Multiplicative δ - derivations of standard algebras*

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Abstract. Every multiplicative δ - derivation of a standard algebra U is additive if there exists an idempotent e', $(e' \neq 0, 1)$ in U satisfying the following conditions: (i) $aU = 0$ implies a $= 0$ for $e' = \delta(e)$; (ii) $e'Ua = 0$ implies U= 0; (iii) $\tilde{e}a(e'' - 1)U\ e'(1-e)$ implies $e'ae = 0$ where $\tilde{e} = \delta^2 e'$ and $e'' = 2\delta(e)$; (iv) $e'a(1-e')\ Ue = 0$ where $e' = \delta(e), e'' = 2\delta(e)$. In particular, every δ - derivation of a standard algebra U with a non - trivial idempotent is additive. As an application the concept is applied to multiplicative δ - derivation to a standard complex algebra $M_n(\mathcal{C})$ of all $n \times n$ complex matrices to see how it decomposes into a sum of δ - inner derivation and a δ - derivation on $M_n(\mathcal{C})$ given by an additive derivation λ on $\mathcal{C}$.

Keywords: Standard algebra, Derivation, δ - derivation, Peirce decomposition.

1 Introduction

Standard algebra U was first studied by Albert in 1948 which includes all associative algebras and commutative Jordan algebras. Later this theory was developed by Schafer in the study of Pierce decomposition and derivations of standard algebras by proving Weddeburn principal theory for these algebras of a char.$\neq 2$ and for char. 0 by proving analogues of the Malcev - Harish - Chandra theorem and the first Whitehead Lemma. Whereas on the other hand several other authors have studied multiplicative mapping and stated conditions for the additivity of the mapping viz. Martindale [5], Ji [4], Ferreira [3] etc. The authors showed that under the additional conditional the bijectivity of such maps with non-trivial idempotents are additive. Simultaneously, if these mappings are defined on an arbitrary nonassociative ring then it is very difficult to analyse the results. However, the present paper takes up a special case of standard algebra with multiplicative δ - derivation exhibiting conditions of when a multiplicative δ - derivation must be an additive map.

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2 standard algebra

Let U be a standard algebra containing nontrivial idempotent. If δ is a fixed element of Φ, then a δ - derivation of the φ - algebra U is a linear map $\varphi : U \to U$ satisfying the equality $\varphi(ab) = \delta(a\varphi(b) + \varphi(a)b)$ for arbitrary elements $a, b \in U$. For any algebra, the multiplication operator by element of the ground field F is a $\frac{1}{2}$ - derivation. We are interested in the behaviour of non-trivial δ - derivations of an standard algebras. As in [1], let $\delta e' = e, \delta^2 e' = \tilde{e}$ and $2\delta(e) = e''$.

The Peirce decomposition of U related to an idempotent $\tilde{e}$ and e takes the form $U = U_{11}(e) \oplus U_{12}(e) \oplus U_{\frac{1}{2}\frac{1}{2}}(e) \oplus U_{21}(e) \oplus U_{22}(e)$,
where $U_{ij}(e) = \{a_{ij} \in U/ea_{ij} = (-\frac{4}{3}i^2 + 3i - \frac{2}{3})a_{ij}$ and $a_{ij}e = (-\frac{4}{3}j^2 + 3j - \frac{2}{3})a_{ij}\}(i, j = 1, 1/2, 2)$ satisfying the multiplicative relations:
(i) $U_{ii}(e)U_{kl}(e) \subseteq \delta_{jk}U_{i1}(e)$ $(i, j, k, l = 1, 2)$ (δ_{jk} = Kroneckar delta);
(ii) $U_{ii}(e)U_{\frac{1}{2}\frac{1}{2}}(e) \subseteq U_{\frac{1}{2}\frac{1}{2}}(e)$ *and* $U_{\frac{1}{2}\frac{1}{2}}(e)U_{ii}(e) \subseteq U_{\frac{1}{2}\frac{1}{2}}(e)(i = 1, 2)$;
(iii) $U_{ii}(e)U_{\frac{1}{2}\frac{1}{2}}(e) = 0$ *and* $U_{\frac{1}{2}\frac{1}{2}}(e)U_{ii}(e) = 0$ $(i, j = 1, 2; i \neq j)$;
(iv) $U_{\frac{1}{2}\frac{1}{2}}(e)U_{\frac{1}{2}\frac{1}{2}}(e) \subseteq U_{11}(e) \oplus U_{22}(e)$;
(v) $[U, U_{\frac{1}{2}\frac{1}{2}}(e)] = 0$.

