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Optimal solution of Bi-level Programming Problem using fuzzy rule base constraints

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Abstract In this work, a linguistic bi-level programming problem has been developed where the functional relationship linking decision variables and the objective functions of the leader and the follower are not utterly well known to us. Because of the uncertainty in practical life decision-making situation most of the time it is inconvenient to find the veracious relationship between the objective functions of leader, follower and the decision variables. It is expected that the source of information which gives some command about the objective functions of leader and follower, is composed by a block of fuzzy if-then rules. In order to analyze the model, a dynamic programming approach with a suitable fuzzy reasoning scheme is applied to calculate the deterministic functional relationship linking the decision variables and the objective functions of the leader as well as the follower. Thus a bi-level programming problem is constructed from the actual fuzzy rule-based to the conventional bi-level programming problem. To solve the final problem, we use the lingo software to find the optimal of the objective function of the follower first and using its solution we optimize the objective function of the leader. A numerical example has been solved to signify the computational procedure.

Keywords Dynamic Programming · Linguistic Variable · Bi-level Programming · Fuzzy Rule-Base · Fuzzy Inference Schemes.

1 Introduction

This paper contains a linguistic bi-level optimization problem and provides a fuzzy rule base structure of this hierarchical class of optimization problem.

In a classical optimization problem, bi-level programming problem (BLPP) has been used in a vast domain of practice. In the fields of management, it has been used to deal with facility location, environmental regulation, energy policy, etc. In the fields of economic planning, it has been used to deal with oil production, electric power pricing etc. In engineering, to solve optimal design, shape, and structure. Decision-makers (DMs) often deal with conflicting objectives in a hierarchical administrative structure. A decision maker has his own objective and decision space at one level and due to other levels of the hierarchy, it may be influenced by the choice of other decision makers. There are two levels with two decision makers in bi-level programming problem. Decision makers of both levels control the variables of its own level. The DM of the upper level is called the leader and by his decision, the objective function of another level may be affected. Decision maker of the lower level is called the follower. Decision makers of both levels want to optimize their objective function with the restriction of the decision of one another. The hierarchical structure of the final problem needed an optimal to the follower's problem first then a solution to the leader's problem is feasible and then the optimal is selected. The major segment of research in bi-level programming problem is still concerned on the deterministic case. BLPP were initially considered by Candler et. al. [6] and Fortuny et. al. [10] as a two player game where the first player can affect the resources available to a second player, this game is known as Stackelberg game. For a given movement of the first player, the other player will maximize a linear program, subject to the available resources.

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BLPP [1,5,6] can be formulated as:

$$\max_x F_l(x,y) = ax + by \quad \text{where y solves}$$
$$\max_y F_f(x,y) = cx + dy$$
$$s.t \qquad Ax + By \le r,$$

Here $a, c \in \mathbb{R}^{n_1}$, $b, d \in \mathbb{R}^{n_2}$, $r \in \mathbb{R}^m$, A is an $m \times n_1$ matrix, B is an $m \times n_2$ matrix. $(x, y) \in \mathbb{R}^n$ is a vector of decisions which can be controlled by the decision makers. $x \in \mathbb{R}^{n_1}$ is a vector control by leader and $y \in \mathbb{R}^{n_2}$ by follower, where $n_1 + n_2 = n$. F_l denotes the objectives functions of leader while F_f denotes the objectives functions of the follower.

In the past, different approaches have been developed by Bard [1–3], Karwan [5], Bialas and Wen and Hsu [19,20], Dempe [8,9] , Zhang et.al. and LU et. al. [13,25] and others to solve deterministic bi-level programming problem. However, in real-world situations, uncertainty and impreciseness are involved in defining the parameters and to form a mathematical model. It is difficult to define the imprecise and uncertain parameters in the objective functions and constraints in this case. Both the decision-makers need to take a decision even if they do not know the parameter of the problem with full certainty hence bi-level programming problem with fuzzy parameter and Stochastic bi-level programming problem has been introduced. Sakawa et. al. [16] designed the bi-level programming problem with fuzzy variable and introduced a fuzzy programming method to solve it. Zang and Lu [23] has given a fuzzy number based Kuhn-Tucker condition to solve bi-level programming problem with the fuzzy variable. Some multi-objective bi-level programming problem also has been investigated with fuzzy parameter [11,24]. In 2016 Zhang et. al. [22] presented a survey on Fuzzy Bi-level Decision-Making Techniques. For the randomness, Nishizaki et. al [15] introduced the two-level programming problem with parameters which are a random variable. To solve this, they considered the variance of the objective function of the leader (DM) and means of the objective function of the follower (DM) to find the deterministic programming problem. Stochastic two-level programming problem has been investigated by Kosuch et al. [12] with probabilistic knapsack constraints, which can be used to jointly optimize service pricing and network resources. Singh and Chakraborty [17] presented the bi-level programming problem with fuzzy random variable coefficients to capture impreciseness and randomness together.

In this work, a mathematical tool based on fuzzy rule base is used to build a linguistic bi-level programming problem in a hierarchical administrative system. The antecedent part of the rule is defined as fuzzy inputs and its consequent part is a linear input-output relation. The problem has been formulated for a bi-level system where a leader and a follower both deals with there own decision space but the functional relationship between the objectives and decision space has not been known exactly due to the imprecise nature of the information they have. Most of the fuzzy bi-level programming problem may be modeled mathematically to find the solution which optimizes the objective functions of the leader as well as of follower. The decision space may contain $\widetilde{C}x\widetilde{\le}\widetilde{d}$ type of fuzzy inequality, where $\widetilde{C}, \widetilde{d}$ are fuzzy parameter and the objectives may be demonstrated as functions of decision variables. However, the information which is available may not be sufficient to form such type of model. In this situation, we can only form a linguistical relation between the leader, follower's objective functions and the constraints from the past data. Lu et. al. [14] developed a rule-set based bi-level decision approach, which models a bilevel decision problem by creating, transforming and reducing related rule sets. Chakraborty and Guha [7] formed a multi-objective optimization model in which they have used fuzzy rule base to define the exact relationship between the objective function and constraints. In this work, we have defined a linguistic bi-level programming problem in which the exact relationship between the objective functions and the constraints has been defined from the past imprecise information to develop a real-life hierarchical model. In designing such type of programming problem, the problem emerges as the functional connection between the objective functions and constraints cannot be found directly in the given information. The framework has been defined linguistically to tackle such type of problem. It is expected that the source of information from where some command may be acquired about the objective functions of leader and follower which is composed by a block of fuzzy if-then rules. At both the level of the model, we have fuzzy if-then rules that connect the objective functions and constraints. At first level, the leader's fuzzy rules are defined and at second level the follower's fuzzy rules are defined. To solve the problem, we have developed a dynamic programming approach using Takagi and Sugeno fuzzy reasoning scheme [18] to solve the problem.

