Connectivity

(Assume no loops)

Vertex cut or **separating set of G:**
subset S of $V(G)$ such that $G - S$ is disconnected.

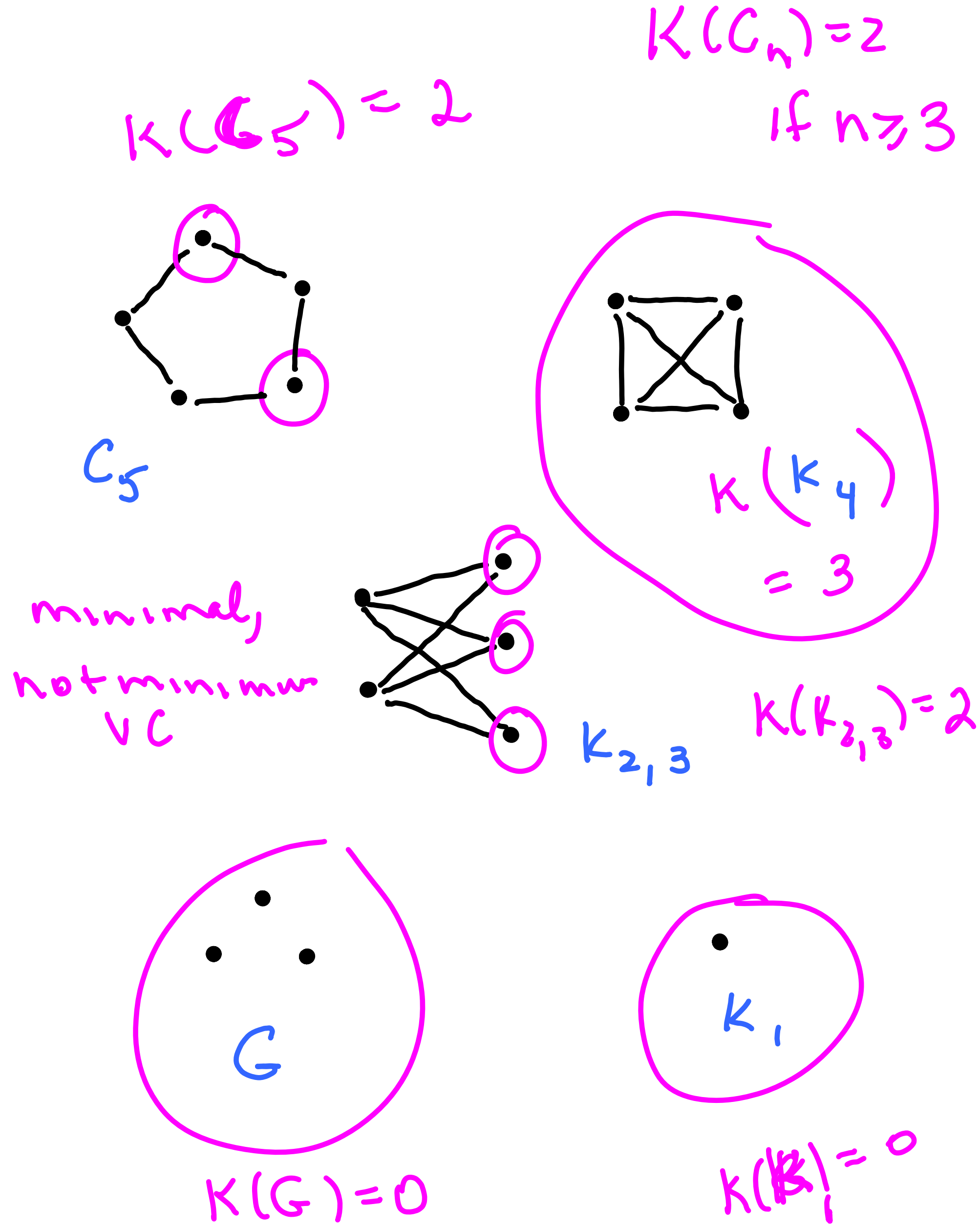
Could a vertex cut be empty?

Must every graph have a vertex cut?

$\kappa(G)$ - **connectivity** of G

$$\kappa(G) = \begin{cases} n-1 & \text{if } G \text{ has } K_n \text{ as a spanning subgraph} \\ \text{otherwise, the minimum } k \text{ for which } G \text{ has a } k\text{-vertex cut} \end{cases}$$

G is **k-connected** if $\kappa(G) \geq k$.



