Proceedings of the National Conference on Computational Modeling of Fluid Dynamics Problems (CMFDP-2019)
NIT Warangal, India – Jan 18-20, 2019
Paper No. XXX (The number assigned when the papaer is accepted)

Steady Flow of Couple Stress Fluid through a Rectangular Channel Under Transverse Magnetic Field

M. Pavankumar Reddy[a*], J.V. Ramana Murthy[b]
[a, b]*Department of Mathematics, National Institute of Technology Warangal, Warangal-506004, Telangana, India*

*Corresponding author Email: mprnitw@gmail.com

Received: Date? Accepted: Date?

Abstract

In this paper, we have considered the steady and an incompressible conducting couple stress fluid flow in the presence of transverse magnetic field through a rectangular channel with uniform cross–section. The induced magnetic field is neglected. We consider the case that there is no externally applied electric field. Under these conditions, we get 4^{th} order PDE for velocity w along the axis of the rectangular tube. The usual no slip and hyper stick boundary conditions are used to obtain the solution for w. We obtained the velocity w in terms of Fourier series. Skin friction on the walls and volumetric flow rate are obtained in terms of physical parameters like couple stress parameter and Hartmann number. The effects of these parameters on skin friction and volumetric flow rate are studied through graphs.
Keywords: Couple stress fluid, couple stresses, skin friction, rectangular channel, magnetic field.

1. Introduction

The steady flow of a conducting fluid through a straight avenue under a uniform transverse magnetic field presents one of the elementary problems in magneto hydrodynamics. Magnetic flow in a rectangular channel is a humanistic problem that has significant applications in magneto hydrodynamic power generators and pumps *etc*. Nowadays, magnetic field has earned great value due to widespread applications in industry and bioengineering, such as electrostatic precipitation, power generators, petroleum industry, aerodynamic heating, the purification of molten metals from non-metallic inclusions, polymer technology and fluid droplet sprays. Hartmann[1] was the first person to obtain a solution for this type of flows to compare with his experimental results on mercury. Hartmann and Lazarus[2] studied the impact of a transverse uniform magnetic field on the flow of a viscous incompressible electrically conducting fluid between two infinite parallel stagnant and insulating plates. An approximate method of solution has given by Tani[3] for the steady laminar incompressible flow of an electrically conducting fluid through a straight avenue of arbitrary cross section with conducting or non-conducting walls in the presence of a uniform transverse magnetic field based on a minimum principle. Ahmed and Attia[4] further studied the viscous and joule dissipation effects under an external uniform magnetic field in an eccentric annulus of an electrically conducting incompressible fluid. Abel et al.[5] studied the momentum, mass and heat transfer past a stretching sheet using the Walters-B viscoelastic model in the presence of a transverse magnetic field. Ahmed and Attia[6] and Attia[7] studied the MHD flow and heat transfer of a viscous incompressible fluid through a rectangular duct. Hassan and Attia[8] considered the transient Hartmann flow of a dusty incompressible fluid in a rectangular channel under the influence of an applied uniform magnetic field. The steady flow of Micropolar Fluid with Suction under transverse magnetic field in a rectangular channel was studied by Ramana Murthy et al.[9]. Srnivasacharya and Shiferaw[10] studied the steady flow of an electrically conducting and incompressible micropolar fluid flow through a rectangular channel taking into consideration the Hall and ionic effects.

In the above studies of non-Newtonian fluids with MHD effect, couple stress fluids have not been considered. Stokes[11] introduced the theory of Couple stresses and gave the simplest generalization of the humanistic viscous fluid theory that maintains the couple stresses and body couples. One of the applications of couple-stress fluid is its use to the study the mechanism of lubrication of synovial joints[12], which has grown to be the object of scientific research. In Recent past, the significance of couple stress fluid flows in chemical engineering applications involving liquid crystals, polymeric suspensions[13], polymer-thickened oils and physiological fluid mechanics[14] were attracted by researchers. These fluids are also used vigorously in the tribology of thrust bearings[15] and the lubrication of engine rod bearings[16]. Couple stress fluids are not much complicated than micropolar fluids[17]. As the microstructure was not available at the kinematic level, hence kinematics of such fluids were explained using the velocity field. Stokes' problems were studied by Devakar and Iyengar[18] under the isothermal conditions for an incompressible couple stress fluid. The magnetic field effects in 3D flow subject to convective boundary condition were investigated by Hayat et al.[19] for couple stress nanofluid over a nonlinear stretched surface. Srinivasacharya and Kaladhar[20] studied the mixed convection flow of couple stress fluid with Soret and Dufour effects in a non-Darcy porous medium. The inclined magnetic field characteristics of couple stress material in a porous medium was recently inspected by Ramesh[21] in peristaltic flow. The peristaltic flows were investigated by Reddy et al.[22] in a rectangular duct. The information available on the topic endorsed that the magneto hydrodynamic flow of couple stress fluid has not been treated analytically through a rectangular channel. In this paper, we have studied the magneto hydrodynamic couple stress fluid flow through a rectangular channel. We have used Cartesian co-ordinate system for formulating the mathematical equations and obtained the exact solution for velocity. Skin friction on the walls and volumetric flow rate are obtained in terms of physical parameters like couple stress parameter and Hartmann number. We have studied the effects of these parameters on volumetric flow rate, skin friction and illuminated the results through graphs.