Dynamic programming has been used to define the n variable objective into $n-stage$ single variable problem then Takagi and Sugeno fuzzy reasoning schemes have been used for finding the value of the objective function of the leader as well as the follower at each stage. In dynamic programming, there is a physical system whose state at any time t is determined by a set of quantities which are called as state parameters, or state variables. At certain times, which may be prescribed in advance, or which may be determined by the process itself, one has to make decisions which will affect the state of the system. These decisions are equivalent to transformations of the state variables, the choice of a decision being identical with the choice of a transformation. The outcome of the preceding decisions is to be used to guide the choice of future ones, with the purpose of the whole process that of maximizing some function of the parameters describing the final state. Here we have a bi-level programming problem based on the fuzzy rule as a physical system. Time denotes the stages at time $t = 1$, we solve the stage 1 problem which is determined by the state variable x_1 and y_1, in our case we can also call it as stage variable. At a certain time say $t = r$ i.e r^{th} stage we are called upon to take a decision which will affect the stage of the system. These decisions are equivalent to transformations of the state variables, the choice of a decision being identical with the choice of a transformation. The outcome of the preceding decisions is to be used to guide the choice of future ones, with the purpose of the whole process that of finding the pareto optimal solution of some optimization problem of the parameters describing the final stage. We have given a recurrence relation for the leader as well as for the follower's objective function as a stage transformation equation which connects two stages.

The paper is organized as follows. Linguistic variable and Fuzzy reasoning scheme have been defined in section 2. Next, in section 3, Fuzzy rule-base bi-level programming problem and methodology of solving this problem is considered. Section 4 contains a numerical example and the conclusion has been given in section 5.

2 Preliminaries

2.1 Linguistic variable

In our natural or real life language, linguistic variables are those variables which are real-life words or sentences instead of crisp numbers [21]. Each value of the linguistic variable can be represented by using a triangular fuzzy number which are defined with the assist of membership functions. In particular, a linguistic variable can be evaluated either as a value of a fuzzy number or values which are defined in linguistic terms.

Definition 1 (Linguistic variable): A quintuple $(y, \Im(y), Y, G, M)$ denotes a linguistic variable. Here y denotes the linguistic variable's name; $\Im(y)$ denotes the term set of y, i.e., the set of names of linguistic values of y with each value being a fuzzy number defined on Y; M is a semantic rule for associating with each value its meaning and G is a syntactic rule for developing the names of values of x. $\Im(Y)$ denotes the family of all fuzzy sets in Y.

Here, it is considered that the values of each term of the linguistic variables are defined in the interval $[c, d] \subset \mathbb{R}$. Let $Y = [c, d]$ and $\Im(y)$ consists of $p + 1$, $(P \geq 2)$, terms as given in the Fig.(1).

$$\Im(y) = \{low_1, around(c+\beta), around(c+2\beta), \dots, around(c+(p-1)\beta), high_P\}; \text{where} \beta = (d-c)/p,$$

Here each value is defined as a triangular fuzzy number. It may be illustrated by the membership functions $\mu_{\widetilde{B}_1}, \dots, \mu_{\widetilde{B}_{P+1}}$ of triangular fuzzy number of the following form:

$$\mu_{\widetilde{B}_1}(y) = \mu_{low_1}(y)$$
$$\mu_{\widetilde{B}_1}(y) = \begin{cases} 1 - (y-c)/(d-c), & \text{for } c \leq y \leq d \\ 0, & otherwise \end{cases} \tag{1}$$

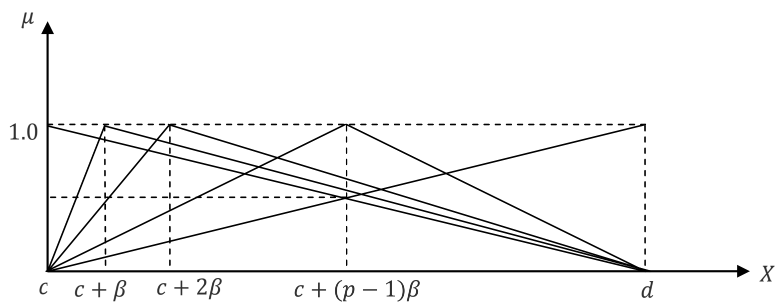

Fig. 1: Membership value for the linguistic variable with (P+1) terms

A fuzzy number $\widetilde{B}_1$ is denoted by $\widetilde{B}_1 = (c; 0, d-c)$.

$$\mu_{\widetilde{B}_p}(y) = \mu_{around(c+p\beta)}(y)$$
$$\mu_{\widetilde{B}_p}(y) = \begin{cases} 1-(c+p\beta-y)/p\beta, & \text{for } c \le y \le c+p\beta \\ 1-(y-c-p\beta)/(d-c-p\beta), & \text{for } c+p\beta \le y \le d \\ 0, & otherwise \end{cases} \tag{2}$$

for $1 \le p \le (P-1)$ and each of the corresponding fuzzy number $\widetilde{B}_k$ is denoted as $\widetilde{B}_k = (c+p\beta; p\beta, d-c-p\beta)$.

$$\mu_{\widetilde{B}_{P+1}}(y) = \mu_{High_P}(y)$$
$$\mu_{\widetilde{B}_{P+1}}(y) = \begin{cases} 1-(d-y)/(d-c), & \text{for } c \le y \le d \\ 0, & otherwise \end{cases} \tag{3}$$

The triangular fuzzy number $\widetilde{B}_{P+1}$ is denoted by $\widetilde{B}_{P+1} = (d; d-c, 0)$. In this work, to represent a triangular fuzzy number in $[0,1]$ by its membership function, the standard notation, $\widetilde{B} = (m; \alpha, \beta)$ is used. Where m represents the middle value α represents the left spread and β represents the right spread. When a very little knowledge is known about the boundaries of each value, each value is extended over the complete domain through the middle values of each of the fuzzy number which are located at a fixed distance apart.

