Coloring Planar Graphs

(so assume graphs are simple)

ex 1 Show if G is simple, planar then $\delta(G) \leq 5$

ex. 2 Show that planar graphs are 6-colorable

The Five Color Theorem

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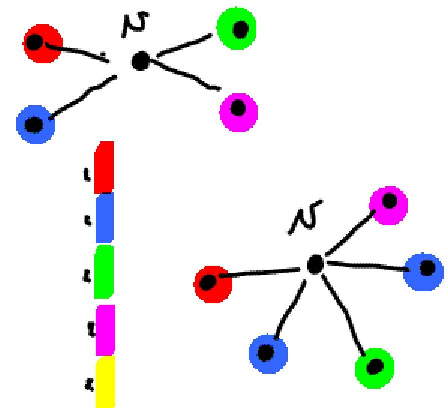

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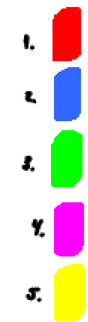
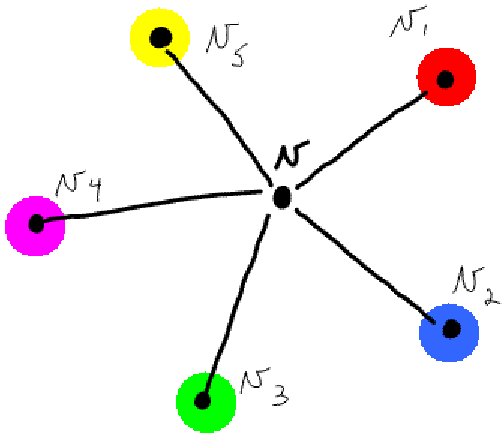
If $d_G(v) \leq 4$ or if $d_G(v) = 5$, but neighbors of v use only 4 colors, then G is 5-colorable.

So, suppose $d_G(v) = 5$ and each nbr of v has different color.

Planar embed G and label nbrs of v clockwise v_1, v_2, v_3, v_4, v_5





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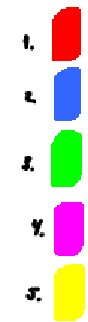

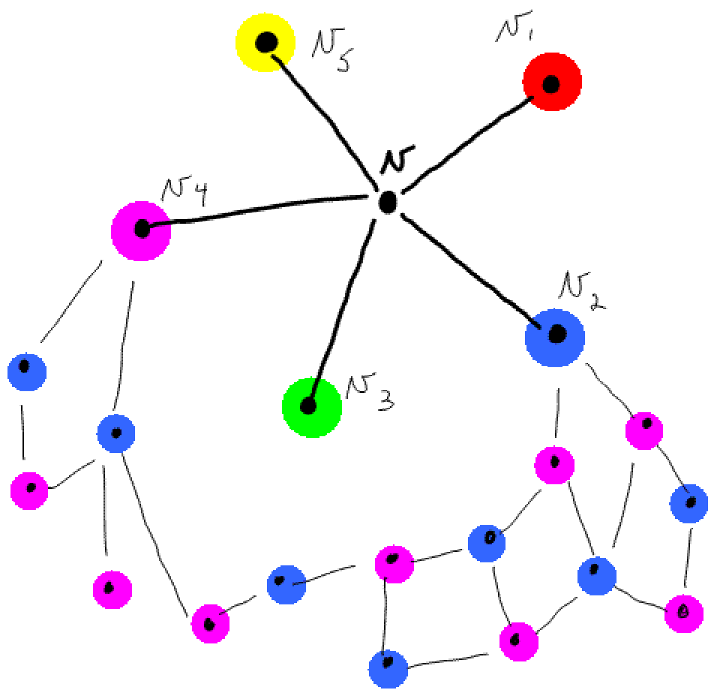

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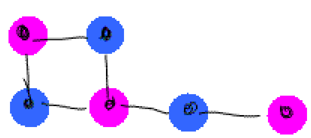
e.g. $G_{2,4}$



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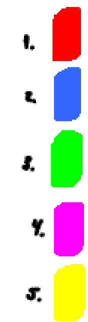