2. Mathematical Formulation

The coupled equations for steady, incompressible and couple stress fluid flow with transverse magnetic field are given by

$$\nabla_0 . \bar{Q} = 0 \tag{1}$$

$$\rho \bar{Q} . \nabla_0 \bar{Q} = -\nabla_0 \mathrm{P} + \mu \nabla_0^2 \bar{Q} - \eta \nabla_0^4 \bar{Q} + \bar{J} \times \bar{H} \tag{2}$$

where $\bar{Q}$ is the velocity, P is the pressure, ρ is the density, μ is the viscosity coefficient, η is the couple stress viscosity parameter.

An incompressible and couple stress fluid flow through a channel is considered with uniform rectangular cross section with side lengths *a* and *b*. Using a Cartesian co-ordinate system (X,Y,Z) with center of rectangular cross section as origin and the axis of the tube as Z axis along which the flow is assumed. H_0, a constant magnetic field in the perpendicular direction to the flow is applied. Along the rectangular tube a constant pressure gradient causes generation of the flow in it. We made an assumption that the induced magnetic and electrical fields are negligible.

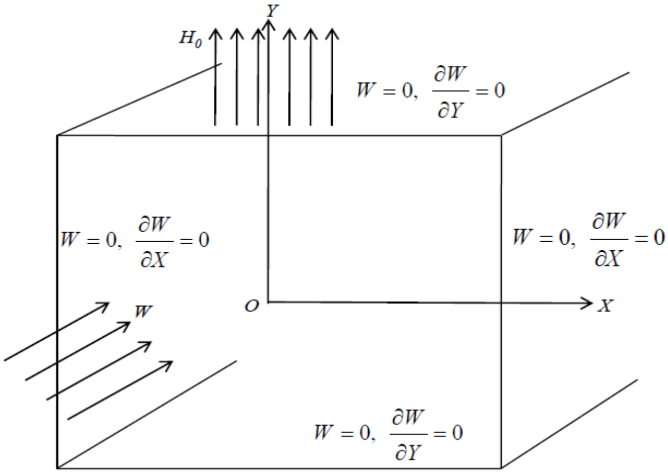

Fig.1 Flow configuration in a rectangular channel

We take by the geometry of the problem given in fig. 1 and nature of the flow

$\bar{Q} = U_0 W \bar{k}, \quad \bar{H} = H_0 \bar{j}, \quad \bar{J} = \frac{\sigma}{c} \bar{Q} \times H_0 \bar{j} = -\frac{\sigma}{c} U_0 H_0 W \bar{i}$ where W=W(X, Y)

Hence $\bar{J} \times \bar{H} = -\frac{\sigma}{c} U_0 H_0^2 W \bar{k}$ and $\bar{Q}.\nabla_0 \bar{Q} = 0$

Now equation (2) reduces to

$$\nabla_0 \mathrm{P} = \mu \nabla_0^2 \bar{Q} - \eta \nabla_0^4 \bar{Q} - \frac{\sigma}{c} H_0^2 \bar{Q} \tag{3}$$

The following non-dimensional scheme is introduced:

$X = ax,\ Y = ay,\ W = U_0 w,\ Z = az$ and $P = \rho U_0^2 p$

where U_0 an average entrance velocity.Substituting these in equation (3) we obtain

$$\nabla^4 w - S\nabla^2 w + SM^2 w = -L_0 \tag{4}$$

The equation (4) is solved with no slip boundary conditions:

$$w = 0 \ on \ x = \pm 1 \ and \ y = \pm y_0 \ where \ y_0 = \frac{b}{a} \tag{5}$$

and hyper stick boundary conditions:

$$\frac{1}{2}\nabla \times \bar{Q} = \frac{1}{2}\frac{\partial w}{\partial y}\bar{i} - \frac{1}{2}\frac{\partial w}{\partial x}\bar{j} = 0 \text{ on } y = \pm y_0 \text{ and } x = \pm 1 \quad \text{respectively} \tag{6}$$

where Couple stress parameter $S = \frac{\mu a^2}{\eta}$, Reynolds number $\mathrm{Re} = \frac{\rho U_0 a}{\mu}$,

Hartmann number $M = \sqrt{\frac{\sigma H_0^2 a^2}{c\mu}}$ and $L_0 = SL = \mathrm{Re}.S\frac{dp}{dz} =$ constant

Equation (4) can be written as;

$$(\nabla^2 - \lambda_1^2)(\nabla^2 - \lambda_2^2) w = -L_0$$

where $\quad \lambda_1^2 + \lambda_2^2 = S, \quad \lambda_1^2 \lambda_2^2 = SM^2$

3. Solution of the Problem

Let us choose

$$w = -\frac{L}{M^2} + \sum_{n=1}^{\infty} f_n(y)\cos r_n x + \sum_{n=1}^{\infty} g_n(x)\cos t_n y \tag{7}$$

where $r_n = \frac{(2n-1)\pi}{2}$, $t_n = \frac{r_n}{y_0}$. Substituting (7) in (4) we get,

$$\sum_{n=1}^{\infty}\left(r_n^4 f_n - 2r_n^2 f_n^{ii} + f_n^{(iv)}\right)\cos r_n x + \sum_{n=1}^{\infty}\left(t_n^4 g_n - 2t_n^2 g_n^{ii} + g_n^{(iv)}\right)\cos t_n y$$

$$- S\left[\sum_{n=1}^{\infty}\left(-r_n^2 f_n + f_n^{ii}\right)\cos r_n x + \sum_{n=1}^{\infty}\left(-t_n^2 g_n + g_n^{ii}\right)\cos t_n y\right] + SM^2 \sum_{n=1}^{\infty}\left(f_n \cos r_n x + g_n \cos t_n y\right) = 0$$

$$\Rightarrow f_n^{(iv)} - (2r_n^2 + S) f_n^{ii} + (r_n^4 + Sr_n^2 + SM^2) f_n = 0 \tag{8}$$

and $g_n^{(iv)} - (2t_n^2 + S) g_n^{ii} + (t_n^4 + St_n^2 + SM^2) g_n = 0$ (9)

Equations (8) and (9) can be written as

$$\left(D^2 - u_n^2\right)\left(D^2 - v_n^2\right) f_n = 0 \text{ and } \left(D_1^2 - \alpha_n^2\right)\left(D_1^2 - \beta_n^2\right) g_n = 0 \text{ where } D = \frac{d}{dy} \text{ and } D_1 = \frac{d}{dx} \tag{10}$$

$$\therefore u_n^2 + v_n^2 = S + 2r_n^2 = \lambda_1^2 + \lambda_2^2 + 2r_n^2, \quad u_n^2 v_n^2 = r_n^4 + Sr_n^2 + SM^2 = r_n^4 + (\lambda_1^2 + \lambda_2^2) r_n^2 + \lambda_1^2 \lambda_2^2$$

$$\therefore \alpha_n^2 + \beta_n^2 = S + 2t_n^2 = \lambda_1^2 + \lambda_2^2 + 2t_n^2, \quad \alpha_n^2 \beta_n^2 = t_n^4 + St_n^2 + SM^2 = t_n^4 + (\lambda_1^2 + \lambda_2^2) t_n^2 + \lambda_1^2 \lambda_2^2$$

$$\therefore u_n^2 = \lambda_1^2 + r_n^2, \quad v_n^2 = \lambda_2^2 + r_n^2 \text{ and } \alpha_n^2 = \lambda_1^2 + t_n^2, \quad \beta_n^2 = \lambda_2^2 + t_n^2$$

