Characterization of Planar Graphs

<u>**Kuratowski's Theorem**</u> [1930]: G is planar iff G contains no subdivision of K_5 or $K_{3,3}$.

(one way easy)

Method: Prove a slightly stronger formulation due to Tutte via a proof due to Thomassen.

Definitions

Kuratowski subgraph of G

subgraph of G which is a subdivision of K_5 or $K_{3,3}$

minimal nonplanar graph

nonplanar graph such that every proper subgraph is planar

S-lobe

Suppose S is a separating set of G and H is a component of $G - S$. Then $G[S \cup V(H)]$ is an S-lobe of G.

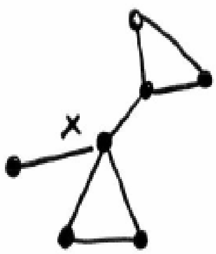
S-lobe of G,

$S \subseteq V(G)$

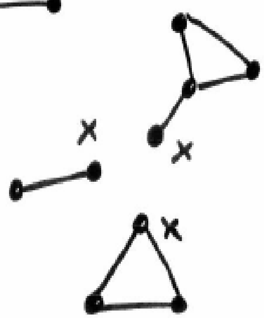
Subgraph of G induced by vertices of S and vertices of some component of $G-S$.

EX1

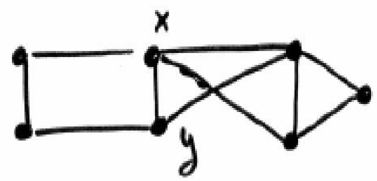
G



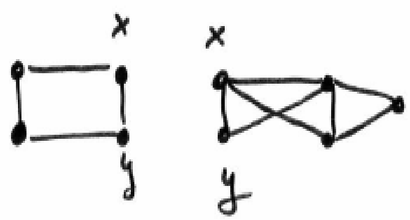
$\{x\}$-lobes of G



EX2



$\{x, y\}$-lobes



Kuratowski's Theorem [1930]: G is planar iff G contains no subdivision of K_5 or $K_{3,3}$.

Hard part: If G has no subdivision of K_5 or $K_{3,3}$ then G is planar

Lemma 6.2.7. Suppose G is a nonplanar graph with no subdivision of K_5 or $K_{3,3}$, and G has the fewest edges among such graphs. Then G is 3-connected.

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Theorem 6.2.11 (Tutte) If G is a 3-connected graph with no subdivision of K_5 or $K_{3,3}$, then there exists a (convex) planar embedding of G (with no 3 vertices on a line).

Proof. [Thomassen 1980]

Lemma 6.2.9. A 3-connected graph with at least 5 vertices contains an edge whose contraction leaves a 3-connected graph.

Lemma 6.2.10. If $G \cdot e$ has a Kuratowski subgraph, then G also has a Kuratowski subgraph.

Lemma 6.2.6 Suppose $S = \{x, y\}$ is a separating set of G of size 2. If G is nonplanar, then adding xy to some S-lobe of G yields a nonplanar graph.

Lemma 6.2.5. Every minimal nonplanar graph is 2-connected.

Lemma 6.2.4. If E is the edge set of a face in a planar embedding of G, then G has an embedding in which E is the edge set of the unbounded face.

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Proof. Embed G on the sphere (with no crossings) so that face f contains the north pole.

Then do stereographic projection

http://mathworld.wolfram.com/StereographicProjection.html

Lemma 6.2.5. Every minimal nonplanar graph is 2-connected.

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Proof. If v is a cut vertex of G,

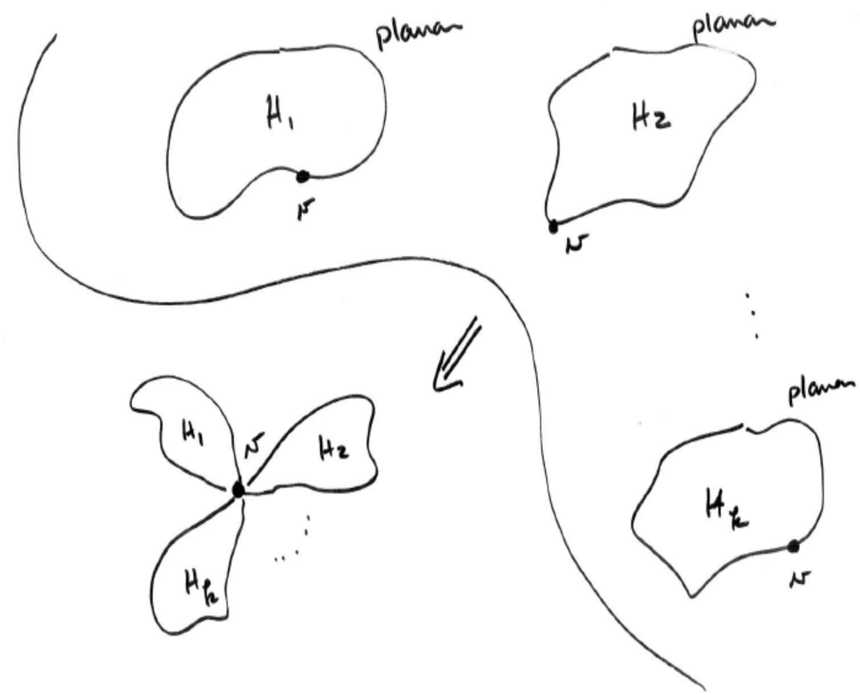
consider the $\{v\}$-lobes $H_1, H_2, \ldots, H_k$.

Each must be planar (why?). (Since G was minimal)

Each has planar embedding with v on unbounded (by 6.2.4)

Combine to get a planar embedding of G

$\Rightarrow\Leftarrow$



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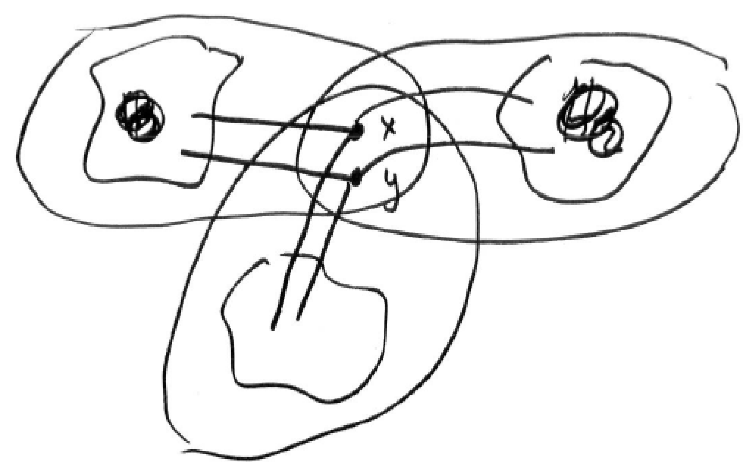

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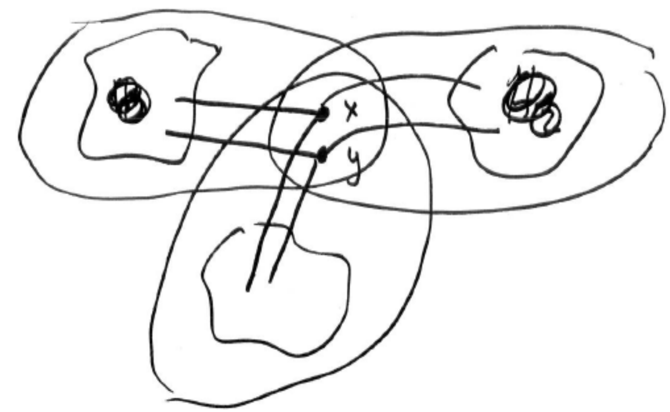

Pf. Let $H_1, H_2, \ldots, H_k$ be the $\{x,y\}$-lobes.

Suppose every $H_i + xy$ is planar

(*) Embed each $H_i + xy$ with xy on the boundary of the unbounded face

But then can combine these to get a planar embedding of G.

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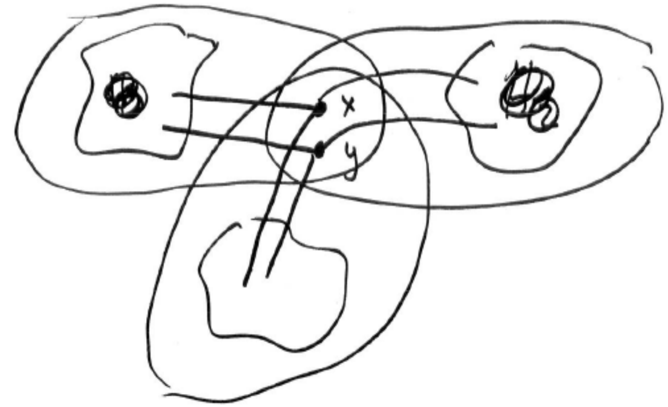

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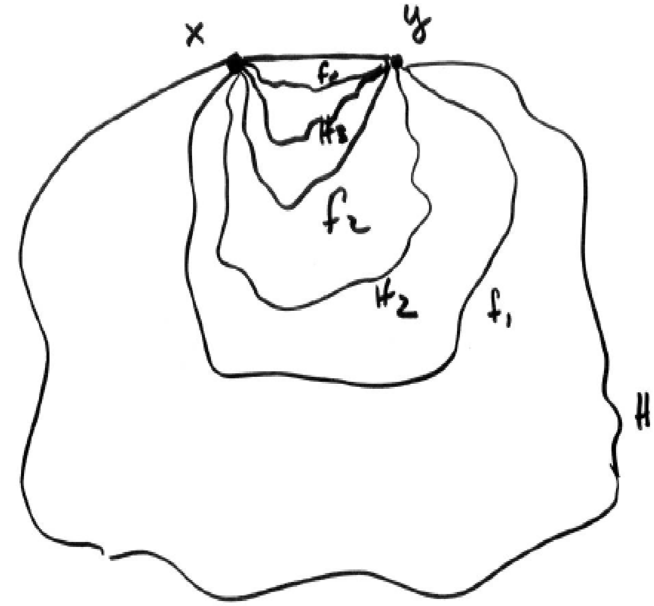

Let f_i be a face with xy on boundary.

Embed H_i in f_{i-1}, making copies of xy coincide

Get planar embedding of G

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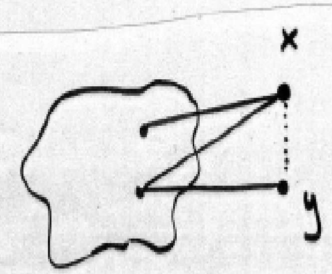
Proof. By 6.2.5 G is 2-connected. (since minimal nonplanar)

If G is not 3-connected, it has a 2-vertex cut $\{x,y\}$.

By 6.2.6, for some $\{x,y\}$-lobe H of G,

H+xy is not planar:



H+xy

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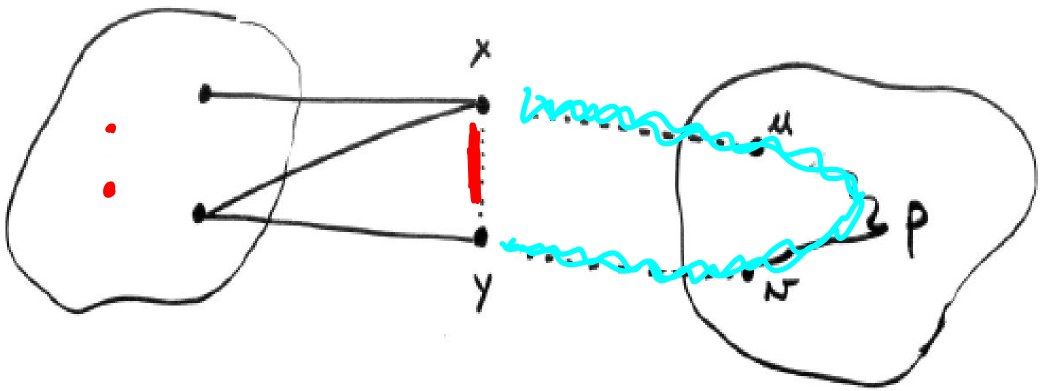
Let C be another component of $G - \{x,y\}$



$H + xy$, not planar

but fewer edges than G

$\therefore$ must have Kurat. subg. K

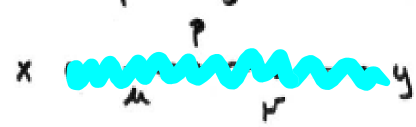
But G did not,

so K must use edge xy

x & y each have at least one nbr in C (since G 2-connected)

C has (u,v) path P

Replacing x — y in K with



gives

Kuratowski subgraph in G $\Rightarrow\Leftarrow$

10

Lemma 6.2.9. A 3-connected graph with at least 5 vertices contains an edge whose contraction leaves a 3-connected graph.

Suppose not.

Then for any edge xy, $G \cdot xy$ has a

2-vertex cut $\{xy, z\}$

(Then $\{x, y, z\}$ is a 3-vertex cut of G)

So, every edge xy has a "companion" vertex, z

s.t. $G - \{x, y, z\}$ is disconnected.

Consider all possible $\{x, y, z\}$ where:

xy is edge and z is "companion" of xy.

Choose $\{x, y, z\}$ so that largest <u>component</u> of $G - \{x, y, z\}$ is maximum. Let H be largest component

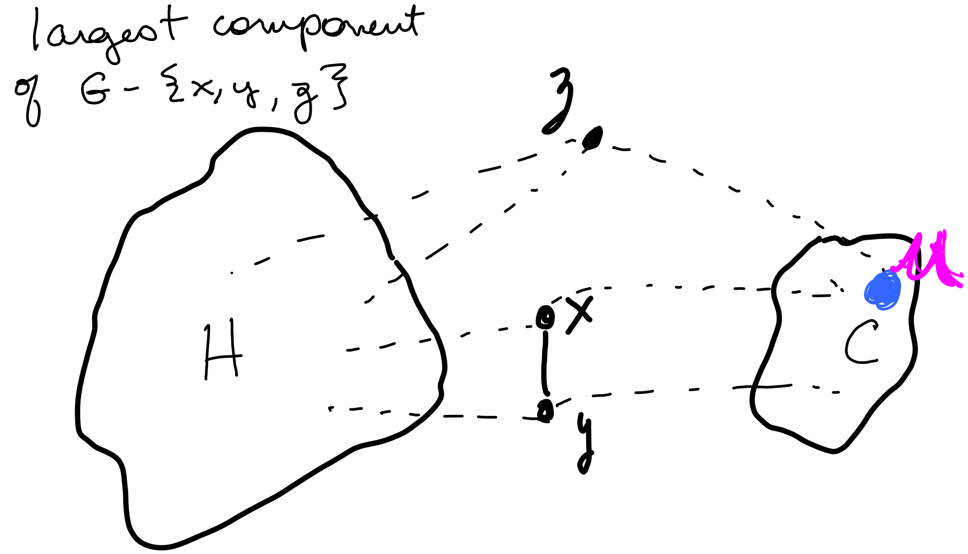
Let C be another component.

z must have a u neighbor in C (why?)

