An Improved Secant-like Method and its Convergence for Univariate Unconstrained Optimization

R. Bhavani and P. Paramanathan

Department of Mathematics,
Amrita School of Engineering - Coimbatore
Amrita Vishwa Vidyapeetham, India. bhavaniramar13@gmail.com
p_paramanathan@cb.amrita.edu

Abstract. We describe an improved Secant-like method based on divided difference approximation of third derivative for optimization of univariate unconstrained function. The convergence rate of the method is discussed with sufficient number of numerical examples to show the applications of our method. The collation of our consequence with subsisting models are given.

Keywords: Unconstrained optimization · Univariate Optimization · Iterative method · Secant-like method.

1 Introduction

In the optimization of univariate unceasingly differentiable function $f(x)$, the minimization operation can still be executed just using function evaluation when the derivatives are not available at the evaluating time. Many optimization algorithms developed using various quasi newton approaches, aim to refining an initial rough approximation [5],[6],[8],[10]. For rectifying the disadvantages of these methods, different versions of secant methods are introduced by various authors for optimizing univariate minimization problem

$$min\ f(x) \tag{1}$$

where f is described on a real space $\mathbb{R}$.
In that, modification in the classical Secant method stationed on the development of cubic approximation gave markable improved results than the existing one [1]. Also replacing the second derivative of Newton's method using taylor series expansion we can get an variant Secant method,

$$x_{d+1} = x_d - \frac{f'(x_d)(x_d - x_{d-1})}{2f'(x_d) - \frac{f(x_d)-f(x_{t-d})}{(x_d - x_{d-1})}} \tag{2}$$

In order to overcome the drawbacks of variant of Secant method we again use the taylor series expansion and second derivative of variant secant method to replace the third derivative and get modified Secant method.

$$x_{d+1} = x_d - \frac{1}{f'''(x_d)}[f''(x_d) + \sqrt{f''(x_d)^2 - 2f'''(x_d)f'(x_d)}] \tag{3}$$

This method converges faster compared to Secant method and variant of Secant method. But this technique can also be improved in a better way using divided difference property.
A unified convergence analysis for Wang's method at $\theta = -2$ with quadratical convergence is found for variant of Secant method using Divided difference property is excellent compared Secant method and variant of Secant method [2]

$$x_{d+1} = x_d - \frac{f'(x_d)(x_d - x_{d-1})}{4f'(x_d) - 6f[x_d, x_{d-1}] + 2f'(x_{d-1})} \tag{4}$$

But we can get faster convergence if the divided difference property is applied on f'''.

For solving nonlinear equations, a new secant-like method is found where no derivative is required and survey of the confluence manifest that the asymptotical order of convergence is $1 + \sqrt{3}$ [7]

$$y_{d+1} = x_{d+1} - \frac{(x_d - y_d)f(x_{d+1})}{f(x_d) - f(y_d)}$$

It is efficient only when the evaluation cost of the derivative is high.

A modification of Secant method, helps to upgrade the felicitous of the Secant method that changes the aspiration of a linear system in every step. They does not utilize derivatives and has R-order of convergence $\frac{1+\sqrt{5}}{2}$ [9]. Only the quantitative firmness of the Secant method is enhanced in

A modification Secant-like method is demonstrated to have confluence order of 2.732 [11] and it is effective than the order of convergence 2.414 [3]. But increasing the order of derivative can give a faster and better convergence rate.

A unaccustomed technique for minimisation of univariate functions which evince an vitally quadratic convergence and whose confluence interim is only restricted by the actuality of closeby maxima is found [12]. Entailing only the first and second-order derivatives may help to find minimum but increasing the order helps to converge smoothly.

This paper introduces an improved secant-like method for finding the solution of univariate minimization problem by replacing third derivative f''' using divided difference property. Since the logic of any algorithm relatively depends on one-dimensional optimization [4] and they are absolutely necessary, we discuss one-dimensional optimization [1]. In most problems, certain degree of smoothness is possessed by functions. To utilize this smoothness, execution on polynomial approximation is designed. Order of convergence is greater than unity in this method [1]. Newton's method is prepossessing with regard to its convergence amid the techniques of polynomial approximation. Here using divided difference property in the modified Secant method, an improved Secant-like method is derived. Using x_d and x_{d-1} as our initial values for d=1,2,3,... we will find x_{d+1} and until it satisfies the terminating condition (i.e) $|f'(x_{d+1})| < \epsilon$ we will iterate by replacing this x_{d+1} value with x_d if $f'(x_{d+1}) > 0$ else replace x_{d-1} if $f'(x_{d+1}) < 0$.

