Edge Coloring
(Assume no loops)

k-edge-coloring of G: an assignment of colors $1, \dots, k$ to the edges of G.

Edge coloring is**proper** if incident edges are assigned different colors.

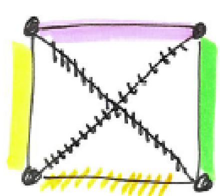
G is **k-edge-colorable** if it has a proper k-edge-coloring.

Edge chromatic number of G : $\chi'(G)$: minimum k for which G is k-edge-colorable.

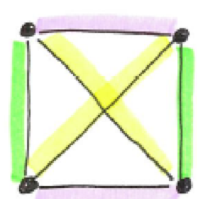
G is **k-edge-chromatic** iff $\chi'(G) = k$.

Example

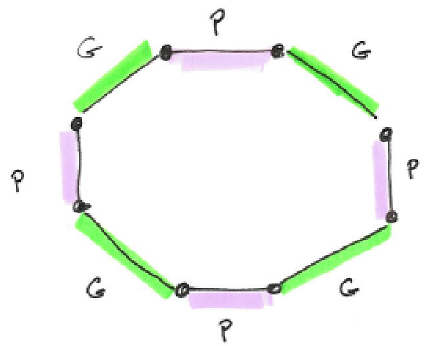
K_4 is 5-edge-colorable:



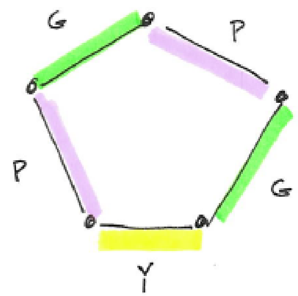
But $\chi'(K_4) = 3$:



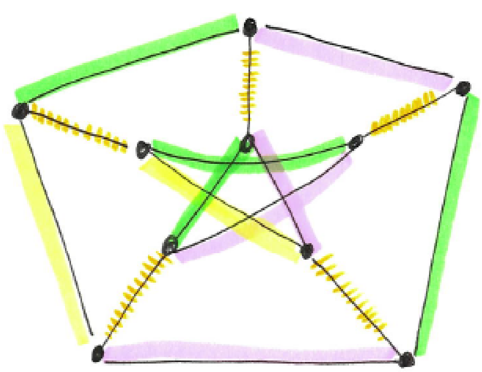
Even Cycles:



Odd Cycles:



Petersen Graph:



Examples

$\chi'(C_n)$ (n **even**) $=$ 2

$\chi'(C_n)$ (n **odd**) $=$ 3

$\chi'(K_n)$ (n **even**) $=$ $n-1$ (proof attached)

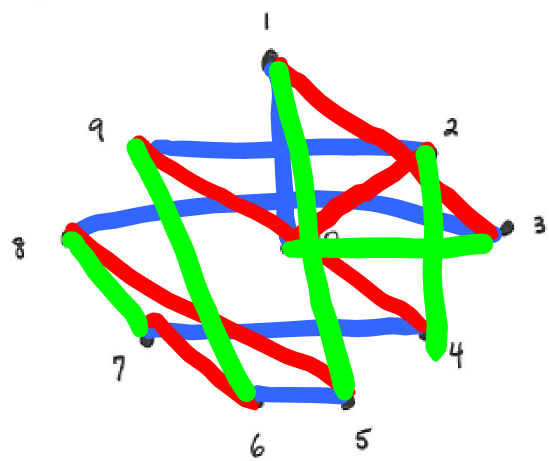
$\chi'(K_n)$ (n **odd**)$=$ n? (See attached)

$\chi'(K_{n,m})$ $\cancel{(n \text{ odd})}$$=$ (Rec. exercise)

χ'(Petersen graph) $=$ 4 (It is easy to see you need <u>at least</u> 3 colors. It is harder to see that a 4th color is required. (Rec: think about it))

Edge coloring K_n with $n-1$ colors when n is even

1. Arrange vertices like this: (e.g. K_{10})

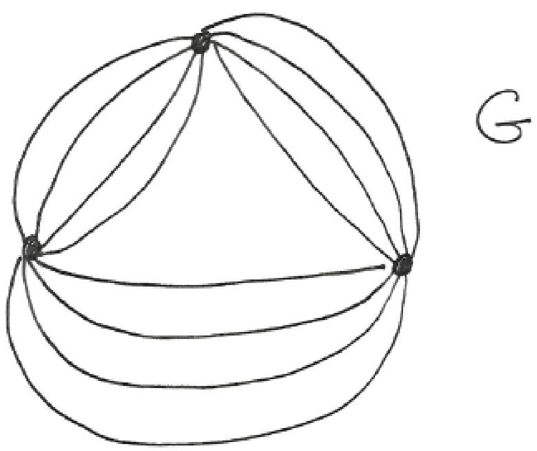


What can you say about the relationship between $\chi'(G)$ and $\Delta(G)$?

① Clearly $\chi'(G) \geq \Delta(G)$.

② It could be much bigger.

Example:



$\Delta(G) =$ ______

$\chi'(G) =$ ______

What can you say about the relationship between $\chi'(G)$ and $\Delta(G)$?

① Clearly $\chi'(G) \geq \Delta(G)$.

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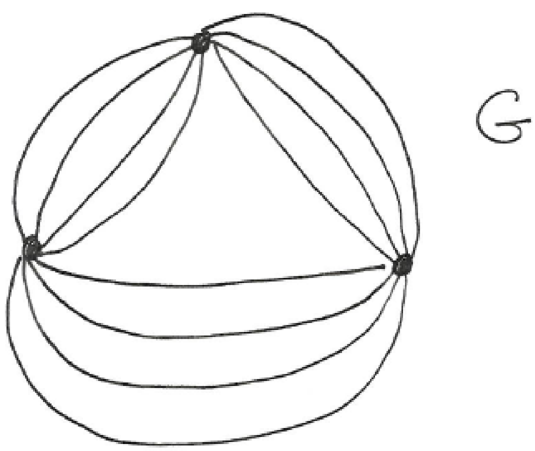
Example:



$\Delta(G) =$ ______

$\chi'(G) =$ ______

However, if G is simple, there is an amazingly close relationship between $\chi'(G)$ and $\Delta(G)$.

Relationship Between χ' and Δ

$\chi'(G) \geq \Delta(G)$. (clear)

*

Theorem 7.1.7 (Konig) If G is bipartite (multigraph) then $\chi'(G) = \Delta(G)$.

*

Theorem 7.1.10 (Vizing) If G is simple then $\chi'(G)$ is $\Delta(G)$ or $\Delta(G)+1$.

Not necessarily true if G is not simple - example:

(previous page)

(Can show in general multigraph that
$\chi'(G) \leq 3\Delta(G)/2$.)

(But we will not show it here.)

Theorem 7.1.7 (Konig) If G is bipartite (multi-graph) then $\chi'(G) = \Delta(G)$.

Proof (It suffices to show $\chi'(G) \leq \Delta(G)$ (why?))

(a) For k-regular graphs, use induction on k and Corollary 3.1.8 to Hall's Theorem.

(See next pages)

(b) Then show G of the theorem is a subgraph of a $\Delta(G)$-regular bipartite graph.

(See next pages)

(a) Suppose G is k-regular.

By Cor. 3.1.8 to Hall's Theorem,

G has a perfect matching M.

Color all the edges in M with color "k".

If $k=1$, this is a 1-edge coloring of G.

Otherwise, $G-M$ is a $k-1$ regular

bipartite graph. If we assume inductively

that $k-1$-regular bipartite graphs have a

$k-1$-edge-coloring, this gives a

k-edge-coloring of G.

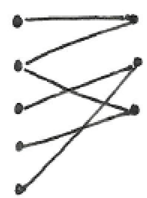
(b) If G is not ~~bipartite~~ regular, let $k = \Delta(G)$.

Show G is a subgraph of a k-regular bipartite graph H.

By (a) H has a proper k-edge-coloring $\therefore$ so does G.

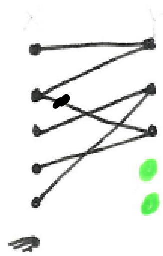
To construct H from G:

① Suppose G has bipartition (X, Y)



$H = G$

② If $|X| \neq |Y|$, add vertices to X or Y to make $|X| = |Y|$

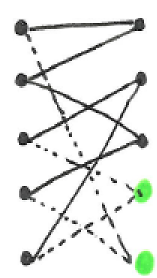


H

③ While H is not k-regular do

Find $x \in X$ and $y \in Y$ s.t. $d(x) < k$, $d(y) < k$

Join them.



H

(Note that if some $x \in X$ has $d(x) < k$, then some $y \in Y$ must also have $d(y) < k$ (why?). So ③ terminates with k-regular H.)

Theorem 7.1.10 (Vizing) If G is simple then $\chi'(G)$ is $\Delta(G)$ or $\Delta(G)+1$.

However

Here is the amazing thing.

Although Vizing's Theorem makes edge-coloring seem easy (you only have 2 choices for a simple graph G), it is NP-complete to decide whether $\Delta(G)$ or $\Delta(G)+1$ is the correct answer!!

* **Theorem 7.1.10** (Vizing) If G is simple then $\chi'(G)$ is $\Delta(G)$ or $\Delta(G)+1$.

Proof.

Suppose we have a partial, proper $\Delta+1$-edge coloring of G.
Here is how to extend it to get one more edge colored.

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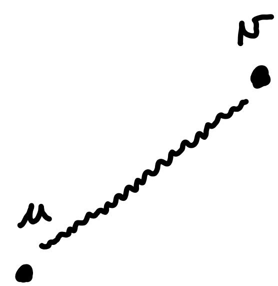
Here is how to extend it to get one more edge colored.

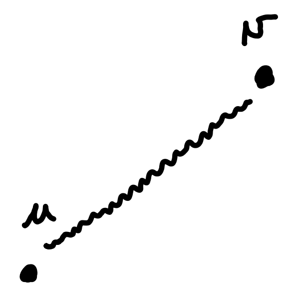
Suppose edge uv is not colored.



Suppose we have a partial, proper $\Delta+1$-edge coloring of G.
Here is how to extend it to get one more edge colored.

