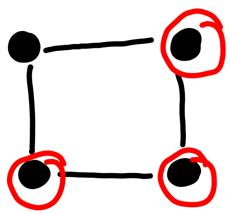
Matchings & Covers (Ch 3)

I. Matchings ✓

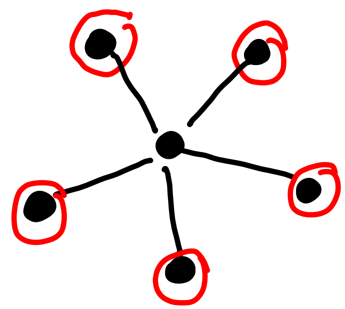
II. Covers

III. Independent Sets & Edge Covers

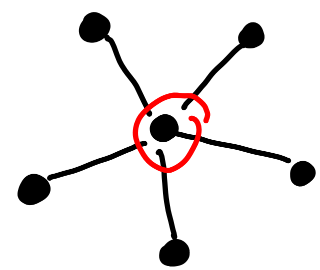
A vertex cover of G is a subset S of $V(G)$ such that every edge of G has at least one end-point in S.



A vertex cover, not minimal



A minimal vertex cover, not minimum



A minimum vertex cover

Problem: Given a graph G, find a minimum vertex cover

Preview:

No polynomial time algorithm known (NP-Hard)

Can approximate within a factor of 2 in polynom. time (easy)

In a bipartite graph, solvable in polynomial time (relate to maximum matching.)

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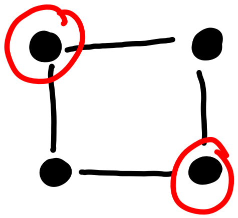

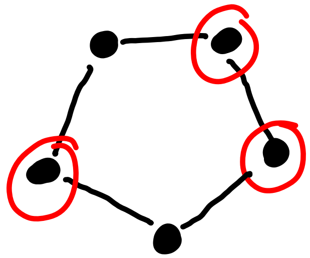

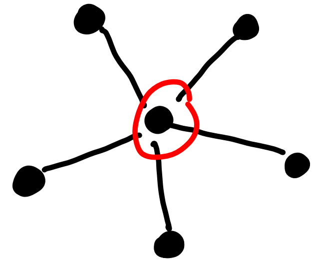

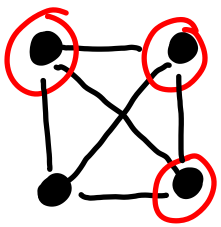

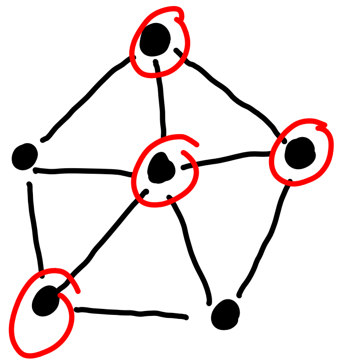

Minimum vertex covers

<u>Note</u>: If S is any vertex cover of G and M is any matching of G, then

$$|S| \geq |M|. \qquad \textbf{(Why?)}$$

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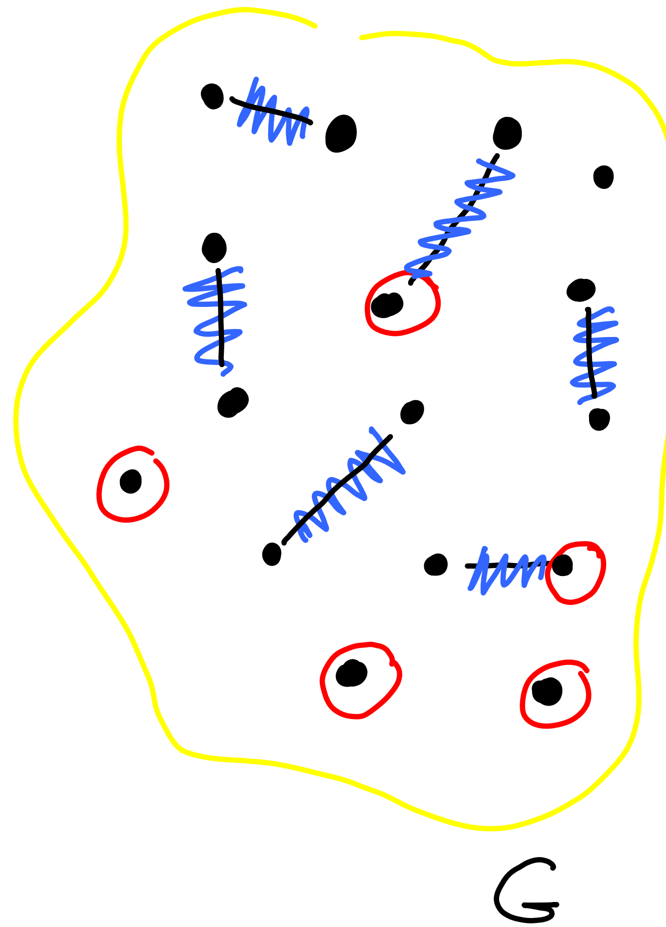

