Matchings & Covers (Ch. 3)

I. Matchings

II. Covers

III. Independent Sets & Edge Covers

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II. Covers

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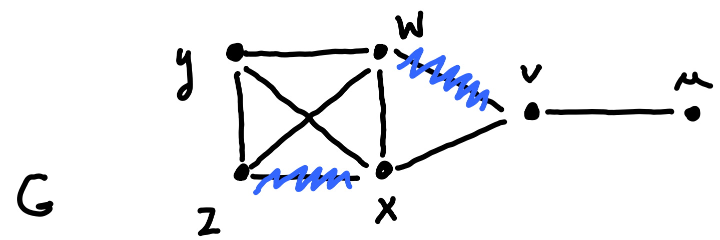
Matching in G: set of non-loop **edges** with **no common endpoints.**



$M = \{wv, xz\}$ is a matching in G

A vertex is **M-saturated** if it is the endpoint of an edge in M

otherwise, **M-unsaturated.**

In the example above, w, v, x, and z are M-saturated

y and u are M-unsaturated

Problem: Given G, find **largest** possible matching.

(e.g. G models "compatability" of programmers)

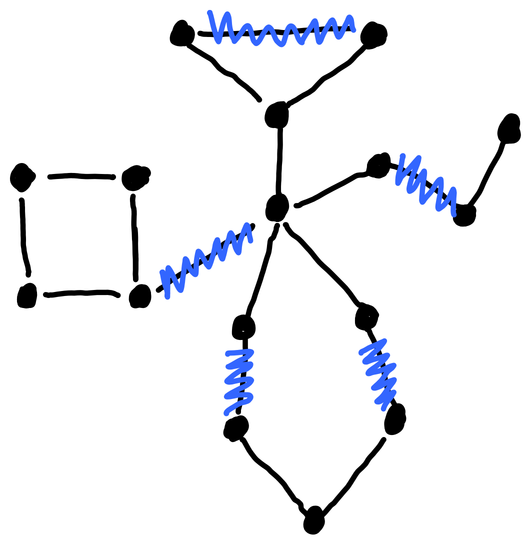
Preview:

- "Greedy" does <u>not</u> work
- but problem <u>is</u> solvable in <u>polynomial</u> time
- <u>bipartite</u> case is easier
- <u>weighted</u> versions are also polynomial!

Matching M in graph G:

M is maximal if M is not a proper subset of another matching in G.

M is maximum if G has no larger matching than M.



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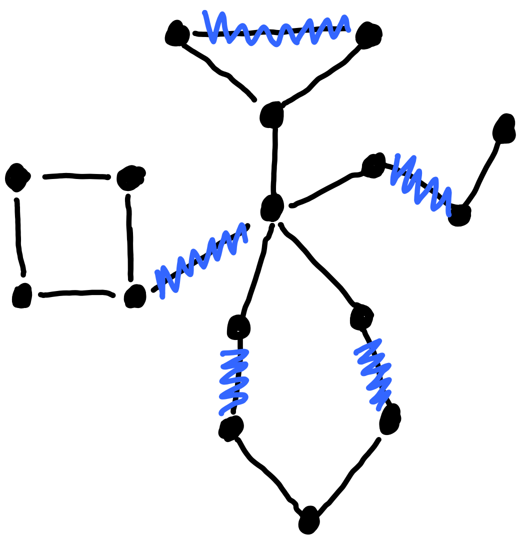

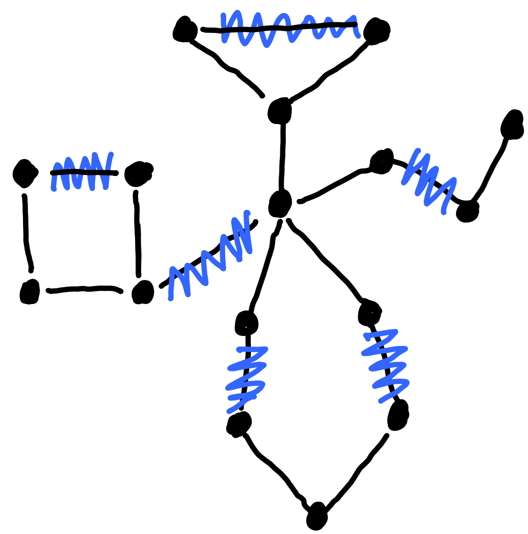

<u>not</u> maximal

Matching M in graph G:

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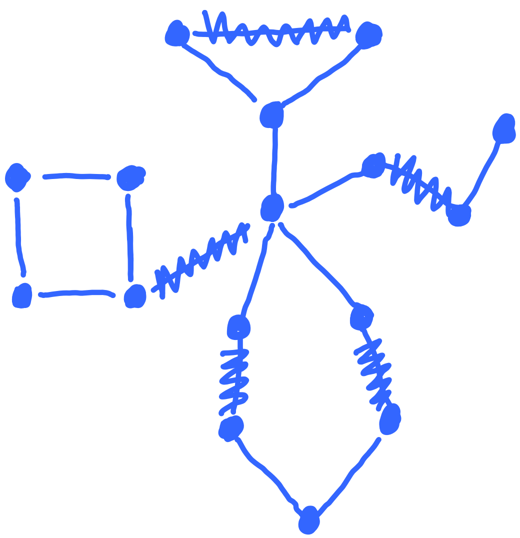
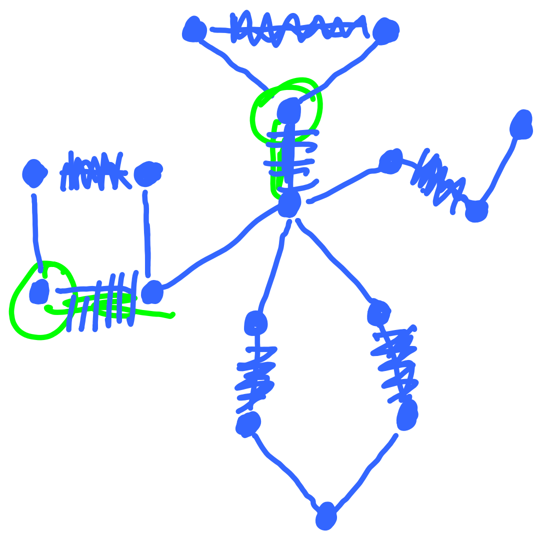
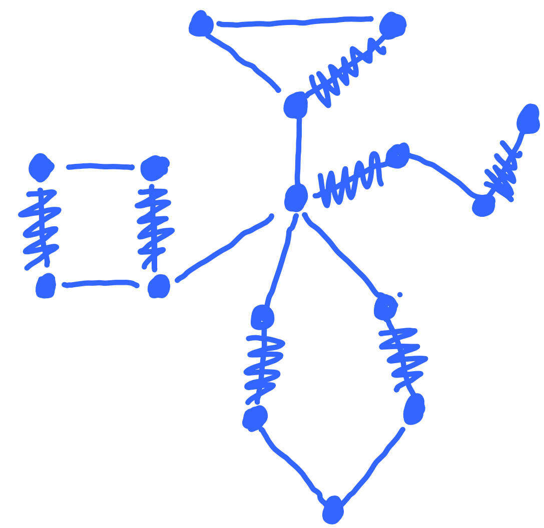
$E(P) \,\Delta\, M = M'$



