Counting Trees

Let $[n] = \{1, 2, 3, \ldots, n\}$

1. How many simple graphs with vertex set $[n]$?

2. How many paths with vertex set $[n]$?

3. How many stars with vertex set $[n]$?

4. How many trees with vertex set $[n]$?

Counting Trees

Let $[n] = \{1, 2, 3, \ldots, n\}$

① How many simple graphs with vertex set $[n]$?

$2^{\binom{n}{2}}$

② How many paths with vertex set $[n]$?

$\frac{n!}{2}$

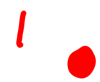
③ How many stars with vertex set $[n]$?

n if $n>2$

④ How many trees with vertex set $[n]$?

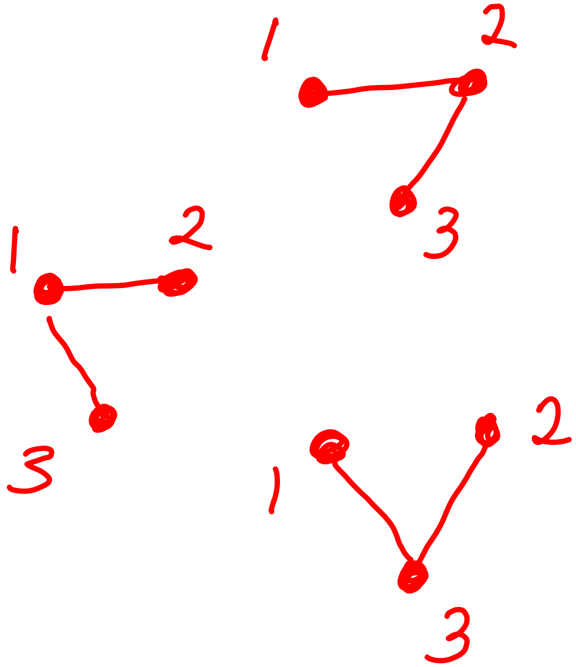
Trees with vertex set $[n] = \{1, 2, \ldots, n\}$

$n=1$

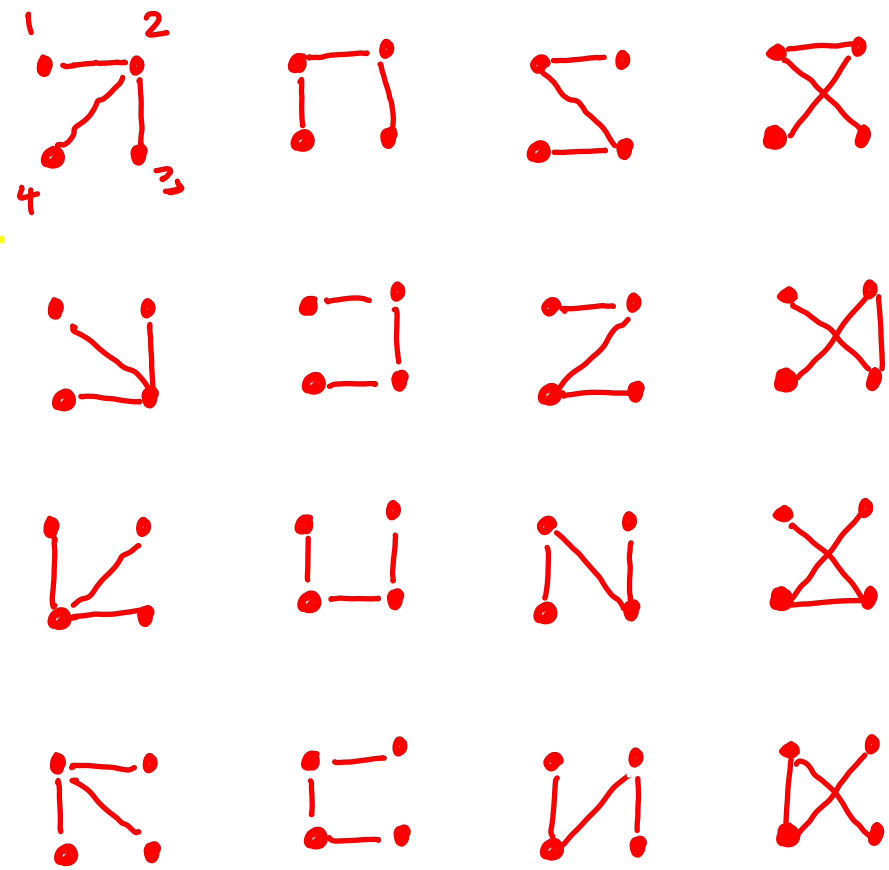


$n=2$

$n=3$



$n=4$



Assume K_n has vertex set $[n]$

Then "tree with vertex set $[n]$" is same as

"spanning tree of K_n"

Define $\tau(G)$ to be # of spanning trees of G

n	$\tau(K_n)$
1	1
2	1
3	3
4	16
5	?
6	?