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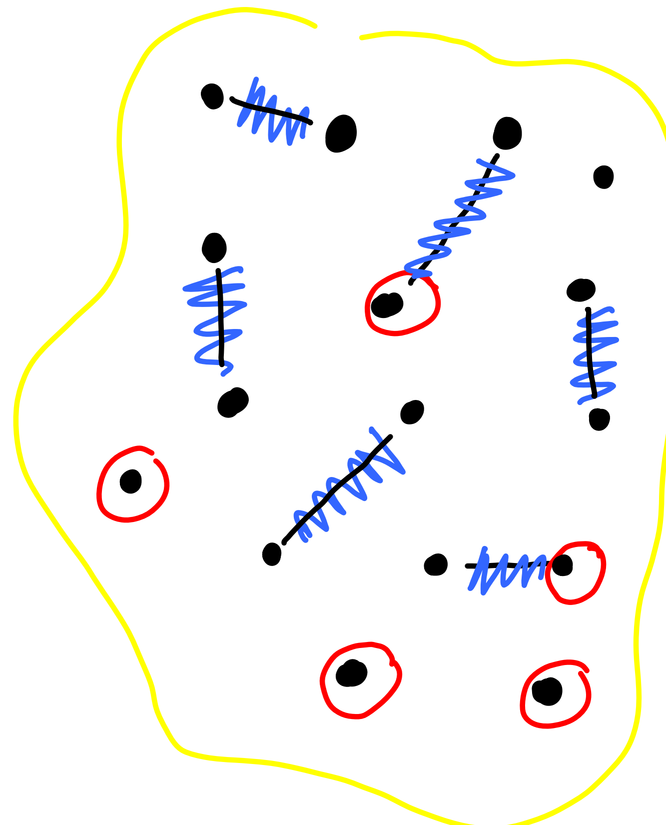
(At least one endpoint of every edge in M must be in S)



G

In particular,

|minimum cover| $\geq$ |maximum matching|.

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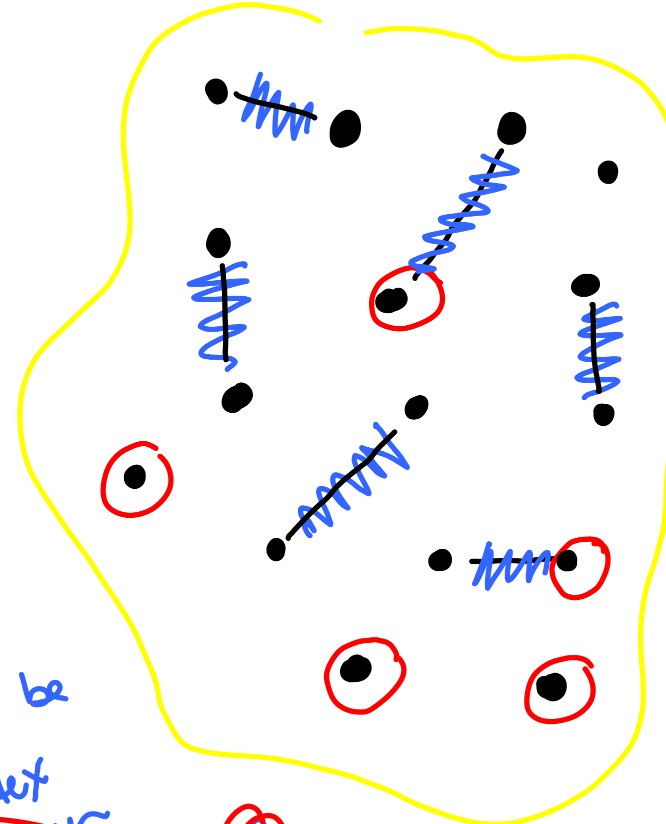

