Graph Theory HW 4

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Problem 2

Use network flows to prove Menger's theorem for internally disjoint paths in digraphs: $\kappa(x, y) = \lambda(x, y)$ when xy is not an edge .

$\kappa(x, y)$ is the minimum size of the (x, y) cut and $\lambda(x, y)$ is the maximum size of the pairwise internally disjoint paths from x to y, in a digraph G with vertex set $V(G)$ and edge set $E(G)$, $x, y \in V(G)$ and $xy \notin E(G)$. It is apparent from their definitions that $\kappa(x, y) \geq \lambda(x, y)$. Now, using Ford-Fulkerson theorem for Max-flow min-cut duality, we can also prove that $\kappa(x, y) \leq \lambda(x, y)$ as follows:

Let G' be a modified version of the digraph G such that $\forall v \in V(G) - \{x, y\}$, v is expanded into 2 vertices v^- and v^+, and they are joined by an edge v^-v^+. All such vertices are called intra-vertices. All the edges incident on v in G will be incident on v^- in G' and all the edges that are outgoing from v will now be outgoing edges from v^+ in G'. Let the capacity of the edge v^-v^+ be 1 for all such intra-vertices, and let the capacity be ∞ for all other edges. For this graph G', if k is the value of the max flow from source x to sink y, then by integrality theorem, it corresponds to k pairwise internally disjoint paths from x to y in G which can be obtained by shrinking the intra-vertex edges of G', giving us $\lambda(x, y) \geq k$. By duality max-flow min-cut duality, we also have that k is the capacity of the minimum capacity cut from source $X = x \cup \{v^- : v \neq x, y\}$ to sink $Y = y \cup \{v^+ : v \neq x, y\}$. Since all the intra-vertex edges have capacity 1, every edge of the min cut will be an intra-vertex edge. The vertices that we get by shrinking these intra-vertex edges will from the vertex cut for the graph G, since all min-cut paths pass through these vertices. Thus, these k vertices form the $x, y - cut$, giving $\kappa(x, y) \leq k$. Therefore, we have $\kappa(x, y) \leq k \leq \lambda(x, y)$ and $\kappa(x, y) \geq \lambda(x, y)$ which implies

$$\kappa(x, y) = \lambda(x, y)$$

. Hence proved.

Problem 3

Find the chromatic number of the graphs in exercise 8.1 of these notes of Frederic Havet: http://www-sop.inria.fr/members/Frederic.Havet/Cours/coloration.pdf **Are either of the graphs critical?**

(a)

Max degree $\Delta = 6 \implies \chi(G) \leq \Delta(G) + 1 = 7$

Min degree $\delta = 3$

Since this is not a complete graph nor itself a cycle of odd length, by Brooks' theorem we have $\chi(G) \leq \Delta(G) \implies \chi(G) \leq 6$

It containes K_4 as subgraph, hence $\chi(G) \geq 4$.

Its chromatic number is 5, as a proper coloring can be obtained for $k = 5$. And it is not critical since on removing the bottom right vertex the chromatic number of the graph obtained is still 5.

(b)

Max degree $\Delta = 5 \implies \chi(G) \leq \Delta(G) + 1 = 6$

Since this is not a complete graph nor itself a cycle of odd length, by Brooks' theorem we have $\chi(G) \leq \Delta(G) \implies \chi(G) \leq 5$.

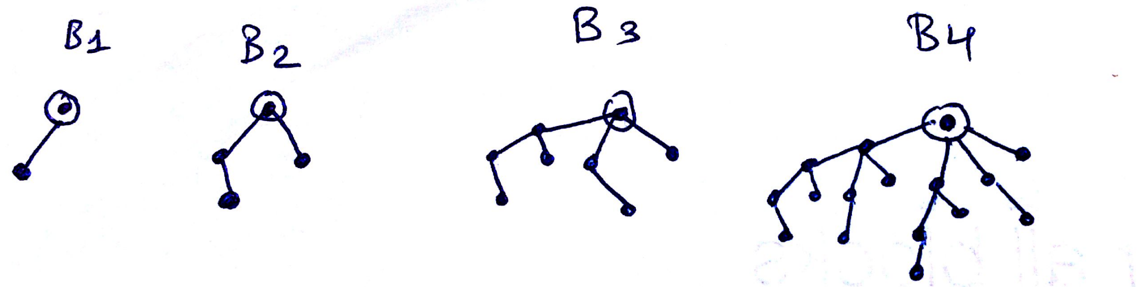
It contains an odd cycle as a subgraph. Hence, $\chi(G) \geq 3$.