Throughout his section U is a standard algebra of *char.* $\neq 2$ and $U = U_{11}(e) \oplus U_{12}(e) \oplus U_{\frac{1}{2}\frac{1}{2}}(e) \oplus U_{21}(e) \oplus U_{22}(e)$ the Peirce decomposition of U, relative to idempotent e.

Lemma 1. *(i)* $a_{11} = 2\, e'(e'ae') - e'ae'$, $a \in U$;
(ii) $a_{\frac{1}{2}\frac{1}{2}}(e') = 2^2(e'a - e'(e'a))$, $a \in U$;
(iii) $a_{22} = a - e'a - ae' - e'ae' + 2e'(e'ae')$, $a \in U$.

The proof of above Lemma is based only on simple calculations from the properties (i) - (v) of the Peirce decomposition and the conditions stated in the abstract.

Lemma 2. $\varphi(U_{ij}) \subseteq U_{ij}$.

Proof. To show that $a_{11} \in U_{11}$, One can simply apply the definition and $\varphi(e')$ $= 0$. For $a_{12} \in U_{12}$, we have $\varphi(a_{12}) = \varphi(e'a_{12}) = \delta e'\varphi(a_{12}) = e\varphi(a_{12})$ and $0 = \varphi(0) = \varphi(a_{12}e') = \delta\varphi(a_{12})e'$. Since $\delta \neq 0$, $\varphi(a_{12})e' = 0$. So that $(1 - e)\varphi(a_{12}) = e\varphi(a_{12})$, which implies that $\varphi(a_{12}) \in U_{12}$. For $a_{21} \in U_{21}$, we have $\varphi(a_{21}) = \varphi(a_{21}e') = \delta\varphi(a_{21})e' = \varphi(a_{21})e'$ and $0 = \varphi(e'a_{21}) = \delta e'(a_{21}) = e\varphi(a_{21})$. Hence $a_{21} \in U_{21}$. For $a_2 \in U_{22}$, we have $0 = \varphi(e'a_{22}) = e\varphi(a_{22})$ and $0 = \varphi(a_{22}e') = \varphi(a_{22})e'$. Hence $a_{22} \in U_{22}$. For $\varphi a_{\frac{1}{2}\frac{1}{2}}$, we have $\varphi a_{\frac{1}{2}\frac{1}{2}} = \varphi(2e'(e'a_{\frac{1}{2}\frac{1}{2}}e') - e'a_{\frac{1}{2}\frac{1}{2}}e') = \tilde{e}\varphi a_{\frac{1}{2}\frac{1}{2}}e'(e'' - 1)$ which implies $a_{\frac{1}{2}\frac{1}{2}} \in U_{\frac{1}{2}\frac{1}{2}}$.

Lemma 3. *For each* a_{ii} *in* U_{ii} *and* a_{jk} *in* U_{jk} *with* $1 \leq i, j, k \leq 2$ *and* $j \neq k$ *we have* $\varphi(a_{ii} + a_{jk}) = \varphi(a_{ii}) + \varphi(a_{jk})$.

Proof. Obviously, one needs only to show that $\varphi(a_{ii}) + \varphi(a_{jk}) - \varphi(a_{ii} + a_{jk}) = 0$. By the hypothesis, we consider eight cases.
Case 1: $i = j = 1$ and $k = 2$. Using the relation (i), one has to show that $U(\varphi(a_{11}) + \varphi(a_{12}) - \varphi(a_{11} + a_{12})) = 0$. For $u_1 \in e'U$, using Lemma 2, we have