<u>not</u> maximal



maximal, but <u>not</u> maximum



maximum ✓

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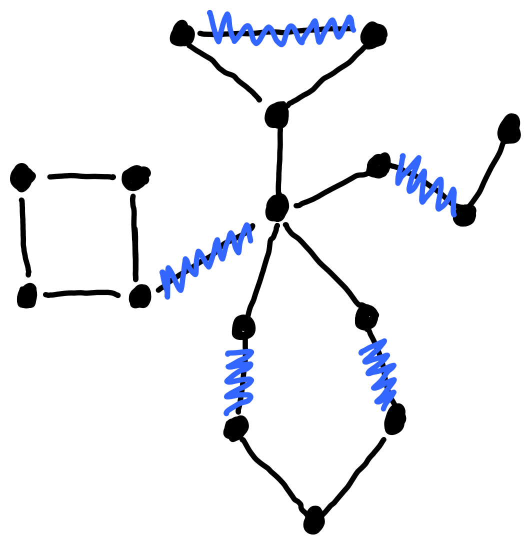
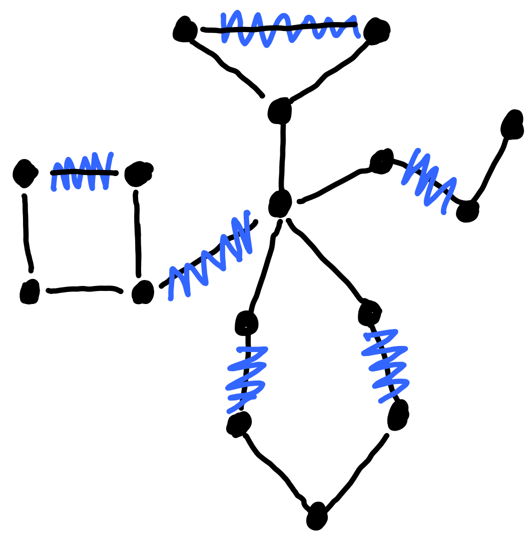
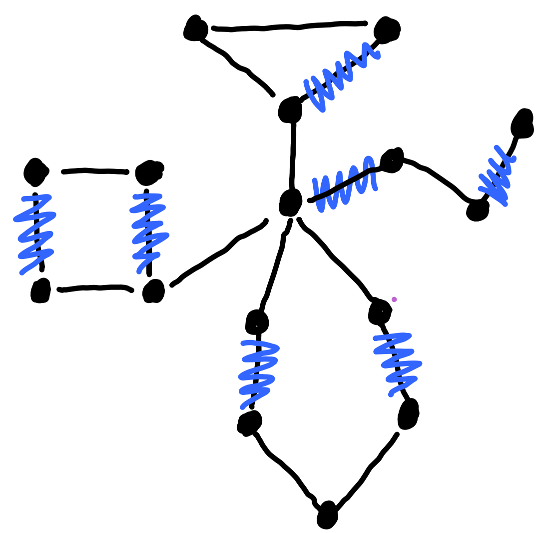


not maximal



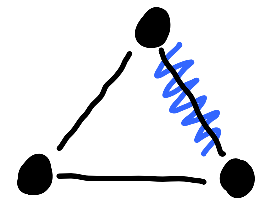
maximal, but not maximum



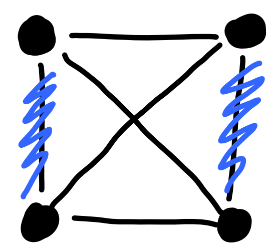
maximum ✓

Matching M in G is **perfect** if every vertex of G is M-saturated.

Perfect matching in:



K_3



K_4

K_n?

n odd, no

n even, yes

(How many perfect matchings in K_n?)

$K_{n,m}$?

Q_k?

Petersen graph?

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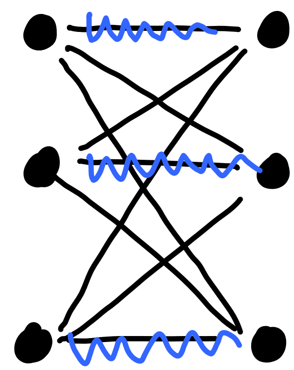
Perfect matching in:



$K_{3,3}$

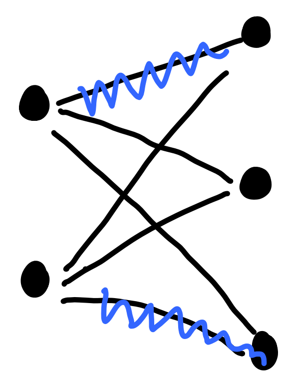
K_n?

$K_{n,m}$?

$n=m$, yes

$n \neq m$, no

(How many in $K_{n,n}$?)



$K_{2,3}$

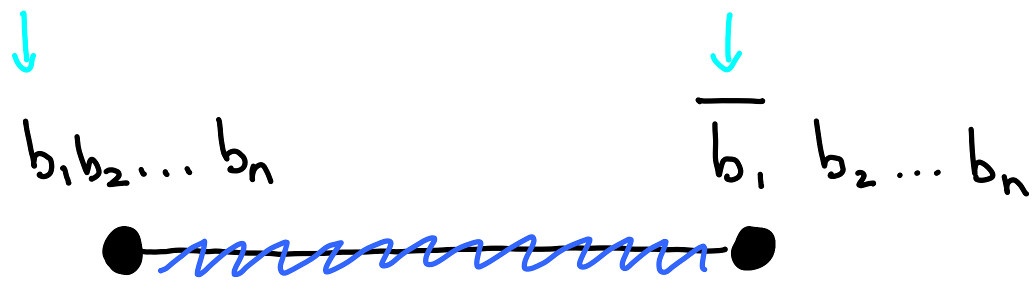
Q_k?

Petersen graph?

Matching M in G is **perfect** if every vertex of G is M-saturated.

Perfect matching in:

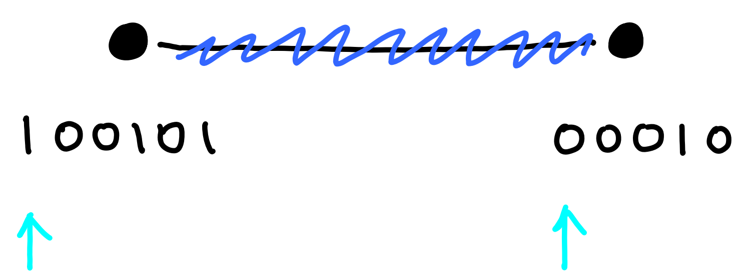
K_n?



$K_{n,m}$?

Q_k? yes ✓

e.g.



Petersen graph?

Matching M in G is **perfect** if every vertex of G is M-saturated.

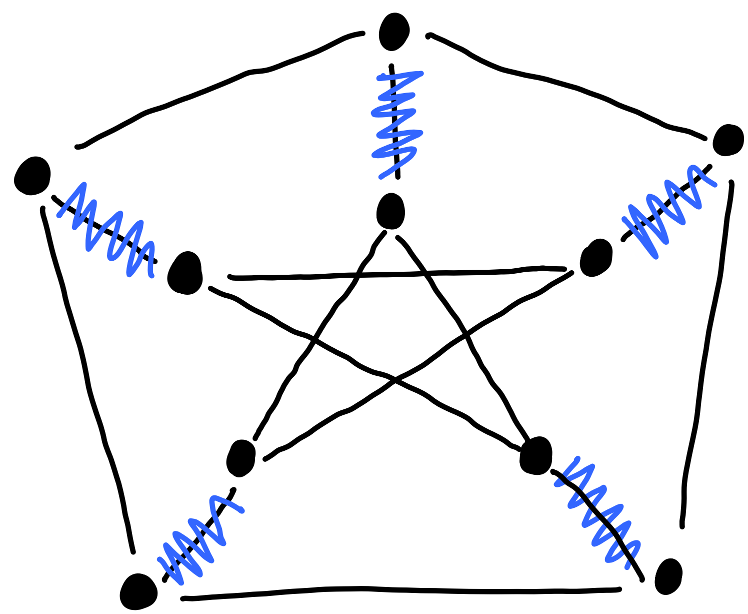
Perfect matching in:

K_n?

$K_{n,m}$?

Q_k?

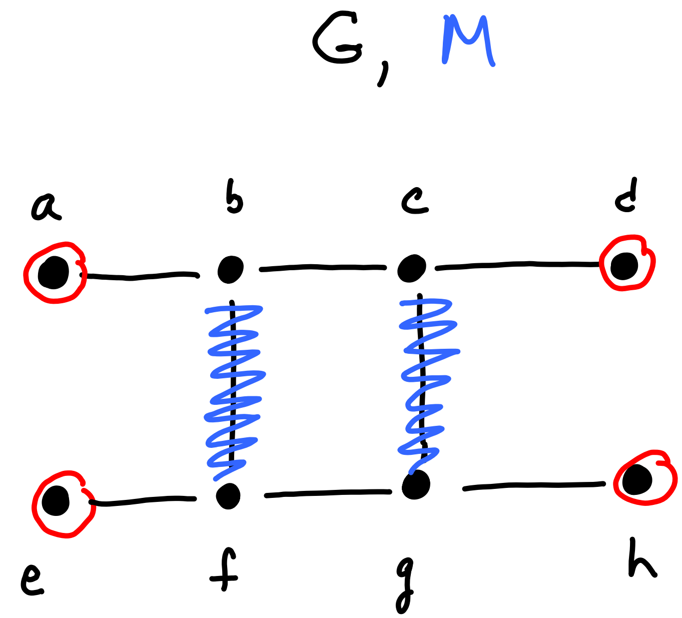
Petersen graph? ✓ yes



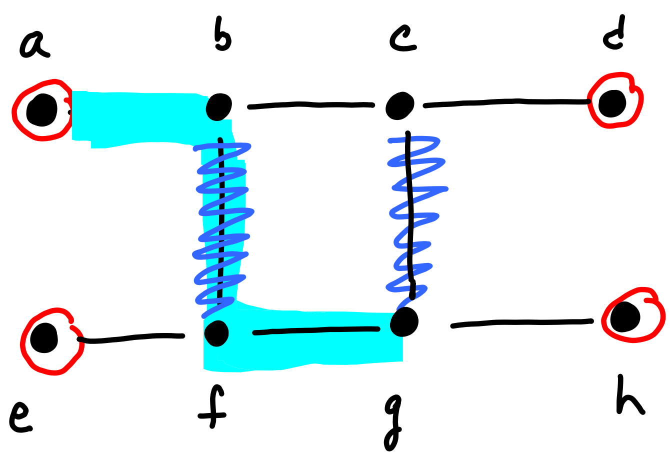
G graph, M matching in G

M-alternating path: path whose edges are alternately in M, not in M.

