Graph Theory HW 4

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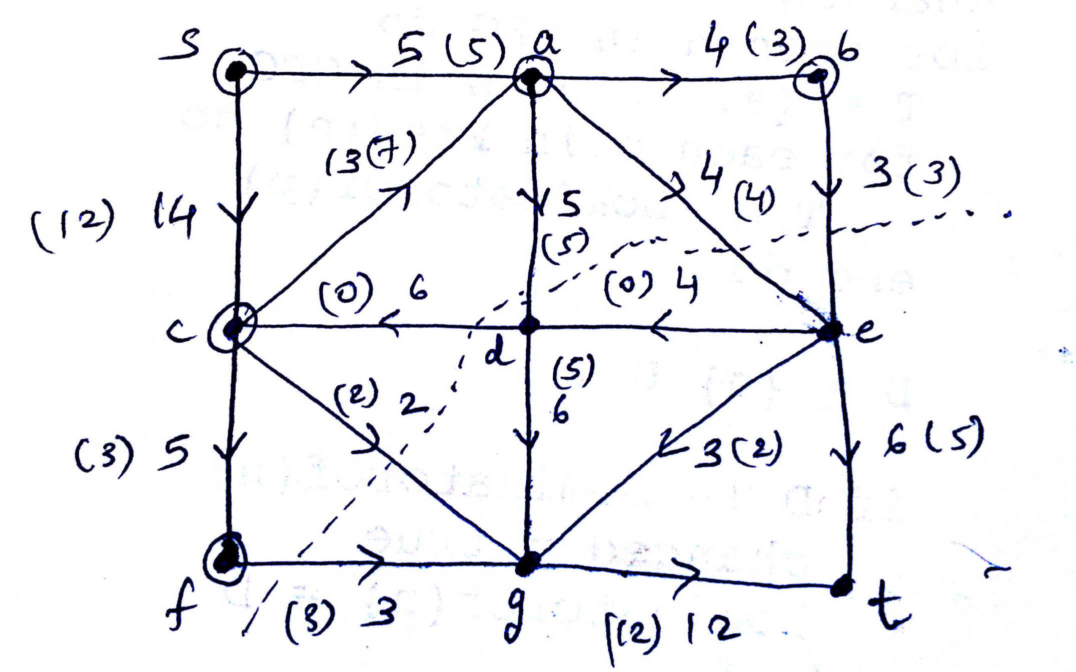
March 4, 2016

Problem 1

In the network below, find the maximum feasible flow from s to t. Prove that your answer is optimal by using the duality and explain why this proves optimality.



The given flowvalues form the maximum feasible flow for the network. they satisfy the capacity constraint and conservation constraint at each vertex. Its value is 17 and this is the maximum value which can be proven by the min-cut theorem. Consider the cut with the source set $\{s, a, b, c, f\}$ and the sin set consisting of the remaining vertices. The edges of this cut have the total capacity 17. This proves optimality because no feasible flow can have capacity more than this cut. Hence it is proved that the maximum flow value for this network is also 17.

Problem 2

Use network flows to prove Menger's theorem for internally disjoint paths in digraphs: $\kappa(x, y) = \lambda(x, y)$ **when** xy **is not an edge** .

$\kappa(x, y)$ is the minimum size of the (x, y) cut and $\lambda(x, y)$ is the maximum size of the pairwise internally disjoint paths from x to y, in a digraph G with vertex set $V(G)$ and edge set $E(G)$, $x, y \in V(G)$ and $xy \notin E(G)$. It is apparent from their definitions that $\kappa(x, y) \geq \lambda(x, y)$. Now, using Ford-Fulkerson theorem for Max-flow min-cut duality, we can also prove that $\kappa(x, y) \leq \lambda(x, y)$ as follows:

