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Working Vacation in Queueing Inventory System with Server Breakdown and Repair

M. Sangeetha · **N.Anbazhagan**

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Abstract In this paper, we contemplate a continuous review queueing-inventory system along with the poisson arrival process. The magnitude of waiting space is N. We speculate that customers check in according to a Poisson process with parameter $\lambda > 0$ and claim absolutely one unit of item. The server furnishes two types of service. An order for $Q(= S-s > s+1)$ items is placed whenever the inventory level plunges to s and the items are acquired only after a arbitrary time which is distributed as exponential with parameter $\beta(> 0)$. The server commences a working vacation each time, when waiting space be reduced to empty or inventory turn into zero or both cases happen, which is also exponentially distributed. The server may breakdown at any instance whether busy or working vacation period, i.e., server is declining for a short interval of time, breakdown time is exponentially distributed with parameter α_b for busy period as well as for α_v for working vacation period. The server instantly started for repair action and repair time follows an exponentially distributed with parameter η_b for busy period as well as η_v for working vacation period. The sensitive analysis is also accomplished to procure stationary measures of system performance together with we computed total expected cost rate. The results are illustrated with numerical examples in order to find out the convexity of the total expected cost rate.

Keywords Queueing-inventory · Working Vacation · Server breakdown and repair.

1 Introduction

Servi and Finn [14] have introduced a new kind of semi-vacation policy called working vacation (WV) policy, where the server works at a lower rate rather than completely terminates during a vacation period in the M/M/1 queue, they derived the probability generating function of the queue length and sojourn time in steady state, and applied these results to evaluate the performance measures of gateway router in fiber communication networks. Sigman and Simchi-Levi[15] were proposed inventory models with positive service time and they assumed that the processing of inventory require an arbitrarily distributed positive amount of time, thus leading to the formation of queue. Since then numerous studies on inventory models with positive service time are reported.

Maike Schwarz et al.[9] has proposed the model M/M/1 Queueing systems with inventory. They derived M/M/1-systems with inventory under continuous review and different inventory management policies, and with lost sales. Here demand time points follow Poisson distribution, service times and lead times exponential distribution. Anbazhagan et al.[1] considered an inventory system with service facility. They assumed that the demand realizing according to a poission process that are issued to the customer after a random time of service. Rajukumar et al.[11] considered an inventory system in which the customers who arrive during the stock out period join an orbit of infinite size. The condition for ergodicity of the system and joint probability distribution of the number of customers in the system and the inventory level were obtained.

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Rajkumar[12] has derived the model for continuous review inventory system with two types of customers arrived according to a poission process with exponential lead time and ordering policy(s,S). Sivakumar[16] considered a continuous review inventory system with Markovian demand. The server goes for a vacation of an exponentially distributed duration whenever the inventory level reaches zero and he was calculated about long run total expected cost rate. Saravanarajan and Chandrasekaran[17] were analysed an M/G/1 feedback queueing system providing two types of services with vacations and breakdowns. The vacation times are exponentially distributed and the system may breakdown at random and repair time follows exponential distribution. Rajadurai et al.[13] considered a single server feedback retrial queueing system with multiple working vacations and vacation interruption. Here the server works at a lower service rate during working vacation (WV) period.

Berman and Kim [3] analyzed a queueing inventory system with Poisson arrivals, exponential service times and zero lead times. Berman and Kim [4] inspected about internet based supply chains with Poisson arrivals, exponential service times, the Erlang lead times and found that the optimal ordering policy has a monotonic threshold structure. Berman and Sapna [5] considered queueing inventory systems with Poisson arrivals, arbitrary distribution service times and zero lead times. Berman, Kaplan and Shimshak [2] analysed about various types of inventory management which is used at service facilities.