Suppose edge uv is not colored.

Some color, say a_0, is not represented at u.



Suppose we have a partial, proper $\Delta+1$-edge coloring of G.

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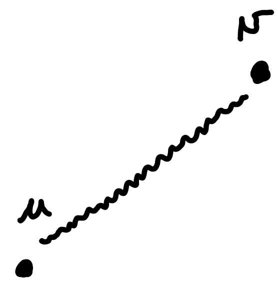
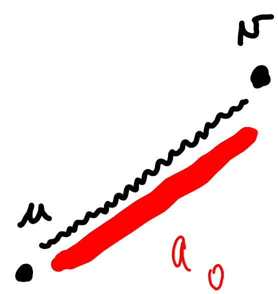


If a_0 is not represented at v,

Suppose we have a partial, proper $\Delta+1$-edge coloring of G.
Here is how to extend it to get one more edge colored.

Suppose edge uv is not colored.

Some color, say a_0, is not represented at u.



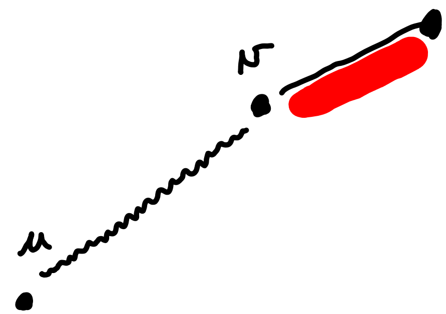
If a_0 is not represented at v,

Then color uv with a_0 and stop.

Suppose we have a partial, proper $\Delta+1$-edge coloring of G.
Here is how to extend it to get one more edge colored.

Suppose edge uv is not colored.

Some color, say a_0, is not represented at u.

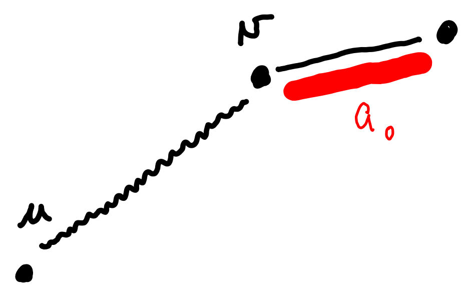


If a_0 is not represented at v,

Then color uv with a_0 and stop.

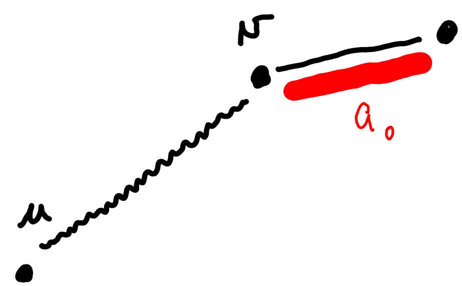
Otherwise, go on

a_0 not represented at u



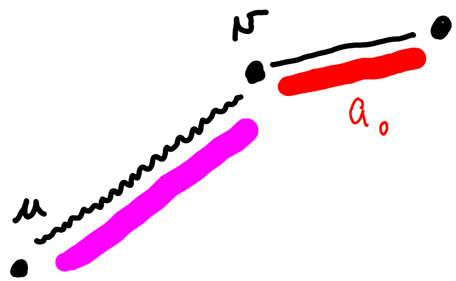
a_0 not represented at u

Some a_1 not represented at v



a_0 not represented at u

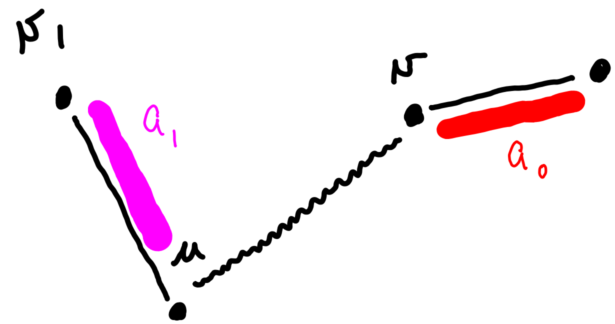
Some a_1 not represented at v



If a_1 not rep'd at u, then color uv with a_1 and stop.

a_0 not represented at u

<u>Some</u> a_1 not represented at v

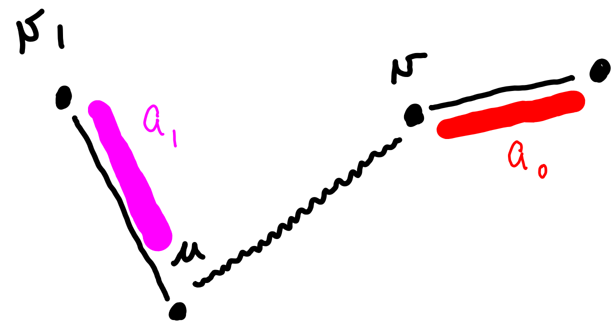


If a_1 not rep'd at u, then color uv with a_1 and stop.

Otherwise, go on.

a_0 not represented at u

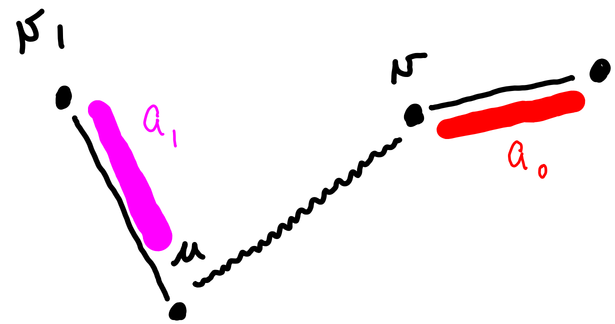
a_1 not represented at v



a_0 not represented at u

a_1 not represented at v

<u>Some</u> a_2 not represented at v_1

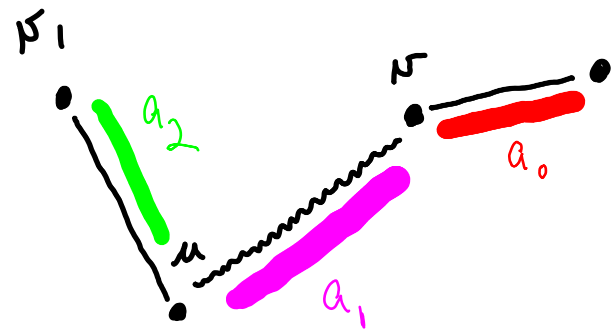


If a_2 is not represented at u,

a_0 not represented at u

a_1 not represented at v

<u>Some</u> a_2 not represented at v_1

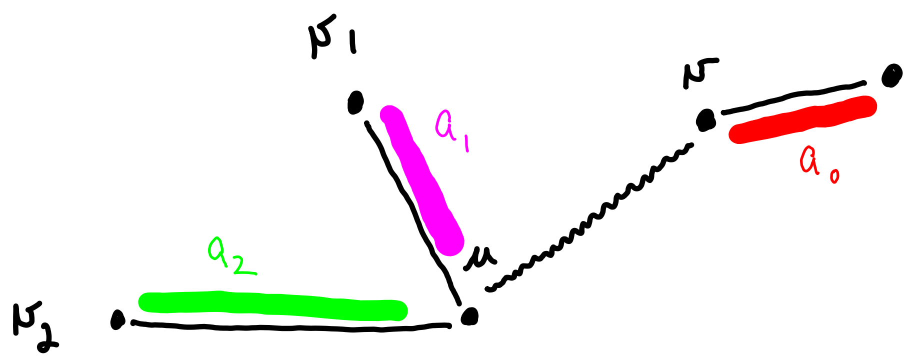


If a_2 is not represented at u, "downshift" from uv_1 and color uv_1 with a_2, then STOP.

a_0 not represented at u

a_1 not represented at v

<u>Some</u> a_2 not represented at v_1



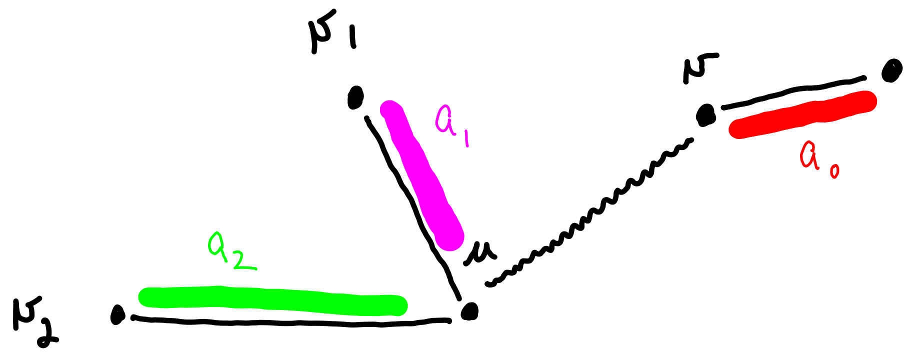
If a_2 is not represented at u, "downshift" from uv_1 and color uv_1 with a_2, then STOP.

Otherwise, go on...

a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

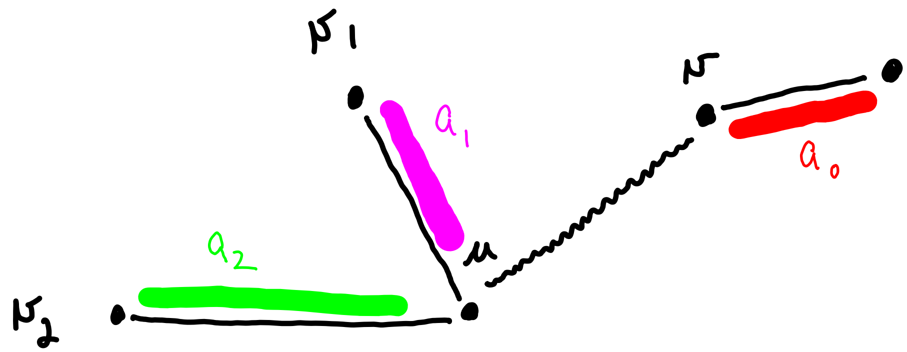


a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

Some color a_3 not represented at v_2

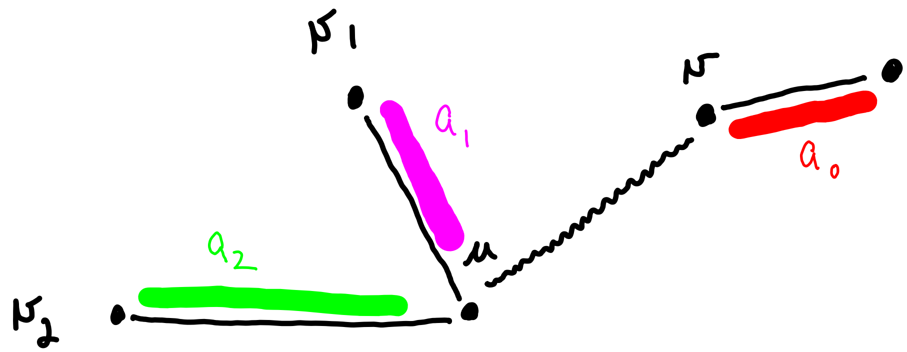


a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

Some color a_3 not represented at v_2



If a_3 not rep'd at u, then downshift from uv_2 and recolor uv_2 with a_3 and stop.