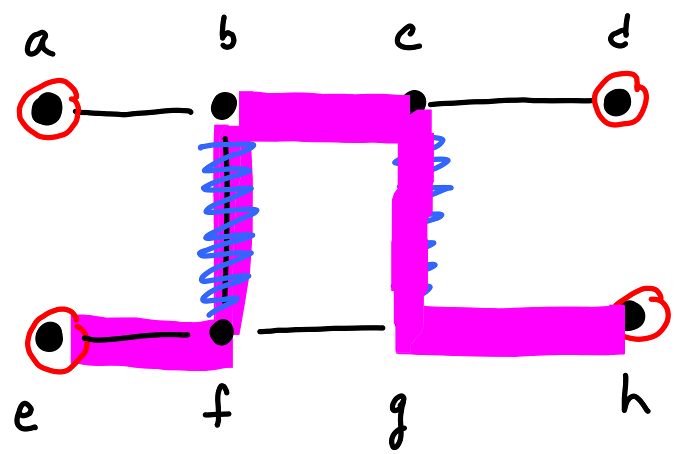
M-augmenting path: M-alternating path (of length ≥ 1) which starts and ends at M-unsaturated vertices.



M- alternating:



M- augmenting:



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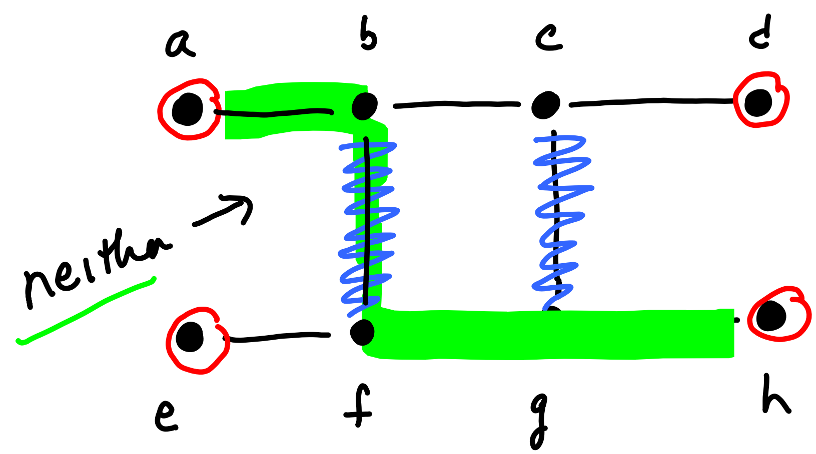
G, M



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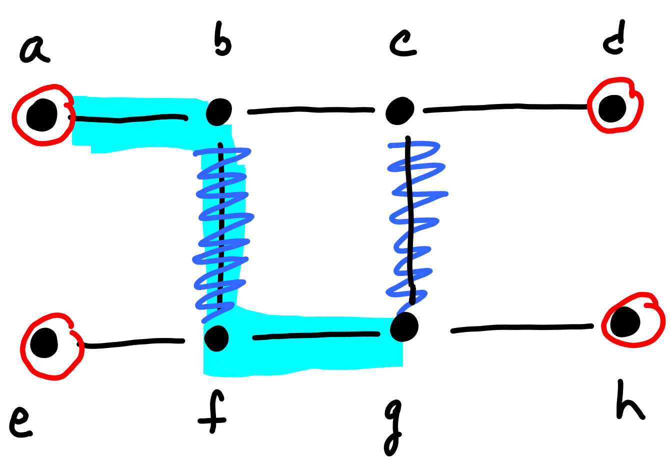

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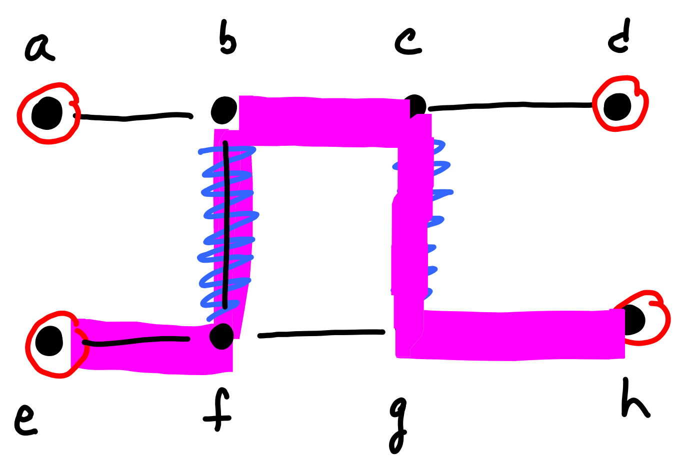
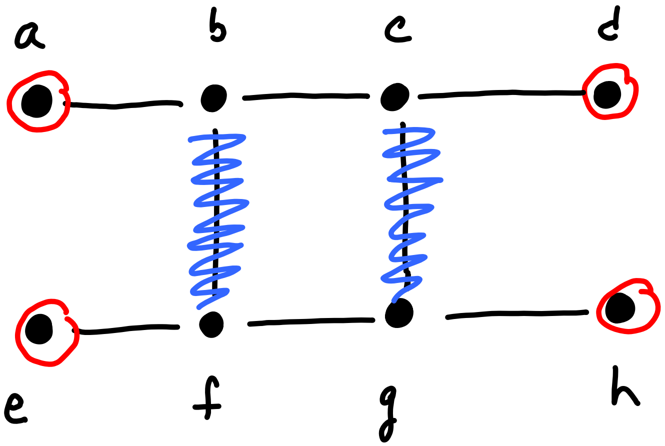
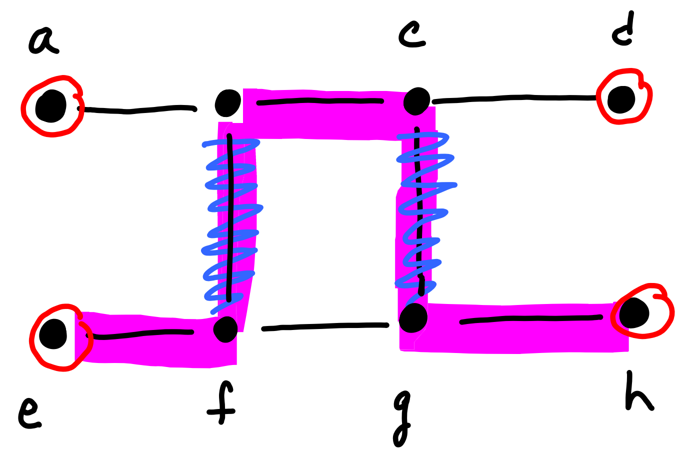


Matching of size 2:



Why **augmenting**?

M-augmenting path:



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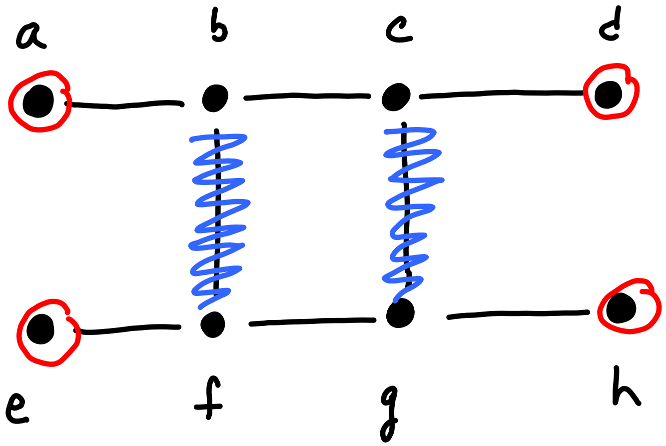