Example 1 Suppose the amount of water during flood in a particular region is interpreted as a linguistic variable, and then its term set is of the following type:

$$\Im(Amount\ of\ water) = \{very\ low,\ low,\ medium,\ high,\ veryhigh\}$$

where every term in $\Im(amount\ of\ water)$ is characterized by a fuzzy set in the universe of discourse [0,100]. Then, using the fuzzy partition given in 4,5,6 for the interval [0,100], each term can be transformed to associated fuzzy numbers as provided in the table 1 given below:

Table 1: The term set $\Im(Amount\ of\ water)$

$\Im(y_3)$	Fuzzy numbers	Normalized fuzzy numbers
Very Low (VL)	(0; 0, 100)	(0; 0, 1)
Low (L)	(25; 25, 75)	(0.25; 0.25, 0.75)
Medium (M)	(50; 50, 50)	(0.5; 0.5, 0.5)
High (H)	(75; 75, 25)	(0.75; 0.75, 0.25)
Very High (VH)	(100; 100, 0)	(1; 1, 0)

2.2 Fuzzy Inference Schemes

Fuzzy Mathematicians have been introduced different types of fuzzy reasoning schemes. The distinction between different reasoning schemes consist in the illustration of the consequents of theirs fuzzy implication. A fuzzy if-then rule base system has been developed so that it joins the input variables to the output variable by means of $If - Then$ rules. For some given term of the input values, the degree of fulfillment of a rule is acquired by grouping the membership functions of these input values into the respective fuzzy number. The output is obtained by the degrees of fulfillment and the consequence of the implications. Several fuzzy inference schemes are available in the literature. In this work, to make a decision from fuzzy $if - then$ rule based system, we have used Takagi and Sugeno [18] fuzzy inference scheme, which is given below:

2.2.1 Sugeno and Takagi fuzzy reasoning scheme

Sugeno and Takagi uses the following architecture of the fuzzy rule-base implications:

$\Re_1$: If r_1 is $\widetilde{B}_{11}$ and r_2 is $\widetilde{B}_{12}$ and …and r_n is $\widetilde{B}_{1n}$ then $z = b_{11}r_1 + b_{12}r_2 + \cdots + b_{1n}r_n + c_1$

$\Re_2$: If r_1 is $\widetilde{B}_{21}$ and r_2 is $\widetilde{B}_{22}$ and …and r_n is $\widetilde{B}_{2n}$ then $z = b_{21}r_1 + b_{22}r_2 + \cdots + b_{2n}r_n + c_2$

......

$\Re_p$: If r_1 is $\widetilde{B}_{p1}$ and r_2 is $\widetilde{B}_{p2}$ and ... and r_n is $\widetilde{B}_{pn}$ then $z = b_{p1}r_1 + b_{p2}r_2 + \cdots + b_{pn}r_n + c_p$

Input: r_1 is $\overline{y}_1$ and r_2 is $\overline{y}_2$ and ... and r_n is $\overline{y}_n$

Output: z is z_{TgS}.

Where $\widetilde{B}_{jk} \in \Im\{R_k\}$ is the term set of the linguistic variable r_k which has been defined in the universe of discourse $R_k \subset \mathbb{R}$, and b_{jk} and c_j are real numbers for $j = 1, 2, \ldots, p$ and $k = 1, 2, \ldots, n$. The methodology for find the deterministic output z_{TgS}, from the crisp input vector $y = \{y_1, y_2, \ldots, y_n\}$ and fuzzy impications $\{\Re_1, \Re_2, \ldots, \Re_p\}$ are given below:

(i) The degree to which input matches the j^{th} rule $\Re_j$ is typically computed using the relation

$$l_j = T_p(\mu_{\widetilde{B}_{j1}}(y_1), \mu_{\widetilde{B}_{j2}}(y_2), \ldots, \mu_{\widetilde{B}_{jn}}(y_n)), \quad for \;\; j = 1, 2, \ldots, p. \tag{4}$$

(ii) Then the outputs of the individual rule calculated from the implications

$$z_j(y) = \sum_{k=1}^{n} b_{jk}y_k + c_j \tag{5}$$

(iii) Finally, the output of the fuzzy reasoning scheme is calculated by the following expression

$$z_{TgS} = \frac{l_1z_1 + l_2z_2 + \cdots + l_pz_p}{l_1 + l_2 + \cdots + l_p} \tag{6}$$

3 Fuzzy Rule-based Bi-level Programming Problem

In a two level administrative structure, to built a mathematical model we need deterministic type information. Where both the decision maker have their own objective function to optimize under some crisp inequalities. If the information source from where the crisp inequalities has to be built, is not known precisely i.e we know the linguistic relationship between the objective function and the constraints. In that case it is difficult to build a deterministic bi-level programming model. Here, we present a linguistic bi-level programming model which overcomes this difficulty. The model is represented is as follows:

$$\begin{aligned} &\widetilde{\max_x}\ \widetilde{f}_l(x, y) \text{ where y solves} \\ &\quad \widetilde{\max_y}\ \widetilde{f}_f(x, y) \\ &\quad \text{s.t. } \{\Re_1(x, y), \Re_2(x, y), \ldots, \Re_p(x, y) \mid x, y \in \widetilde{R} \subset \mathbb{R}^m\} \end{aligned} \tag{7}$$

$\widetilde{f}_l(x, y)$ and $\widetilde{f}_f(x, y)$ are the leader and followers' objective functions respectively. x is the linguistic variable controlled by leader and y is the linguistic variable controlled by follower.

Depending upon the nature of the information available for the objective functions of leader and follower, we can write our problem as follows:

$\Re_j(x, y)$: If x_1 is $\widetilde{A}_{j1}$ and x_2 is $\widetilde{A}_{j2}$ ... x_m is $\widetilde{A}_{jm}$ and y_1 is $\widetilde{B}_{j1}$ and y_2 is $\widetilde{B}_{j2}$ ... y_m is $\widetilde{B}_{jm}$ then $\widetilde{f}_{lj}(x, y) = \sum_{k=1}^{m} a_{jk}x_k + \sum_{k=1}^{m} b_{jk}y_k$.

where y solves

$\Re_j(y)$: If x_1 is $\widetilde{A}_{j1}$ and x_2 is $\widetilde{A}_{j2}$ ... x_m is $\widetilde{A}_{jm}$ and y_1 is $\widetilde{B}_{j1}$ and y_2 is $\widetilde{B}_{j2}$ ... y_m is $\widetilde{B}_{jm}$ then $\widetilde{f}_{fj}(X, Y) = \sum_{k=1}^{m} c_{jk}x_k + \sum_{k=1}^{m} d_{jk}y_k$.

for $j = 1, 2, \ldots, p$.

After normalization, the fuzzy model takes the form as follows:

$\Re_j(x)$: If x_1 is $\widetilde{NA}_{j1}$ and x_2 is $\widetilde{NA}_{j2}$ $\ldots x_m$ is $\widetilde{NA}_{jm}$ and y_1 is $\widetilde{NB}_{j1}$ and y_2 is $\widetilde{NB}_{j2}$ $\ldots y_m$ is $\widetilde{NB}_{jm}$ then $\widetilde{f}_{lj}(x,y) = \sum_{k=1}^{m} a_{jk}x_k + \sum_{k=1}^{m} b_{jk}y_k$.

where y solves

$\Re_j(y)$: If x_1 is $\widetilde{NA}_{j1}$ and x_2 is $\widetilde{NA}_{j2}$ $\ldots x_m$ is $\widetilde{NA}_{jm}$ and y_1 is $\widetilde{NB}_{j1}$ and y_2 is $\widetilde{NB}_{j2}$ $\ldots y_m$ is $\widetilde{NB}_{jm}$ then $\widetilde{f}_{fj}(X,Y) = \sum_{k=1}^{m} c_{jk}x_k + \sum_{k=1}^{m} d_{jk}y_k$.
for $j = 1, 2, \ldots, p$.