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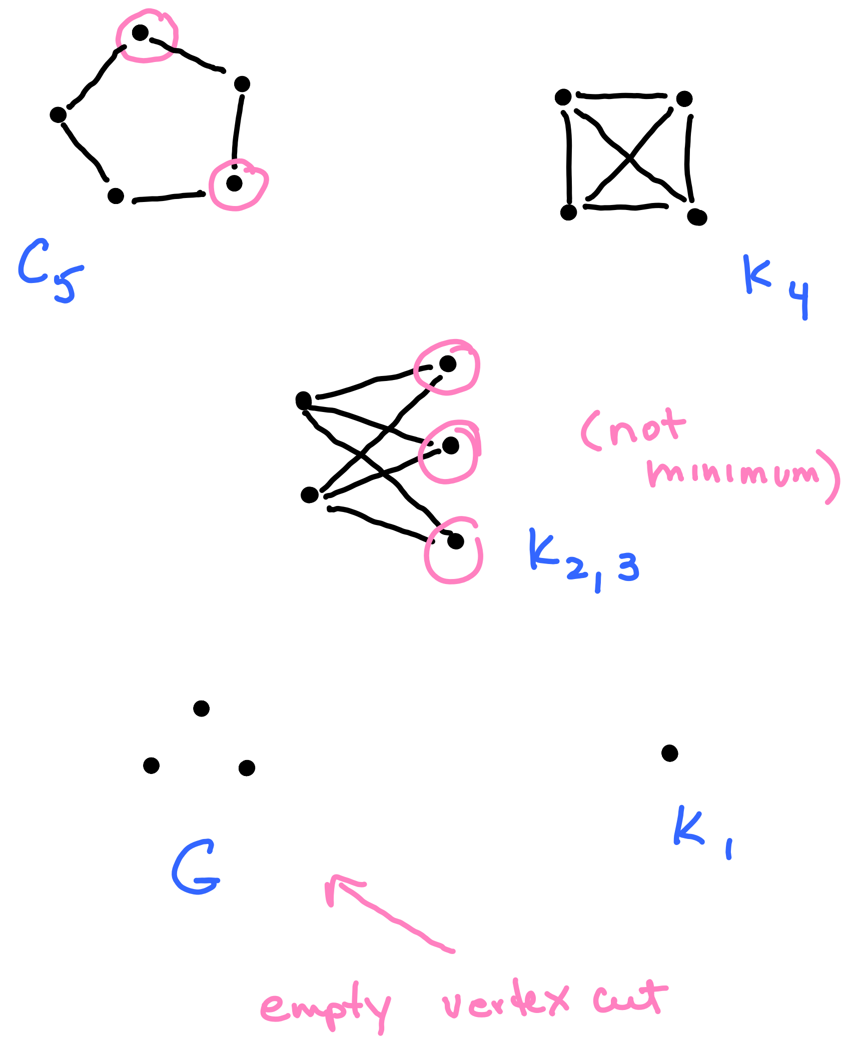


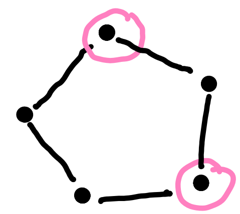
$\kappa(C_5) = 2$

$\kappa(K_4) = 3$

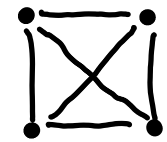
$\kappa(K_{2,3}) = 2$

$\kappa(K_1) = 0$

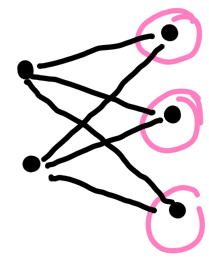
$\kappa(\text{disconnected graph}) = 0$



C_5

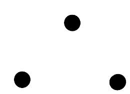


K_4



(not minimum)

$K_{2,3}$



G

K_1

empty vertex cut

$\kappa(C_5) = 2$

$\kappa(K_4) = 3$

$\kappa(K_{2,3}) = 2$

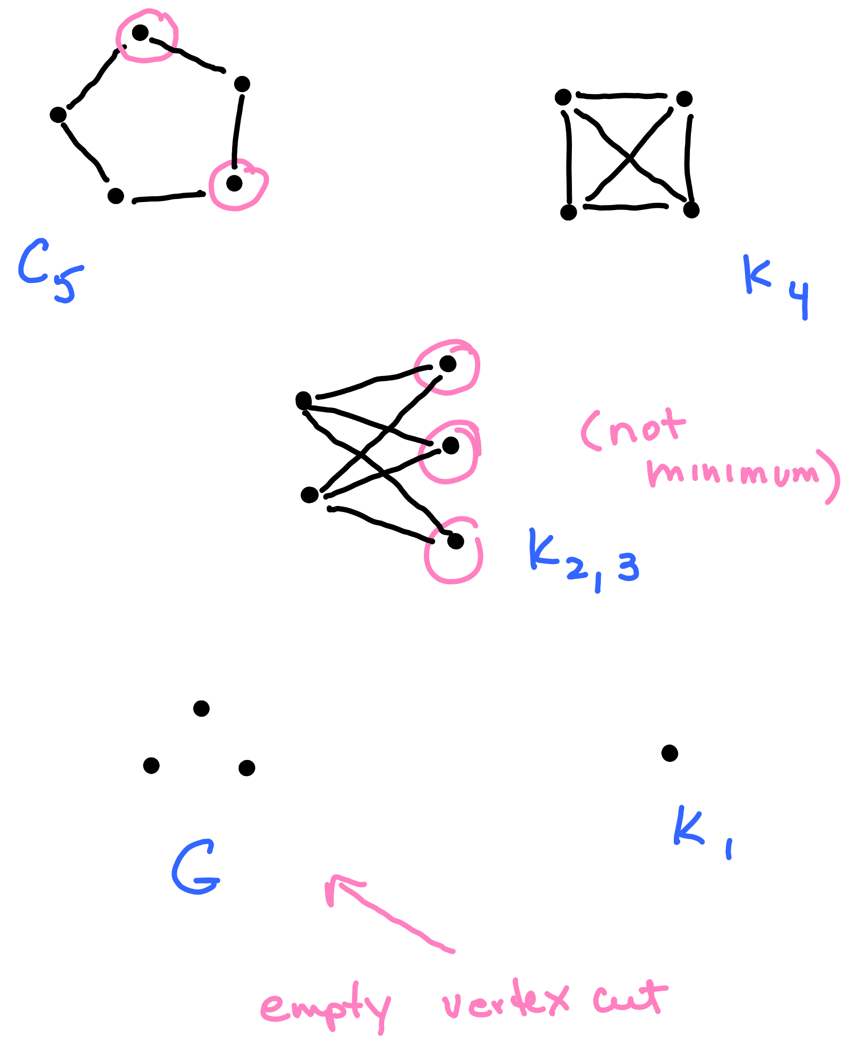
$\kappa(K_1) = 0$

$\kappa(\text{disconnected graph}) = 0$

The connectivity of K_4 is 3

K_4 is 3-connected

K_4 is also 2-connected and
1-connected



Claim: If $m \leq n$, $\kappa(K_{m,n}) = m$.