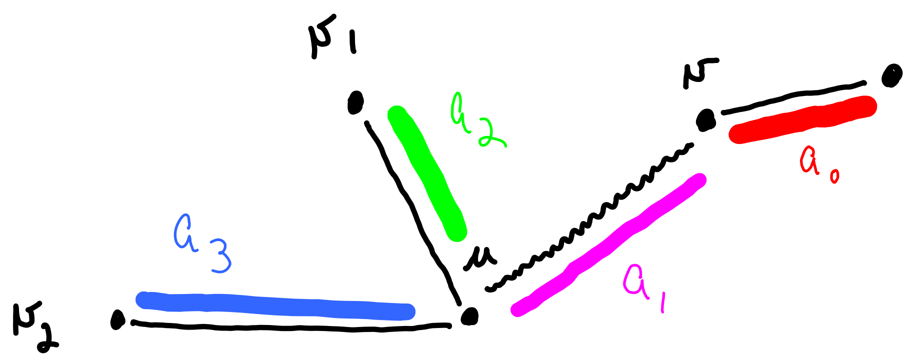
a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

a_3 not represented at v_2

Some color



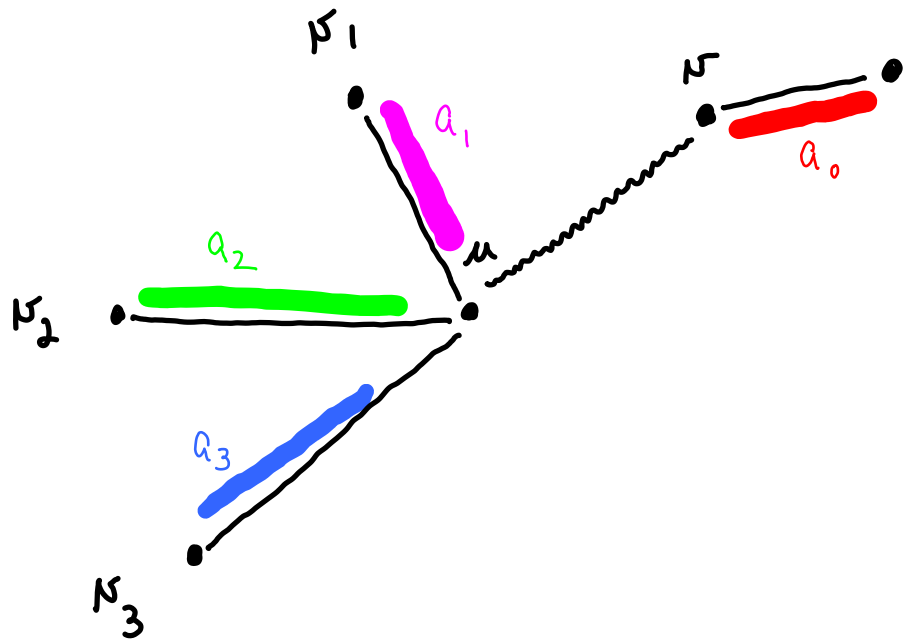
If a_3 not rep'd at u, then downshift from uv_2 and recolor uv_2 with a_3 and STOP.

a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

<u>Some</u> color a_3 not represented at v_2



If a_3 not rep'd at u, then downshift from uv_2 and recolor uv_2 with a_3 and stop.

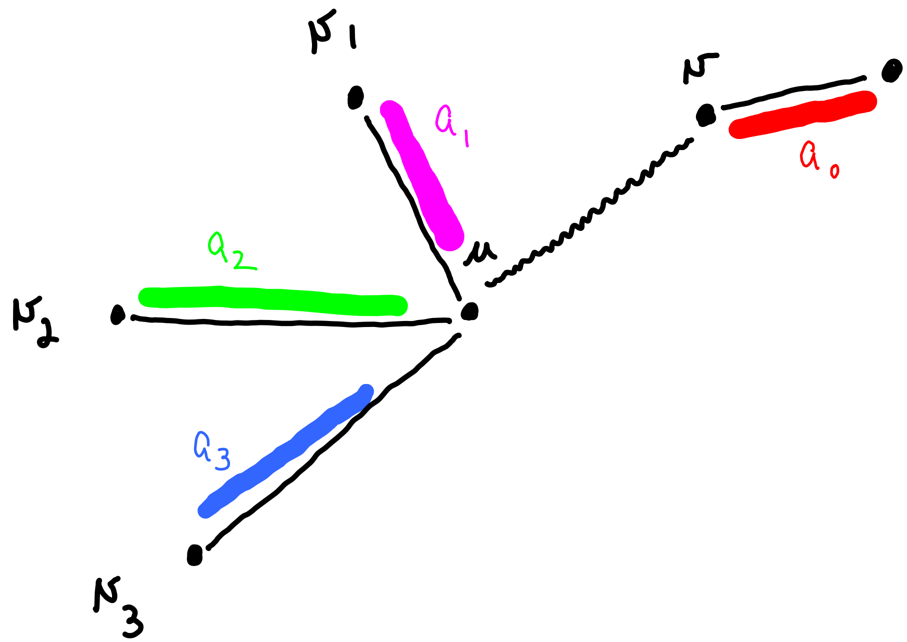
Otherwise, go on

a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

a_3 not represented at v_2



Continue ...