The execution of improved Secant-like method is compared and concluded based on its faster convergence using six standard test functions and its convergence rate is found. The best part of this method is for each value of θ ,a new class is obtained and its convergence is found and compared. Since this method is used to solve non-linear functions we have taken functions of degree 2 or more and transcendental functions like exponential and cosine functions.

In this paper, we entrust an improved Secant-like method (9). Section 2 describes the details of the proposed method and its convergence properties. Section 3 manifest the numerical outcome of the proposed technique and its comparison with Secant method, variant of Secant method, Modified Secant method and Wang's method.

1.1 An Improved Secant-Like Method

Modified secant method [1] is given below as follows:

$$x_{d+1} = x_d - \frac{1}{f'''(x_d)}[f''(x_d) + \sqrt{f''(x_d)^2 - 2f'''(x_d)f'(x_d)}]$$

It has an order of convergence $p = 1.618...$

Using the divided difference property (see[2]), we have

$$\begin{aligned} f'''(x_d) &\approx \frac{f''(x_d) - f''(x_{d-1})}{x_d - x_{d-1}} \\ &= \frac{M - f[x_d, x_d, x_{d-1}] + f[x_d, x_d, x_{d-1}] - f[x_d, x_{d-1}, x_{d-1}] + f[x_d, x_{d-1}, x_{d-1}]}{x_d - x_{d-1}} \\ &= f[x_d, x_d, x_d, x_{d-1}] + f[x_d, x_d, x_{d-1}, x_{d-1}] + f[x_d, x_{d-1}, x_{d-1}, x_{d-1}] \qquad (5) \end{aligned}$$

where,

$$M = \frac{f[x_d, x_d, x_d] - f[x_{d-1}, x_{d-1}, x_{d-1}]}{x_d - x_{d-1}}$$

Generally, in the backward difference of $f'''(x_d)$ the possibility of getting an error is high so in order to reduce the error we are implementing divided difference property.
and

$$f'''(x_d) \approx \frac{3f''(x_d) - h_d}{x_d - x_{d-1}} \tag{6}$$

This $f'''(x_d)$ is proposed in [1]
where,

$$f''(x_d) = f[x_d, x_d, x_{d-1}] + f[x_d, x_{d-1}, x_{d-1}]$$

and

$$\begin{aligned} h_d &= \frac{2f'(x_d) - [\frac{f(x_d) - f(x_{d-1})}{x_d - x_{d-1}}]}{x_d - x_{d-1}} \\ &= 2\frac{f[x_d, x_d] - f[x_d, x_{d-1}]}{x_d, x_{d-1}} \\ &= 2f[x_d, x_d, x_{d-1}] \end{aligned}$$

Therefore,

$$\begin{aligned} f'''(x_d) &= 3\frac{f[x_d, x_d, x_{d-1}] + f[x_d, x_{d-1}, x_{d-1}] - 2f[x_d, x_d, x_{d-1}]}{x_d - x_{d-1}} \\ &= -3f[x_d, x_d, x_{d-1}, x_{d-1}] \end{aligned} \tag{7}$$

where $f \underbrace{.....}_{d+1}$ are divided difference of order d for d=1,2,... [2]

From [2], suppose $\theta \in R$ is a persistent. By linear fusion of (4) and (6), we acquire

$$\begin{aligned} f'''(x_d) \approx\ & \theta(f[x_d, x_d, x_d, x_{d-1}] + f[x_d, x_d, x_{d-1}, x_{d-1}] + f[x_d, x_{d-1}, x_{d-1}, x_{d-1}]) - \\ & - 3(1-\theta)(f[x_d, x_d, x_{d-1}, x_{d-1}]) \\ =\ & \frac{\theta f''(x_d) - 3(1+\theta)(f[x_d, x_d, x_{d-1}] - f[x_d, x_{d-1}, x_{d-1}]) - \theta f''(x_{d-1})}{x_d - x_{d+1}} \end{aligned} \tag{8}$$

