Graph Theory HW 5

Akash Agarwal(aagrawa6), Shalki Shrivastava (spshriva), Tanay Pande (tpande)

March 27, 2016

Problem 1

Prove that every simple planar graph has a vertex of degree at most 5.

Proof by contradiction:

Suppose G does not have a vertex of degree at most 5. This means that every vertex of G has degree more than 5. Therefore, if e is the number of edges, then $2e = \sum_{v \in V(G)} d(v) > 5n$. Therefore, $e > 2.5n \implies e \geq 3n$. But the given graph is planar, so we have $e \leq 3n - 6 < 3n$. Thus we have reached a contradiction. Hence, our hypothesis that all vertices have degree greater than 5 is false. Hence proved.

Therefore, in a planar simple graph, there is a vertex of degree at most 5.

Problem 2

Using only the above result, prove that if G is a simple planar graph, then $\chi(G) \leq 6$.

Proof by induction:

We will do induction on the number of vertices n.

Base case: For $n \leq 6$, we can color each vertex of the simple planar graph a different color using ≤ 6 colors.

Induction hypothesis: Let the given hypothesis be true for a simple planar graph G with n vertices. Thus, we can properly color a graph with at most 6 colors.

Now, consider a simple planar graph G' with $n+1$ vertices. From the previous problem, we know that a simple planar graph has a vertex v with degree at most 5. Remove the vertex v from the graph, and properly color the remaining graph with atmost 6 colors using the induction hypothesis. Now add vertex v to the graph. Since, its degree is atmost 5, one of the 6 colors will not be used for any of its neighbors, and that color can be used to color v thus giving us proper coloring for G'.

Hence proved.

Problem 3

Prove that the complement of a simple planar graph with atleast 11 vertices is non-planar. Construct a self-complementary simple planar graph with 8 vertices.

Let G be a simple planar graph. Therefore, we have $e \leq 3n - 6$. If G' is the complement of G, then number of edges in G', $e' = (n * (n-1))/2 - e$.

