Fuzzy solutions for Impulsive Neutral Functional Integrodifferential Equations

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Abstract

In this manuscript, the existence and uniqueness for impulsive fuzzy neutral functional integrodifferential equations are studied using Banach fixed point theorem.

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1 Introduction

Fuzzy theory provides a suitable way to objectively account for parameter uncertainty in models. Fuzzy logic approaches appear promising in preclinical applications and might be useful in drug discovery and design. Fuzzy integrodifferential equations plays a significant role in the analysis of phenomena with memory where imprecision is inherent. Balasubramaniam and Muralisankar [?] proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equations with nonlocal initial condition. Kwun et al. [?] proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equations by using successive iteration.

Neutral functional differential equations have been studied extensively by many authors. These type of equations occur in the study of heat conduction in materials with memory and in many other physical phenomena. Balasubramaniam and Muralisankar [?] established the existence and uniqueness of a fuzzy solution for the nonlinear fuzzy neutral functional differential equation via Banach fixed point analysis. For more on fuzzy differential equation we can refer [? ? ? ?].

The theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects. For more details on this theory and on its applications, we refer to the monographs of Lakshmikantham et al. [?].

In this paper, we study the existence and uniqueness for the following fuzzy impulsive neutral functional integrodifferential equations using Banach fixed point theorem.

$$\frac{d}{dt}\left[x(t)+g(t,x_t)\right] = Ax(t) + f\left(t, x_t, \int_0^t q(t,s,x_s)ds\right),\ t \in J=[0,T] \tag{1}$$

$$x_0 = \phi \in E^n \tag{2}$$

$$\Delta x(t_k) = I_k(x(t_k)),\quad t \neq t_k,\quad k=1,2,\cdots,m. \tag{3}$$

where $A : J \to E^n$ is fuzzy coefficient , E^n is the set of all upper semicontinuously convex fuzzy numbers on R, $f : J \times E^n \times E^n \to E^n$ and $q : J \times J \times E^n \to E^n$ are nonlinear regular fuzzy functions, $g : J \times E^n \to E^n$ is a nonlinear continuous function and $I_k \in C(E^n, E^n)$ are bounded functions, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^-)$ and $x(t_k^+)$ represent the left and right limits of $x(t)$ at $t = t_k$ respectively. Here, $x_t(\cdot)$ represents the history where $x_t(\theta) = x(t+\theta), \quad \theta < 0$.

This paper has three sections. In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used in the later sections. In section 3, we prove the existence of fuzzy impulsive neutral functional integrodifferential equation.

2 Preliminaries

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper.

Definition 2.1. *{fuzzy set} Let X be a nonempty set. A fuzzy set A in X is characterized by its membership function $A : X \to [0,1]$ and $A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$.*

Let $CC(R^n)$ denotes the set of all nonempty compact, convex subsets of R^n. Denote by, $E^n = \{u : R^n \to [0,1]$ such that u is normal ,u is fuzzy convex, u is upper semicontinuous. For $0 < \alpha \leq 1$, we denote $[u]^\alpha = \{x \in R^n : u(x) \geq \alpha\}$. If $g : R^n \times R^n \to R^n$ is a function, then by using Zadeh's extension principle we can extend g to $E^n \times E^n \to E^n$ by the equation

$$[g(u,v)(z)] = \sup_{z=g(x,y)} \min\{u(x), v(y)\}.$$

It is well known that $[g(u,v)]^\alpha = g([u]^\alpha, [v]^\alpha)$ for all $u, v \in E^n, 0 \leq \alpha \leq 1$ and continuous function g. Further, we have $[u+v]^\alpha = [u]^\alpha + [v]^\alpha, [ku]^\alpha = k[u]^\alpha$, where $u, v \in E^n, k \in R$, $0 \leq \alpha \leq 1$.

