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Hamiltonian based solutions of Certain PDE in Plasma Flows

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Plasma Transport through magnetic field lines is a domain with implications in hot fusion plasmas. It also has application with the sources of plasma which work at low temperature and pressure. Their working often depends on the external magnetic field. The understanding of important principles applicable for transportation of plasma, such as magnetized plasma diffusion depends on the underlying partial differential equations. One such PDE is the modified Kdv Zhakharov Kuznetsov equation. These equations has solutions that are best described by solitons, kinks and periodic wave type travelling wave solutions. We here try to obtain Hamiltonian based solutions of these evolution equations. We compare the solutions with those obtained by Jacobi-Elliptic Function Method, Rational Trigonometric Expansion Method and Petrov Galerkien Method and present our results.
***Keywords*:** Solitons, Hamiltonians, mKdv-Zakarov-Kuznetsov equations, Petrov Galerkien Method.

1. Introduction

The solitons are one of the most useful structures that widely occur in nature. They have a very unique characteristic of shape preserving. Due to this characteristic, they have potentially useful applications in different domains such as nonlinear optics, plasmas, fluid mechanics, condensed matter, electro-magnetic and many more. We here study the problem of determining of useful techniques for a large pool of nonlinear partial differential equations which describe different physical systems. This problem is of great importance from both the theoretical and experimental point of view.
Recently, a rich pool of exact and numerical methods have been developed to study PDEs. One Such Method is the Hamiltonian based method to find soliton solutions suggested by Zhang et al.[1]. We also find travelling wave solution using a recently introduced Rational Trigonometric expansion method by Darvishi et al[2]. He used it to find exact solutions to some variants of Boussinesq Equations. We use these methods to find travelling wave solutions to (3+1)-dimensional modified Kdv-Zakharov- Kuznetsov Equations. We also compare our solutions with the solutions obtained by Zia Yun Zhang [3] by Jacobi-Elliptic Function Method.

2. The Hamiltonian Based Method Solutions to mKdv-Zk equations

The preserving of hamiltonian indicates that any truncated fluid flow model will not given any unphysical terms. Hence, Hamiltonian in itself, becomes an essential tool for studying fluid flows. The Modified Kdv- Zakharov Kuznetsov equations are given by

$$\mathrm{u_t} + \beta u^2 u_x + u_{xxx} + u_{xxy} + u_{xzz} = 0 \tag{1}$$

Considering $u(-ct + kz + x + y)] = U(\vartheta)$ with x, y z and c being constants and ϑ as wave variable, we can re-write the above equation (1) as

$$(2 + k)\mathrm{U}''' + \beta \mathrm{U}^2\mathrm{U}' - \mathrm{C}\mathrm{U}' = 0 \tag{2}$$

Integrating the above equation with respect to ϑ gives

$$(2+k)U'' + \beta\frac{U^3}{3} - CU' = \text{Constant} = \vartheta_0 \tag{3}$$

We obtain the solutions to above non-linear differential equation reductions of the mKdv-ZK equation using Hamiltonian method suggested by Zhang et al.[1], where they studied the hamiltonian and the bifurcations induced by a generalized non-linear evolution equations of the form (4) and present our results.

$$Q'' = a_3Q^3 + a_2Q^2 + a_1Q + a_0 \tag{4}$$

Equations (3) can be thought of in terms of the generalized system (4) with $a_3 = \frac{-\beta}{3(2+k)}$, $a_2 = 0, a_1 = \frac{C}{(2+k)}$ and $a_0 = \frac{\vartheta_0}{(2+k)}$.

If $(2+k) \neq 0$, the system (3) can be re-written in a modified form as

$$U'' - (P\,U^3 + RU + S) = 0 \tag{5}$$

with $P = \frac{-\beta}{3(2+k)}$, $R = \frac{C}{(2+k)}$ and $S = \frac{\vartheta_0}{(2+k)}$.

