Graph Coloring

(Assume graph simple) why?

1.
2.
3.
4.
5.

k-coloring of G: an assignment of colors $1, \ldots, k$ to the vertices of G.

$f : V(G) \to \{1, 2, \ldots, k\}$
(not necessarily onto)

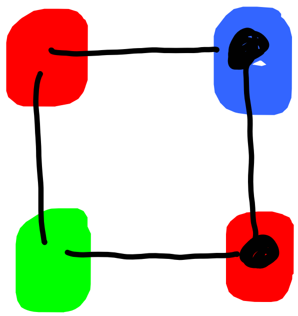
Coloring is **proper** if adjacent vertices are assigned **different** colors.

G is **k-colorable** if it has a proper k-coloring.

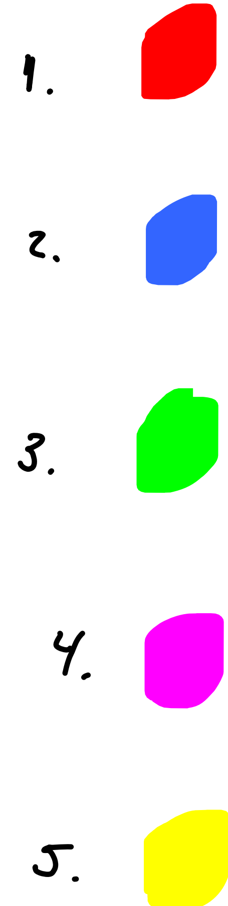
Chromatic number of G : $\chi(G)$: minimum k for which G is k-colorable.

G is **k-chromatic** iff $\chi(G) = k$.

C_4

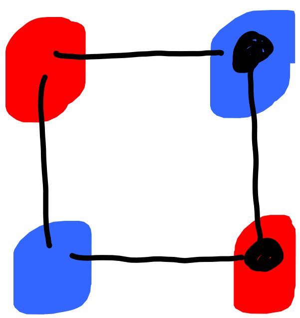


C_4 is 3-colorable, since it has a proper 3-coloring



but

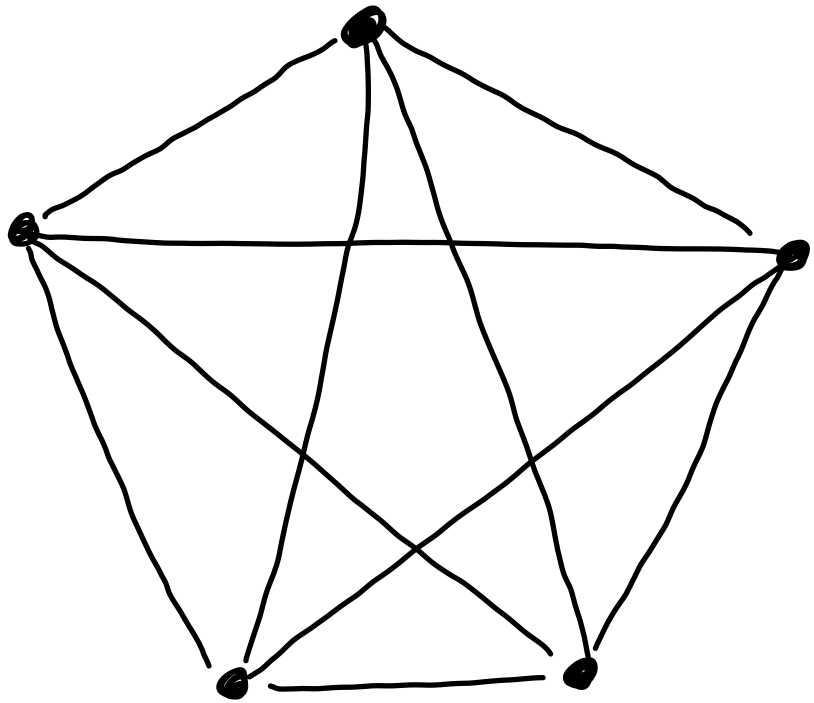
C_4



$$\chi(C_4) = 2$$

(The chromatic number of C_4 is 2.)

K_5



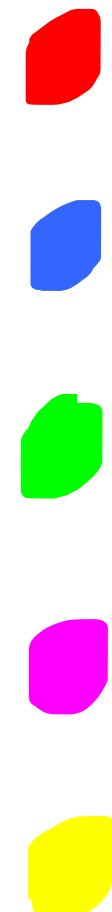
1.

2.

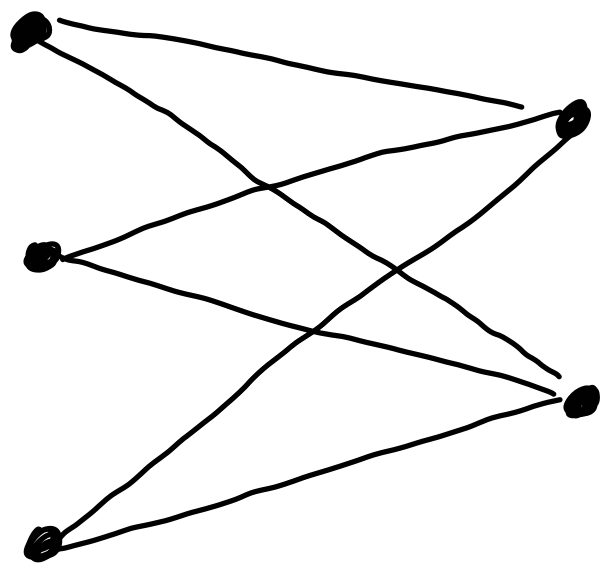
3.

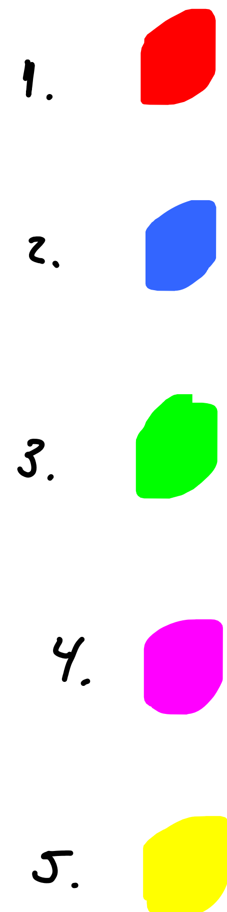
4.

5.

