Vibrations of membranes with generalized finite Hankel transformation

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Abstract. In this study, it is shown that the vibrations of membranes are governed by the two-dimensional wave equation and can be solved partial differential equations using generalized finite Hankel transformation. The graphical representation of modes of vibrating membranes is included at the end of the section.

Keywords: generalized finite Hankel transformation, Fourier-Bessel series type, operator, vibrating membrane, wave equation.

1 Introduction

Malgonde [1] investigated the following variant of the generalized Hankel-Clifford transform defined by

$$\left(h_{\alpha,\beta} f\right)(y) = F(y) = \int_0^\infty (y/x)^{-(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\sqrt{xy}\right) f(x)\,dx\,,\ (\alpha-\beta)\geq -1/2$$

$$\left(h_{\alpha,\beta} f\right)(y) = y^{-\alpha-\beta}\int_0^\infty \mathrm{J}_{\alpha,\beta}(xy)\, f(x)\,dx\,,\ \ (\alpha-\beta)\geq -1/2\,, \tag{1.1}$$

where $\mathrm{J}_{\alpha,\beta}(x) = x^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\sqrt{x}\right)$, $J_{\alpha-\beta}(x)$ being the Bessel function of the first kind of order $(\alpha-\beta)$, in spaces of generalized functions. Note that (1.1) reduces to a well-known Hankel-Clifford transformation for suitable values of the parameters $\alpha = 0$ and $\beta = -\mu$, a transform studied in [5].

Following [2], a finite version of (1.1), firstly developed from a classical point of view. Later the classical finite generalized Hankel-Clifford transform of a function $f(x)$ defined on the interval $(0,a)$ was introduced in [3] as

$$\left(\hbar_{\alpha,\beta} f\right)(n) = F_{\alpha,\beta}(n) = \int_0^a x^{-(\alpha+\beta)} \mathrm{J}_{\alpha,\beta}(\lambda_n x)\, f(x)\,dx \tag{1.2}$$

where $\mathrm{J}_{\alpha,\beta}(z) = z^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\sqrt{z}\right)$, $J_{\alpha-\beta}(z)$ being the special Bessel function of the first kind of order $(\alpha-\beta)\geq -1/2$, for any complex z; and $\lambda_1, \lambda_2, \lambda_3, \ldots$ are positive zeros of $\mathrm{J}_{\alpha,\beta}(z)$ arranged in the ascending order of magnitude. Note that (1.2) reduces to well-known Hankel-Clifford transform for suitable values of the parameters viz. for α=0 and $\beta=-\mu$ a transform studied in [4].

Theorem 1.1: Let $f(x)$ be a function defined and absolutely integrable on $(0,a)$. Assume that $(\alpha-\beta)\geq-\frac{1}{2}$ and $a_m=\frac{1}{a^{2-\alpha-\beta}\lambda_m \mathrm{J}^2_{\alpha,\beta-1}(\lambda_m a)}\int_0^a t^{-(\alpha+\beta)}\mathrm{J}_{\alpha,\beta}(\lambda_m t)f(t)dt, m=1,2,3...$

If $f(t)$ is of the bounded variation in $(a_1,a_2),(0<a_1<a_2<a)$ and if $t\in(a_1,a_2)$,

then the series $\sum_{m=1}^{\infty}a_m\mathrm{J}_{\alpha,\beta}(\lambda_m x)$ converges to $\frac{1}{2}\left[f(x+0)+f(x-0)\right]$.

Some particular cases of orthonormal series expansions including Fourier-Bessel series expansion of generalized functions have also been studied by Zemanian [6]. The method involved in his work is very much related to Hilbert space techniques. Vibrations of membranes are of great practical importance which occur in telephones, microphones, pumps, and other devices. We assume that the membrane is made of elastic material of constant mass per unit area without resistance to (slight) bending. It is stretched along all of its boundary in the xy-plane generating constant tension T per unit length in all relevant directions, which does not change while the membrane vibrates. Note that, in our derivation of the wave equation, we have tacitly assumed no damping of the membrane.

2 Preliminary results

Considering the Sturm-Liouville problem

$$\left(\Delta_{\alpha,\beta,\nu,\mu}+\lambda^2\right)y=0 \tag{2.1}$$

$$a_1 y(a)+a_2 y'(a)=0 \quad ; \quad b_1 y(b)+b_2 y'(b)=0 \tag{2.2}$$

where $\Delta_{\alpha,\beta,\nu,\mu}=x^{-\mu\nu+\alpha+1-2\nu}Dx^{2\mu\nu+1}Dx^{-\mu\nu-\alpha}$ and a,a_1,a_2,b,b_1,b_2 are real constants and $D=\frac{d}{dx}$.

