Homework 6
Graph Theory CSC/MA/OR 565
Sketch of Solutions

1. We know that $\chi'(K_n) \geq \Delta(K_n) = n-1$. If $\chi'(K_n) = n-1$ then $E(K_n)$ can be decomposed into $n-1$ edge-disjoint matchings. Each of these matchings has at most $\lfloor n/2 \rfloor = (n-1)/2$ edges, totaling $(n-1)(n-1)/2 < \binom{n}{2} = e(K_n)$, contradicting that the $n-1$ matchings include every edge of K_n.

The next few questions refer to the *line graph* $L(G)$ of a graph G.

2. a. The $d(v)$ edges incident with v induce a clique in $L(G)$ and these cliques partition $E(L(G))$.

b. If G is 2-regular, then every component of G is isomorphic is a cycle. The line graph of a cycle is the cycle itself.

Conversely, if G is isomorphic to $L(G)$, then they must have the same number of edges and the same number of vertices. So if G has degree sequence $(d_1, \ldots, d_n$, then by part (a) and the degree sum formula

$$e(L(G)) \; = \; \sum_{i=1}^{n} \frac{d_i(d_i-1)}{2} \; = \; \sum_{i=1}^{n} \frac{d_i^2}{2} - \frac{1}{2}\sum_{i=1}^{n} d_i \; = \; \sum_{i=1}^{n} \frac{d_i^2}{2} - e(G)$$

and since, because they are isomorphic, $e(L(G)) = e(G)$ and $n = n(G) = n(L(G)) = e(G)$, we have

$$n = \sum_{i=1}^{n} \frac{d_i^2}{2} - n$$

or, equivalently,

$$\sum_{i=1}^{n} {d_i}^2 = 4n. \tag{1}$$

At this point, you can try to use graph theory to show that $d_i > 1$ for every i. The last equation tells you that $d_i \leq 2$ and therefor G is 2-regular.

However, a sneaky way that avoids graph theory is to use the Cauchy-Schwarz inequality (see `https://en.wikipedia.org/wiki/Cauchy?Schwarz_inequality`).

If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are real vectors in n-space, the Cauchy-Schwarz theorem says that

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2, \tag{2}$$

and (what will be useful to us) equality holds if and only if $(x_1, \ldots, x_n) = c * (y_1, \ldots, y_n)$ for some constant c.

So, apply (2) with $x_i = d_i$ and $y_i = 1$ to get:

$$\left(\sum_{i=1}^{n} d_i\right)^2 = \left(\sum_{i=1}^{n} d_i * 1\right)^2 \leq \sum_{i=1}^{n} d_i^2 \sum_{i=1}^{n} 1^2 = n * \sum_{i=1}^{n} d_i^2$$

Now combine with (1) and use the fact that $\sum_{i=1}^{n} d_i = 2e(G) = 2n(L(G)) = 2n(G) = 2n$ to get:

$$(2n)^2 = \left(\sum_{i=1}^{n} d_i\right)^2 \leq n * \sum_{i=1}^{n} d_i^2 = n * 4n = 4n^2.$$

This means that equality must hold and therefore, by the Cauchy-Schwarz theorem, $x = c * y$ for some constant c, i.e. $(d_1, \ldots, d_n) = c(1, \ldots, 1) = (c, \ldots, c)$. Substituting in the previous equation,

$$(2n)^2 = \left(\sum_{i=1}^{n} c\right)^2 = c^2 n^2,$$

and therefore $c = 2$, that is, $d_i = 2$ for all i and G is 2-regular.

3. (Note that this result is good for the football scheduling application.)

If $e(G) < \Delta + 1$, color each edge a different color. Then $0 \leq e(G)/(\Delta + 1) < 1$ and each color is used 0 or 1 times, so the result holds.

If $e(G) \geq \Delta + 1$, by Vizing's theorem there is a $\Delta + 1$ coloring. First make sure all colors are used. At least Δ colors are used. If some color i is not used, then another color j is used on at least two edges. Re-color one of those edges with i. Now all $\Delta + 1$ colors are represented.

For $1 \leq k \leq \Delta + 1$, let m_k be the number of edges colored with color k. Let $a = \lfloor e(G)/(\Delta + 1) \rfloor$ and $b = \lceil e(G)/(\Delta + 1) \rceil$. Note that $(\Delta + 1) * b \geq e(G)$ and $(\Delta + 1) * a \leq e(G)$.

Suppose it is not the case that m_k is a or b for every k.

Let s be the most used color and t be the least used color. Then, because of the previous statement, at least one of these must be true: $m_s > b$ or $m_t < a$.