$e \leq 3n - 6$

$e' = \frac{n(n-1)}{2} - e$

$\therefore e' \geq \frac{n(n-1)}{2} - (3n - 6)$

For G' to be planar, we should have $e' \leq 3n - 6$

$\therefore e' \leq (3n - 6)$

$\therefore \frac{n(n-1)}{2} - e \leq (3n - 6)$

$\therefore \frac{n(n-1)}{2} - (3n - 6) \leq e$ and $e \leq (3n - 6)$

$\therefore \frac{n(n-1)}{2} - (3n - 6) \leq e \leq (3n - 6)$

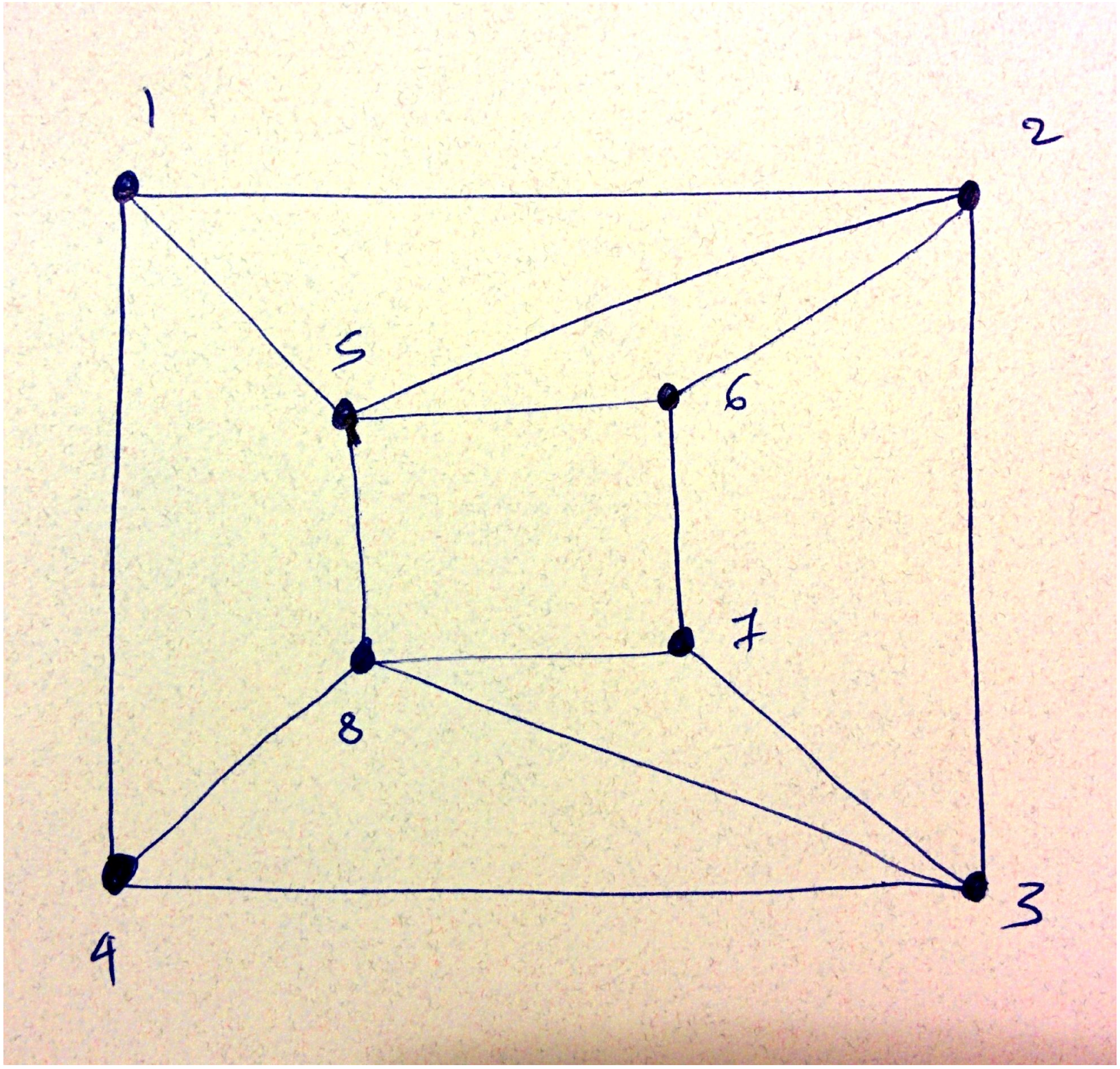
$\therefore \frac{n(n-1)}{2} - 2(3n - 6) \leq 0$

$\therefore n^2 - 13n + 24 \leq 0$

$\therefore n \in [2, 11)$

Thus, for a simple planar graph G, its complement raph is planar iff the number of vertices is in the interval $[2, 11)$. Therefore, if the number of vertices in a simple planar graph is 11 or more, then its complement is non-planar. Hence proved.

This is a self-complementary graph with 8 vertices.



Problem 4

Let G be a triangulation, and let n_i be the number of vertices of degree i in G. Prove that $\sum(6-i)n_i = 12$.

Since G is a triangulation, then this implies that G is a maximal planar graph. Thus, addition of even a single edge will make G non-planar.

Therefore, $e = 3n - 6$.

$\therefore \sum_i n_i(i) = 2e = 2(3n-6)$ where n_i is the number of vertices of degree i.

Also, $n = \sum_i n_i$

$\therefore \sum_i n_i(i) = 2(3n-6) = 6n-12$

$\therefore \sum_i n_i(i) = 6\sum_i n_i - 12$

$\therefore \sum_i n_i(i) - 6\sum_i n_i = -12$

$\therefore \sum_i n_i(6-i) = 12$

Hence proved.

Question 5

What is the maximum number of edges (as a function of the number of vertices) in a simple planar graph of girth 5? Use this to prove that the Petersen graph is not planar.

Solution

Let G be a n vertex simple planar graph of girth k. Since adding edges will not change the girth, the graph can be assumed to be connected. Let the graph have f faces and each face will have a length of at least k(i.e. The girth). The edges on the face boundaries will count twice.

So total number of edges e including the in face edges will follow the inequality 2e $\geq$ kf. Since G is connected we will use Eulers formula n-e+f = 2.