The general solution of the (2.1) is

$$y=\phi_\lambda(x)=A(\lambda)\left[(x)^\alpha J_\mu\left(\beta(x\lambda)^\nu\right)\right]+B(\lambda)(x)^\alpha Y_\mu\left(\beta(x\lambda)^\nu\right)$$

$$y=\phi_{\lambda_n}(x)=A(\lambda_n)\mathrm{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x)+B(\lambda_n)\mathrm{Y}_{\alpha,\beta,\nu,\mu}(\lambda_n x)$$

where $\mathrm{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x)=(x)^\alpha J_\mu\left(\beta(x\lambda)^\nu\right);\mathrm{Y}_{\alpha,\beta,\nu,\mu}(\lambda_n x)=(x)^\alpha Y_\mu\left(\beta(x\lambda)^\nu\right)$

$\mathrm{Y}_{\alpha,\beta,\nu,\mu}(\lambda_n x)$ is the Bessel-type function of the second kind of order μ.

Equation (2.1) may be written as

$$x^{2\alpha}\frac{d}{dx}\left[x^{1-2\alpha}y'\right]^2+\left[\lambda^2\beta^2\nu^2x^{2\nu-2\alpha}+\frac{\alpha^2-\mu^2\nu^2}{x^{2\alpha}}\right]\frac{d}{dx}\left(y^2\right)=0. \tag{2.3}$$

Taking $y=\phi_n(x)$, which correspond to the non-zero eigenvalues λ_n becomes

$$\beta^2\nu^2\int_0^a x^{2\nu-2\alpha-1}\left[\phi_n(x)\right]\left[\phi_n(x)\right]dx$$

$$=\begin{cases}\dfrac{1}{2\nu\lambda_n^{\,2}}\left[\begin{array}{l}x^{2-2\alpha}\left\{\dfrac{d}{dx}\phi_n(x)\right\}^2+\lambda_n^{\,2}\beta^2\nu^2x^{2\nu-2\alpha}\left(\phi_n(x)\right)^2\\ +\left(\alpha^2-\nu^2\mu^2\right)x^{-2\alpha}\left(\phi_n(x)\right)^2-2\alpha x^{1-2\alpha}\left(\phi_n(x)\right)\left[\dfrac{d}{dx}\phi_n(x)\right]\end{array}\right]_0^a & ;\ if\ \ m\neq n\\ 0 & ;\ if\ \ m=n\end{cases} \tag{2.4}$$

may be derived from (2.3) and Sturm-Liouville theory.
The problem may be considered as

$$\begin{aligned}&\left(\Delta_{\alpha,\beta,\nu,\mu}+\lambda^2\right)\phi(x)=0, 0\le x\le a\\ &\phi(a)=0\end{aligned} \tag{2.5}$$

whose solution is obtained in the form of

$$\phi_n(x)=\mathrm{J}_{\alpha,\beta,\nu,\mu}\left(\lambda_n x\right) \tag{2.6}$$

where $\lambda_1,\lambda_2,\ldots$ represent the positive zeros arranged in ascending order of magnitude of the equation [4, p.479]

$$\mathrm{J}_{\alpha,\beta,\nu,\mu}\left(\lambda_n x\right)=0. \tag{2.7}$$

The above orthogonality condition (2.4) now becomes

$$\beta^2\nu^2\int_0^a x^{2\nu-2\alpha-1}\left[(x)^\alpha J_\mu\left(\beta(x\lambda)^\nu\right)\right]\left[(x)^\alpha J_\mu\left(\beta(x\lambda)^\nu\right)\right]dx$$

$$=\ \beta^2\nu^2\int_0^a x^{2\nu-2\alpha-1}\mathrm{J}_{\alpha,\beta,\nu,\mu}\left(\lambda_n x\right)\mathrm{J}_{\alpha,\beta,\nu,\mu}\left(\lambda_n x\right)dx$$

$$=\begin{pmatrix}\dfrac{\nu\beta^2}{2}a^{2\nu-2\alpha}\mathrm{J}^2_{\alpha,\beta,\nu,\mu+1}\left(\lambda_n a\right);\ m=n\\ 0 \qquad\qquad\qquad\qquad ;m\neq n\end{pmatrix}. \tag{2.8}$$