Let v be a "companion" of zu (then $G - \{z, u, v\}$ is disconnected)

Now ask "where is v" and reach a contradiction.

③



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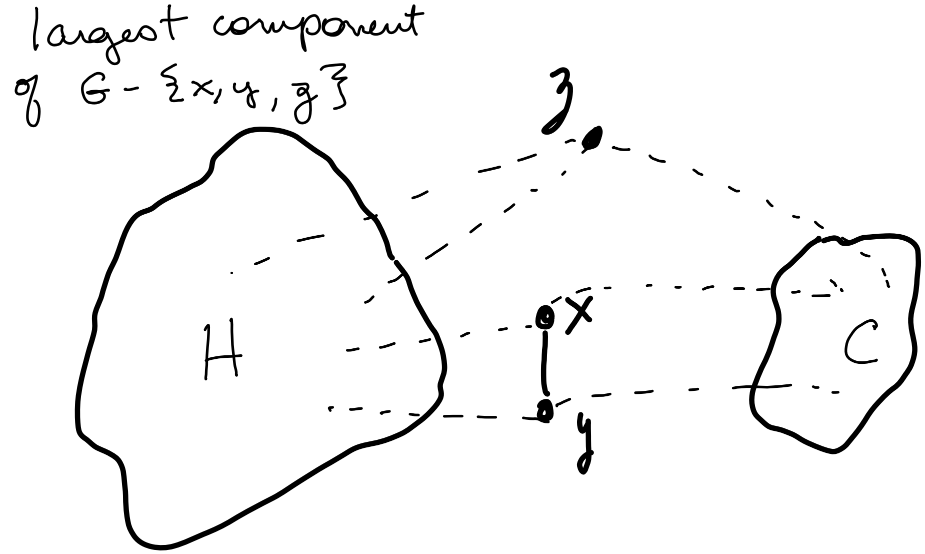
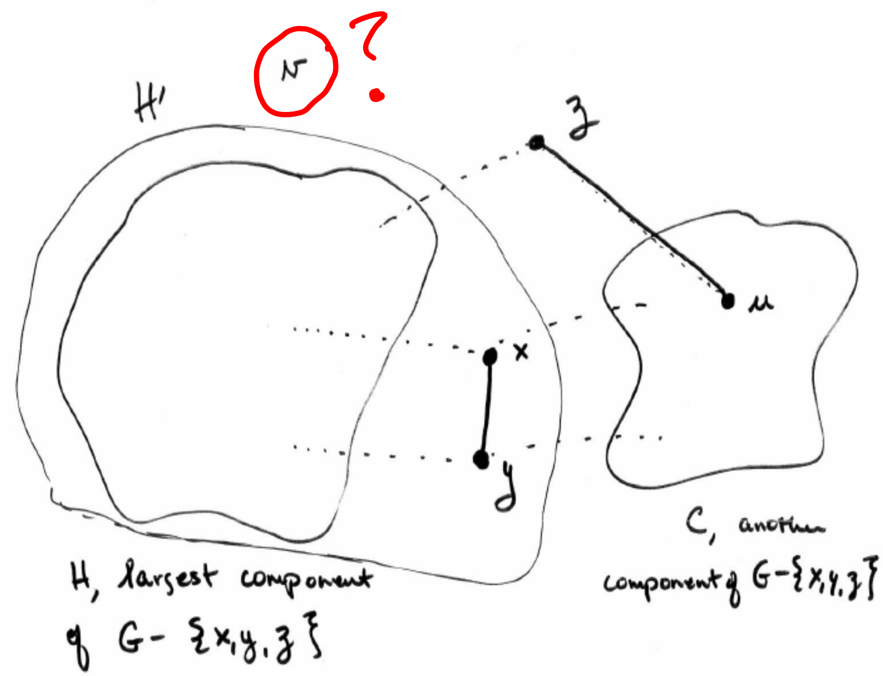
(3)

Let $H' = G[V(H) \cup \{x, y\}]$

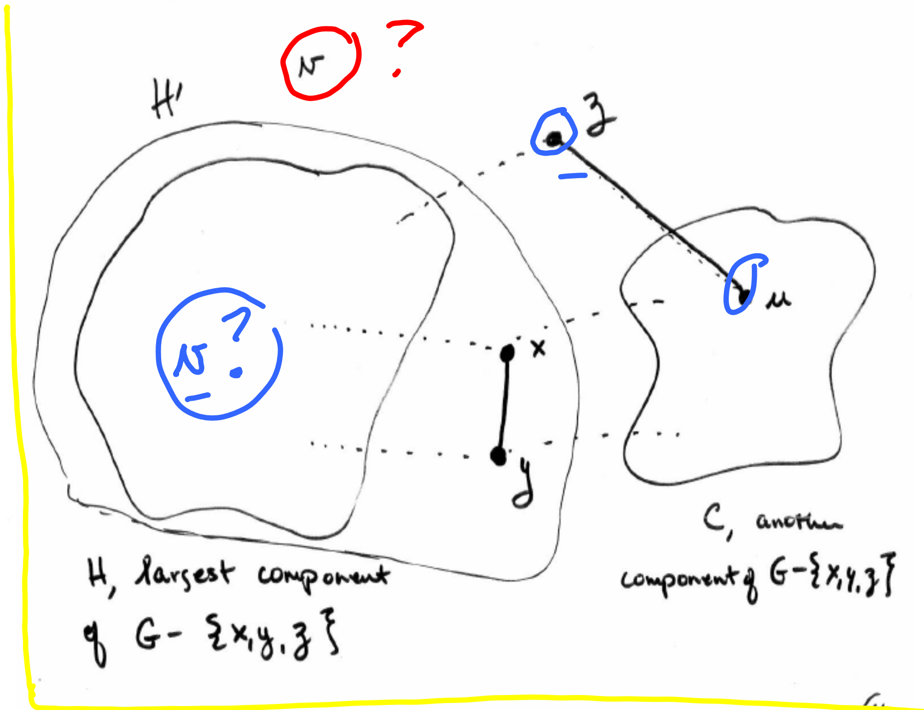




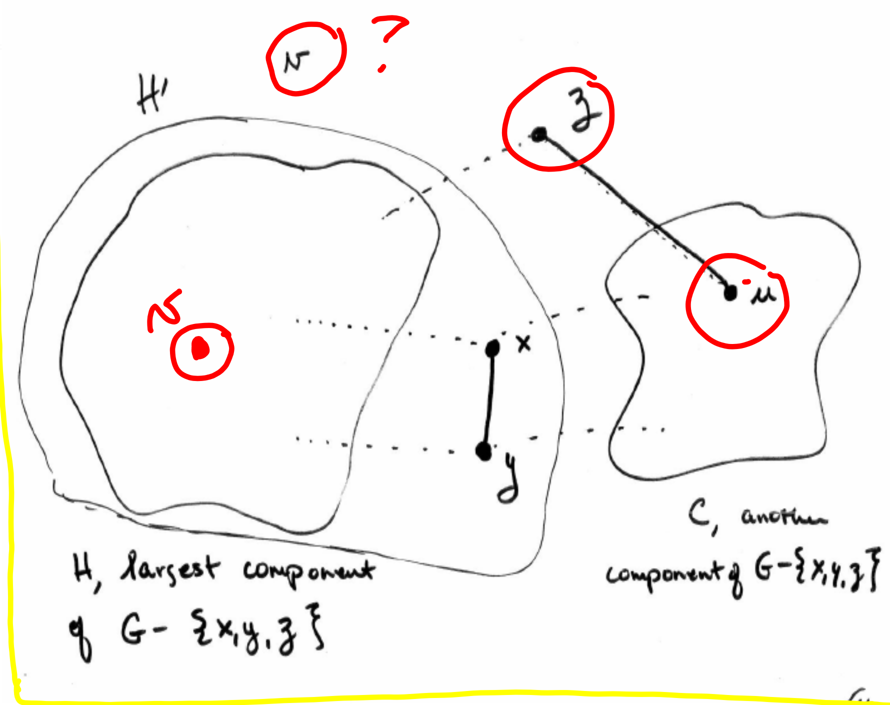
If $v \notin \overset{H'}{\boxed{V(H) \cup \{x,y\}}}$, then $\overset{H'}{\boxed{V(H) \cup \{x,y\}}}$ is induces a connected subgraph of $G - \{z,u,v\}$ larger than H. Contradicts choice of $\{x,y,z\}$.

Similarly, if v is not a cut vertex of $\overset{H'}{\boxed{G[V(H) \cup \{x,y\}]}}$, then $G - \{z,u,v\}$ contains the connected subgraph $\underset{H'-v}{G[V(H) \cup \{x,y\} - \{v\}]}$, larger...

However, if v is a cut vertex of $\overset{H'}{\boxed{G[V(H) \cup \{x,y\}]}}$, then $\{v,z\}$ is a 2-vertex cut of G, contradicting 3-connectivity



If $v \notin \overbrace{\boxed{V(H) \cup \{x,y\}}}^{H'}$, then $\overbrace{\boxed{V(H)\cup\{x,y\}}}^{H'}$ is induces a connected subgraph of $G - \{z,u,v\}$ larger than H. Contradicts choice of $\{x,y,z\}$.



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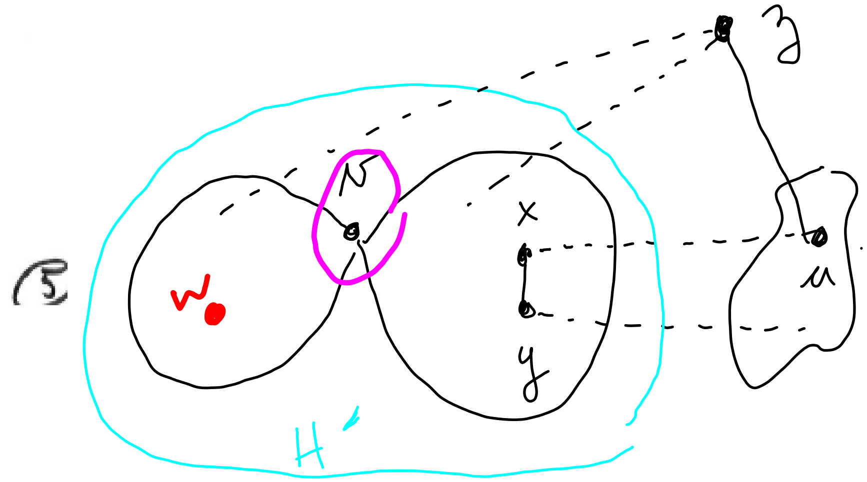

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Proof of 6.2.10 [redacted])

Suppose $G \cdot xy$ has a Kurat. subgraph H.

Let $z = xy$ (vertex)

If $z \notin V(H)$, then $H \subseteq G$.

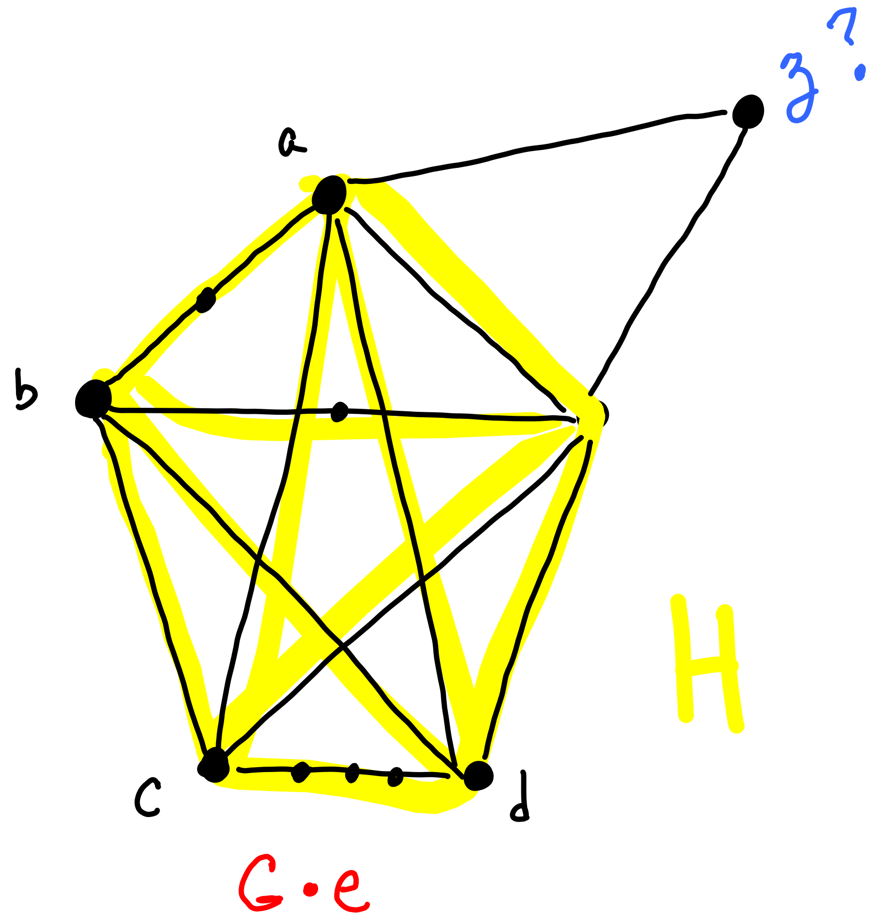
Else look at $\deg_H(z)$ (2, 3, or 4)

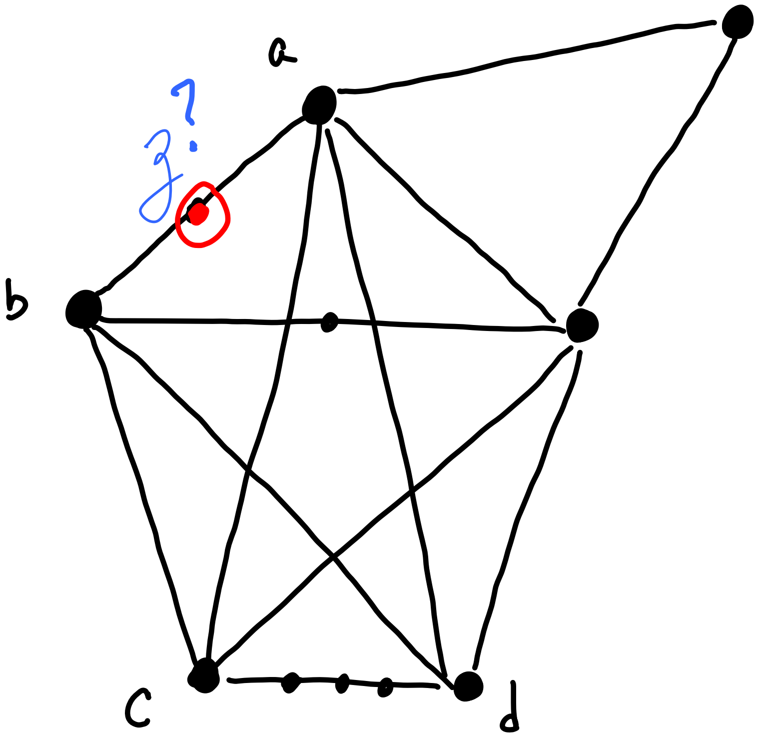
(case by case)