Berman and Sapna[6] discussed about the problem of inventory control of service parts at a service facility where there is only a limited waiting space for customers. Elango [8] considered a continuous review perishable (s, S) inventory system with a service facility consisting of finite waiting room and a single server. The life time of each item and the lead time of reorders are assumed to have independent exponential distributions. Paul Manuel et al.[10] analysed a continuous review perishable inventory system with multi-server service facility. The joint probability distribution of the number of busy servers, the inventory level and the number of customers in the orbit is obtained. Choi and park [7] derived insensitive bounds for various performance measures of a single server retrial queue with generally distributed inter retrial times and Bernoulli schedule, under the special assumption that only the customer at the head of the orbit queue.

2 Notation

$[A]_{ij}$: The element/submatrix at $(i,j)^{th}$ position of A
$\mathbf{0}$: Zero matrix
I_N : Identity matrix of order N
e : A column vector of $1's$ appropriate dimension

$$Y(t) : \begin{cases} 0 & \text{if server is on normal working vacation} \\ 1 & \text{if server is on breakdown working vacation} \\ 2 & \text{if server is on breakdown busy} \\ 3 & \text{if sever is on normal busy} \end{cases}$$

3 Mathematical Model Description

In this paper, we contemplate a continuous review queueing inventory system alongwith Poisson arrival process. The magnitude of waiting space is N. An arriving customers form a single waiting line based on their order of arrivals and served respectively. Moreover we assume that the arrival of customers follow a Poisson process with $\lambda > 0$ and claim absolutely one unit of item. The server furnishes two types of service. Upon the completion of type-1 service induces one customer exit the system with probability p_1 and type-2 service induces one customer exit the system and diminish inventory by one item with probability p_2. The service times follow exponential distribution in both cases. The type-1 service accomplished with parameter $p_1\mu_b$ and type-2 service accomplished with parameter $p_2\mu_b$. The customer may choose either types of service which is stated above. An order $Q(= S - s > s + 1)$ item is placed whenever the inventory level plunges to s and the items are received after a arbitrary time which is distributed as exponential with parameter $\beta(> 0)$. The server commences a working vacation each time, when waiting space be reduced to empty or inventory turn into zero or both cases happen, which is exponentially distributed. whether busy or working vacation period, (i.e.) server is declining for a short interval of time and breakdown time is exponentially distributed with parameter α_b for busy period as well as α_v for working vacation period. The server immediately started for repair and repair time follows an exponentially distributed with parameter η_b for busy periodas well as η_v for working vacation period.

Whenever the server returns from vacation, if it is find atleast one customer in the waiting space and positive inventory with server is normal, then the server starts to serve the waiting customer. Service completion on working vacation period with parameter μ_v. The lead time is assumed to be exponentially distributed.

4 Analysis

Let $I(t)$, $X(t)$, $Y(t)$ denote, respectively, the inventory level, number of customers (waiting and being served) in the system and server status at time t .
From the assumptions made on the input and out processes, it may be shown that, $(I, X, Y) = (I(t), X(t), Y(t))$; t $\geq$ 0 on the state space E = $E_1 \cup E_2$, where
$E_1 = \{(i, j, k)/i = 0, 1, 2, 3....S,\ j = 0, 1, 2, ...N,\ k = 0, 1\}$
$E_2 = \{(i, j, k)/i = 1, 2, 3.......S,\ j = 1, 2, 3.....N,\ k = 2, 3\}$
To determine the infinitesimal generator,

$$A = (a((i, j, k)(i', j', k'))), (i, j, k)(i', j', k') \in E \tag{1}$$

of this process we use the following arguments :

- The arrival of customer makes a transition with intensity λ from (i, j, k) to (i, j+1, k), i = 0, 1, 2, ..., S, j = 0, 1, 2, ..., N, k = 0, 1 (or) from (i, j, k) to (i, j+1, k), i = 1, 2, ..., S, j = 1, 2, ..., N, k = 2, 3.

- Type 1 service completion causes a transition with intensity $p_1\mu_b$ from (i, j, k) to (i, j-1, k), i = 1, 2, ..., S, j = 2, 3, ..., N, k = 3 (or) from (i, j, 3) to (i, j-1, 1), i = 1, 2, ..., S , j = 1.