a_0 not represented at u

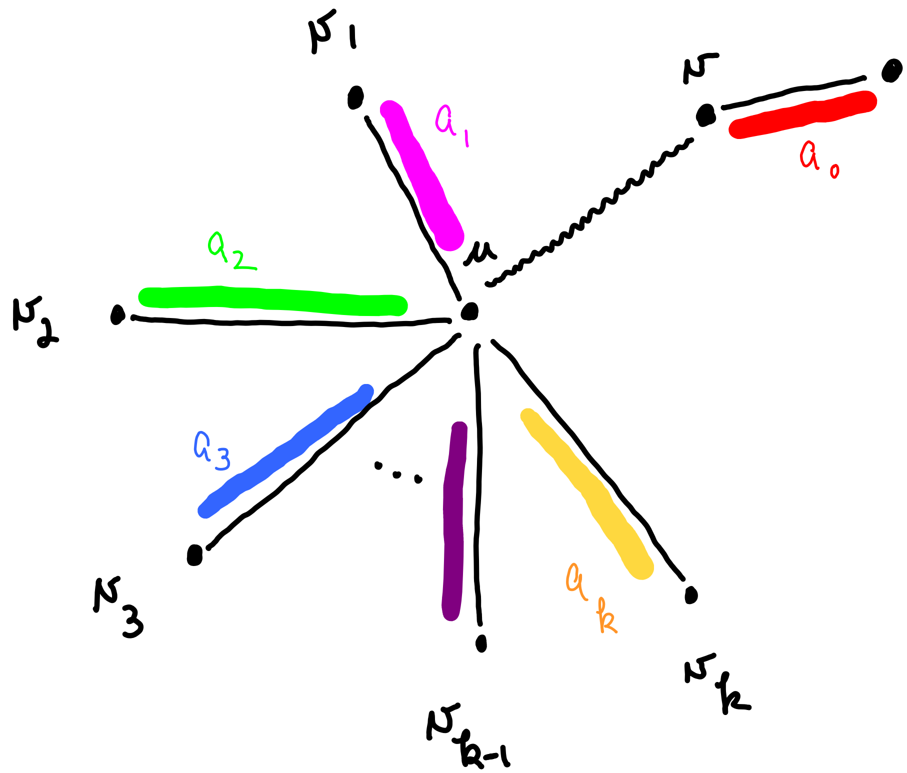
a_1 not represented at v

a_2 not represented at v_1

a_3 not represented at v_2

⋮

a_k not rep'd at v_{k-1}



Until the first time a color is repeated, say $a_{\ell+1} = a_k$

a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

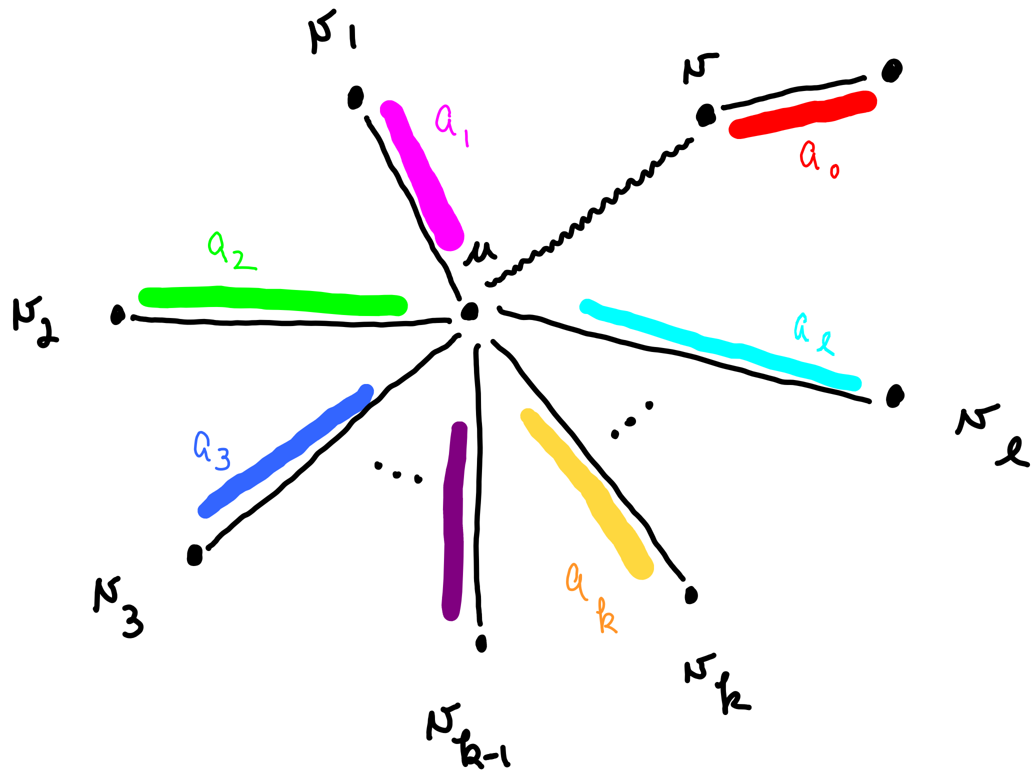
a_3 not represented at v_2

$\vdots$

a_k not rep'd at v_{k-1}

$\vdots$

$a_{\ell+1}$ not rep'd at v_ℓ



$a_{\ell+1} = a_k$

Ask: Is a_0 rep'd at v_ℓ?

a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

a_3 not represented at v_2

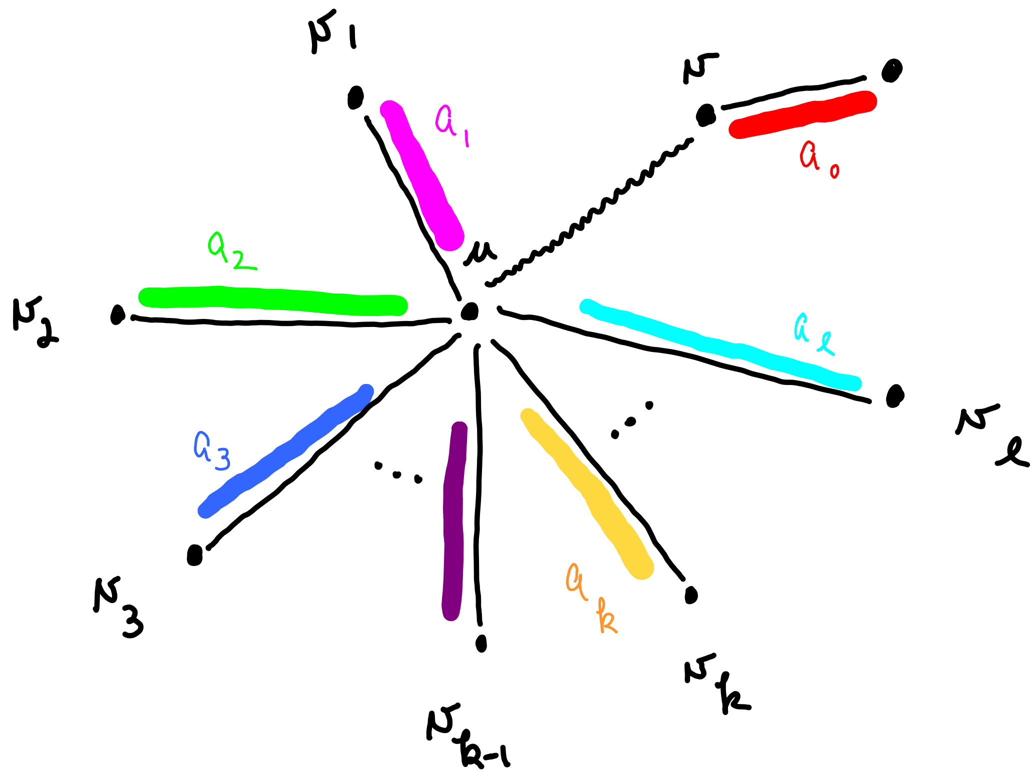
$\vdots$

a_k not rep'd at v_{k-1}

$\vdots$

$a_{\ell+1}$ not rep'd at v_ℓ

$a_{\ell+1} = a_k$



Ask: Is a_0 rep'd at v_ℓ?

If <u>No</u> recolor uv_ℓ with a_0

and downshift to get uv colored

a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

a_3 not represented at v_2

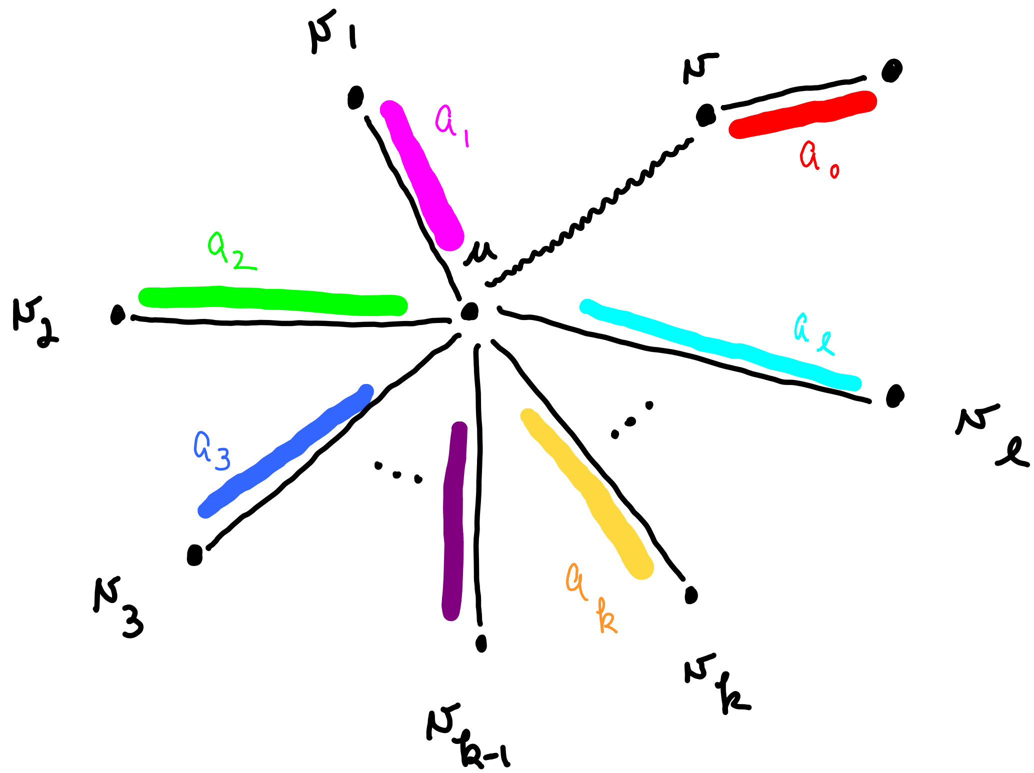
⋮

a_k not rep'd at v_{k-1}

⋮

$a_{\ell+1}$ not rep'd at v_ℓ

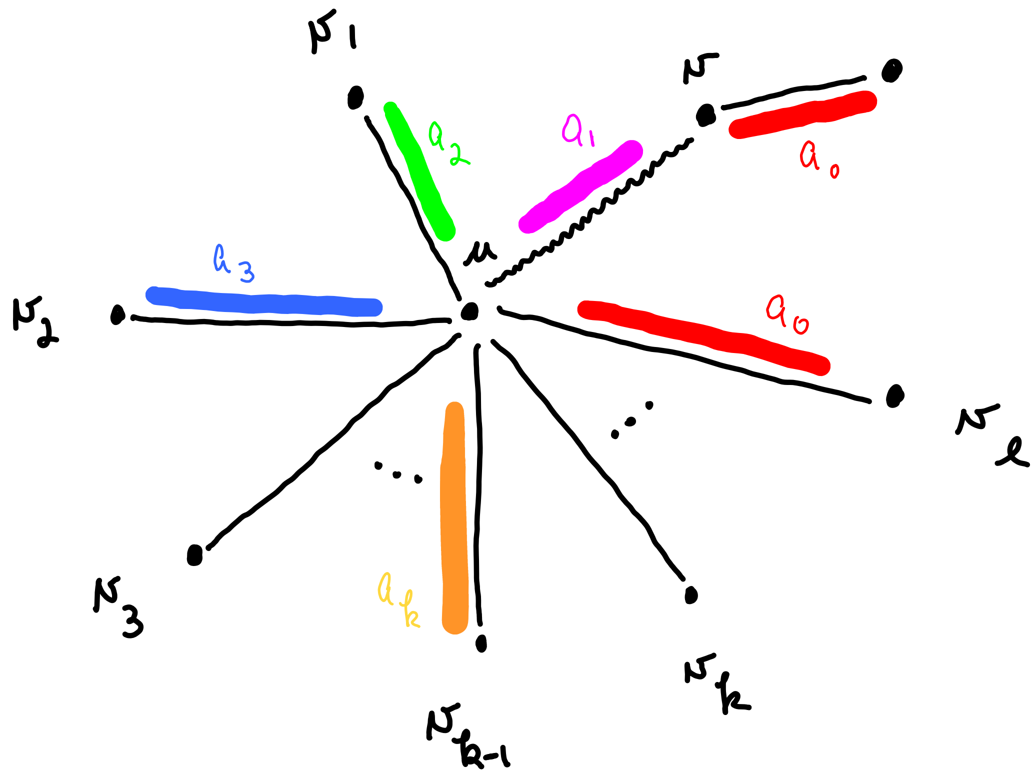
$a_{\ell+1} = a_k$



Ask: Is a_0 rep'd at v_ℓ?