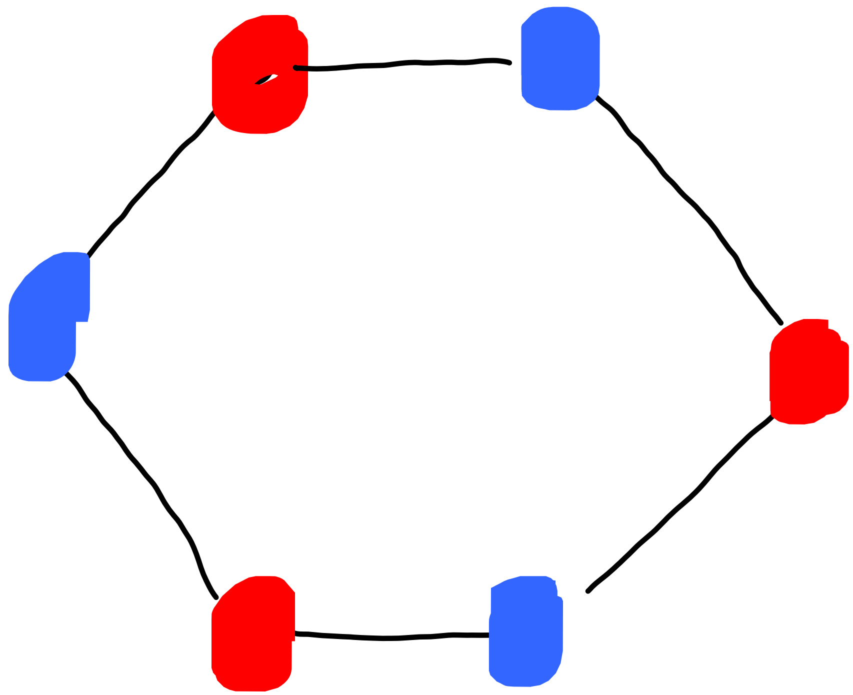


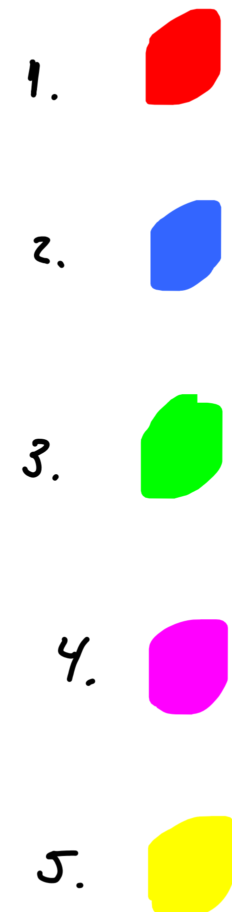
$K_{3,2}$



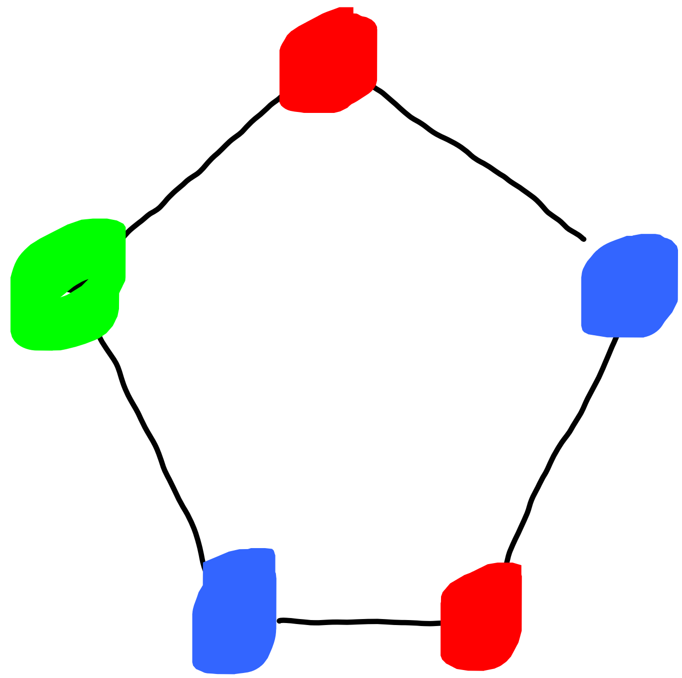


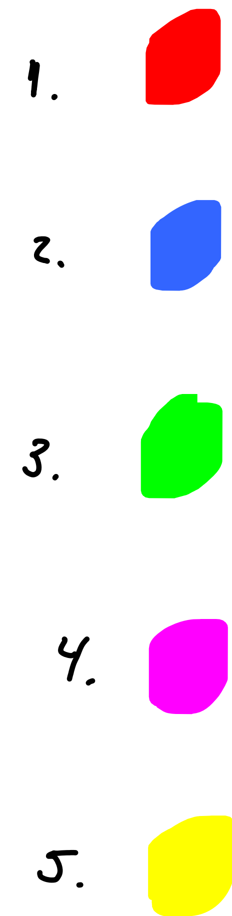
C_6



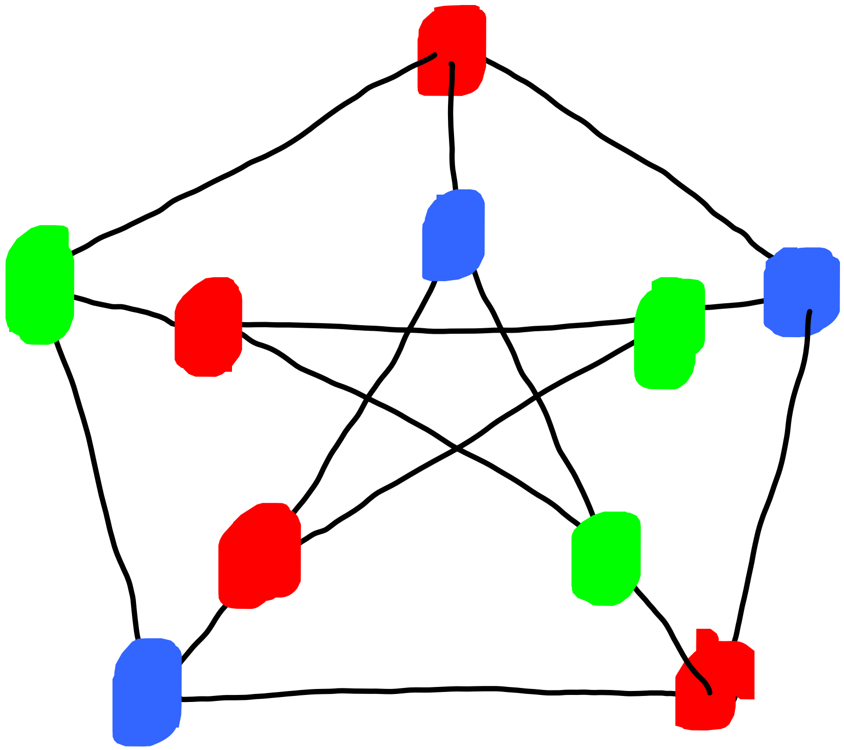


C_5





Petersen Graph



1.

2.

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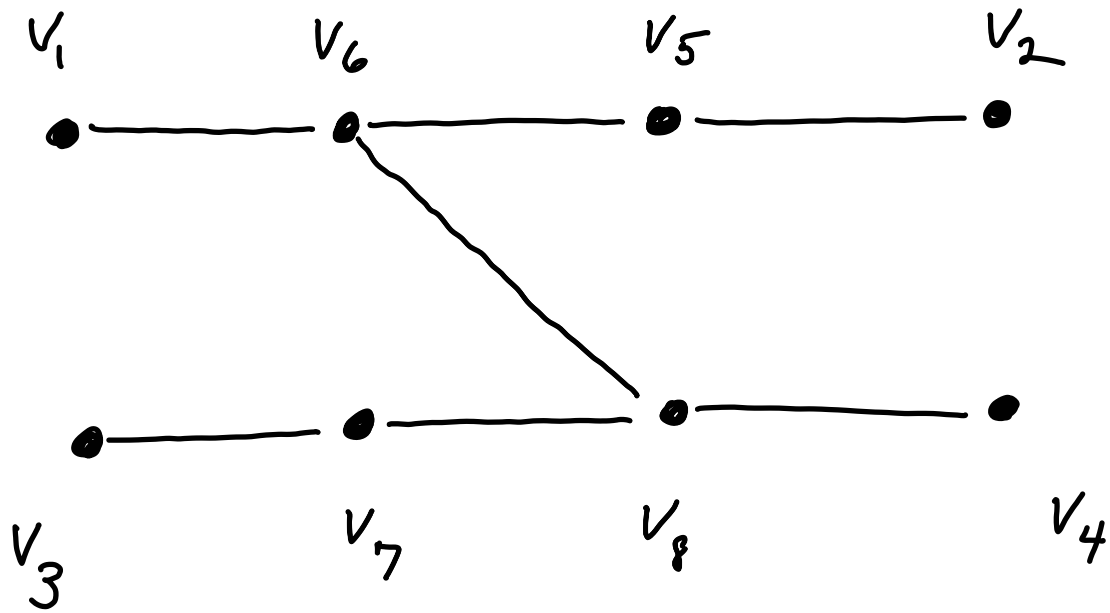
5.



1.

2.