The Hamiltonian for following two dimensional system of differential equations obtained from (5)

$$\frac{dU}{d\vartheta} = V, \frac{dV}{d\vartheta} = (P\,U^3 + RU + S) \tag{6}$$

is given by

$$H(U,V) = \frac{V^2}{2} - \frac{P\,U^4}{4} - \frac{R\,U^2}{2} \tag{7}$$

We thus obtain two cases for phase portraits, $Case\ i. When\ \ S \neq 0\quad Case\ ii. When\ S = 0$.

i. When $S \neq 0$

We take the right hand side of second equation in the system (6) as zero and obtain

$$U = \frac{\frac{2}{3}R}{G} + \frac{G}{2^{\frac{1}{3}}3^{\frac{2}{3}}}\ , U = \frac{(1+i\sqrt{3})R}{G2^{\frac{1}{3}}3^{\frac{2}{3}}} - \frac{(1-i\sqrt{3})G}{2^{\frac{1}{3}}3^{\frac{2}{3}}\,2P}\ , U = \frac{(1-i\sqrt{3})R}{G2^{\frac{1}{3}}3^{\frac{2}{3}}} - \frac{(1+i\sqrt{3})G}{2^{\frac{1}{3}}3^{\frac{2}{3}}\,2P} \tag{8}$$

Where, $G^3 = -9RP^2 + \sqrt{3}\sqrt{4R^3P^3 + 27S^2R^4}$. If $R = 0$. The system (8) reduces to

$$U = \frac{G}{2^{\frac{1}{3}}3^{\frac{2}{3}}}\ , U = \frac{(-1+i\sqrt{3})G}{2^{\frac{1}{3}}3^{\frac{2}{3}}\,2P}\ , U = -\frac{(1+i\sqrt{3})G}{2^{\frac{1}{3}}3^{\frac{2}{3}}\,2P} \tag{9}$$

The equations (8) and (9) indicates that the system (6) does not have unique equilibrium point. Which shows that, the real roots of the equation $f(U) = PU^3 + RU + S = 0$ is the abscissa of the equilibrium points. The only equilibrium point of (6) is $S_1 = \left[\frac{\frac{2}{3}R}{G} + \frac{G}{2^{\frac{1}{3}}3^{\frac{2}{3}}},\ 0\right]$, which is a saddle, indicating unbounded solution. If we change the parameters groups of P, R and S we can derive different phase portraits for (6) along with the condition $S \neq 0$. Some Phase Diagrams for $P = R = S = -1$ and $P = R = S = 1$ are plotted to show that the first condition gives periodic solutions and the second condition gives unbounded solutions respectively. In this context, a recent work of Wang et al.[4] shows that bifurcation and phase portrait study gives a whole lot of solutions and forms a necessary field for exploration, where as Zhang et al.[1], have provided with detailed exposition on the conditions of classification and bifurcation of some class of ODEs of the form

$$U'' - (P\,U^3 + QU^2 + RU + S) = 0 \tag{10}$$

It is also clear from Zhang et al.[1] work that, the subclasses of solutions applicable to equation (6) are the following **7** subclasses:

1. $P = 1$ and $R = 1$ will give unbounded solutions. **2.** With $P = 1$ and $R = \text{Any Value}$ will give bounded Solutions. These bounded solutions will have following three sub categories

$i.$ P< 0, $ii.\ 0 < P < \frac{2}{9}$, $iii.\ \ P = \frac{2}{9}$ and $iv. \frac{2}{9} < P < \frac{1}{4}$.

3. If $\mathrm{P} = 1$ and $\mathrm{R} = -1$, we have heteroclic and periodic orbits. **4.** $\mathrm{P} = -1$ and $\mathrm{R} = -1$, gives periodic orbits. **5.** $\mathrm{P} = -1$ and $\mathrm{R} = -1$ gives homoclinic orbits and bounded periodic solutions corresponding to periodic orbits in-side and outside the homoclinic orbits. **6.** $\mathrm{P} = 1$ and $\mathrm{R} = 0$ gives no bounded solution, as it implies that there exists only one equilibrium point,which is a saddle.**7.** If $\mathrm{P} = -1$ and $\mathrm{R} = 0$, we have bounded periodic solutions corresponding to periodic orbits.