Substituting previous inequality we get

2e $\geq$ k(2+e-n)

2e $\geq$ 2k+ke-kn

2e-ke $\geq$ 2k-kn

Ke-2e $\leq$ kn-2k

e(k-2) $\leq$ k(n-2)

e $\leq$ k(n-2)/ (k-2)

We are given girth as 5. So maximum number of edges (in terms of number of vertices) is e $\leq$ 5/3 (n-2)

Petersen graph has 10 vertices, 15 edges and girth of 5.

By putting the the number of vertices in above equation we get

e $\leq$ 5/3 (8) = 40/3 = 13.33

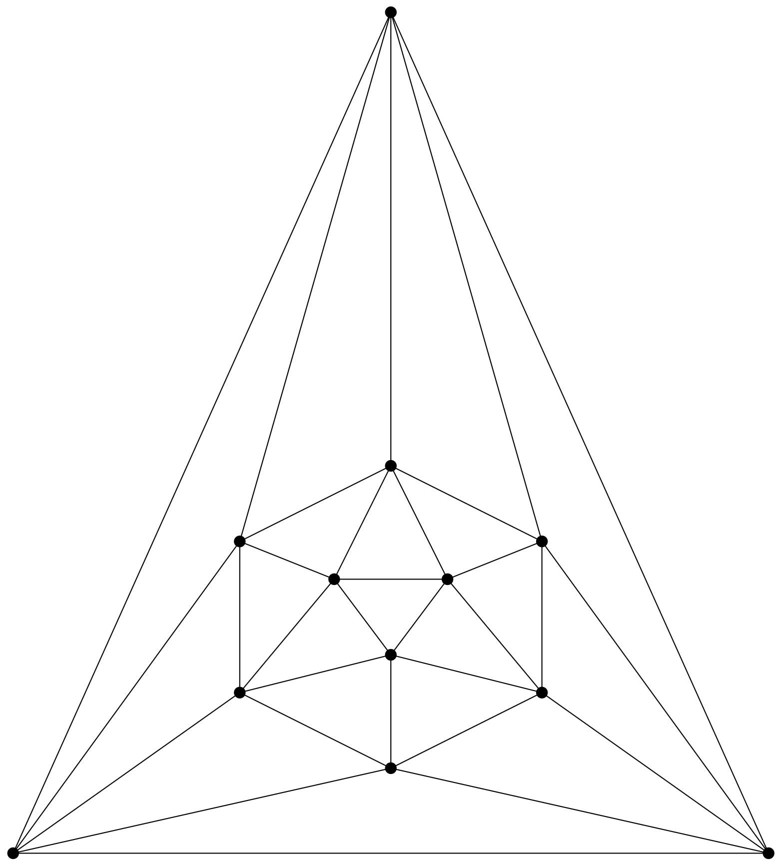
So, maximum number of edges for a simple planar graph with 10 vertices and girth 5 can be 13. Petersen graph has 15 edges. Hence Petersen graph cannot be planar

Question 6

Give an example of a simple planar graph with minimum degree 5.

Solution

An icosahedron graph gives us a simple planar graph with minimum degree of 5.