$\varphi(a_{12})u_1 = 0$. Let $u_1 = \delta(v_1)$ and thus $v_1 a_{12} = 0$. Since $e' a_{12} = 0$, it follows that $\varphi(v_1 e')a_{12} = \delta\varphi(v_1)a_{12} = 0$. Hence $(\varphi(a_{11}) + \varphi(a_{12})) = u_1\varphi(a_{11} + a_{12})$. Thus, $u_1(\varphi(a_{11}) + \varphi(a_{12}) - \varphi(a_{11} + a_{12})) = 0$, for each $u_1 \in eU$. On the other hand, for $u_2\varphi(1-e)U$, it follows from Lemma 2 that $u_2\varphi(a_{11}) = 0$. Let $v_2 = \delta^{-1}(u_2)$. Then $v_2 = \delta^{-1}(u_2(1-e')) = v_2(1-e')$ and hence $v_2 a_{11} = 0$. Since $0 = \varphi(0) = \varphi(v_2 e') = \varphi(v_2)e' = 0$. Hence $u_2(\varphi(a_{11}) + \varphi(a_{12})) = u_2\varphi(a_{11} + a_{12})$. Thus, $u_2(\varphi(a_{11}) + \varphi(a_{12}) - \varphi(a_{11} + a_{12})) = 0$. Since u_1 and u_2 are arbitrary, we obtain that $u(\varphi(a_{11}) + \varphi(a_{12}) - \varphi(a_{11} + a_{12})) = 0$ for each $u \in U$.

Case 2: $i = j = 2$, $k = 1$. Similar to case 1, it remains to show that $U(\varphi(a_{22}) + \varphi(a_{21}) - \varphi(a_{22} + a_{21})) = 0$. For $u_1 \in e'U$, from Lemma 2, we have $u_1(\varphi(a_{22})) = 0$. Let $v_1 = \delta^{-1}(u_1)$. Then $v_1 = \delta^{-1}(e'u_1) = ev_1$, and thus $v_1 a_{22} = 0$. Also since $ea_{22} = 0$, it follows that $\varphi(v_1 e')a_{22} = \delta\varphi(v_1)a_{22} = 0$, therefore, $u_1(\varphi(a_{22}) + \varphi(a_{21})) = u_1\varphi(a_{22} + a_{21})$. For $u_2 \in (1-e')U$, from Lemma 2, we have $u_2(\varphi(a_{21})) = 0$. Let $v_2 = \delta^{-1}(u_2)$. Then $v_2 = \delta^{-1}(1-e')u_2 = v_2(1-e)$ and thus $v_2 e = 0$, which implies that $\varphi(v_2 e)a_{21} = \delta\varphi(v_2)a_{21} = 0$. Hence $u_2(\varphi(a_{22}) + \varphi(a_{21})) = u_2\varphi(a_{22} + a_{21})$.Since u_1 and u_2 are arbitrary, we have $U(\varphi(a_{22}) + \varphi(a_{21}) - \varphi(a_{22}) + \varphi(a_{21}))) = 0$.

Case 3 is $i = k = 1$, $j = 2$ and **Case 4** is $i = k = 2$, $j = 1$. The proofs of these two cases is similar to that of the case 1. Hence we omit the proofs.

Case 5: $i = \frac{1}{2}$, $j = 1$, $k = 2$. By a similar way to case 1, we show $U(\varphi(a_{\frac{1}{2}\frac{1}{2}}) + \varphi(a_{12}) - \varphi(a_{\frac{1}{2}\frac{1}{2}} + a_{12})) = 0$. For $u_1 \in e'U$, it follows from Lemma 2, we have $u_1(\varphi(a_{\frac{1}{2}\frac{1}{2}}))) = 0$. Let $v_1 = \delta^{-1}(u_1)$. Then $v_1 = \delta^{-1}(e'u_1) = ev_1$, and thus $v_1 a_{\frac{1}{2}\frac{1}{2}} = 0$. Also since $ea_{\frac{1}{2}\frac{1}{2}} = 0$, it follows that $\varphi(v_1 e')a_{\frac{1}{2}\frac{1}{2}} = \delta\varphi(v_1)a_{\frac{1}{2}\frac{1}{2}} = 0$, implies $u_1(\varphi(a_{\frac{1}{2}\frac{1}{2}}) + \varphi(a_{12})) = u_1\varphi(a_{\frac{1}{2}\frac{1}{2}} + a_{12})$. For $u_2 \in (e''-1)U$, using Lemma 2, we have $u_2(\varphi(a_{12})) = 0$. Let $v_2 = \delta^{-1}(u_2)$. Then from relation (ii) $v_2 = \delta^{-1}(e''-1)u_2 = u_2(e-1)$ and thus $u_2 e = 0$, which implies that $\varphi(v_2 e)a_{12} = \delta\varphi(v_2)a_{12} = 0$. Hence $u_2(\varphi(a_{\frac{1}{2}\frac{1}{2}}) + \varphi(a_{12})) = u_2\varphi(a_{\frac{1}{2}\frac{1}{2}} + a_{12})$. Since u_1 and u_2 are arbitrary, we have $U(\varphi(a_{\frac{1}{2}\frac{1}{2}})) + \varphi(a_{12}) - \varphi(a_{\frac{1}{2}\frac{1}{2}}) + a_{12})) = 0$.