Proof. Let X, Y be partite sets with $|X| = m$. If $n = 1$, then $m = 1$ and $\kappa(K_{1,1}) = 1$. Otherwise, $|Y| > 2$, so X is a vertex cut of size m.

If $S \subseteq V(K_{m,n})$ with $|S| < |X|$, neither $X - S$ nor $Y - S$ is empty.
So, $K_{m,n} - S$ is a complete bipartite graph and therefore connected.

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Proving connectivity is m requires both

Proof. Let X, Y be partite sets with $|X| = m$. If $n = 1$, then $m = 1$ and $\kappa(K_{1,1}) = 1$. Otherwise, $|Y| > 2$, so X is a vertex cut of size m.

(i) Showing there is a vertex cut of size m (or K_{m+1} is spanning subgraph)

and

If $S \subseteq V(K_{m,n})$ with $|S| < |X|$, neither $X - S$ nor $Y - S$ is empty. So, $K_{m,n} - S$ is a complete bipartite graph and therefore connected.

(ii) Showing there is no vertex cut smaller than m.

Claim: If $m \leq n$, $\kappa(K_{m,n}) = m$.

Proving connectivity is m requires both

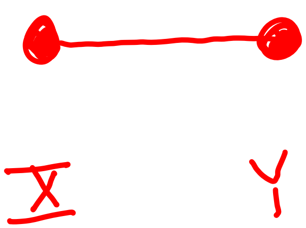
$\kappa(K_{m,n}) \leq m$

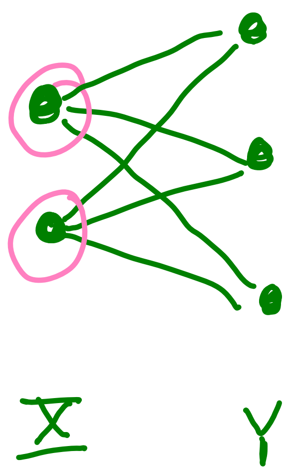
(i) Showing there is a vertex cut of size m

(or K_{m+1} is spanning subgraph)

and

Proof. Let X, Y be partite sets with $|X| = m$. If $n = 1$, then $m = 1$ and $\kappa(K_{1,1}) = 1$. Otherwise, $|Y| \geq 2$, so X is a vertex cut of size m.





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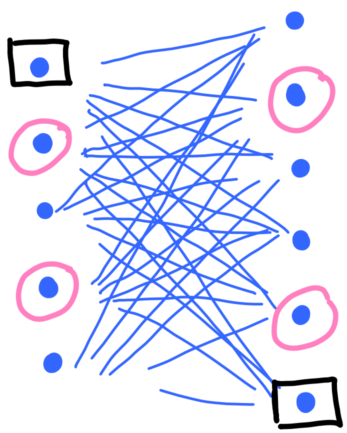


If $S \subseteq V(K_{m,n})$ with $|S| < |X|$, neither $X - S$ nor $Y - S$ is empty. So, $K_{m,n} - S$ is a complete bipartite graph and therefore connected.

(ii) Showing there is no vertex cut smaller than m.