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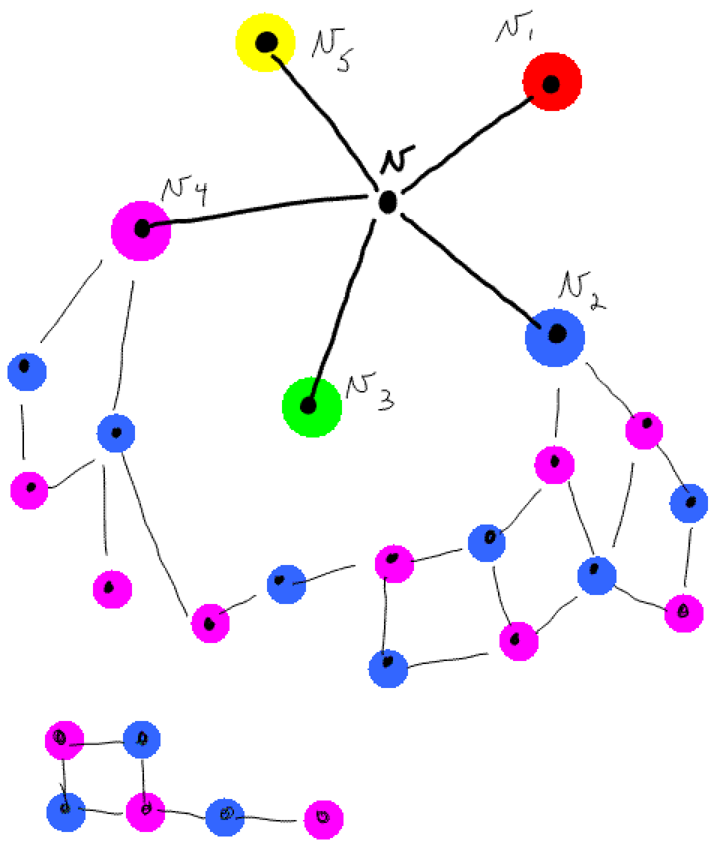

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Are v_2 + v_4 in same component?

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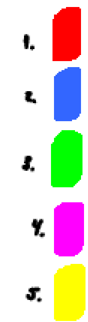

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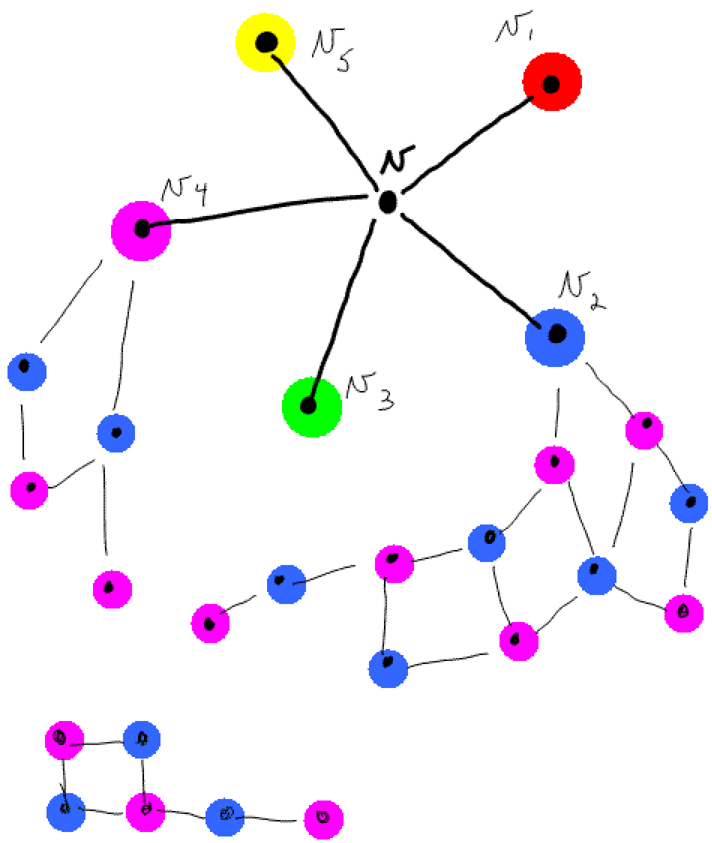

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Are v_2 & v_4 in same component?