Case 6 is $i = \frac{1}{2}, j = 2$, $k = 1$ and **Case 7** is $i = 1$, $j, k = \frac{1}{2}$. The proofs are similar to that of the case 5. We omit the proofs.

Case 8: $i = \frac{1}{2}, j, k = 2$. Using the relation (iii), it suffices to show that $U(\varphi(a_{\frac{1}{2}\frac{1}{2}}) + \varphi(a_{22}) - \varphi(a_{\frac{1}{2}\frac{1}{2}} + a_{22})) = 0$. For, $u_1 \in (e''-1)U$, from Lemma 2, we have $u_2(\varphi(a_{\frac{1}{2}\frac{1}{2}})) = 0$. Let $v_1 = \delta^{-1}(u_1)$. Then $v_1 = \delta^{-1}(e''-1)u_1 = u_1(e-1)$ and thus $u_1 e = 0$, which implies that $\varphi(v_2 e)a_{12} = \delta\varphi(v_2)a_{21} = 0$. Hence $u_1(\varphi(a_{\frac{1}{2}\frac{1}{2}}) + \varphi(a_{22})) = u_1\varphi(a_{\frac{1}{2}\frac{1}{2}} + a_{22})$. Thus, $u_1(\varphi(a_{\frac{1}{2}\frac{1}{2}}) + \varphi(a_{22}) - \varphi(a_{\frac{1}{2}\frac{1}{2}} + a_{22})) = 0$ for each $u_1 \in (e''-1)U$. On the other hand, for $u_2 \in (1-e)U$, it follows from Lemma 2 that $u_2\varphi(a_{\frac{1}{2}\frac{1}{2}}) = 0$. Let $v_2 = \delta^{-1}(u_2)$. Then $v_2 = \delta^{-1}(u_2(1-e')) = v_2(1-e')$ and hence $v_2 a_{11} = 0$. Since $0 = \varphi(0) = \varphi(v_2 e') = \varphi(v_2)e' = 0$. Hence $u_2(\varphi(a_{\frac{1}{2}\frac{1}{2}}) + \varphi(a_{22})) = u_2\varphi(a_{\frac{1}{2}\frac{1}{2}} + a_{22})$. Thus, $u_2(\varphi(a_{\frac{1}{2}\frac{1}{2}}) + \varphi(a_{22}) - \varphi(a_{\frac{1}{2}\frac{1}{2}} + a_{22})) = 0$. Since u_1 and u_2 are arbitrary, we obtain that $U(\varphi(a_{\frac{1}{2}\frac{1}{2}}) + \varphi(a_{22}) - \varphi(a_{\frac{1}{2}\frac{1}{2}} + a_{22})) = 0$ for each $u \in U$.