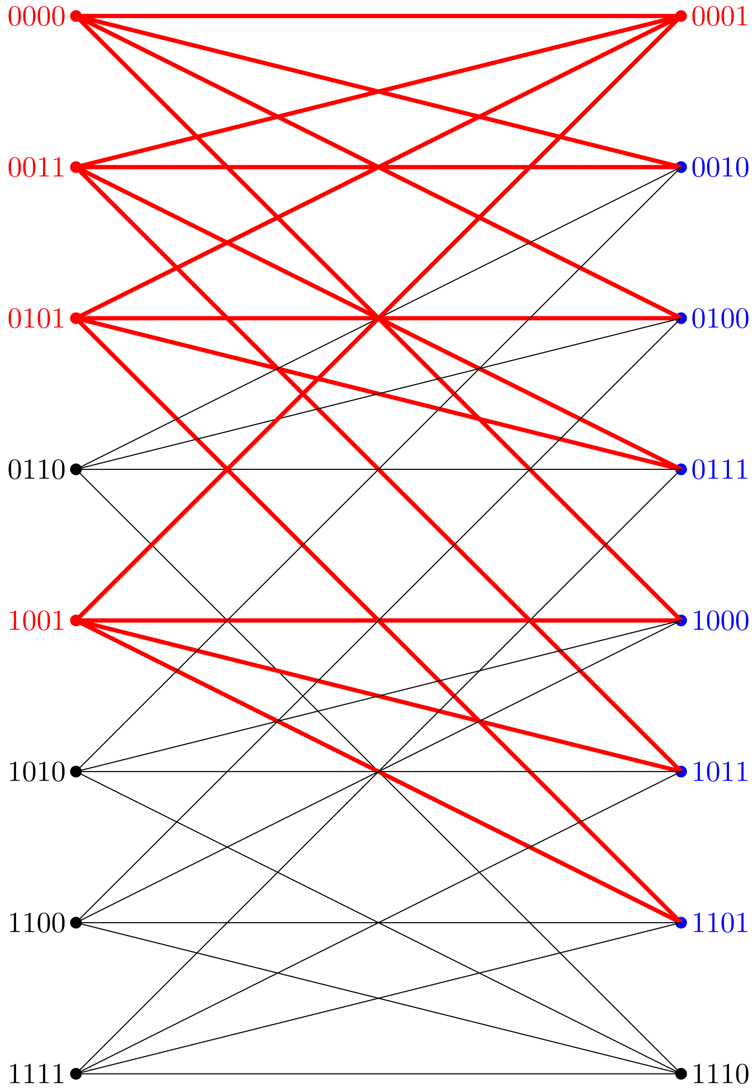


Question 7

Find, if possible, a subdivision of K5 in Q4.

Solution

In the graph of Q4, we can find a subdivision of K5 using vertex set [0001,0000,0011,0101,1001] (red vertices)