$\tau(K_n) =$

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Then "tree with vertex set $[n]$" is same as

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Define $\tau(G)$ to be # of spanning trees of G

n	$\tau(K_n)$
1	1
2	1
3	3
4	16
5	?
6	?

$$\tau(K_n) = n^{n-2}$$

(what else does this count?)

Thm 2.2.3 $\tau(K_n) = n^{n-2}$ [Cayley 1889]

pf. [Prüfer correspondence] Show bijection:

$$\boxed{\text{spanning trees of } K_n} \longleftrightarrow \boxed{\text{sequences of } n-2 \text{ elements from } [n]}.$$

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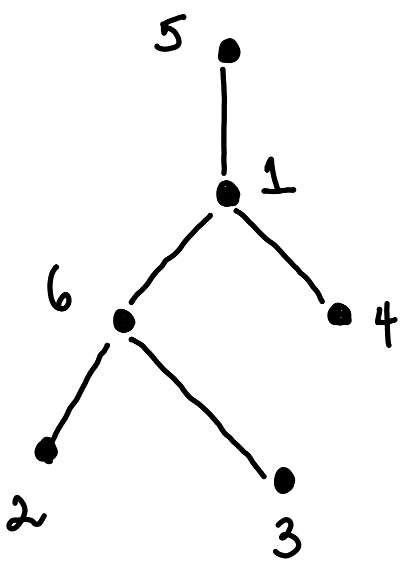
(1) spanning tree → sequence



smallest remaining leaf	vertex adjacent to it

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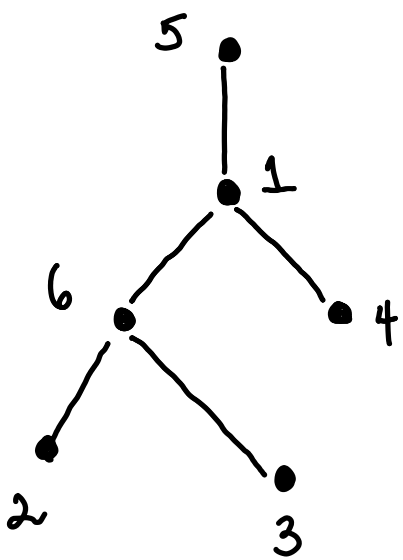

smallest remaining leaf	vertex adjacent to it
2 (remove)	6

Thm 2.2.3 $\tau(K_n) = n^{n-2}$ [Cayley 1889]

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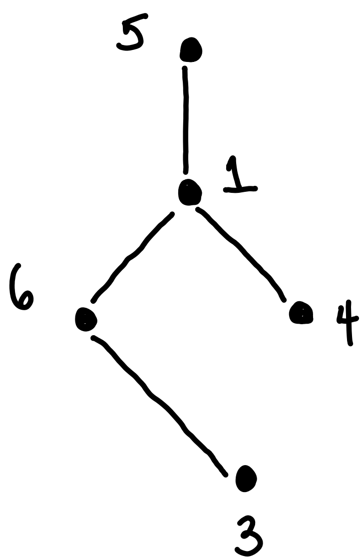
(1) spanning tree → sequence



smallest remaining leaf	vertex adjacent to it
2 (remove)	6
3 (remove)	6

Thm 2.2.3 $\tau(K_n) = n^{n-2}$ [Cayley 1889]

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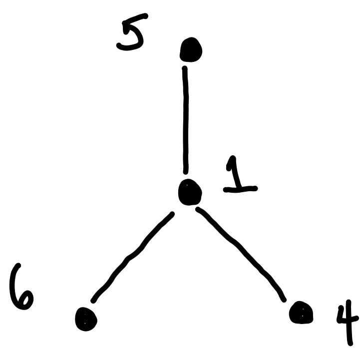
(1) spanning tree → sequence



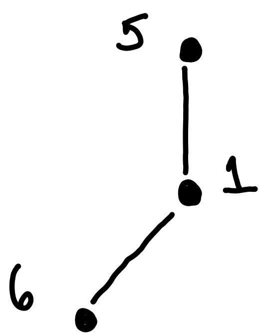
smallest remaining leaf	vertex adjacent to it
2 (remove)	6
3 (remove)	6
4 (remove)	1

Thm 2.2.3 $\tau(K_n) = n^{n-2}$ [Cayley 1889]

pf. [Prüfer correspondence] Show bijection:

$$\boxed{\text{spanning trees of } K_n} \longleftrightarrow \boxed{\text{sequences of } n-2 \text{ elements from } [n]} \dashrightarrow n^{n-2}$$

