Homework 2
Graph Theory CSC/MA/OR 565
Sketch of Solutions

The **Johnson graph** $J_{n,k}$ has as its vertex set the set of k-element subsets of $\{1, 2, \ldots, n\}$ with two vertices A and B joined by an edge if $|A \cap B| = k - 1$.

1. Draw $J(5, k)$ for $k = 1, \ldots, 5$. Draw the complement of $J(5, 2)$. Show that it is isomorphic to the Petersen graph. (I think this is easy and everyone should have done it correctly.)

2. $J_{n,k}$ is regular. Show how to use the degree-sum formula to find its number of edges. Use one of our results to show that $J(9, 5)$ is not bipartite.

For the first part, we need to find the degree of a vertex. To get a neighbor of vertex A, we can replace any one of the k elements in A with any of the $n - k$ elements not in A, for a total of $k(n - k)$ neighbors. The number of vertices of $J_{n,k}$ is $\binom{n}{k}$, so by the degree sum formula

$$e(J_{n,k}) = \binom{n}{k}\frac{k(n-k)}{2}.$$

For the second part, use the result that a graph is bipartite if and only if it has no odd cycles and find an odd cycle in $J(9, 5)$. For example, take the 3-cycle on the vertices $\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 4, 6\}$, $\{1, 2, 3, 4, 7\}$.

3. Prove that $J_{n,k}$ is isomorphic to $J_{n,n-k}$.

For $A \subseteq \{1, 2, \ldots, n\}$, let $\overline{A}$ denote the complement of A in $\{1, 2, \ldots, n\}$. The function $\Theta : V(J_{n,k}) \to V(J_{n,n-k})$ defined by $\Theta(A) = \overline{A}$ is a bijection. To see that it is an isomorphism, note that, by the algebra of sets, for $A, B \in J_{n,k}$,

$$\begin{aligned}
|\Theta(A) \cap \Theta(B)| &= |\overline{A} \cap \overline{B}| \\
&= |\overline{A \cup B}| \\
&= n - |A \cup B| \\
&= n - (|A| + |B| - |A \cap B|) \\
&= n - 2k - |A \cap B|
\end{aligned}$$

So, $|A \cap B| = k - 1$ if and only if $|\Theta(A) \cap \Theta(B)| = n - k - 1$. That is, AB is an edge in $J_{n,k}$ if and only if $\Theta(A)\Theta(B)$ is an edge in $J_{n,n-k}$.

4. For which pairs (n, k) is $J(n, k)$ claw free? Answer: if and only if $k < 3$ or $n < k + 3$. Proof below.

Suppose $J(n,k)$ has an induced claw, that is, has a subgraph $K_{1,3}$ on vertices x, a, b, c, with x adjacent to each of a, b, c and suppose that a, b, c are pairwise nonadjacent. The x has $k-1$ elements in common with each of a, b, c, but each pair among a, b, c differ by at least two elements. Then there must be at least 3 elements of $\{1, 2, \ldots, n\}$ that are not in x, so $n \geq k+3$. Since $J_{n,k}$ is isomorphic to $J_{n,n-k}$ this means that we also must have $k \geq 3$.

Conversely, if $n \geq k+3$ and $k \geq 3$, the subgraph induced by these four vertices is an induced claw:

$$\{1,2,3\} \cup S,\ \{1,2,4\} \cup S,\ \{1,3,5\} \cup S,\ \{2,3,6\} \cup S,$$

where $S = \{n-k+4, \ldots n\}$. (That is, S is the last $k-3$ elements of $\{1, 2, \ldots, n\}$ that we throw in to make sure each of the four sets has k elements.)

5. Let G_k be the graph whose vertices are (all of) the subsets of $\{1, 2, \ldots, k\}$. Vertices A and B of G_k are joined by an edge if either $A \subseteq B$ or $B \subseteq A$ and $|A| - |B| = \pm 1$. Find the number of vertices and edges of G_k. Prove that G_k is bipartite. Prove that G_k is isomorphic to Q_k.

The number of vertices is the number of subsets of a k element set which is 2^k. Each vertex has degree k since for each $A \subseteq \{1, 2, \ldots, k\}$ and for each $i \in \{1, 2, \ldots, k\}$, you find a neighbor of A by either removing i from A (if it belongs to A) or adding it to A (if it does not already belong). So by the degree sun formula, the number of edges is $k2^{k-1}$.