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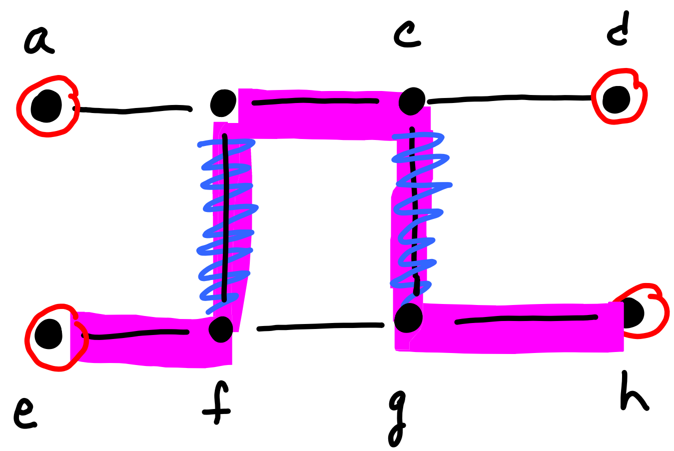
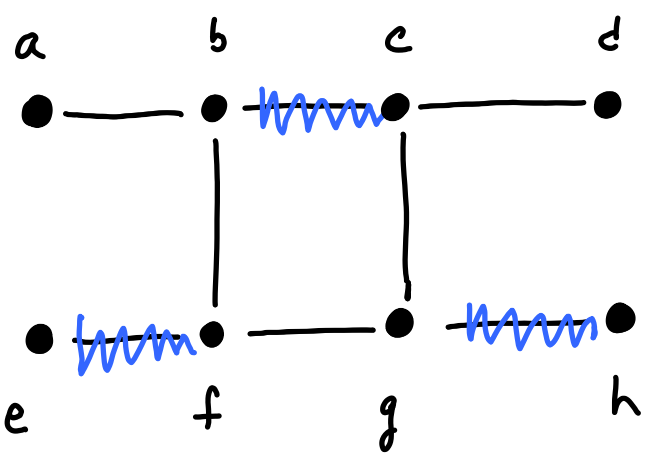


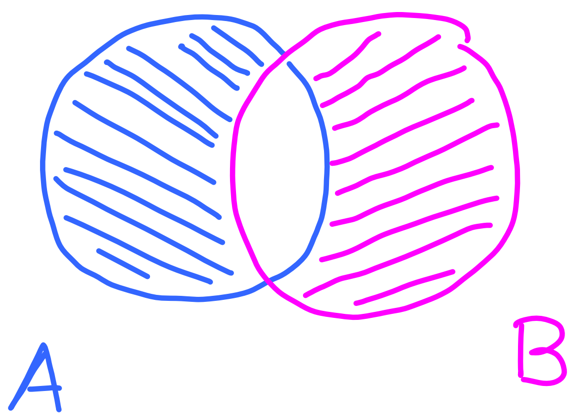


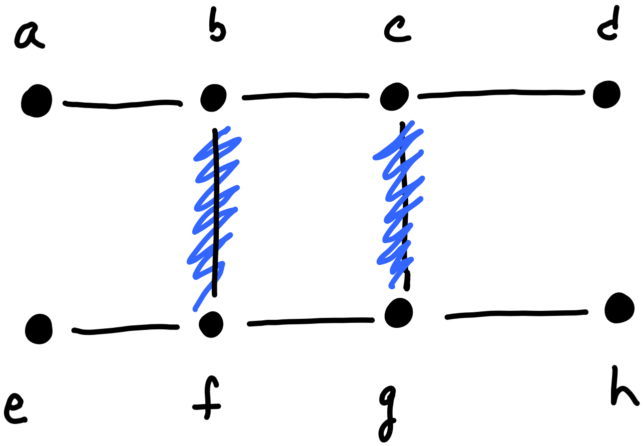
Matching of size 3

Reminder: symmetric difference of 2 sets

$$A \,\triangle\, B = (A \cup B) - (A \cap B)$$

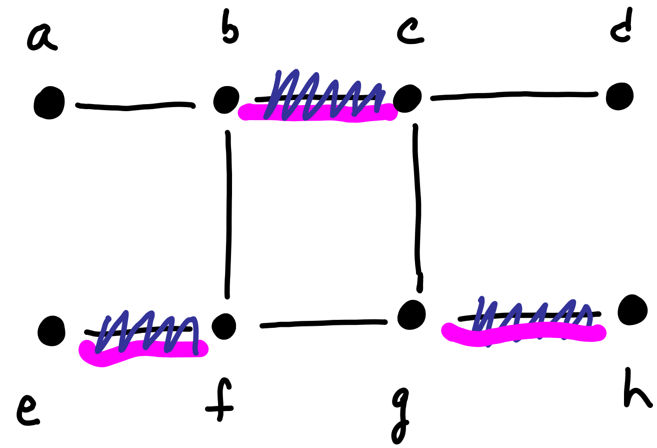
$$= (A - B) \cup (B - A)$$

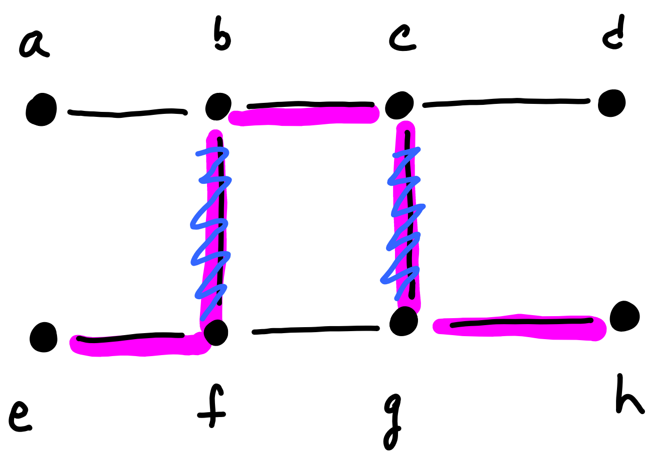




M

M'





p

$$\boxed{M' = M \,\triangle\, E(p)}$$

Theorem 3.1.10. [Berge 1957] Matching M in G is maximum iff G contains no M-augmenting path.

Proof. $\Rightarrow$ (by contrapositive)

If p is an M-augmenting path,

$M \triangle E(p)$ is still a matching and

$|M \triangle E(p)| > |M|$ since

$|E(p) \cap M| < |E(p) - M|$

matched edges on p

unmatched edges on p

(So, M is not a maximum matching.)

Theorem 3.1.10. [Berge 1957] Matching M in G is maximum iff G contains no M-augmenting path.

$\boxed{\text{No M-aug Path}} \Rightarrow \boxed{\text{M max}}$

$\boxed{\text{M not max}} \rightarrow \boxed{\text{M-aug path exists}}$

Proof. $\Leftarrow$ (by contrapositive)

Suppose M' is a matching in G with $|M'| > |M|$. Consider the spanning subgraph F of G with $E(F) = M \triangle M'$.

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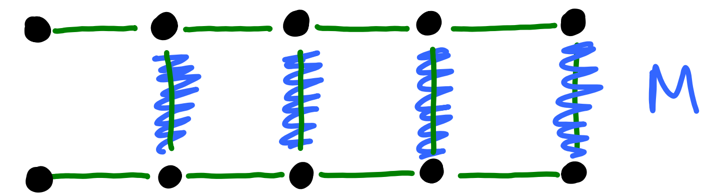

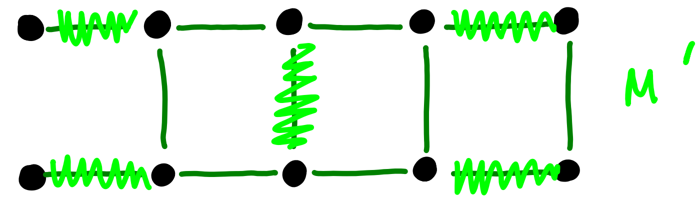

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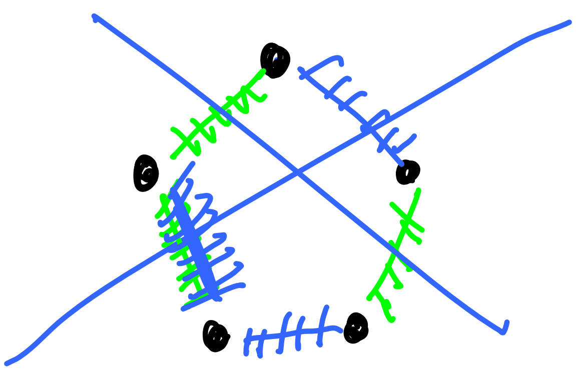