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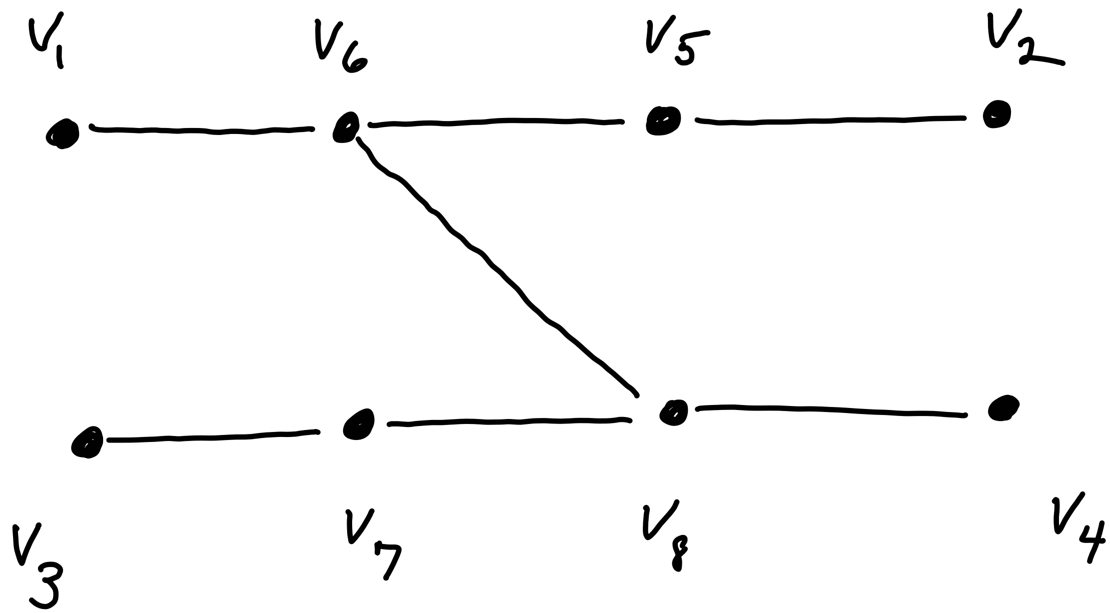
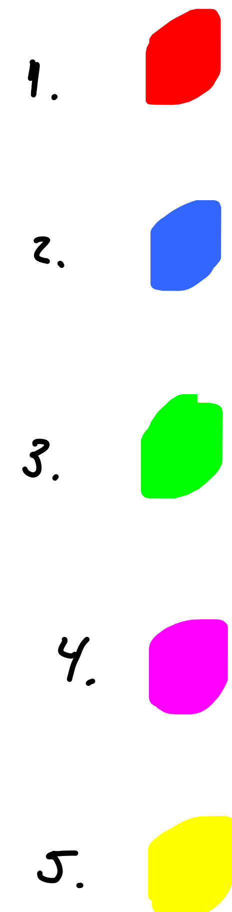


V_1
V_6
V_5
V_2
V_3
V_7
V_8
V_4



1.
2.
3.
4.
5.

Examples

$\chi(K_n) =$

$\chi(K_{n,m}) =$

$\chi(C_n)$ (n even) $=$

$\chi(C_n)$ (n odd) $=$

$\chi(\text{Petersen graph}) =$

G is <u>k-**critical**</u> if

(i) $\chi(G) = k$, but

(ii) $\chi(H) < \chi(G)$ for every proper subgraph H of G.

Which are critical:

K_n**?**

$K_{n,m}$**?**

even cycles?

odd cycles?

Petersen graph?

<u>Note</u>: Every k-chromatic graph has a k-critical subgraph.

Note: Every k-chromatic graph has a k-critical subgraph.

Let G be k-chromatic.

If G is critical, done.

Else G has a proper subgraph H with $\chi(H) = k$

If H is critical, done.

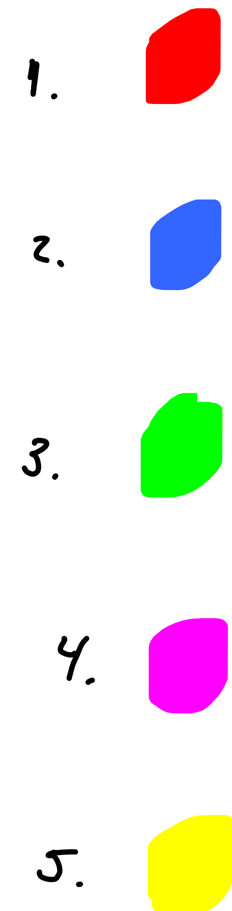
Else H has a proper subgraph H' with $\chi(H') = k$.

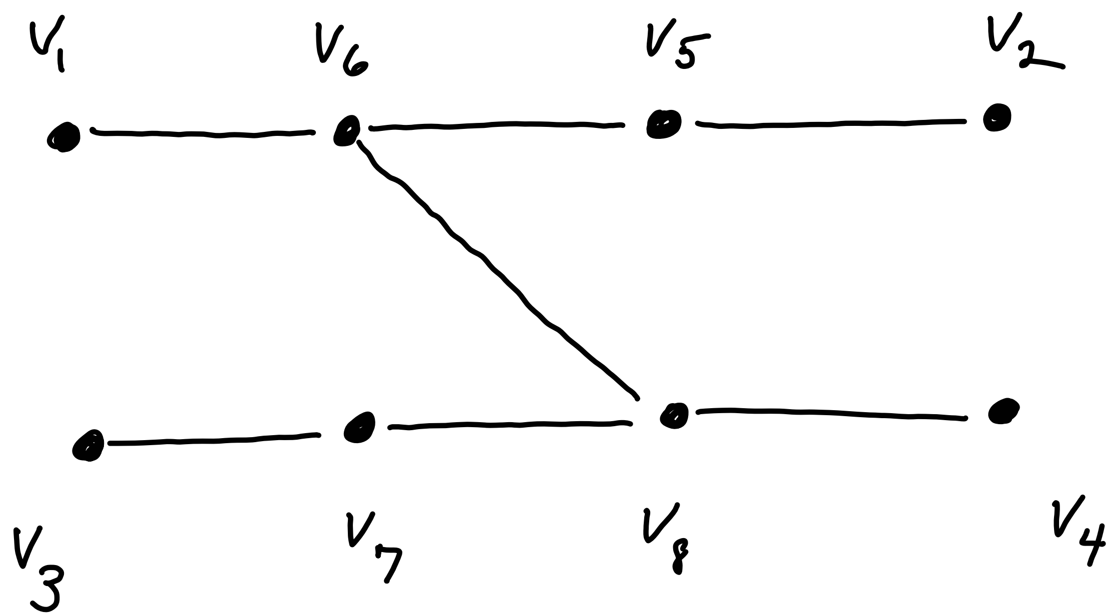
etc.

Process halts (since graph is finite)

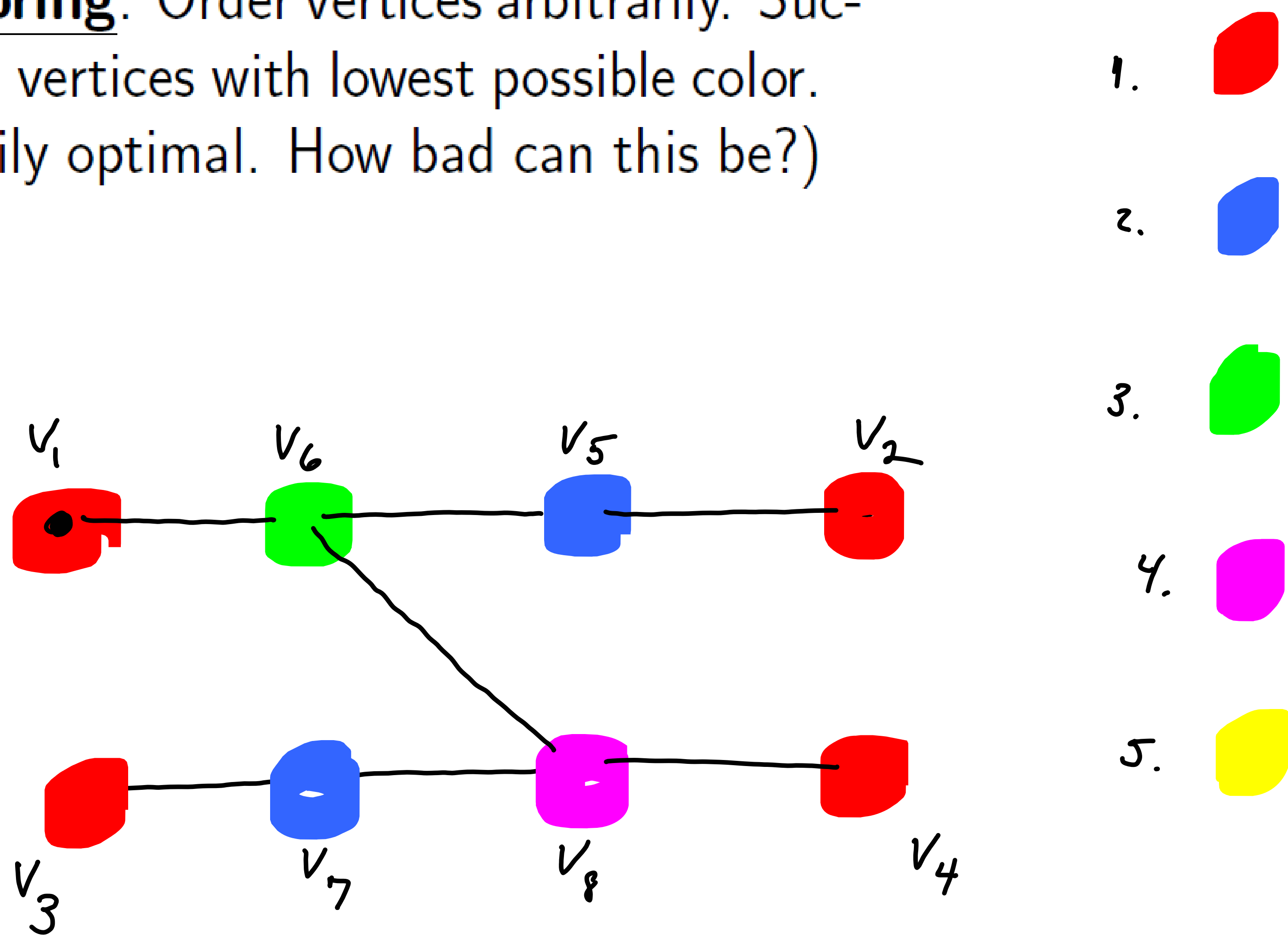
with k-critical graph.