Lemma 4. *φ is additive on U_{12}.*

Proof. Let a_{12}, $b_{12} \in U_{12}$. Using Lemma 2, we have $\varphi(a_{11}), \varphi(b_{12}, \varphi(a_{11}+b_{12}) \in U_{12}$. Hence $\varphi(a_{12})+\varphi(b_{12}-\varphi(a_{11}+b_{12}) = 0$ for each $u_1 \in eU_{12}$. For $u_2 \in (1-e')U$. Let $v_2 = \delta^{-1}(u_2)$, then $v_2 = v_2(1-e')$, which implies that $v_2b_{12} = v_2b_{12}$, that is $e'(v_2+v_2b_{12}) = v_2b_{12}$. Therefore $\varphi(v_2b_{12}) = \delta e'\varphi(v_2+v_2b_{12})$. Since $e' \in U_{11}$ and $a_{12} \in U_{12}$, then from Lemma 3 one has $\varphi(e'+a_{12}) = \varphi(a_{12})$. Also since $v_2(a_{12}+b_{12}) = (v_2+v_2b_{12})(e'+a12)$ one has $\varphi(v_2(a_{12}+b_{12})) = \varphi v_2a_{12}+\varphi(v_2b_{12})$. Hence $u_2(\varphi(a_{12})+\varphi(b_{12})-\varphi(a_{12}+b_{12}))$= 0. Since u_1 and u_2 are arbitrary, we have $U(\varphi(a_{11})+\varphi(b_{12})-\varphi(a_{11}+b_{12})) = 0$. It follows from relation(i) that $\varphi(a_{12}+b_{12}) = \varphi(a_{11})+\varphi(b_{12})$. Hence φ is additive on U_{12}.

Lemma 5. *φ is additive on U_{11}.*

Proof. Let $a_{11}, b_{11} \in U_{11}$. Then, $\varphi(a_{11}), \varphi(b_{11}), \varphi(a_{11}+b_{11}) \in U_{11}$ from Lemma 2. Hence $u_1(\varphi(a_{11})+\varphi(b_{11})-\varphi(a_{11}+b_{11})) = 0$ for each $u_1 \in U_{11}$. For $u_2 \in U_{11}$. Let $\delta v_2 = u_2$. Then $\delta v_2 = (1-e)u_2 = (1-e')\delta v_2 \in (1-e')U$. Hence $a_{11}v_2$ and $b_{11}v_2$ are in U_{11}. Therefore, from Lemma 4 that $\varphi(a_{11}v_2+b_{11}v_2) = \varphi(a_{11}v_2)+\varphi(b_{11}v_2)$. So, $u_2(\varphi(a_{11})+\varphi(b_{11})-\varphi(a_{11}+b_{11})) = 0$. Since u_1 and u_2 are arbitrary, we have $(1-e)U(\varphi(a_{11})+\varphi(b_{11})-\varphi(a_{11}+b_{11})) = 0$. By Lemma 2, it is noted that $(\varphi(a_{11})+\varphi(b_{11})-\varphi(a_{11}+b_{11})) \in U_{11}$. Hence it follows from the relation (ii) that $\varphi(a_{11}+b_{11}) = \varphi(a_{11})+\varphi(b_{11})$. Hence φ is additive on U_{11}.