- Type 2 service completion causes a transition with intensity $p_2\mu_b$ from (i, j, k) to (i-1, j-1, k), i = 2, 3, ..., S, j = 2, 3, ..., N, k = 3 (or) from (i, j, 3) to (i-1, j-1, 1), i = 1, 2, ..., S, j = 1 , i = 1, j = 2, 3, ..., N.

- The completion of service from primary demands makes a transition with intensity μ_v from (i, j, 1) to (i, j, 1), i = 0, j = 1, 2, ..., N

- When a replenishment occur takes a transition with intensity β from (i, j, k) to (i+Q, j, k), i = 0, 1, 2, ..., s, j = 0, 1, 2, ..., N, k = 0, 1 or from (i, j, k) to (i+Q, j, k) i = 1, 2, 3, ..., s, j = 1, 2, ..., N, k = 2, 3.

- Server changes from working vacation to normal with intensity η_v from (i, j, 0) to (i, j, 1), i = 0, 1, 2, ..., S, j = 0, 1, 2, ..., N

- Server changes from breakdown busy period to normal busy period with intensity η_b from (i, j, 2) to (i, j, 3), i = 1, 2, 3, ..., S, j = 1, 2, ..., N.

- Server changes from normal working vacation period to breakdown busy period with intensity α_v from (i, j, 1) to (i, j, 0), i = 0, 1, 2, ..., S, j = 0, 1, 2, ..., N.

- Server changes from normal busy period to breakdown busy period with intensity α_b from (i, j, 3) to (i, j, 2), i = 1, 2, ..., S, j = 1, 2, 3, ..., N.

- Server changes from normal working vacation period to busy period with intensity θ from (i, j, 1) to (i, j, 3), i = 0, 1, 2, ..., S, j = 0, 1, 2, ..., N.

For other transition from (i, j, k) to $(i^{'}, j^{'}, k^{'})$ except (i, j, k) $\neq$ $(i^{'}, j^{'}, k^{'})$,the rate is zero.
Finally, note that

$$a((i, j, k), (i^{'}, j^{'}, k^{'})) = - \sum_{i} \sum_{j} \sum_{\substack{k \\ (i,j,k) \neq (i^{'}, j^{'}, k^{'})}} a((i, j, k), (i^{'}, j^{'}, k^{'}))$$

Hence $a((i,j,k)(\mathrm{i}^{'},\mathrm{j}^{'},\mathrm{k}^{'}))$ can be written as,

$$
\begin{cases}
\lambda, & i'=i, i=0,1,2...,S \\
 & j'=j+1, j=0,1,2....N-1 \\
 & k'=k, k=0,1 \\
 & or \\
 & i'=i, i=1,2,3...,S \\
 & j'=j+1, j=1,2,3....N-1 \\
 & k'=k, k=2,3 \\
 & \\
p_1\mu_b, & i'=i, i=1,2,3...S \\
 & j'=j-1, j=2,3,4...N \\
 & k'=k, k=3 \\
 & or \\
 & i'=i, i=1,2,3...S \\
 & j'=j-1, j=1 \\
 & k'=k-2, k=3 \\
 & \\
p_2\mu_b, & i'=i-1, i=2,3,4,...S \\
 & j'=j-1, j=2,3,4,...N \\
 & k'=k, k=3 \\
 & or \\
 & i'=i-1, i=1 \\
 & j'=j-1, j=1 \\
 & k'=k-2, k=3 \\
 & \\
\mu_v, & i'=i, i=0 \\
 & j'=j, j=1,2,3,...N \\
 & k'=k, k=1 \\
 & \\
\beta, & i'=i+Q, i=0,1,2,...s \\
 & j=j, j=0,1,2,...N \\
 & k'=k, k=0,1 \\
 & or \\
 & i'=i+Q, i=1,2,...s \\
 & j=j, j=1,2,...N \\
 & k'=k, k=2,3 \\
 & \\
\eta_v, & i'=i, i=0,1,2,...S \\
 & j'=j, j=0,1,2,...N \\
 & k=k+1, k=0
\end{cases}
$$