(1) spanning tree → sequence



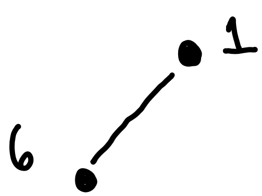
smallest remaining leaf	vertex adjacent to it
2 (remove)	6
3 (remove)	6
4 (remove)	1
5 (remove)	1

Thm 2.2.3 $\tau(K_n) = n^{n-2}$ [Cayley 1889]

pf. [Prüfer correspondence] Show bijection:

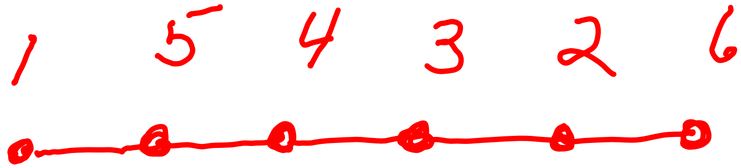
$\boxed{\text{spanning trees of } K_n} \longleftrightarrow \boxed{\text{sequences of } n-2 \text{ elements from } [n]}$ n^{n-2}

(1) spanning tree → sequence



smallest remaining leaf	vertex adjacent to it
2 (remove)	6
3 (remove)	6
4 (remove)	1
5 (remove)	1

Sequence: $(6, 6, 1, 1)$


1
5
4
3
2
6

What about the reverse?

$\boxed{\text{spanning trees of } K_n}$ $\longleftarrow$ $\boxed{\text{sequences of } n-2 \text{ elements from } [n]}$ $\dashrightarrow$ n^{n-2}

(2) sequence $\longrightarrow$ spanning tree (reverse the process)

Given $(t_1, t_2, \ldots t_{n-2})$	Construct $(s_1, \ldots, s_{n-2})$ s_i = smallest v <u>not</u> in $\{t_i, \ldots t_{n-2}\} \cup \{s_1, \ldots, s_{i-1}\}$	and tree edges $t_i s_i$
$t_1 = 5$		
$t_2 = 4$		
$t_3 = 3$		
$t_4 = 2$		

(2) sequence $\longrightarrow$ spanning tree (reverse the process)

Given $(t_1, t_2, \ldots t_{n-2})$	Construct $(s_1, \ldots, s_{n-2})$ s_i = smallest v **not** in $\{t_i, \ldots t_{n-2}\} \cup \{s_1, \ldots, s_{i-1}\}$	and tree edges $t_i s_i$
$t_1 = 5$	$s_1 = \underline{1}$	5 •—• 1
$t_2 = 4$		
$t_3 = 3$		
$t_4 = 2$		

(2) sequence $\longrightarrow$ spanning tree (reverse the process)

Given $(t_1, t_2, \dots t_{n-2})$	Construct $(s_1, \dots, s_{n-2})$ s_i = smallest v not in $\{t_i, \dots t_{n-2}\} \cup \{s_1, \dots, s_{i-1}\}$	and tree edges $t_i s_i$
$t_1 = 5$	$s_1 = 1$	5 — 1
$t_2 = 4$	$s_2 = 5$	4 — 5
$t_3 = 3$		
$t_4 = 2$		

(2) sequence $\longrightarrow$ spanning tree (reverse the process)

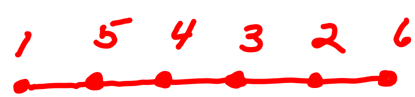
Given $(t_1, t_2, \dots, t_{n-2})$	Construct $(s_1, \dots, s_{n-2})$ $s_i =$ smallest v <u>not</u> in $\{t_i, \dots t_{n-2}\} \cup \{s_1, \dots, s_{i-1}\}$	and tree edges $t_i s_i$
$t_1 = 5$	$s_1 = \underline{1}$	5 — 1
$t_2 = 4$	$s_2 = 5$	4 — 5
$t_3 = 3$	$s_3 = 4$	3 — 4
$t_4 = 2$		

(2) sequence $\longrightarrow$ spanning tree (reverse the process)