G

Also

In particular,
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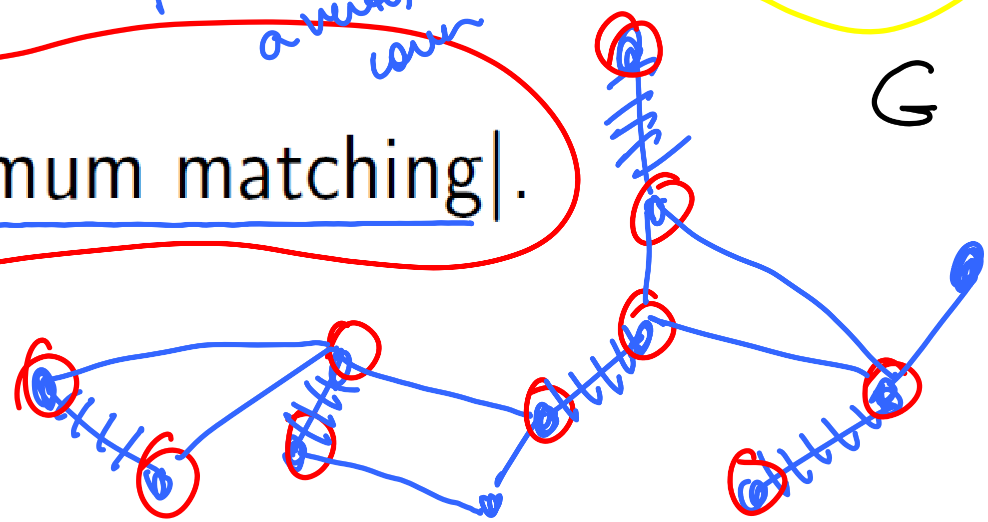
take both endpoints – this must be a vertex cover



So,

$2 * |\text{maximum matching}| \geq |\text{minimum cover}| \geq |\text{maximum matching}|$

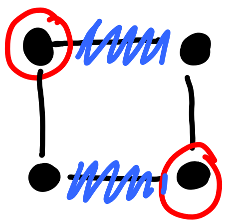
(This means we can approximate minimum vertex cover within a factor of 2!)

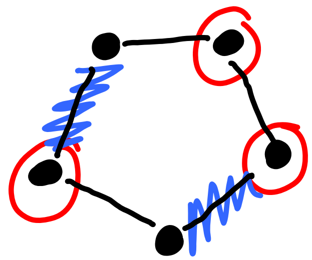
* Note further: "maximum" can be replaced with "maximal" giving an even easier factor of 2 approximation

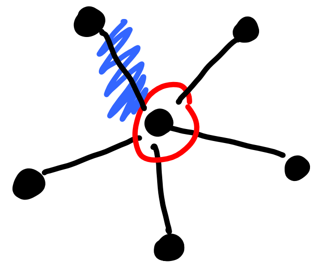
Maximum matching

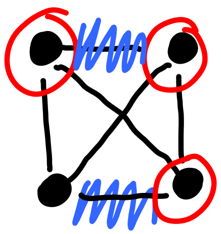
$|MM| \leq |MVC|$

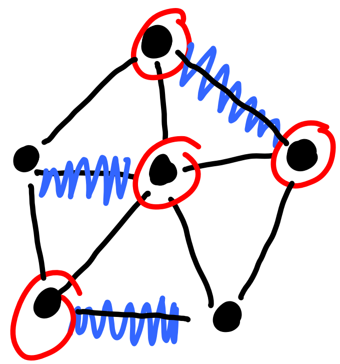
(not always equal)











Minimum vertex covers

but ...

Theorem 3.1.16. [König 1931, Egerváry 1931]: If G is bipartite, the size of a maximum matching in G is the same as the size of a minimum vertex cover of G.

bipartition (X, Y)

Show $|MM| \geq |MVC|$

(We already know reverse)

Proof. Start with a MVC, C.

Find matching M with $|M| = |C|$.

