Complete multipartite graph

$$K_{n_1,n_2,\ldots,n_k}$$

is a simple graph G with $V(G)$ consisting of disjoint sets

$$V_1, V_2, \ldots, V_k$$

of sizes $n_1, \ldots, n_k$ and

$$E(G) = \{uv \mid u \in V_i,\ v \in V_j,\ i \neq j\}.$$

$\boldsymbol{K_{1,2,3}}$**?**

$\boldsymbol{K_{2,4,2}}$**?**

$\overline{\boldsymbol{K}_{n_1,n_2,\ldots,n_k}}$?

$e(\boldsymbol{K}_{n_1,n_2,\ldots,n_k})$?

Turán graph - $T_{n,r}$ $(n \div r = a \text{ remainder } b)$

complete r-partite graph with
n vertices and
partite set sizes equal as possible.

(b partite sets of size $a+1$)
($r-b$ partite sets of size a)

$T_{5,3}$?

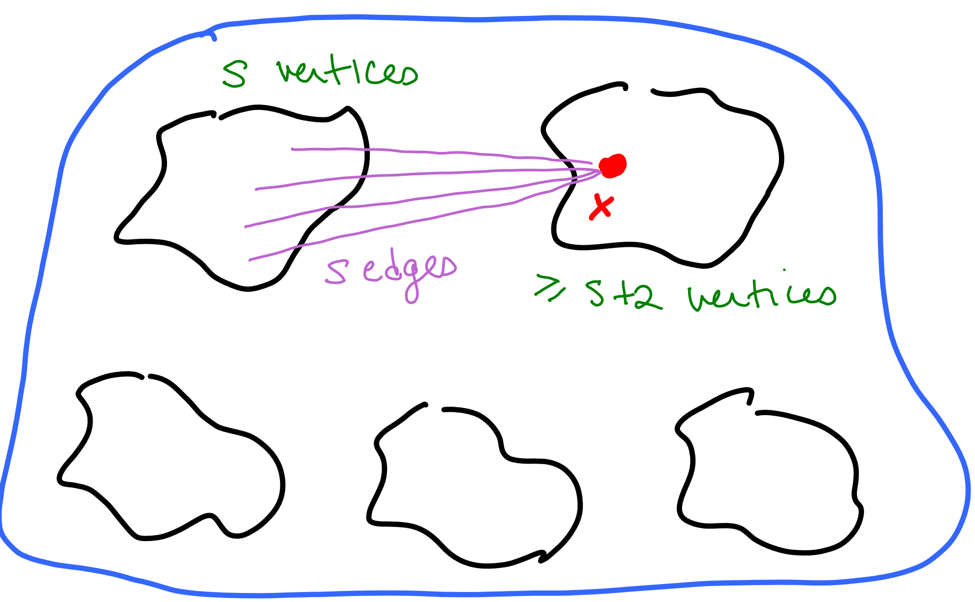
$T_{8,4}$?

$T_{7,3}$?

Lemma 5.2.8. If H is a simple r-partite graph with n vertices, then $e(H) \leq e(T_{n,r})$

Proof. wlog, assume H is complete r-partite.