Solving (10) we get

$$f_n(y) = A_n \frac{\cosh u_n y}{\cosh u_n y_0} + B_n \frac{\cosh v_n y}{\cosh v_n y_0} \text{ and } g_n(x) = C_n \frac{\cosh \alpha_n x}{\cosh \alpha_n} + D_n \frac{\cosh \beta_n x}{\cosh \beta_n}$$

$$\therefore w = -\frac{L}{M^2} + \sum_{n=1}^{\infty}\left(A_n \frac{\cosh u_n y}{\cosh u_n y_0} + B_n \frac{\cosh v_n y}{\cosh v_n y_0}\right)\cos r_n x + \sum_{n=1}^{\infty}\left(C_n \frac{\cosh \alpha_n x}{\cosh \alpha_n} + D_n \frac{\cosh \beta_n x}{\cosh \beta_n}\right)\cos t_n y \quad (11)$$

By no slip condition on $x = \pm 1$, $w = 0$ gives $\frac{L}{M^2} = \sum_{n=1}^{\infty}\left(C_n + D_n\right)\cos t_n y$ (12)

Again by no slip condition on $y = \pm y_0$, $w = 0$ gives $\frac{L}{M^2} = \sum_{n=1}^{\infty}\left(A_n + B_n\right)\cos r_n x$ (13)

By hyper-stick condition, on $y = \pm y_0$, $\frac{\partial w}{\partial y} = 0$ which gives

$$\sum_{n=1}^{\infty}\left(A_n u_n \tanh u_n y_0 + B_n v_n \tanh v_n y_0\right)\cos r_n x + \sum_{n=1}^{\infty}\left(C_n \frac{\cosh \alpha_n x}{\cosh \alpha_n} + D_n \frac{\cosh \beta_n x}{\cosh \beta_n}\right) t_n (-1)^n = 0 \quad (14)$$

Similarly by hyper-stick condition on $x = \pm 1$, $\frac{\partial w}{\partial x} = 0$ which gives

$$\sum_{n=1}^{\infty}\left(A_n \frac{\cosh u_n y}{\cosh u_n y_0} + B_n \frac{\cosh v_n y}{\cosh v_n y_0}\right) r_n (-1)^n + \sum_{n=1}^{\infty}\left(C_n \alpha_n \tanh \alpha_n + D_n \beta_n \tanh \beta_n\right)\cos t_n y = 0 \quad (15)$$

Using the orthogonality property, we have $\frac{1}{y_0}\int_{-y_o}^{y_0} \cos t_n y \ \cos t_m y \ dy = \delta_{\text{mn}}$

From (13) we obtain, $A_n + B_n = \frac{2L}{M^2 r_n}(-1)^{n+1}$

$$\Rightarrow \qquad B_n = \frac{2L}{M^2 r_n}(-1)^{n+1} - A_n \quad (16)$$

From (12) we obtain, $C_n + D_n = \frac{2L}{M^2 r_n}(-1)^{n+1}$

$$\Rightarrow \qquad D_n = \frac{2L}{M^2 r_n}(-1)^{n+1} - C_n \quad (17)$$

From (14) we obtain,

$$A_n u_n \tanh u_n y_0 + B_n v_n \tanh v_n y_0 = \sum_{m=1}^{\infty}(-1)^{m+1} t_m \left(C_m \frac{2r_n(-1)^{n+1}}{\alpha_m^2 + r_n^2} + D_m \frac{2r_n(-1)^{n+1}}{\beta_m^2 + r_n^2}\right) \quad (18)$$

From (15) we obtain,

$$C_n \alpha_n \tanh \alpha_n + D_n \beta_n \tanh \beta_n = \frac{1}{y_0}\sum_{m=1}^{\infty}(-1)^{m+1} r_m \left(A_m \frac{2t_n(-1)^{n+1}}{u_m^2 + t_n^2} + B_m \frac{2t_n(-1)^{n+1}}{v_m^2 + t_n^2}\right) \quad (19)$$

Substituting (16) and (17) in (18) and (19) we get

$$A_n\left(u_n \tanh u_n y_0 - v_n \tanh v_n y_0\right) + \sum_{m=1}^{\infty} C_m \left(\frac{2r_m r_n(\alpha_m^2 - \beta_m^2)(-1)^{m+n}}{y_0(\alpha_m^2 + r_n^2)(\beta_m^2 + r_n^2)}\right) = \sum_{m=1}^{\infty}\left(\frac{4Lr_n(-1)^{n+1}}{y_0 M^2(\beta_m^2 + r_n^2)}\right) - \frac{2Lv_n(-1)^{n+1}\tanh v_n y_0}{M^2 r_n} \quad (20)$$

and

$$C_n\left(\alpha_n \tanh \alpha_n - \beta_n \tanh \beta_n\right) + \sum_{m=1}^{\infty} A_m \left(\frac{2r_m r_n(u_m^2 - v_m^2)(-1)^{m+n}}{y_0^2(u_m^2 + t_n^2)(v_m^2 + t_n^2)}\right) = \sum_{m=1}^{\infty}\left(\frac{4Lr_n(-1)^{n+1}}{y_0^2 M^2(v_m^2 + t_n^2)}\right) - \frac{2L\beta_n(-1)^{n+1}\tanh \beta_n}{M^2 r_n} \quad (21)$$