To show that G_k is bipartite, note that if A and B are adjacent vertices in G_k, then one of A, B has an even number of elements and the other has an odd number. Thus letting X be the vertices of G_k with an even number of elements and letting Y be the ones with an odd number of elements gives a bipartition of G_k.

You can check that the mapping sending $A \subseteq \{1, 2, \ldots, k\}$ to $b_1 b_2 \ldots b_k$ where

$$b_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}$$

is an isomorphism.

6. Problem 1.1.30 in the text. (We talked about the adjacency matrix already, but you'll need to look up the definition of incidence matrix in the text.)

Using the fact that A is symmetric and that every entry is 0 or 1, we have:

$$A^2[i,i] = \sum_{j=1}^{n} A[i,j]A[j,i] = \sum_{j=1}^{n} A[i,j]^2 = \sum_{j=1}^{n} A[i,j] = d_G(v_i).$$

Using the fact that every entry is 0 or 1:

$$MM^T[i,i] = \sum_{j=1}^{e(G)} M[i,j]M^T[j,i] = \sum_{j=1}^{e(G)} M[i,j]^2 = \sum_{j=1}^{e(G)} A[i,j] = d_G(v_i).$$

$A^2[i,j] = \sum_{k=1}^{n} A[i,k]A[k,j]$. This counts the number of paths of length 2 from i to j.

$MM^T[i,j] = \sum_{k=1}^{e(G)} M[i,k]M^T[k,j]$. This is counting the number of edges joining i and j, so for a simple graph, $MM^T[i,j] = A[i,j]$.

7. Problem 1.3.14 in the text.

If all degrees a different, then the degree sequence is $(n-1, n-2, \ldots, 0)$. But this means there is a with no neighbors and a vertex adjacent to every other vertex, which is impossible.

8. Find, if possible, a simple graph with 8 vertices that has neither a clique of size 3 nor an independent set of size 4. What about the other way around: neither a clique of size 4 nor an independent set of size 3?

First observe that if you can find a graph for the first, then its complement will satisfy the second since a clique in a graph G is an independent set in the complement of G.

Make an 8-cycle with vertices equally spaced around the circle. Join pairs of vertices that are opposite on the cycle. This 3-regular graph has no clique of size 3 and no independent set of size 4.

This graph shows that the *Ramsey number* $R(3,4)$ must be at least 9. You can find more information in this article by Imre Leader:

https://plus.maths.org/content/friends-and-strangers

or by looking up Ramsey numbers on Wikipedia.

9. Problem 1.2.20 in the text.

Let v be a cut vertex in a graph G. We show that the complement of $G-v$ is connected. Let $\overline{G-v}$ denote the complement of $G-v$.

Let x and y be vertices in $G-v$. If x and y are not adjacent in $G-v$, they are adjacent in $\overline{G-v}$ and therefore connected in $\overline{G-v}$.

If x and y are adjacent in $G-v$, they are in the same component C of $G-v$. Since $G-v$ is not connected, it has another component C'. Let z be a vertex of C'. Then z is not adjacent to x or y in $G-v$. So in $\overline{G-v}$, z is adjacent to both x and y and therefore x, z, y is an x,y-path in $\overline{G-v}$.

10. Problem 1.3.47 in the text.

Use induction on $n(G)$ to prove that every nontrivial loopless graph G has a bipartite subgraph H such that H has more than $e(G)/2$ edges

The claim is true if G has only one vertex. Let G be a graph with $n > 1$ vertices and assume the claim is true for graphs with fewer vertices. Let $v \in V(G)$. $G - v$ has $n - 1$ vertices and $e(G) - d_G(v)$ edges. By induction $G - v$ has a bipartite subgraph J with more than $(e(G) - d_G(v))/2$ edges. Let X, Y be a bipartition of J. Recall that G is not necessarily simple, although it is loopless. Let A be the set of edges of G joining v to vertices in X. Let B be the set of edges of G joining v to vertices in Y. Without loss of generality, assume $|A| \geq |B|$. Then $|A| \geq d_G(v)/2$. Construct H by adding to J the vertex v together with the edges in A. Then H is bipartite with bipartition X, $Y \cup \{v\}$ and the number of edges of H is

$$e(H) = e(J) + |A| > (e(G) - d_G(v))/2 + d_G(v)/2 = e(G)/2,$$

so $e(H) > e(G)/2$.