$\deg_H(z) = 2$:

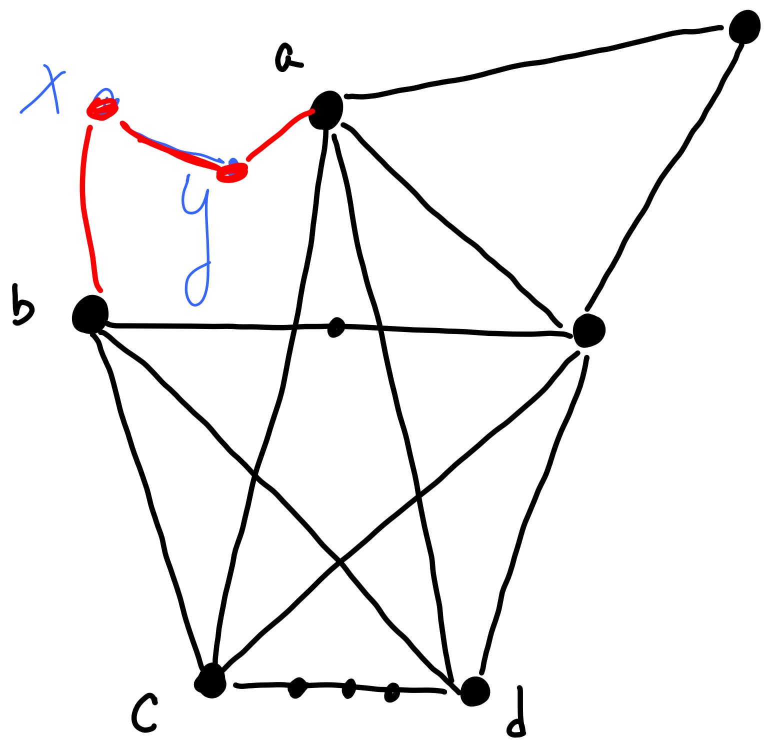
3 :

4 :





$G \cdot e$



G

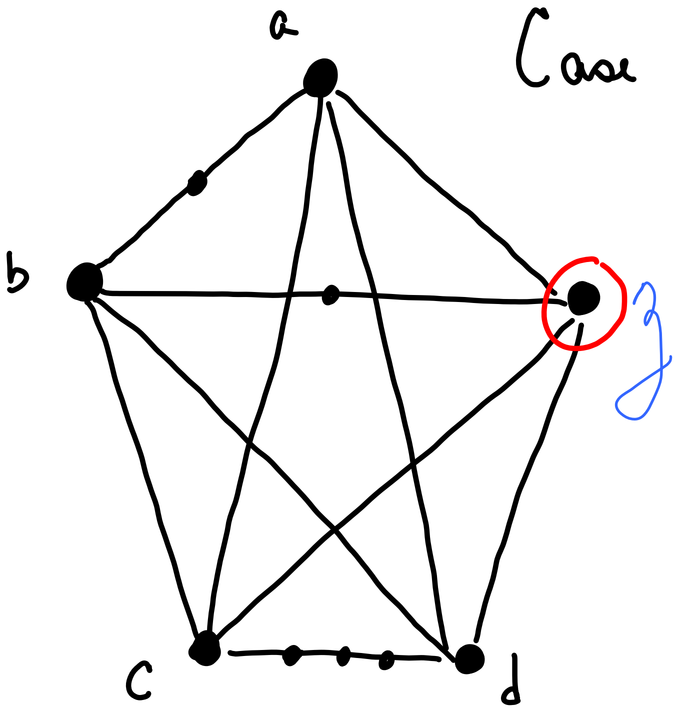
Case 1: $\deg_H(z) = 2$

Case 2: $d_H(z) = 3$

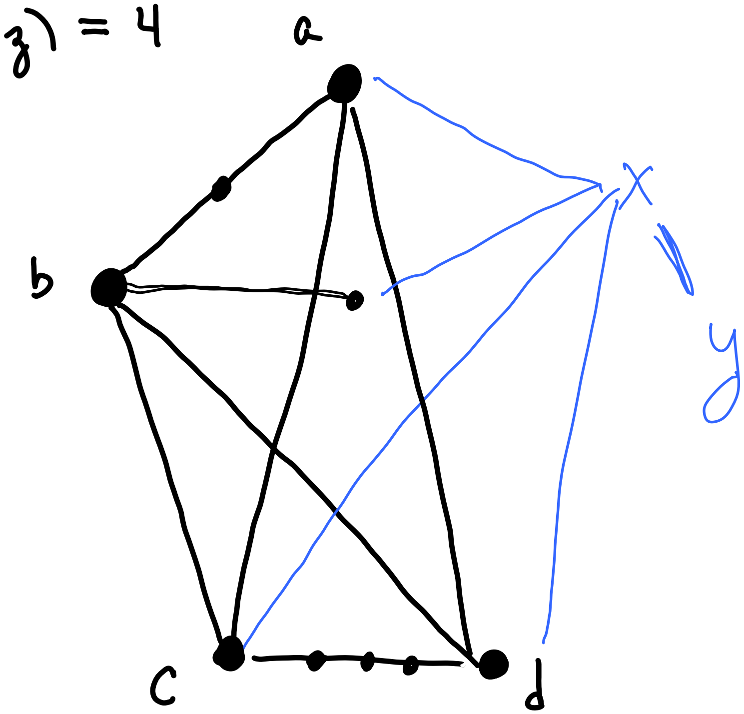
Then H is a subdivision of $K_{3,3}$.

Do this case as an exercise

Case 3: $d_H(z) = 4$



G • e



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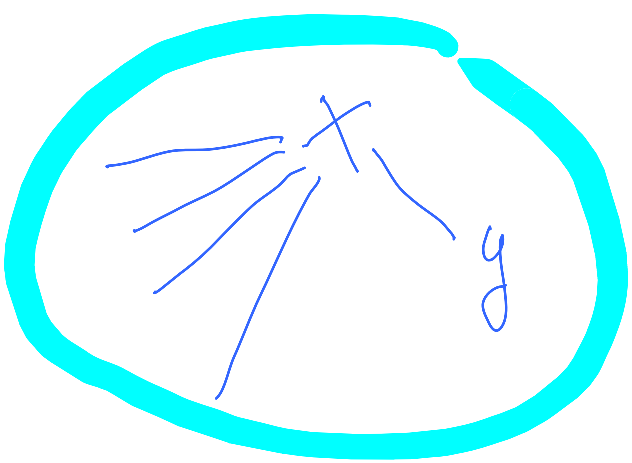

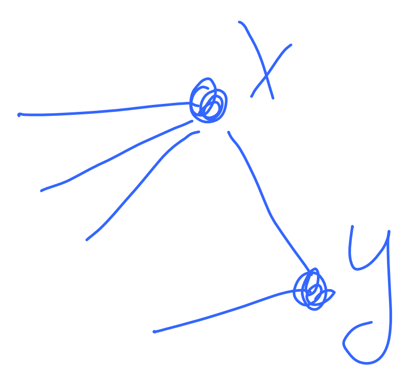

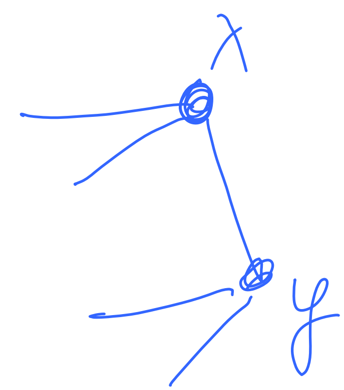

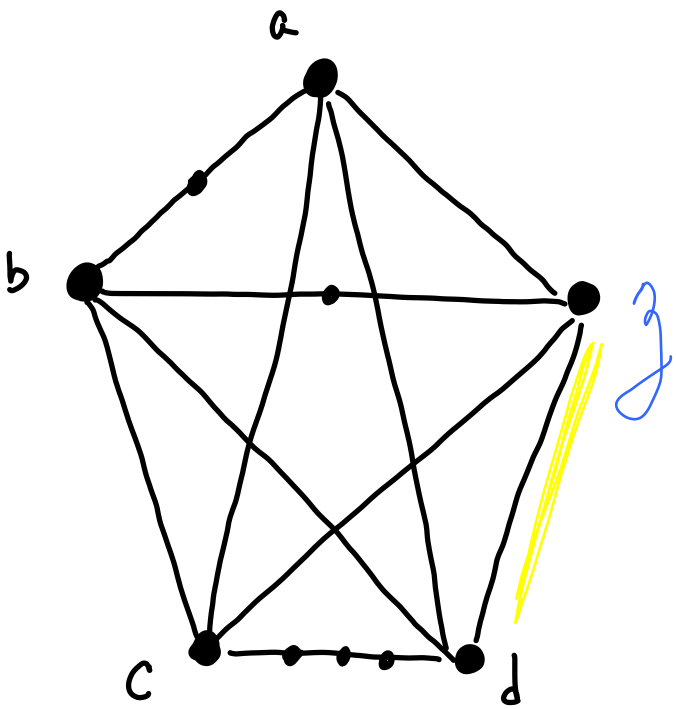

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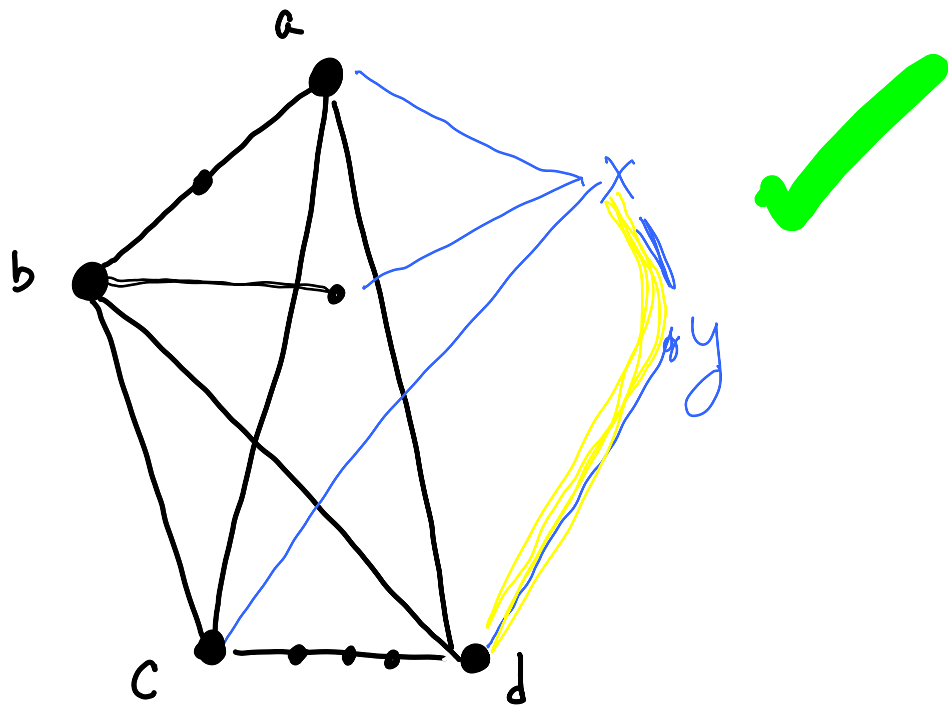