<u>Greedy Coloring</u>: Order vertices arbitrarily. Successively color vertices with lowest possible color. (Not necessarily optimal. How bad can this be?)





Greedy Coloring: Order vertices arbitrarily. Successively color vertices with lowest possible color. (Not necessarily optimal. How bad can this be?)

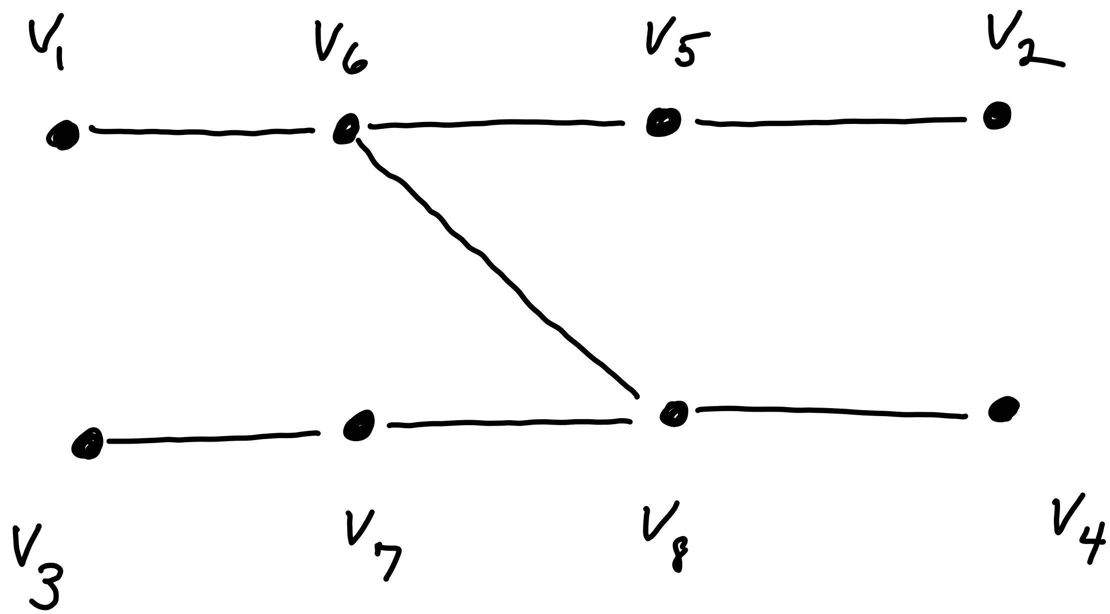


(But $\chi = 2$)

Proposition 5.1.13.

$$\chi(G) \leq \Delta(G) + 1.$$

Example :



Proposition 5.1.13.

$$\chi(G) \leq \Delta(G) + 1.$$

(In fact, greedy coloring always colors with $\leq \underline{\Delta+1}$ colors)

Start colouring:

$v_1 \quad v_2 \quad v_3 \quad \cdots\cdots \quad v_k \quad \cdots$

$\leq \Delta$ nbrs

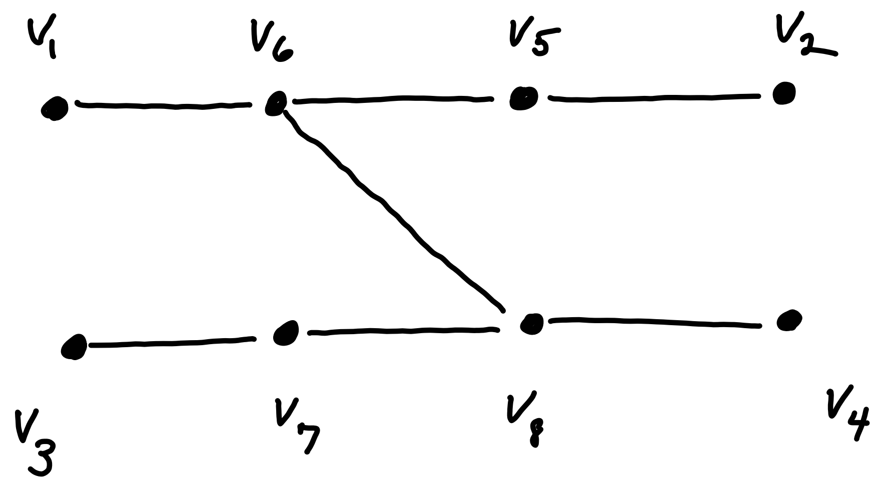
so at least 1 color

is available

Proposition 5.1.14. If G has degree sequence $(d_1, d_2, \ldots, d_n)$ (nonincreasing) then

$$\chi(G) \leq 1 + \max_i \min\{d_i, i-1\}.$$

Example:



Proposition 5.1.14. If G has degree sequence $(d_1, d_2, \ldots, d_n)$ (nonincreasing) then

$$\chi(G) \leq 1 + \max_i \min\{d_i, i-1\}.$$

(This is what greedy coloring guarantees if you color in decreasing order of degrees)

pf. Start coloring:

v_1 v_2 v_3 . . . . v_i

d_i nbrs
but only
$i-1$ forbidden colors

Lemma 5.1.18. If H is k-critical,

$$\delta(H) \geq k - 1.$$

Restated:

If H is critical then

$$\chi(H) \leq \delta(H) + 1$$

Lemma 5.1.18. If H is k-critical,

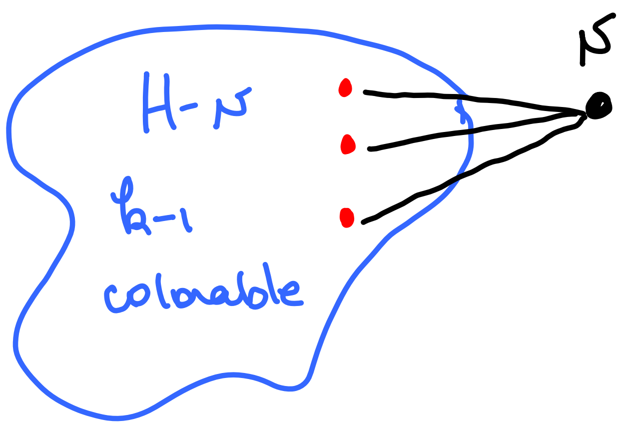
$$\delta(H) \geq k-1.$$

Proof Assume H is k-critical.

Let v be a vertex of minimum degree. (Show $d(v) \geq k-1$)

$\chi(H-v) < k$ (since H is critical).