If NO recolor uv_ℓ with a_0

and downshift to get uv colored



a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

a_3 not represented at v_2

⋮

a_k not rep'd at v_{k-1}

⋮

$a_{\ell+1}$ not rep'd at v_ℓ

$a_{\ell+1} = a_k$

But if a_0 is rep'd at v_ℓ let p be a maximal path starting at v_ℓ and coloured a_0 and a_k

a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

a_3 not represented at v_2

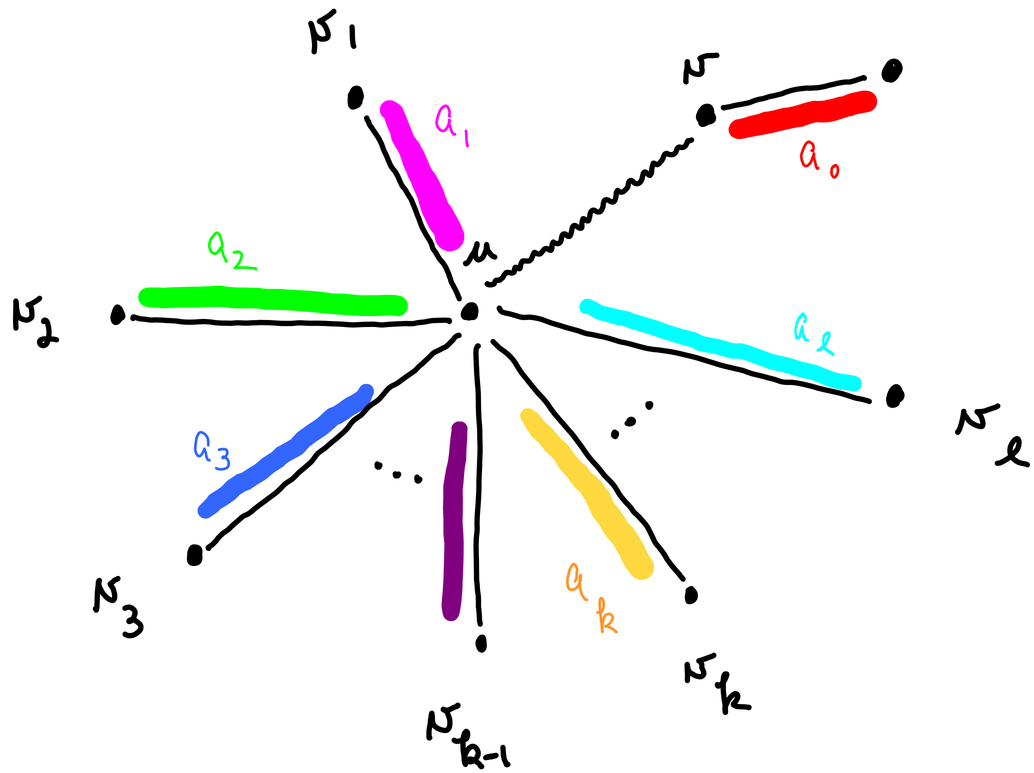
$\vdots$

a_k not rep'd at v_{k-1}

$\vdots$

$a_{\ell+1}$ not rep'd at v_ℓ

$a_{\ell+1} = a_k$



But if a_0 is rep'd at v_ℓ let p be a maximal path starting at v_ℓ and colored a_0 and a_k

Case 1 p reaches v_k

a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

a_3 not represented at v_2

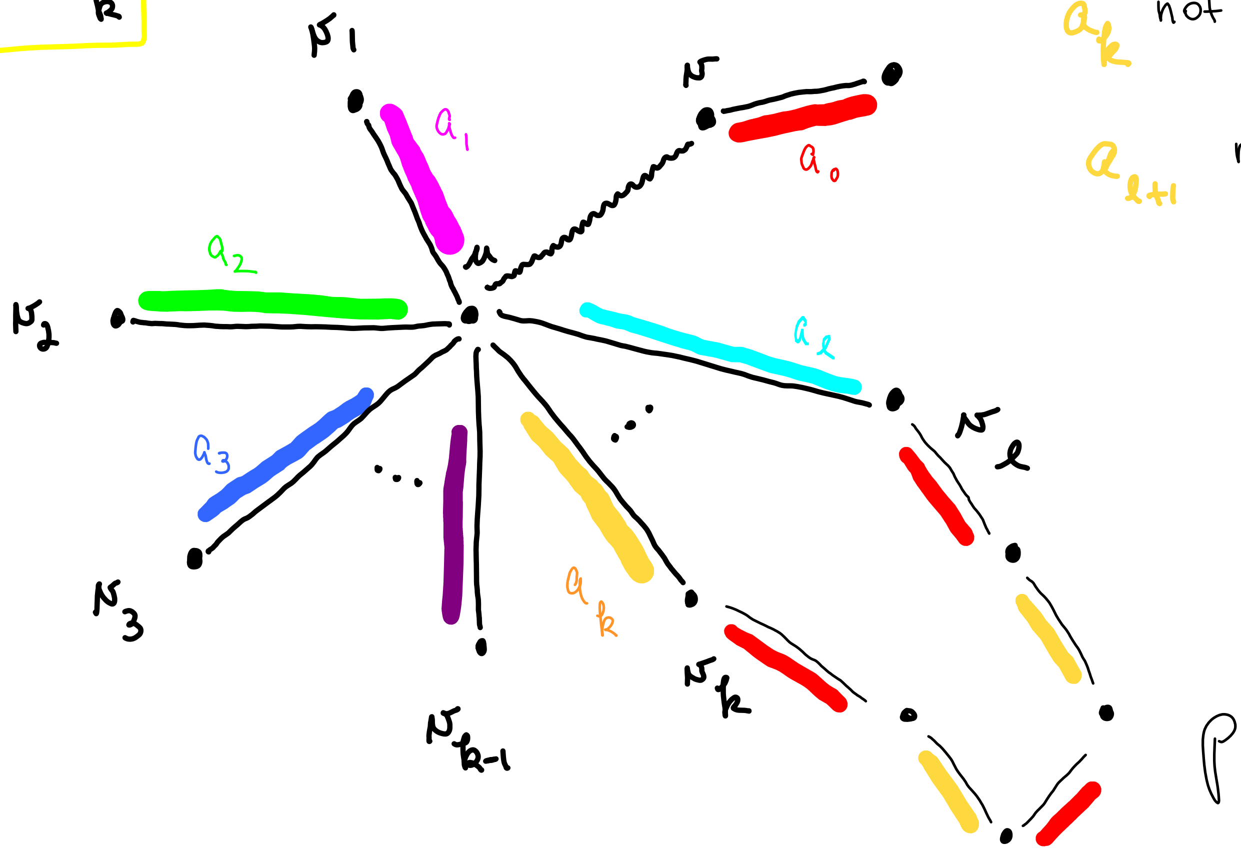
$\vdots$

a_k not rep'd at v_{k-1}

$\vdots$

$a_{\ell+1}$ not rep'd at v_ℓ

$a_{\ell+1} = a_k$

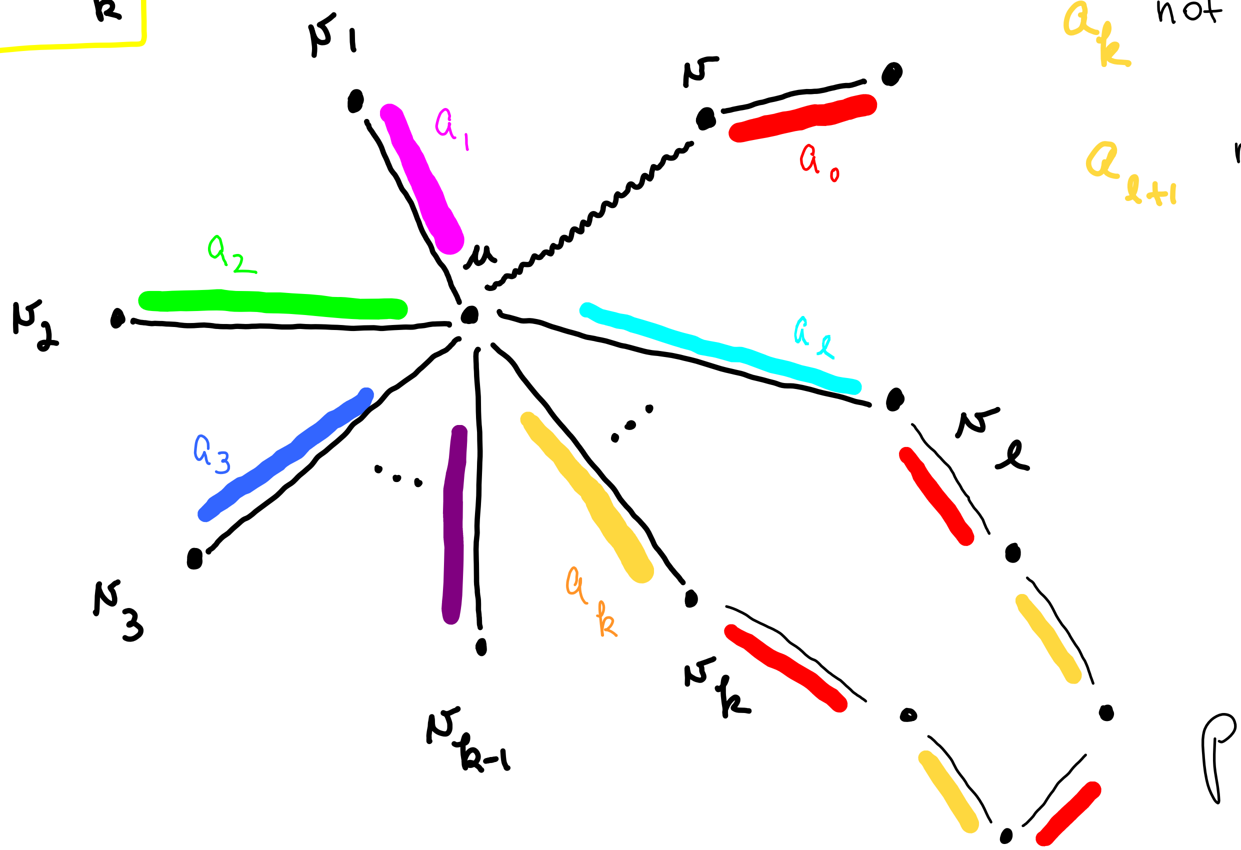


But if a_0 IS rep'd at v_ℓ let p be a maximal path starting at v_ℓ and colored a_0 and a_k