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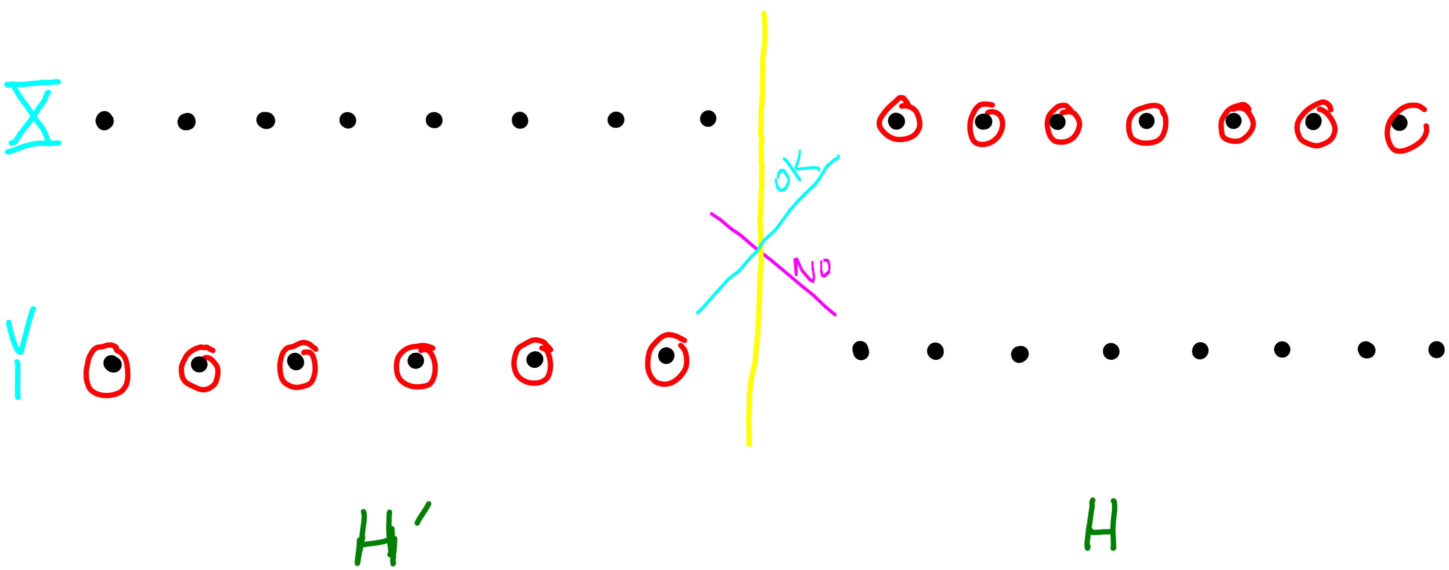

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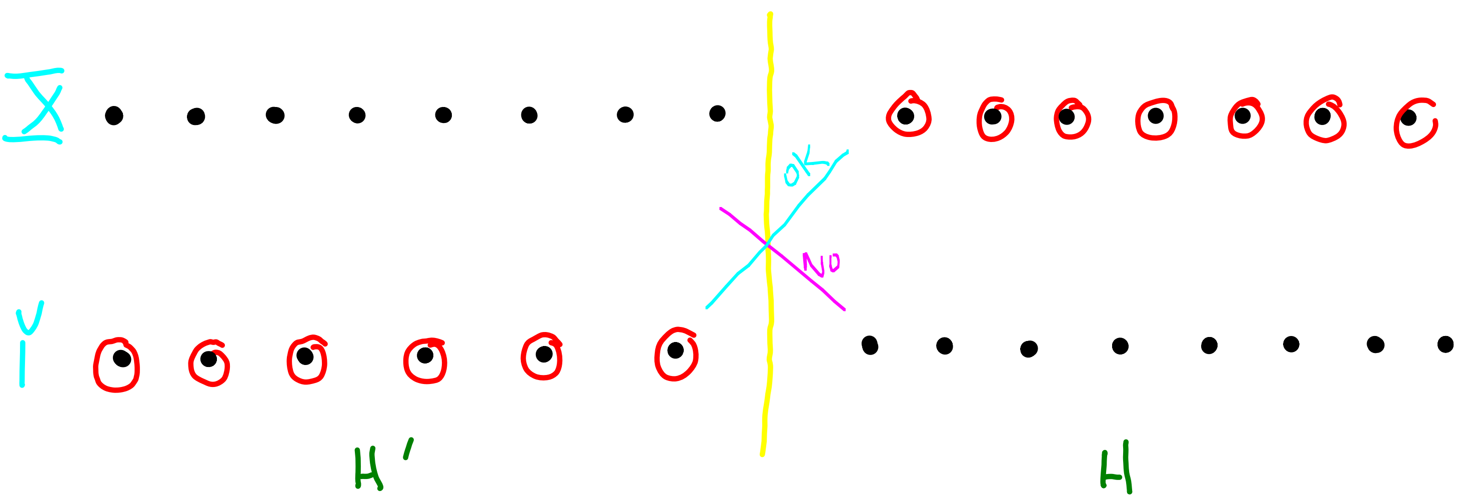