If not

Swap colors in one of the components

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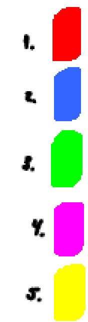
Planar embed G and label nbrs of v clockwise v_1, v_2, v_3, v_4, v_5



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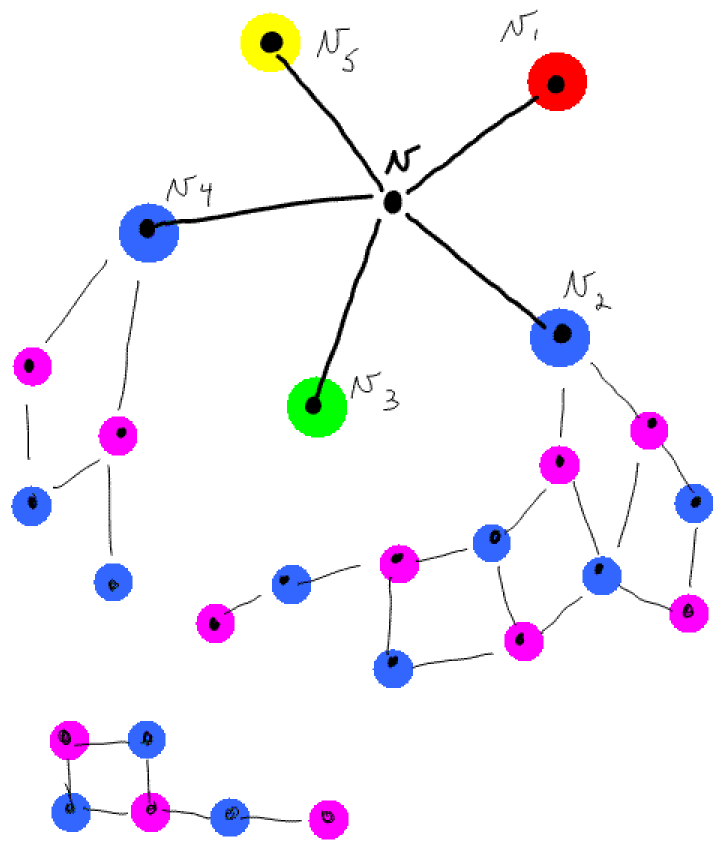

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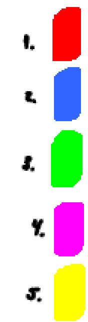

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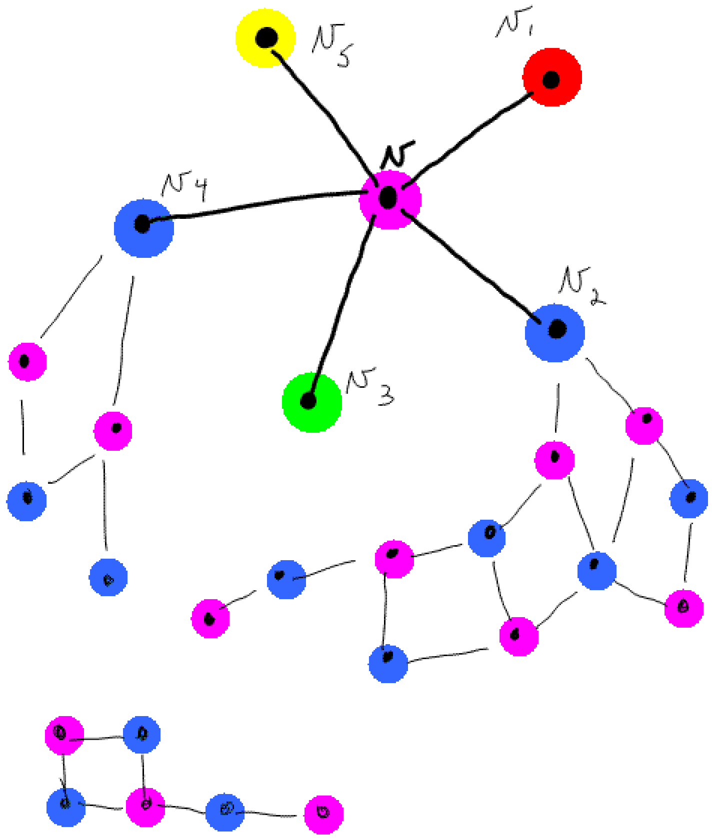

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Are v_2 & v_4 in same component?