$$
\begin{cases}
\eta_b, & \begin{array}{l} i' = i, i = 1, 2, 3, ...S \\ j' = j, j = 1, 2, 3....N \\ k' = k - 1, k = 3 \\ or \end{array} \\
\alpha_v, & \begin{array}{l} i' = i, i = 0, 1, 2, ...S \\ j' = j, j = 0, 1, 2, ...N \\ k' = k - 1, k = 1 \end{array} \\
\alpha_b, & \begin{array}{l} i' = i, i = 1, 2, 3, ...S \\ j' = j, j = 1, 2, 3, ...N \\ k' = k - 1, k = 3 \end{array} \\
\theta, & \begin{array}{l} i' = i, i = 1, 2, 3, ...S \\ j' = j, j = 1, 2, 3, ...N \\ k' = k + 2, k = 1 \end{array} \\
\varepsilon_1 = -(\lambda + \eta_v), & \begin{array}{l} i' = i, i = S, S - 1..., s + 1 \\ j' = j, j = 0, 1, 2....N - 1 \\ k' = k, k = 0 \end{array} \\
\varepsilon_2 = -(\alpha_v + \lambda), & \begin{array}{l} i' = i, i = S, S - 1, ..., s + 1 \\ j' = j, j = 0 \\ k' = k, k = 1 \end{array} \\
\varepsilon_3 = -(\alpha_v + \lambda + \theta), & \begin{array}{l} i' = i, i = S, S - 1, ..., s + 1 \\ j' = j, j = 1, 2, ..., N - 1 \\ k' = k, k = 1 \end{array} \\
\varepsilon_4 = -(\lambda + \eta_b), & \begin{array}{l} i' = i, i = S, S - 1, ..., s + 1 \\ j' = j, j = 0, 1, 2....N - 1 \\ k' = k, k = 2 \end{array} \\
\varepsilon_5 = -(\alpha_b + \lambda + p_1\mu_b + p_2\mu_b), & \begin{array}{l} i' = i, i = S, S - 1...s + 1 \\ j' = j, j = 1, 2, 3, 4...N - 1 \\ k' = k, k = 3 \end{array} \\
\varepsilon_6 = -(\eta_v), & \begin{array}{l} i' = i, i = S, S - 1, ..., s + 1 \\ j' = j, j = N \\ k' = k, k = 0 \end{array}
\end{cases}
$$

$$
\begin{cases}
\varepsilon_7 = -(\alpha_v + \theta), & \begin{array}{l} i' = i, i = S, S-1, ..., s+1 \\ j' = j, j = N \\ k' = k, k = 1 \end{array} \\[2ex]
\varepsilon_8 = -(\eta_b), & \begin{array}{l} i' = i, i = S, S-1, ..., s+1 \\ j' = j, j = N \\ k' = k, k = 2 \end{array} \\[2ex]
\varepsilon_9 = -(\alpha_b + p_1\mu_b + p_2\mu_b), & \begin{array}{l} i' = i, i = S, S-1, ...s+1 \\ j' = j, j = N \\ k' = k, k = 3 \end{array} \\[2ex]
\varepsilon_{10} = -(\lambda + \eta_v + \beta), & \begin{array}{l} i' = i, i = s, s-1, ...1 \\ j' = j, j = 0, 1, 2....N-1 \\ k' = k, k = 0 \\ or \\ i' = i, i = 0 \\ j' = j, j = 0, 1, 2, 3....N-1 \\ k' = k, k = 0 \end{array} \\[2ex]
\varepsilon_{11} = -(\alpha_v + \lambda + \beta), & \begin{array}{l} i' = i, i = s, s-1, ...1 \\ j' = j, j = 0 \\ k' = k, k = 1 \\ \\ or \\ i' = i, i = 0 \\ j' = j, j = 0 \\ k' = k, k = 1 \end{array} \\[2ex]
\varepsilon_{12} = -(\alpha_v + \lambda + \theta + \beta), & \begin{array}{l} i' = i, i = s, s-1, ...1 \\ j' = j, j = 1, 2, ..., N-1 \\ k' = k, k = 1 \end{array} \\[2ex]
\varepsilon_{13} = -(\lambda + \eta_b + \beta), & \begin{array}{l} i' = i, i = s, s-1, ...1 \\ j' = j, j = 0, 1, 2....N-1 \\ k' = k, k = 2 \end{array} \\[2ex]
\varepsilon_{14} = -(\alpha_b + \lambda + p_1\mu_b + p_2\mu_b + \beta), & \begin{array}{l} i' = i, i = i = s, s-1, ...1 \\ j' = j, j = 2, 3, 4...N-1 \\ k' = k, k = 3 \end{array}
\end{cases}
$$

