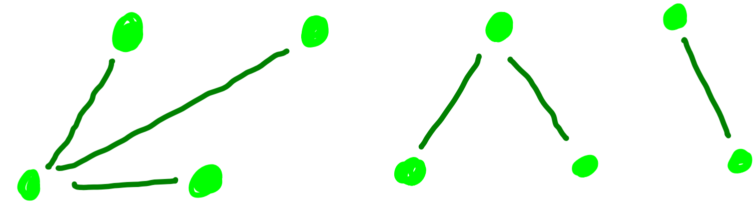
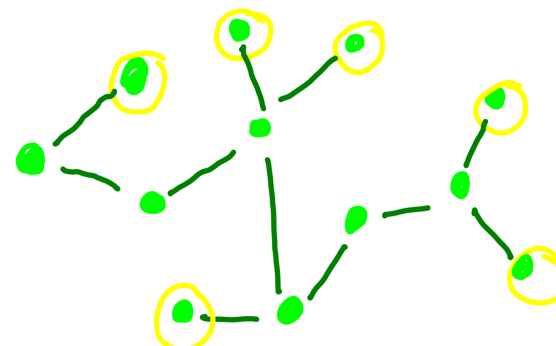
Trees

G is **acyclic** if no cycles.



forest: acyclic graph

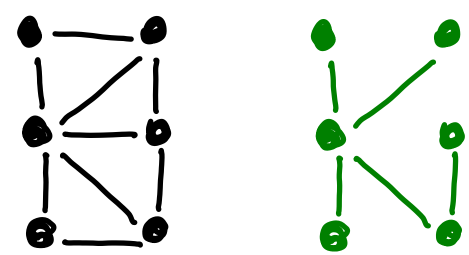
tree: connected acyclic graph



leaf: vertex of degree 1

spanning tree of connected graph G:

spanning subgraph of G which is a tree.



Lemma 2.1.3. If T is an n-vertex tree with $n > 1$, then G has at least two leaves.

[redacted]

Proof.

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[redacted]

Proof. Let p be a longest path ...

Lemma 2.1.3. If T is an n-vertex tree [redacted]

If v is a leaf of T, then $T - v$ is a tree.

Proof. Show $T-v$ is acyclic and connected.

Since T is acyclic, so is any subgraph, $\therefore T-v$ is acyclic.

Let x, y be vertices in $T-v$. Since T is connected, there is an x,y path p in T. Since v is a leaf, v is not on p. Thus p is an x,y path in $T-v$, so $T-v$ is connected

Characterization of Trees

Theorem 2.1.4. For an n-vertex graph G, the following are equivalent.

A) G is connected and has no cycles.

B) G is connected and has $n - 1$ edges.

C) G has $n - 1$ edges and no cycles.

D) Every pair of vertices of G is connected by a unique path.

Proof Structure: (in text)

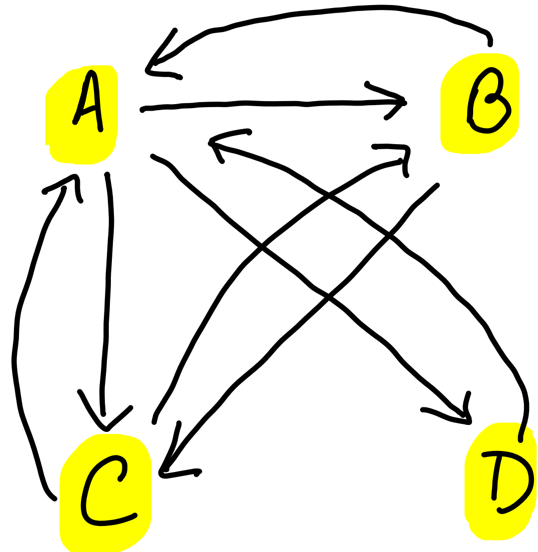
$$A \Rightarrow B, C;$$

$$B \Rightarrow A, C;$$

$$C \Rightarrow A, B;$$

$$A \Rightarrow D;$$

$$D \Rightarrow A;$$



Corollary 2.1.5.

a) Every edge in a tree is a cut edge.

b)Adding an edge e not in T to tree T creates a unique cycle containing e.

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Pf. No edge lies on a cycle (there are none)

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a) Every edge in a tree is a cut edge.