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Swap colors in one of the components and make a color available for v, giving a 5-coloring of G.

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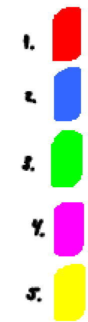

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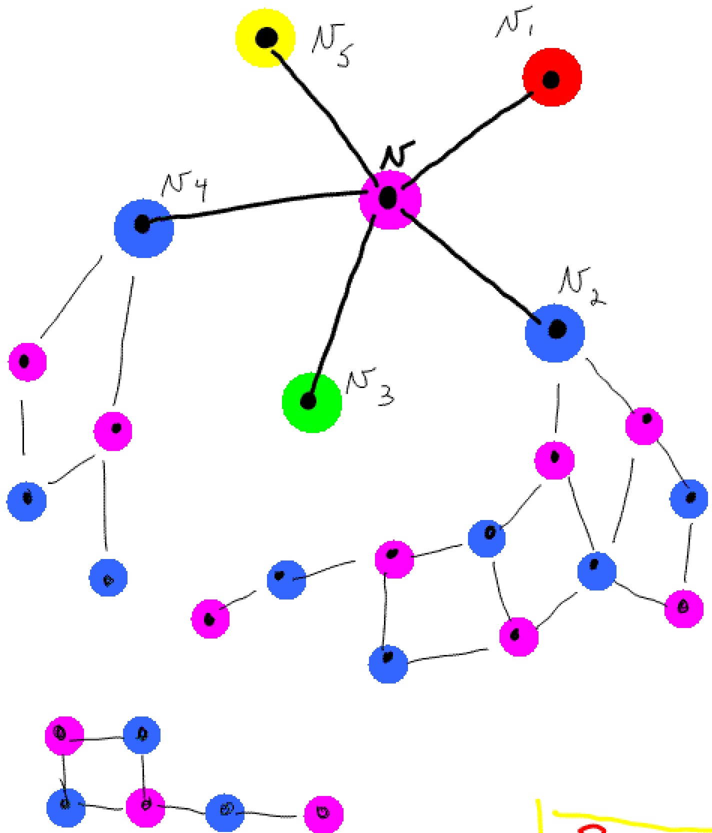


Let $G_{i,j}$ be subgraph induced by vertices colored i and j

e.g. $G_{2,4}$

Are v_2 & v_4 in same component?

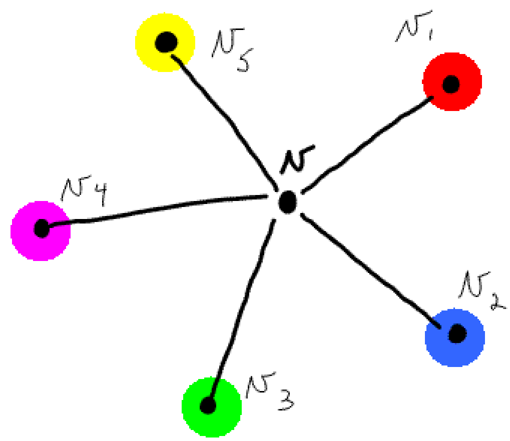
If not

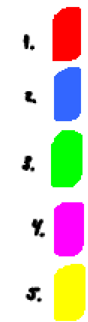
Swap colors in one of the components and make a color available for v, giving a 5-coloring of G.

But what if v_i connected to v_j in every $G_{i,j}$?