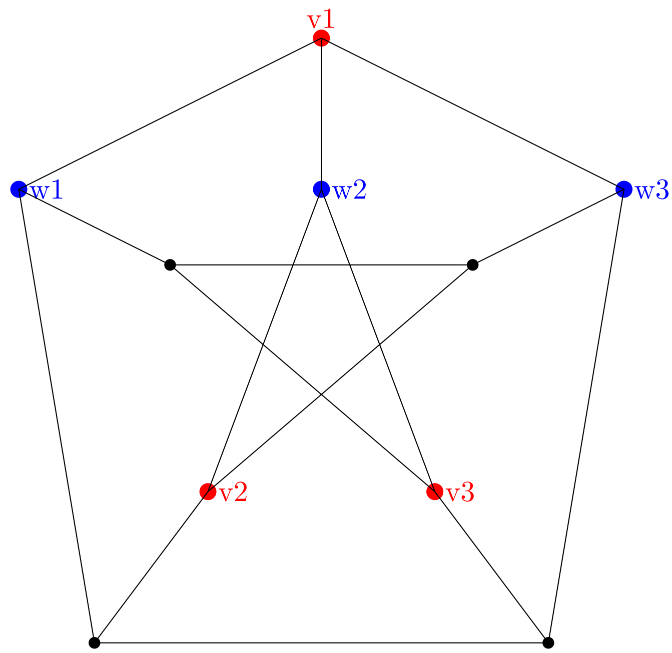


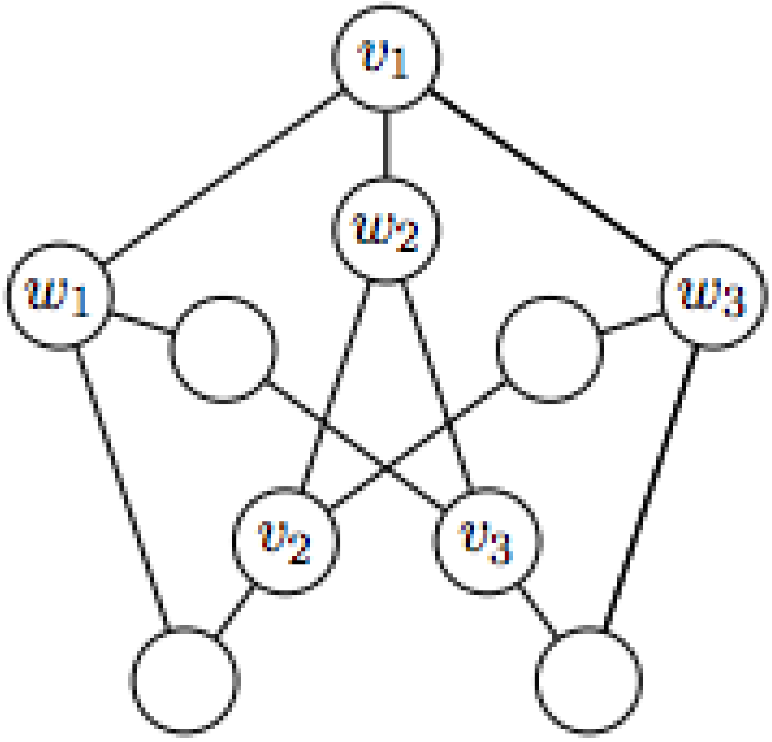
Question 8

Find, if possible, a subdivision of K3,3 in the Petersen graph.

Solution

Subdivision [v1,v2,v3,w1,w2,w3] will give us a K3,3 for a Petersen graph.


v1
w1
w2
w3
v2
v3



Problem 9

For $r \geq 5, \forall n$, Turan graph will contain K_5 as subgraph, so it will not be planar since it will have at least 5 complete partite sets of size at least 1.

For $r \geq 2$, if any 2 of the subsets contain at least 3 vertices, then it will contain $K_{3,3}$ as sub-graph, so that won't be planar again.

Thus, Turan graph is planar for:

$r = 1, \forall n$,

$r = 2, n \leq 5$,

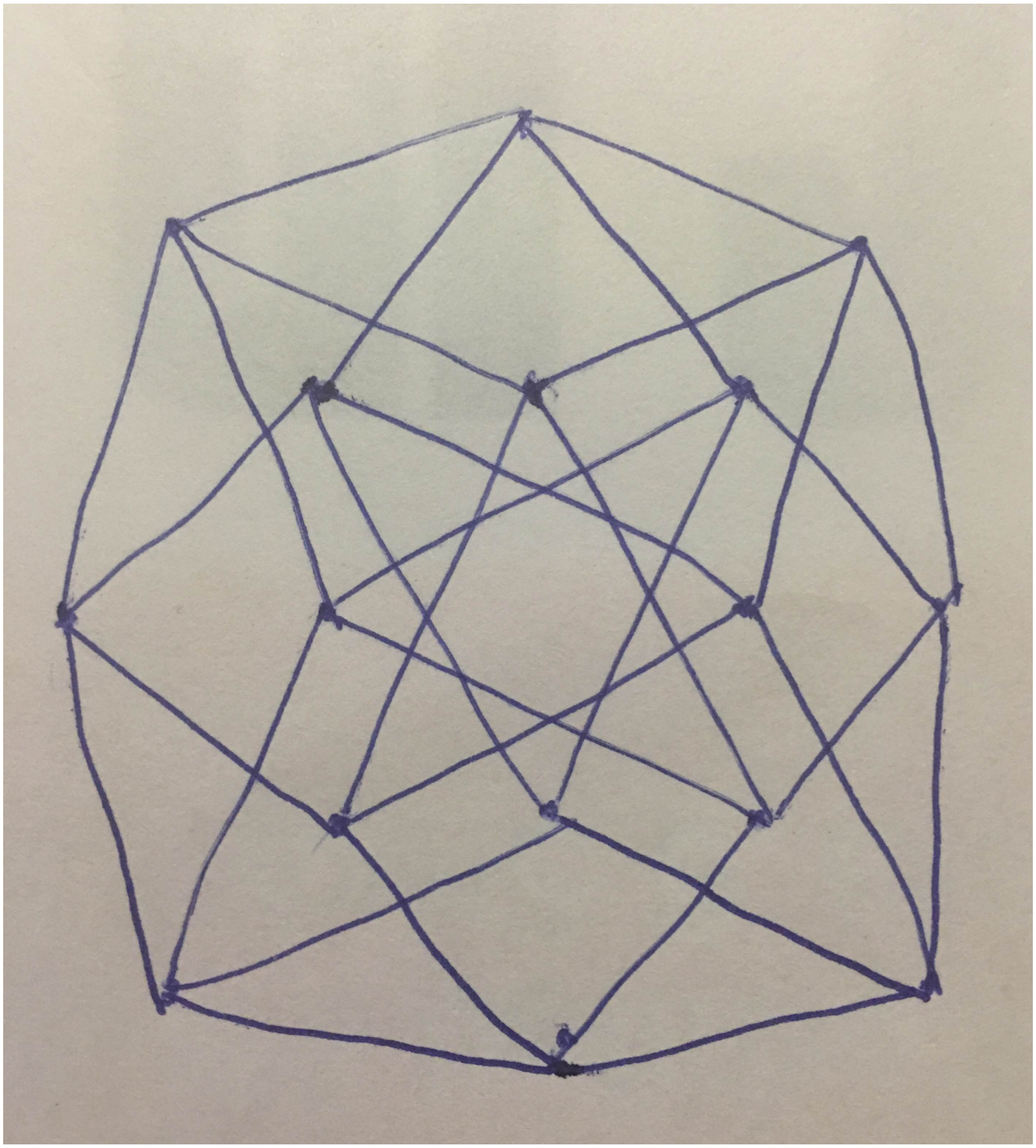
$r = 3, n \leq 7$, and

$r = 4, n \leq 9$.

Problem 10

Thickness of a graph G h is the minimum number of planar graphs in a decomposition of G into planar graphs. So, thickness of every non planar graph must be at least 2. Every planar graph has thickness 1. Also, by Proposition 6.3.10, A simple graph G with n vertices and m edges with no triangles has thickness at least $m/(2n-4)$.

Thickness of Q_4

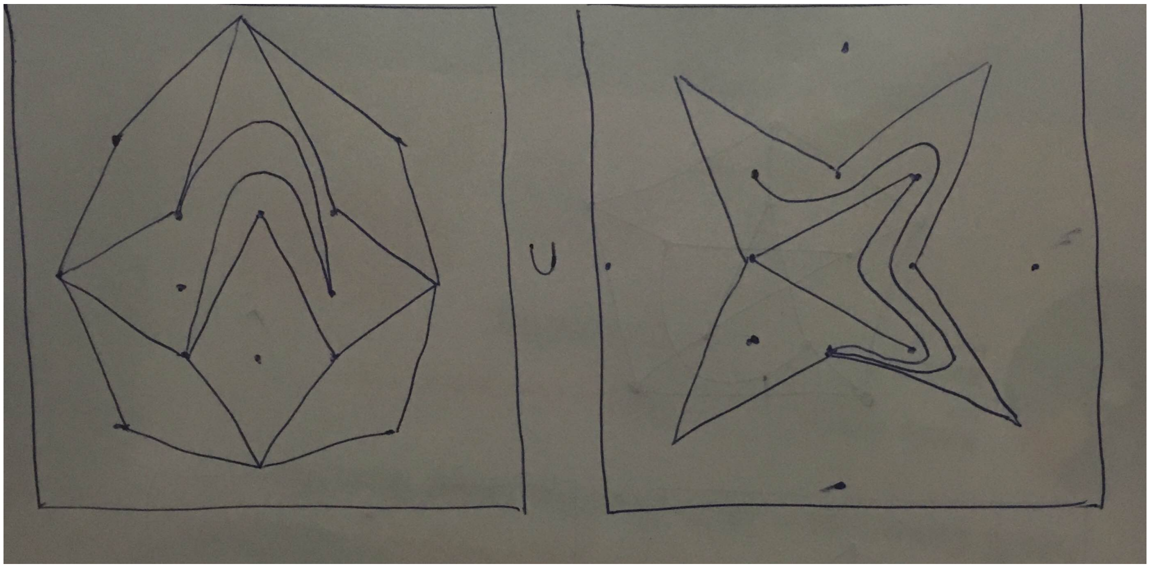


Number of vertices in Q_4 are 16 and number of edges are 32. By theorom 6.1.23, if G is a simple planar graph with at least 3 vertices and if G is also triangle free, then G can have at most 2n-4 edges.

Since Q_4 is triangle free, 2n - 4 = 2*16 - 4 = 28. But since Q_4 have 32 edges so we know that Q_4 is not planar and hence the thickness of Q_4 is greater than 1.