$$
\begin{cases}
\varepsilon_{15} = -(\eta_v + \beta), & \begin{array}{l} i' = i, i = s, s-1, ...1 \\ j' = j, j = N \\ k' = k, k = 0 \\ \\ or \\ i' = i, i = 0 \\ j' = j, j = N \\ k' = k, k = 0 \end{array} \\[2ex]
\varepsilon_{16} = -(\alpha_v + \theta + \beta), & \begin{array}{l} i' = i, i = s, s-1, ...1 \\ j' = j, j = N \\ k' = k, k = 1 \end{array} \\[2ex]
\varepsilon_{17} = -(\eta_b + \beta), & \begin{array}{l} i' = i, i = s, s-1, ...1 \\ j' = j, j = N \\ k' = k, k = 2 \end{array} \\[2ex]
\varepsilon_{18} = -(\alpha_b + p_1\mu_b + p_2\mu_b + \beta), & \begin{array}{l} i' = i, i = s, s-1, ...1 \\ j' = j, j = N \\ k' = k, k = 3 \end{array} \\[2ex]
\varepsilon_{19} = -(\alpha_v + \lambda + \beta + \mu_v), & \begin{array}{l} i' = i, i = 0 \\ j' = j, j = 1, 2, ..., N-1 \\ k' = k, k = 1 \end{array} \\[2ex]
\varepsilon_{20} = -(\alpha_v + \beta + \mu_v), & \begin{array}{l} i' = i, i = 0 \\ j' = j, j = N \\ k' = k, k = 1 \end{array}
\end{cases}
$$

The more general form is;

$$
A = \begin{array}{c} \\ S \\ S-1 \\ S-2 \\ \vdots \\ s \\ s-1 \\ \vdots \\ 1 \\ 0 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} S & S-1 & S-2 & \dots & s & s-1 & \dots & 1 & 0 \end{array} \\
\begin{pmatrix}
L & M & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\
0 & L & M & \dots & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & L & \dots & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
R & 0 & 0 & \dots & L_1 & M & \dots & 0 & 0 \\
0 & R & 0 & \dots & 0 & L_1 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & R & \dots & 0 & 0 & \dots & L_1 & M_1 \\
0 & 0 & 0 & \dots & R_1 & 0 & \dots & 0 & L_2
\end{pmatrix}
\end{array}
$$

where,

$$L = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \dots & N-1 & N \end{array} \\ \begin{pmatrix} A_0 & B_0 & 0 & 0 & \dots & 0 & 0 \\ C_0 & A & B & 0 & \dots & 0 & 0 \\ 0 & C & A & B & \dots & 0 & 0 \\ 0 & 0 & C & A & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & A & B \\ 0 & 0 & 0 & 0 & \dots & C & A_1 \end{pmatrix} \end{array}$$