Let $G_{i,j}$ be subgraph induced by vertices colored i and j

Suppose in every G_{ij} there is path P_{ij} joining v_i and v_j





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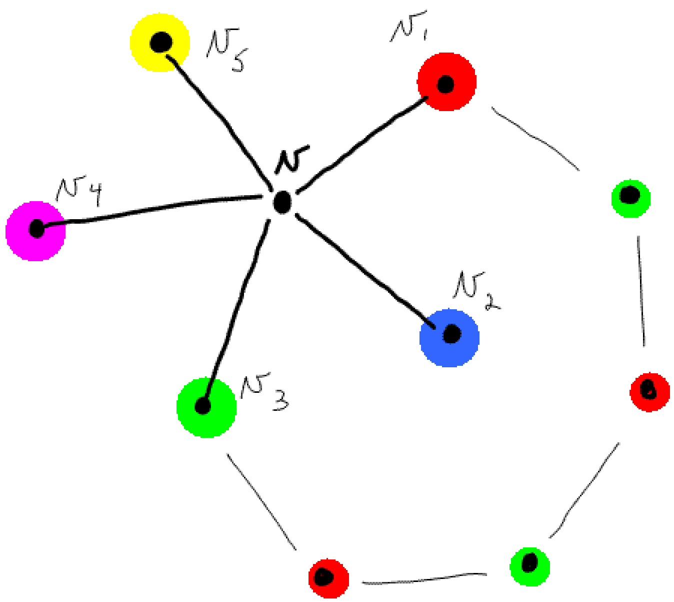

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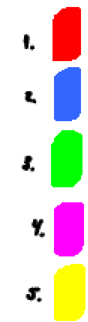

$P_{13} + vv_1 + vv_3$ forms a cycle C

v_2 and v_4 are on opposite sides (int/ext)

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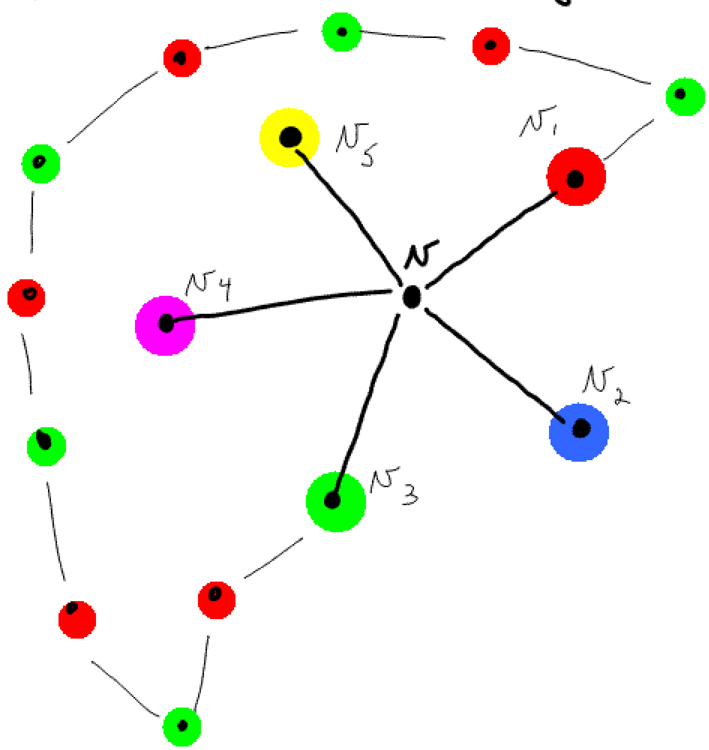

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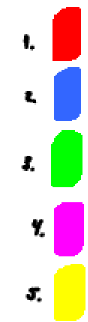

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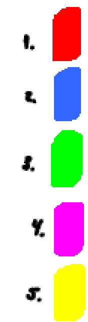

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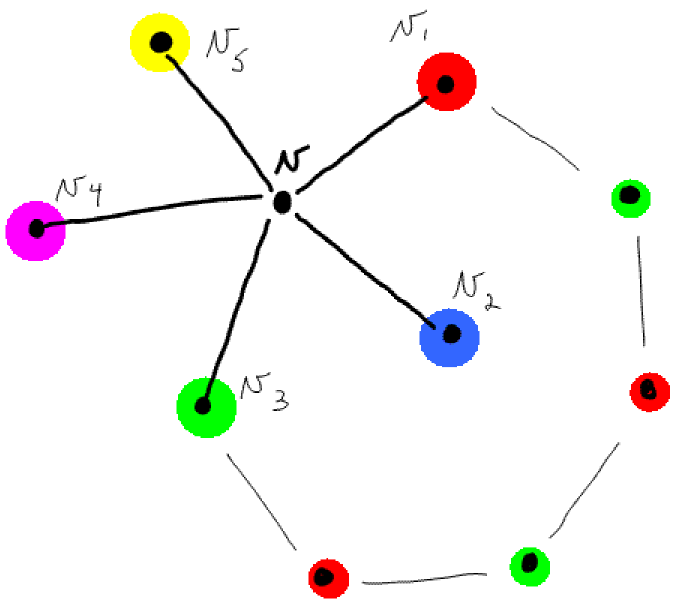