Let G' be a modified version of the digraph G such that $\forall v \in V(G) - \{x, y\}$, v is expanded into 2 vertices v^- and v^+, and they are joined by an edge v^-v^+. All such vertices are called intra-vertices. All the edges incident on v in G will be incident on v^- in G' and all the edges that are outgoing from v will now be outgoing edges from v^+ in G'. Let the capacity of the edge v^-v^+ be 1 for all such intra-vertices, and let the capacity be ∞ for all other edges. For this graph G', if k is the value of the max flow from source x to sink y, then by integrality theorem, it corresponds to k pairwise internally disjoint paths from x to y in G which can be obtained by shrinking the intra-vertex edges of G', giving us $\lambda(x, y) \geq k$. By duality max-flow min-cut duality, we also have that k is the capacity of the minimum capacity cut from source $X = x \cup \{v^- : v \neq x, y\}$ to sink $Y = y \cup \{v^+ : v \neq x, y\}$. Since all the intra-vertex edges have capacity 1, every edge of the min cut will be an intra-vertex edge. The vertices that we get by shrinking these intra-vertex edges will from the vertex cut for the graph G, since all min-cut paths pass through these vertices. Thus, these k vertices form the $x, y - cut$, giving $\kappa(x, y) \leq k$. Therefore, we have $\kappa(x, y) \leq k \leq \lambda(x, y)$ and $\kappa(x, y) \geq \lambda(x, y)$ which implies

$$\kappa(x, y) = \lambda(x, y)$$

. Hence proved.

Problem 3

Find the chromatic number of the graphs in exercise 8.1 of these notes of Frederic Havet: http://www-sop.inria.fr/members/Frederic.Havet/Cours/coloration.pdf **Are either of the graphs critical?**

(a)

Max degree $\Delta = 6 \implies \chi(G) \leq \Delta(G) + 1 = 7$

Min degree $\delta = 3$

Since this is not a complete graph nor itself a cycle of odd length, by Brooks' theorem we have $\chi(G) \leq \Delta(G) \implies \chi(G) \leq 6$

It containes K_4 as subgraph, hence $\chi(G) \geq 4$.

Its chromatic number is 4, as a proper coloring can be obtained for $k = 4$. And the given graph is critical. -2

(b)

Max degree $\Delta = 5 \implies \chi(G) \leq \Delta(G) + 1 = 6$

Since this is not a complete graph nor itself a cycle of odd length, by Brooks' theorem we have $\chi(G) \leq \Delta(G) \implies \chi(G) \leq 5$.

It contains an odd cycle as a subgraph. Hence, $\chi(G) \geq 3$.

Its chromatic number is 5, as a proper coloring can be obtained for $k = 5$. And the given graph is critical. -2

Problem 4

Let G be a k-chromatic graph. Since, every k-chromatic graph has a k-critical subgraph so let H be the k-critical subgraph of G. Then by lemma 5.1.18, degree of every vertex of H is at least k-1. And since H is k-critical so there are at least k vertices. Hence, in G there are at least k vertices with degree at least k-1.

Problem 5

a. Let the chromatic number $\chi(G)$ of G be k.

Case I: If G is also critical then the chromatic number of every subgraph of G obtained by eliminating any one vertex v is $\chi(G - v)$ of G is k-1.

Case II: If G is not critical then the chromatic number of every subgraph of G obtained by eliminating any one vertex v is $\chi(G - v)$ of G $\leq$ k-1(i.e., either k or k-1)

So combining the results of the two cases we get $\chi(G) \leq \chi(G - v) + 1$ for any vertex v.

b. Let the chromatic number $\chi(G)$ of G be k.
Given, $\chi(G) = \chi(G - v) + 1$

-2; there exist a vertex v in the graph so that the condition is true. The vertex v does not mean any vertex in the graph

If G is also critical then the chromatic number of every subgraph of G obtained by eliminating any one vertex v is $\chi(G - v)$ of G is k-1.Therefore, G is k critical and $\chi(G - v)$ of G = k-1
By lemma 5.1.18, if G is k critical, minimum degree δ(G) $\geq$ k-1.
So, δ(G) $\geq \chi(G - v)$ i.e., $d_G(v) \geq \chi(G - v)$

c. From (a) and (b) $\chi(G) \leq \chi(G - v) + 1$

-3; so that you can not use the assumption that G is critical from part b.