If $m_s > b$, then $m_t < b$, otherwise

$$e(G) = m_1 + m_2 + \ldots + m_{\Delta+1} \geq m_s + \Delta * m_t \geq b + 1 + \Delta * b = (\Delta + 1) * b + 1 \geq e(G) + 1.$$

Similarly, if $m_t < a$, then $m_s > a$, otherwise

$$e(G) = m_1 + m_2 + \ldots + m_{\Delta+1} \leq m_t + \Delta * m_s \leq a - 1 + \Delta * a = (\Delta + 1) * a - 1 \leq e(G) - 1.$$

Note that in either case we have $m_s - m_t \geq 2$.

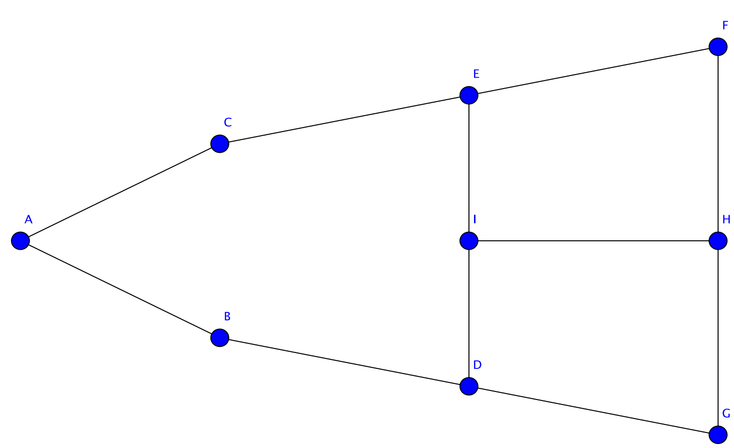
Then perform the following "rebalancing step": Look at the subgraph induced by the edges colored s and t. Every component of this subgraph is an even cycle or a path. Therefore colors s and t can be swapped in some of the components to balance the count so that $m_s - m_t \leq 1$. (Note, however, that it is not necessarily true that after this one iteration of rebalancing m_s and m_t are in $\{a, b\}$.)

Repeat the rebalancing step until m_k is a or b for every k. To see that the procedure terminates with this condition, note that each iteration either increases by one a count m_t that was smaller than a or decreases by one a count m_s that was larger than b.

4. The Harary graph $H_{4,7}$ is class 2, since each edge color class can have at most 3 edges and the number of edges is 14. The graph $H_{3,7}$ is class 1 (its maximum degree is 4). The Harary graphs $H_{3,6}$, $H_{4,6}$, and $H_{5,6}$ are class 1. ($H_{5,6}$ is K_6).

5. a. If G is Eulerian then $L(G)$ is Hamiltonian: If $v_0, e_1, \ldots, v_{k-1}, e_k, v_k = v_0$ is an Eulerian circuit in G, then $e_1, e_2, \ldots, e_k, e_1$ is a Hamiltonian cycle in $L(G)$.

b. Example below. There are four vertices of degree 3, so it is not Eulerian. But you can check that the line graph is Hamiltonian. (I constructed it with help from exercise 7.2.2, text: it suffices to show that the graph below has a closed circuit containing at least one endpoint of every edge of the graph: A,B,D,G,H,I,E,C,A. Note that it need not contain every vertex or every edge.)



6. The graph is Hamiltonian (you should exhibit the Hamiltonian cycle) but our necessary

conditions (Dirac, Ore Chvatal) do not apply.

7. Let G_n the the graph whose vertices are the squares of a $4 \times n$ chessboard and vertices u and v are joined by a edge in G_n if you can get from square u to square v by a knight's move. A knight's tour is a Hamiltonian cycle in G_n.

We want to show that G_n is not Hamiltonian by finding a set S_n of vertices (squares) so that $G_n - S_n$ has more than $|S_n|$ components.

To describe S_n, assume that the squares of the $4 \times n$ chessboard are colored white and black in the usual way, with the top left square colored white. Let S_n be the set of white squares in rows 2 and 3. Then $|S_n| = n$. Note that all neighbors of the black squares in rows 1 and 4 are in S_n. Thus in $G_n - S_n$, every black square in rows 1 and 4 is in its own 1-vertex component. There is at least one other component since the white squares in rows 1 and 4 and the black squares in rows 2 and 3 must be in some component. There are n black squares in rows 1 and 4, so the number of components of $G_n - S_n$ is at least $n + 1 > n = |S_n|$.

8. a. NO b. YES