with,

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \varepsilon_1 & \eta_v & 0 & 0 \\ \alpha_v & \varepsilon_3 & 0 & \theta \\ 0 & 0 & \varepsilon_4 & \eta_b \\ 0 & 0 & \alpha_b & \varepsilon_5 \end{pmatrix} \end{array} \quad A_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \varepsilon_6 & \eta_v & 0 & 0 \\ \alpha_v & \varepsilon_7 & 0 & \theta \\ 0 & 0 & \varepsilon_8 & \eta_b \\ 0 & 0 & \alpha_b & \varepsilon_9 \end{pmatrix} \end{array}$$

$$A_0 = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \varepsilon_1 & \eta_v \\ \alpha_v & \varepsilon_2 \end{pmatrix} \end{array} \quad C_0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & p_1\mu_b \end{pmatrix} \end{array} \quad B_0 = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \end{pmatrix} \end{array}$$

$$C = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_1\mu_b \end{pmatrix} \end{array} \quad B = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \end{array}$$

$$L_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \dots & N-1 & N \end{array} \\ \begin{pmatrix} D_0 & B_0 & 0 & 0 & \dots & 0 & 0 \\ C_0 & D & B & 0 & \dots & 0 & 0 \\ 0 & C & D & B & \dots & 0 & 0 \\ 0 & 0 & C & D & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & D & B \\ 0 & 0 & 0 & 0 & \dots & C & D_1 \end{pmatrix} \end{array}$$

with,

$$D_0 = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \varepsilon_{10} & \eta_v \\ \alpha_v & \varepsilon_{11} \end{pmatrix} \end{array} \quad D = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \varepsilon_{10} & \eta_v & 0 & 0 \\ \alpha_v & \varepsilon_{12} & 0 & \theta \\ 0 & 0 & \varepsilon_{13} & \eta_b \\ 0 & 0 & \alpha_b & \varepsilon_{14} \end{pmatrix} \end{array} \quad D_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \varepsilon_{15} & \eta_v & 0 & 0 \\ \alpha_v & \varepsilon_{16} & 0 & \theta \\ 0 & 0 & \varepsilon_{17} & \eta_b \\ 0 & 0 & \alpha_b & \varepsilon_{18} \end{pmatrix} \end{array}$$

$$L_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \dots & N-1 & N \end{array} \\ \begin{pmatrix} D_0 & B_0 & 0 & 0 & \dots & 0 & 0 \\ C_1 & E_0 & B & 0 & \dots & 0 & 0 \\ 0 & C_1 & E_0 & B & \dots & 0 & 0 \\ 0 & 0 & C_1 & E_0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & E_0 & B_0 \\ 0 & 0 & 0 & 0 & \dots & C_1 & E_0 \end{pmatrix} \end{array}$$

with,

$$E_0 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \varepsilon_{10} & \eta_v & 0 & 0 \\ \alpha_v & \varepsilon_{19} & 0 & 0 \end{pmatrix} \end{array} \quad E_1 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \varepsilon_{15} & \eta_v & 0 & 0 \\ \alpha_v & \varepsilon_{20} & 0 & 0 \end{pmatrix} \end{array} \quad C_1 = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & \mu_v \end{pmatrix} \end{array}$$

$$R = \begin{array}{c c} & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \dots & N-1 & N \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \\ N \end{array} & \begin{pmatrix} F_0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & F_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & F_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & F_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & F_1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & F_1 \end{pmatrix} \end{array}$$

with,

$$F_0 = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} \end{array} \quad F_1 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix} \end{array}$$

$$R_1 = \begin{array}{c c} & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \dots & N-1 & N \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \\ N \end{array} & \begin{pmatrix} F_0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & F_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & F_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & F_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & F_2 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & F_2 \end{pmatrix} \end{array}$$

with,

$$F_2 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{pmatrix} \end{array}$$

$$M = \begin{array}{c c} & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \dots & N-1 & N \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \\ N \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ G_0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & G_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & G_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & G_1 & 0 \end{pmatrix} \end{array}$$

with,

$$G_0 = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & p_2\mu_b \end{pmatrix} \end{array} \quad G_1 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_2\mu_b \end{pmatrix} \end{array}$$

$$M_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \dots & N-1 & N \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ G_0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & G_0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & G_0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & G_0 & 0 \end{pmatrix} \end{array}$$