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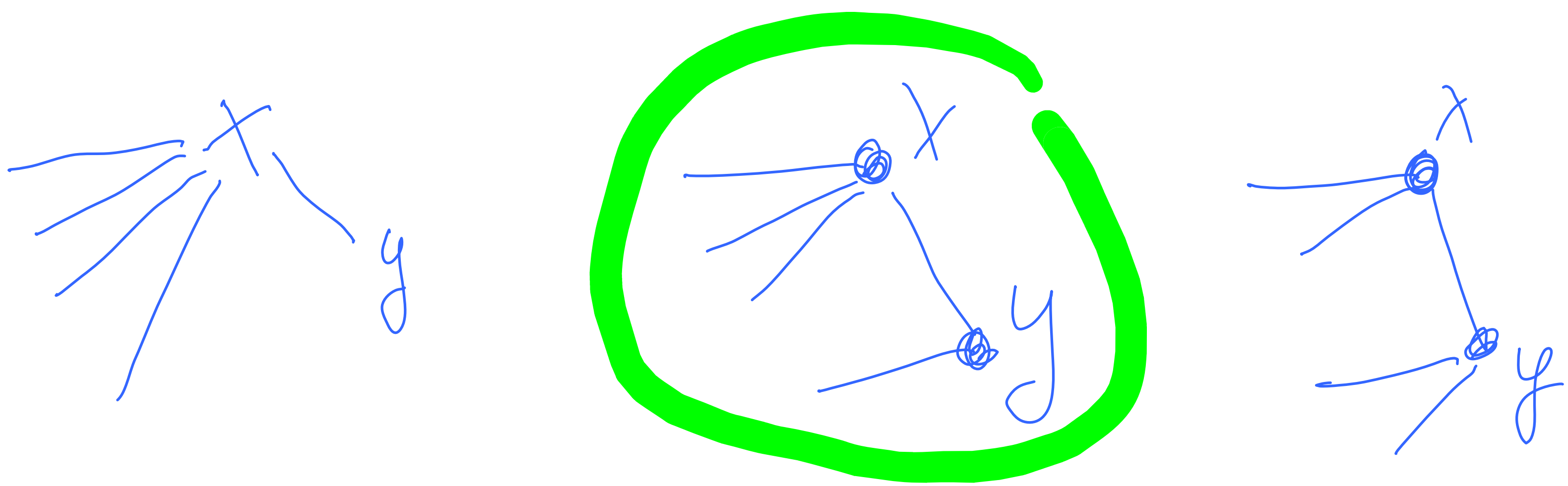
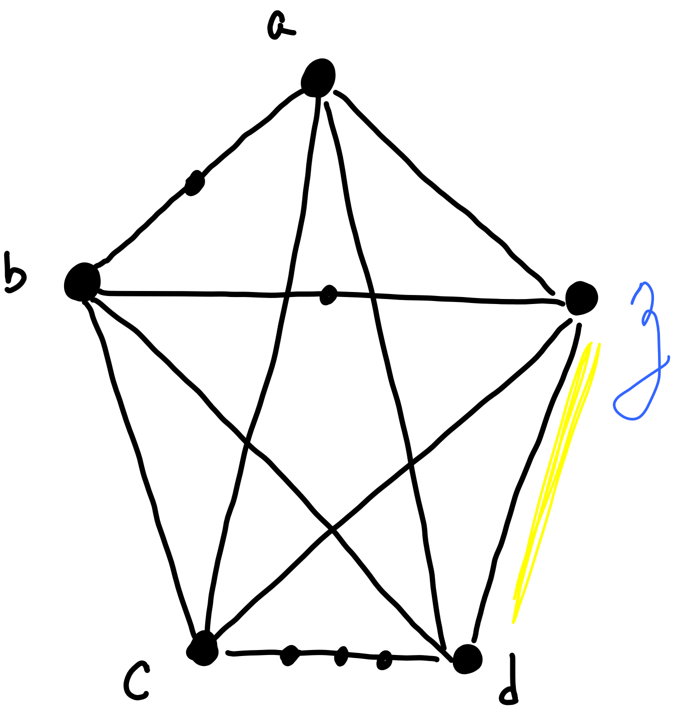
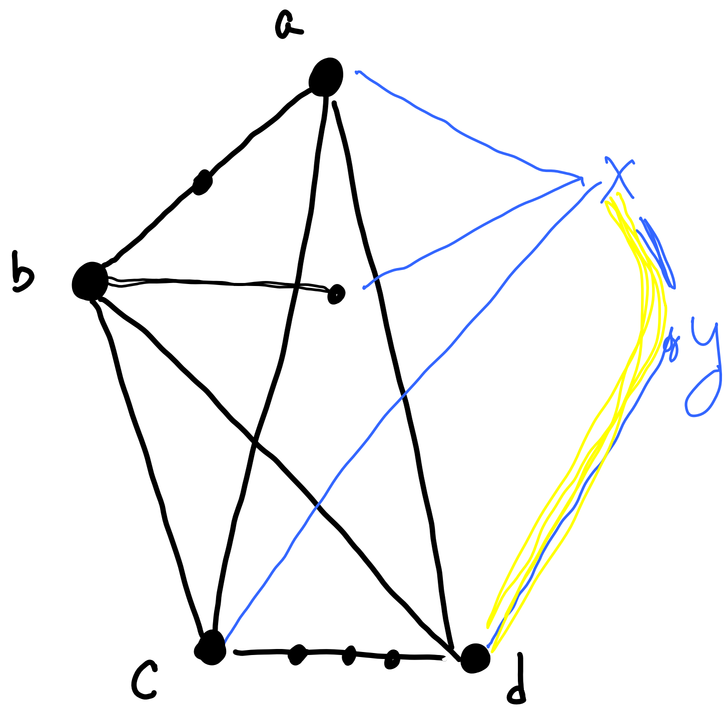




G · e



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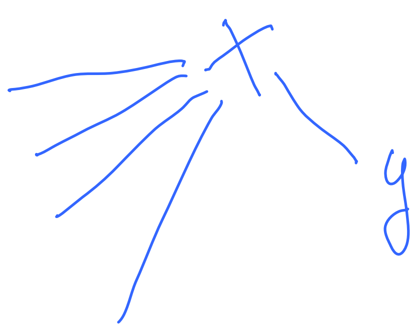

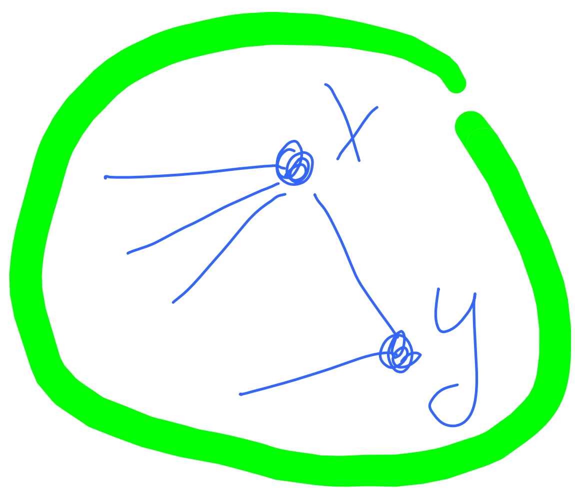

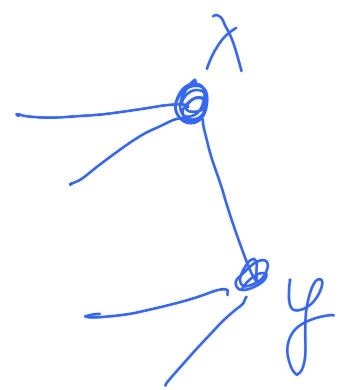

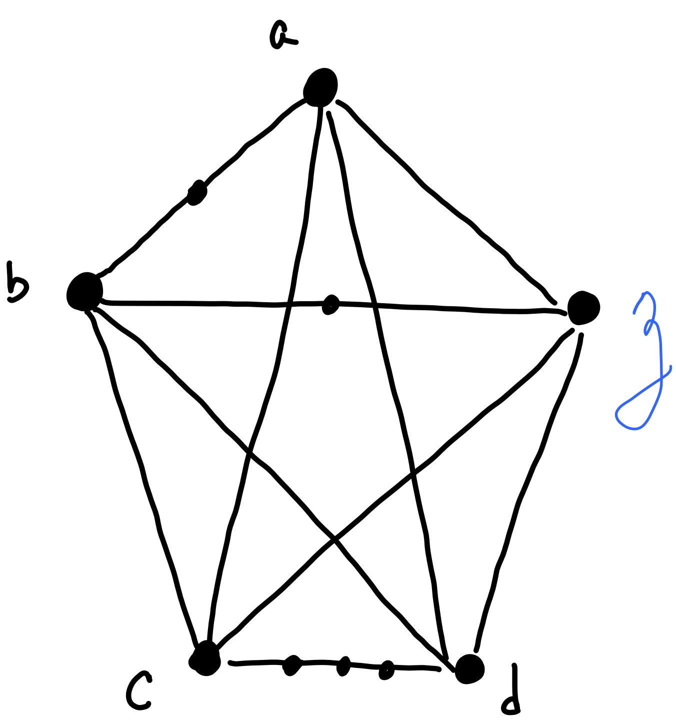

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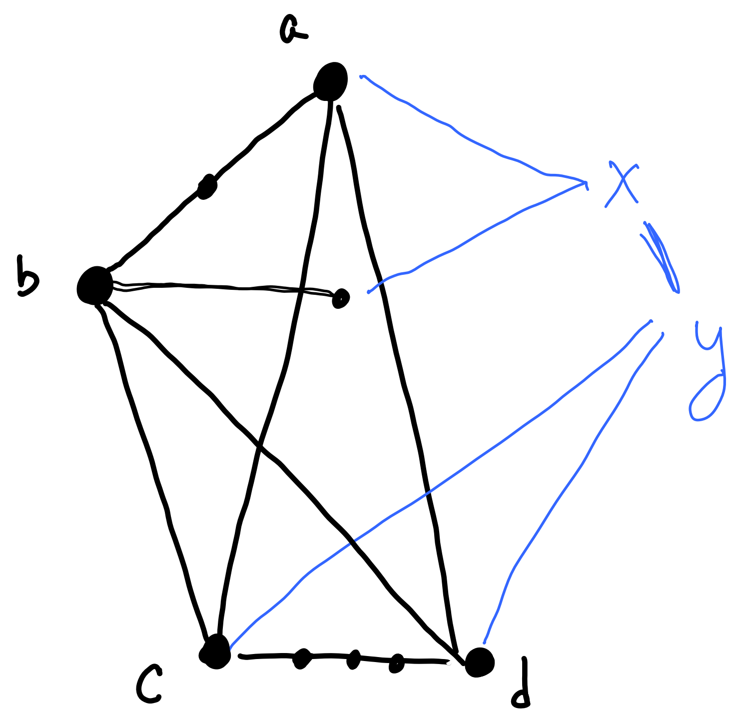

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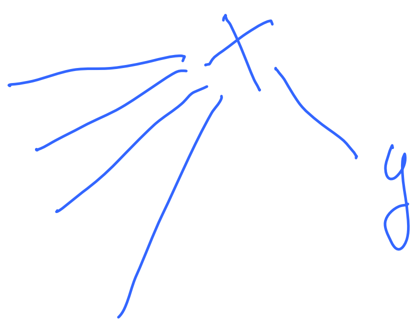
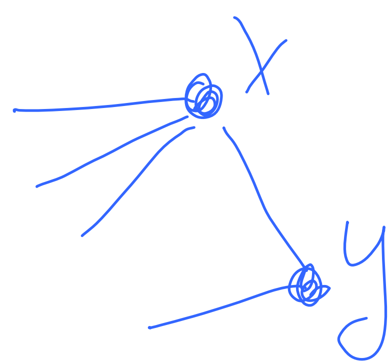
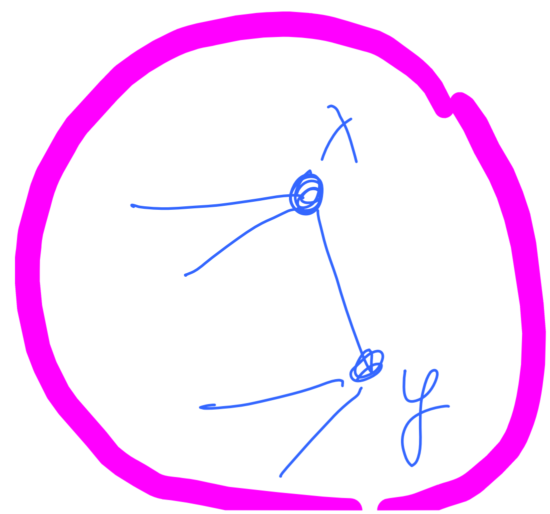
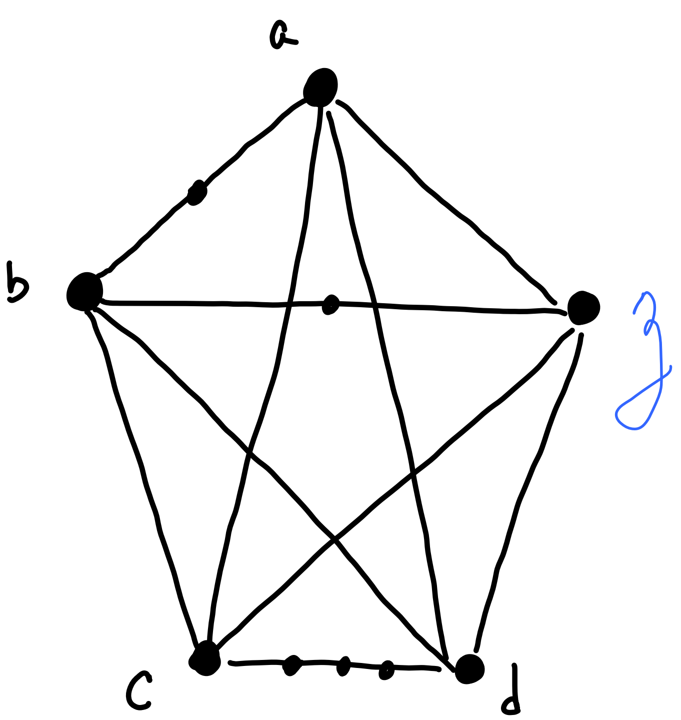
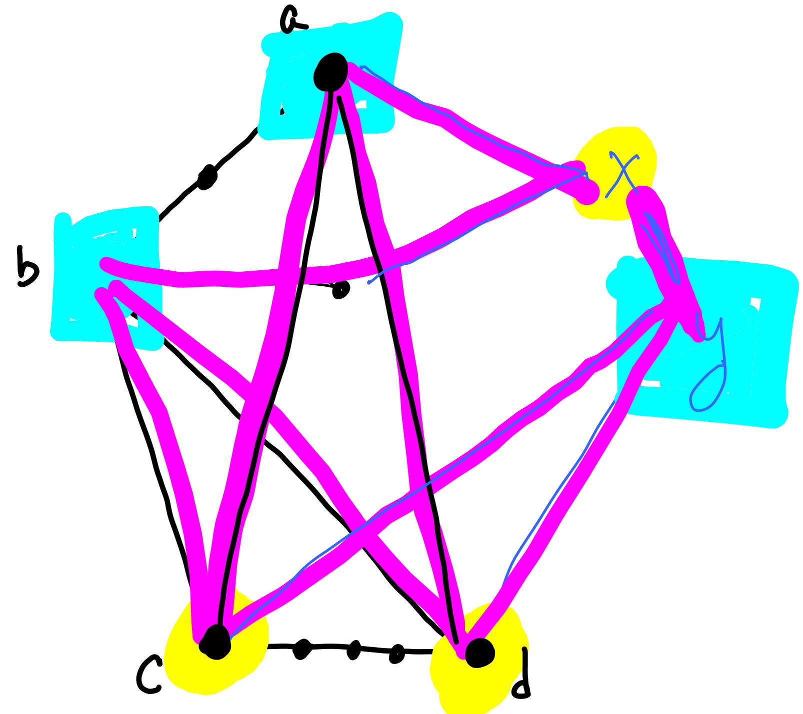