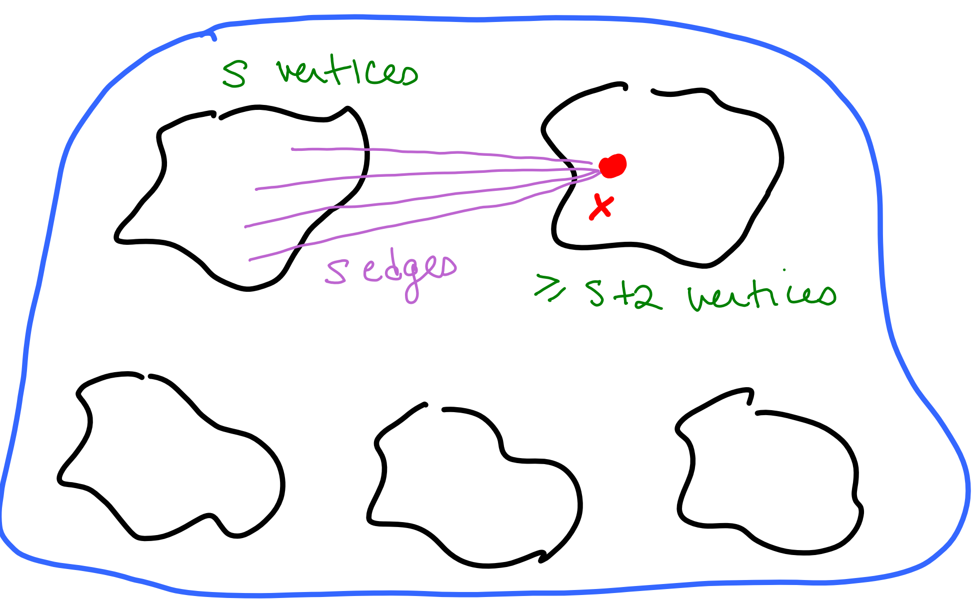
If $H \not\cong T_{n,r}$ then H has 2 partite sets that differ by more than 1. Moving a vertex from larger to smaller increases # of edges but at least 1:

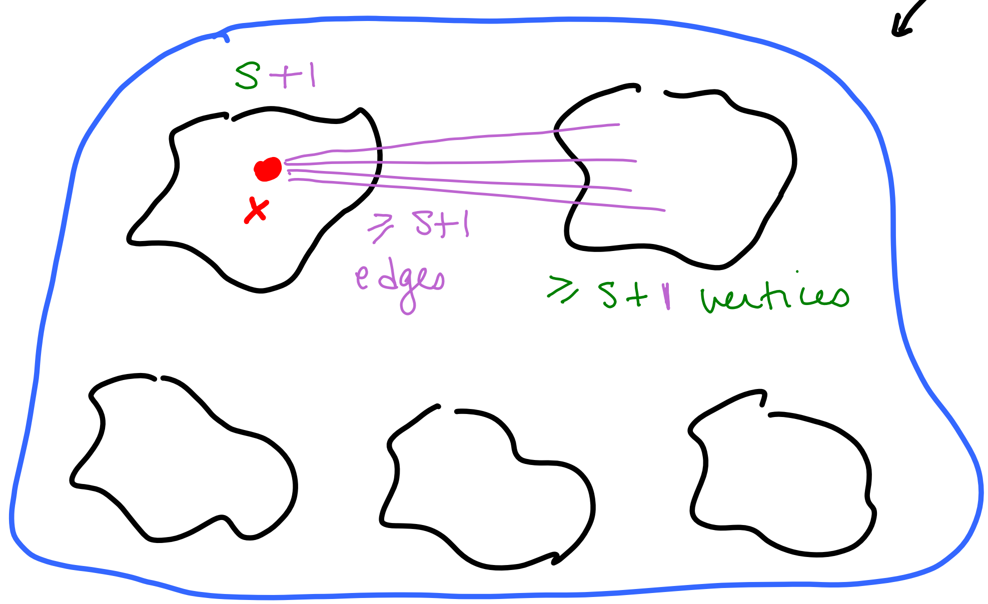


Lemma 5.2.8. If H is a simple r-partite graph with n vertices, then $e(H) \le e(T_{n,r})$

Proof. wlog, assume H is complete r-partite.

If $H \not\cong T_{n,r}$ then H has 2 partite sets that differ by more than 1. Moving a vertex from larger to smaller increases # of edges but at least 1:





Theorem 5.2.9. [Turán] Among the n-vertex simple graphs with no $r+1$-clique, $T_{n,r}$ has the maximum number of edges.

Pf Show that if G is a simple n-vertex graph with no $r+1$ clique, there is an n-vertex r-partite graph H with $e(G) \le e(H)$ (Then apply previous lemma)

Induction on r : If $r=1$: no 1-clique $\Rightarrow$ no edges $\Rightarrow e(G)=0$

So let $r \geq 2$ and assume true for $r-1$.