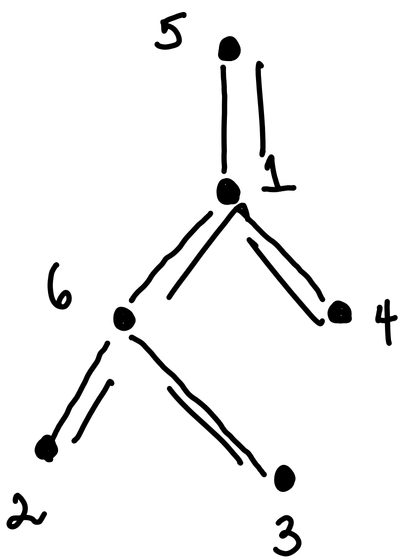
Given $(t_1, t_2, \dots, t_{n-2})$	Construct $(s_1, \dots, s_{n-2})$ s_i = smallest v <u>not</u> in $\{t_i, \dots t_{n-2}\} \cup \{s_1, \dots, s_{i-1}\}$	and tree edges $t_i s_i$
$t_1 = 5$	$s_1 = \underline{1}$	5 — 1
$t_2 = 4$	$s_2 = 5$	4 — 5
$t_3 = 3$	$s_3 = 4$	3 — 4
$t_4 = 2$	$s_4 = 3$	2 — 3

(2) sequence $\longrightarrow$ spanning tree (reverse the process)

Given $(t_1, t_2, \ldots, t_{n-2})$	Construct $(s_1, \ldots, s_{n-2})$ $s_i =$ smallest v <u>not</u> in $\{t_i, \ldots, t_{n-2}\} \cup \{s_1, \ldots, s_{i-1}\}$	and tree edges $t_i s_i$
$t_1 = 5$	$s_1 = \underline{1}$	$5 - 1$
$t_2 = 4$	$s_2 = 5$	$4 - 5$
$t_3 = 3$	$s_3 = 4$	$3 - 4$
$t_4 = 2$	$s_4 = 3$	$2 - 3$
Last tree edge joins 2 vertices not in $\{s_1, \ldots, s_{n-2}\}$		$2 - 6$

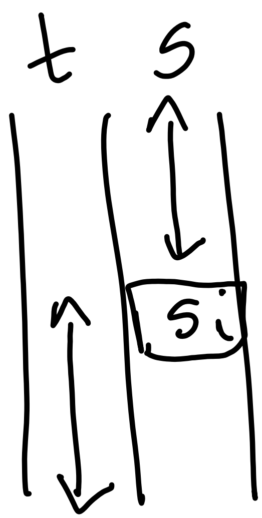


Start with (6,6,1,1) + see if you get back tree in first example



t	s		
6	2		26
6	3		36
1	4		14
1	5		15

16



Ex

$$(2, 2, 2, 2, 2, 2) \longrightarrow \ ?$$

$$(5, 8, 1, 2, 7, 1, 3, 9) \longrightarrow \ ?$$

NEXT:

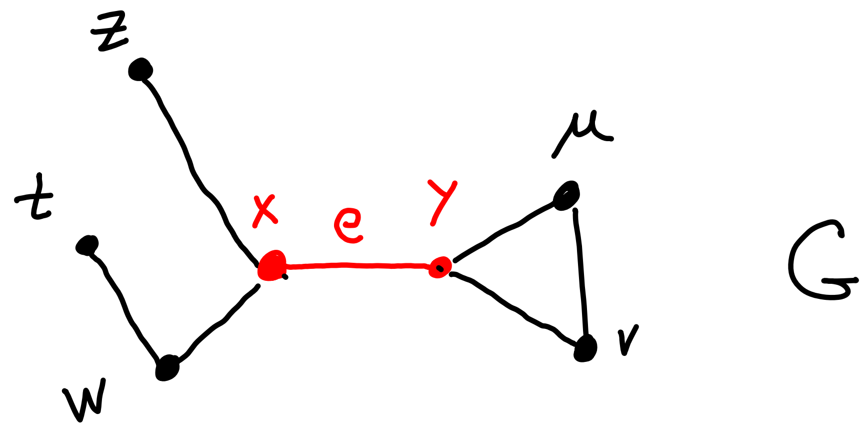
Finding $\tau(G)$ for arbitrary G

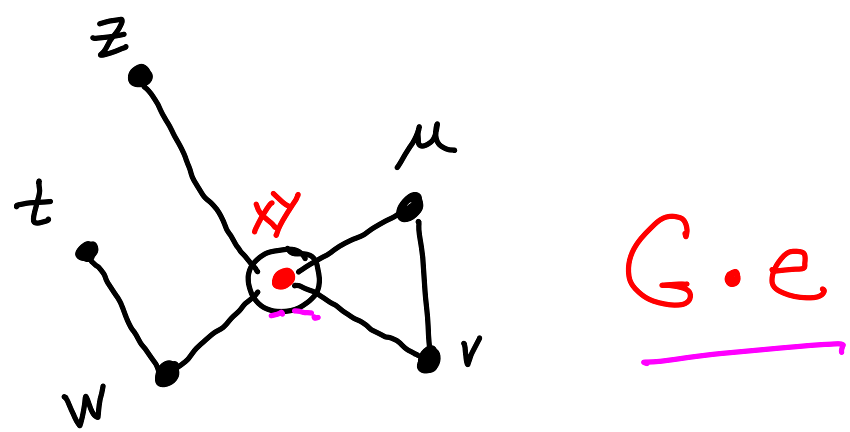
Edge Contraction

For non-loop, e, $G \cdot e$ denotes the graph obtained from G by <u>**contracting**</u> edge e.