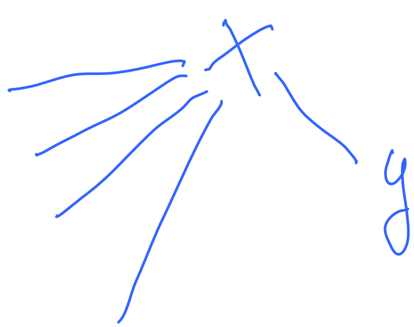


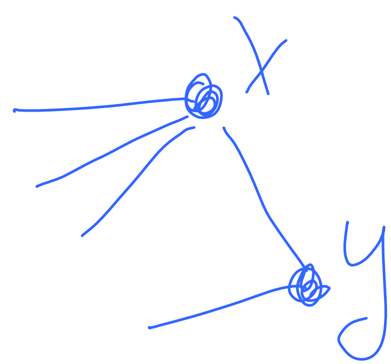
$G \cdot e$

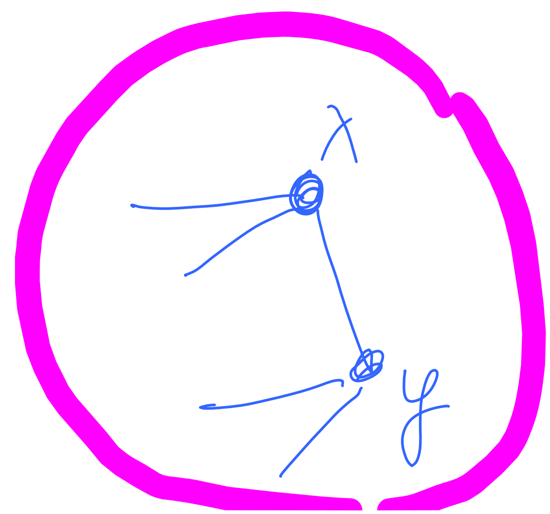


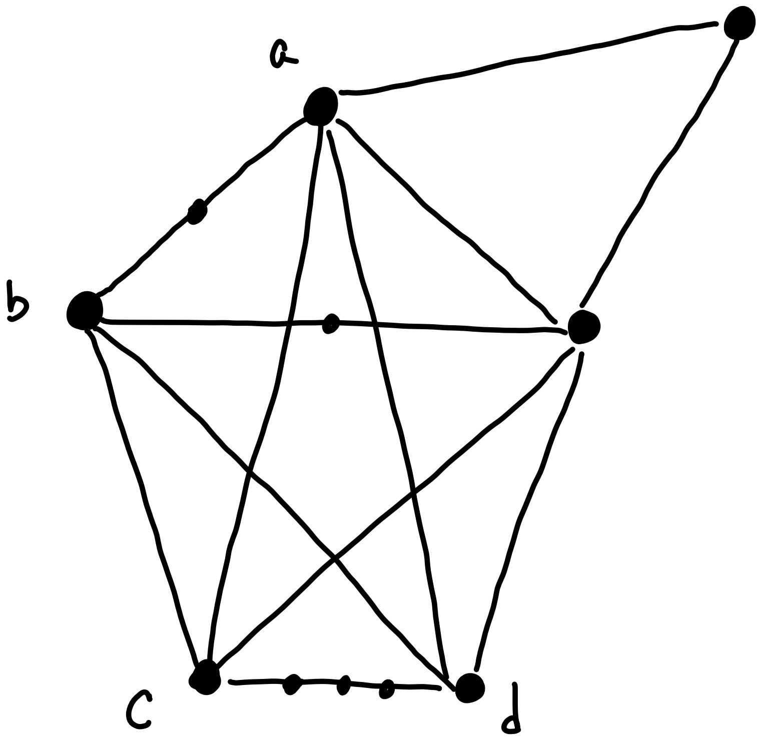
G

Get subdivision of $K_{3,3}$ in G

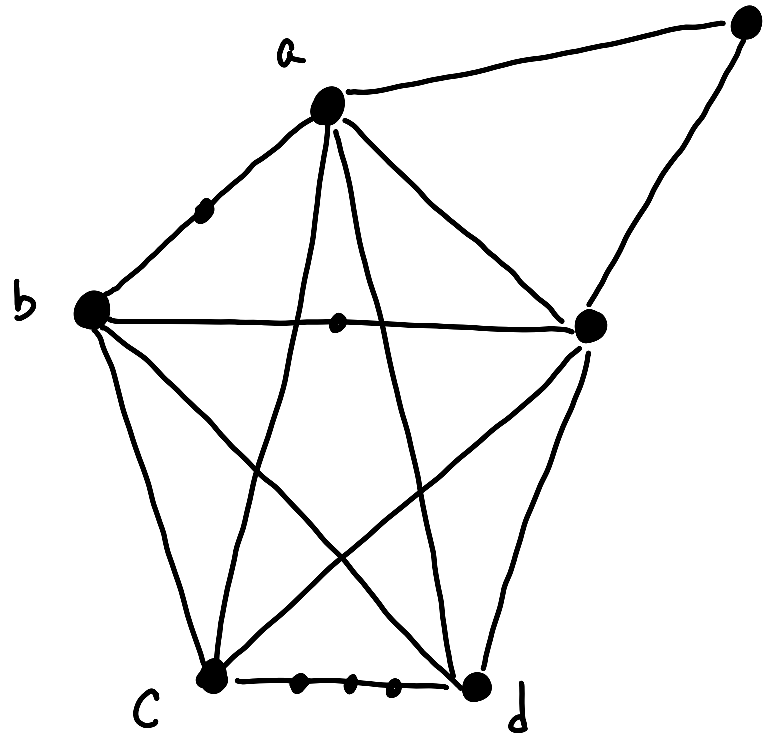








G • e



G • e

Theorem 6.2.11 (Tutte) If G is a 3-connected graph with no subdivision of K_5 or $K_{3,3}$, then there exists a (convex) planar embedding of G (with no 3 vertices on a line).

Proof. [Thomassen 1980]

<u>Theorem 6.2.11</u> (Tutte) If G is 3-connected, with no Kuratowski subgraph, then G is planar

<u>Proof</u> [Thomassen 1980] Induction on # of vertices.

Since $\kappa(G) \geq 3$, $n(G) \geq 4$.

If $n(G) = 4$, then $G = K_4$, planar.

Else $n(G) \geq 5$.

Apply Lemma 6.2.9 to find an edge e s.t.

$H = G \cdot e$ is 3-connected.

By Lemma 6.2.10, H has no Kuratowski subgraph.

$\therefore$ By induction H is planar

Let H' be a planar embedding of H.

Let $e = xy$ (the contracted edge)

Let z be the contracted vertex.

Since $H' - z$ is 2-connected $(G \cdot e - z)$ since 3-connected

the boundary of the face containing z is a cycle C

Rest of proof breaks into 3 cases :

Let $x_1, x_2, \ldots, x_k$ be neighbors of x in cyclic order

Case 0 All neighbors of y lie in interval x_i, x_{i+1} for some i ....

else

Case 1 y shares 3 neighbors with x

else

Case 2 y has neighbors u & v that alternate with some x_i, x_{i+1}

pictures ...

$H = G \cdot e$ (planar)

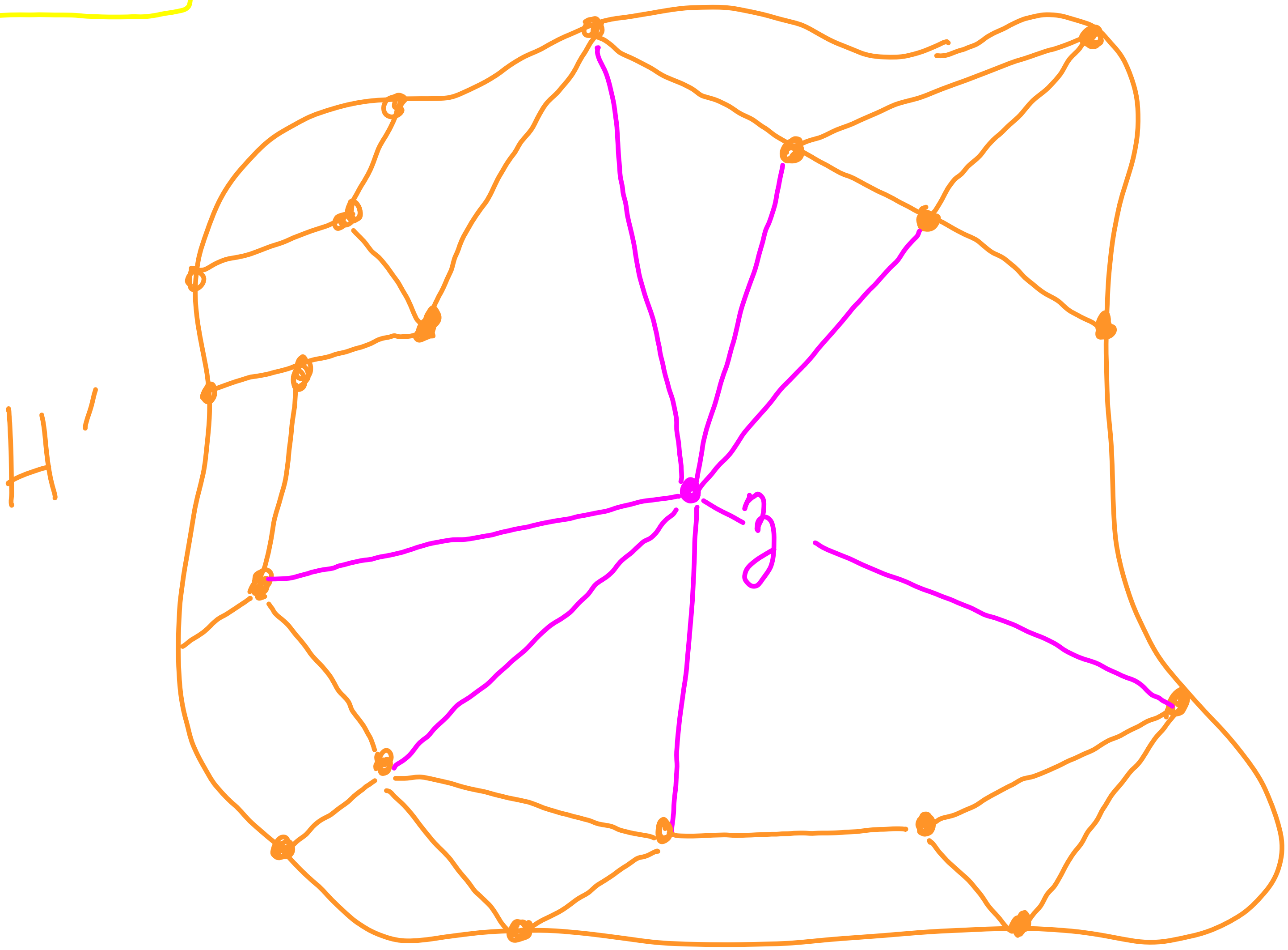
$e = xy$

z

H' planar embedding

$\downarrow$

$H' - z$ 2-connected



$H = G \cdot e$ (planar)

$e = xy$

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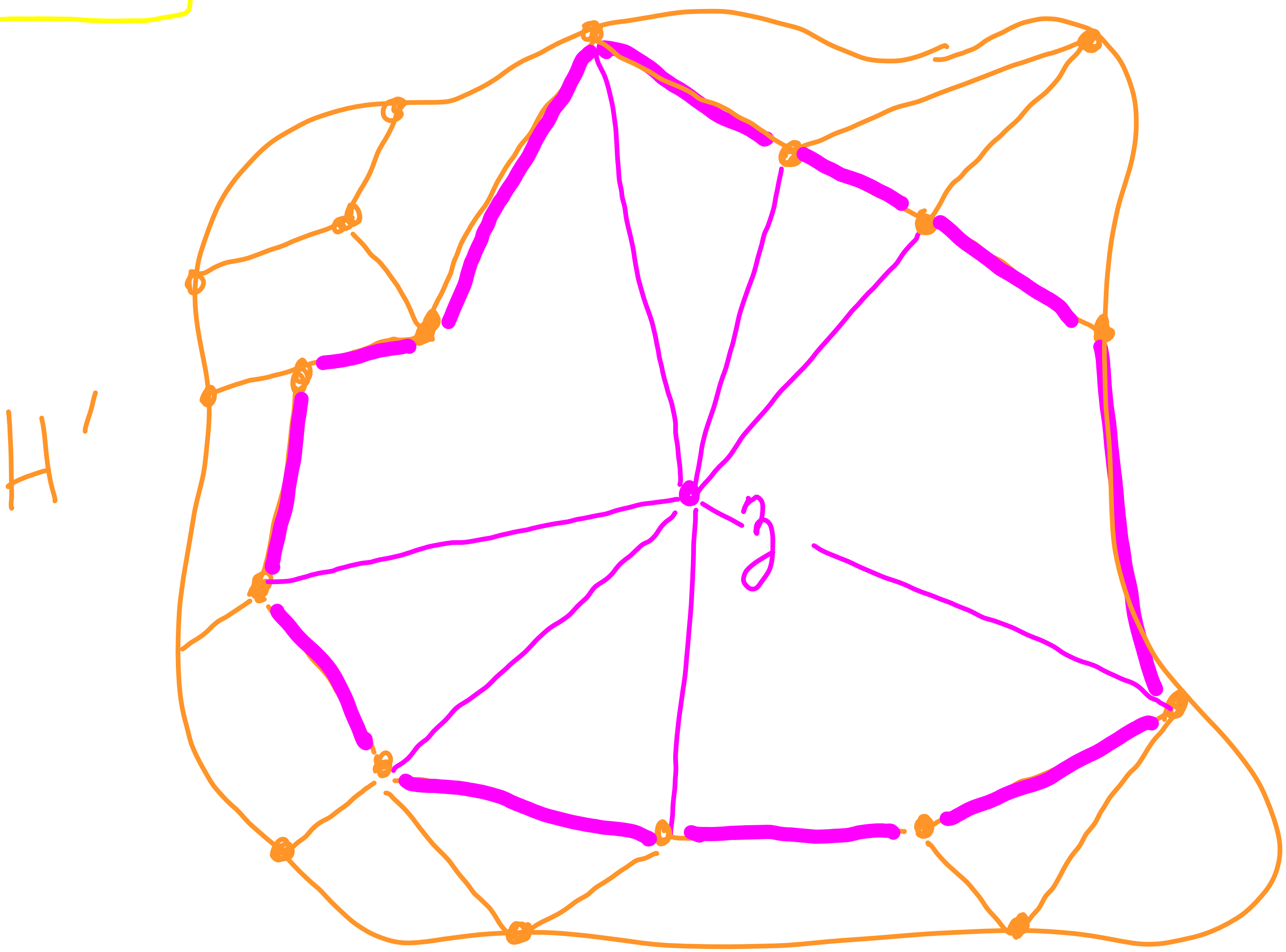
H' planar embedding

$\downarrow$

$H' - z$ 2-connected



$H = G \cdot e$ (planar)