Pf. No edge lies on a cycle

(there are none)

b) Adding an edge $\boldsymbol{e}$ not in $\boldsymbol{T}$ to tree $\boldsymbol{T}$ creates a unique cycle containing $\boldsymbol{e}$.

By (D), T has a unique u,v path.
Adding uv creates a cycle. If not unique, u,v path not unique.

(c) Every connected graph contains a spanning tree.

while G is not acyclic do

let C be a cycle of G

let e be an edge of C

delete e from G

At the end of each iteration:

(loop invariant)

(c) Every connected graph contains a spanning tree.

while G is not acyclic do

let C be a cycle of G

let e be an edge of C

delete e from G

At the end of each iteration:

$\boxed{\text{G is connected}}$

(loop invariant)

Proposition 2.1.6. If T and T' are two spanning trees of connected graph G and

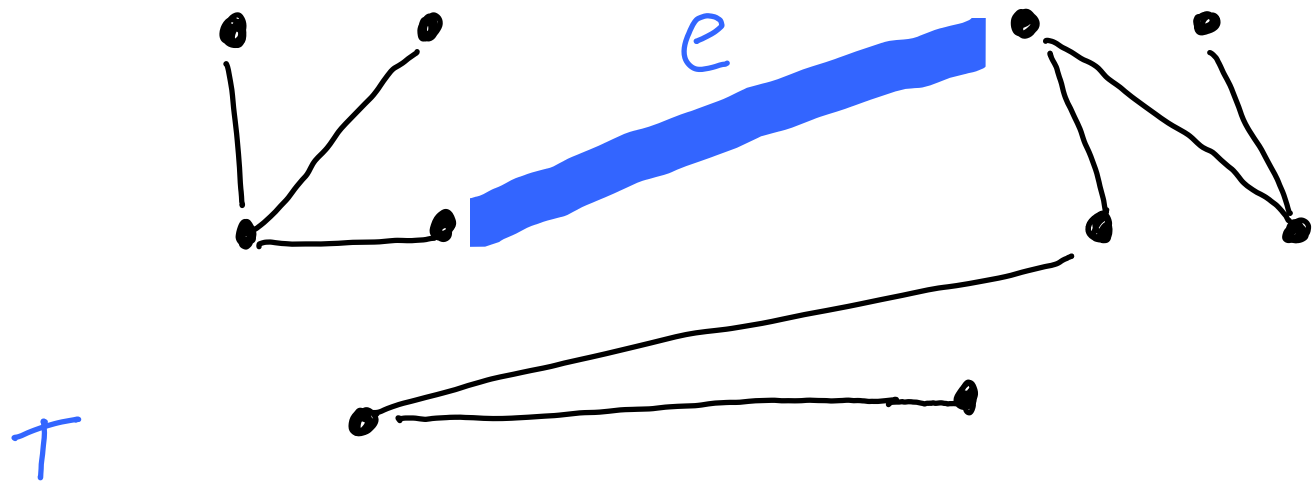
$$e \in E(T) - E(T'),$$

then there is an edge

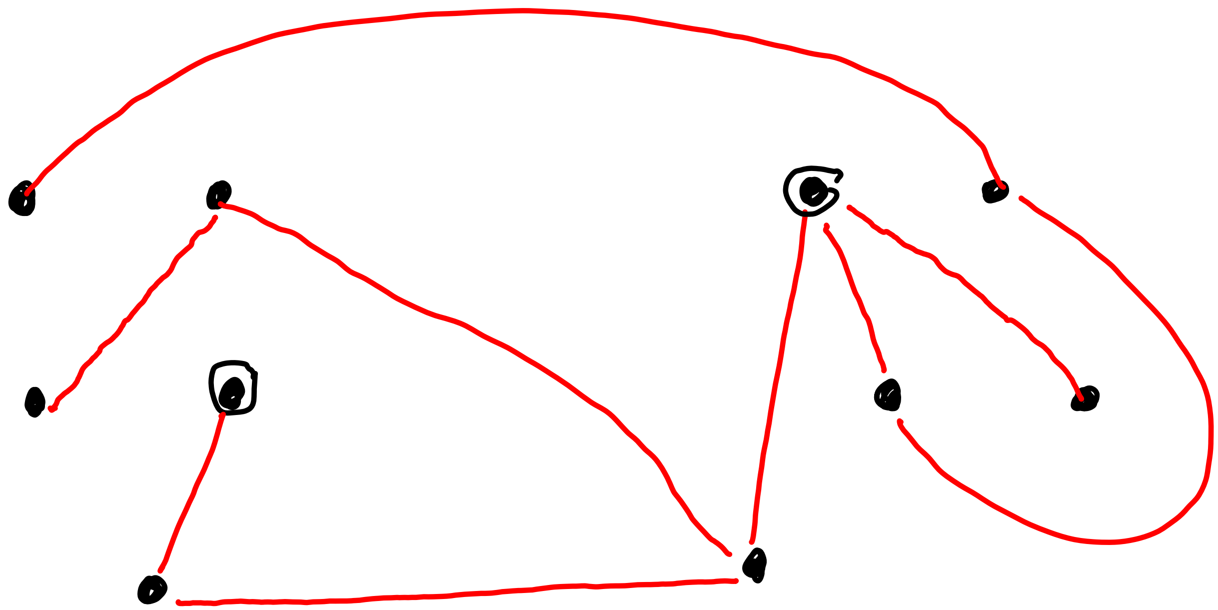
$$e' \in E(T') - E(T)$$

such that $T - e + e'$ is a spanning tree of G.

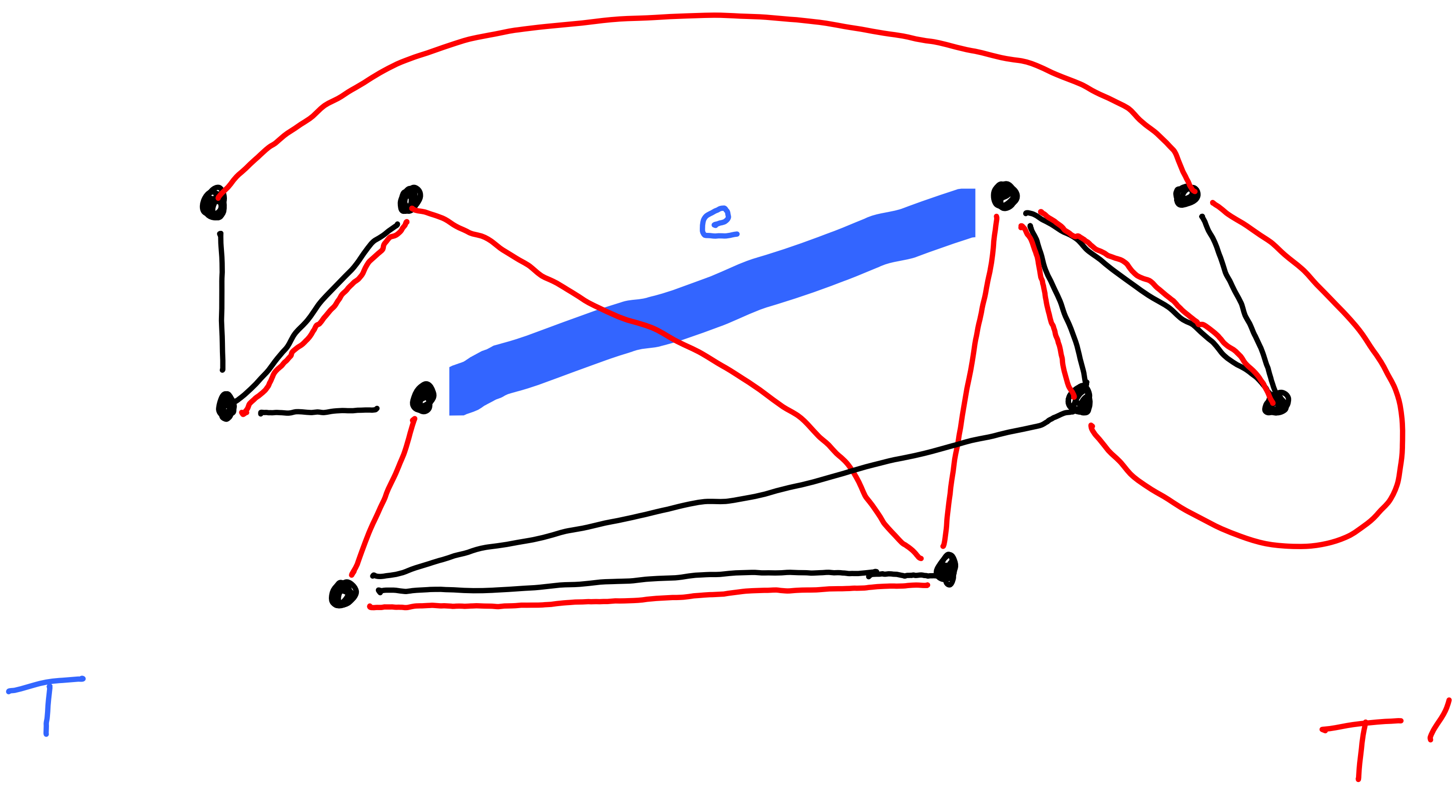
Proof.