The details of the solutions for each condition can be found using $Case\ ii. When\ S = 0$.

ii. When $\boldsymbol{S = 0}$, a variety of solutions for the system (6) and hence (3) can be obtained as follows: 1. When $P = 1\ and\ R = any\ arbitrary\ value$, gives bounded solutions with four sub-catagories: $i. When\ R$ <0 gives following bounded solutions:

$$P(\vartheta) = L - \frac{6L(2L+1)}{2+6L+\sqrt{4+6L}\cosh[L(2L+1)](\vartheta-\vartheta_0)}, \qquad (11)$$

With $L = \frac{\sqrt{-1+4R}}{2}$ corresponding to homoclinic orbit.

$ii. When\ 0 < R < 2/9$, the bounded solution for homoclinic orbit is

$$P(\vartheta) = \frac{-6R}{2+\sqrt{4-18R}\cosh[\sqrt{R}](\vartheta-\vartheta_0)} \qquad (12)$$

$iii. When\ R = 2/9,\ \ P(\vartheta) = \frac{1}{3}\tanh\left[\frac{\sqrt{2}}{6}(\vartheta-\vartheta_0)\right] - \frac{1}{3}\ and\ P(\vartheta) = -\frac{1}{3}\tanh\left[\frac{\sqrt{2}}{6}(\vartheta-\vartheta_0)\right] - \frac{1}{3}$ (13)

are the solutions.

$iv. When \frac{2}{9} < R < \frac{1}{4}$, gives family of periodic solutions as discussed in [1].

2. When $P = 1\ and\ R = -1$ gives heteroclinic and periodic orbits. The bounded solutions of the two heteroclinic orbits are $P(\vartheta) = \tanh\left[\frac{\sqrt{2}}{2}(\vartheta-\vartheta_0)\right]\ and\ P(\vartheta) = -\tanh\left[\frac{\sqrt{2}}{2}(\vartheta-\vartheta_0)\right]$ (14)

and the bounded periodic solutions for the periodic orbits are

$$P(\vartheta) = -\sqrt{2-P_0^2} + \frac{\sqrt{2-2P_0^2}}{P_0+\sqrt{2-P_0^2}-2P_0 sn^2[\tau(\vartheta-\vartheta_0),p]} \qquad (15)$$

Where, $\tau = \frac{\sqrt{2}}{4}[P_0+\sqrt{2-P_0^2}\,]$ and $p = \frac{2\sqrt{P_0\sqrt{2-P_0^2}}}{P_0+\sqrt{2-P_0^2}}$ for any arbitrary $0 < P_0 < 1$.

3.When $P = -1\ and\ R = -1$, gives periodic orbits and the bounded solutions for these orbits are $P(\vartheta) = \frac{2P_0}{1+[ns[\tau(\vartheta-\vartheta_0),l]+ns[\tau(\vartheta-\vartheta_0),l]]^2} - P_0$ (16)

Where $\tau = \sqrt{1+P_0^2}$, $l = \frac{P_0}{\sqrt{1+P_0^2}}$, for arbitrary P_0 with $ns = \frac{1}{sn}\ and\ cs = \frac{sn}{cn}$ are usual Jacobi elliptic functions.

4.When $P = -1\ and\ R = 1$, this case may be referred to in [1].

5.When $P = -1\ and\ R = 0$, this case may also be referred to in [1].

The solutions obtained by us are improvement over those obtain by zia-yun zhang [3], where he has used Jacobi elliptic functions to find exact solutions to mKdv-ZK equation. The solutions obtained by him, forms a particular case of those obtained by us when S = 0.