3 Operational properties

To consider the expansion of an arbitrary function $f(x)$ defined in the interval $(0,a)$ as a Fourier-Bessel series type expansion

$$f(x)=\sum_{n=1}^{\infty}a_n\mathrm{J}_{\alpha,\beta,\nu,\mu}\left(\lambda_n x\right) \tag{3.1}$$

where $a_n=\dfrac{2\beta^2\nu^2}{\nu\beta^2a^{2\nu-2\alpha}\mathrm{J}^2_{\alpha,\beta,\nu,\mu+1}\left(\lambda_n a\right)}\displaystyle\int_0^a x^{2\nu-2\alpha-1}\mathrm{J}_{\alpha,\beta,\nu,\mu}\left(\lambda_n x\right)f(x)dx$ (3.2)

and λ_n denote the positive zeros of the functions $\mathrm{J}_{\alpha,\beta,\nu,\mu}(xa)$, i.e.

$$\mathrm{J}_{\alpha,\beta,\nu,\mu}(\lambda_n a)=0\,. \tag{3.3}$$

Expressions (3.1) and (3.2) with theorem 1.1 suggests to introduce the integral transformation

$$\left(\hbar_{\alpha,\beta,\nu,\mu} f\right)(n)=\beta^2\nu^2\int_0^a x^{2\nu-2\alpha-1}\mathrm{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x)\, f(x)\,dx \tag{3.4}$$

which will be called the generalized finite Hankel transformation of the first kind.
Its inversion formula is given by

$$\left(\hbar^{-1}{}_{\alpha,\beta,\nu,\mu} f\right)(n)=\frac{2\nu}{a^{2\nu}}\sum_{n=1}^{\infty}\frac{\hbar_{\alpha,\beta,\nu,\mu}(n)\,\mathrm{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x)}{\mathrm{J}^2_{\alpha,\beta,\nu,\mu+1}(\lambda_n a)}\,. \tag{3.5}$$

We point out the following operational rules:

i) If $f(x)\in C^2(0,a)$, upon integrating by parts we get the formula

$$\begin{aligned}&\hbar_{\alpha,\beta,\nu,\mu}\left[x^{2-2\nu} f''(x)+(1-2\alpha)x^{1-2\nu} f'(x)-x^{-2\nu}\left\{(\mu\nu)^2-\alpha^2\right\} f(x)\right]\\ &= f(a)(\nu\beta)^3\lambda_n{}^{\nu}a^{\nu-2\alpha}\mathrm{J}_{\alpha,\beta,\nu,\mu+1}(\lambda_n a)-(\nu\beta)^4\lambda_n{}^{2\nu}\left(\hbar_{\alpha,\beta,\nu,\mu}\right)\left[f(x)\right].\end{aligned} \tag{3.6}$$

ii) If $f(x)\in C^{2r}(0,a)$, $f^{(i)}(0)$ are finite and $f^{(i)}(a)=0; i=(1,2,3,\ldots 2r-2)$, then

$$\begin{aligned}&\hbar_{\alpha,\beta,\nu,\mu}\left[x^{2-2\nu} f''(x)+(1-2\alpha)x^{1-2\nu} f'(x)-x^{-2\nu}\left\{(\mu\nu)^2-\alpha^2\right\} f(x)\right]^r\\ &=(-1)^r(\nu\beta)^{4r}\lambda_n^{2\nu r}\left(\hbar_{\alpha,\beta,\nu,\mu}\right)\left[f(x)\right].\end{aligned} \tag{3.7}$$

4 Applications

The vibrations of vibrating membrane are governed by the two-dimensional wave equation. The wave equation using generalized finite Hankel transformation of first kind is demonstrated. For radially symmetric and not radially symmetric vibrations the solutions have been obtained.
Find a function $u(r,t)$ on the domain $\{(r,t):0<r<1, t>0\}$ that satisfies the differential equation

$$r^{2-2\nu}\frac{\partial^2 u}{\partial r^2}+(1-2\alpha)r^{1-2\nu}\frac{\partial u}{\partial r}-r^{-2\nu}\left\{(\mu\nu)^2-\alpha^2\right\}u+\frac{\partial^2 u}{\partial t^2}=0,\mu\geq 0 \tag{4.1}$$

satisfying boundary conditions

i) As $t\to\infty+$, $u(r,t)$ converges to zero in the sense of $D'(I)$

ii) As $r\to 0+$, $u(r,t)$ converges in the sense of $D'(I)$ to $g(r)\in V'_{\alpha,\beta,\nu,\mu}(I)$

iii) As $r\to 1-$, $u(r,t)$ converges to zero on $c\leq t<\infty$ for each $c<0$

iv) As $r\to 0+$, $u(r,t)=O(1)$ on $c\leq t<\infty$.