Case 1 p reaches v_k

Swap colors on p

Downshift from uv_k



a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

a_3 not represented at v_2

⋮

a_k not rep'd at v_{k-1}

⋮

$a_{\ell+1}$ not rep'd at v_ℓ

$a_{\ell+1} = a_k$

But if a_0 is rep'd at v_ℓ let p be a maximal path starting at v_ℓ and colored a_0 and a_k

a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

a_3 not represented at v_2

$\vdots$

a_k not rep'd at v_{k-1}

$\vdots$

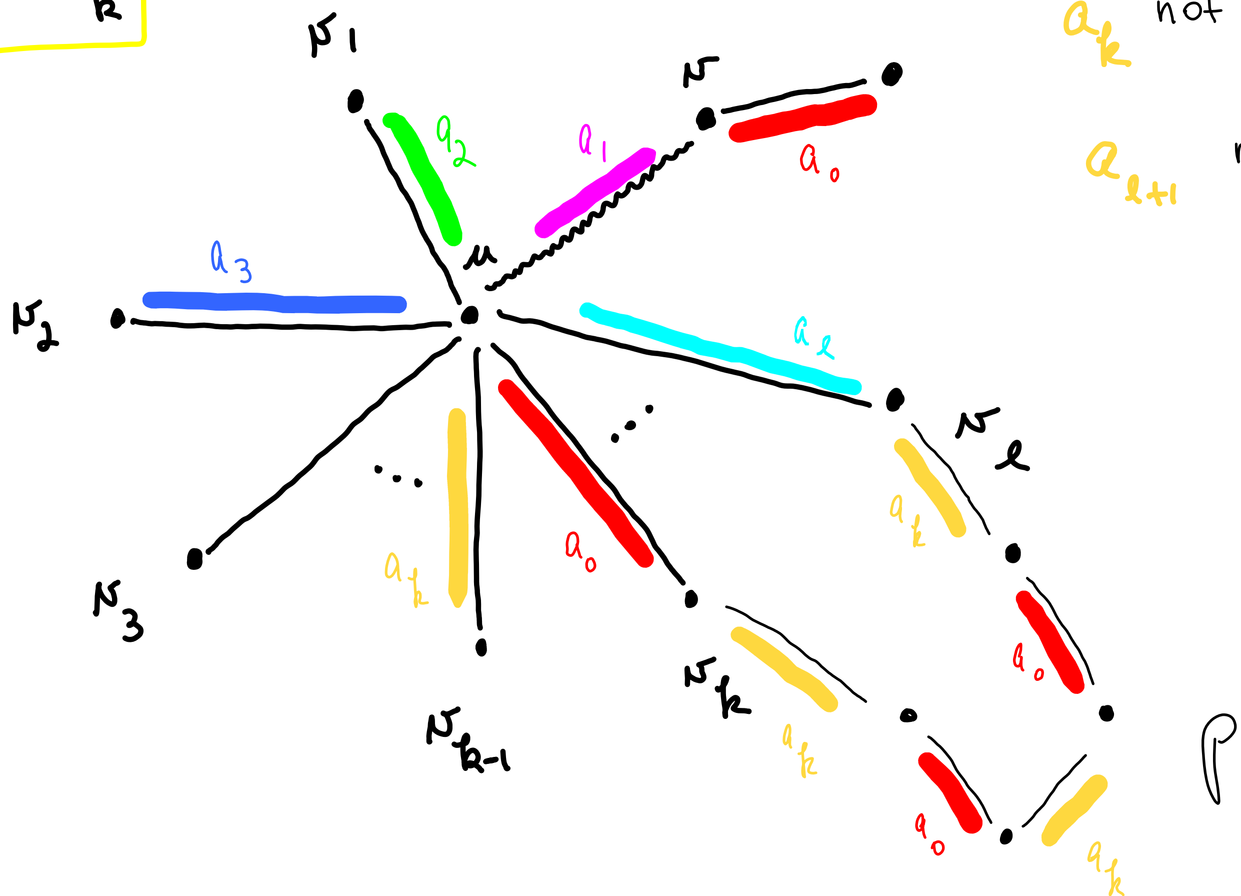
$a_{\ell+1}$ not rep'd at v_ℓ

$a_{\ell+1} = a_k$

Case 1 p reaches v_k

Swap colors on p

Downshift from uv_k

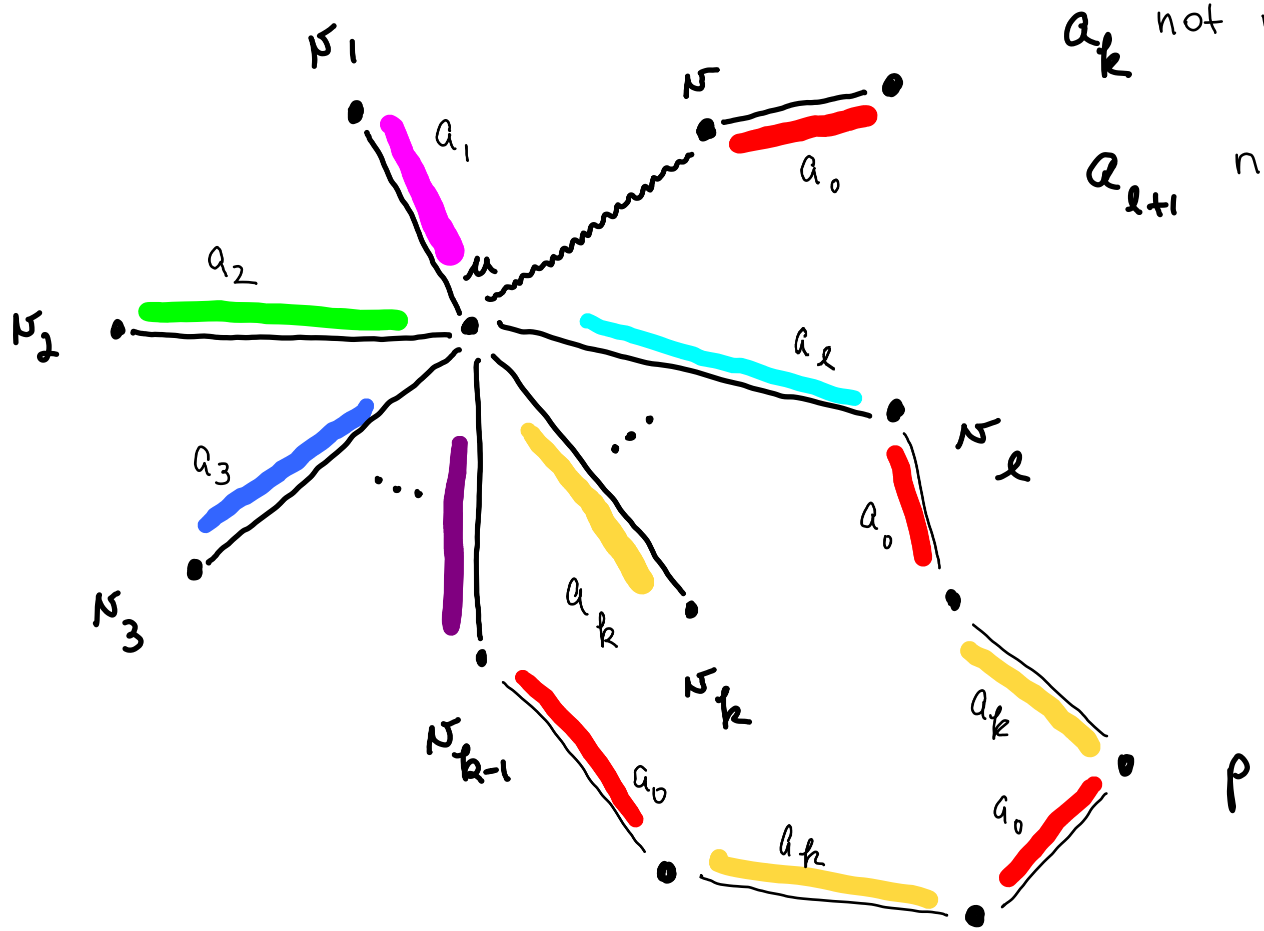


But if a_0 <u>IS</u> rep'd at v_ℓ let p be a maximal path starting at v_ℓ and colored a_0 and a_k

Case 2:
p reaches v_{k-1}

a_0 not represented at u
a_1 not represented at v
a_2 not represented at v_1
a_3 not represented at v_2
$\vdots$
a_k not rep'd at v_{k-1}
$\vdots$
$a_{\ell+1}$ not rep'd at v_ℓ