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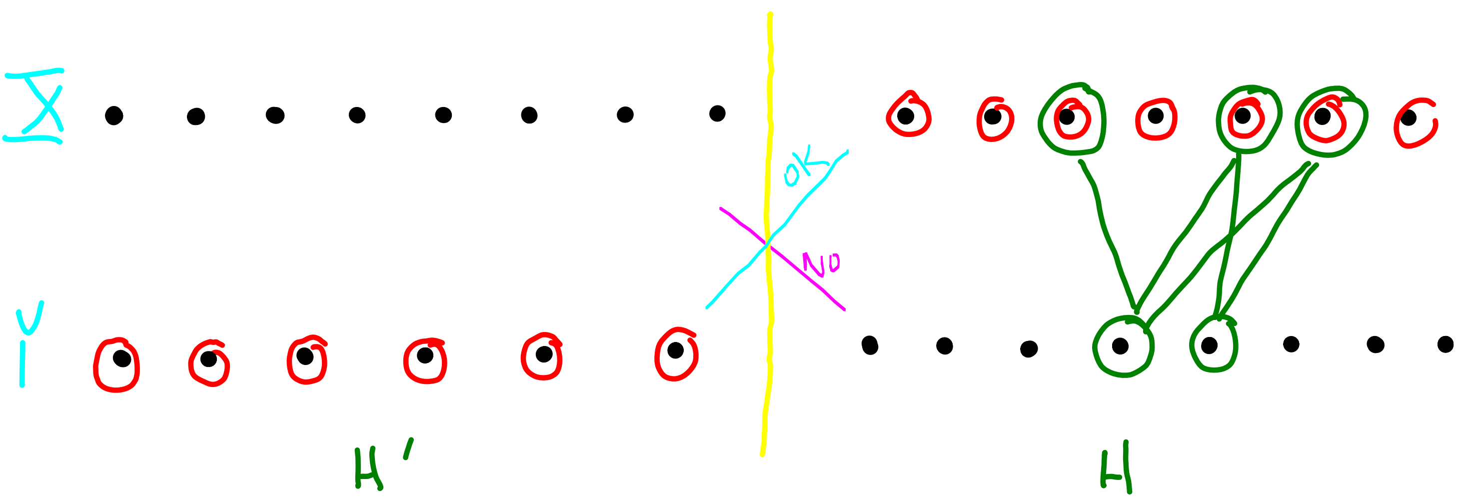

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No (else smaller VC)

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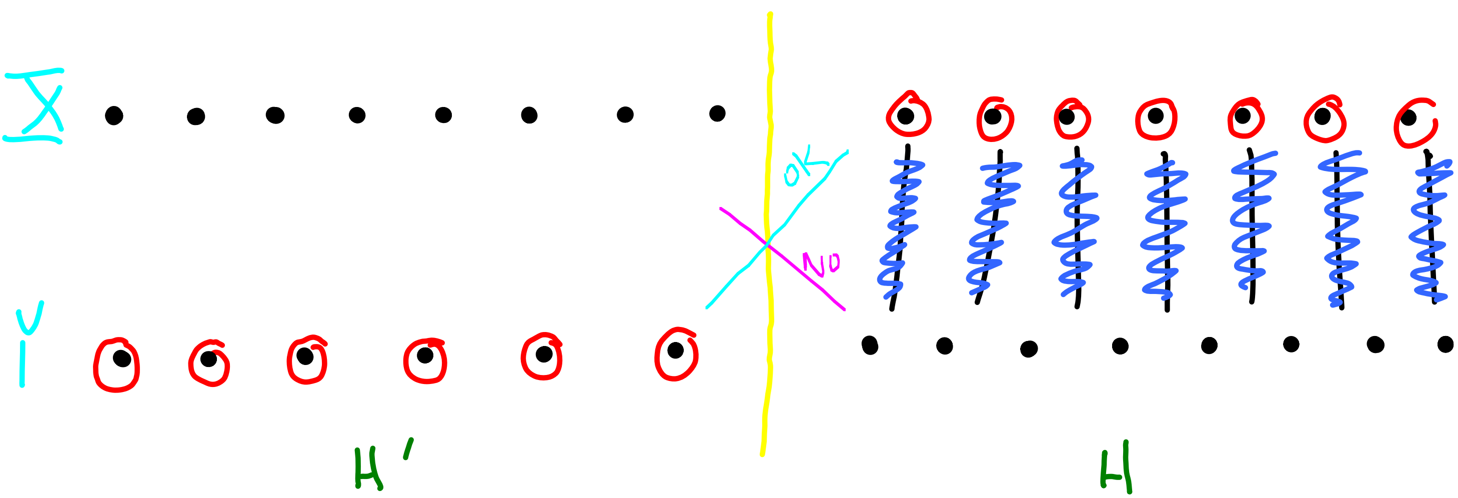