Let A, B be two nonempty bounded subsets of R^n. The distance between A and B is defined by the Hausdorff metric

$$H_d(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}$$

where $\|.\|$ denotes the usual Euclidean norm in R^n. Then $(CC(R^n), H_d)$ is a complete and seperable metric space [**?**].

Definition 2.2. *[**?**] We define the complete metric d_∞ on E^n by*

$$d_\infty(u,v) = \sup_{0<\alpha\leq 1} H_d([u]^\alpha, [v]^\alpha) = \sup_{0<\alpha\leq 1}\{|u_l^\alpha - v_l^\alpha|, |u_r^\alpha - v_r^\alpha|\}$$

for any $u, v \in E^n$, which satisfies $H_d(u+w, v+w) = H_d(u,v)$. Hence, (E^n, d_∞) is a complete metric space.

Definition 2.3. *[**?**] The supremum metric H_1 on $C(J, E^n)$ is defined by*

$$H_1(u,v) = \sup_{0\leq t\leq T} d_\infty(u(t), v(t)).$$

Hence $(C(J,E^n), H_1)$ is a complete metric space.

Definition 2.4. *[?] A mapping $f : J \to E^n$ is strongly measurable if, for all $\alpha \in [0,1]$ the set-valued map $f_\alpha : J \to CC(R^n)$ defined by $f_\alpha(t) = [f(t)]^\alpha$ is Lebesgue measurable when $CC(R^n)$ has the topology induced by the Hausdorff metric.*

Definition 2.5. *[?] A map $f : J \to E^n$ is called levelwise continuous at $t_0 \in J$ if the multi-valued map $f_\alpha(t) = [f(t)]^\alpha$ is continuous at $t = t_0$ with respect to the Hausdorff metric for all $\alpha \in [0,1]$.*

A map $f : J \to E^n$ is said to be integrably bounded if there is an integrable function $h(t)$ such that $\|x(t)\| \leq h(t)$ for every $x(t) \in f_0(t)$.

3 Existence Results

Let us consider the existence and uniqueness of the fuzzy solution for the problem $(1)-(3)$ $(u \equiv 0)$. In order to define the solution, consider the space $\Omega = \{x : J \to E^n\}$ and there exist $x(t_k^+)$ and $x(t_k^-),\ k = 1, 2, \cdots, m$ with $x(t_k^-) = x(t_k)$. Let $\Omega' = \Omega \cap C(J, E^n)$.

Assume the following hypotheses:
(H1) $S(t)$ is a fuzzy number, where $[S(t)]^\alpha = [S_l^\alpha(t), S_r^\alpha(t)],\ S(0) = I$ and $S_j^\alpha(t)(j = l, r)$ is continuous with $|S_j(t)| \leq M,\ M > 0$, $|AS(t)| \leq M_1$, for all $t \in J = [0, T]$.
(H2) The nonlinear function $g : J \times E^n \to E^n$ is continuous and there exists constant $d_1 > 0$ satisfying global Lipschitz condition such that

$$H_d([g(t,x)]^\alpha, [g(t,y)]^\alpha) \leq d_1 H_d([x(\theta)]^\alpha, [y(\theta)]^\alpha)$$

for all $t \in J$ and $x, y \in E^n$.
(H3) The nonlinear function $f : J \times E^n \times E^n \to E^n$ is continuous and there exist constants $d_2 > 0, d_3 > 0$ satisfying global Lipschitz condition such that

$$\begin{aligned} &H_d([f(s,\xi_1(s),\eta_1(s))]^\alpha, [f(s,\xi_2(s),\eta_2(s))]^\alpha) \\ &\quad \leq d_2 H_d([\xi_1(\theta)]^\alpha, [\xi_2(\theta)]^\alpha) + d_3 H_d([\eta_1(\theta)]^\alpha, [\eta_2(\theta)]^\alpha), \end{aligned}$$

where $\xi_j(s), \eta_j(s) \in E^n, \quad (j = 1, 2)$
(H4) The nonlinear function $q : J \times J \times E^n \to E^n$ is continuous and there exists a constant $d_4 > 0$ satisfying global Lipschitz condition such that