In the next section, we explore another method recently introduced by Darvishi et al. [2] to find exact solutions to Boussinesq equations and their variants, which form fundamental equations in atmospheric and climate modelling.

3. The Rational Trigonometric Expansion Method Solutions to mKdv-Zk equations

Darvishi et al. [2] have suggested a extended rational trigonometric expansion method for obtaining exact solutions in the form of non-linear travelling waves for Boussinesq equations and it's variants. They have used Rational form of Trigonometric and Hyperbolic Sine-Cosine functions and obtained a rich pool of solutions for the Boussinesq Equations.We apply the same method, and obtain certain travelling wave solutions to the modified Kdv-Zk equations given by

(1) and (3). We consider $u(-ct+kz+x+y)] = U(\vartheta)$ with x, y z and c being constants and ϑ as wave variable as the solution of (1). Following the work of Darvishi et al.[2], the general form of solutions of (1) as travelling waves can be represented as following 4 extended rational trigonometric types

$$i.\ u(\vartheta) = \frac{a_0 Sin[\mu\,\vartheta]}{[a_2 + a_1 Cos[\mu\,\vartheta]]},\ ii.\ u(\vartheta) = \frac{a_0 Cos[\mu\,\vartheta]}{[a_2 + a_1 Sin[\mu\,\vartheta]]},$$
$$\text{with} \quad Cos[\mu\,\vartheta] \neq -\frac{a_2}{a_1} \qquad \text{with} \quad Sin[\mu\,\vartheta] \neq -\frac{a_2}{a_1}$$

$$iii.\ u(\vartheta) = \frac{a_0 Sinh[\mu\,\vartheta]}{[a_2 + a_1 Cosh[\mu\,\vartheta]]}\ and \quad iv.\ u(\vartheta) = \frac{a_0 Cosh[\mu\,\vartheta]}{[a_2 + a_1 Sinh[\mu\,\vartheta]]}$$
$$\text{with} \quad Cosh[\mu\,\vartheta] \neq -a_2/a_1 \qquad \text{with} \quad Sinh[\mu\,\vartheta] \neq -a_2/a_1 \quad (17)$$

Solving the equation (3) using approximation $i.$ in rational sine-cosine form of the equation (17) and proceeding in the similar way as Darvishi et al. [2] and using Mathematica-9 software, we obtain following travelling wave solutions to (1)

$$u_{11}(\vartheta) = \frac{Sin[(\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]}{\sqrt{\beta}[\frac{2}{\sqrt{6c+3a1c}} + \frac{1}{\sqrt{3c}} Cos[(\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]]} \quad u_{12}(\vartheta) = \frac{Sin[(\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]}{\sqrt{\beta}[\frac{2}{\sqrt{6c+3a1c}} + \frac{1}{\sqrt{3c}} Cos[(\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]]}$$

$$u_{13}(\vartheta) = \frac{Sin[(\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]}{\sqrt{\beta}[\frac{-2}{\sqrt{6c+3a1c}} + \frac{1}{\sqrt{3c}} Cos[(\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]]} \quad u_{14}(\vartheta) = \frac{Sin[(-\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]}{\sqrt{\beta}[\frac{-2}{\sqrt{6c+3a1c}} + \frac{1}{\sqrt{3c}} Cos[(-\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]]} \quad (18)$$

$$u_{15}(\vartheta) = \frac{Sin[(\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]}{\sqrt{\beta}[\frac{2}{\sqrt{6c+3a1c}} - \frac{1}{\sqrt{3c}} Cos[(\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]]} \quad u_{16}(\vartheta) = \frac{Sin[(-\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]}{\sqrt{\beta}[\frac{-2}{\sqrt{6c+3a1c}} - \frac{1}{\sqrt{3c}} Cos[(-\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]]}$$

$$u_{17}(\vartheta) = \frac{Sin[(\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]}{\sqrt{\beta}[\frac{-2}{\sqrt{6c+3a1c}} + \frac{1}{\sqrt{3c}} Cos[(\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]]} \quad u_{18}(\vartheta) = \frac{Sin[(-\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]}{\sqrt{\beta}[\frac{-2}{\sqrt{6c+3a1c}} - \frac{1}{\sqrt{3c}} Cos[(-\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]]}$$