Equations (20) and (21) are in the form

$$a1_n A_n + \sum_{m=1}^{\infty} C_m e_{nm} = b1_n \quad (22)$$

$$c1_n C_n + \sum_{m=1}^{\infty} A_m f_{nm} = d1_n \tag{23}$$

where $a1_n = u_n \tanh u_n y_0 - v_n \tanh v_n y_0$, $c1_n = \alpha_n \tanh \alpha_n - \beta_n \tanh \beta_n$,

$$e_{nm} = \frac{2 r_m r_n (\alpha_m^2 - \beta_m^2)(-1)^{m+n}}{y_0(\alpha_m^2 + r_n^2)(\beta_m^2 + r_n^2)}, \quad f_{nm} = \frac{2 r_m r_n (u_m^2 - v_m^2)(-1)^{m+n}}{y_0^2(u_m^2 + t_n^2)(v_m^2 + t_n^2)},$$

$$b1_n = \sum_{m=1}^{\infty}\left(\frac{4 L r_n (-1)^{n+1}}{y_0 M^2 (\beta_m^2 + r_n^2)}\right) - \frac{2 L v_n (-1)^{n+1} \tanh v_n y_0}{M^2 r_n},$$

$$d1_n = \sum_{m=1}^{\infty}\left(\frac{4 L r_n (-1)^{n+1}}{y_0^2 M^2 (v_m^2 + t_n^2)}\right) - \frac{2 L \beta_n (-1)^{n+1} \tanh \beta_n}{M^2 r_n}$$

Eliminating C_n from equations (22) and (23) we get

$$\sum_{m=1}^{\infty} a_{nm} A_m = b_n \tag{24}$$

where $a_{nm} = \begin{cases} a1_n - \sum_{k=1}^{\infty} e_{nk} \dfrac{f_{kn}}{c1_k} & \text{if } n = m \\ -\sum_{k=1}^{\infty} e_{nk} \dfrac{f_{km}}{c1_k} & \text{if } n \neq m \end{cases}$ and $b_n = b1_n - \sum_{m=1}^{\infty} e_{nm} \dfrac{d1_m}{c1_m}$

The equation (24) is an infinite system of equations in A_n. We truncate the system to n =10 and solve for A_n. Then from (22) C_n can be found. From (16), B_n can be found and from (17) D_n can be found. Hence all the coefficients in (11) for the velocity w are now known.

4. Results and Discussions:

For particular value of physical parameters S and M, the values of λ_1 and λ_2 are calculated using the quadratic equation

$$\lambda^2 - S\lambda + SM^2 = 0.$$

Then u_n, v_n, α_n and β_n are found. Now velocity w is computed using (11). The effects of physical parameters S and M on velocity, Volumetric flow rate and skin friction are found. We can observe that for a fixed S value, to get real values of λ, $S \geq 4M^2$.

Velocity w: In fig. 2, velocity contours at different values of M for a fixed value of S=50 are shown.