Lemma 6. *φ is additive on $U_{\frac{1}{2}\frac{1}{2}}$.*

Proof. Let $a_{\frac{1}{2}\frac{1}{2}}, b_{\frac{1}{2}\frac{1}{2}} \in U_{\frac{1}{2}\frac{1}{2}}$. Then, $\varphi(a_{\frac{1}{2}\frac{1}{2}}), \varphi(b_{\frac{1}{2}\frac{1}{2}}), \varphi(a_{\frac{1}{2}\frac{1}{2}}+b_{\frac{1}{2}\frac{1}{2}}) \in U_{\frac{1}{2}\frac{1}{2}}$ from Lemma 2. Hence $u_1(\varphi(a_{\frac{1}{2}\frac{1}{2}})+\varphi(b_{\frac{1}{2}\frac{1}{2}})-\varphi(a_{\frac{1}{2}\frac{1}{2}}+b_{\frac{1}{2}\frac{1}{2}})) = 0$ for each $u_1 \in e(1-e)U$. For $u_2 \in Ue'(e''-1)$, let $v_2 = \delta^{-1}(u_2)$, then $v_2 = v_2e'(e''-1)$, which implies that $v_2b_{\frac{1}{2}\frac{1}{2}} = v_2b_{\frac{1}{2}\frac{1}{2}} - v_2b_{\frac{1}{2}\frac{1}{2}}e' = e'(v_2+v_2b_{\frac{1}{2}\frac{1}{2}})$. Therefore $\varphi(v_2b_{\frac{1}{2}\frac{1}{2}}) = \varphi(e'(v_2+v_2b_{\frac{1}{2}\frac{1}{2}})) = \delta e'\varphi(v_2+v_2b_{\frac{1}{2}\frac{1}{2}})$. Since $e' \in U_{-\frac{1}{2}\frac{1}{2}}$ and $\varphi(e'+a_{\frac{1}{2}\frac{1}{2}})) = \varphi(e')+\varphi(a_{\frac{1}{2}\frac{1}{2}})) = \varphi(a_{\frac{1}{2}\frac{1}{2}}))$ from Lemma 4. Also since $v_22(a_{\frac{1}{2}\frac{1}{2}})+b_{\frac{1}{2}\frac{1}{2}})) = (v_2+v_2b_{\frac{1}{2}\frac{1}{2}}))(e'+a_{\frac{1}{2}\frac{1}{2}}))$, one has $\varphi(v_2(a_{\frac{1}{2}\frac{1}{2}}+b_{\frac{1}{2}\frac{1}{2}})) = \varphi(v_2a_{\frac{1}{2}\frac{1}{2}})+\varphi(v_2b_{\frac{1}{2}\frac{1}{2}})$. Hence $u_2(\varphi(a_{\frac{1}{2}\frac{1}{2}})+\varphi(b_{\frac{1}{2}\frac{1}{2}})-\varphi(a_{\frac{1}{2}\frac{1}{2}}+b_{\frac{1}{2}\frac{1}{2}}))$= 0. Since u_1 and u_2 are arbitrary, $U(\varphi(a_{\frac{1}{2}\frac{1}{2}})+\varphi(b_{\frac{1}{2}\frac{1}{2}})) = 0$. Thus, from relation (iii) one obtains $\varphi(a_{\frac{1}{2}\frac{1}{2}}+b_{\frac{1}{2}\frac{1}{2}}) = \varphi(a_{\frac{1}{2}\frac{1}{2}})+\varphi(b_{\frac{1}{2}\frac{1}{2}})$. Hence φ is additive on $U_{\frac{1}{2}\frac{1}{2}}$.

Lemma 7. *φ is additive on $e'U = U_{11} \oplus U_{12} \oplus U_{\frac{1}{2}\frac{1}{2}}$.*

Proof. By Lemmas 4, 5 and 6, φ is additive on U_{11}, U_{12} and $U_{\frac{1}{2}\frac{1}{2}}$, respectively. Using Lemma 3, for each $a_{11} \in U_{11}$, $a_{12} \in U_{12}$ and $a_{\frac{1}{2}\frac{1}{2}} \in U_{\frac{1}{2}\frac{1}{2}}$, we have $\varphi(a_{11}+a_{12}+a_{\frac{1}{2}\frac{1}{2}}) = \varphi(a_{11})+\varphi(a_{12})+\varphi(a_{\frac{1}{2}\frac{1}{2}})$. Hence φ is additive on $e'U = U_{11} \oplus U_{12} \oplus U_{\frac{1}{2}\frac{1}{2}}$

Theorem 1. *Let φ be a multiplicative δ - derivation containing a non-trivial idempotent e satisfying the following conditions:*
(i) $aU = 0$ implies $a = 0$; where $e' = \delta(e)$,
(ii) $e'Ua = 0$ implies $a = 0$ (and hence $U\ a = 0$ implies $a = 0$);
(iii) $\tilde{e}ae'(e''-1)Ue(1-e)$ implies $e'ae = 0$ where $\tilde{e} = \delta^2e'$ and $e'' = 2\delta(e)$;

(iv) $e'a(1-e)Ue$ = 0 implies eae = 0(and hence $e'aeU$ =0 implies eae = 0); where $e' = \delta(e), e'' = 2\delta(e)$.
then every multiplicative δ - derivation onto an arbitrary standard algebra U is additive.

Proof. Let $a, b \in U$. For each $u \in e'U$, let $v = \delta^{-1}(u)$. Then $v = \delta^{-1}(eu) = e'v$, and hence $va, vb \in e'U$. But, $\varphi(va + vb) = \varphi(va) + \varphi(vb)$ by Lemma 7. Thus, $u(\varphi(a)+\varphi(b)) = \delta v\varphi(a+b) = u\varphi(a+b)$. Since u is arbitrary, one has $e'U(\varphi(a+b) - \varphi(a) - \varphi(b)) = 0$. From the relation (ii) one obtains $\varphi(a+b) = \varphi(a) + \varphi(b)$. Hence φ is additive.