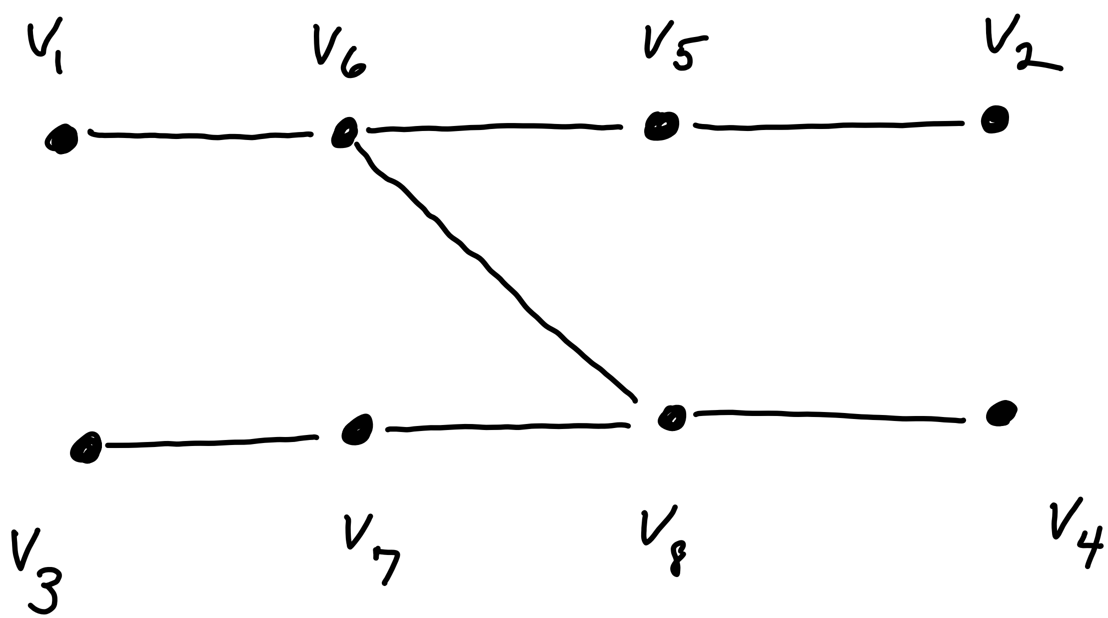
So:



If $d(v) \leq k-2$, one of the $k-1$ colors is available for v, giving a $k-1$ coloring of H. Contradiction, since $\chi(H) = k$.

Brooks' Theorem (5.1.22): If G is a (simple) connected graph other than a clique or an odd cycle, then $\chi(G) \leq \Delta(G)$.

Example:



<u>**Brooks' Theorem**</u> (5.1.22): If G is a (simple) connected graph other than a clique or an odd cycle, then $\chi(G) \leq \Delta(G)$.

<u>Proof</u>. Let $t = \Delta(G)$

<u>$t = 0$?</u>  K_1

<u>$t = 1$?</u>  K_2

G cannot be these.

<u>$t = 2$?</u> P_n $(n \geq 3)$ or C_n (n even)

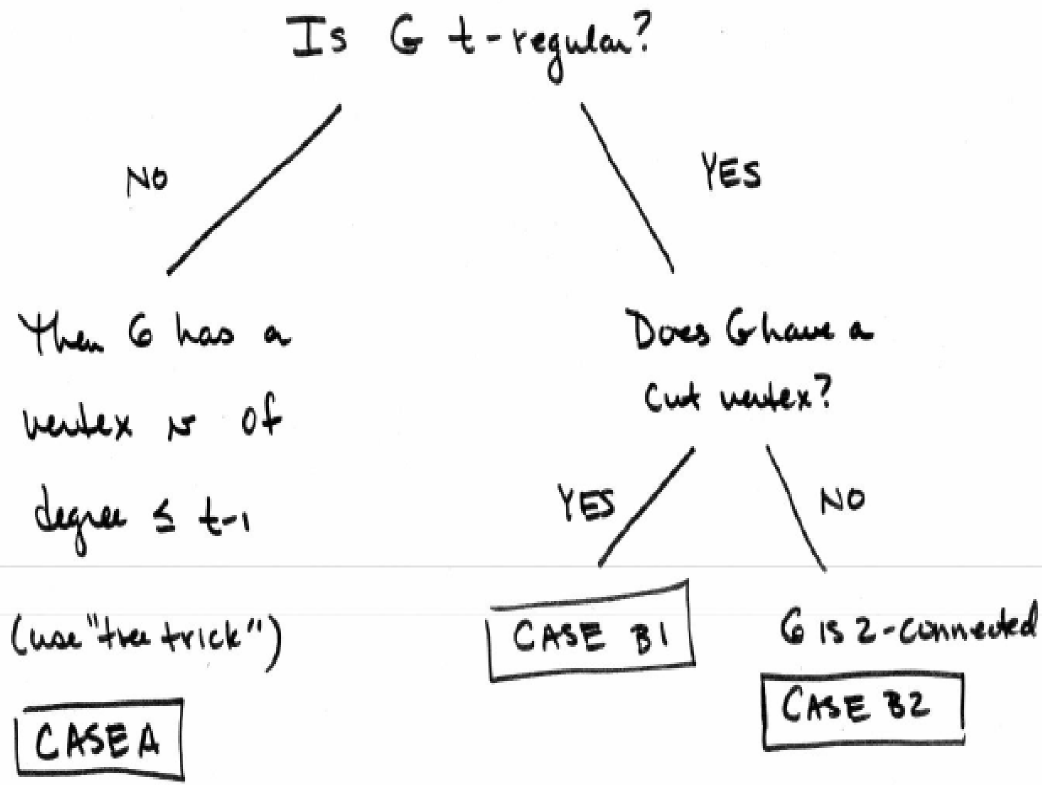
$$\chi = 2 = \Delta \checkmark$$

So, assume $t \geq 3$ ...

Brooks' Theorem (5.1.22): If G is a (simple) connected graph other than a clique or an odd cycle, then $\chi(G) \leq \Delta(G)$.

Assume $t \geq 3$ $\quad (t = \Delta(G))$

Outline of Proof

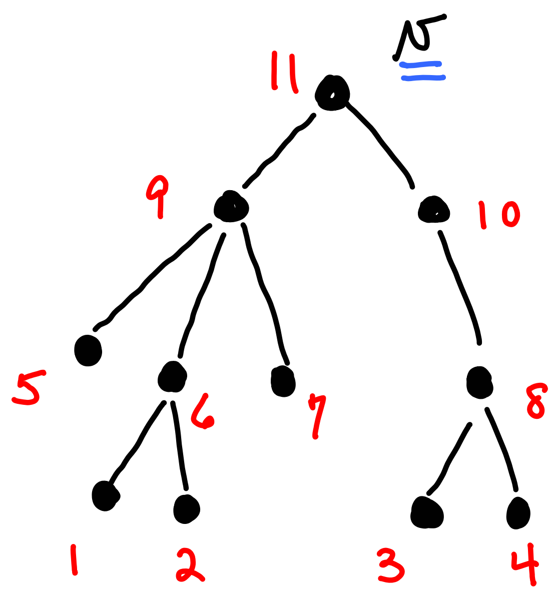


Brooks' Theorem (5.1.22): If G is a (simple) connected graph other than a clique or an odd cycle, then $\chi(G) \leq \Delta(G)$.

$t = \Delta(G) \geq 3$

Case A: G has a vertex v of degree $\leq t-1$

We will order the vertices using the "tree trick" and then do greedy coloring with t colors.



Tree trick

Let T be a spanning tree of G rooted at v.

Order vertices so that every vertex occurs earlier than its parent.

Claim: you never get stuck

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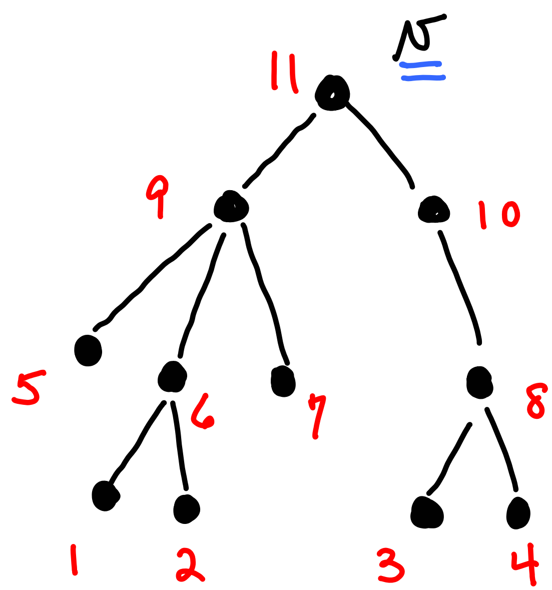