3.1 Stage-wise decomposition

In a two-level programming problem, the feasible solution of the first level is acceptable only if it is the optimal solution of the second level problem. To find the optimal solution of both the level, we can update the above $n-stage$ bi-level programming problem in which the constraints are the fuzzy rule of both the decision makers i.e it contains the variable which is controlled by the leader and the variable which is controlled by the follower.

The equivalent problem can be given as:

Stage 1 :

Leader's Fuzzy rule:

$\Re_j(x_1, y_1)$: If x_1 is $\widetilde{NA}_{j1}$ and y_1 is $\widetilde{NB}_{j1}$ then $\widetilde{f}_{lj}(x_1, y_1) = a_{j1}x_1 + b_{j1}y_1$, $j = 1, 2, \ldots, p$ where y_1 solves

Follower's fuzzy rule :

$\Re_j(x_1, y_1)$:x_1 is $\widetilde{NA}_{j1}$ and If y_1 is $\widetilde{NB}_{j1}$ then $\widetilde{f}_{fj}(x_1, y_1) = c_{j1}x_1 + d_{j1}y_1$, $j = 1, 2, \ldots, p$

Stage 2 :

Leader's Fuzzy rule :

$\Re_j(x_1, x_2, y_1, y_2)$: If x_1 is $\widetilde{NA}_{j1}$, x_2 is $\widetilde{NA}_{j2}$ and y_1 is $\widetilde{NB}_{j1}$,y_2 is $\widetilde{NB}_{j2}$ then $\widetilde{f}_{lj}(x_2, y_2) = a_{j2}x_2 + b_{j2}y_2$, $j = 1, 2, \ldots, p$ where y_1, y_2 solves

Follower's fuzzy rule :

$\Re_j(x_1, x_2, y_1, y_2)$: If x_1 is $\widetilde{NA}_{j1}$, x_2 is $\widetilde{NA}_{j2}$ and y_1 is $\widetilde{NB}_{j1}$,y_2 is $\widetilde{NB}_{j2}$ then $\widetilde{f}_{fj}(x_1, y_1) = c_{j1}x_1 + d_{j1}y_1$, $j = 1, 2, \ldots, p$

$\ldots$

Stage r :

Leader's Fuzzy rule :

$\Re_j(x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_r)$: If x_1 is $\widetilde{NA}_{j1}$, x_2 is $\widetilde{NA}_{j2}$,$\ldots$, x_r is $\widetilde{NA}_{jr}$ and y_1 is $\widetilde{NB}_{j1}$, y_2 is $\widetilde{NB}_{j2}$,$\ldots$, y_r is $\widetilde{NB}_{jr}$ then $\widetilde{f}_{lj}(x_r, y_r) = a_{jr}x_r + b_{jr}y_r$, $j = 1, 2, \ldots, p$ where $y_1, y_2, \ldots, y_r$ solves

Follower's fuzzy rule :

$\Re_j(x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_r)$: If x_1 is $\widetilde{NA}_{j1}$, x_2 is $\widetilde{NA}_{j2}$,$\ldots$, x_r is $\widetilde{NA}_{jr}$ and y_1 is $\widetilde{NB}_{j1}$, y_2 is $\widetilde{NB}_{j2}$,$\ldots$, y_r is $\widetilde{NB}_{jr}$ then $\widetilde{f}_{fj}(x_r, y_r) = c_{jr}x_r + d_{jr}y_r$, $j = 1, 2, \ldots, p$

$\ldots$

Stage m :

Leader's Fuzzy rule :

$\Re_j(x_1, x_2, \ldots, x_r, \ldots, x_m, y_1, y_2, \ldots, y_r, \ldots, y_m)$: If x_1 is $\widetilde{NA}_{j1}$, x_2 is $\widetilde{NA}_{j2}$,..., x_r is $\widetilde{NA}_{jr}$,..., x_m is $\widetilde{NA}_{jm}$ and y_1 is $\widetilde{NB}_{j1}$, y_2 is $\widetilde{NB}_{j2}$,..., y_r is $\widetilde{NB}_{jr}$,..., y_m is $\widetilde{NB}_{jm}$ then $\widetilde{f}_{lj}(x_m, y_m) = a_{jm}x_m + b_{jm}y_m$, $j = 1, 2, \ldots, p$ where Y solves

Follower's fuzzy rule :

$\Re_j(x_1, x_2, \ldots, x_r, \ldots, x_m, y_1, y_2, \ldots, y_r, \ldots, y_m)$: If x_1 is $\widetilde{NA}_{j1}$, x_2 is $\widetilde{NA}_{j2}$,..., x_r is $\widetilde{NA}_{jr}$,..., x_m is $\widetilde{NA}_{jm}$ and y_1 is $\widetilde{NB}_{j1}$, y_2 is $\widetilde{NB}_{j2}$,..., y_r is $\widetilde{NB}_{jr}$,..., y_m is $\widetilde{NB}_{jm}$ then $\widetilde{f}_{fj}(x_m, y_m) = c_{jm}x_m + d_{jm}y_m$, $j = 1, 2, \ldots, p$

3.2 Stage-wise computational procedure of fuzzy rule-base bi-level programming problem

Here, we have taken the superscript in each notation to denote the stage number.