Claim $\kappa(G) \le \delta(G)$

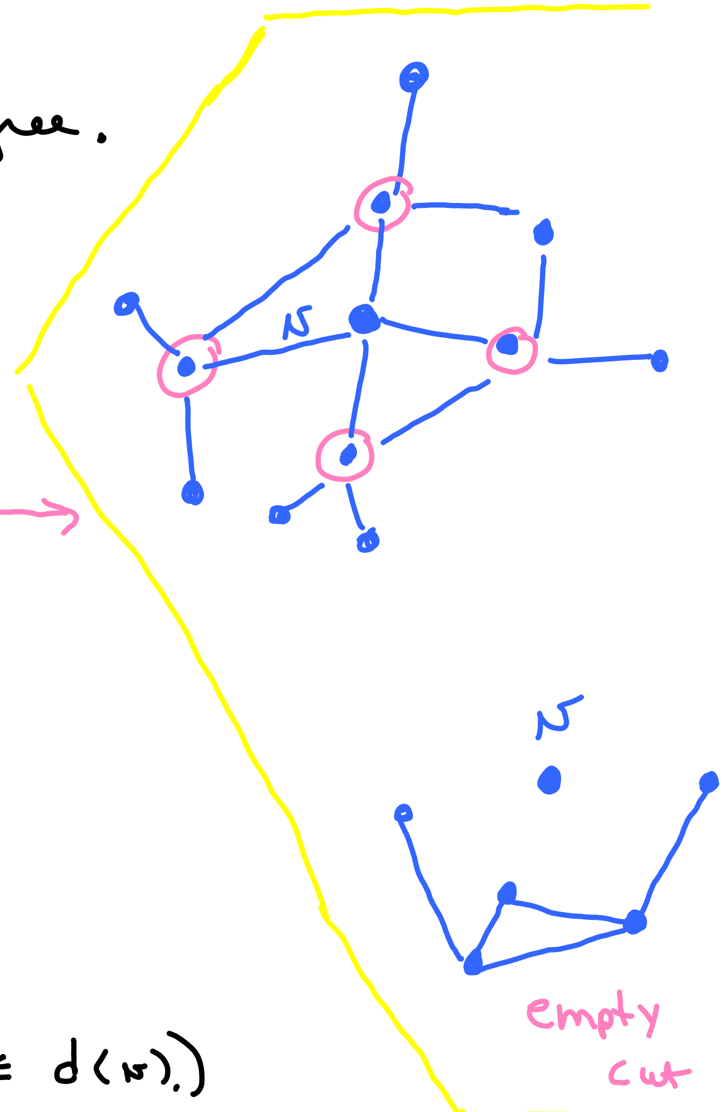
$\delta(G)$ is minimum degree

Why?

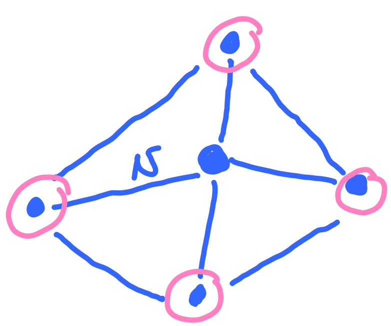
Let v be a vertex of minimum degree.

Then $N(v)$ has size $\delta(G)$

and either $N(v)$ is a vertex cut →



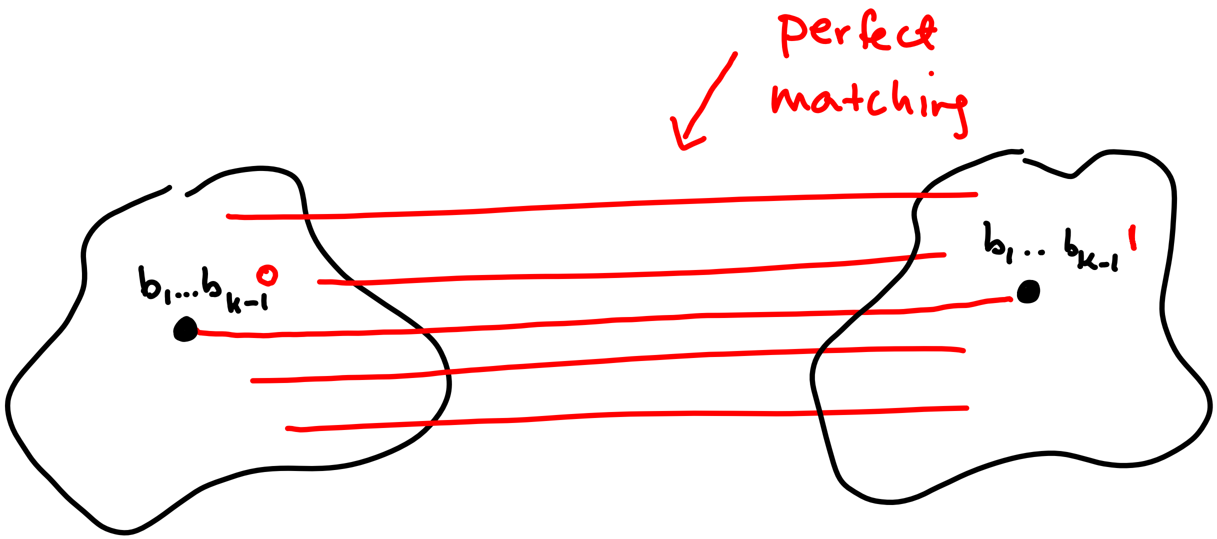
or G had a complete spanning subgraph



(Then $\kappa(G) = n(G) - 1 \le d(v)$.)

Connectivity of Q_k?

View: $Q_k =$



$Q = Q_{k-1} \cdot 0$ $\qquad$ $Q' = Q_{k-1} \cdot 1$

for $k \geq 2$

$2^{k-1} \geq k$

If Q_k had a vertex cut S of size $\leq k-1$, where would the vertices of S be?

Claim: $\kappa(Q_k) = k$

Proof.

($\leq$): $\kappa(Q_k) \leq \delta(Q_k) = k$.

($\geq$): **Induction on** k

If $k = 0, 1$, true.

Let $k \geq 2$ and assume true for $k-1$.

Suppose S is a vertex cut in Q_k with $|S| < k$.

Since $|S| < k \doteq 2^{k-1}$, $Q_k - S$ contains an edge of the matching.