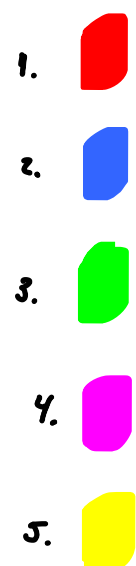

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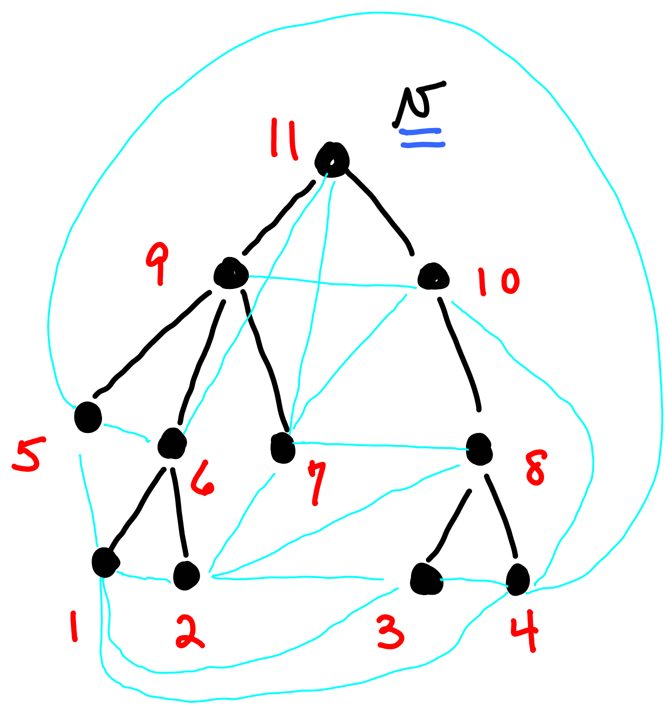

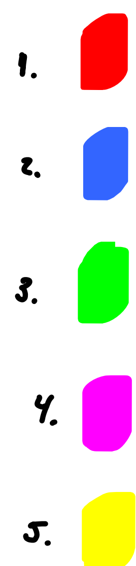

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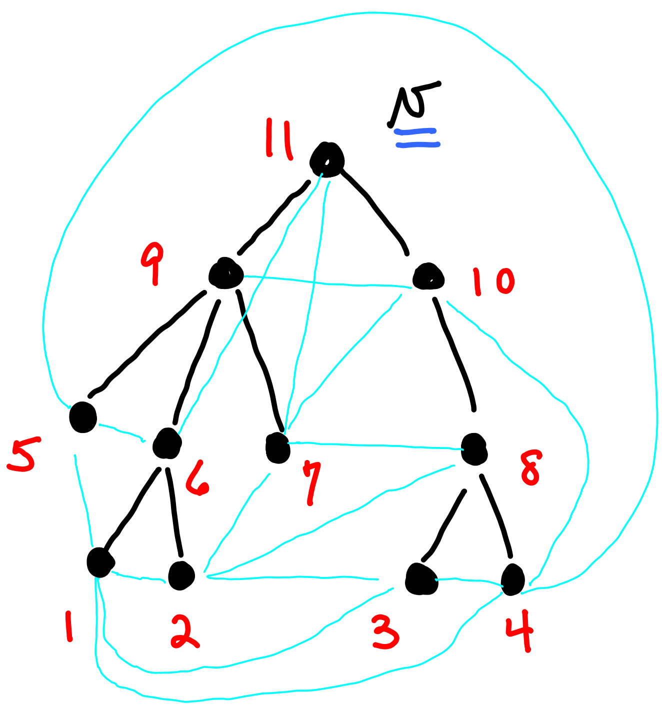
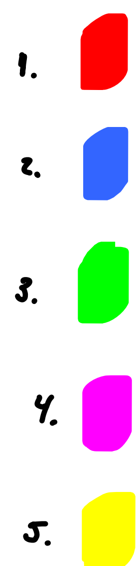
Case A: G has a vertex $\underline{\underline{v}}$ of degree $\leq t-1$

We will order the vertices using the "tree trick" and then do greedy coloring with t colors.





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Let T be a spanning tree of G rooted at $\underline{\underline{v}}$.

Order vertices so that every vertex occurs earlier than its parent.

Claim: you never get stuck

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$t = \Delta(G) \geq 3$

Case B1: G is t-regular and G has a cut vertex v

Let z be a cut vertex.

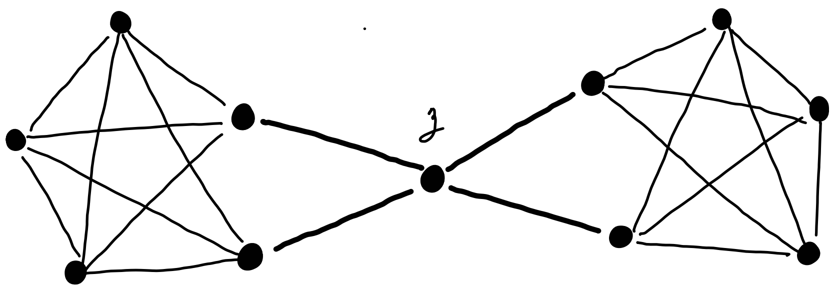
Let $C_1, C_2, \ldots, C_k$ be comps. of $G - z$.

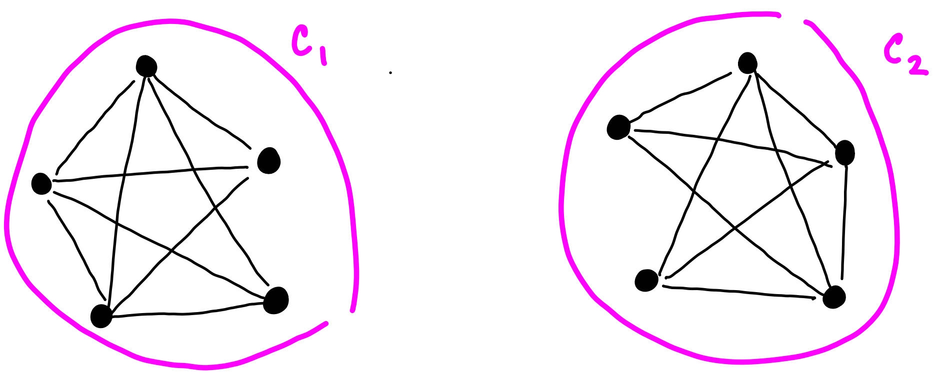
Let $H_i = G[V(C_i) \cup \{z\}]$

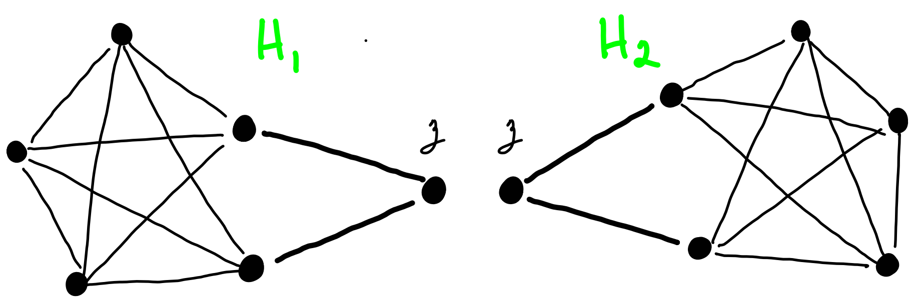
Claim: Each H_i is t-colorable

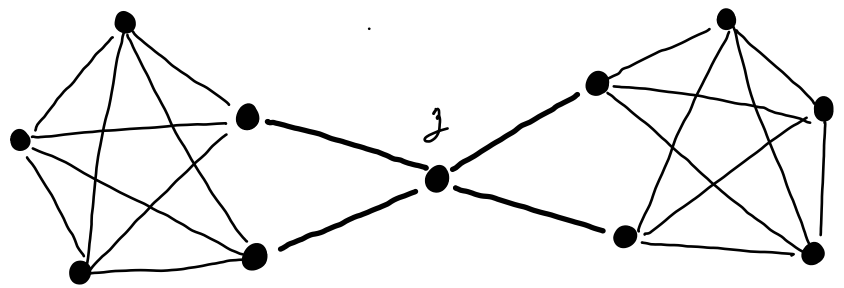
(why? Each H_i is Case A. Can use the trick!)