4.1 Steady state results

It can be seen from the structure of the rate matrix A that the homogeneous Markov process (I(t),X(t),Y(t), t $\geq$ 0) on the finite state space E is irreducible. Hence, the limiting distribution of the Markov process exists. Let ϕ, partitioned as $\phi = (\phi^{(S)}, \phi^{(S-1)}, ..., \phi^{(1)}, \phi^{(0)})$, donote the steady state probability vector of A. That is, ϕ satisfies

$$\phi\ A = 0 \text{ and } \sum_{i} \sum_{j} \sum_{k}_{(i,j,k)\in E} \phi^{(i,j,k)} = 0$$

The first equation of the above yields the following set of equations:

$$\begin{aligned} \phi^i L + \phi^{i-Q} R &= 0,\ i = S \\ \phi^{i+1} M + \phi^i L + \phi^{i-Q} R &= 0,\ i = S-1, S-2, ..., Q+1 \\ \phi^{i+1} M + \phi^i L + \phi^{i-Q} R_1 &= 0,\ i = Q \\ \phi^{i+1} M + \phi^i L &= 0,\ i = Q-1, Q-2, ..., s+1 \\ \phi^{i+1} M + \phi^i L_1 &= 0,\ i = s, s-1, ...1 \\ \phi^{i+1} M_1 + \phi^i L_2 &= 0,\ i = 0 \end{aligned}$$

After long simplifications, the above equations,except the third one yield

$$\begin{aligned} \phi^i &= (-1)^Q \phi^Q M^{Q-1} M_1 \left(\frac{1}{L_2}\right)\left(\frac{1}{L_1}\right)^s \left(\frac{1}{L}\right)^{Q-s-1}, \quad i = 0 \\ \phi^i &= (-M)^{Q-i} \phi^Q \left(\frac{1}{L_1}\right)^{s+1-i} \left(\frac{1}{L}\right)^{Q-s-1}, \quad i = 1, 2, ...s \\ \phi^i &= (-M)^{Q-i} \phi^Q \left(\frac{1}{L}\right)^{Q-i}, \quad i = s+1, s+2, ..., Q-1 \\ \phi^i &= (-1)^{i-1} \phi^Q (M)^{2Q-i} R \left(\sum_{n=0}^{S-i} \left(\frac{1}{L_1}\right)^{n+1} \left(\frac{1}{L}\right)^{2Q-(i+n)}\right), \quad i = Q+1, Q+2, ..., S \end{aligned}$$

Where ϕ^Q can be obtained by solving,

$$\phi^{Q+1} M + \phi^Q L + \phi^0 R_1 = 0,$$

and

$$\sum_{i=0}^{S} \phi^i\ e = 1,$$

that is

$$\phi^Q\ [(-1)^Q\ M^Q \sum_{n=0}^{S-Q-1} \left(\tfrac{1}{L_1}\right)^{n+1} \left(\tfrac{1}{L}\right)^{Q-(1+n)} + L + (-1)^Q\ M^{Q-1}\ M_1\ R_1 \left(\tfrac{1}{L_2}\right)\left(\tfrac{1}{L_1}\right)^s \left(\tfrac{1}{L}\right)^{Q-s-1}]=0$$

and

$$\phi^{Q}\,[\,(-1)^{Q}\,M^{Q-1}\,M_1\left(\tfrac{1}{L_2}\right)\left(\tfrac{1}{L_1}\right)^{s}\left(\tfrac{1}{L}\right)^{Q-s-1}+\sum_{i=1}^{s}(-M)^{Q-i}\left(\tfrac{1}{L_1}\right)^{s+1-i}\left(\tfrac{1}{L}\right)^{Q-s-1}+\sum_{i=s+1}^{Q-1}(-M)^{Q-i}$$

$$\left(\tfrac{1}{L}\right)^{Q-i}+\mathrm{I}+\sum_{i=Q+1}^{S}(-1)^{i-1}\,(M)^{2Q-i}\,R\,(\sum_{n=0}^{S-i}\left(\tfrac{1}{L_1}\right)^{n+1}\left(\tfrac{1}{L}\right)^{2Q-(i+n)})]\;e=1.$$