"Contract e" means identify the endpoints of e and delete e.

Example:





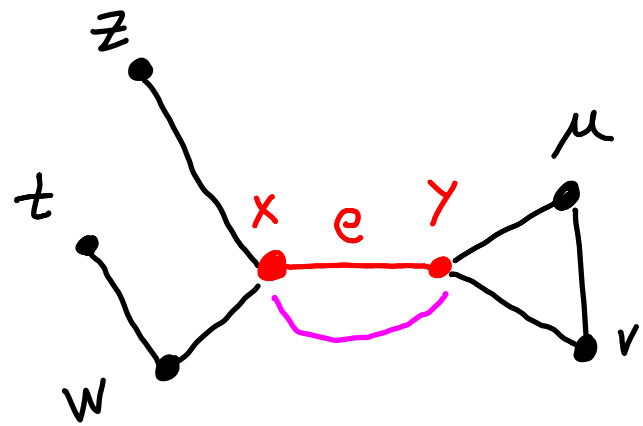
Edge Contraction

For non-loop, e, $G \cdot e$ denotes the graph obtained from G by **contracting** edge e.

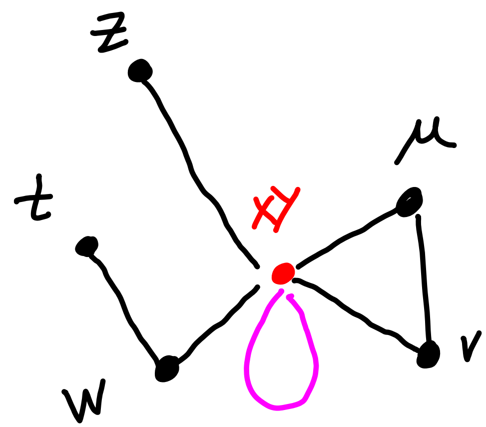
"Contract e" means identify the endpoints of e and delete e.

Example:

(if there were another edge joining x & y it would become a loop.)



G



$G \cdot e$

Graph G, e non-loop edge

(i) T is a spanning tree of G <u>not</u> containing e

$\Leftrightarrow$ T is a spanning tree of $G - e$.

(ii) T is a spanning tree of G <u>containing e</u>

$\Leftrightarrow$ $T \cdot e$ is a spanning tree of $G \cdot e$

Graph G, e non-loop edge

(i) T is a spanning tree of G not containing e

$\Longleftrightarrow$ T is a spanning tree of $G-e$.

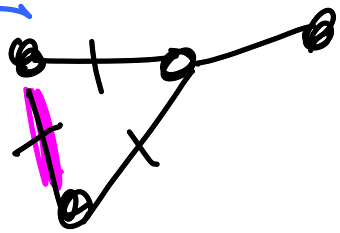
(ii) T is a spanning tree of G containing e

$\Longleftrightarrow$ T is a spanning tree of $G \cdot e$

$\boxed{\text{Proposition 2.2.8}}$ For a non-loop edge, e,

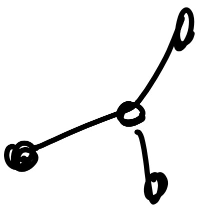
$$\boxed{\tau(G) = \tau(G-e) + \tau(G \cdot e)}$$

Example

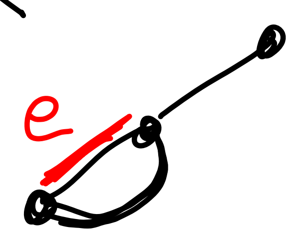


G $\quad$ 3

c



$+$

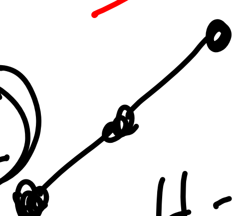


e

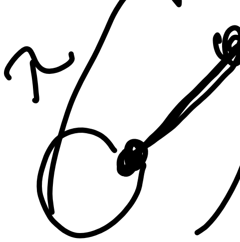
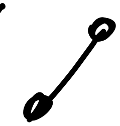
$G - e$ $\quad$ 1