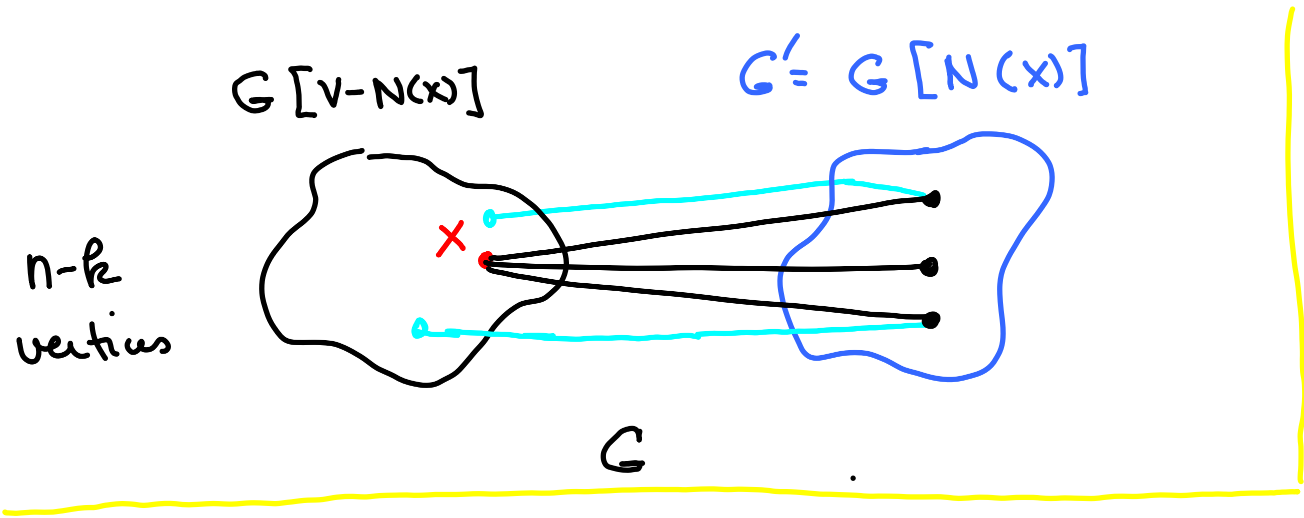
Let G be an n-vertex simple graph with no $r+1$ clique.

Let $k = \Delta(G)$. Let x be a vertex of degree k.

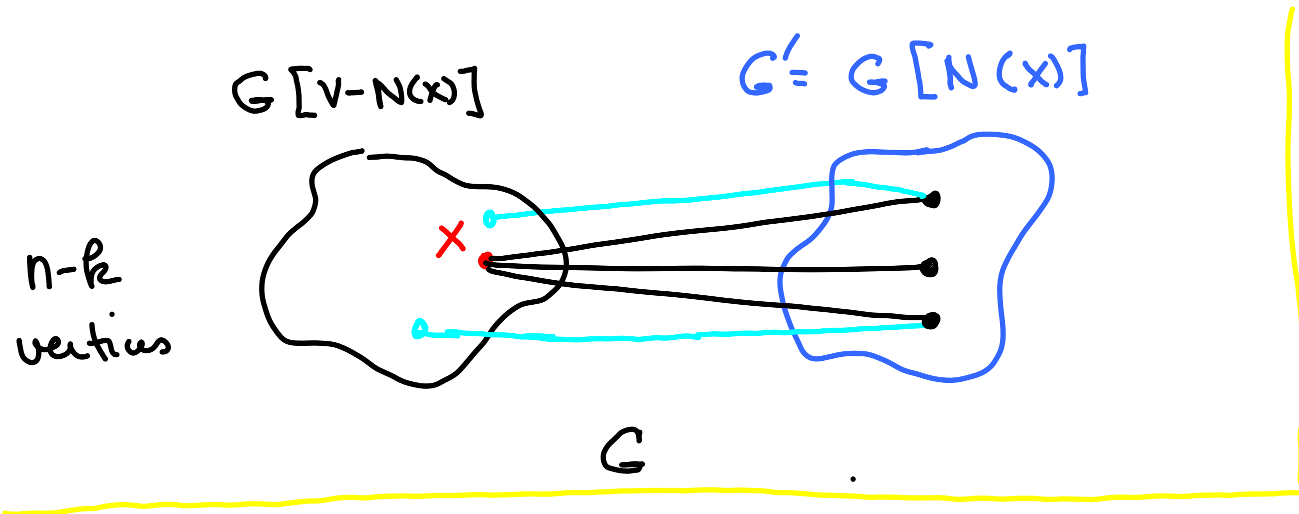
Let $G' = G[N(x)]$.