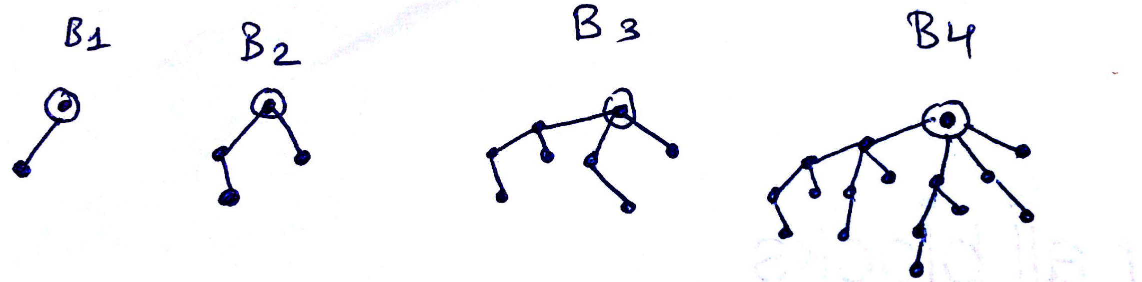
$\chi(G) - 1 \leq \chi(G - v)$
So, $\chi(G) - 1 \leq \delta$(G) -(1)
Similarly, $\chi(\overline{G}) - 1 \leq \delta(\overline{G})$ -(2)
Adding (1) and (2) we get, $\chi(G) + \chi(\overline{G}) - 2 \leq \delta(G) + \delta(\overline{G})$
$\chi(G) + \chi(\overline{G}) \leq \delta(G) + \delta(\overline{G})$ + 2

Let $\delta(G)$ be x. Therefore, in $\overline{G}$ it can be connected to a maximum of n(G)-x-1 vertices.
So, $\delta(G) + \delta(\overline{G})$ = x+n(G)-x-1 = n(G)-1
$\chi(G) + \chi(\overline{G}) \leq n(G) + 1$

Problem 6

Draw B_1, B_2, B_3 and B_4. And prove by induction that for every k there is an ordering of vertices of B_k for which greedy coloring uses $k + 1$ colors.

A binomial tree B_k of order k ($k \geq 0$) is an ordered tree defined recursively as: (i) B_0 is a one-vertex graph. (ii) B_k consists of two copies of B_{k-1} such that the root of one is the left most child of the root of the other.



Proof by Induction:

Base case: This is true for $k = 1$. B_1 will always be colored with 2 colors by the greedy algorithm for every ordering of the vertices.

Induction hypothesis: Let there be an ordering of vertices for B_k such that the greedy algorithm uses $k + 1$ colors.

Induction step: For B_{k+1}, which is formed by merging 2 B_k, let the new ordering of vertices be formed using the following sequence of ordering: 1) ordering of vertices of the B_k whose root becomes the root of B_{k+1} 2) ordering of the vertices of the second B_k tree except its root 3) the root of the seconf B_k tree is colored last

Following the above sequence of coloring always ensures that greedy algorithm takes $k + 2$ colors for B_{k+1} given that B_k takes $k + 1$ colors.

Hence proved.

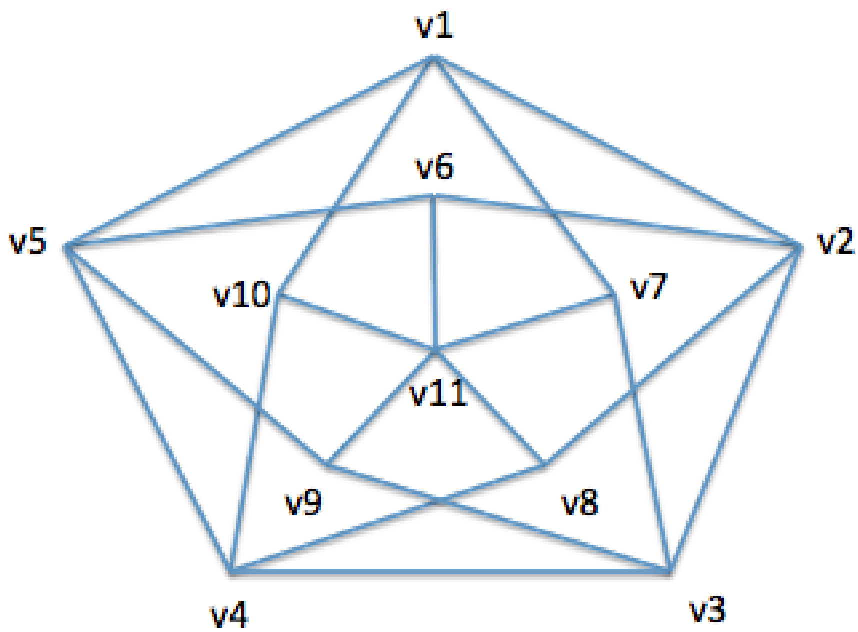
Problem 7

Without using Brooks' theorem, prove that if G is a simple connected graph which is not regular, then $\chi(G) \leq \Delta(G)$