$a_{\ell+1} = a_k$



But if a_0 is rep'd at v_ℓ let p be a maximal path starting at v_ℓ and colored a_0 and a_k

a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

a_3 not represented at v_2

$\vdots$

a_k not rep'd at v_{k-1}

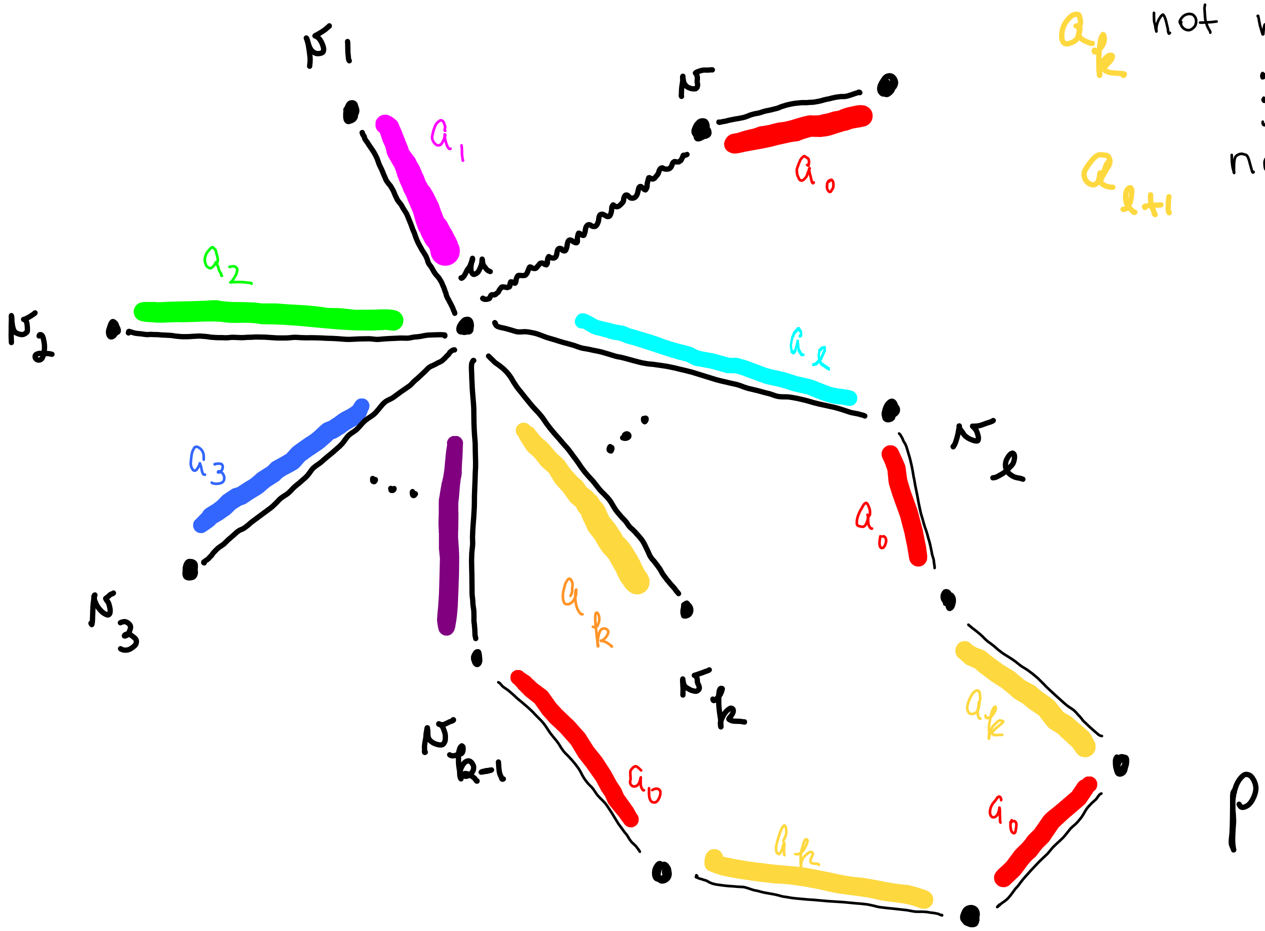
$\vdots$

$a_{\ell+1}$ not rep'd at v_ℓ

$a_{\ell+1} = a_k$

Case 2: p reaches v_{k-1}

Recolor p,
Downshift from uv_{k-1}
& color uv_{k-1} with a_0



But if a_0 is rep'd at v_ℓ let p be a maximal path starting at v_ℓ and colored a_0 and a_k

a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

a_3 not represented at v_2

$\vdots$

a_k not rep'd at v_{k-1}

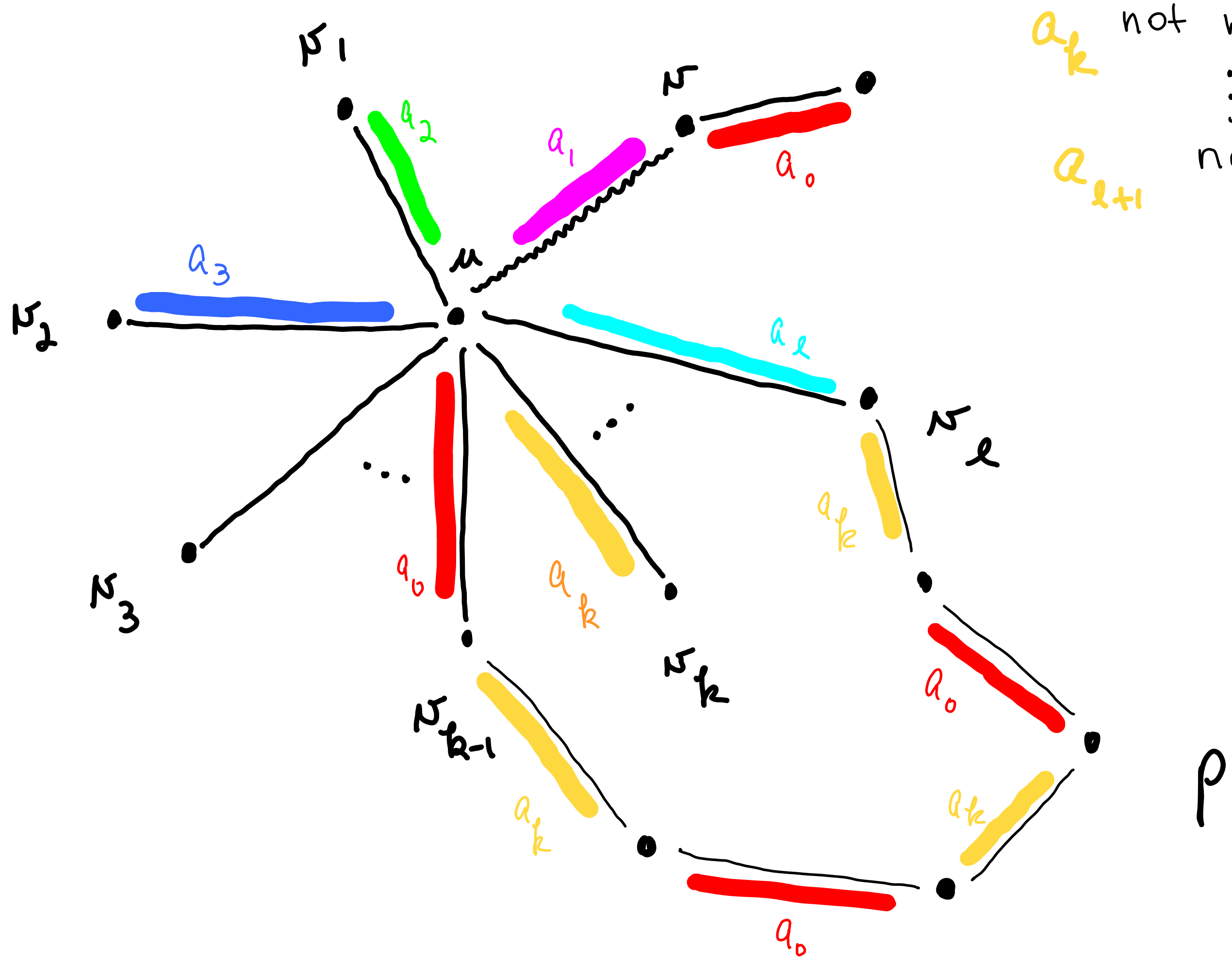
$\vdots$

$a_{\ell+1}$ not rep'd at v_ℓ

$a_{\ell+1} = a_k$

Case 2: p reaches v_{k-1}

Recolor p, Downshift from uv_{k-1} & color uv_{k-1} with a_0



But if a_0 is rep'd at v_ℓ let p be a maximal path starting at v_ℓ and coloured a_0 and a_k

a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

a_3 not represented at v_2

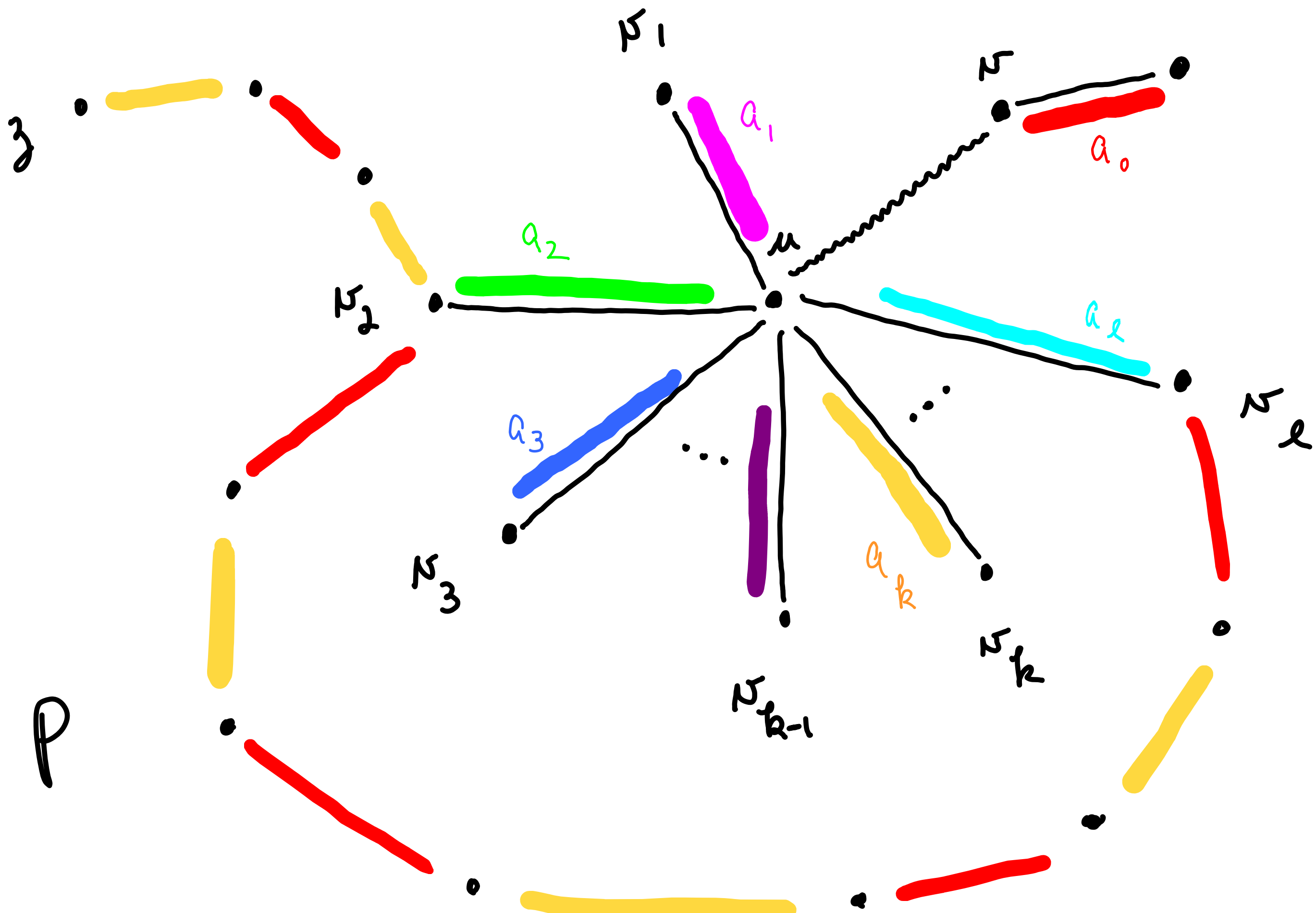
$\vdots$

a_k not rep'd at v_{k-1}

$\vdots$

$a_{\ell+1}$ not rep'd at v_ℓ

$a_{\ell+1} = a_k$



Case 3: p ends at a vertex other than v_ℓ, v_k, v_{k-1}, u

But if a_0 is rep'd at v_ℓ let p be a maximal path starting at v_ℓ and colored a_0 and a_k