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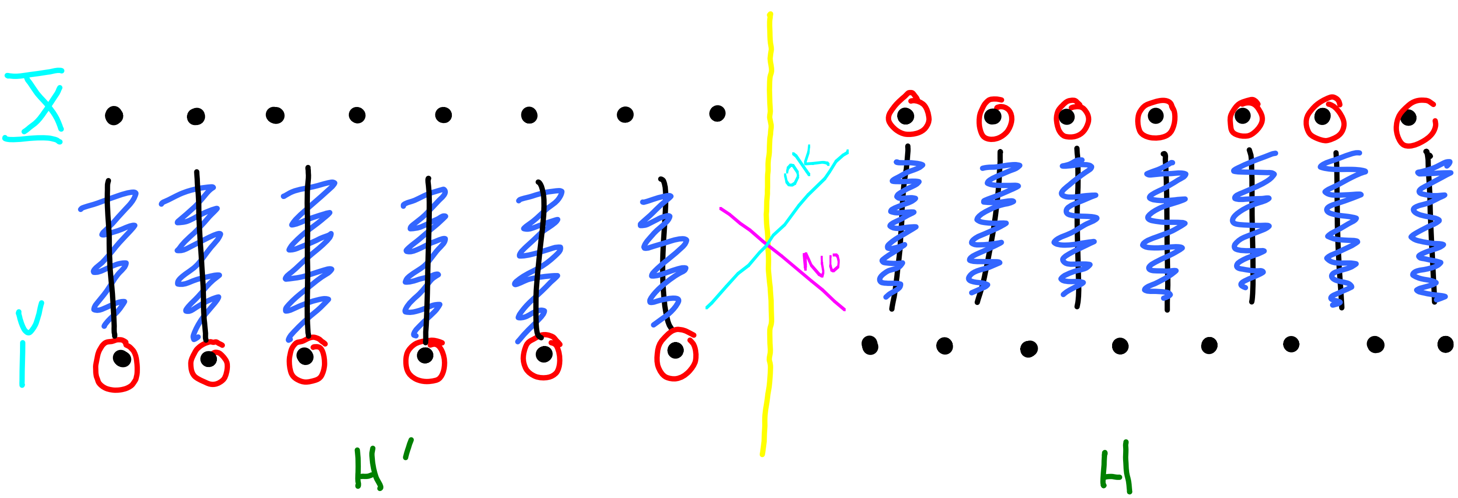

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Similarly for H'. This gives desired M.

Vertices	Edges	
$\alpha(G)$ maximum independent set	$\alpha'(G)$ maximum matching	independence
$\beta(G)$ minimum vertex cover	$\beta'(G)$ minimum edge cover	covering

Matchings & Covers (Ch 3)

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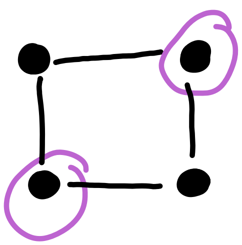
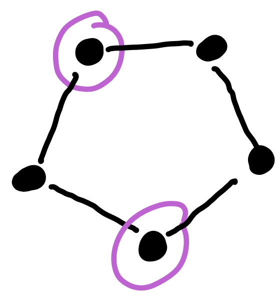
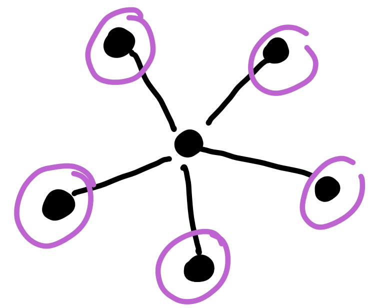
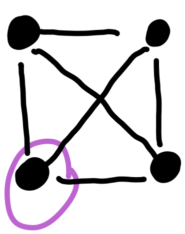
Recall An independent set in G is a subset $S \subseteq V(G)$, no two vertices of S joined by an edge