Stage 1 :

Leader's Fuzzy rule:

$\Re_j^1(x_1, y_1)$: If x_1 is $\widetilde{NA}_{j1}$ and y_1 is $\widetilde{NB}_{j1}$ then $\widetilde{f}_{lj}^1(x_1, y_1) = a_{j1}x_1 + b_{j1}y_1$, $j = 1, 2, \ldots, p$ where Y solves

Follower's fuzzy rule :

$\Re_j^1(x_1, y_1)$: If x_1 is $\widetilde{NA}_{j1}$ and y_1 is $\widetilde{NB}_{j1}$ then $\widetilde{f}_{fj}^1(x_1, y_1) = c_{j1}x_1 + d_{j1}y_1$, $j = 1, 2, \ldots, p$

For solving above fuzzy bi-level programming model. First, we consider the follower's problem:

Follower's fuzzy rule :

$\Re_j^1(x_1, y_1)$: If x_1 is $\widetilde{NA}_{j1}$ and y_1 is $\widetilde{NB}_{j1}$ then $\widetilde{f}_{fj}^1(x_1, y_1) = c_{j1}x_1 + d_{j1}y_1$, $j = 1, 2, \ldots, p$

We take x_{c1} and y_{c1} as crisp input corresponding to x_1 and y_1. The membership degree up to which the j^{th} rule $\Re_j^1(x_1, y_1)$ matches the given input, is calculated by the following expression:

$$l_j^1 = T(\mu_{\widetilde{NA}_{j1}}(x_1), \mu_{\widetilde{NB}_{j1}}(y_1)) \tag{8}$$

Objective functions of the follower for the individual rule are given by

$$f_{fj}^1(x_{c1}, y_{c1}) = c_{j1}x_{c1} + d_{j1}y_{c1} \tag{9}$$

The crisp objective function for the follower's fuzzy model has been given by

$$F_f^1(x_{c1}, y_{c1}) = \frac{l_1 f_{f1}^1 + l_2 f_{f2}^1 + \cdots + l_p f_{fp}^1}{l_1 + l_2 + \cdots + l_p} \tag{10}$$

The follower's objective function is the piece-wise continuous function in the interval $x_{c1}, y_{c1} \in [0, 1]$. To calculate the follower's optimal solution under the optimal of leader, we calculate the optimal of this function with the general value of $0 < y_{c1} <= x_{c1}$ for each subinterval in which the function is defined.

We use the value of optimal y_{c1}^* in the leader's objective function. To calculate the leader's crisp objective function, we follow the same steps after putting the value $y_{c1} = y_{c1}^*$.

Objective functions of the leader for the individual rule are given by

$$f_{lj}^1(x_{c1}, y_{c1}^*) = c_{j1}x_{c1} + d_{j1}y_{c1}^* \tag{11}$$

Leader's objective function will be of the form

$$F_l^1(x_{c1}) = \frac{l_1 f_{l1}^1 + l_2 f_{l2}^1 + \cdots + l_p f_{lp}^1}{l_1 + l_2 + \cdots + l_p} \tag{12}$$

To solve the second stage problem, we need to define a recurrence relation to connect the two stages, which satisfies the principle of optimality [4]. The leader's and follower's recurrence relation has been given by:

Follower's recurrence relation:

$$F_f^r(x_{cr}, y_{cr}) = F_f^{r-1}(x_{cr-1}^*, y_{cr-1}^*) + \frac{l_1 f_{f1}^r + l_2 f_{f2}^r + \cdots + l_p f_{fp}^r}{l_1 + l_2 + \cdots + l_p} \tag{13}$$

Leader's recurrence relation

$$F_l^r(x_{cr}) = F_l^{r-1}(x_{cr-1}^*) + \frac{l_1 f_{l1}^r + l_2 f_{l2}^r + \cdots + l_p f_{lp}^r}{l_1 + l_2 + \cdots + l_p} \tag{14}$$

To solve the $r^{th} - stage$ problem:

Leader's Fuzzy rule:

$\Re_j^r(x_r, y_r)$: If x_1 is $\widetilde{NA}_{jr}$ and y_1 is $\widetilde{NB}_{jr}$ then $\widetilde{f}_{lj}^r(x_r, y_r) = a_{jr}x_r + b_{jr}y_r$, $j = 1, 2, \ldots, p$ where Y solves

Follower's fuzzy rule :

$\Re_j^r(x_r, y_r)$: If x_1 is $\widetilde{NA}_{jr}$ and y_1 is $\widetilde{NB}_{jr}$ then $\widetilde{f}_{fj}^1(x_r, y_r) = c_{jr}x_r + d_{jr}y_r$, $j = 1, 2, \ldots, p$

For solving above fuzzy bi-level programming model. First, we consider the follower's problem:

Follower's fuzzy rule :

$\Re_j^r(x_r, y_r)$: If x_r is $\widetilde{NA}_{jr}$ and y_r is $\widetilde{NB}_{jr}$ then $\widetilde{f}_{fj}^r(x_r, y_r) = c_{jr}x_r + d_{jr}y_r$, $j = 1, 2, \ldots, p$

We take x_{cr} and y_{cr} as crisp input corresponding to x_r and y_r. The membership degree up to which the j^{th} rule $\Re_j^1(x_1, y_1)$ matches the given input, is calculated by the following expression:

$$l_j^r = T(\mu_{\widetilde{NA}_{jr}}(x_r), \mu_{\widetilde{NB}_{jr}}(y_r)) \tag{15}$$

Objective functions of the follower for the individual rule are given by

$$f_{fj}^r(x_{cr}, y_{cr}) = c_{jr}x_{cr} + d_{jr}y_{cr} \tag{16}$$

The crisp objective function for the follower's fuzzy model has been given by

$$F_f^r(x_{cr}, y_{cr}) = F_f^{r-1}(x_{cr-1}^*, y_{cr-1}^*) + \frac{l_1 f_{f1}^r + l_2 f_{f2}^r + \cdots + l_p f_{fp}^r}{l_1 + l_2 + \cdots + l_p} \tag{17}$$

The follower's objective function is the piece-wise continuous function in the interval $x_{cr}, y_{cr} \in [0, 1]$. To calculate the follower's optimal under the optimal of leader, we calculate the optimal of this function with a general value of $0 <= x <= y$ for each subinterval in which the function is defined.

We use the value of optimal y_{cr}^* in the leader's objective function. To calculate the leader's crisp objective function, we follow the same steps by putting the value $y_{cr} = y_{cr}^*$

Objective functions of the leader for the individual rule are given by

$$f_{lj}^r(x_{cr}, y_{cr}^*) = c_{jr}x_{cr} + d_{jr}y_{cr}^* \tag{18}$$

Leader's objective function will be of the form

$$F_l^r(x_{cr}) = F_l^{r-1}(x_{cr-1}^*) + \frac{l_1 f_{l1}^r + l_2 f_{l2}^r + \cdots + l_p f_{lp}^r}{l_1 + l_2 + \cdots + l_p} \tag{19}$$

To find the optimal solution of the fuzzy bi-level model, We proceed in a similar manner up to $n^{th} - stage$. At the last stage, The optimal solution contains the optimal return from each stage.

3.3 Algorithm for linguistic bi-level programming problem

3.3.1 Algorithm

Require: Linguistic relationship between the decision variables and objectives of leader's and follower's.
Ensure: $F_l^m(x_{cr}), F_f^m(x_{cr}, y_{cr})$.