Let us denote $U(n,t)=\hbar_{\alpha,\beta,\nu,\mu}(u(r,t))$. According to (4.1), (2.1) becomes

$$\Delta_{\alpha,\beta,\nu,\mu}u+\frac{\partial^2 u}{\partial t^2}=0\,. \tag{4.2}$$

By applying $\hbar_{\alpha,\beta,\nu,\mu}$ to (4.2), follows

$$(-1)\left(\nu\beta\lambda_n^{\nu}\right)^2\left(\hbar_{\alpha,\beta,\nu,\mu}\right)\left[u(r,t)\right]+\frac{\partial^2}{\partial t^2}\hbar_{\alpha,\beta,\nu,\mu}\left[u(r,t)\right]=0.$$

$$(-1)\left(\nu\beta\lambda_n^{\nu}\right)^2 U(n,t)+\frac{\partial^2}{\partial t^2}U(n,t)=0\,. \tag{4.3}$$

The solution of becomes

$$U(n,t)=\mathsf{F}_{\alpha,\beta,\nu,\mu}(n)e^{-\nu\beta t\lambda_n} \tag{4.4}$$

because of the boundary conditions (i) and (ii).
Also the inversion formulae can be obtained

$$u(r,t)=\sum_{n=1}^{\infty}\frac{2\nu G(n)e^{-\nu\beta\lambda_n t}\mathsf{J}_{\alpha,\beta,\nu,\mu}(\lambda_n r)}{(\lambda_n)^{2\nu}\,\mathsf{J}^2_{\alpha,\beta,\nu,\mu}(\lambda_n)}\,. \tag{4.5}$$

We end this section pointing out that, in addition to the importance that the equations (4.4) and (4.5) have by themselves, many other problems in Mathematical Physics have the same form. Thus, given a partial differential equation involving the *n*-dimensional Laplacian operator $\Delta_{\alpha,\beta,\nu,\mu}u=D_{x_1}^2u+D_{x_2}^2u+\ldots+D_{x_n}^2u$ if we seek solutions depending only on $r=\left(x_1^2+x_2^2+x_3^2+\ldots+x_{n-1}^2\right)^{\frac{1}{2}}$ and $z=x_n$ and it follows that $u(r,z)$ must satisfy equations analogous to (4.4) and (4.5) with $\mu=(n-3)/2; n\geq 3$ [7]. But neither (4.4) nor (4.5) can directly be solved by applying the finite Hankel transformation, except when $\mu=0$ (i.e., when $n=3$). Nevertheless, the finite generalized finite Hankel transformation of first kind provides an elegant and straightforward method to solve both equations for any value of $\mu\geq 0$ (i.e., for each $n\geq 3$).

5 Radially symmetric and not symmetric modes

i) Radially symmetric modes

For a circular membrane, the representation is done in polar coordinates is $x=r\cos\theta; y=r\sin\theta$.
Assume $u(r,t)=R(r)T(t)$. (5.1)

Wave equation for $u=RT$

$$\frac{1}{v^2}RT''=R''T+\frac{1}{r}R'T$$

$$\frac{1}{v^2}\frac{T''}{T}=\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}=-\lambda_n^2.$$

Obtaining solution for

$$T''+\lambda_n^2v^2T=0$$

is given by

$$T(t)=c_1\cos(\lambda_n vt)+c_2\sin(\lambda_n vt). \tag{5.2}$$

Here solution of

$$R(r)=\mathsf{J}_0(\lambda_n r)$$

$$T(t)=\cos(\lambda_n vt)\,.$$

Thus from (5.1) the equation now becomes

$$u(r,t) = \mathrm{J}_0(\lambda_n r)\cos(\lambda_n v t). \tag{5.3}$$

Eigenvalues $\mathrm{J}_0(\lambda) = 0$; implies that the

$\lambda_{01} = 1.445796..., \lambda_{02} = 7.617815..., \lambda_{03} = 18.721751...$ etc.