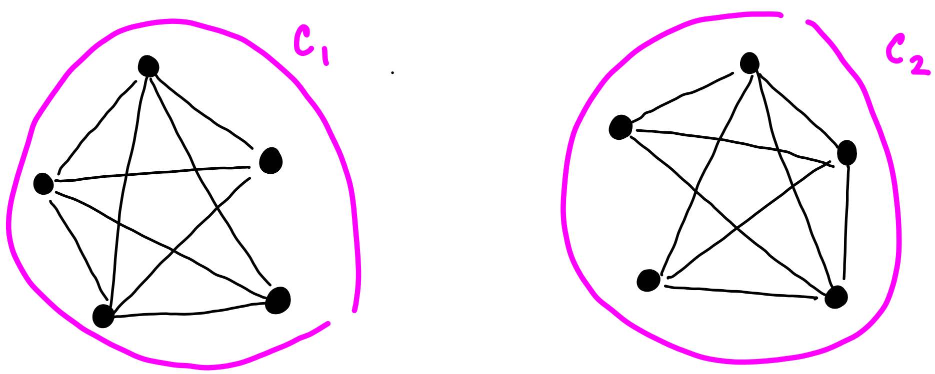
Making t-colorings agree on z gives a t-coloring of G

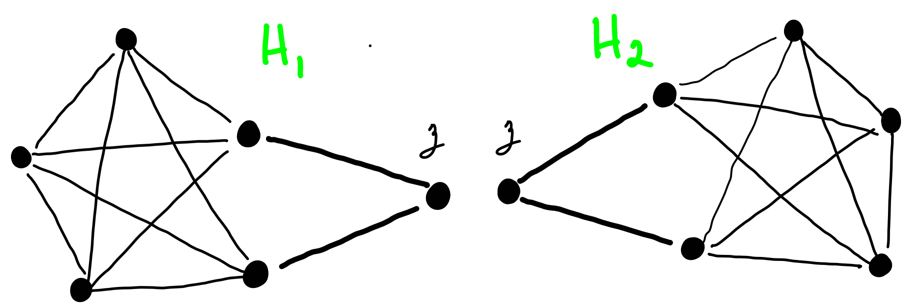

z

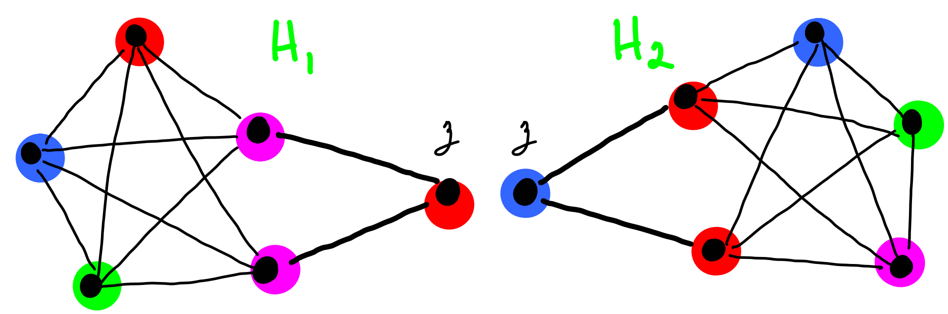

C1
C2


H1
H2
z
z

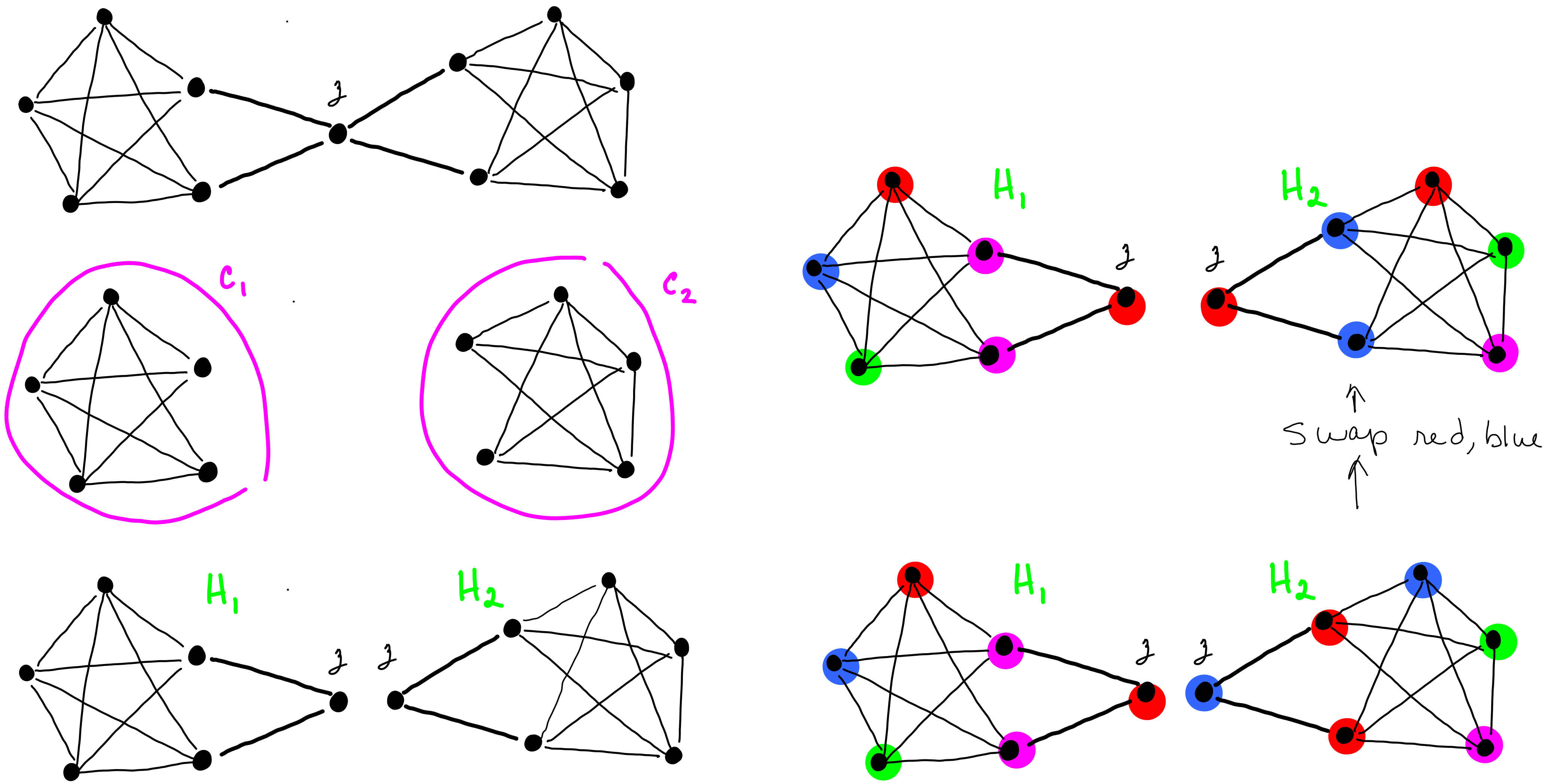








t-color H_1, H_2 (tree trick)