Start
$\widetilde{NA}_{ji}, \widetilde{NB}_{ji}, j = 1, 2, \ldots, p$ and $i = 1, 2, \ldots, m$;
$F_l^0(x_{cr}) = 0$;
$F_f^0(x_{cr}) = 0$;
for $r := 1 \to m$ **do**
$l_j^r = T(\mu_{\widetilde{NA}_{jr}}(x_{cr}), \mu_{\widetilde{NB}_{jr}}(y_{cr}))$;
$f_{fj}^r(x_{cr}, y_{cr}) = c_{jr}x_{cr} + d_{jr}y_{cr}$;
$F_f^r(x_{cr}, y_{cr}) = F_f^{r-1}(x_{cr-1}^*, y_{cr-1}^*) + \frac{l_1 f_{f1}^r + l_2 f_{f2}^r + \cdots + l_p f_{fp}^r}{l_1 + l_2 + \cdots + l_p}$;
$f_{lj}^r(x_{cr}, y_{cr}^*) = c_{jr}x_{cr} + d_{jr}y_{cr}^*$;
$F_l^r(x_{cr}) = F_l^{r-1}(x_{cr-1}^*) + \frac{l_1 f_{l1}^r + l_2 f_{l2}^r + \cdots + l_p f_{lp}^r}{l_1 + l_2 + \cdots + l_p}$;
end for
End

4 Numerical Example

Let us consider a company, say company ABC, manufactures a product P. The owner of the company wants to maximize his profit which depends upon the production cost and the supply of the product. The buyers look at the product quality, pricing to maximize their utility. The company finds that the sales of its product are falling and the company is operated at a loss. In this scenario, mainly based on the past experience, the review committee of the company reviews its productions of the past several years. Which can be provided in the form of the fuzzy rule-based bi-level programming problem rather than any mathematical formulation. Without imprecision and uncertainty, the past information cannot be accessed accurately, therefore the available information has a form of linguistic descriptors. Let us consider that $x = (x_1, x_2)$ denotes the production cost and the supply and $y = (y_1, y_2)$ denotes the price and product quality for the customer. Let f_l and f_f denotes the objective function of the Owner (leader) and follower (customer) respectively then the linguistic bi-level programming problem can be defined as:

Leader's fuzzy rules :

$\Re_1(x, y)$: If x_1 is high and x_2 is very high and y_1 is medium and y_2 is low then $\widetilde{f}_l(x_1, x_2, y_1, y_2) = -x_1 + x_2 + y_1/2 - y_2$

$\Re_2(x, y)$: If x_1 is low and x_2 is high and y_1 is very low and y_2 is high then $\widetilde{f}_l(x_1, x_2, y_1, y_2) = x_1 + x_2 + y_1 + y_2$

where $Y = (y_1, y_2)$ solves the follower's fuzzy rules given by

$\Re_1(x, y)$: If x_1 is high and x_2 is very high and y_1 is medium and y_2 is low then $\widetilde{f}_f(x_1, x_2, y_1, y_2) = x_1 + x_2/2 - y_1 + y_2$

$\Re_2(x, y)$: If x_1 is low and x_2 is high and y_1 is very low and y_2 is high then $\widetilde{f}_f(x_1, x_2, y_1, y_2) = -x_1 + x_2 + y_1 - y_2$

$\Im(x_1)$	Fuzzy numbers	Normalized fuzzy numbers
Very Low (VL)	(0; 0, 10)	(0; 0, 1.0)
Low (L)	(2.5; 2.5, 7.5)	(0.25; 0.25, 0.75)
Medium (M)	(5.0, 5.0, 5.0)	(0.5; 0.5, 0.5)
High (H)	(7.5; 7.5, 2.5)	(0.75; 0.75, 0.25)
Very High (VH)	(10.0; 10.0, 0)	(1.0; 1.0, 0)

Table 2: Linguistic value for variable x_1

$\Im(x_2)$	Fuzzy numbers	Normalized fuzzy numbers
Very Low (VL)	(10; 0, 40)	(0.2; 0.2, 0.8)
Low (L)	(20; 10, 30)	(0.4; 0.2, 0.6)
Medium (M)	(30, 20, 20)	(0.6; 0.4, 0.4)
High (H)	(40; 30, 10)	(0.8; 0.6, 0.2)
Very High (VH)	(50; 40, 0)	(1.0; 0.8, 0)

Table 3: Linguistic value for variable x_2

$\Im(x_1)$	Fuzzy numbers	Normalized fuzzy numbers
Very Low (VL)	(2; 1, 8)	(0.2; 0.1, 0.8)
Low (L)	(10; 10, 20)	(0.334; 0.334, 0.667)
Medium (M)	(20; 10, 30)	(0.4; 0.2, 0.6)
High (H)	(50; 20, 30)	(0.625; 0.25, 0.375)
Very High (VH)	(100; 50, 50)	(0.667; 0.334, 0.334)

Table 4: Linguistic value for variable y_1

$\Im(x_2)$	Fuzzy numbers	Normalized fuzzy numbers
Very Low (VL)	(50; 0, 50)	(0.2; 0, 0.2)
Low (L)	(60; 10, 40)	(0.6; 0.1, 0.4)
Medium (M)	(70, 20, 30)	(0.7; 0.2, 0.3)
High (H)	(85; 35, 15)	(0.85; 0.35, 0.15)
Very High (VH)	(100; 50, 0)	(1; 0.5, 0)

Table 5: Linguistic value for variable y_2

First, we calculate the normalized linguistic fuzzy number from the given linguistic variables which are given in the above tables.

Stage 1 Problem:

Leader's fuzzy rules :

$\Re_1^1(x,y)$: If x_1 is high and y_1 is medium then $\widetilde{f}_l^1(x_1,y_1) = -x_1 + y_1/2$

$\Re_2^1(x,y)$: If x_1 is low and y_1 is very low then $\widetilde{f}_l^1(x_1,y_1) = x_1 + y_1$

where y_1 solves the follower's fuzzy rules by

$\Re_1^1(x,y)$: If x_1 is high and y_1 is medium then $\widetilde{f}_f^1(x_1,y_1) = x_1 - y_1$

$\Re_2^1(x,y)$: If x_1 is low and y_1 is high then $\widetilde{f}_f^1(x_1,y_1) = -x_1 + y_1$