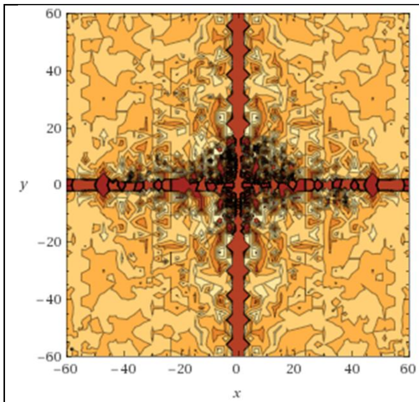

Fig 1. Contour plot of J0(((x*y)))*Cos[x] showing roots

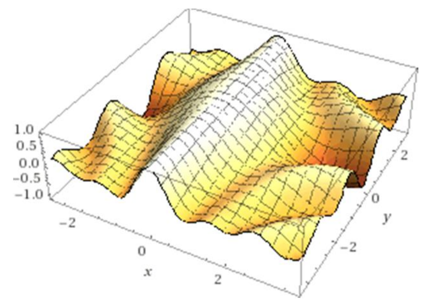

Fig 2. 3D plot of J0(((x*y)))*Cos[x]

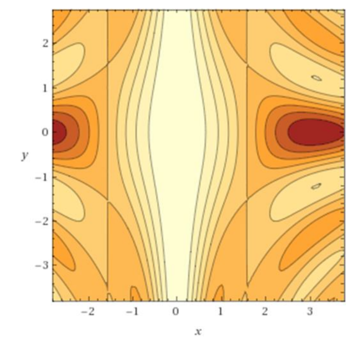

Fig 3. Plot of J0(((x*y)))*Cos[x] showing contour lines

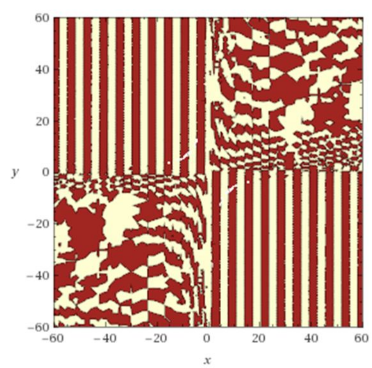

Fig 4. Contour plot J0(2*((x*y)^0.5))*Cos[x] showing roots

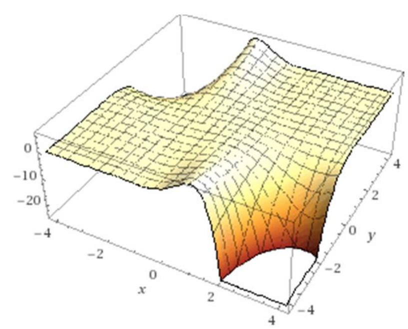

Fig 5. 3D plot of J0(2*((x*y)^0.5))*Cos[x]

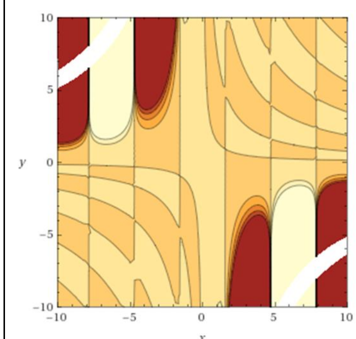

Fig 6. Plot of J0(2*((x*y)^0.5))*Cos[x] showing contour lines

Here Fig.4, 5, 6 are representation of Dorta [2], a finite version of (1.1) from a classical point of view, which was further extended by Malgonde [3].

ii) Not radially symmetric modes:

$$u(r,\theta,t) = R(r)\Theta(\theta)T(t). \tag{5.4}$$

Three ODE's with two eigenvalue problems for R and θ. Solutions with eigenvalues n, λ

$$\Theta = \cos(n\theta), n = 0,1,2,3,...$$
$$\mathrm{R} = \mathrm{J}_n(\lambda x),$$
$$T = \cos(\lambda v t)$$

such that

$$u(r,\theta,t) = \mathrm{J}_n(\lambda r)\cos(n\theta)\cos(\lambda v t). \tag{5.5}$$

Eigenvalues $\lambda = \lambda_{nk}$ is the k^{th} zero of $\mathrm{J}_n(\lambda) = 0$ are given by

$\lambda_{21} = 0, \lambda_{22} = 6.593654..., \lambda_{23} = 17.712499...$ etc.