Hence substituting $f'''(x_d)$ in equation (2), we get an improved Secant-like method as follows,

$$x_{d+1} = x_d - \frac{1}{f'''(x_d)}[f''(d_t)^2 + N] \tag{9}$$

where,

$$f'''(x_d) = \frac{\theta f''(x_d) - 3(1+\theta)(f[x_d, x_d, x_{d-1}] - f[x_d, x_{d-1}, x_{d-1}]) - \theta f''(x_{d-1})}{x_d - x_{d-1}}$$

$$N = \sqrt{f''(x_d)^2 - 2(\frac{\theta f''(x_d) - 3(1+\theta)(f[x_d, x_d, x_{d-1}] - f[x_d, x_{d-1}, x_{d-1}]) - \theta f''(x_{d-1})}{x_d - x_{d-1}})f'(x_d)}$$

For each value of θ we will get a new class and each class is compared with secant method, variant of Secant method and Modified Secant method based on its faster convergence.
This method has an order of convergence atleast $p = 1.618...$

Algorithm 1 This is the algorithm helps to calculate the optimal minimum point x^*
.

1. Choose the value of θ, $\theta \in R$
2. Choose two points u and v such that $f'(u) < 0$ and $f'(v) > 0$ and also choose a small integer ϵ
   set $x_{d-1} = u$ for $d = 1, 2, ...$
   set $x_d = v$ for $d = 1, 2, ...$
3. Calculate the new point x_{d+1} using Eq.(2) and evaluate $f'(x_{d+1})$
4. If $|f'(x_{d+1})| < \epsilon$, Terminate
   elseif $f'(x_{d+1}) < 0$, set $x_{d-1} = x_{d+1}$ and go to Step 3.
   elseif $f'(x_{d+1}) > 0$, set $x_d = x_{d+1}$ and go to Step 3.

Numerical test We will compare the execution of improved Secant-like method with Secant method, Variant of Secant method, Modified Secant method, Wang's

method based on its number of iterations using six Test functions which were used in [1],[2],[3],[4]. In Table 1, the test function and its minimum point x^* is given. Table 2 and Table 3, will give the improved Secant-like method results. Table 4 gives the final results by comparing the following equations (10),(2),(3),(4),(9) using their initial points.
Secant method:

$$x_{d+1} = x_d - \frac{f'(x_d)(x_d - x_{d-1})}{f'(x_d) - f'(x_{d-1})} \tag{10}$$

Table 1. Standard Optimization Problems

No	Function	minimum point
1	$x^4 - 8.5 * x^3 - 31.0625 * x^2 - 7.50 * x + 45$	8.2784
2	$(x+2)^2(x+4)(x+5)(x+8)(x-16)$	12.6791
3	$e^x - 3 * x^2$	2.8331
4	$Cosx + (x-2)^2$	2.3542
5	$3774.522(1/x) + 2.27x - 181.529, x > 0$	40.7772
6	$10.2(1/x) + 6.2 * x^3, x > 0$	0.8605

Table 2. Secant-like method results(1)

No	θ=5	θ=10	θ=15	θ=20	θ=25
1	4	3	4	3	4
2	12	6	8	6	8
3	9	5	9	6	6
4	6	4	5	4	5

Table 2 represents the first four test functions with its number of iterations for θ values (5,10,15,20,25), they converges and at $\theta = 10$ the number of iterations are less compared to other θ values. For θ values (0,-5,-10,-15,-20) these four functions are increasing function and they does not converge.

Table 3 represents the last two test functions with its number of iterations for θ

Table 3. Secant-like method results(2)

No	$\theta=-5$	$\theta=-10$	$\theta=-15$	$\theta=-20$	$\theta=-25$
5	11	5	6	5	6
6	10	5	8	6	12

values (-5,-10,-15,-20,-25), they converges and at $\theta=-10$ the number of iterations are less compared to other θ values. For θ values (0,5,10,15,20) these four functions are increasing function and they does not converge.

Table 4. The results

Functions	x_0	x_{-1}	(10)	(2)	(3)	(4)	(9)
1	8.0	7.8	6	6	4	6	3
2	12.0	12.1	7	7	4	5	6
3	2.6	2.5	4	4	3	6	5
4	2.0	1.8	6	5	4	4	4
5	32.0	30.0	7	7	5	5	5
6	0.5	0.4	7	7	4	5	5

Conclusion (Fourth Level) We have developed an improved Secant-like method to solve univariate unconstrained optimization problems, based on modified Secant method from [1],[2]. The Proposed method examines using divided difference property and shares the same convergence properties with modified secant method[1]. The improved Secant-like method is excellent compared to proposed methods .

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