5 System perfomance measures

In this section, we derive some system performance measures with respect to the steady state analysis.

5.1 Mean Inventory Level

let ε_I denote the average inventory level in the steady state, which is given by,

$$\varepsilon_I=\sum_{i=1}^{S}\sum_{j=1}^{N}\sum_{k=0}^{3}i\phi^{(i,j,k)}+\sum_{i=1}^{S}\sum_{k=0}^{1}i\phi^{(i,0,k)}$$

5.2 Mean Reorder Rate

Let ε_R denote the mean reorder rate, which is given by,

$$\varepsilon_R=\sum_{j=1}^{N}P_2\mu_b\phi^{(s+1,j,3)}$$

5.3 Mean Repair Rate

Let ε_{REP} denote the mean repair rate, which is given by,

$$\varepsilon_{REP}=\sum_{i=0}^{S}\sum_{j=0}^{N}\sum_{k=0}^{1}\alpha_v\phi^{(i,j,k)}+\sum_{i=1}^{S}\sum_{j=1}^{N}\sum_{k=1}^{2}\alpha_b\phi^{(i,j,k)}$$

5.4 Mean Number of Customers Waiting while Server is on Vacation and Breakdown Period

Let ε_W denote the mean number of customers waiting while server is on working vacation and breakdown working vacation and breakdown busy period in the steady state and is given by,

$$\varepsilon_W=\sum_{i=1}^{S}\sum_{j=1}^{N}\sum_{k=0}^{2}j\phi^{(i,j,k)}+\sum_{j=1}^{N}\sum_{k=0}^{1}j\phi^{(0,j,k)}$$

5.5 Mean Number of Customers in the System When the Server is Providing Service

Let ε_B denote the number of customers in the waiting room, when the server is busy in the steady state and is given by,

$$\varepsilon_B=\sum_{i=1}^{S}\sum_{j=1}^{N}j\phi^{(i,j,3)}$$

6 Cost Analysis

In order to compute the total expected cost per unit time, we introduce the following notations:
C_h : The inventory holding cost per unit per unit time.
C_s : The setup cost per order.
C_w : Waiting time cost of a customer per unit time.
C_R : Repair cost per unit item per unit time.
Then the long-run expected cost rate is given by,

$$T(S,s) = C_h\ \varepsilon_I + C_s\ \varepsilon_R + C_w\ \varepsilon_W + C_R\ \varepsilon_{REP}$$

7 Numerical analysis

We have not exhibited analytically the covexity of the function TC(S, s, N).We have scrutinized the behaviour of this function by taking into account of (S, N) in order to shown the convexity through numerical illustration.

S	N				
	14	15	16	17	18
21	6.8116328177	6.4330649079	6.3632807870	6.4846177857	6.4868629391
22	**6.6989630524**	**6.2584626834**	**6.3080667962**	**6.3926950467**	**6.4203639560**
23	6.7954729238	6.7047044242	6.3596057809	6.5255339292	6.7427577632
24	6.9217716870	6.7625438136	6.7315070362	6.6844511608	6.7897318290

8 Conclution

In this paper, we analysed a continuous review queueing inventory system (S,s) with Poisson arrival process and the magnitute of waiting room is N. We investigated about two types of service with working vacation along with server breakdown and repair. We have also obtained various stationary measures. The results are illustrated with numerical examples in order to find out the convexity of the total expected cost rate.

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