$$H_d\big([q(t,s,\zeta_1(s))]^\alpha, [q(t,s,\zeta_2(s))]^\alpha\big) \leq d_4 H_d\big([\zeta_1(\theta)]^\alpha, [\zeta_2(\theta)]^\alpha\big),$$

where $\zeta_j(s) \in E^n \quad (j = 1, 2)$.
(H5) There exists a constant $d_5 > 0$ such that

$$H_d\big([I_k(x(t_k^-))]^\alpha, [I_k(y(t_k^-))]^\alpha\big) \leq d_5 H_d([x(t)]^\alpha, [y(t)]^\alpha),$$

where $x(t), y(t) \in \Omega'$.
(H6) If $M\left(\frac{d_1}{M} + M_1 d_1 + d_5 + \left(d_2 + d_3 d_4 \frac{T}{2}\right) T\right) < 1$, then the initial value problem $(1)-(3)$ has a unique fuzzy solution.

Definition 3.1. *If x is an integral solution of the problem* $(1)-(3)$ $(u \equiv 0)$*, then x is given by*

$$\begin{aligned} x(t) &= S(t)[\phi(0) + g(0,\phi)] - g(t,x_t) - \int_0^t AS(t-s) g(s,x_s) ds \\ &+ \int_0^t S(t-s) f\left(s, x_s, \int_0^s q(s,\tau,x_\tau) d\tau\right) ds + \sum_{0<t_k<t} S(t-t_k) I_k(x(t_k^-)). \end{aligned} \tag{4}$$

Theorem 3.1. *Let $T > 0$. If hypotheses $(H1) - (H6)$ hold, then for every $x_0 \in E^n$, (4) has a unique fuzzy solution $x \in \Omega'$.*

Proof. For each $x(t) \in \Omega'$ and $t \in [0, T]$, define $\Phi x(t) \in \Omega'$ by

$$\begin{aligned}\Phi x(t) = {} & S(t)[\phi(0) + g(0, \phi)] - g(t, x_t) - \int_0^t AS(t-s)g(s, x_s)ds \\ & + \int_0^t S(t-s)f\left(s, x_s, \int_0^s q(s, \tau, x_\tau)d\tau\right)ds + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)).\end{aligned}$$

Hence, $\Phi x : [0, T] \to \Omega'$ is continuous, so Φ is a mapping from Ω' into itself. By hypotheses $(H1) - (H6)$, we have the following inequalities. Now, for $x, y \in \Omega'$, we have

$$\begin{aligned} & H_d\big([\Phi x(t)]^\alpha, [\Phi y(t)]^\alpha\big) \\ & \le H_d\Bigg(\Bigg[S(t)[\phi(0) + g(0, \phi)] - g(t, x_t) - \int_0^t AS(t-s)g(s, x_s)ds \\ & \quad + \int_0^t S(t-s)f\left(s, x_s, \int_0^s q(s, \tau, x_\tau)d\tau\right)ds + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-))\Bigg]^\alpha, \\ & \quad \Bigg[S(t)[\phi(0) + g(0, \phi)] - g(t, y_t) - \int_0^t AS(t-s)g(s, y_s)ds \\ & \quad + \int_0^t S(t-s)f\left(s, y_s, \int_0^s q(s, \tau, y_\tau)d\tau\right)ds + \sum_{0<t_k<t} S(t-t_k)I_k(y(t_k^-))\Bigg]^\alpha\Bigg) \\ & \le H_d\big([g(t, x_t)]^\alpha, [g(t, y_t)]^\alpha\big) \\ & \quad + H_d\left(\left[\int_0^t AS(t-s)g(s, x_s)\right]^\alpha, \left[\int_0^t AS(t-s)g(s, y_s)\right]^\alpha\right)ds \\ & \quad + H_d\Bigg(\left[\int_0^t S(t-s)f\left(s, x_s, \int_0^s q(s, \tau, x_\tau)d\tau\right)\right]^\alpha, \\ & \qquad \left[\int_0^t S(t-s)f\left(s, y_s, \int_0^s q(s, \tau, y_\tau)d\tau\right)\right]^\alpha\Bigg)ds \\ & \quad + H_d\Bigg(\left[\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-))\right]^\alpha, \\ & \qquad \left[\sum_{0<t_k<t} S(t-t_k)I_k(y(t_k^-))\right]^\alpha\Bigg) \\ & \le d_1 H_d([x(t+w)]^\alpha, [y(t+w)]^\alpha) \\ & \quad + MM_1 d_1 H_d([x(s+w)]^\alpha, [y(s+w)]^\alpha)ds \\ & \quad + M\int_0^t \Bigg(d_2 H_d([x(s+w)]^\alpha, [y(s+w)]^\alpha) \\ & \quad + d_3 d_4 \int_0^s H_d([x(\tau+w)]^\alpha, [y(\tau+w)]^\alpha)d\tau\Bigg)ds \\ & \quad + Md_5 H_d([x(t)]^\alpha, [y(t)]^\alpha).\end{aligned}$$