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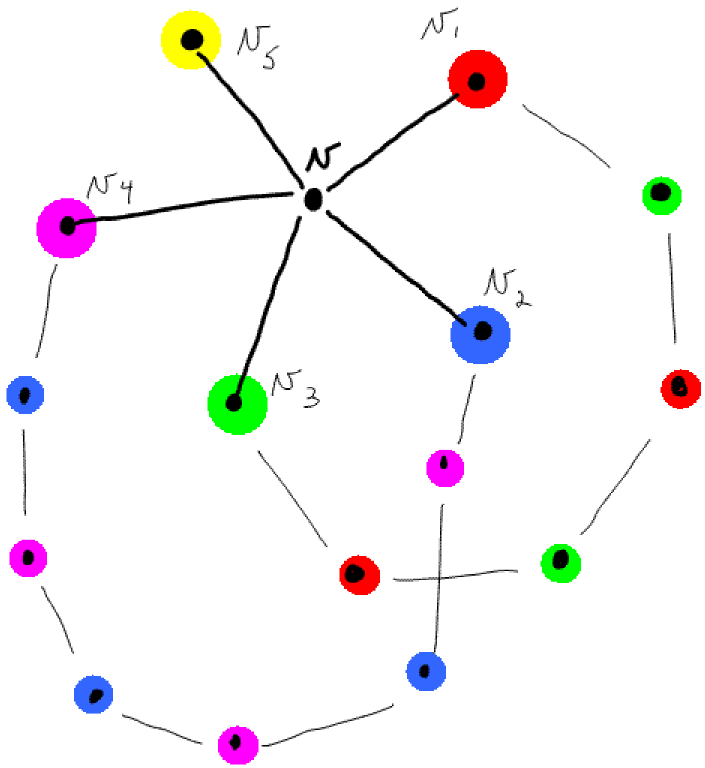

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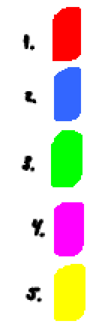

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So P_{24} must cross C (not at a vertex) contradicting that the embedding is planar.

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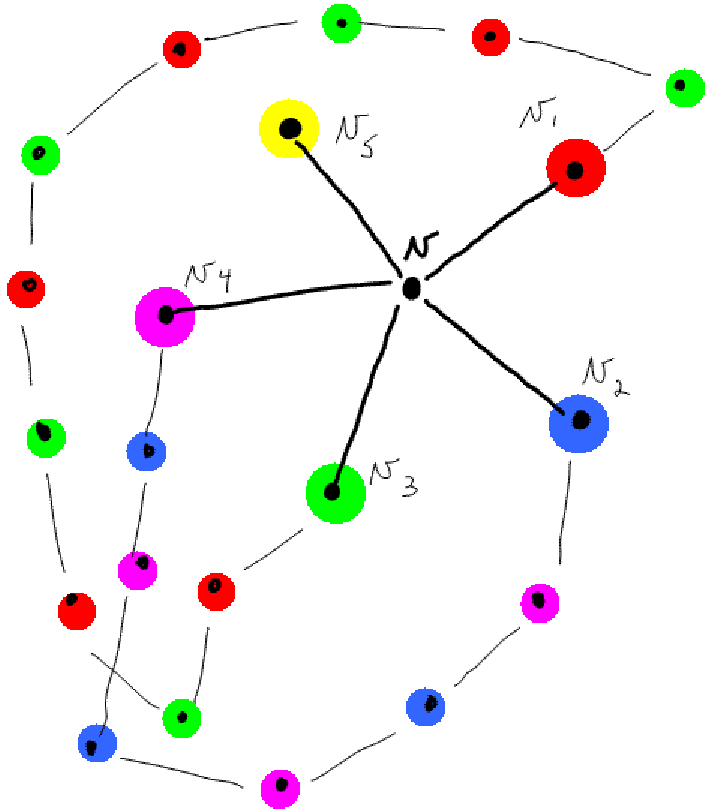

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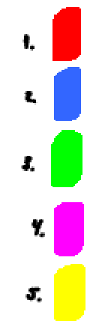

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