Thus,

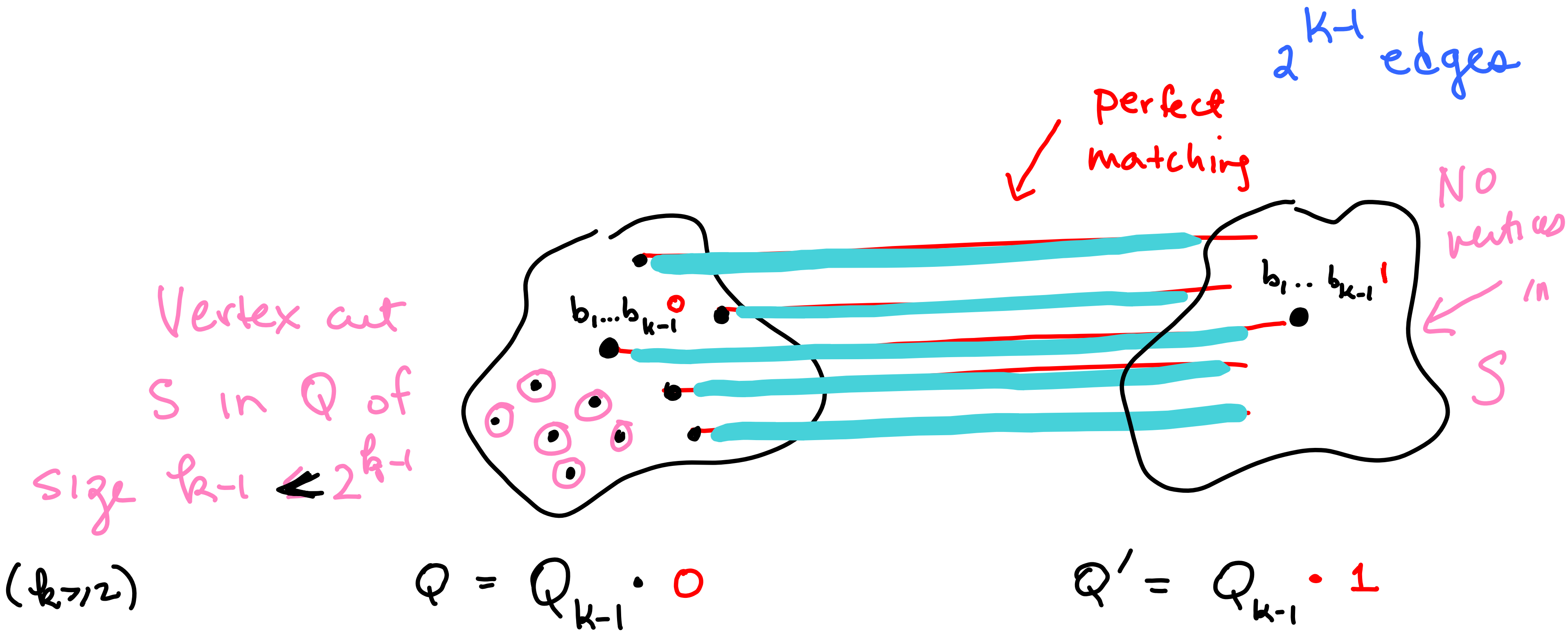
if $Q-S$ and $Q'-S$ are both connected, $Q_k - S$ is connected

So one of them, say $Q-S$ is disconnected. By induction, $|S| \geq k-1$

But then $|S| = k-1$, so $S \subseteq V(Q)$ and $S \cap V(Q') = \emptyset$.

So, all vertices in $Q - S$ are still adjacent to vertices in Q' via edges in the matching.

Thus $Q_k - S$ is still connected.



Next How to construct a graph G

with connectivity k,

but with the minimum possible # of edges?

Next

How to construct an n-vertex graph, G with connectivity k, but with the minimum possible # of edges?

$\overline{2}$

Use:

$$\kappa(G) \leq \delta(G)$$

$$e(G) = \frac{1}{2} \sum_{v \in G} d(v)$$

Next

How to construct an n-vertex graph, G

with connectivity k,

but with the minimum possible # of edges?

Use:

$k = \kappa(G) \leq \delta(G)$

$$e(G) = \frac{1}{2} \sum_{v \in G} d(v) \geq \frac{n * \delta(G)}{2} \geq \frac{nk}{2}$$

How to construct an n-vertex graph, G with connectivity k, but with the minimum possible # of edges?

$$\boxed{e(G) \geq \frac{nk}{2}}$$

Harary graphs $H_{n,k}$ achieve $e(G) = \left\lceil \frac{nk}{2} \right\rceil$

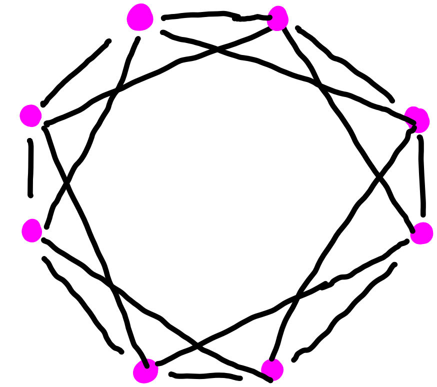
Harary was widely recognized as the "father" of modern graph theory, a discipline of mathematics he helped found, popularize and revitalize. He wrote numerous books and articles, including the 1969 book, "Graph Theory," which has become a modern classic that helped define, develop, direct and shape the field of modern graph theory.

http://www.ur.umich.edu/0405/Jan31_05/obits.shtml

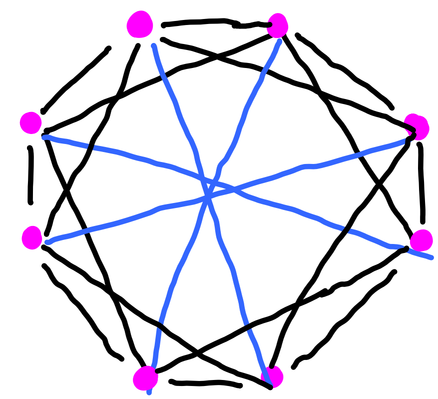
Harary (Photo by Marcia Ledford, U-M Photo Services)