*******Chromatic number coming out to be 6 on coloring. Please check this.*****
This graph is critical.

Problem 6

Draw B_1, B_2, B_3 and B_4. And prove by induction that for every k there is an ordering of vertices of B_k for which greedy coloring uses k colors.

A binomial tree B_k of order k ($k \geq 0$) is an ordered tree defined recursively as: (i) B_0 is a one-vertex graph. (ii) B_k consists of two copies of B_{k-1} such that the root of one is the left most child of the root of the other.



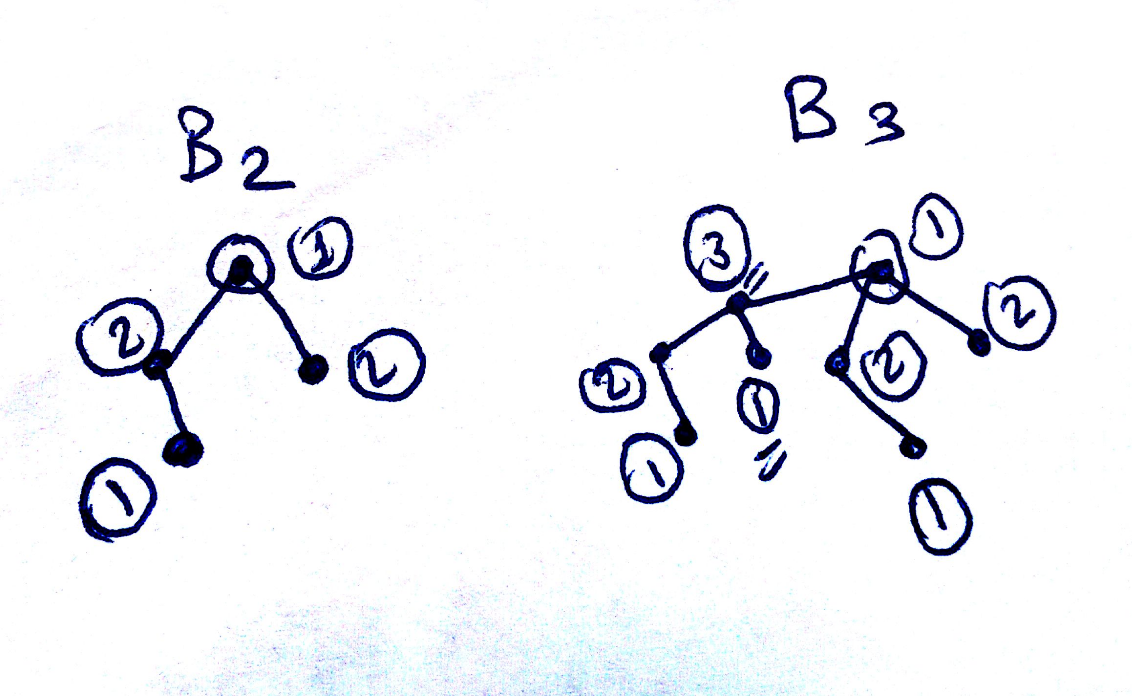
Proof by Induction:

Base case: This is true for $k = 2$. B_2 will always be colored with 2 colors by the greedy algorithm for every ordering of the vertices.

Induction hypothesis: Let there be an ordering of vertices for B_k such that the greedy algorithm uses k colors.

Induction step: For B_{k+1}, which is formed by merging 2 B_k, let the new ordering of vertices be formed using the following sequence of ordering: 1) ordering of vertices of the B_k whose root becomes the root of B_{k+1} 2) rightmost child of the second B_k 3) maintaining the relative ordering of the remaining vertices of B_k except its original root 4) The root of the second B_k is colored last

For example:



Following the above sequence of coloring always ensures that greedy algorithm takes $k+1$ colors given that B_k takes k colors.

Hence proved.

Problem 7

Without using Brooks' theorem, prove that if G is a simple connected graph which is not regular, then $\chi(G) \leq \Delta(G)$

Let Δ, the maximum degree of a vertex in the graph be k. This case is possible since the graph is given to be non-regular. Let v be a vertex of degree less than k. Construct a spanning tree with v as root and assign indices in decreasing order as the vertices are reached. Thus, here v will be the vertex with the highest index. Hence, every other vertex will have a neighbor with higher index in the ordering. Thus, every vertex will have at most $k-1$ vertices with lower indices. Hence, greedy coloring will use atmost k colors.