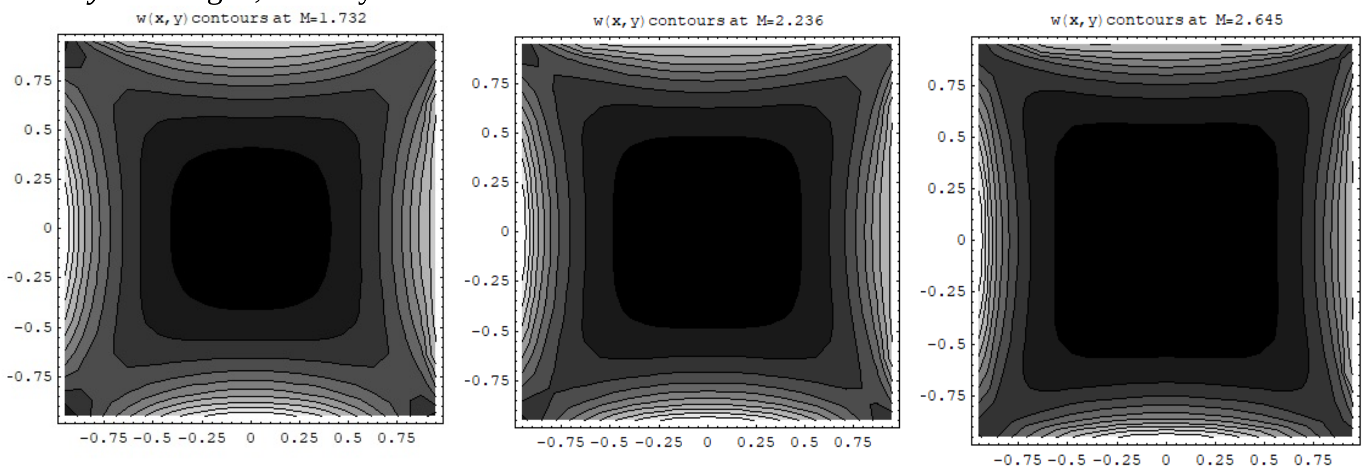

Fig. 2. For S = 50, Velocity $w(x, y)$ at M = 1.732, M = 2.236, M = 2.645

We notice that as M increases, fluid is having high velocity near the walls and more and more fluid is drifted towards walls of the channel and the center of the channel being maintained flat. In the figure, black region shows low values and bright region indicates high values of w. To show this clearly w is plotted in fig. 3 at fixed values of cross-sections for y=0.25, y=0.5, y=0.75 and y=0.9.

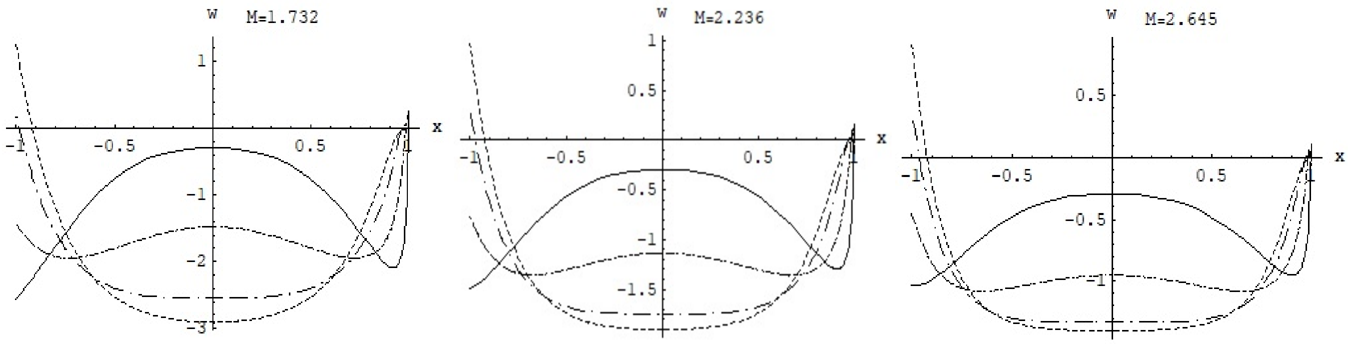

Fig.3.At S=50 & M=1.732, M=2.236, M=2.645, $w(x, y)$ at cross-sections y=0.25, y=0.5, y=0.75 & y=0.9

Volumetric Flow rate:

Volumetric flow rate V (non-dimensional) is given by

$$V = \int_{-1}^{1}\int_{-y_0}^{y_0} w\,dy\,dx$$

$$= -\frac{4Ly_0}{M^2} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{r_n}\left\{A_n\frac{Tanhu_n y_0}{u_n} + B_n\frac{Tanhv_n y_0}{v_n} + y_0 C_n\frac{Tanh\alpha_n}{\alpha_n} + y_0 D_n\frac{Tanh\beta_n}{\beta_n}\right\}$$

In fig. 4, volumetric flow rate V is shown at different values of magnetic parameter M. It is observed that as M increases, Volumetric flow rate decreases drastically. But when M is fixed, as S increases, Volumetric flow rate is almost constant.