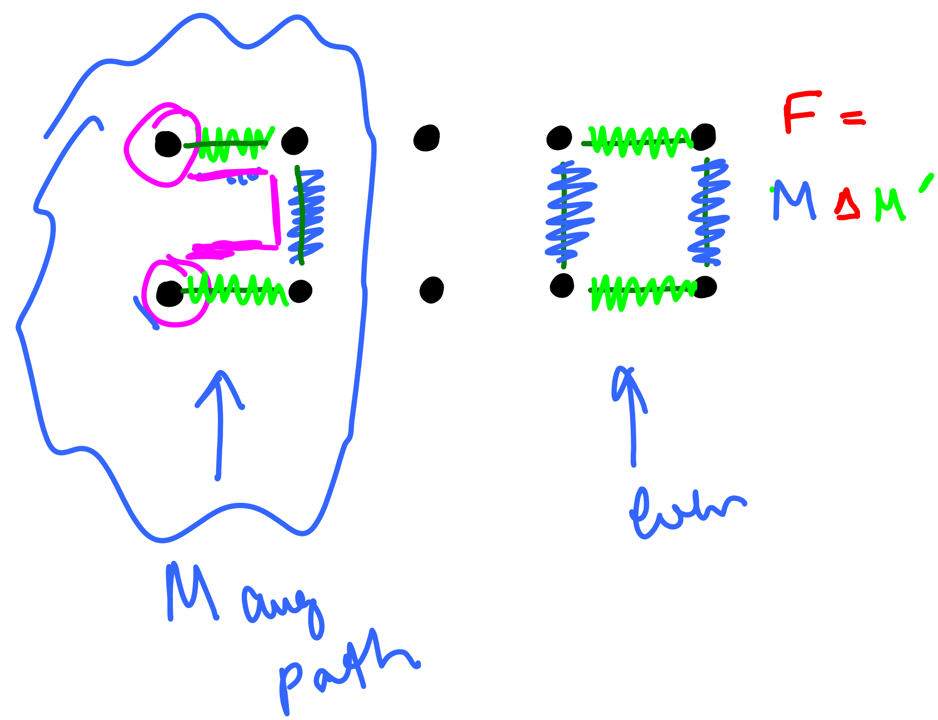

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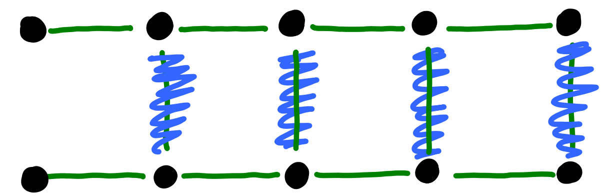
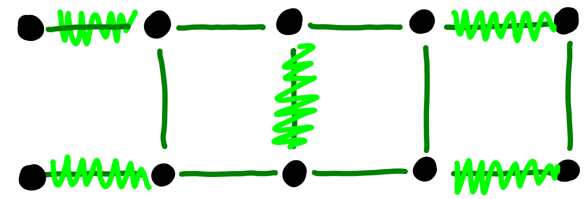


M



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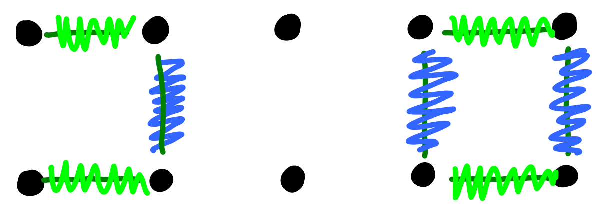
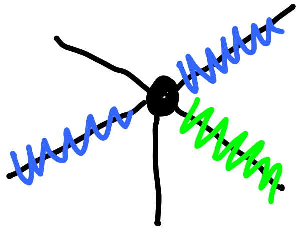


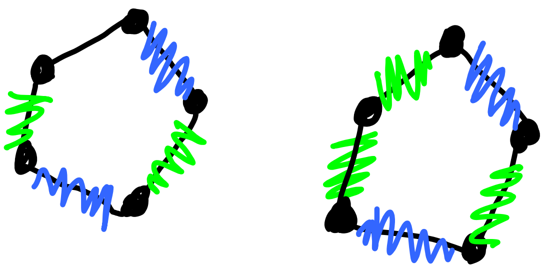
F = M △ M'

Each component of F is a path or an even cycle (why?).

Why can't F have any of these?

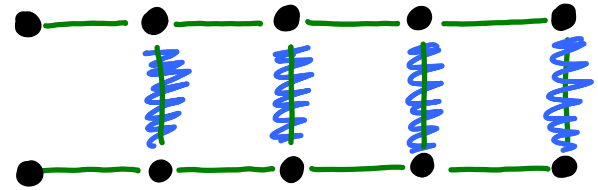


$d_F(v) > 2$?

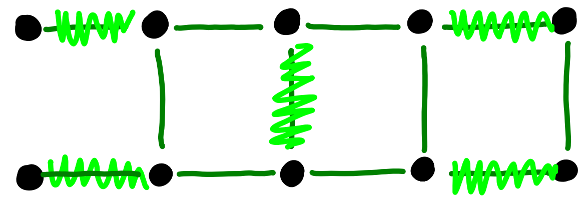


odd cycle?

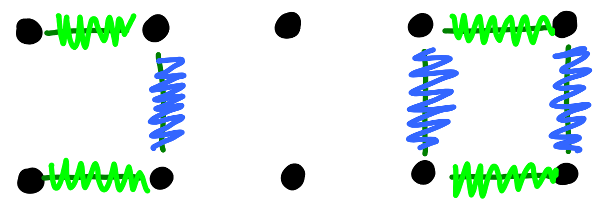
Each component of F is a path or an <u>even</u> cycle (why?).



M



M'



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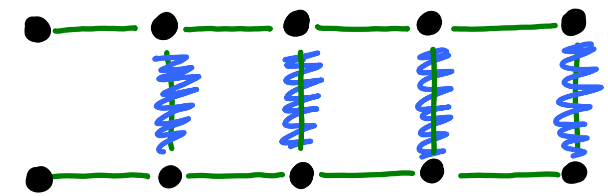
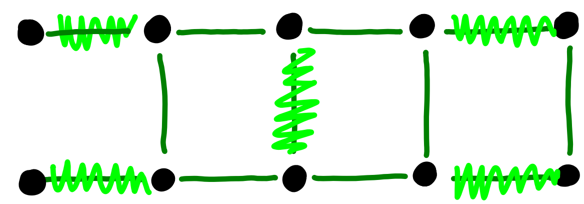
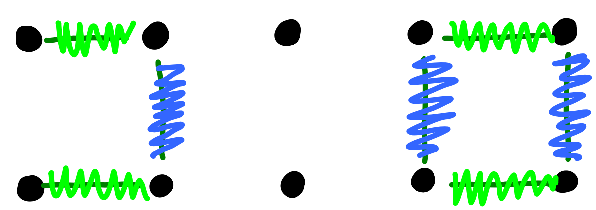