$G \cdot e = H$

1



$H - e$

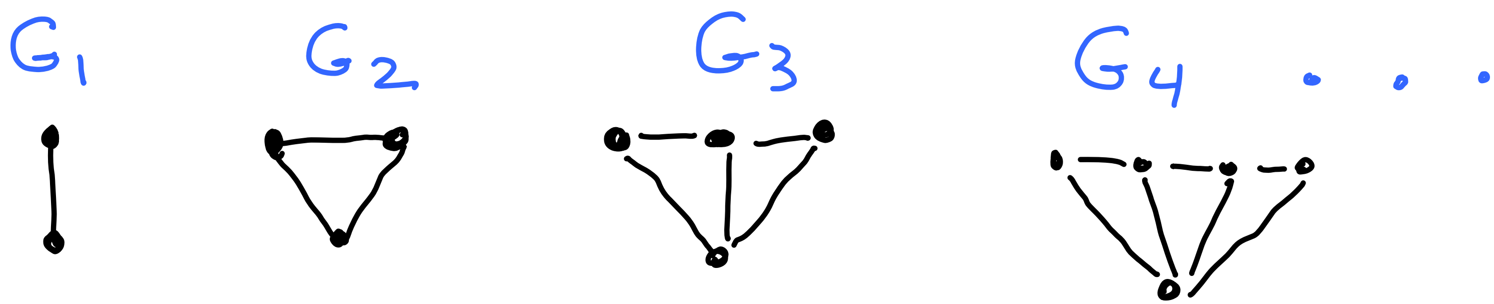
$+ \ \tau\Big($  $\Big) = \tau\Big($  $\Big) = 1$

Proposition 2.2.8. For non-loop $e \in E(G)$,

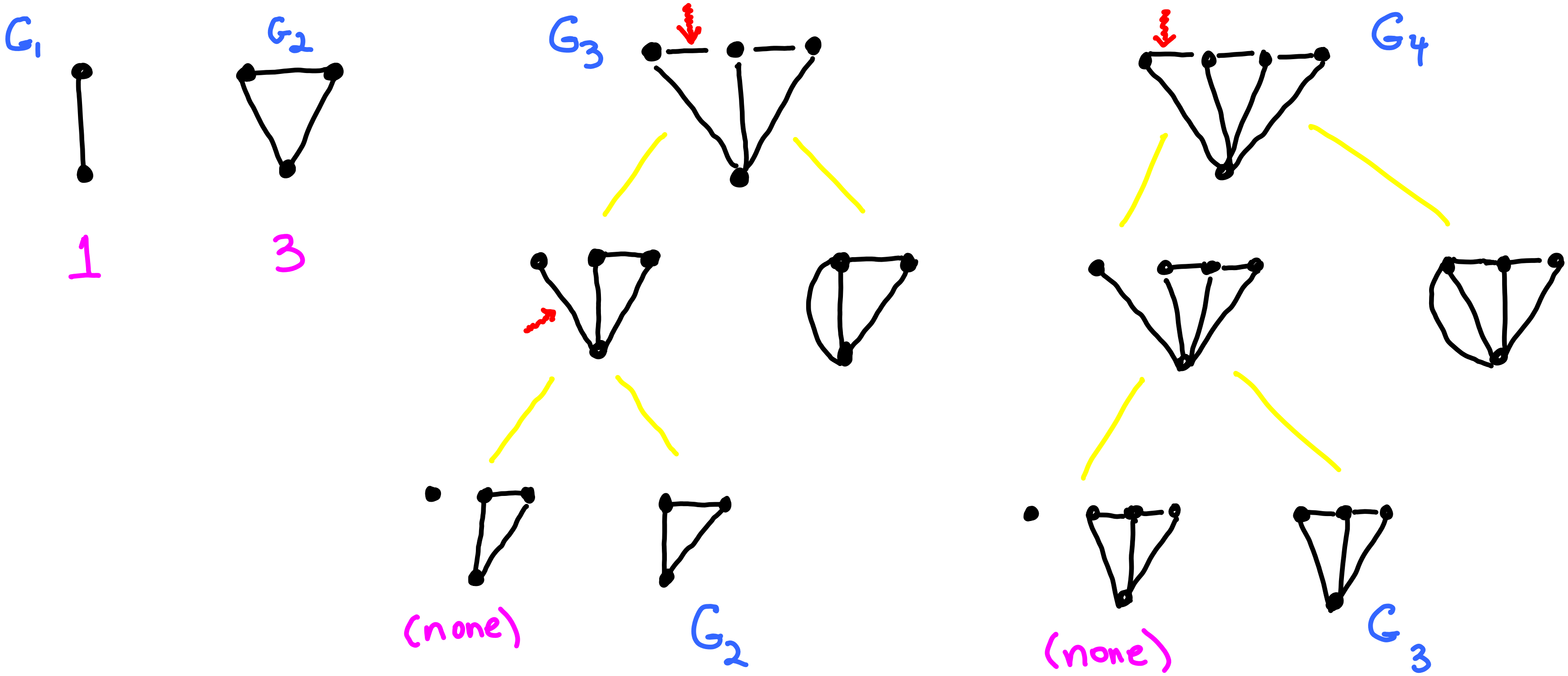
$$\tau(G) = \tau(G - e) + \tau(G \cdot e)$$

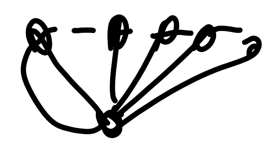
(Gives recursion for counting spanning trees of a graph.)