<u>Solution:</u> Harary's $H_{k,n}$ graphs

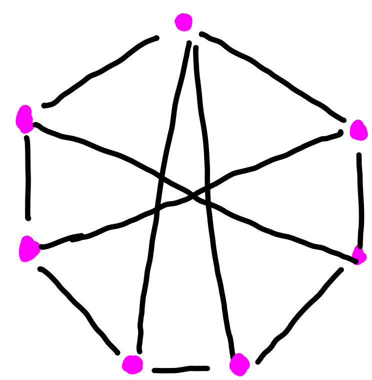
(a) k even: (e.g. $H_{4,8}$)



(b) k odd, n even: (e.g. $H_{5,8}$)



(c) k odd, n odd: (e.g. $H_{3,7}$)



Edge Connectivity

(Assume no loops)

Disconnecting set of edges

subset F of $E(G)$ such that $G - F$ is disconnected.

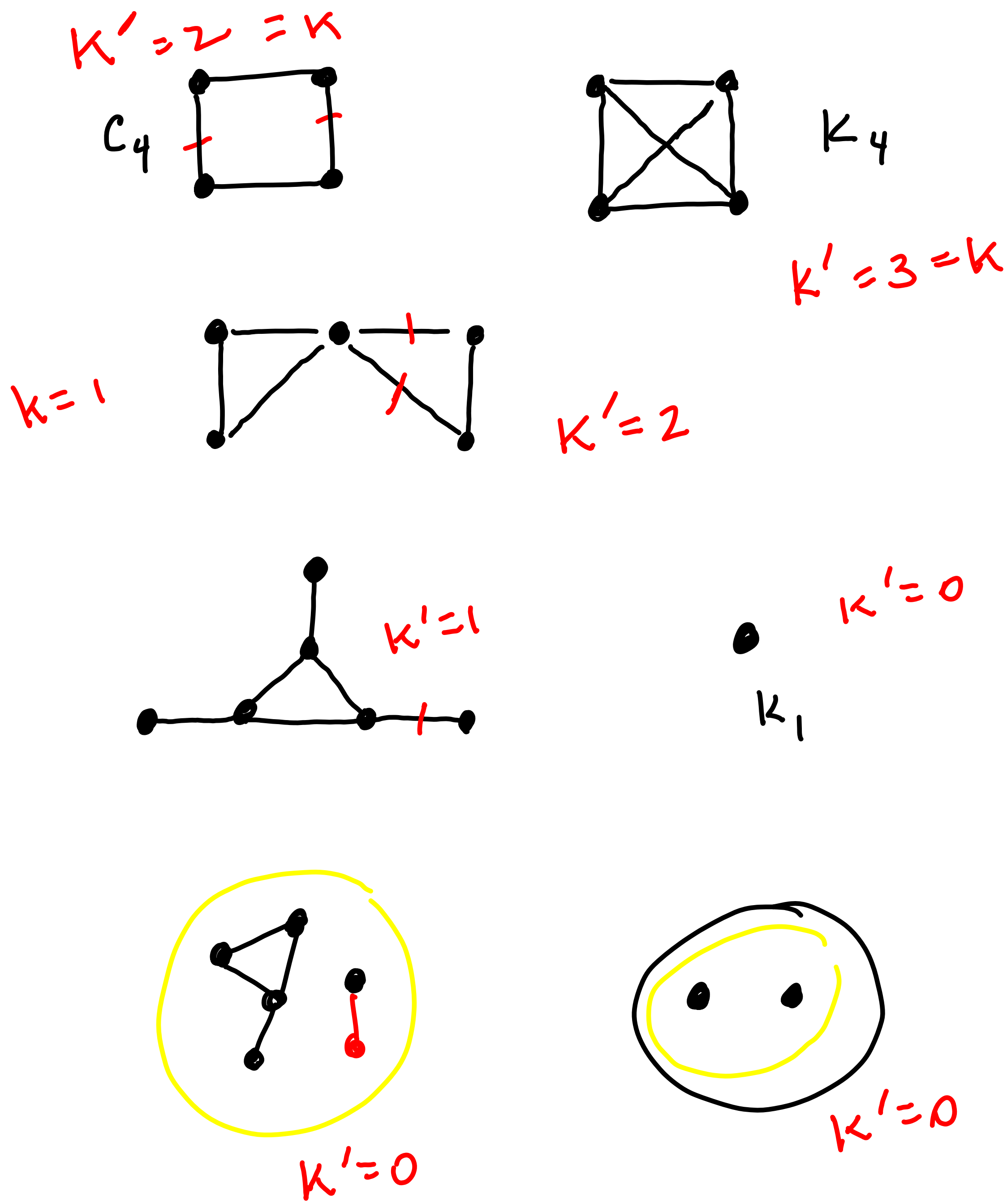
Could a disconnecting set be empty?

Must every graph have a disconnecting set?

$\kappa'(G)$ - **edge connectivity** of G

$$\kappa'(G) = \begin{cases} 0 & \text{if } n(G) = 1 \\ \text{otherwise, the minimum } k \text{ such} \\ \quad \text{that } G \text{ has a disconnecting} \\ \quad \text{set of size } k \end{cases}$$

G **is** k**- edge-connected** if $\kappa'(G) \geq k$.