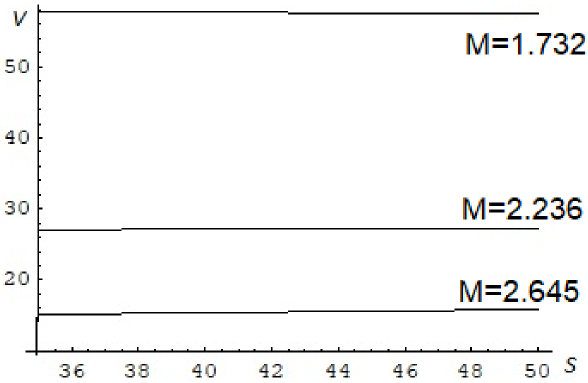

Fig. 4. Volumetric flow rate vs Couple stress parameter at different values of magnetic parameter M.

Skin friction: Skin friction is the force acting on the surface per unit area. It is obtained from constitutive equation of couple stress fluid.

$$T^* = -Ip + 2\mu E + \frac{1}{2}I \times div\, M$$

$M = mI + 4\eta\nabla\boldsymbol{\omega} + 4\eta'(\nabla\boldsymbol{\omega})^T$ with $\boldsymbol{\omega} = \frac{1}{2}\nabla \times \boldsymbol{Q}$

For our problem, $\boldsymbol{M} = \begin{bmatrix} m + 4(\eta+\eta')\omega_{1,1} & 4\eta\omega_{1,2} + 4\eta'\omega_{2,1} & 0 \\ 4\eta\omega_{2,1} + 4\eta'\omega_{1,2} & m + 4\omega_{2,2} & 0 \\ 0 & 0 & m \end{bmatrix}$

Hence $div\,\boldsymbol{M} = 4\eta'\left\{\frac{\partial\nabla^2 W}{\partial y}\boldsymbol{i} - \frac{\partial\nabla^2 W}{\partial x}\boldsymbol{j}\right\}$ and $\boldsymbol{I} \times div\boldsymbol{M} = 4\begin{bmatrix} 0 & 0 & -\frac{\partial\nabla^2 W}{\partial x} \\ 0 & 0 & -\frac{\partial\nabla^2 W}{\partial y} \\ \frac{\partial\nabla^2 W}{\partial x} & \frac{\partial\nabla^2 W}{\partial y} & 0 \end{bmatrix}$

These equations give non-dimensional stress $T = T^* a/\mu U_0$ and

$$T_{13} = \frac{\partial}{\partial x}\left(w - \frac{e}{S}\nabla^2 w\right) \;\; and \; T_{23} = \frac{\partial}{\partial y}\left(w - \frac{e}{S}\nabla^2 w\right)$$

Hence the skin friction on faces $x = \pm 1$ is $c_f = T_{13}$ and on faces $y = \pm y_0$ is $c_f = T_{23}$.

On $x = \pm 1$,

$$c_f = \frac{e}{S}\sum_{n=1}^{\infty}\left\{r_n\left[A_n\lambda_1^2.\frac{coshu_n y}{coshu_n} + B_n\lambda_2^2\frac{coshv_n y}{coshv_n}\right](-1)^{n+1} - [C_n\lambda_1^2\alpha_n Tanh\alpha_n + D_n\lambda_2^2\beta_n Tanh\beta_n]cost_n y\right\}$$

This skin friction is function of y locally. Hence we find average skin friction = $\frac{1}{2y_0}\int_{-y_0}^{y_0} c_f \, dy$

At $S = 50$, $e = 0.5$, at different values of M^2 the average skin friction is tabulated in table 1.

Table 1. Average skin friction values for different values of M^2 at $S = 50$, $e = 0.5$

M^2	3	5	7
Average c_f	171.483	104.032	75.0971

From this we observe that as Hartmann number M increases, skin friction decreases.

5. Conclusions:

1. By applying Magnetic field, for couple stress fluids the volumetric flow rate and skin friction on the walls are controlled i.e decrease.
2. For a fixed value of M, the effect of Couple stress parameter on volumetric flow rate is almost nil.
3. Skin friction is inversely proportional to couple stress parameter.