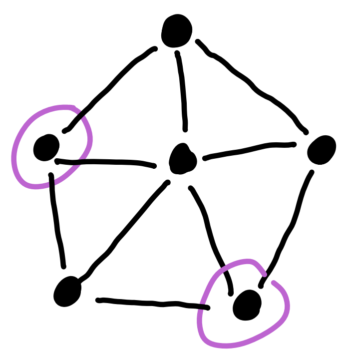
Maximum independent sets











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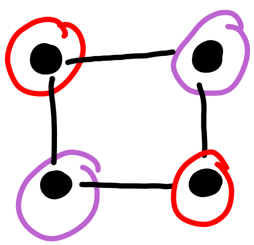
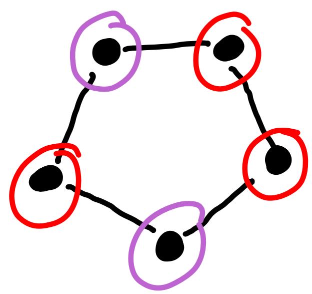
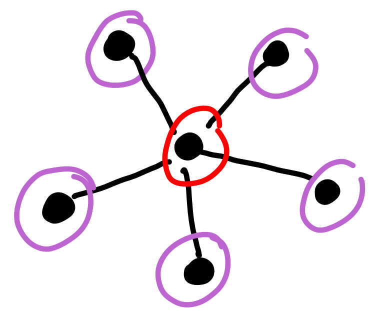
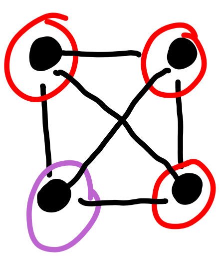
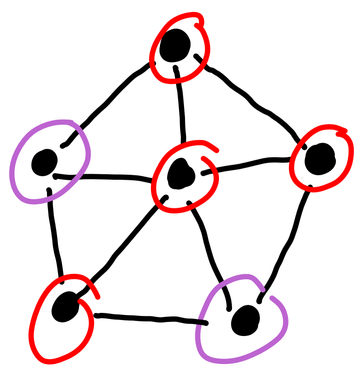
Maximum independent sets











Minimum vertex covers

Lemma 3.1.21

In a graph G, $S \subseteq V(G)$ is an independent set if and only if $V-S$ is a vertex cover.

Pf. S is an independent set iff no edge has both of its endpoints in S, i.e. iff every edge has at least one endpoint in $V-S$; i.e. $V-S$ is a vertex cover.

Vertices	Edges
$\alpha(G)$ maximum independent set	$\alpha'(G)$ maximum matching
$\beta(G)$ minimum vertex cover	$\beta'(G)$ minimum edge cover

independence $\leftarrow$

covering $\leftarrow$

Any graph:

$$\alpha(G) + \beta(G) = n(G)$$

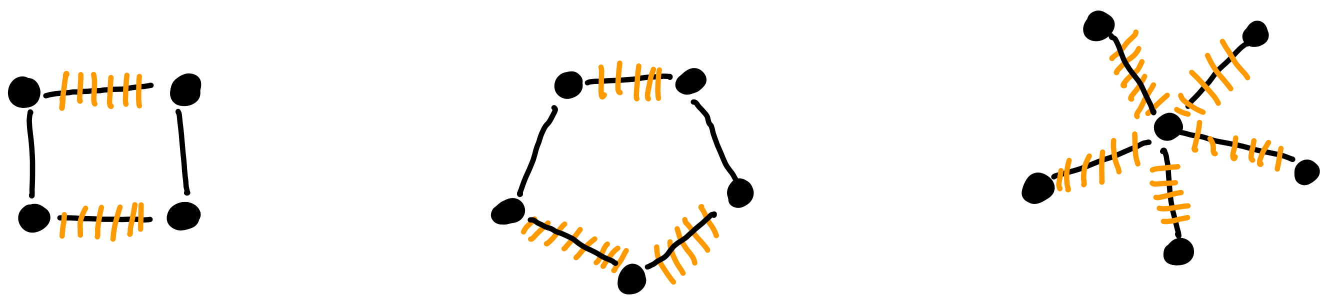
Bipartite:

$$\alpha'(G) = \beta(G)$$

Summary

Edge cover L of G: Subset L of $E(G)$ such that every vertex of G is incident with an edge of L.

Minimum Edge covers



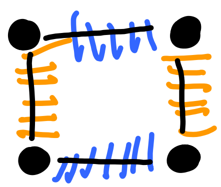
What if G has isolated vertices?

Edge cover L of G: Subset L of $E(G)$ such that every vertex of G is incident with an edge of L.

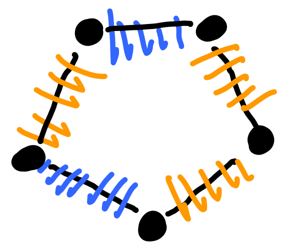
////// Minimum Edge covers

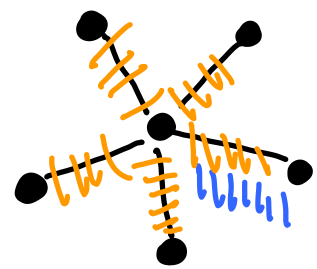
////// Maximum matchings

2 + 2 = 4



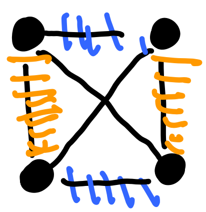
3+2=5

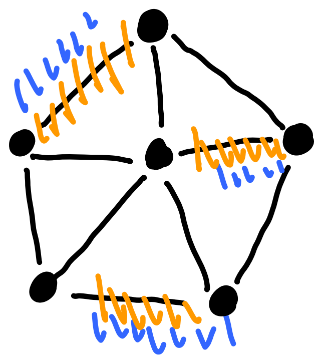




5+1 = 6

2+2 = 4





3+3 = 6

Theorem 3.1.22 [Gallai 1959]