Therefore,

$$\begin{aligned}
d_\infty\big(\Phi(x)(t), \Phi(y)(t)\big) &= \sup_{0<\alpha\leq 1} H_d([\Phi x(t)]^\alpha, [\Phi y(t)]^\alpha) \\
&\leq d_1 \sup_{0<\alpha\leq 1} H_d([x(t+w)]^\alpha, [y(t+w)]^\alpha) \\
&\quad + MM_1 d_1 \sup_{0<\alpha\leq 1} H_d([x(s+w)]^\alpha, [y(s+w)]^\alpha ds \\
&\quad + M \int_0^t \Bigg(d_2 \sup_{0<\alpha\leq 1} H_d([x(s+w)]^\alpha, [y(s+w)]^\alpha) \\
&\quad + d_3 d_4 \int_0^s \sup_{0<\alpha\leq 1} H_d([x(\tau+w)]^\alpha, [y(\tau+w)]^\alpha d\tau \Bigg) ds \\
&\quad + M d_5 \sup_{0<\alpha\leq 1} H_d([x(t)]^\alpha, [y(t)]^\alpha) \\
&\leq d_1 d_\infty(x(t+w), y(t+w)) \\
&\quad + MM_1 d_1 d_\infty(x(s+w), y(s+w)) ds \\
&\quad + M \int_0^t \Bigg(d_2 d_\infty(x(s+w), y(s+w)) \\
&\quad + d_3 d_4 \int_0^s d_\infty(x(\tau+w), y(\tau+w)) d\tau \Bigg) ds \\
&\quad + M d_5 d_\infty(x(t), y(t))
\end{aligned}$$

Hence,

$$\begin{aligned}
H_1(\Phi x, \Phi y) &\leq d_1 \sup_{0\leq t\leq T} d_\infty(x(t+w), y(t+w)) \\
&\quad + MM_1 d_1 \sup_{0\leq t\leq T} d_\infty(x(s+w), y(s+w)) ds \\
&\quad + M \int_0^t \Bigg(d_2 \sup_{0\leq t\leq T} d_\infty(x(s+w), y(s+w)) \\
&\quad + d_3 d_4 \int_0^s \sup_{0\leq t\leq T} d_\infty(x(\tau+w), y(\tau+w) d\tau \Bigg) ds \\
&\quad + M d_5 \sup_{0\leq t\leq T} d_\infty(x(t), y(t)) \\
&\leq M \left(\frac{d_1}{M} + M_1 d_1 + d_5 + \left(d_2 + d_3 d_4 \frac{T}{2} \right) T \right) H_1(x, y).
\end{aligned}$$

By the hypothesis $(H6)$, Φ is a contraction mapping. By applying the Banach fixed point theorem, it is concluded that (4) has a unique fixed point.

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