Modes nk with $n = 2$ at $t = 1$, (5.5) will be as

$$u(r,\theta) = \mathrm{J}_2(\lambda_{2k} r)\cos(2\theta). \tag{5.6}$$

The Taylor Series expansion at $\lambda = 0$, (5.5) is given by:

$$\lambda/2 - \lambda^2/6 + \lambda^3/48 - \lambda^4/720 + \lambda^5/17280 + O(\lambda^6). \tag{5.7}$$

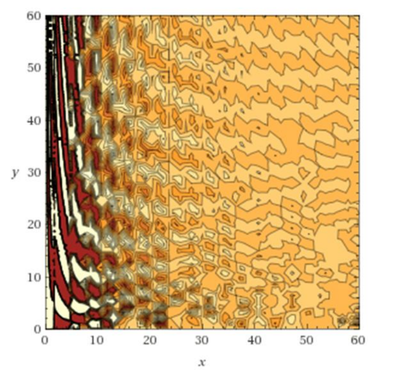

Fig 7. Contour plot x^(-1)*J2(2*((x*y)^0.5))*Cos[x] showing roots

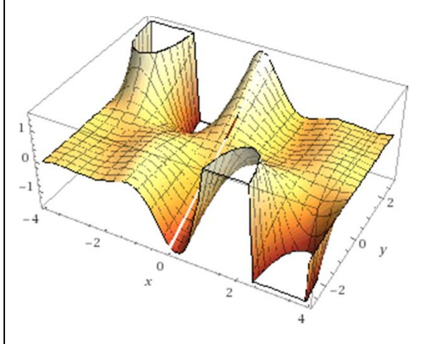

Fig 8. 3D plot of x^(-1)*J2(2*((x*y)^0.5))*Cos[x]

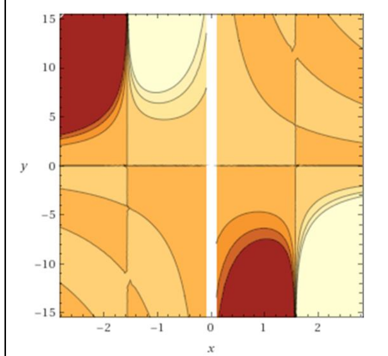

Fig 9. Plot of x^(-1)*J2(2*((x*y)^0.5))*Cos[x] mesh lines

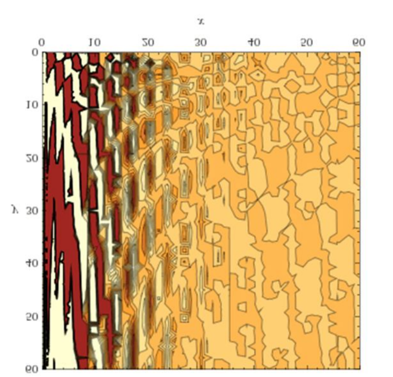

Fig 10. Contour plot x^(-2)*J2(3*((x*y)^0.33))*Cos[2*x] roots

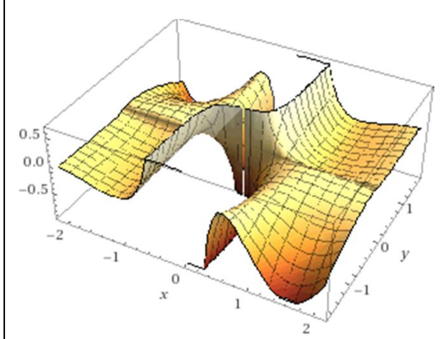

Fig 11. 3D plot of x^(-2)*J2(3*((x*y)^0.33))*Cos[2*x] only for R.P

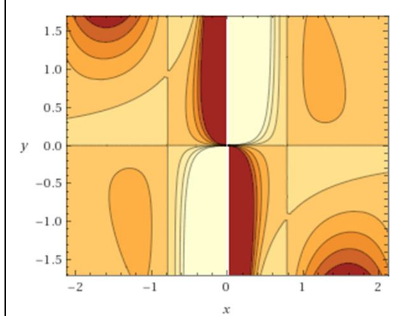

Fig 12. Plot of x^(-2)*J2(3*((x*y)^0.33))*Cos[2*x] only for R.P mesh lines

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