If G has no isolated vertices,

size of maximum matching
+ size of minimum edge cover

= $n(G)$

Corollary 3.1.24 [Konig 1916]

If G is bipartite with no isolated vertices then

size of maximum independent set
= size of minimum edge cover

Vertices	Edges
$\alpha(G)$ maximum Independent set	$\alpha'(G)$ maximum matching
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independence $\longleftarrow$

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Any graph:

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No isolated vertex:

$$\alpha'(G) + \beta'(G) = n(G)$$

Bipartite + no isolated vertex:

$$\alpha(G) = \beta'(G)$$

Summary

Theorem 3.1.22 [Gallai 1959]

If G has no isolated vertices,

size of maximum matching $\alpha'(G)$

+ size of minimum edge cover $\beta'(G)$

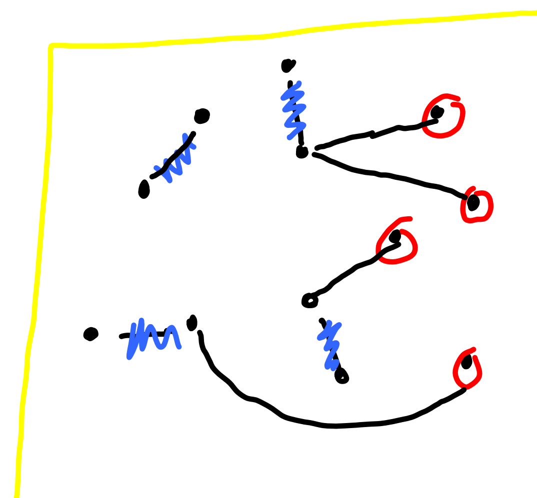
$= n(G)$

(i) $\alpha' + \beta' \leq n$

(ii) $\alpha' + \beta' \geq n$

Proof (i) Let M be a maximum matching in G $(|M| = \alpha')$

Form edge cover L from M and one edge incident to every M-unsaturated vertex.



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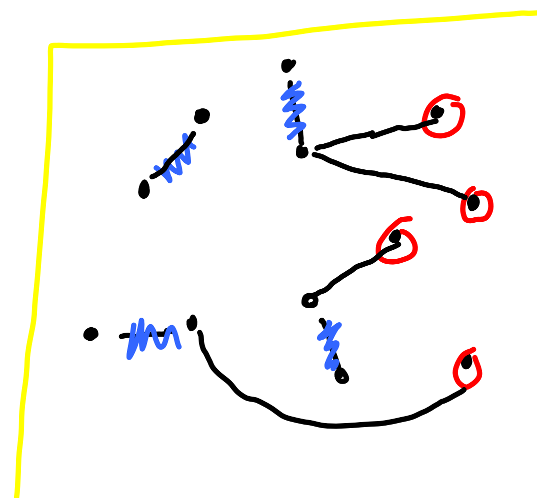
$|L| = |M| + n - 2|M| = n - \alpha'$ so

$\beta' \leq |L| = n - \alpha'$ ✓



Theorem 3.1.22 [Gallai 1959]

If G has no isolated vertices,

$$\begin{array}{rl} & \text{size of maximum matching } \alpha'(G) \\ + & \text{size of minimum edge cover } \beta'(G) \\ \hline = & n(G) \end{array}$$

✓ (i) $\alpha' + \beta' \leq n$

→ (ii) $\alpha' + \beta' \geq n$

Proof (ii) Let L be a minimum edge cover in G $(|L| = \beta')$

Form matching M from L.

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Let H be the subgraph of G induced by the edges of L.

What can H be?

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- no cycles
- no paths of length > 2
- so every component is a star with ≥ 1 edge.

For matching M, take 1 edge from each star.

Now just count

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For matching M, take 1 edge from each star.

Now just count

$|M| = \#$ components of H

$= n - E(H)$

$= n - L = n - \beta'$

so

$\alpha' \geq |M| = n - \beta'$

i.e.

$\alpha' + \beta' \geq n$