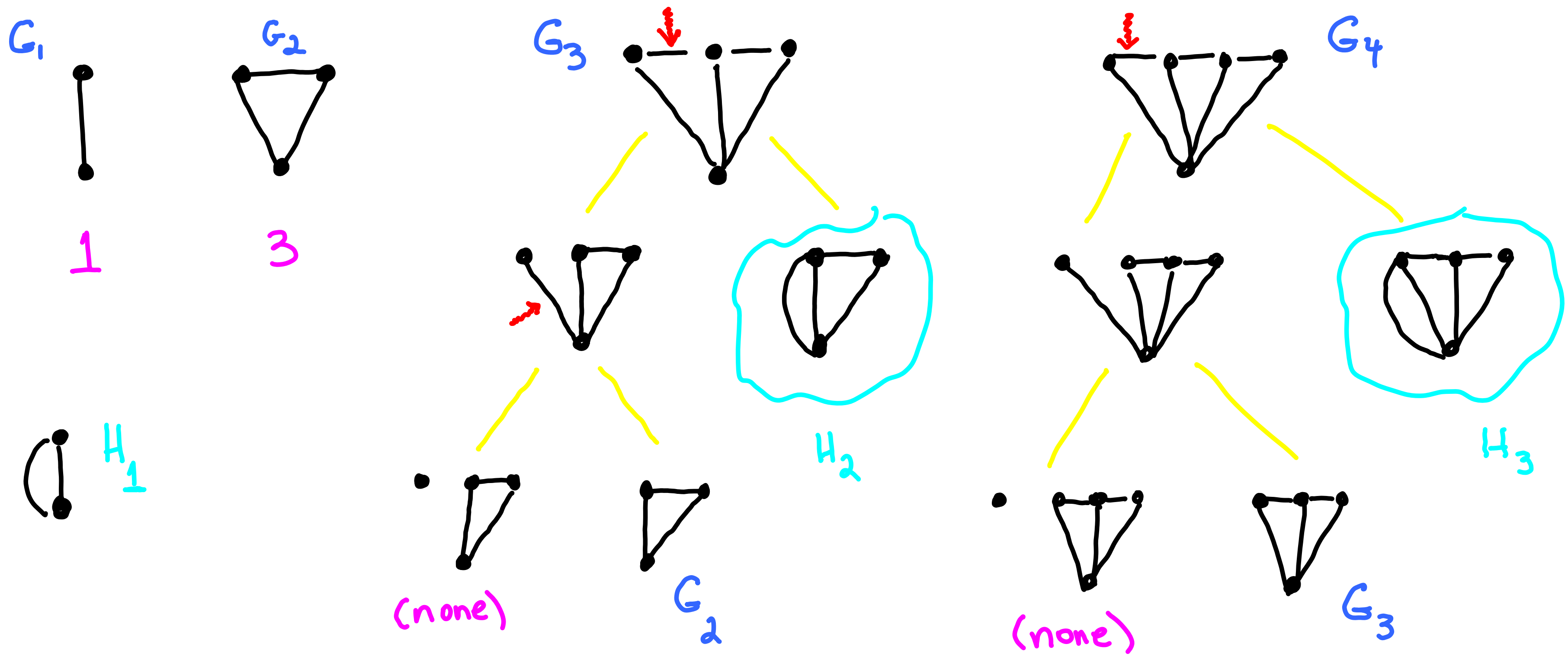
Example:



$$\tau(G_n) = ?$$

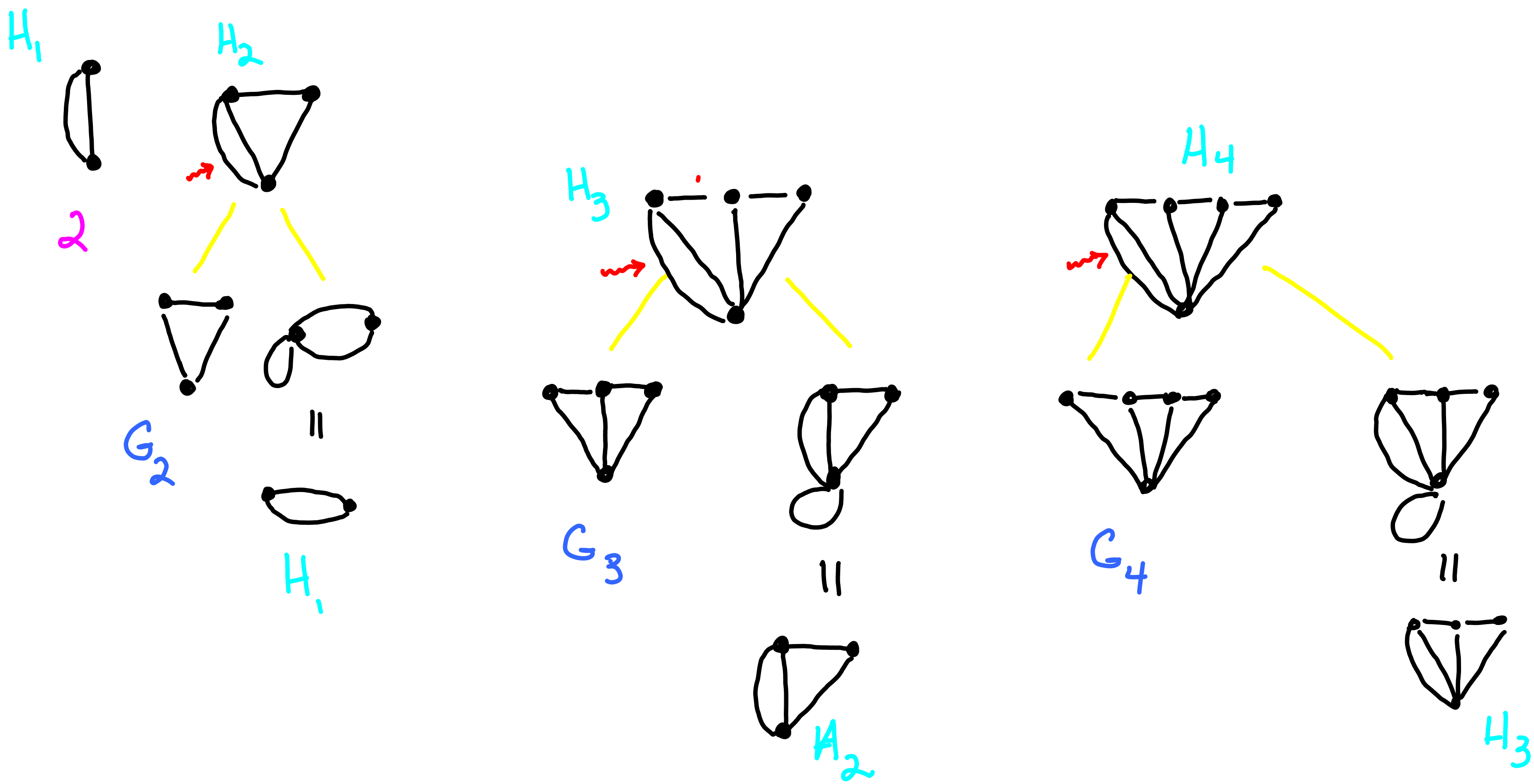

G1
G2
1
3
G3
G4
(none)
G2
(none)
G3





$$\tau(G_n) = \tau(G_{n-1}) + \tau(H_n)$$

???



$$\tau(H_n) = \tau(G_n) + \tau(H_{n-1})$$

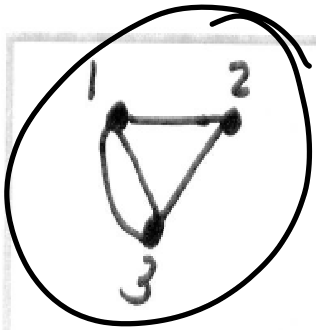
Matrix Tree Theorem 2.2.12.

G - loopless multigraph
D - diagonal matrix with $d_{ii} = d_G(v_i)$
$Q = D - A$ $\longrightarrow$ Laplacian

Then, for any $s, t, \in [n]$:

$$\tau(G) = (-1)^{s+t} det(Q^{(s,t)})$$

where $Q^{(s,t)}$ is obtained from Q by deleting row s and column t.



$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$Q = D - A = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 2 & -1 \\ -2 & -1 & 3 \end{bmatrix}$$

$s = 1, \ t = 2$

$$Q^{(1,2)} = \begin{bmatrix} -1 & -1 \\ -2 & 3 \end{bmatrix} \longleftarrow \text{det: } -3-2 = -5$$

$$(-1)^{s+t} = -1$$

$$\tau(G) = 5$$