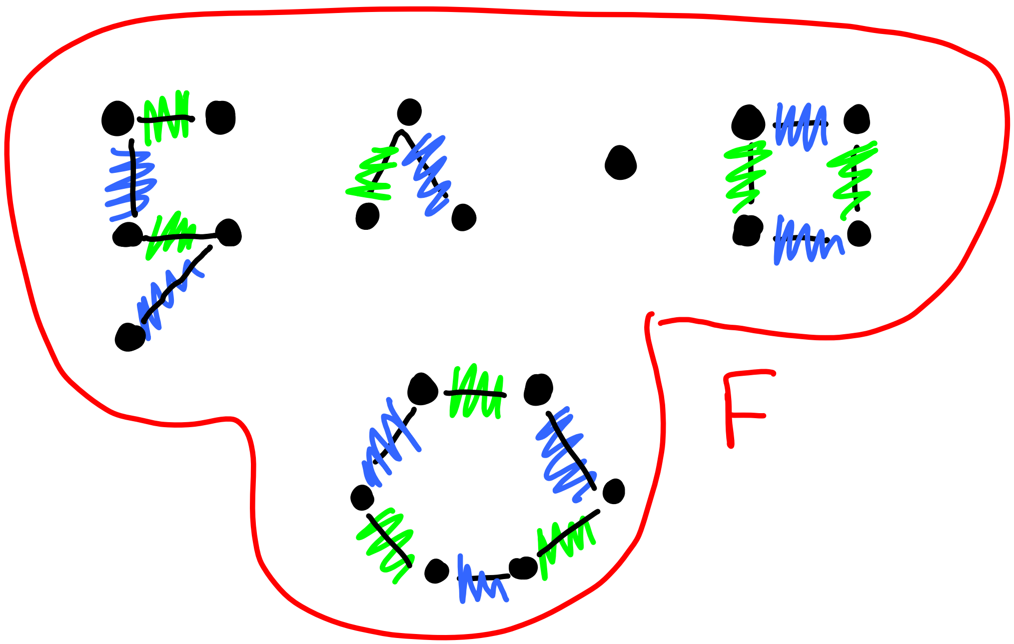


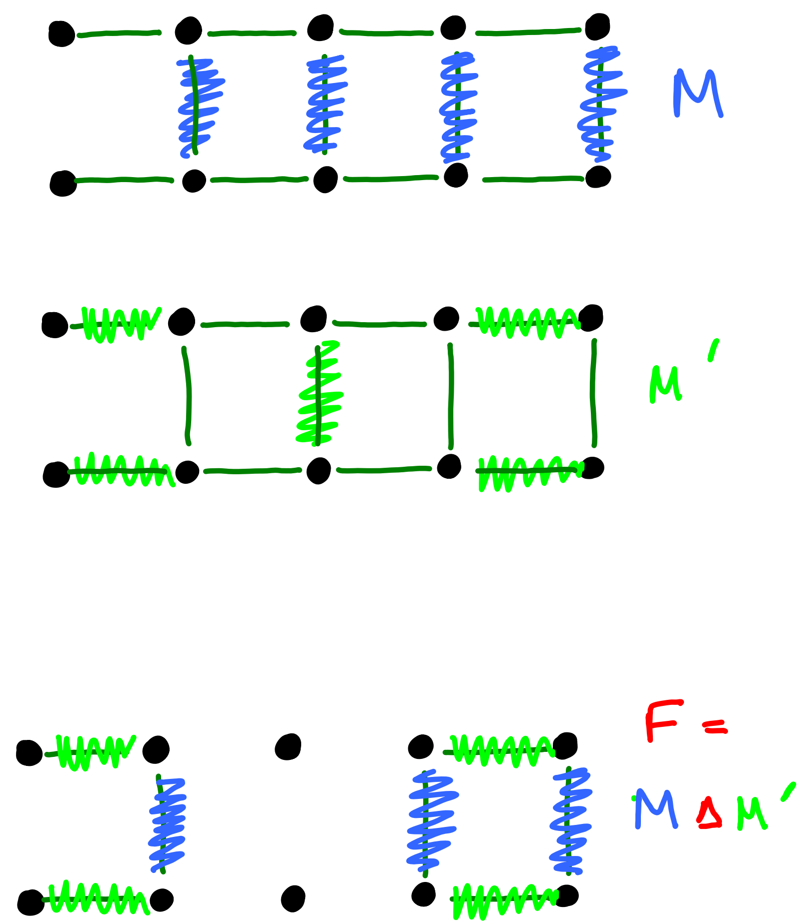


Each component of F is a path or an even cycle (why?).

Some component of F is an odd-length path (why?)

Why can't you just have this:





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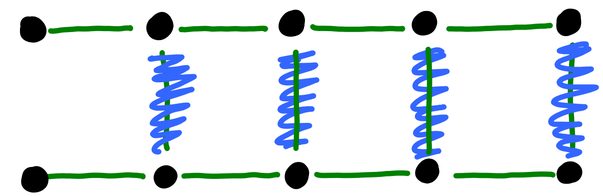
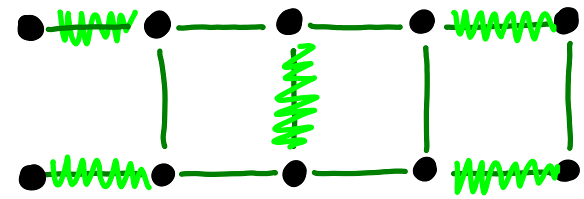

M


M'

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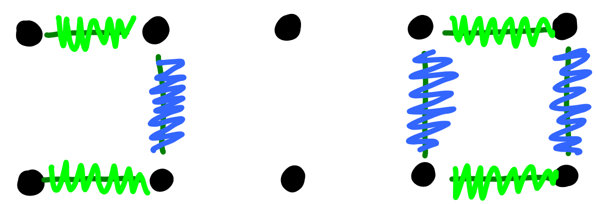
F = M Δ M'

Each component of F is a path or an even cycle (why?).

Some component of F is an odd-length path (why?) with more M' edges than M edges

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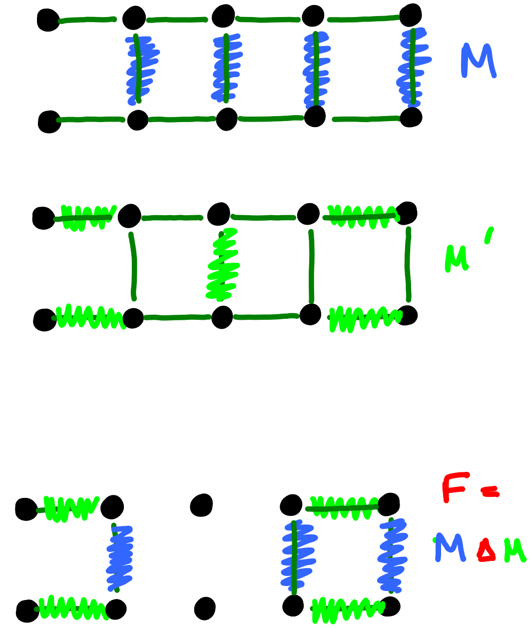

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Each component of F is a path or an even cycle (why?)

Some component of F is an odd-length path (why?) ✓ with more M' edges than M edges

This is an M-augmenting path
≡

Personnel Assignment Problem

Given workers $W_1, W_2, \ldots, W_n$ and jobs $J_1, J_2, \ldots, J_m$, and some specifications (W_i, J_j) indicating that W_i is qualified to do job J_j;

Can each worker be assigned to a job for which she is qualified, if no two workers can be assigned the same job?

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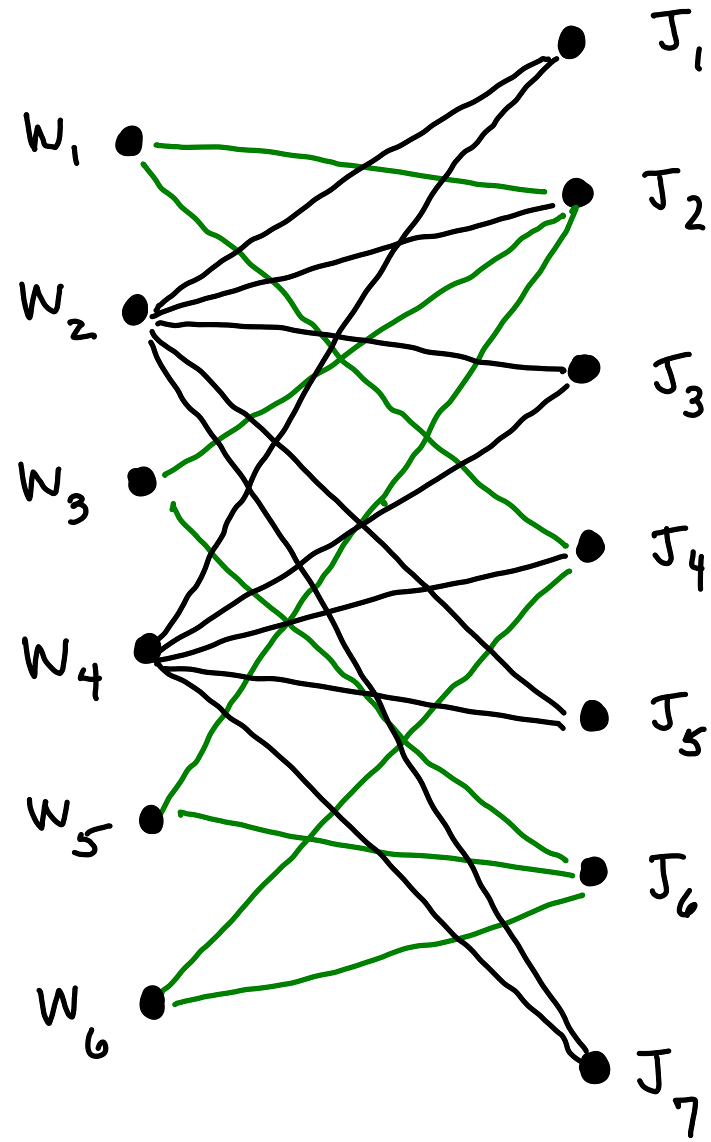
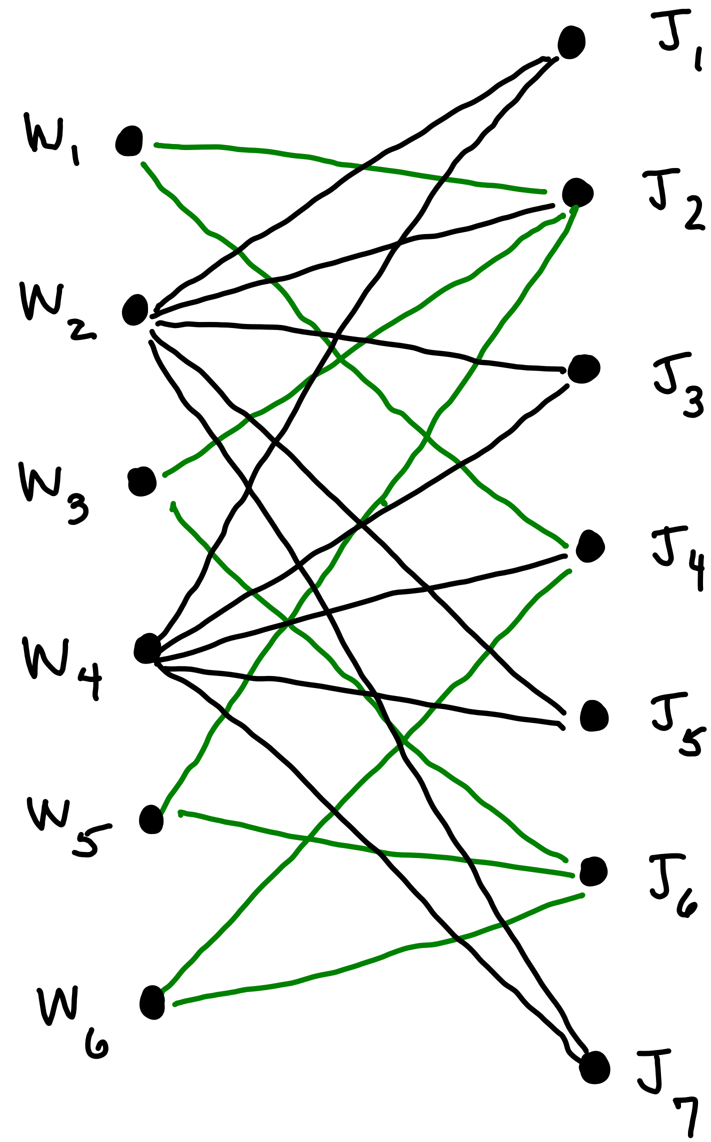