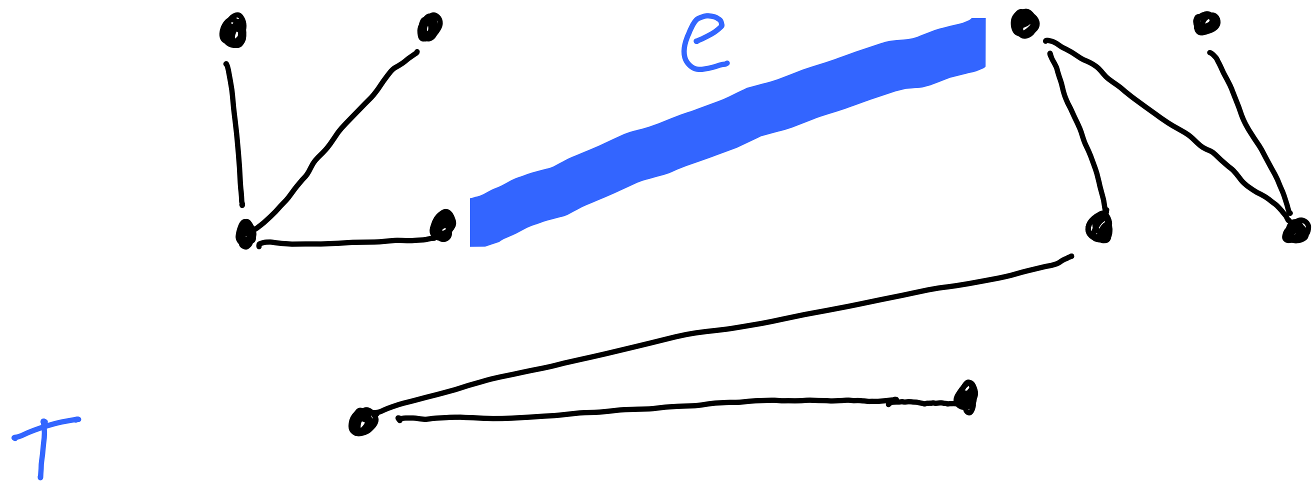
e
T

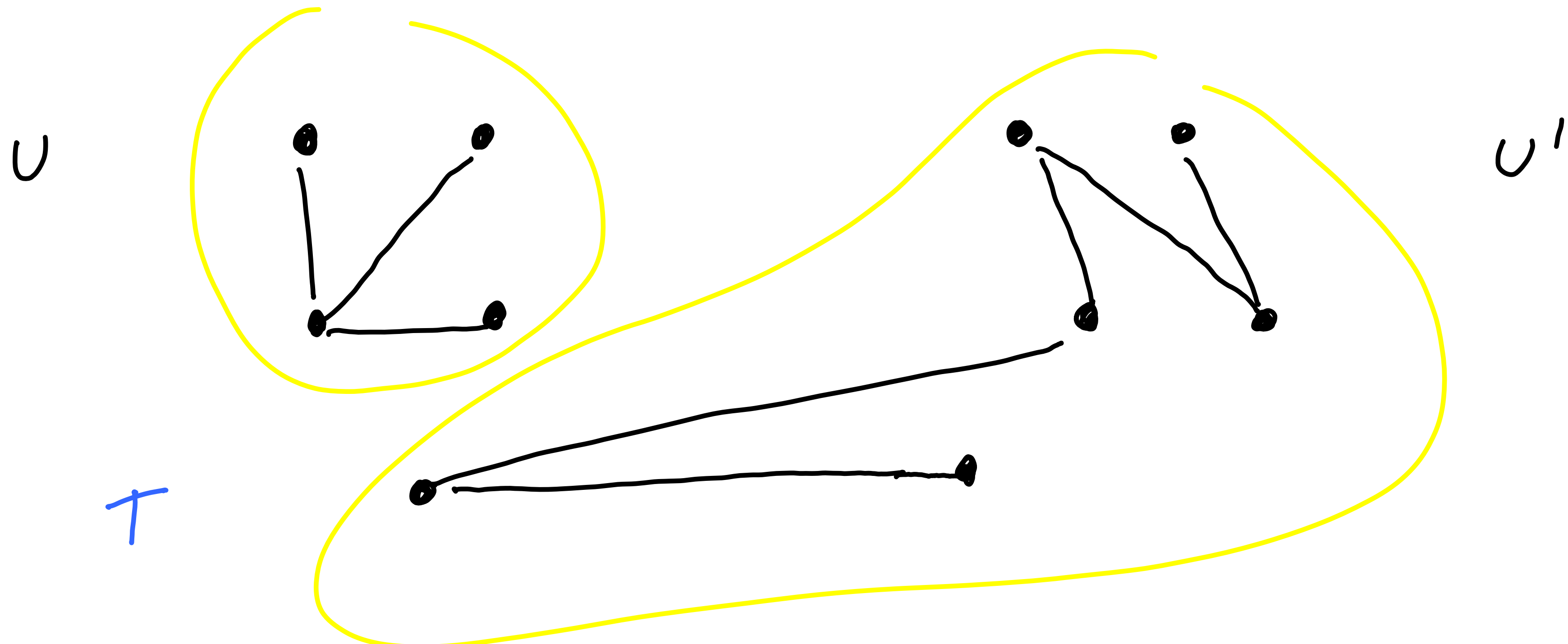