Remark 1. Corollaries 1.17, 1.18, 1.19 of [3] are also applicable for multiplicative δ - derivations in standard algebras. Thus we omit the proofs of those corollaries.

3 Multiplicative δ - derivation on $M_n(\mathcal{C})$

Let U be a standard algebra over a Complex field $\mathcal{C}$, δ be algebraic endomorphism of U. φ is called inner if there exists u_0 in U such that $\varphi(a) = \delta\varphi(a)a_0 + \delta a_0\varphi(a)$ for all $a \in U$. obviously, if U has an identity I, then $\varphi(I) = 0$. In this section, we consider the linearity problems of multiplicative δ - derivation on $M_n(\mathcal{C})$. It follows from Corollary 2 of [2] that every multiplicative δ-derivation on $M_n(\mathcal{C})$ is additive. It is well known that each algebraic endomorphism δ on $M_n(\mathcal{C})$ is inner, i.e.,there is an invertible matrix T_0 in $M_n(\mathcal{C})$ such that $\delta(U) = T_0UT_0^{-1}$ for each U in $M_n(\mathcal{C})$. Like the ordinary derivation, we can show that every linear multiplicative δ - derivation is inner, which may be known fact.

Theorem 2. *Let φ be a multiplicative δ - derivation on $M_n(\mathcal{C})$. If φ is linear, then φ is inner.*

Proof. Let $H_{ij}, i, j = 1, 2..., n$ be the standard matrix on $M_n(\mathcal{C})$. Let $T_0 = \sum_{j=1}^{n} \delta(H_{j1})\varphi(H_{1j})$ Then, for each H_{kl}, using that $\varphi(I) = 0$, we have $\delta(H_{kl})T_0 - T_0\delta(h_{kl}) = \delta(H_{kl}) \sum_{j=1}^{n} \delta(H_{j1})\varphi(H_{1j}) - \sum_{j=1}^{n} \delta(H_{j1})\varphi H_{1j})\delta(H_{kl}) = \delta(H_{k1})\varphi(H_{kl}) - \varphi(I)H_{kl} + \varphi(H_{1j})\delta(H_{kl})\delta E_{kl} = \delta(H_{k1})\varphi(H_{1l}) + \varphi(H_{kl})\delta(H_{k1})\varphi(H_{k1}H_{1l}) = \varphi(H_{kl})$ where I is the identity matrix. Since δ and φ are linear, we have $\varphi(U) = \delta(U)T_0 - T_0\delta(U)$ for each U in $M_n(\mathcal{C})$. Hence φ is inner.

Lemma 8. *Let φ be a multiplicative δ - derivation on $M_n(\mathcal{C})$. Then there exist an additive derivation $\gamma : \mathbf{C} \to \mathbf{C}$ and an invertible matrix V_0 in $M_n(\mathbf{C})$ such that $\varphi(uI) = \gamma(u)V_0$ holds for all u in $\mathbf{C}$.*

Proof. For arbitrary $U \in M_n(\mathcal{C})$ and $u \in \mathcal{C}$, one has $\varphi(uU) = \varphi((mI)U) = \delta u\varphi(U) + \delta\varphi(uI)U$. On the other hand, $\varphi(uU) = \varphi(U(uI)) = \delta U\varphi(uI) + \delta\varphi(U)uI = \delta U\varphi(uI) + \delta\varphi(U)u$. Hence

$$\delta\varphi(uI)U = \delta U\varphi(uI). \tag{1}$$

Since δ is inner, there exists invertible matrices T_0 and S_0 such that (1) can be written as $\delta\varphi(uI)S_0US_0^{-1} = \delta T_0UT_0^{-1}\varphi(uI)$ and hence $\varphi(uI)S_0U = T_0UT_0^{-1}\phi(mI)US_0$, $(T_0^{-1}\varphi(uI)S_0)U = U(T_0^{-1}\varphi(mI)S_0)$ holds for all U in $M_n(\mathcal{C})$, so there exists $\gamma(u) \in \mathcal{C}$ such that $T_0^{-1}\varphi(uI)S_0 = \gamma(u)I$, hence

$$\varphi(uI) = \gamma(u)V_0 \tag{2}$$

where $V_0 = S_0^{-1}T_0$. Since φ is additive, one can see easily that the mapping $\gamma : \mathcal{C} \to \mathcal{C}$ defined by the equation (2) is an additive derivation.