Personnel Assignment Problem

Given workers $W_1, W_2, \ldots, W_n$ and jobs $J_1, J_2, \ldots, J_m$, and some specifications (W_i, J_j) indicating that W_i is qualified to do job J_j;

Can each worker be assigned to a job for which she is qualified, if no two workers can be assigned the same job?

i.e. Does the corresponding bipartite graph have a matching that saturates every vertex in W?



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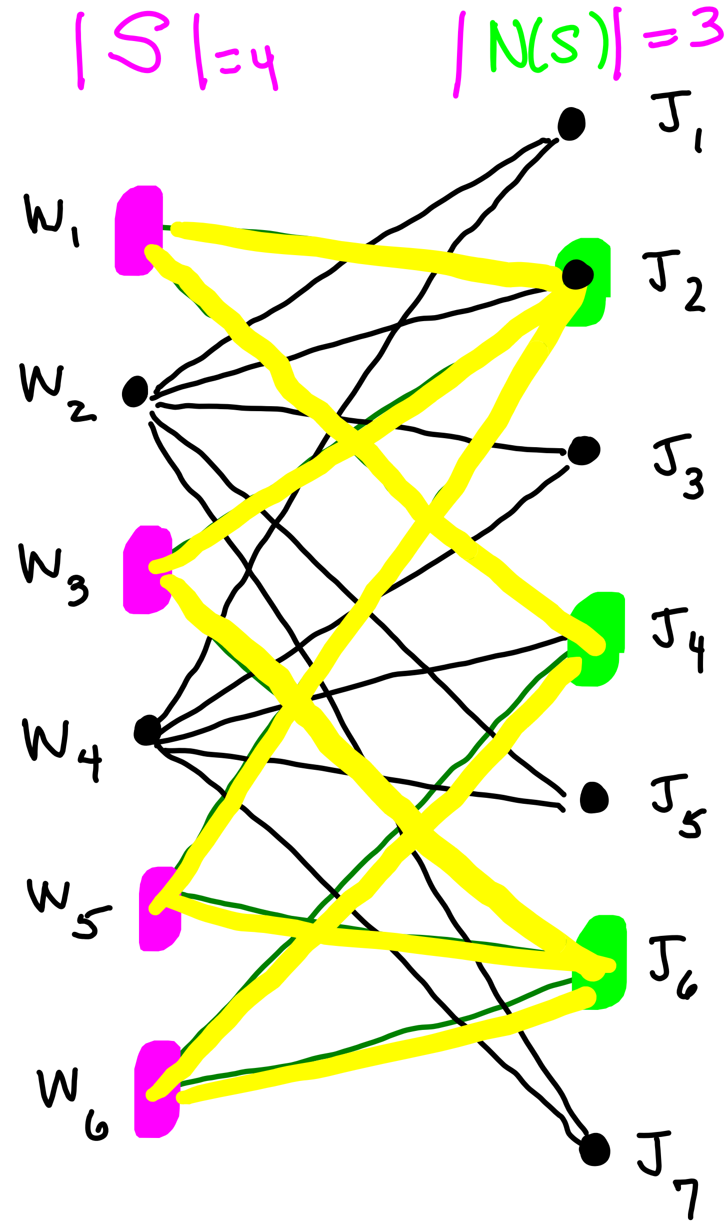
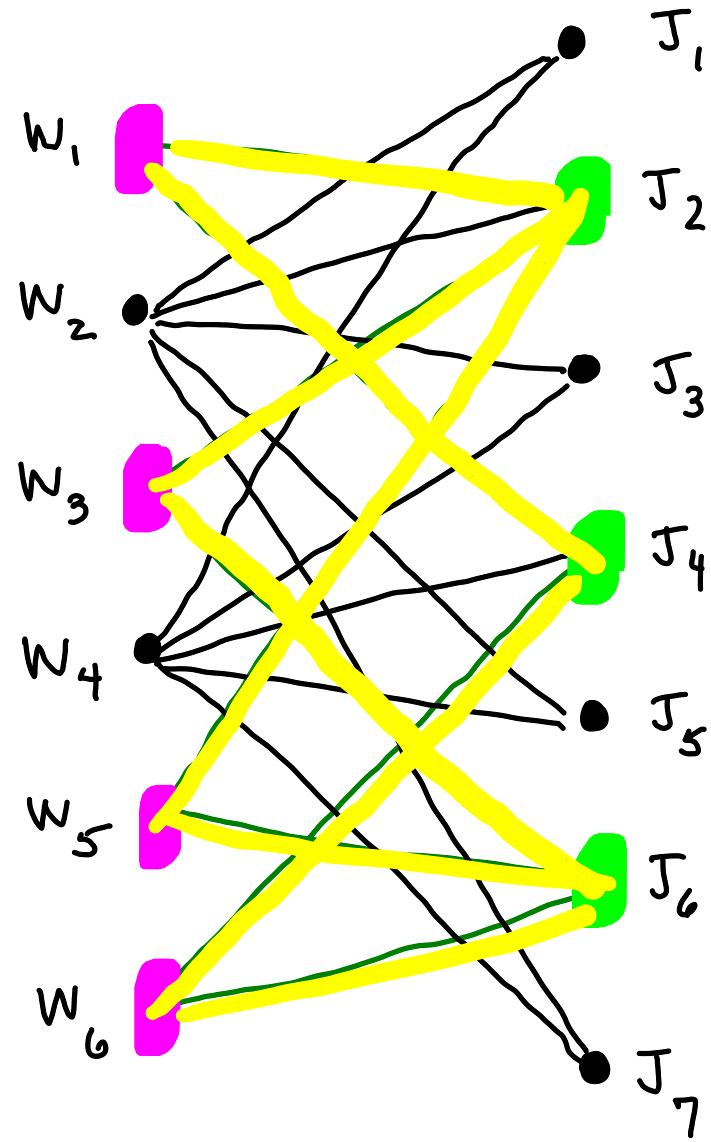


NO Because there is a subset S of W such that $N(S)$ is smaller than S.



i.e. Does the corresponding bipartite graph have a matching that saturates every vertex in W?