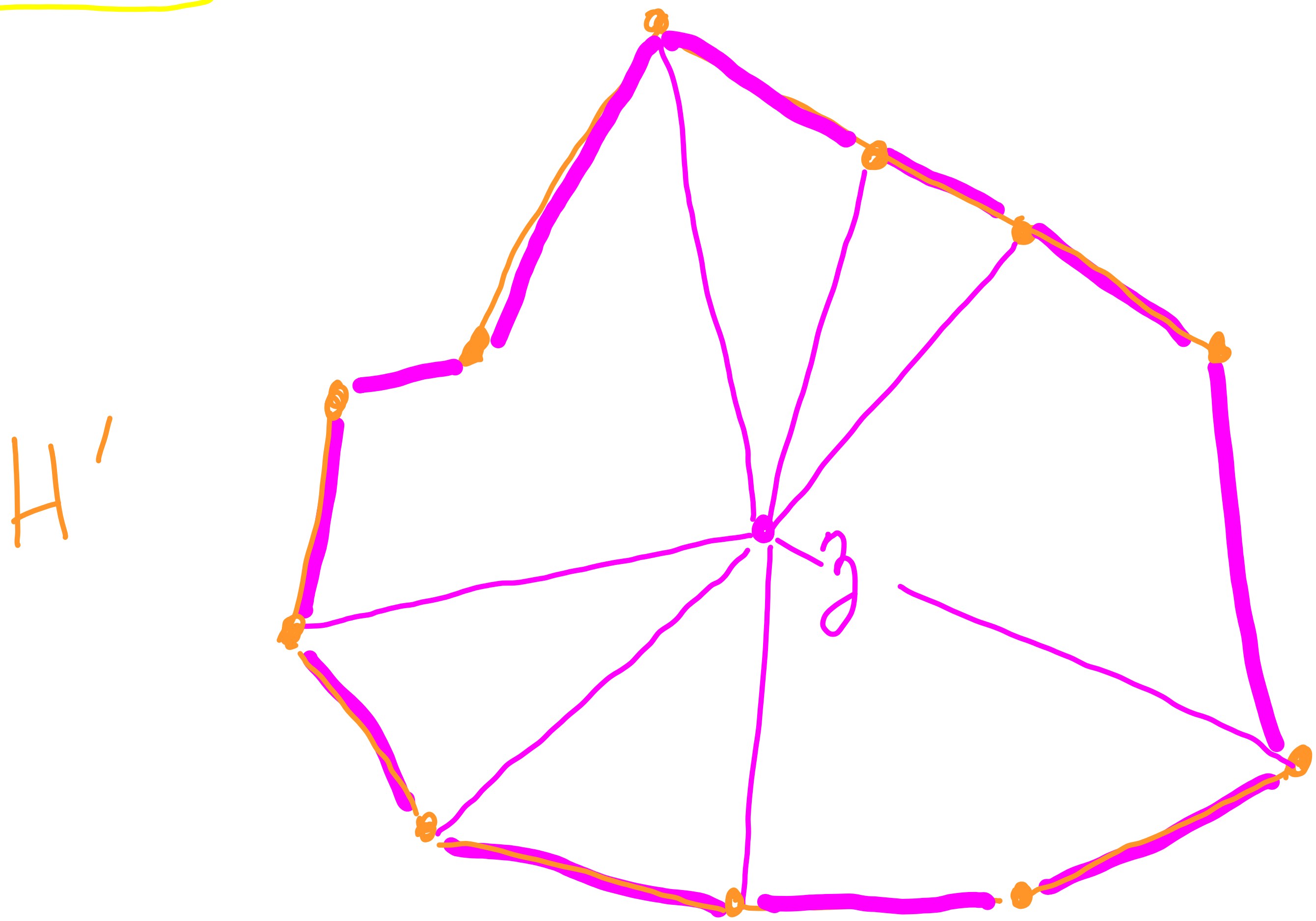
$e = xy$

z

H' planar embedding

$\downarrow$

$H' - z$ 2-connected



$H = G \cdot e$ (planar)

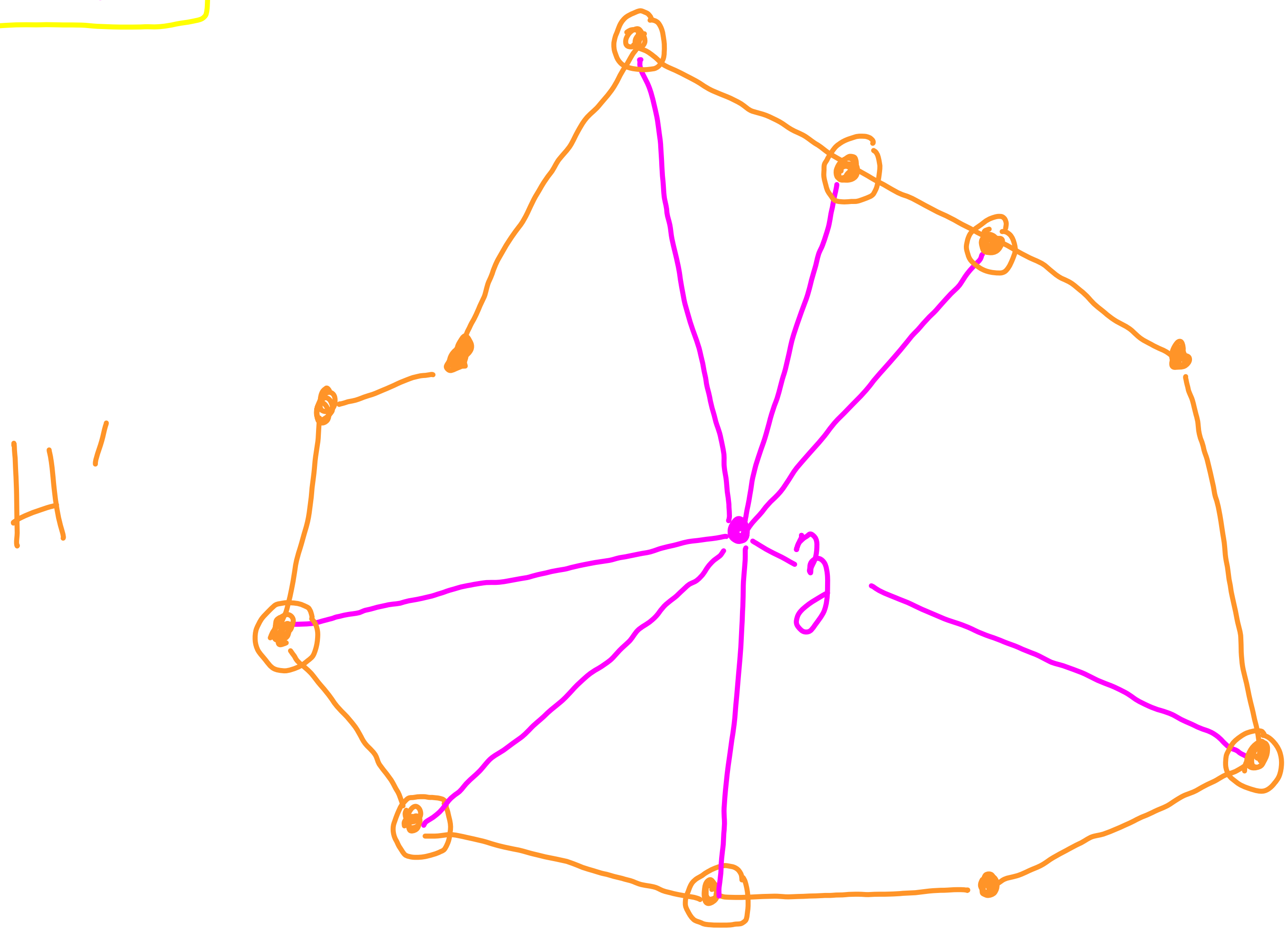
$e = xy$

z

H' planar embedding

$\downarrow$

$H' - z$ 2-connected



$H = G \cdot e$ (planar)

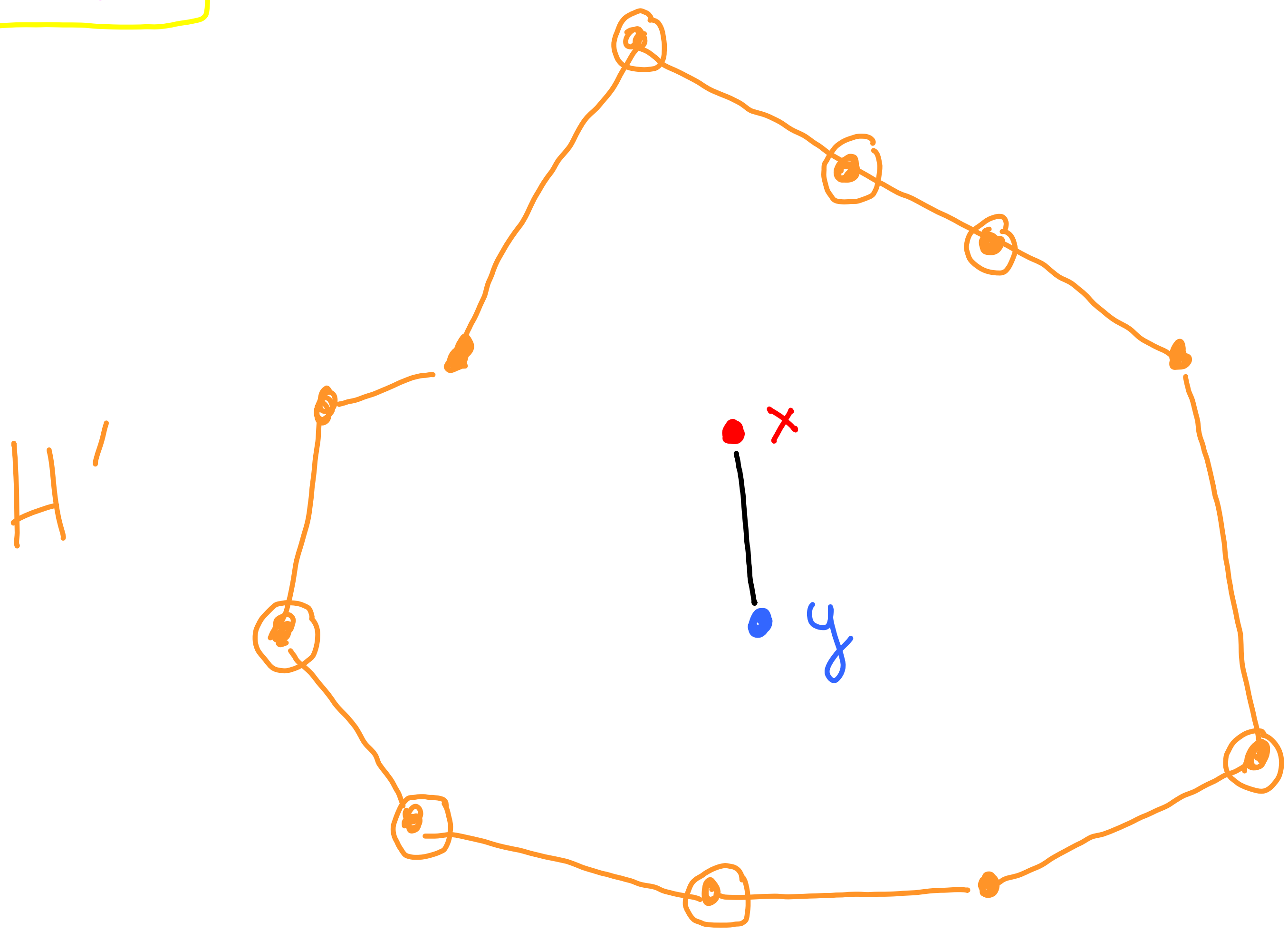
$e = xy$

z

H' planar embedding

$\downarrow$

$H' - z$ 2-connected



$H = G \cdot e$ (planar)

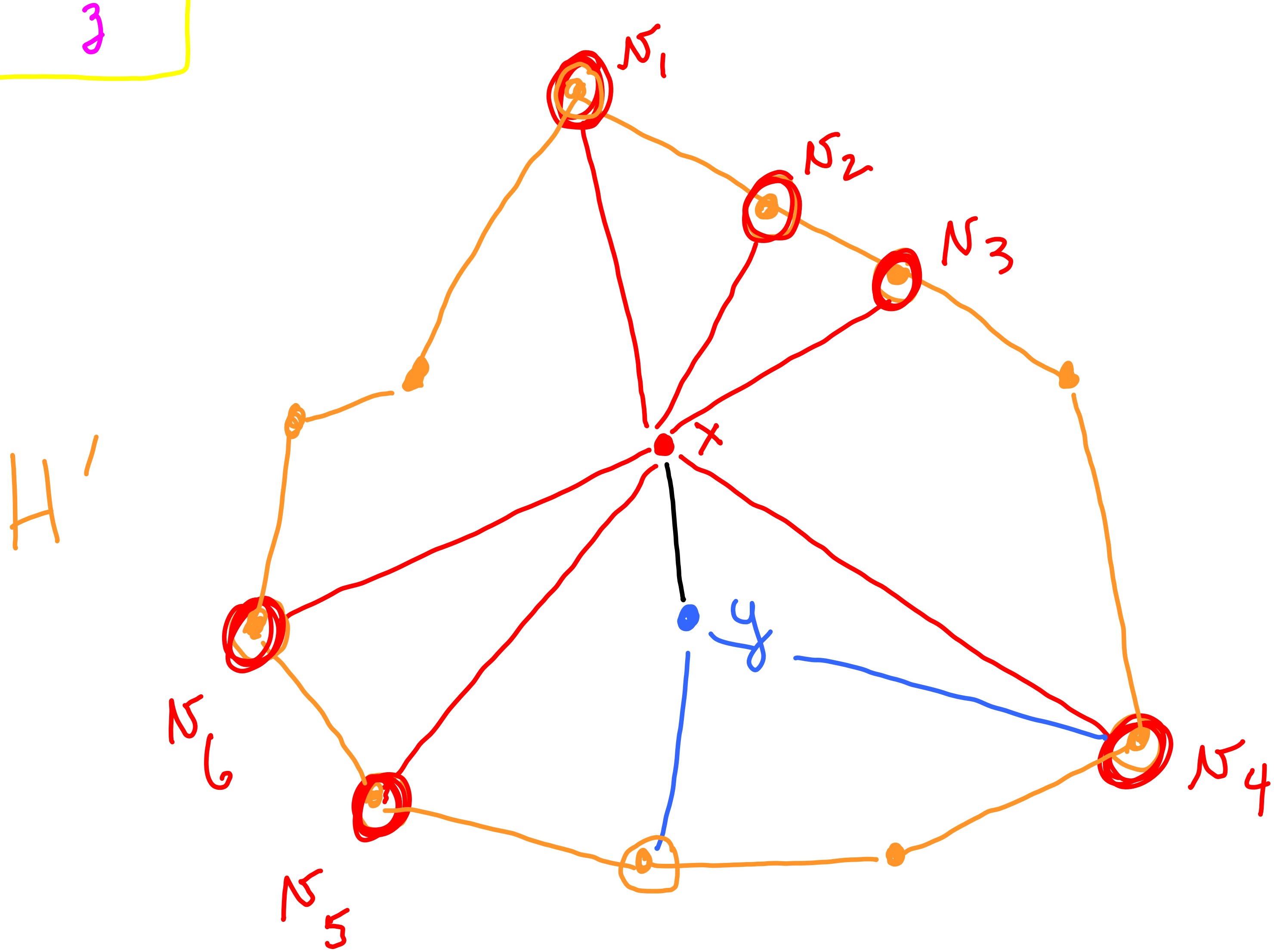
$e = xy$

z

H' planar embedding

$\downarrow$

$H' - z$ 2-connected



Case 0

$H = G \cdot e$ (planar)