First, we solve the follower's problem:

$$l_1^1 = \begin{cases} \frac{x_{c1}}{0.75}.\frac{y_{c1}-0.2}{0.2}, & \text{for } 0.2 \le x_{c1}, y_{c1} \le 0.4 \\ \frac{x_{c1}}{0.75}.\frac{1-y_{c1}}{0.6}, & \text{for } 0.4 \le x_{c1}, y_{c1} \le 0.75 \\ \frac{1-x_{c1}}{0.25}.\frac{1-y_{c1}}{0.6}, & \text{for } 0.75 \le x_{c1}, y_{c1} \le 1 \end{cases} \tag{20}$$

$$l_2^1 = \begin{cases} \frac{x_{c1}}{0.25}.\frac{y_{c1}-0.1}{0.1}, & \text{for } 0.1 \le x_{c1}, y_{c1} \le 0.2 \\ \frac{x_{c1}}{0.25}.\frac{1-y_{c1}}{0.8}, & \text{for } 0.2 \le x_{c1}, y_{c1} \le 0.25 \\ \frac{1-x_{c1}}{0.75}.\frac{1-y_{c1}}{0.8}, & \text{for } 0.25 \le x_{c1}, y_{c1} \le 1 \end{cases} \tag{21}$$

Using our approach the follower's objective function can be calculated as:

$$F_f^1(x_{c1},y_{c1}) = \begin{cases} \frac{(\frac{x_{c1}}{0.75})(\frac{y_{c1}-0.2}{0.2})(x_{c1}-y_{c1})+(\frac{x_{c1}}{0.25})(\frac{1-y_{c1}}{0.8})(y_{c1}-x_{c1})}{(\frac{x_{c1}}{0.75})(\frac{y_{c1}-0.2}{0.2})+(\frac{x_{c1}}{0.25})(\frac{1-y_{c1}}{0.8})} & \text{for } 0.2 \le y_{c1} \le 0.25 \\ \frac{(\frac{x_{c1}}{0.75})(\frac{y_{c1}-0.2}{0.2})(x_{c1}-y_{c1})+(\frac{1-x_{c1}}{0.75})(\frac{1-y_{c1}}{0.8})(y_{c1}-x_{c1})}{(\frac{x_{c1}}{0.75})(\frac{y_{c1}-0.2}{0.2})+(\frac{1-x_{c1}}{0.75})(\frac{1-y_{c1}}{0.8})} & \text{for } 0.25 \le y_{c1} \le 0.4 \\ \frac{(\frac{x_{c1}}{0.75})(\frac{1-y_{c1}}{0.6})(x_{c1}-y_{c1})+(\frac{1-x_{c1}}{0.75})(\frac{1-y_{c1}}{0.8})(y_{c1}-x_{c1})}{(\frac{x_{c1}}{0.75})(\frac{1-y_{c1}}{0.6})+(\frac{1-x_{c1}}{0.75})(\frac{1-y_{c1}}{0.8})} & \text{for } 0.4 \le y_{c1} \le 0.75 \\ \frac{(\frac{1-x_{c1}}{0.25})(\frac{1-y_{c1}}{0.6})(x_{c1}-y_{c1})+(\frac{1-x_{c1}}{0.75})(\frac{1-y_{c1}}{0.8})(y_{c1}-x_{c1})}{(\frac{1-x_{c1}}{0.25})(\frac{1-y_{c1}}{0.6})+(\frac{1-x_{c1}}{0.75})(\frac{1-y_{c1}}{0.8})} & \text{for } 0.75 \le y_{c1} \le 1 \end{cases} \tag{22}$$

Using follower's optimal, the leader optimal can be calculated as:

$$F_l^1(x_{c1},y_{c1}) = \begin{cases} \frac{(\frac{x_{c1}}{0.75})(\frac{y_{c1}-0.2}{0.2})(-x_{c1}+y_{c1}/2)+(\frac{x_{c1}}{0.25})(\frac{1-y_{c1}}{0.8})(y_{c1}+x_{c1})}{(\frac{x_{c1}}{0.75})(\frac{y_{c1}-0.2}{0.2})+(\frac{x_{c1}}{0.25})(\frac{1-y_{c1}}{0.8})} & \text{for } 0.2 \le x_{c1} \le 0.25 \\ \frac{(\frac{x_{c1}}{0.75})(\frac{y_{c1}-0.2}{0.2})(-x_{c1}+y_{c1}/2)+(\frac{1-x_{c1}}{0.75})(\frac{1-y_{c1}}{0.8})(y_{c1}+x_{c1})}{(\frac{x_{c1}}{0.75})(\frac{y_{c1}-0.2}{0.2})+(\frac{1-x_{c1}}{0.75})(\frac{1-y_{c1}}{0.8})} & \text{for } 0.25 \le x_{c1} \le 0.4 \\ \frac{(\frac{x_{c1}}{0.75})(\frac{1-y_{c1}}{0.6})(-x_{c1}+y_{c1}/2)+(\frac{1-x_{c1}}{0.75})(\frac{1-y_{c1}}{0.8})(y_{c1}+x_{c1})}{(\frac{x_{c1}}{0.75})(\frac{1-y_{c1}}{0.6})+(\frac{1-x_{c1}}{0.75})(\frac{1-y_{c1}}{0.8})} & \text{for } 0.4 \le x_{c1} \le 0.75 \\ \frac{(\frac{1-x_{c1}}{0.25})(\frac{1-y_{c1}}{0.6})(-x_{c1}+y_{c1}/2)+(\frac{1-x_{c1}}{0.75})(\frac{1-y_{c1}}{0.8})(y_{c1}+x_{c1})}{(\frac{1-x_{c1}}{0.25})(\frac{1-y_{c1}}{0.6})+(\frac{1-x_{c1}}{0.75})(\frac{1-y_{c1}}{0.8})} & \text{for } 0.75 \le x_{c1} \le 1 \end{cases} \tag{23}$$

Stage 2 Problem:

Leader's fuzzy rules :

$\Re_1^1(x,y)$: If x_2 is very high and y_2 is low then $\widetilde{f}_l^2(x_2,y_2) = x_2 - y_2$

$\Re_2^1(x,y)$: If x_2 is high and y_2 is high then $\widetilde{f}_l^2(x_2,y_2) = x_2 + y_2$

where y_2 solves the follower's fuzzy rules by

$\Re_1^1(x,y)$: If x_2 is very high and y_2 is low then $\widetilde{f}_f^2(x_2,y_2) = x_2/2 + y_2$