- **Bipartite G with bipartition (X, Y)**

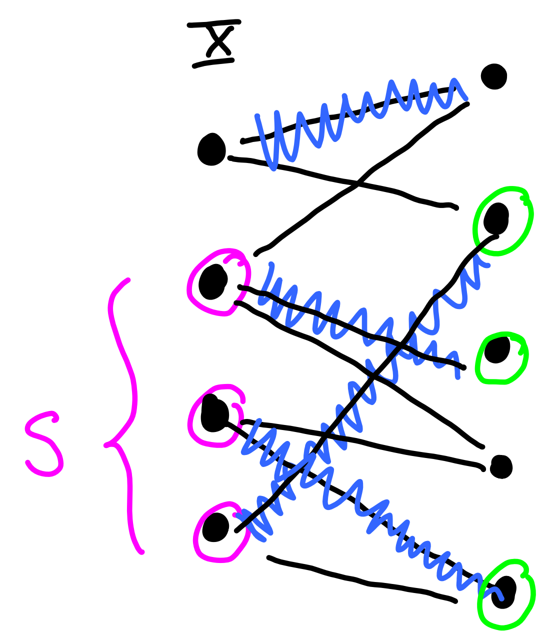
Hall's Theorem

Theorem 3.1.11. [Hall 1935]: G has a matching saturating every vertex in X iff

$$|N(S)| \geq |S|$$

for all $S \subseteq X$.

- Bipartite G with bipartition (X, Y)



Theorem 3.1.11. [Hall 1935]: G has a matching saturating every vertex in X iff

$$|N(S)| \geq |S|$$

for all $S \subseteq X$.

Proof. ($\Rightarrow$) Let M be a matching in G saturating every vertex of X. Let $S \subseteq X$. Each vertex of S is matched to a distinct vertex of N(S).

Therefore $|N(S)| \geq |S|$

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$|N(S)| \geq |S|$ for all $S \subseteq X$

$\Rightarrow$ $\exists$ matching saturating every vertex of X

$\neg \exists$ a matching saturating every vertex

$\Rightarrow$ $\exists$ $S \subseteq X$ s.th

$|N(S)| < |S|$

Proof

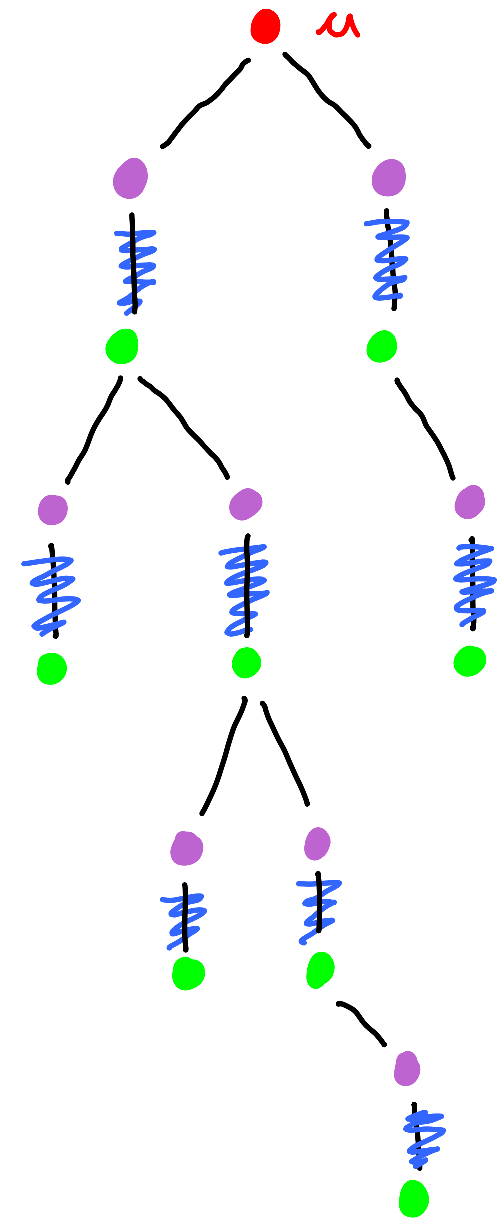
($\Leftarrow$) (by contrapositive)

Let M be a maximum matching and suppose $u \in X$ is M-unsaturated. Find $S \subseteq X$ such that $|N(S)| < |S|$.

Let Z be the set of vertices of G which are connected to u by M-alternating paths.

Let:

$$S = Z \cap X; \quad T = Z \cap Y$$



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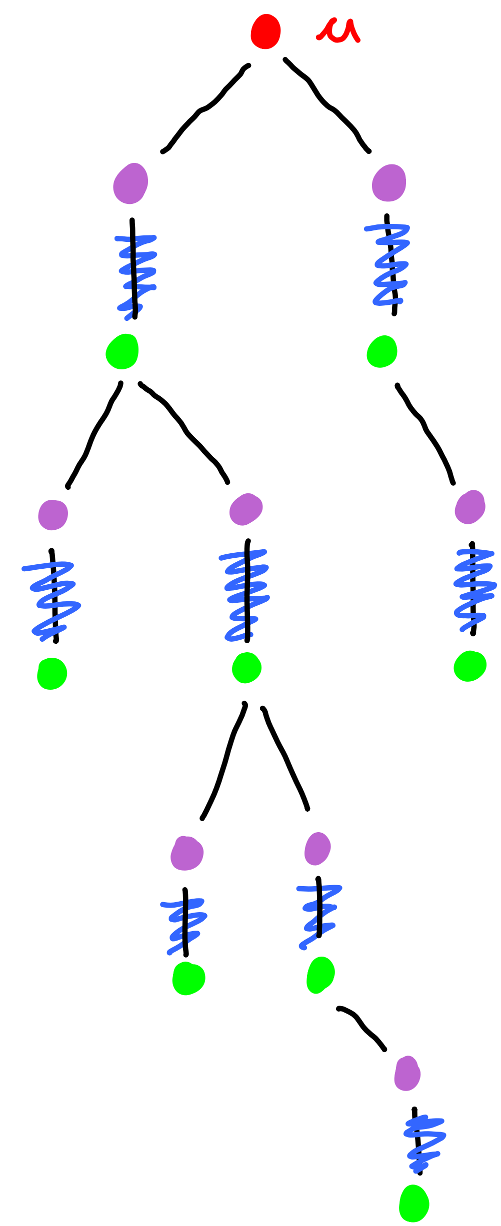

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Let:

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Can show:

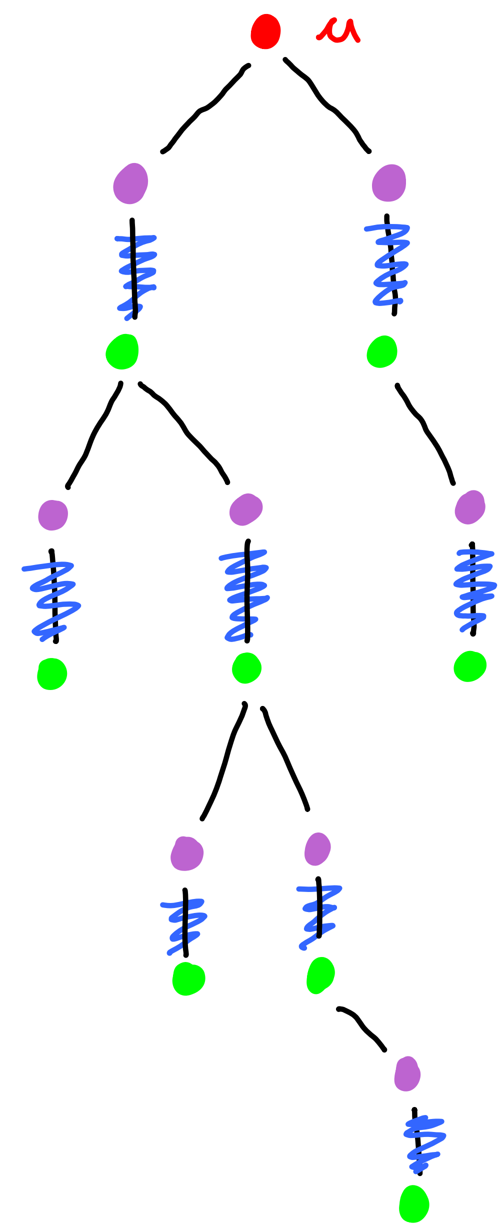
- $|T| = |S| - 1$

- M matches T with $S - u$

- So, $T \subseteq N(S)$

- Furthermore, $N(S) \subseteq T$

- Thus $|N(S)| = |T| = |S| - 1 < |S|$.



Corollary 3.1.13. Every k-regular bipartite graph with $k > 0$ has a perfect matching.

"The Marriage Theorem"

Proof. Let G be a k-regular graph with bipartition (X, Y).

Use Hall's theorem. Show that $|N(S)| \geq |S|$ for any $S \subseteq X$.

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Let E_S be the set of edges incident with vertices in S.

Let $T = N(S)$.

Let E_T be the set of edges incident with vertices in T.

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Let $T = N(S)$.

Let E_T be the set of edges incident with vertices in T.

Then $|E_S| = k|S|$ and $|E_T| = k|T|$

$E_S \subseteq E_T$

So, $|E_S| \leq |E_T|$ and $\therefore$ $|S| \leq |T| = |N(S)|$.