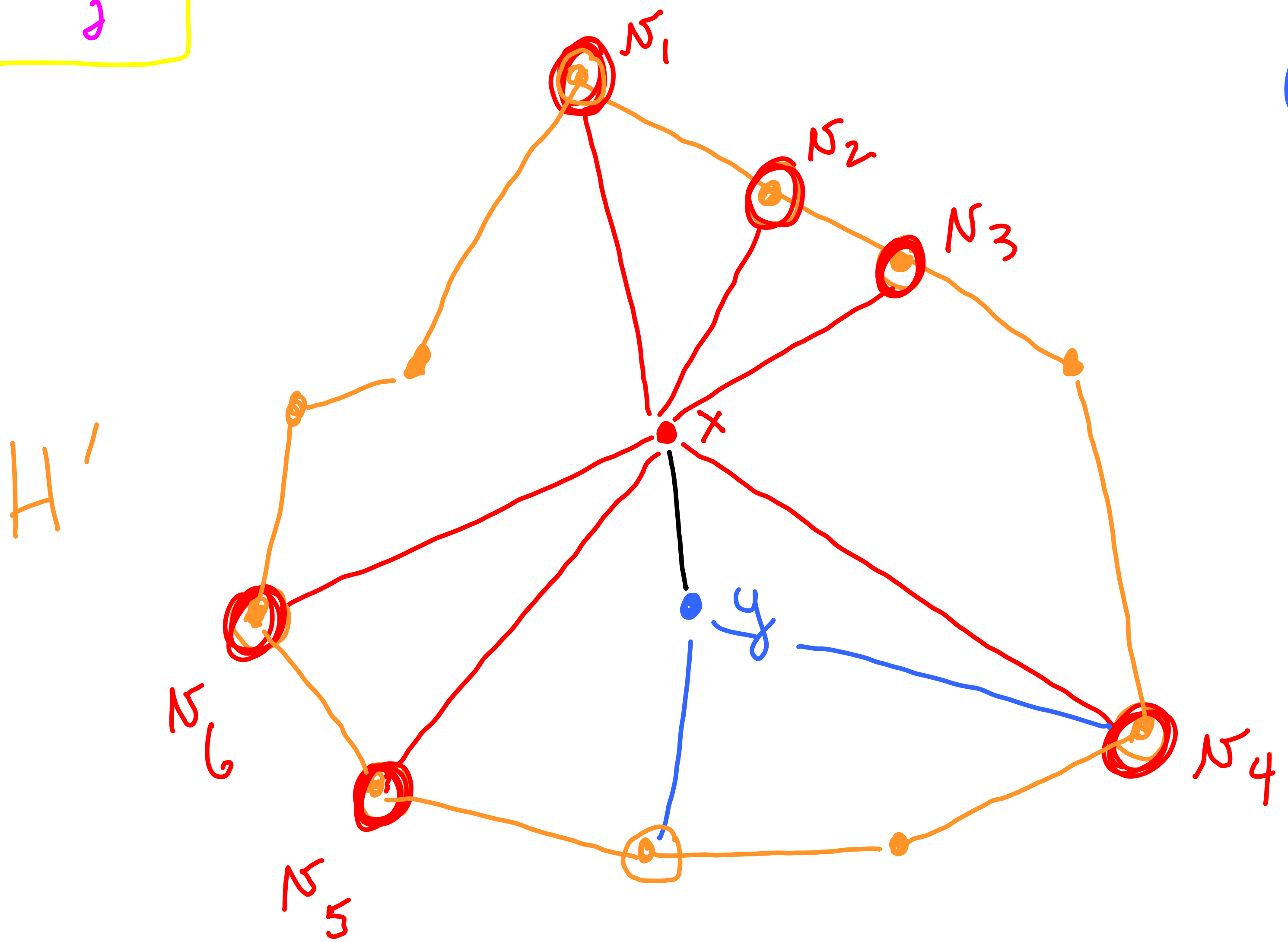
$e = xy$

z

H' planar embedding

$\downarrow$

$H' - z$ 2-connected



Case 0

Can get planar embedding of G

☺

$H = G \cdot e$ (planar)

$e = xy$

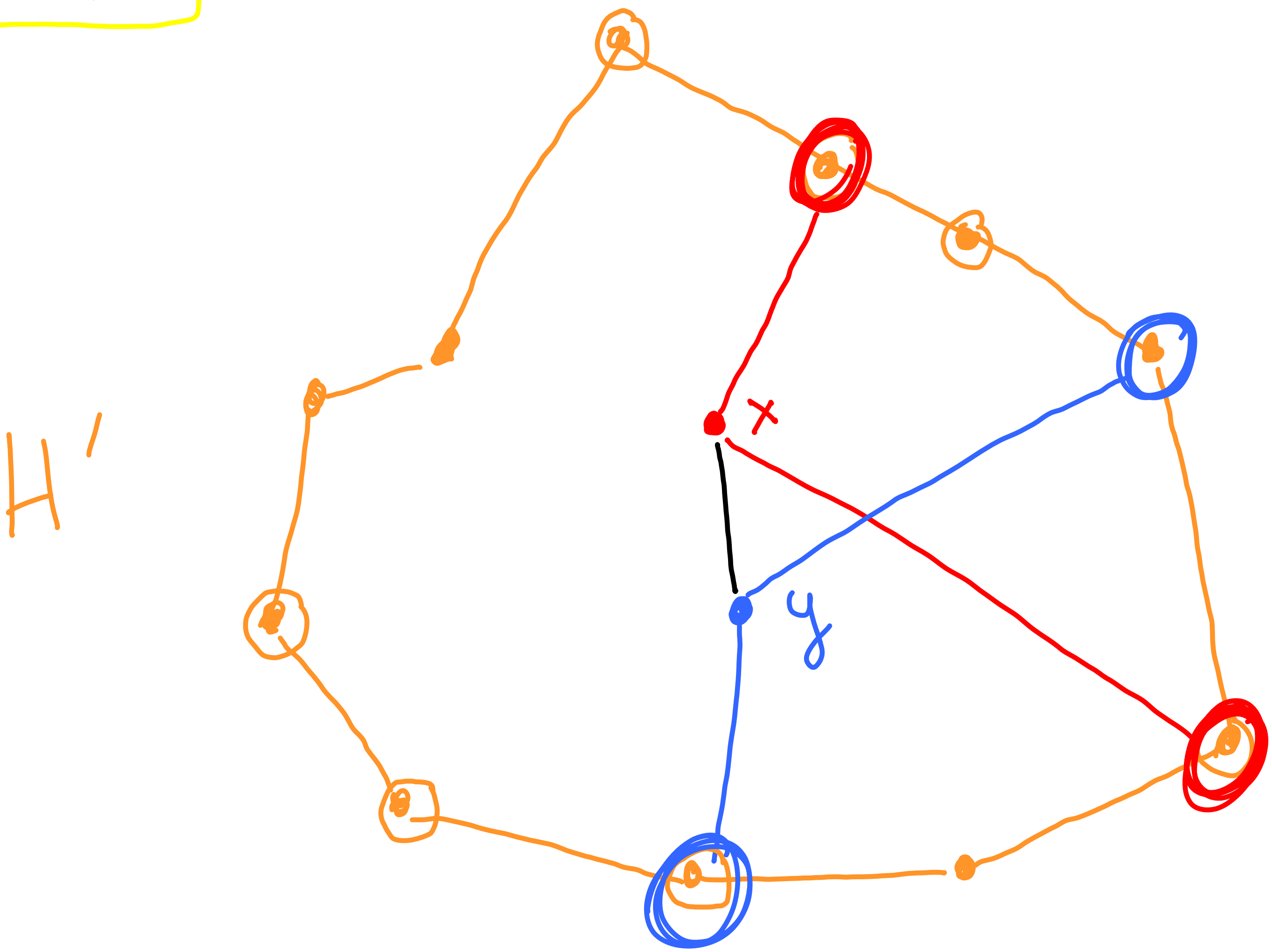
z

H' planar embedding

$\downarrow$

$H' - z$ 2-connected

Case 2



$H = G \cdot e$ (planar)

$e = xy$

z

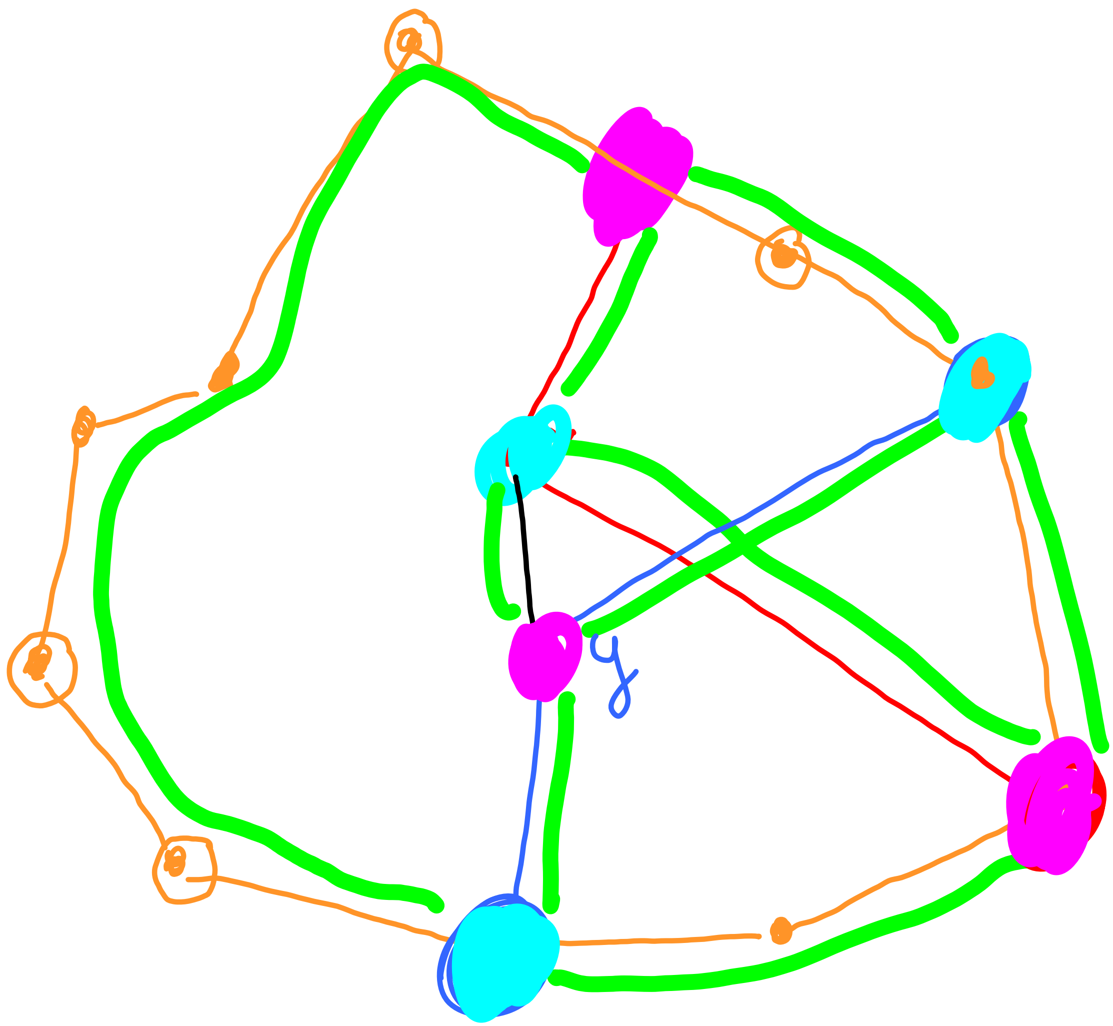
H' planar embedding

$\downarrow$

$H' - z$ 2-connected

Case 2

H'



$H = G \cdot e$ (planar)

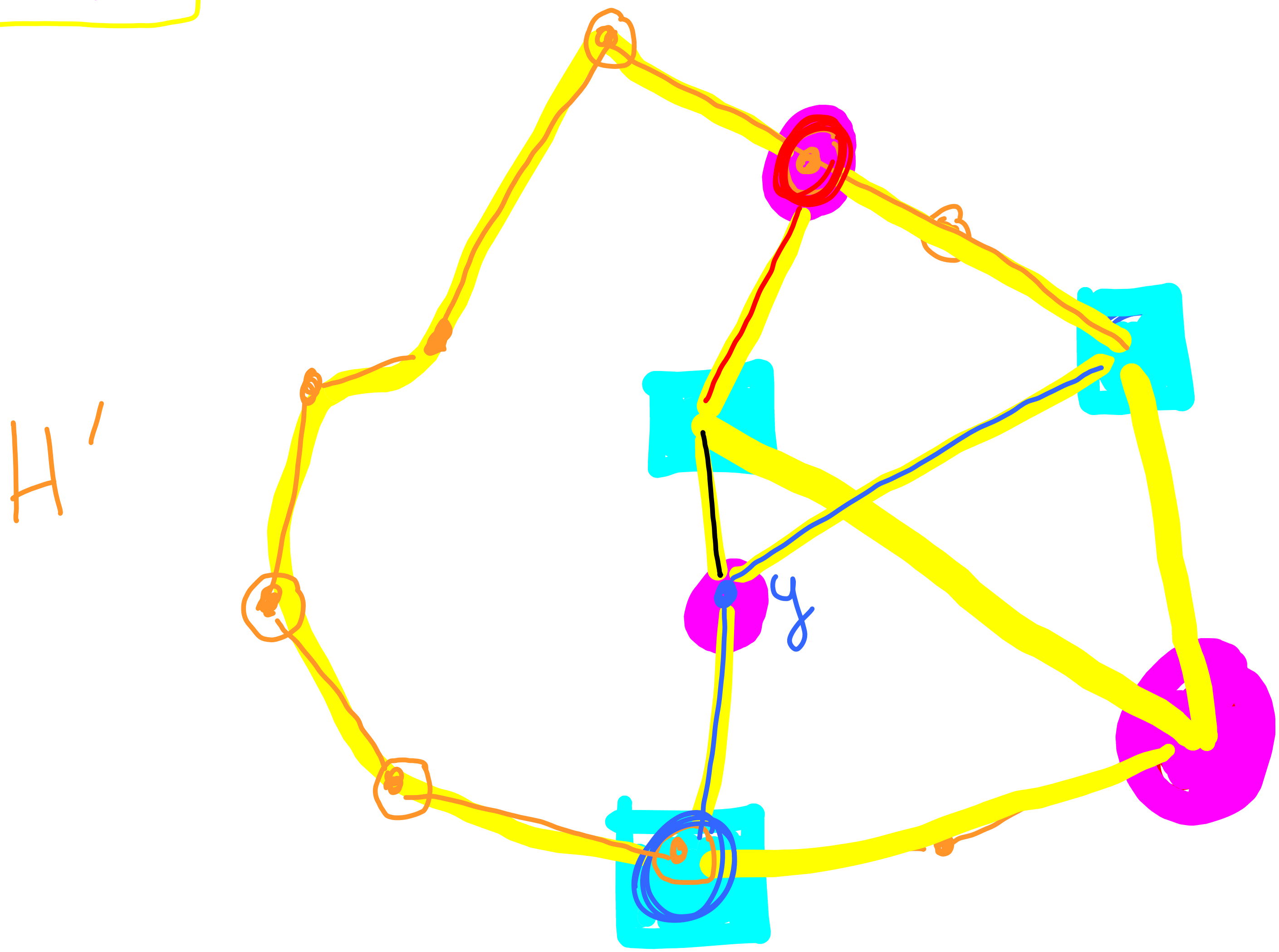
$e = xy$

z

H' planar embedding

↓

$H' - z$ 2-connected



Case 2

Subdivision of $K_{3,3}$ in G

$\Rightarrow\Leftarrow$

$H = G \cdot e$ (planar)