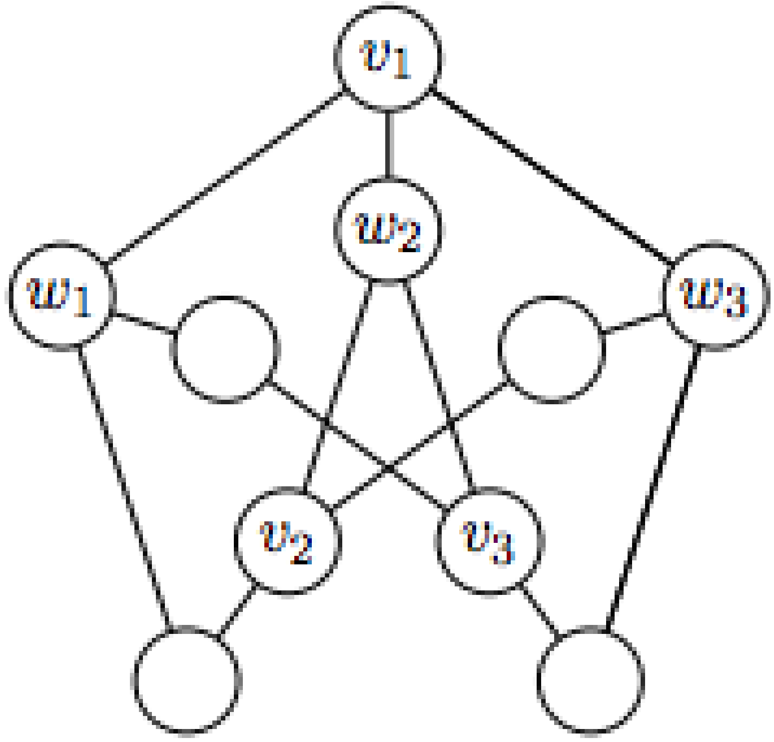
$(m/2n-4) = 32/28 > 1$. So thickness should be at least 2.

Since Q_4 is the union of below two:

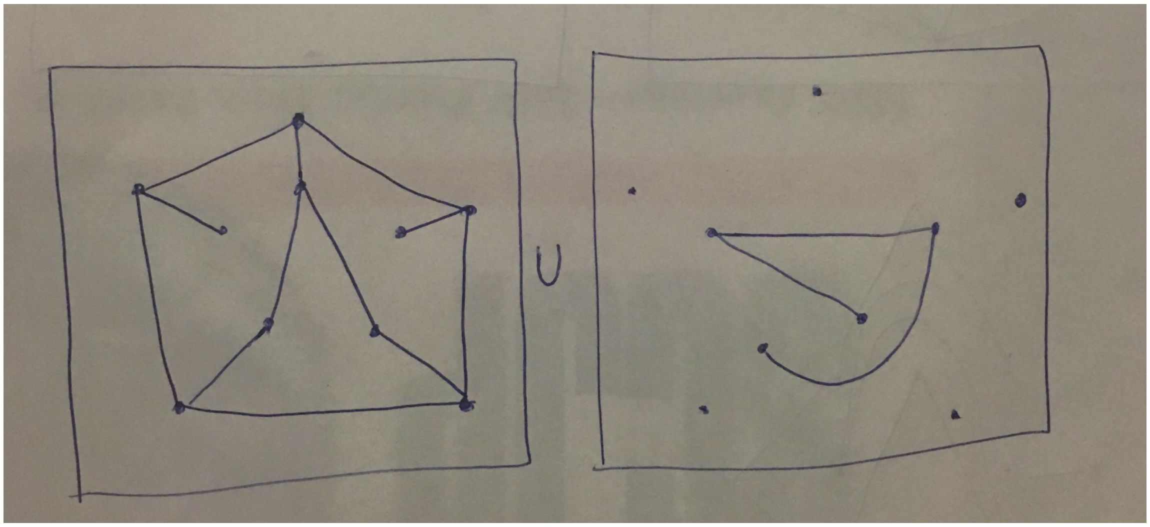


So, thickness of Q_4 is 2.

Thickness of Petersen graph We want to prove that Petersen graph is not planar. Petersen graph contains the subdivision of K_33. If we do the subdivision of petersen graph in the following manner with two partite sets V_1, V_2, V_3 and W_1, W_2,W_3 we can see that it contains K_33.



Hence, by Kuratowski's theorom Petersen graph is non-planar. So, the thickness of Petersen graph should be at least more than one. Since, we are able to decompose Petersen graph into two planar graphs so thickness of Petersen graph is 2. Since Petersen graph is the union of below two:



So, thickness of Petersen graph is 2.