References

1. **Hg-Dynamics I. Theory of the Laminar Flow of an Electrically Conductive Liquid in a Homogeneous Magnetic Field**
J. Hartmann, Kgl. Danske Videnskabernes Selskab, bd. 15, nr. 6, Mathematisk-fysiske Meddelelser, Copenhagen (1937).
2. **Hg-Dynamics II. Experimental Investigations on the Flow of Mercury in a Homogeneous Magnetic Field**
J. Hartmann and F. Lazarus, Kgl. Danske Videnskabernes Selskab, bd. 15, nr. 7, Mathematisk-fysiske Meddelelser, Copenhagen (1937).
3. **Steady Flow of Conducting Fluids in Channels Under Transverse Magnetic Fields With Consideration of Hall Effect**
Itiro Tani, J. of the Aerospace Sci, March, (1962).
4. **Magnetohydrodynamic flow and heat transfer of a non-Newtonian fluid in an eccentric annulus**
M. E. S. Ahmed, and H. A. Attia, Canadian J. of Physics, 76(5), (1998).
5. **Non-Newtonian magneto-hydrodynamic flow over a stretching surface with heat and mass transfer**
S. Abel, P. H. Veena, K. Rajgopal, V. K. Pravin, Int. J. Non-Linear Mech., 39, (2004).
6. **MHD Flow and Heat Transfer in a Rectangular Duct with Temperature Dependent Viscosity and Hall Effect**
M. E. S. Ahmed, and H. A. Attia, Int. Comm. Heat and Mass Transfer, 27(8), (2000).
7. **Magnetic Flow and Heat Transfer in a Rectangular Channel with Variable Viscosity**
H. A. Attia, The Arabian J. for Sci. and Eng., 30(2A), (2005).
8. **Hydromagnetic flow of a dusty fluid in a rectangular channel with Hall current and heat transfer**
A. L. Aboul-Hassan and H. A. Attia, Canadian J. of Physics, 80, (2002).

9. **Steady flow of Micropolar Fluid in a Rectangular Channel Under Transverse Magnetic Field With Suction**
J. V. Ramana Murthy, K. S. Sai and N. K. Bahali, AIP Advances 1, (2011).
10. **MHD Flow of Micropolar Fluid in a Rectangular Duct with Hall and Ion Slip Effects**
D. Srinivasacharya and Mekonnen Shiferaw, J. of the Braz. Soc. of Mech. Sci. & Eng., 4, (2008).
11. **Couple stresses in fluids**
V. K. Stokes, Phys. Fluids, 9, 1709 (1966).
12. **Inertial effect in the squeeze film of couple-stress fluids in biological bearings**
E. Walicki and A. Walicka, Int. J. Appl. Mech. Eng., 4 (1999).
13. **Combined effects of non-Newtonian couple stresses and fluid inertia on the squeeze film characteristics between a long cylinder and an infinite plate**
J. R. Lin, C. R. Hung, Fluid Dynamics Research, 39, (2007).
14. **Couple-stresses in peristaltic transport of fluids**
E. L. E. F. Shehawey and K. S. Mekhemer, J. Phys D: Appl Phys, 27, (1994).
15. **Numerical solution of finite modified Reynolds equation for couple stress squeeze film lubrication of porous journal bearings**
N. B. Naduvinamani and S. B. Patil, Computers and Structures, 87, (2009).
16. **Couple stress effects on the dynamic behavior of connecting rod bearings in both gasoline and diesel engines**
M. Lahmar and B. Bou-Sad, Tribology Transactions, 51, (2008).
17. **Numerical solutions for micropolar transport phenomena over a non linear stretching sheet**
R. Bhargava, S. Sharma, H. S. Takhar, O. A. Be´g and P. Bhargava, Nonlinear Analalysis: Modelling and Control, 12, (2007).
18. **Stokes' problems for an incompressible couple stress fluid**
M. Devakar and T.K.V. Iyengar, Nonlinear Analysis: Modelling and Control, 1(2) (2008).
19. **Influence of magnetic field in three dimensional flow of couple stress nanofluid over a nonlinearly stretching surface with convective condition**
T. Hayat, A. Aziz, T. Muhammad and B. Ahmad, Plos One, 10, (2015).
20. **Mixed convection flow of couple stress fluid in a non-darcy porous medium with Soret and Dufour effects**
D. Srinivasacharya and K. Kaladhar, J Appl Sci and Eng, 15, (2012).
21. **Influence of heat and mass transfer on peristaltic flow of a couple stress fluid through porous medium in the presence of inclined magnetic field in an inclined asymmetric channel**
K. Ramesh, J. Molecular Liquids, 219, (2016).
22. **Influence of lateral walls on peristaltic flow in a rectangular duct**
M. V. Subba Reddy, M. Mishra, S. Sreenadh and A. Ramachandra Rao, J. fluids Engineering, 127, (2005).