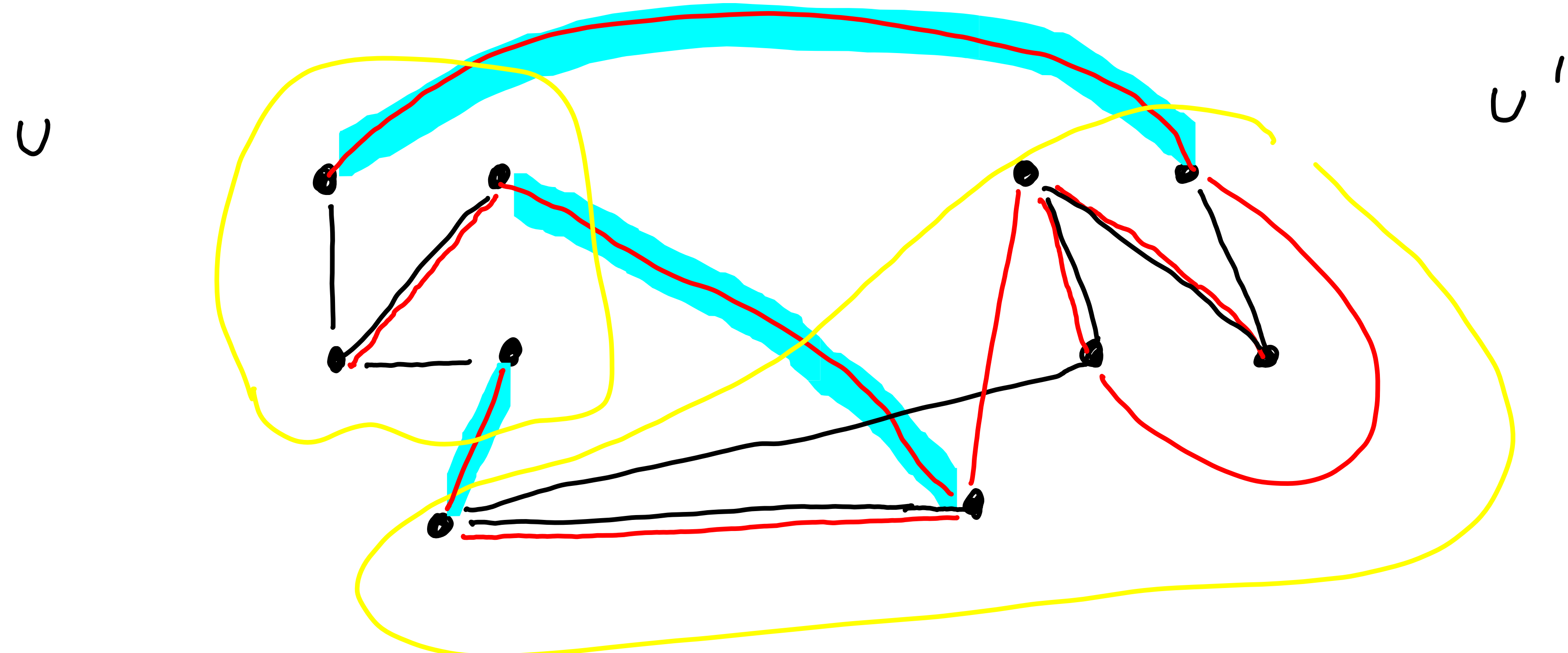
T'



e
T
T'