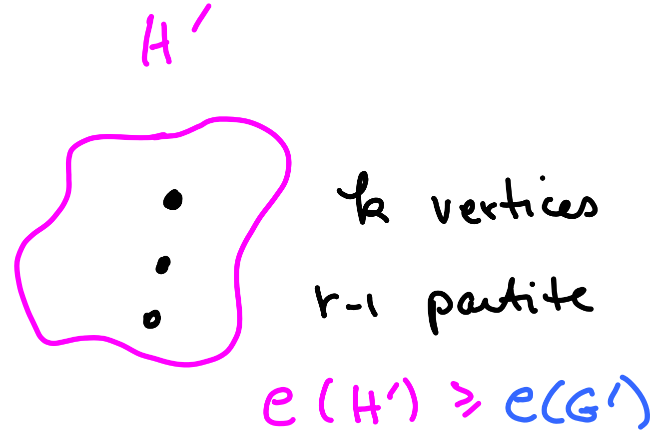
Note G' has no r-clique (why?)

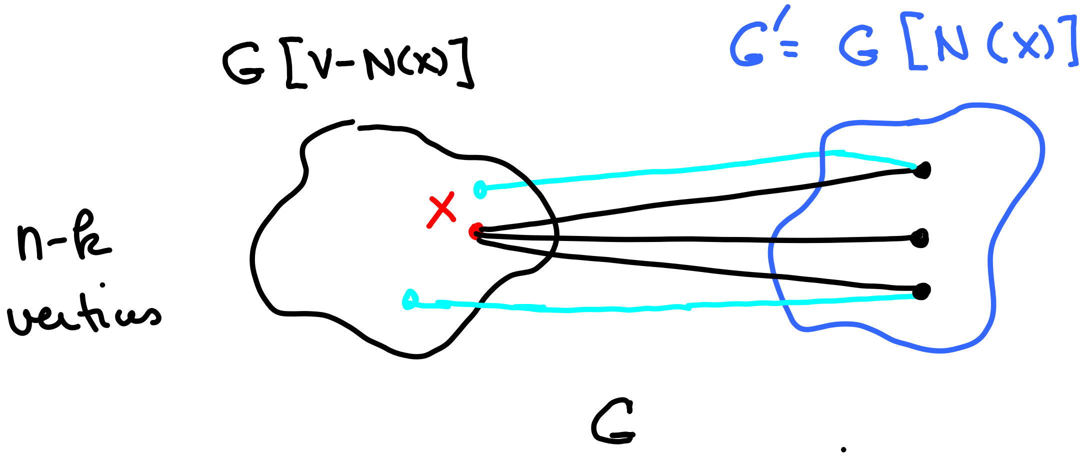
So by induction there is a k-vertex $r-1$ partite graph H' with $e(H') \geq e(G')$

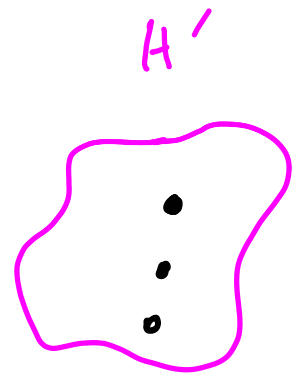


G [V-N(x)]
G' = G [N(x)]
n-k
vertices
x
G


G [V-N(x)]
G' = G [N(x)]
n-k
verties
G


H'
k vertices
r-1 partite
e(H') ≥ e(G')