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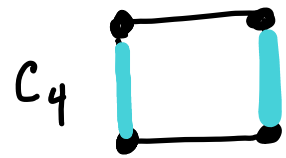
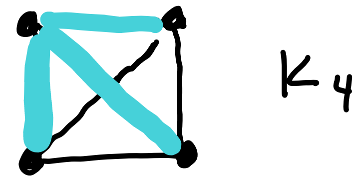
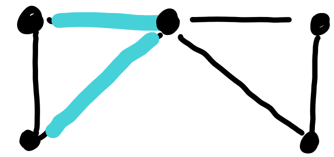
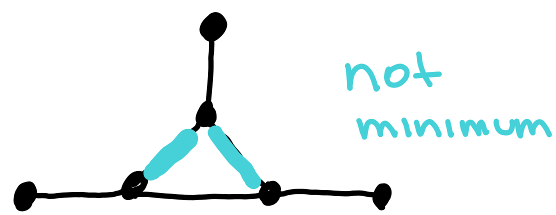
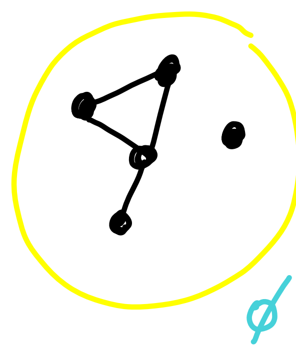
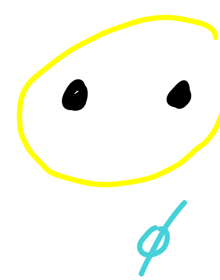




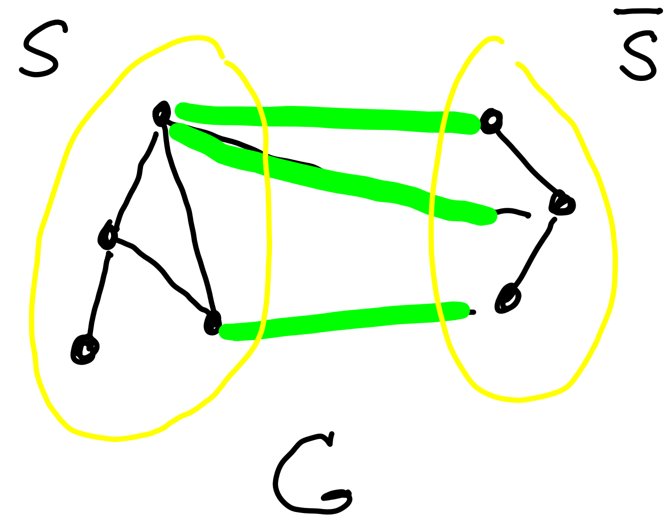








$S, T \subseteq V(G)$

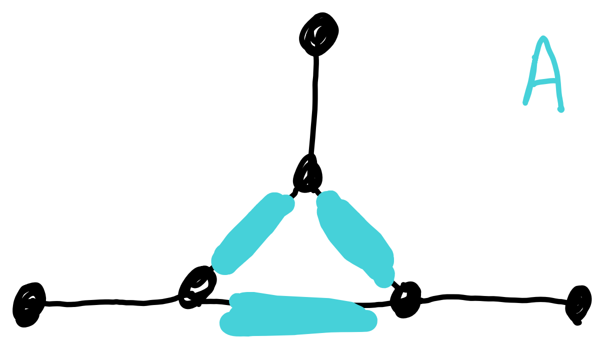


$[S, T] = \{xy \in E(G) \mid x \in S,\ y \in T\}$

edge cut - set of edges $[S, \overline{S}]$ where S is a nonempty proper subset of $V(G)$.

Edge cut $\Rightarrow$ disconnecting set

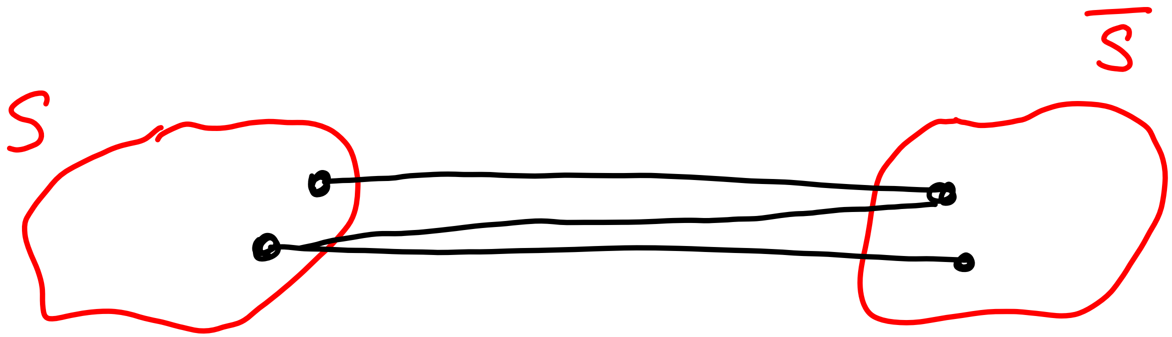
Converse false:



A disconnecting set that is not an edge cut

However: a minimal disconnecting set is an edge cut, if $n(G) > 1$.

Proof. Suppose $F \subseteq E(G)$ is a minimal disconnecting set. Let S be vertex set of one component of $G - F$. Then ...



$$\left.\begin{array}{l}[S,\bar{S}] \subseteq F \\ [S,\bar{S}] \text{ is a disconnecting set}\end{array}\right\} \Rightarrow F = [S,\bar{S}]$$

since F was minimal.