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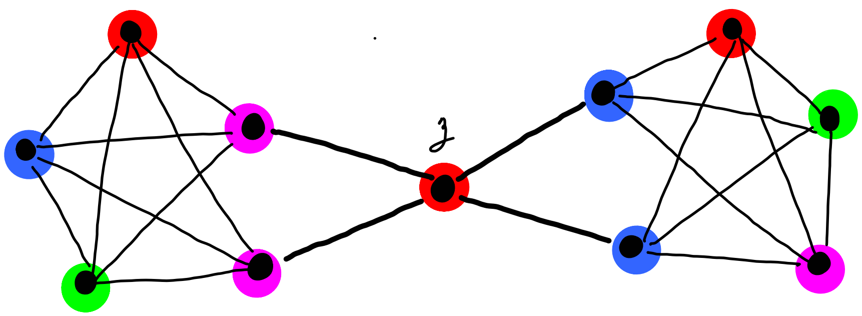
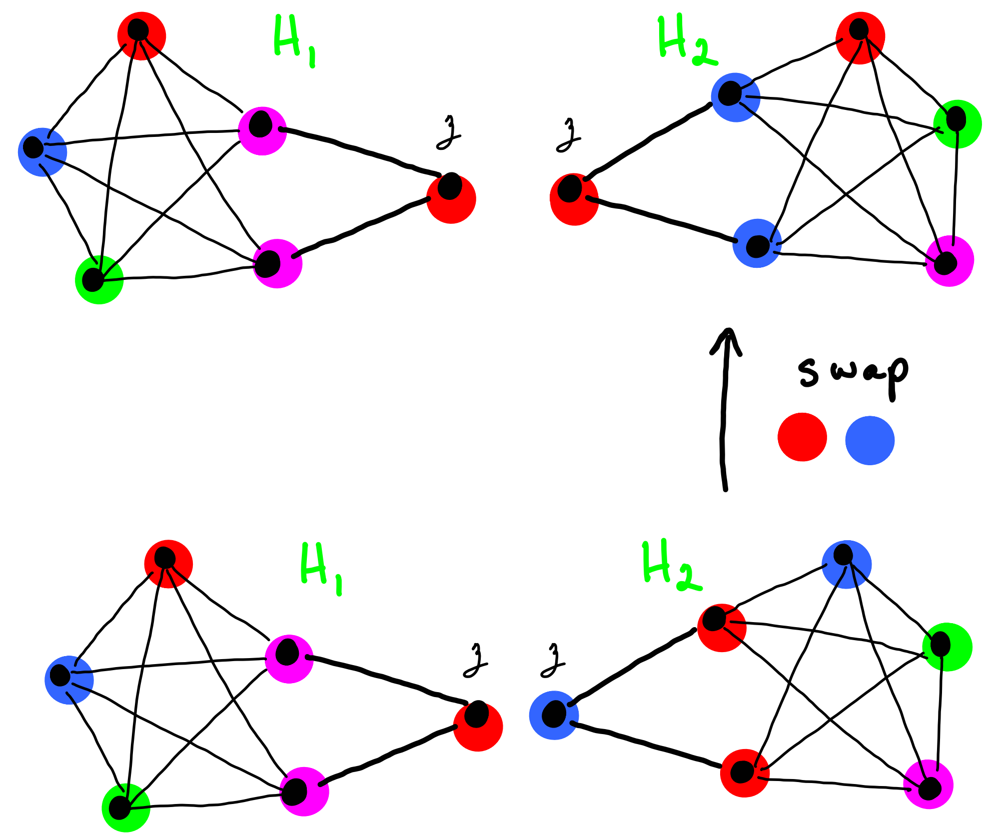
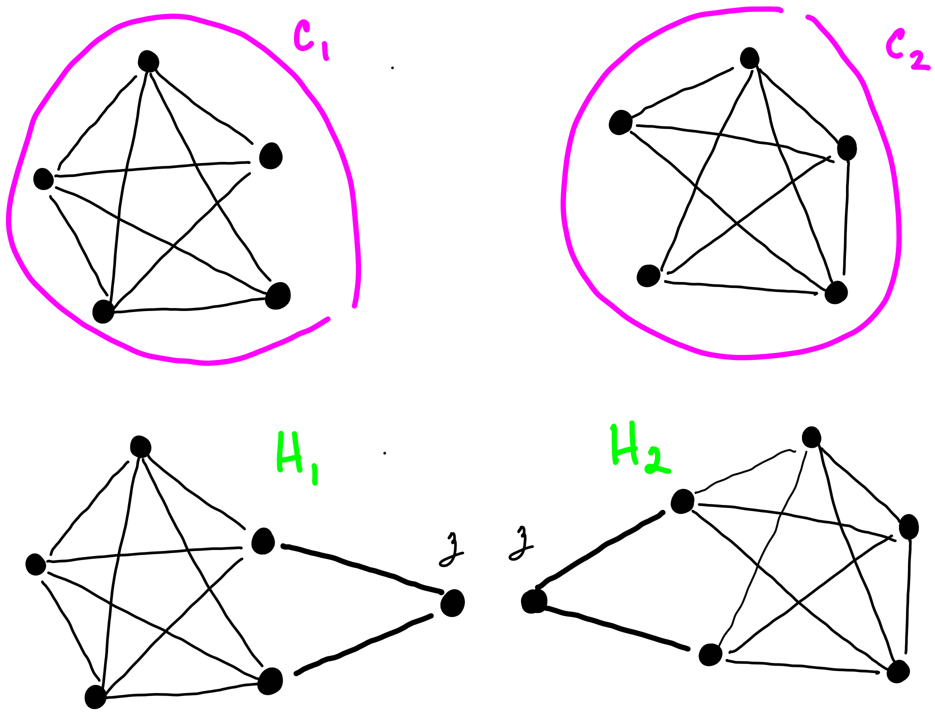


Glue back together to get
t-coloring of G.





t-color H_1, H_2 (tree trick)

Brooks' Theorem (5.1.22): If G is a (simple) connected graph other than a clique or an odd cycle, then $\chi(G) \leq \Delta(G)$.

$t = \Delta(G) \geq 3$

Case B2: G is t-regular and G is 2-connected

Claim. G has a vertex v, with non adjacent neighbors x, y such that $G - \{x,y\}$ is connected.

Using this, Let T be a spanning tree of $G - \{x,y\}$, rooted at v.

Use the tree trick to order the vertices of T so that every node appears earlier than its parent : $v_1, v_2, \ldots, v_{n-2} = v$

Then do greedy t-coloring of G using this vertex ordering :

$$x, y, v_1, v_2, \ldots, v_{n-2} = v$$

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Then do greedy t-coloring of G using this vertex ordering:

$$x, y, v_1, v_2, \ldots, v_{n-2} = v$$

etc

When you reach v, two of its t neighbors (x & y) have been assigned the same color, so there is a color left for v.

Technical Lemma (needed for Brooks' Thm, case B2)

If G is t-regular, $t \geq 3$ and G is not complete, and G is 2-connected, then G has a vertex w, with nonadjacent neighbors x, y such that $G - \{x,y\}$ is connected.

Proof.

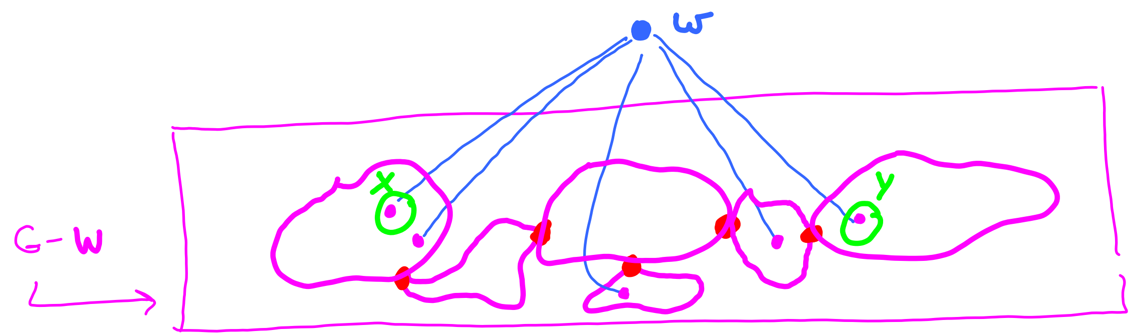
If $N(u)$ is a clique for every u, then G is a clique. (why?)

So some u has nonadjacent neighbors w, z.

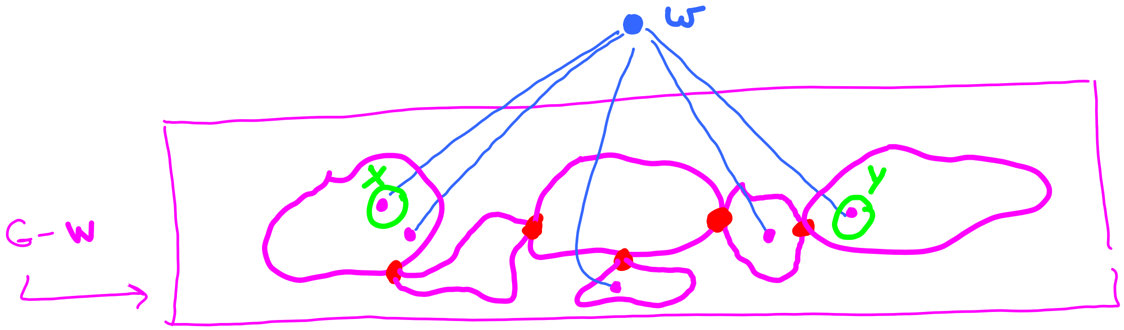
If $G - \{w, z\}$ is connected, let $v = u$, $x = w$, $y = z$ done ✓

Otherwise, we have this:

Choose $x, y \in N(w)$ to be in 2 different "leaf blocks" of $G - w$ and let $v = w$. (Now check)

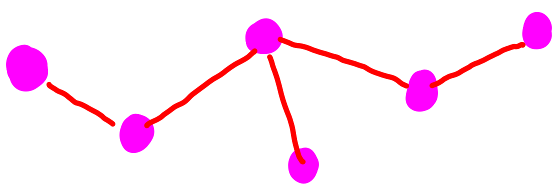


Checking details



G-w has at least one cut vertex (z). Can decompose G into "blocks", i.e. maximal connected subgraphs with no cut vertices.

Can show block-cut vertex graph is a tree, $\therefore$ at least 2 leaves
(if at least 2 blocks)



G-w has ≥ 2 blocks. ✓

Each block has ≥ 2 vertices (G is t-regular, $t \geq 3$).

Each leaf block has only one cut vertex of G-w.

N(w) contains a <u>non-cut</u> vertex from each leaf block of G-w (else w would be a cutvertex.)