$\Re_2^1(x,y)$: If x_2 is high and y_2 is high then $\widetilde{f}_f^2(x_2,y_2) = x_2 - y_2$

$$l_1^2 = \begin{cases} \frac{1-x_{c2}}{0.8}.\frac{y_{c2}-0.5}{0.1}, & \text{for } 0.5 \leq x_{c2}, y_{c2} \leq 0.6 \\ \frac{1-x_{c2}}{0.8}.\frac{1-y_{c2}}{0.4}, & \text{for } 0.6 \leq x_{c2}, y_{c2} \leq 1 \end{cases} \tag{24}$$

$$l_2^2 = \begin{cases} \frac{x_{c2}-0.2}{0.6}.\frac{y_{c2}-0.5}{0.35}, & \text{for } 0.5 \leq x_{c2}, y_{c2} \leq 0.8 \\ \frac{1-x_{c2}}{0.2}.\frac{y_{c2}-0.5}{0.35}, & \text{for } 0.8 \leq x_{c2}, y_{c2} \leq 0.85 \\ \frac{1-x_{c2}}{0.2}.\frac{1-y_{c2}}{0.15}, & \text{for } 0.85 \leq x_{c2}, y_{c2} \leq 1 \end{cases} \tag{25}$$

$$F_f^2(x_{c2},y_{c2}) = \begin{cases} \frac{(\frac{1-x_{c2}}{0.8})(\frac{y_{c2}-0.5}{0.1})(x_{c2}/2+y_{c2})+(\frac{x_{c2}-0.2}{0.6})(\frac{y_{c2}-0.5}{0.35})(x_{c2}-y_{c2})}{(\frac{1-x_{c2}}{0.8})(\frac{y_{c2}-0.5}{0.1})+(\frac{x_{c2}-0.2}{0.6})(\frac{y_{c2}-0.5}{0.35})} & \text{for } 0.51 \leq y_{c2} \leq 0.6 \\ \frac{(\frac{1-x_{c2}}{0.8})(\frac{1-y_{c2}}{0.4})(x_{c2}/2+y_{c2})+(\frac{x_{c2}-0.2}{0.6})(\frac{y_{c2}-0.5}{0.35})(x_{c2}-y_{c2})}{(\frac{1-x_{c2}}{0.8})(\frac{1-y_{c2}}{0.4})+(\frac{x_{c2}-0.2}{0.6})(\frac{y_{c2}-0.5}{0.35})} & \text{for } 0.6 \leq y_{c2} \leq 0.8 \\ \frac{(\frac{1-x_{c2}}{0.8})(\frac{1-y_{c2}}{0.4})(x_{c2}/2+y_{c2})+(\frac{1-x_{c2}}{0.2})(\frac{y_{c2}-0.5}{0.35})(x_{c2}-y_{c2})}{(\frac{1-x_{c2}}{0.8})(\frac{1-y_{c2}}{0.4})+(\frac{1-x_{c2}}{0.2})(\frac{y_{c2}-0.5}{0.35})} & \text{for } 0.8 \leq y_{c2} \leq 0.85 \\ \frac{(\frac{1-x_{c2}}{0.8})(\frac{1-y_{c2}}{0.4})(x_{c2}/2+y_{c2})+(\frac{1-x_{c2}}{0.2})(\frac{1-y_{c2}}{0.15})(x_{c2}-y_{c2})}{(\frac{1-x_{c2}}{0.8})(\frac{1-y_{c2}}{0.4})+(\frac{1-x_{c2}}{0.2})(\frac{1-y_{c2}}{0.15})} & \text{for } 0.85 \leq y_{c2} \leq 1 \end{cases} \tag{26}$$

$$F_l^2(x_{c2},y_{c2}) = \begin{cases} \frac{(\frac{1-x_{c2}}{0.8})(\frac{y_{c2}-0.5}{0.1})(x_{c2}-y_{c2})+(\frac{x_{c2}-0.2}{0.6})(\frac{y_{c2}-0.5}{0.35})(x_{c2}+y_{c2})}{(\frac{1-x_{c2}}{0.8})(\frac{y_{c2}-0.5}{0.1})+(\frac{x_{c2}-0.2}{0.6})(\frac{y_{c2}-0.5}{0.35})} & \text{for } 0.51 \leq x_{c2} \leq 0.6 \\ \frac{(\frac{1-x_{c2}}{0.8})(\frac{1-y_{c2}}{0.4})(x_{c2}-y_{c2})+(\frac{x_{c2}-0.2}{0.6})(\frac{y_{c2}-0.5}{0.35})(x_{c2}+y_{c2})}{(\frac{1-x_{c2}}{0.8})(\frac{1-y_{c2}}{0.4})+(\frac{x_{c2}-0.2}{0.6})(\frac{y_{c2}-0.5}{0.35})} & \text{for } 0.6 \leq y_{c2} \leq 0.8 \\ \frac{(\frac{1-x_{c2}}{0.8})(\frac{1-y_{c2}}{0.4})(x_{c2}-y_{c2})+(\frac{1-x_{c2}}{0.2})(\frac{y_{c2}-0.5}{0.35})(x_{c2}+y_{c2})}{(\frac{1-x_{c2}}{0.8})(\frac{1-y_{c2}}{0.4})+(\frac{1-x_{c2}}{0.2})(\frac{y_{c2}-0.5}{0.35})} & \text{for } 0.8 \leq y_{c2} \leq 0.85 \\ \frac{(\frac{1-x_{c2}}{0.8})(\frac{1-y_{c2}}{0.4})(x_{c2}-y_{c2})+(\frac{1-x_{c2}}{0.2})(\frac{1-y_{c2}}{0.15})(x_{c2}+y_{c2})}{(\frac{1-x_{c2}}{0.8})(\frac{1-y_{c2}}{0.4})+(\frac{1-x_{c2}}{0.2})(\frac{1-y_{c2}}{0.15})} & \text{for } 0.85 \leq y_{c2} \leq 1 \end{cases} \tag{27}$$

By using lingo, we get the solution as $F_l^2 = 7.0776$ and $F_f^2 = 2.657$ at $x_{c2} = 0.6$ and $y_{c2} = 0.8$.

5 Conclusion

In a two-level structure, to build a mathematical model, decision-makers need deterministic type information, where both the decision maker have their own objective function to optimize under some crisp inequalities. If the information source from where the crisp inequalities have to be built, is not known precisely i.e we know the linguistic relationship between the objective function and the constraints. In that case, it is difficult to build a deterministic bi-level programming model. Here, we present the linguistic bi-level programming model which overcomes this difficulty.

In this work, we conclude that, in a hierarchical administrative structure, to solve a linguistic bi-level programming problem where the deterministic functional relationship between the objective functions of decision makers and their decision space is not known exactly, a bi-level programming with fuzzy rule base has been constructed. In order to solve the problem, a suitable fuzzy inference scheme and dynamic programming approach have been used to convert fuzzy rule-base bi-level programming problem into crisp bi-level programming problem of two variables. First, we have normalized the linguistic variable and then using dynamic programming converted the m variable problem into m stage single variable problem, where each stage problem contains linguistic bi-level programming problem. At each stage, first, we have taken the input crisp vector which matches the degree of each rule then using fuzzy inference scheme we have defined the crisp objective function of the leader as well as the follower for each rule. After calculating the crisp objective function of leader and follower for each rule, the procedure used given recurrence relation to calculate the optimal value at that stage. A multi-stage decision-making approach has been used to find the solution of the linguistic bi-level programming problem.

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