Let Δ, the maximum degree of a vertex in the graph be k. This case is possible since the graph is given to be non-regular. Let v be a vertex of degree less than k. Construct a spanning tree with v as root and assign indices in decreasing order as the vertices are reached. Thus, here v will be the vertex with the highest index. Hence, every other vertex will have a neighbor with higher index in the ordering. Thus, every vertex will have at most $k-1$ vertices with lower indices. Hence, greedy coloring will use atmost k colors.

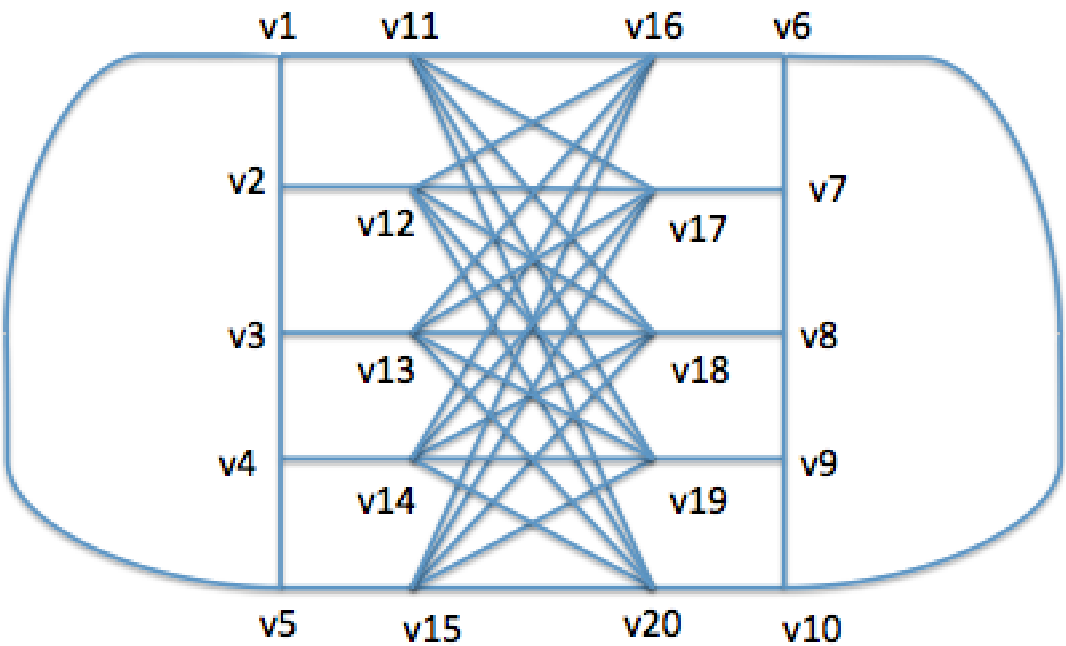
Problem 8

4 chromatic graph with maximum clique of size 2.



Problem 9

4 critical graph which is not regular.



-5; when you remove v15-v20 edge, you still have to color the graph properly with at least 4 colors

Problem 10

By Theorem 5.2.9, among the n-vertex simple graphs with no r+1 clique, $T_{n,r}$ has the maximum number of edges. So no partite sets can differ by more than one. Lets assume that the sizes of partite sets are n/r(almost equal size of partite sets. Since we have to prove maximum edges so maximum edges will occur if the partite sets are of equal size). A vertex present in a partite set is connected to every vertex that does not belong to this partite set. And there are $\binom{r}{2}$ total distinct pairs of partite sets. For each pair of partite set there are a total of $(n/r)^2$ edges. Therefore, total number of edges are:

This is true

-5;This is wrong. What if the n can not be partitioned by r equivalently. so your proof is a special case.

$$\binom{r}{2} * (n/r)^2 = r * (r-1)/2 * (n/r)^2 = (r-1)/r * n^2/2 = (1-1/r) * n^2/2$$

Hence proved.