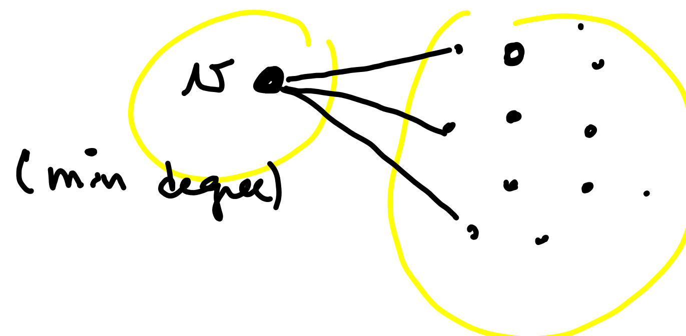
	$\kappa(G)$	$\kappa'(G)$	$\delta(G)$
K_n			
$K_{m,n}$, $m \leq n$			
P_n, $n \geq 2$			
C_n, $n \geq 3$			
Star, $n \geq 2$			

	$\kappa(G)$	$\kappa'(G)$	$\delta(G)$
K_n	$n-1$	$n-1$	$n-1$
$K_{m,n}$, $m \leq n$	m	m	m
P_n, $n \geq 2$	1	1	1
C_n, $n \geq 3$	2	2	2
Star, $n \geq 2$	1	1	1

Relationship

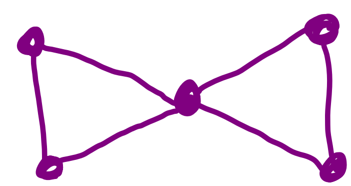
$\kappa(G) \leq \delta(G)$ (done)

$\kappa'(G) \leq \delta(G)$?



yes

$\kappa(G)$ & $\kappa'(G)$?



<u>Theorem 4.1.9</u> [Whitney 1932]

If G is simple, then $\boxed{\kappa(G) \le \kappa'(G) \le \delta(G)}$.

done

Theorem 4.1.9 [Whitney 1932]

If G is simple, then $\boxed{\kappa(G) \leq \kappa'(G) \leq \delta(G)}$.

done

Proof: Remains to show $\kappa(G) \leq \kappa'(G)$

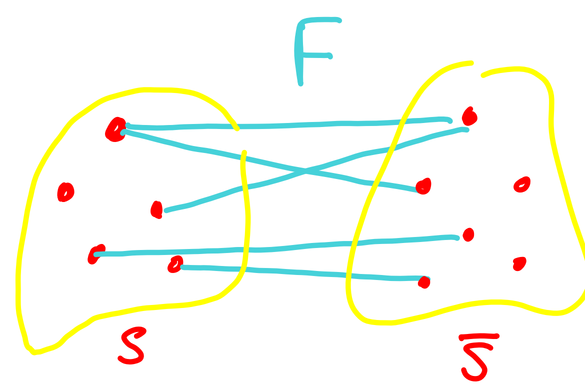
If $\kappa'(G) = 0$, either $n(G) = 1$ or G is disconnected, so $\kappa(G) = 0$.

Else $\kappa'(G) \geq 1$. Let $k = \kappa'(G)$.

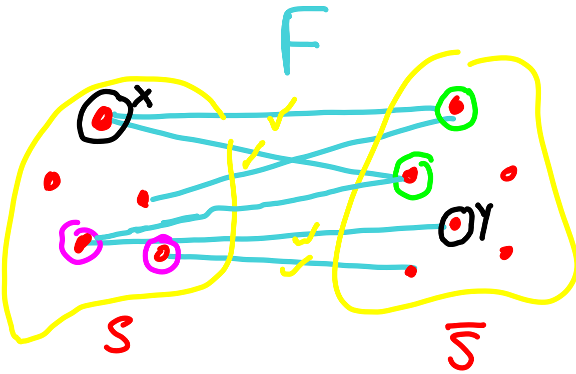
Let F be a disconnecting set of edges of size k.

Then F is minimal (why)

So, F is an edge cut $[S, \bar{S}]$.



(trying to show $\kappa(G) \leq |F|$)

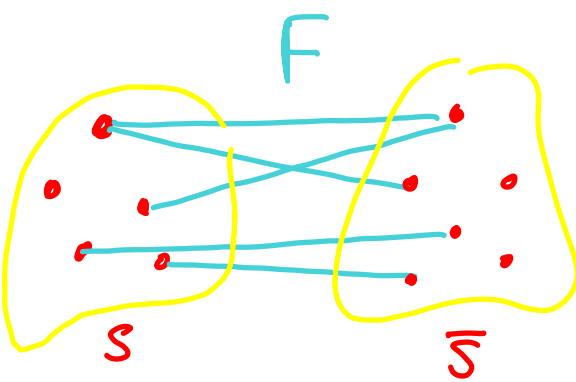


(*) If there is $x \in S$ and $y \in \bar{S}$ s.t. $xy \notin E(G)$, make a vertex cut by taking

$A = N(x) \cap \bar{S}$

B = vertices in $S - \{x\}$ with nbrs in $\bar{S} - A$

Then $A \cup B$ is a vertex cut of size $\leq |F|$



If not (*) then F is all possible edges between S and $\bar{S}$: $|F| = |S|\,|\bar{S}| \geq n-1$

(since $|S|+|\bar{S}|=n$)

and (always) $n-1 \geq \kappa(G)$.

Think: rectangle of perimeter n