$e = xy$

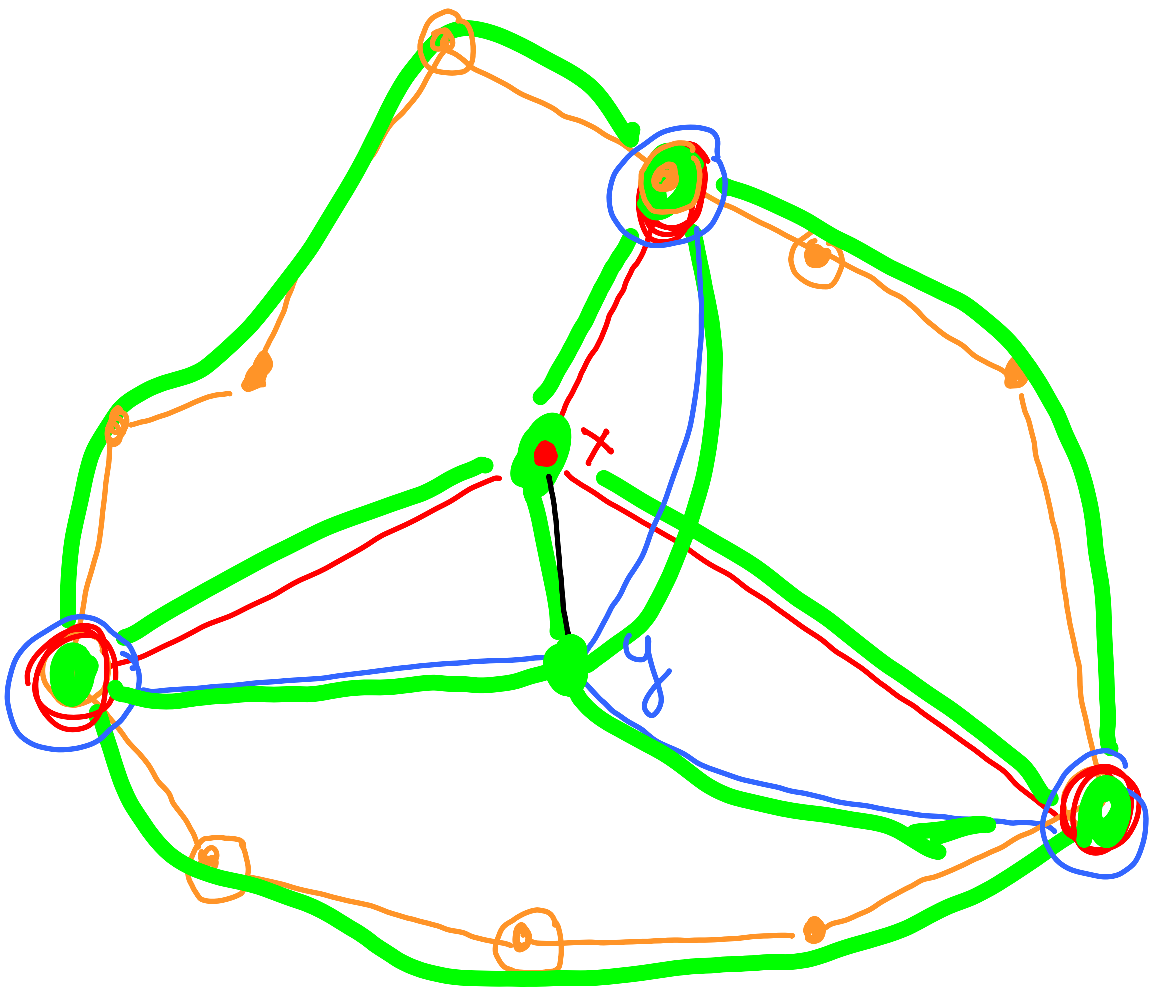
z

H' planar embedding $\downarrow$

$H' - z$ 2-connected

Case 1

H'



$H = G \cdot e$ (planar)

$e = xy$

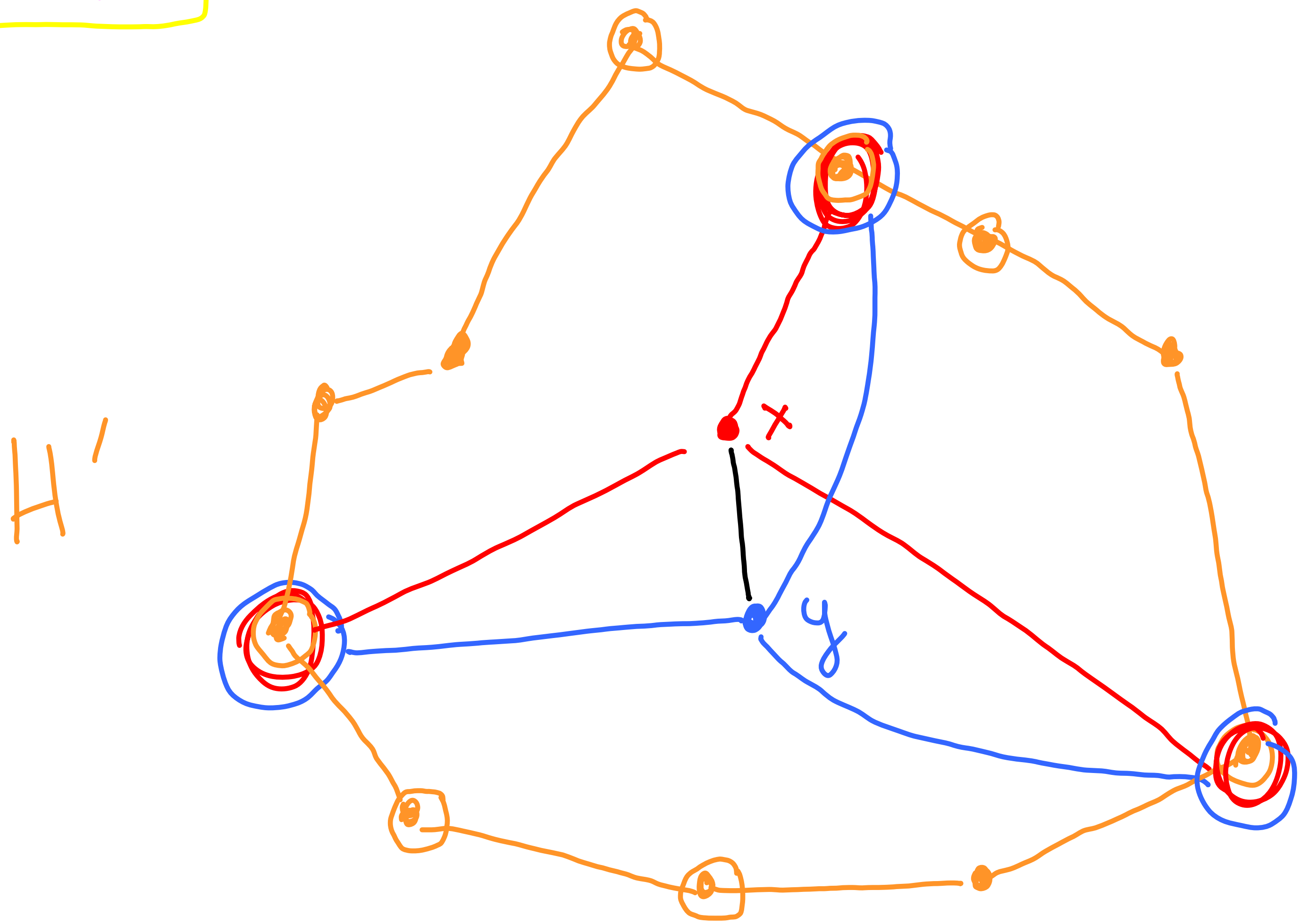
z

H' planar embedding

$\downarrow$

$H' - z$ 2-connected

Case 1



$H = G \cdot e$ (planar)

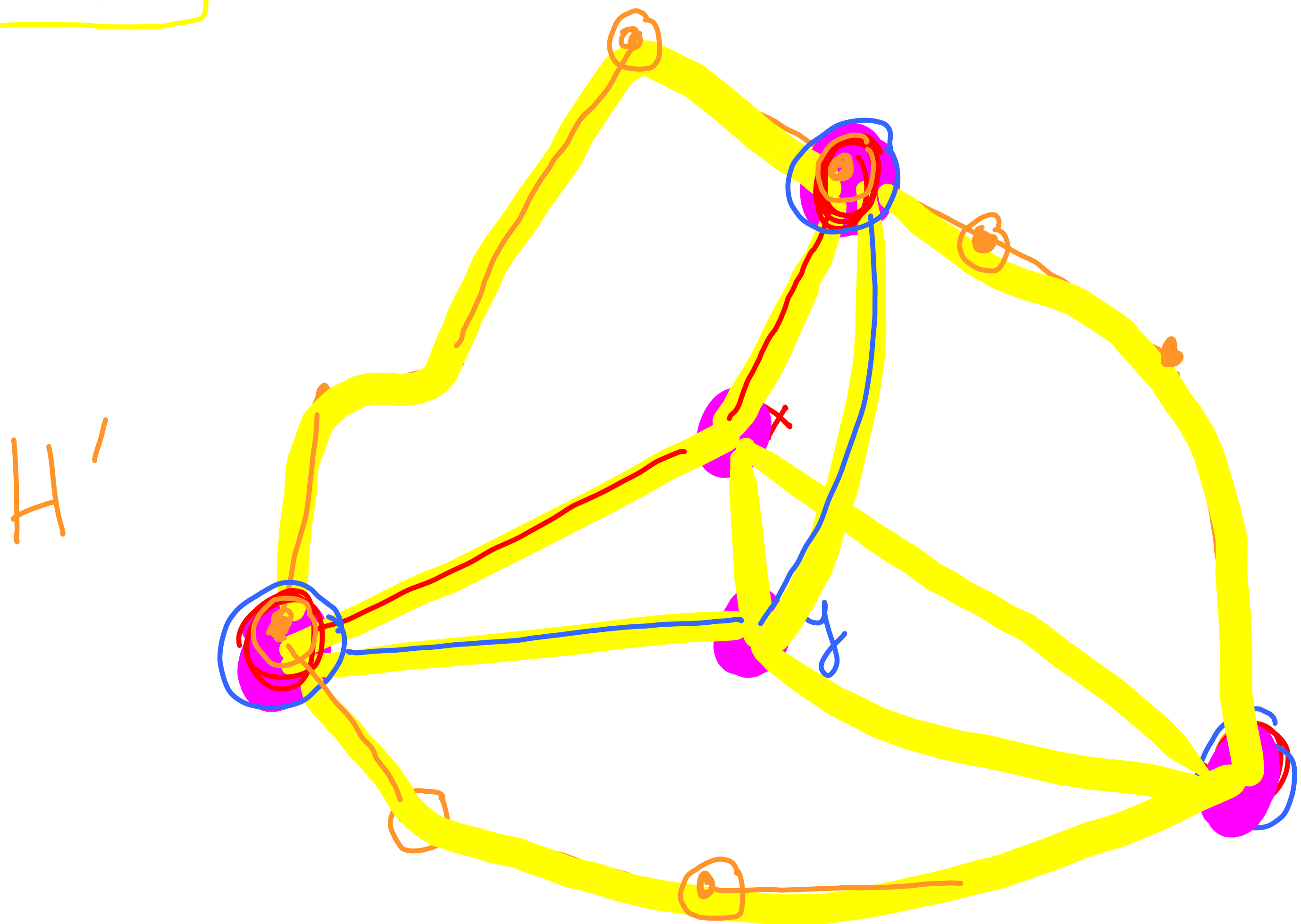
$e = xy$

z

H' planar embedding

$\downarrow$

$H' - z$ 2-connected



Case 1

Subdivision of K_5 in G

$\Rightarrow \Leftarrow$

So, only case 0 can occur, which means G is planar, completing proof

REVIEW

Kuratowski's Theorem [1930]: G is planar iff G contains no subdivision of K_5 or $K_{3,3}$.

Theorem 6.2.11 (Tutte) If G is a 3-connected graph with no subdivision of K_5 or $K_{3,3}$, then there exists a (convex) planar embedding of G (with no 3 vertices on a line).

Proof. [Thomassen 1980]

Lemma 6.2.7. Suppose G is a nonplanar graph with no subdivision of K_5 or $K_{3,3}$, and G has the fewest edges among such graphs. Then G is 3-connected.

Lemma 6.2.9. A 3-connected graph with at least 5 vertices contains an edge whose contraction leaves a 3-connected graph.

Lemma 6.2.6 Suppose $S = \{x, y\}$ is a separating set of G of size 2. If G is nonplanar, then adding xy to some S-lobe of G yields a nonplanar graph.

Lemma 6.2.10. If $G \cdot e$ has a Kuratowski subgraph, then G also has a Kuratowski subgraph.

Lemma 6.2.5. Every minimal nonplanar graph is 2-connected.

Lemma 6.2.4. If E is the edge set of a face in a planar embedding of G, then G has an embedding in which E is the edge set of the unbounded face.