Homework 4
Graph Theory CSC/MA/OR 565
Sketch of Solutions

1. You can find a feasible flow of value 17. It must be the maximum flow, since we can find a source/sink cut of capacity 17 as follows. Label the nine vertices in the diagram as follows: top row (s, a, b), middle row (c, d, e), bottom row (f, g, t). The cut $[\{s, a, b, c, f\}, \{d, e, g, t\}]$ has capacity 17.

2. Clearly $\kappa(x, y) \geq \lambda(x, y)$ since, given a collection of disjoint (x, y)-paths, at least one vertex must removed from each path to destroy all (x, y)-paths. So the interesting part is to use network flows to prove that $\kappa(x, y) \leq \lambda(x, y)$ As suggested by the hint, given vertices x, y in digraph D, construct a network N as follows:

- For each $v \in V(D)$, create two vertices v^- and v^+ in N, with an edge from v^- to v^+ of capacity 1.
- For each edge uv in D, create an edge in N from u^+ to v^- of very large capacity.
- Let x^- be the source of N and let y^+ be the sink.

Note that each directed (x, y)-path in D corresponds to a source-to-sink path in N. Vertex disjoint (x, y)-paths in D correspond to disjoint source-to-sink paths in N.

A collection $\mathcal{P}$ of k disjoint (x, y)-paths in D gives rise to a feasible flow of value k in N by assigning flow value 1 to edges corresponding to paths in $\mathcal{P}$ and 0 flow to all other edges. To see the converse, first note that in any feasible flow, all edges have flow value 0 or 1. By conservation of flow, and the construction of N, if f is a feasible flow of value k, the set of all edges with flow value 1 comprise k disjoint source-to-sink paths in N, corresponding to k disjoint (x, y)-paths in D. Thus, $\lambda(x, y)$ in D is equal to the value of the maximum flow in N.

By the max flow/min cut theorem, if k is the value of the maximum flow in N, then N has a source/sink cut $C = [S, \overline{S}]$ of capacity k and C is a minimum cut. Observe that (because we made the capacities so large), no edge of the form u^+ to v^- can be in C, since moving v^- to S would give a cut of smaller capacity. Thus all edges in C have the form u^-u^+ of capacity 1 and there must be k of them, since C has capacity k. Removing these k edges from N destroys all source/sink paths in N, so removing their corresponding vertices from D destroys all (x, y)-paths in D. Thus $\kappa(x, y) \leq k$. Combining this with the argument in the preceding paragraph shows $\kappa(x, y) \leq \lambda(x, y)$, which was our goal.

3. The first has chromatic number 4, the second has chromatic number 5. Neither is critical.

4. Recall that in a critical graph H, we proved that $\delta(H) \geq \chi(H) - 1$. We also proved that every k-chromatic graph G has a k-critical subgraph H. Putting these together, every

vertex of H has degree at least $k-1$ in H. Clearly, $n(H) \geq k$, so H has at least k vertices of degree at least $k-1$ in H. Since H is a subgraph of G, for any $v \in V(H)$, $d_H(v) \leq d_G(v)$. Thus G has at least k vertices of degree at least $k-1$ in G (namely the vertices of H)

5. a. For any vertex v of G, $\chi(G) \leq \chi(G-v)+1$, since if $G-v$ can be properly colored with k colors, then coloring v a different color from its neighbors would require at most one new color.

b. If $G-v$ can be colored with k colors, and if v has fewer that k neighbors, then one of the k colors is available to color v. Thus $d_G(v) < \chi(G-v)$ implies $\chi(G) = \chi(G-v)$. The contrapositive, which is equivalent is that (using (a)) is that $\chi(G) = \chi(G-v)+1$ implies $d_G(v) \geq \chi(G-v)$.

c. We will use (a) and (b) to show by induction on n that $\chi(G) + \chi(\overline{G}) \leq n(G)+1$.

When $n=1$, G and $\overline{G}$ each have only one vertex are 1-chromatic, so the result holds.