a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

a_3 not represented at v_2

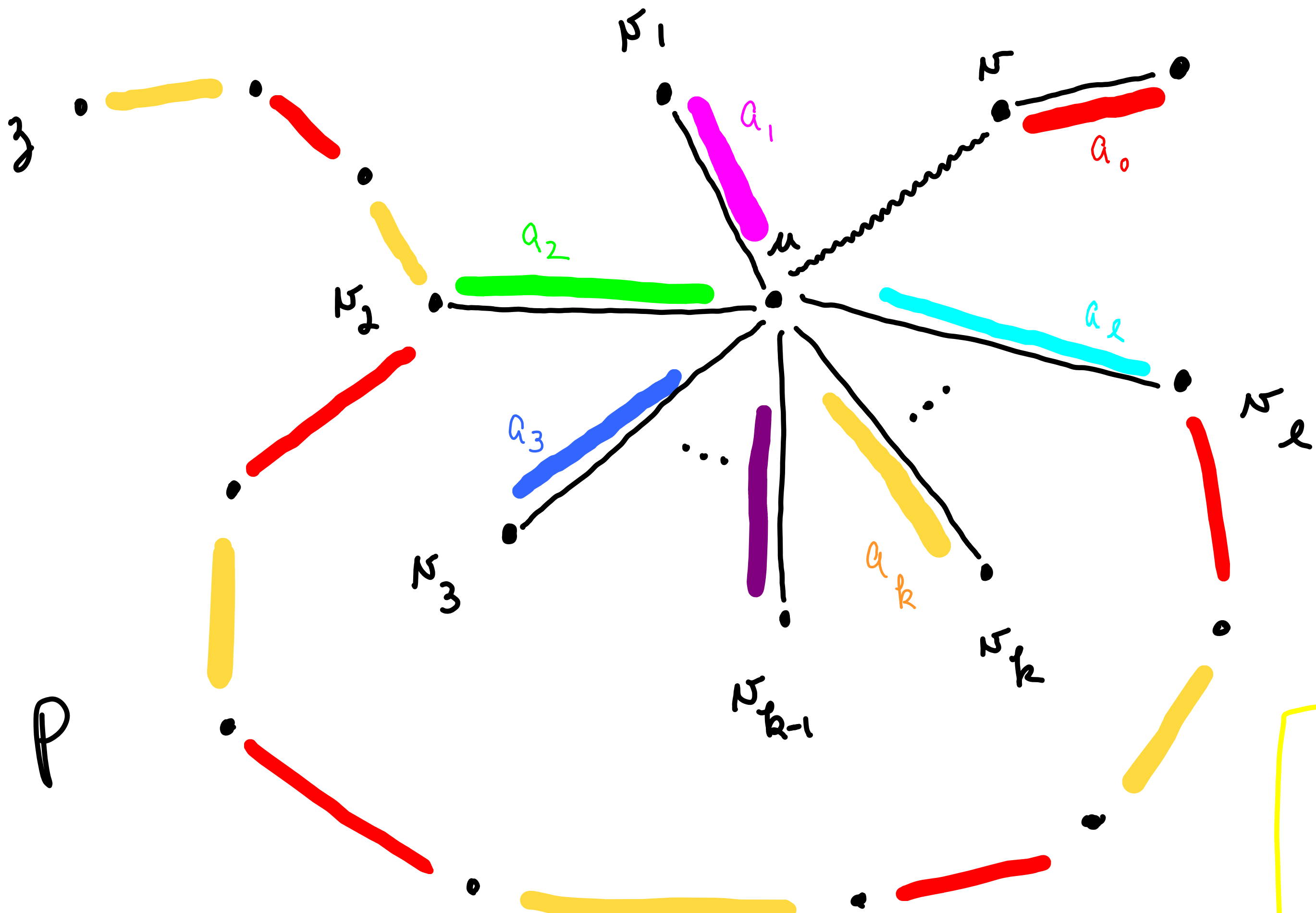
$\vdots$

a_k not rep'd at v_{k-1}

$\vdots$

$a_{\ell+1}$ not rep'd at v_ℓ

$a_{\ell+1} = a_k$



Case 3: p ends at a vertex other than v_ℓ, v_k, v_{k-1}, u

Swap colors on p, downshift from uv_ℓ & recolor uv_ℓ with a_0

But if a_0 IS rep'd at v_ℓ let p be a maximal path starting at v_ℓ and colored a_0 and a_k

a_0 not represented at u

a_1 not represented at v

a_2 not represented at v_1

a_3 not represented at v_2

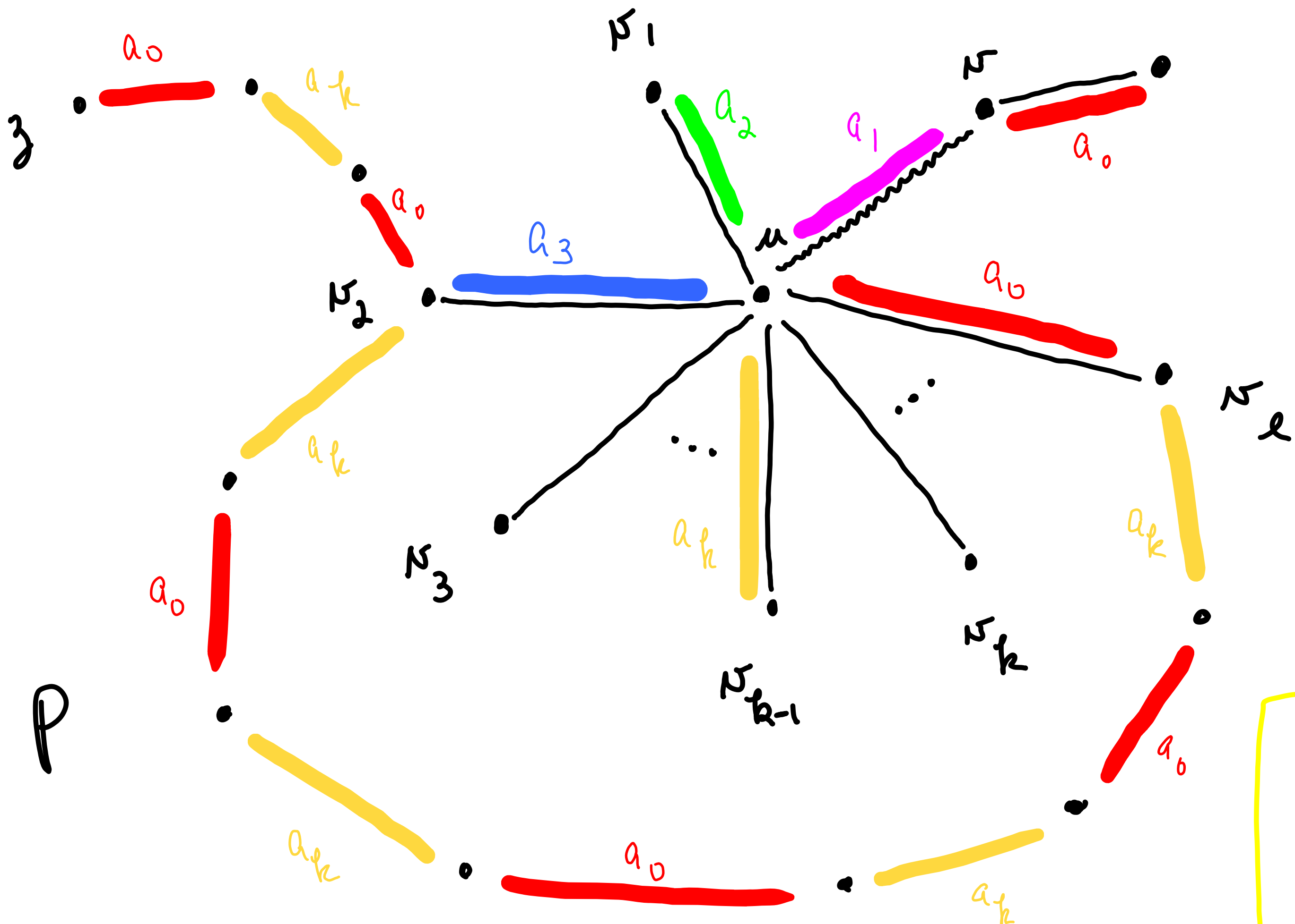
$\vdots$

a_k not rep'd at v_{k-1}

$\vdots$

$a_{\ell+1}$ not rep'd at v_ℓ

$a_{\ell+1} = a_k$



Case 3: p ends at a vertex other than v_ℓ, v_k, v_{k-1}, u

Swap colors on p, downshift from uv_ℓ + recolor uv_ℓ with a_0