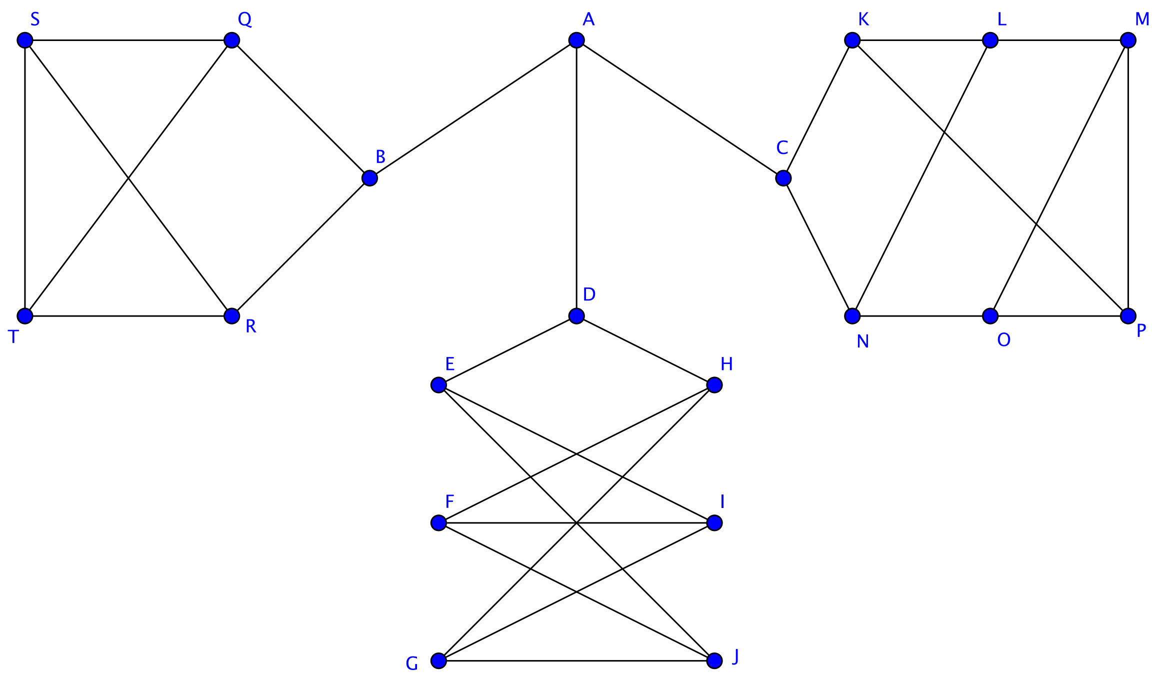
Show that this will not work. Specifically, find a (small) weighted graph G which has some negative weight edges, but no negative cycles, and vertices $u, v \in V(G)$ such that this algorithm incorrectly computes $d(u, v)$.

Use the same graph G as above. Since -4 is the most negative weight, adding 4 to each edge weights give the weights:

$$w(uz) = 10, \quad w(zx) = 6, \quad w(xv) = 0, \quad w(uy) = 6, \quad w(yv) = 7.$$

Dijkstra's algorithm will correctly find the shortest path from u to v in the *new* graph as u, y, v which has weight 13, whereas the path u, w, x, v has weight 16. But u, y, v is not the shortest path in the original graph.

Graph for Problem 1:



HW 3 Teams

TEAM A: aagrawa6 aawellin amthorn arrao bcdutton

TEAM B: bzhong2 cncody csdabral jchen37 jjiang13

TEAM C: drwiner efarhan glingna hguan2 hguo5

TEAM D: bcpilche djzager cghobbs jsduvall rkrish11

TEAM E: lan4 mbushou nshivra rabrown7 rrsizemo

TEAM F: rshah6 rshu rssawyer rzou schinch2

TEAM G: sju2 skukret spshriva thultum tpande

TEAM H: vsharma5 yho yhuang26 ymao4 zbcleghe