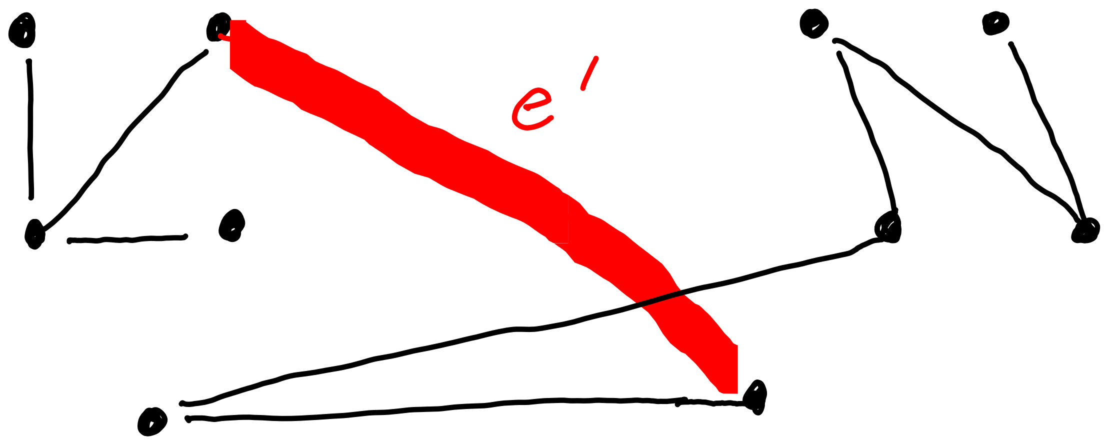

e
T


U
U'
T



Candidates for e'

(edges in T' joining U to U')



$T - e + e'$

still connected

still $n-1$ edges

$\therefore$ by Thm 2.1.4, tree

Proposition 2.1.~~6~~7. If T and T' are two spanning trees of connected graph G and

$$e \in E(T) - E(T'),$$

then there is an edge

$$e' \in E(T') - E(T)$$

$T' + e - e'$

such that $T' + e - e'$ is a spanning tree of G.

Proof.

Proposition 2.1.7 If T and T' are two spanning trees of a connected graph G and if

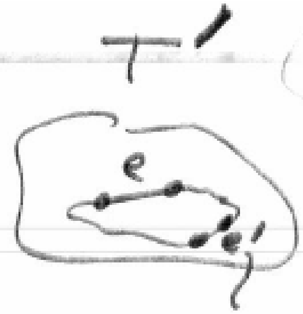
$$e \in E(T) - E(T')$$

then there is an edge

$$e' \in E(T') - E(T)$$

such that $T' + e - e'$ is a spanning tree of G.

Prop. 2.1.7



not in T

Pf $T'+e$ has a (unique) cycle C.

Let e' be an edge of C not in T

(T has no cycles, so there must be such e')

$T'-e'+e$ is connected (e' not a cut edge since on a cycle)

and $T'-e'+e$ has $n-1$ edges

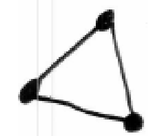
connected and $n-1$ edges $\Rightarrow$ spanning tree

. **distance** from u to v in G, $d_G(u,v)$:

least length of a u, v path in G, if one exists.

diameter of G:

$$\max_{u,v \in V(G)} d(u,v)$$



eccentricity of vertex u of G:

$$\epsilon(u) = \max_{v \in V(G)} d(u,v)$$

radius of G:

$$\min_{u \in V(G)} \epsilon(u)$$

Theorem 2.1.11. If G is a simple graph then diam $G \geq 3$ implies that diam $\overline{G} \leq 3$.

Proof.

Center of a graph G

what is it?

Give some interesting examples

Theorem 2.1.13. The center of a tree is a vertex or an edge.

Proof.