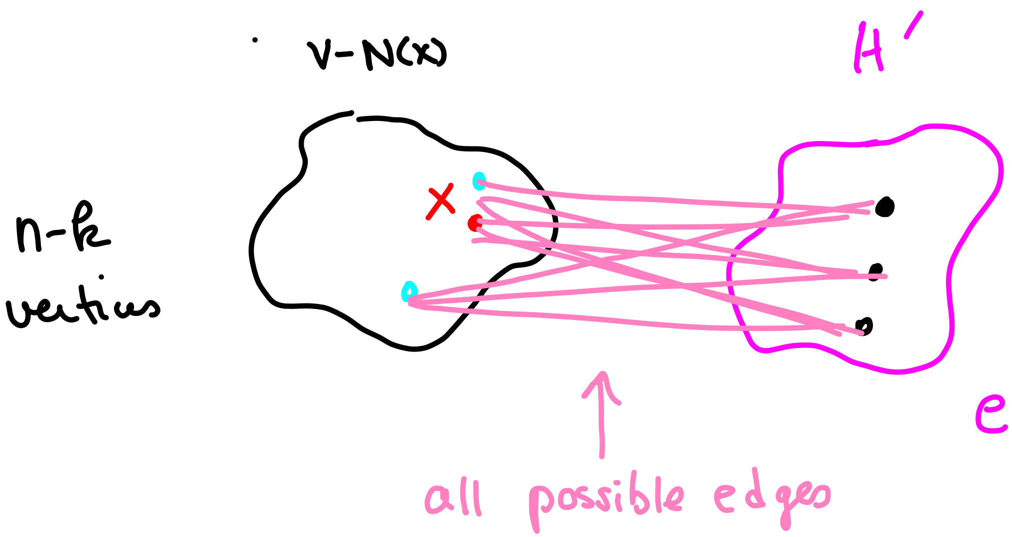




k vertices

r-1 partite

$e(H') \geq e(G')$



k vertices

r-1 partite

$e(H') \geq e(G')$

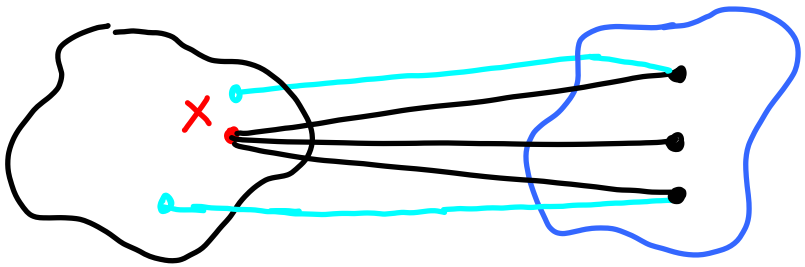
← construct H

like this

(will be r-partite)

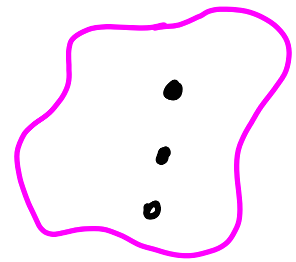
$G[V-N(x)]$ $\quad$ $G' = G[N(x)]$

$n-k$ vertices



G

H'



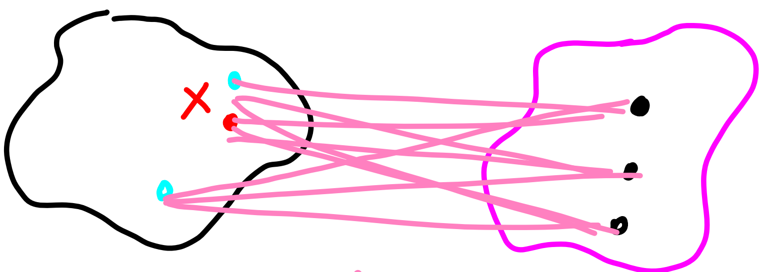
k vertices

$r-1$ partite

$e(H') \geq e(G')$

$V-N(x)$ $\quad$ H'

$n-k$ vertices



k vertices

$r-1$ partite

$e(H') \geq e(G')$

all possible edges

construct H like this

(will be r-partite)

and $e(G) \leq e(G') + \sum_{v \in V-N(x)} d(v) \leq e(H') + (n-k)\Delta(G)$

$= e(H') + (n-k)k = e(H)$.