Let G be a graph with $n>1$ vertices and assume the result holds for graphs with fewer than n vertices. Let v be a vertex of G. By (a) and by induction,

$$\chi(G) + \chi(\overline{G}) \leq \chi(G-v) + 1 + \chi(\overline{G-v}) + 1 \leq n(G-v) + 3 = n(G) + 2.$$

So suppose by contradiction that $\chi(G) + \chi(\overline{G}) > n(G)+1$. Then it must be that $\chi(G) + \chi(\overline{G}) = n(G)+2$. But for this to happen, it must be that all three of these hold:

(i) $\chi(G) = \chi(G-v)+1$;
(ii) $\chi(\overline{G}) = \chi(\overline{G-v})+1$;
(iii) $\chi(G-v) + \chi(\overline{G-v}) = n(G-v)+1 = n(G)$.

But then using (b) it must be that both

(i') $d_G(v) \geq \chi(G-v)$;
(ii')$d_{\overline{G}}(v) \geq \chi(\overline{G-v})$.

But $d_G(v) + d_{\overline{G}}(v) = n(G)-1$, so by (i') and (ii'),

$$\chi(G-v) + \chi(\overline{G-v}) \leq n(G)-1,$$

which contradicts (iii).

6. (Note B_0 is a tree with one vertex.)

Prove by induction on k that $V(B_k)$ has an ordering with the property that the greedy algorithm will color the root of $V(B_k)$ color $k+1$.

This is true for $k=1$: color the leaf first (color 1), then the root (color 2). (You could instead use $k=0$ as the basis)

If $k>1$, assume true for $k-1$. For $k>1$, B_k can be viewed as a root with k children, where those children are the roots of subtrees $B_0, \ldots, B_{k-1}$. Applying the induction hypothesis and for $i = 0, \ldots, k-1$, order the vertices of the subtree B_i of the root of B_k with the "bad" ordering that forces the root of subtree B_i to have color $i+1$. Then apply the greedy algorithm with those orderings to color $B_0, \ldots, B_{k-1}$ one at a time. Finally color the root of B_k. Since its children are colored with all of the colors $1, \ldots, k$, the root of B_k must be colored $k+1$.

7. Use the "tree trick" to order the vertices of G in such a way that the greedy coloring algorithm uses at most $\Delta(G)$ colors, as follows.

Let $k = \Delta(G)$ and let v be a vertex of minimum degree. Then v has fewer than k neighbors. Let T be a spanning tree of G rooted at v. Order the vertices of T so that for every non-root vertex x, x occurs earlier that its parent in the ordering. Now use that ordering to do a greedy coloring of the vertices of G. When it is time to color a non-root vertex x, since x has at most k neighbors, and since its parent has not yet been colored, at most $k-1$ colors are forbidden to x. Therefore at least one of the colors $1, 2, \ldots, k$ is available to properly color x. The root v is colored last. But since v has at most $k-1$ neighbors, at least one of the colors $1, 2, \ldots, k$ is available to properly color v.

8. Use the graph C_4 from the Mycielski construction.

9. Again, use the graph C_4 from the Mycielski construction.

10. Let G be a simple n-vertex graph with no $r+1$ clique. By Turán's theorem, $e(G) \leq e(T_{n,r})$. So it suffices to show that $e(T_{n,r}) \leq (r-1)n^2/(2r)$, or, equivalently, that

$$(2r)\ e(T_{n,r}) \leq (r-1)n^2.$$

Let $n = ar + b$ where $0 \leq b < r$. Then $T_{n,r}$ consists of b independent sets of size $a+1$ and $r-b$ independent sets of size a and two vertices are adjacent if and only if they are in different independent sets. We use the degree sum formula to find the number of edges.

There are $b(a+1)$ vertices in independent sets of size $a+1$ and each has degree $(n-(a+1))$. There are $(r-b)a$ vertices in independent sets of size a and each has degree $(n-a)$.

Using the degree sum formula, we have that

$$\begin{aligned}
(2r)\ e(T_{n,r}) &= r\sum d(v) = r[b(a+1)(n-a-1) + (r-b)a(n-a)] \\
&= rbn - 2rba - rb + anr^2 - a^2r^2 \\
&= rn(ar+b) - (a^2r^2 + 2arb) - rb \\
&= rn^2 - (n^2 - b^2) - rb \\
&= (r-1)n^2 + b(b-r),
\end{aligned}$$

where the next-to-last line follows by using the fact that $n = ar + b$ and also $n^2 = a^2r^2 + 2arb + b^2$.

Note that since $b < r$, $b(b-r)$ is negative, so the result follows.