Remark 2. The proof of the Lemma 8 implies that the multiplicative δ - derivation φ is linear if and only if S_0 in γ i.e., γ is trivial derivative.

Theorem 3. *A mapping φ on $M_n(\mathcal{C})$ is a multiplicative δ - derivation if and only if there exist an additive derivation $\gamma : \mathcal{C} \to \mathcal{C}$, a matrix U_0 and invertible matrices S_0 and T_0 such that $\varphi(x_{ij}) = S_0^{-1}(\gamma x_{ij})T_0 + S_0^{-1}(x_{ij})S_0U_0 - U_0T_0^{-1}(x_{ij})T_0$.*

Proof. Let φ be a multiplicative δ - derivation on $M_n(\mathcal{C}), \gamma$ be the additive derivation on $\mathcal{C}$ defined by $\varphi(uI) = \gamma(u)S_0^{-1}T_0$ as in the proof of Lemma 8. Let $\gamma(X) = \delta S_0(\gamma x_{ij})T_{0^{-1}}$ for each $X = (x_{ij}) \in M_n(\mathcal{C})$. Then γ is additive on $M_n(\mathcal{C})$. For all $X = (x_{ij})$ and $Y = (y_{ij})$ in $M_n(\mathbf{C})$, one has $\delta X\Gamma(Y) + \delta\Gamma(X)Y = \delta S_0XS_0^{-1}S_0\gamma(y_{ij})T_0^{-1} + \delta S_0\gamma(x_{ij})T_0^{-1}T_0YT_0^{-1} = \delta S_0(\gamma \sum_{k=1}^{n}(x_{ik}y_{kj}))T_0^{-1} = \Gamma(XY)$. Thus, Γ is a multiplicative δ-derivation on $M_n(\mathcal{C})$. Define $\tilde{\varphi} = \varphi - \Gamma$. Then $\tilde{\varphi}$ is a multiplicative δ - derivation on $M_n(\mathcal{C})$. Obviously, $\tilde{\varphi}(uI) = \varphi(uI) - \gamma(uI) = \gamma(u)S_0^{-1}T_0^{-1} - \gamma(u)S_0^{-1}T_0^{-1} = 0$. By the remark of Lemma 8 $\tilde{\varphi}$ is linear. It follows from Theorem 2 that there exists U_0 is in $M_n(\mathcal{C})$ such that $\tilde{\varphi}(U) = \delta U\varphi(U_0) - \delta\varphi(U)U_0$ for each U in $M_n(\mathcal{C})$. Hence

$$\varphi(x_{ij}) = S_0^{-1}(\gamma(x_{ij}))T_0 + S_0^{-1}(x_{ij})S_0U_0 - U_0T_0^{-1}(a_{ij})T_0. \tag{3}$$

Remark 3. For fixed δ, putting $(a_{ij}) = uI$ in (3), one can see that the additive derivation Γ is uniquely determined. Hence all such matrices U_0 are different from λI.

References

1. Bharathi, M. V. L., Jayalakshmi, K.:Multiplicative δ-derivation in alternative algebra. Asian-European J. of Mathematics, **11** (2018).
2. Hou *, C., Zhang, W., Meng, Q.: A note on (α, β) - derivations, Linear Algebra and its applications **432**, 2600–2607, (2010).
3. Ferreira, B. L. M., Guzzo Jr. H. and Ferreira, J. C. M: Multiplicative mappings of Standard rings,
4. Peisheng, Ji: Additivity of Jordan maps on Jordan algebras, Linear Algebra Appl. **431** 179–188,(2009).
5. Martindale III, W. S.:When are multiplicative mappings additive? Proc. Amer. Math. Soc., **21** 695–698 (1969).
6. Schafer, R. D.: Standard algebra, Pacific. J. Math, **29**, 203–223 (1969).