Large clique $\Rightarrow$ large chromatic number

Converse true?

No - maybe no clique of size > 2 (!)

Erdös showed that for every $\boldsymbol{k, l \geq 2}$ there is a $\boldsymbol{k}$-chromatic graph with girth $\boldsymbol{l}$.

(The girth of a graph is the length of a shortest cycle, if the graph has a cycle, otherwise ∞.)

Theorem 5.2.3. For any $k > 0$, there is a k-chromatic graph with no triangle.

Pf. (Mycielski construction)

G_1 : 

G_2 : 

For $k \geq 2$, construct G_{k+1} from G_k:

Let $v_1, \ldots, v_n$ be vertices of G_k

Add "shadow vertex" u_i for each v_i.

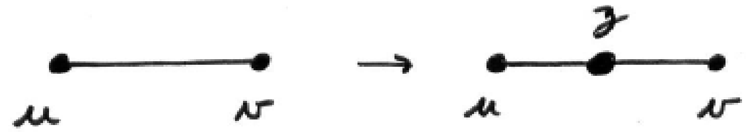
Add edge $u_i v_j \iff v_i v_j \in E(G_k)$

Add new vertex v, adjacent to every v_i

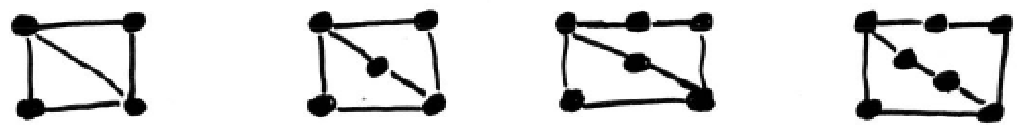
G_{k+1} has no triangles (if G_k does not)

What about $\chi(G_{k+1})$?

Edge subdivision :



Subdivision of a Graph :

G 

Can subdividing a graph ...

increase its chromatic number?

decrease its chromatic number?

Hajós Conjecture

If G has chromatic number k, then G has a subdivision of K_k.

$k=1 \longrightarrow$ no edges $\longrightarrow K_1$

$k=2 \longrightarrow \exists$ edge  $\rightarrow K_2$

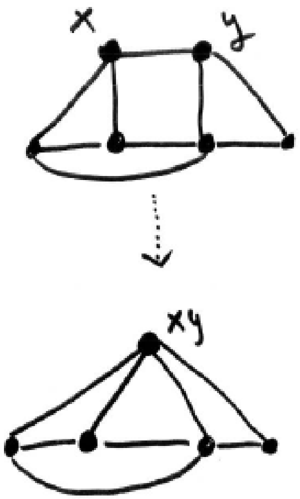
$k=3 \longrightarrow$ not bipartite $\longrightarrow$ odd cycle $\longrightarrow$ Subdivision of K_3

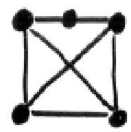
$k=4$ ✓ Dirac (1952)

$k=5,6$?? open

$k=7,8$ disproved - Caitlin (1979)

Contraction vs. Subdivision





contractible
to K_4, but no
subdivision of K_4

Hadwiger Conjecture

If G has chromatic number k, then

G contains a subgraph contractible to K_k

$k \leq 6$ TRUE $\longrightarrow$

$k \geq 7$?? open

$\underline{k \leq 4}$: equiv. to Hajos [Hadwiger 1943]

$\underline{k = 5}$: equiv to Four Color Conj [Wagner 1937]

Four Color Thm [Appel & Haken 1976]

$\underline{k = 6}$: using Four Color Thm [Robertson, Seymour, Thomas 1993]