Similarly, the equation (3) can be solved using approximation $ii.$ in rational cosine-sine form of the equation (17) and using the similar procedure as Darvishi et al. [2] along with Mathematica-9 software, we obtain following travelling wave solutions to (1).

$$u_{21}(\vartheta) = \frac{Cos[(\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]}{\sqrt{\beta}[\frac{2}{\sqrt{6c+3a1c}} + \frac{1}{\sqrt{3c}} Sin[(\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]]} \quad u_{22}(\vartheta) = \frac{Cos\ [(-\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]}{\sqrt{\beta}[\frac{2}{\sqrt{6c+3a1c}} + \frac{1}{\sqrt{3c}} Sin\ [(-\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]]}$$

$$u_{23}(\vartheta) = \frac{Cos[(\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]}{\sqrt{\beta}[\frac{-2}{\sqrt{6c+3a1c}} + \frac{1}{\sqrt{3c}} Sin[(\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]]} \quad u_{24}(\vartheta) = \frac{Cos\ [(-\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]}{\sqrt{\beta}[\frac{-2}{\sqrt{6c+3a1c}} + \frac{1}{\sqrt{3c}} Sin[(-\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]]} \quad (19)$$

$$u_{25}(\vartheta) = \frac{Cos[(\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]}{\sqrt{\beta}[\frac{2}{\sqrt{6c+3a1c}} - \frac{1}{\sqrt{3c}} Sin[(\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]]} \quad u_{26}(\vartheta) = \frac{Cos[(-\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]}{\sqrt{\beta}[\frac{2}{\sqrt{6c+3a1c}} - \frac{1}{\sqrt{3c}} Sin[(-\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]]}$$

$$u_{27}(\vartheta) = \frac{Cos[(-\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]}{\sqrt{\beta}[\frac{-2}{\sqrt{6c+3a1c}} - \frac{1}{\sqrt{3c}} Sin[(-\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]]} \quad u_{28}(\vartheta) = \frac{Cos[(-\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]}{\sqrt{\beta}[\frac{-2}{\sqrt{6c+3a1c}} - \frac{1}{\sqrt{3c}} Sin[(-\frac{\sqrt{2c}}{\sqrt{k}})(-ct+kz+x+y)]]}$$

In terms of unknown parameters, we have obtained 16 solution profiles according to the positive and negative values of μ in the equation (17). Similarly, these calculations can also be extended to rational hyperbolic sine-cosine and cosine-sine approximations $iii.$ and $iv.$ discussed in (17).
These solutions obtained by us are comparable to the solutions in terms of Tan and Cot functions to those obtained by Islam et al [6], only differing by a constant multiple. The only difference being that, they have used $G'/_G$ expansion method, whereas here, we have used the Extended Rational Trigonometric Expansion Method.

Remark 1. The solutions obtained in equations (18) and (19) are checked using Maple by back substation and were found to be correct.
Remark 2. This work can further be Numerically extended using Finite Element Methods such Galerkien and Petrov Galerkin [5]. Also meshfree methods such as Radial Basis Function Method [9] can be applied to this work.

4. Conclusion

We have applied Hamiltonian Method and Extended Rational Trigonometric Function Expansion method to modified Kdv-Zk equations found in modelling of high and low pressure plasma flows. These methods are recently introduced and their applications are not fully explored. Further, Hamiltonian based study of non- linear PDEs in different engineering branches is an analytical approach which gives a rich pool of solutions to the evolution equations describing a particular physical phenomenon, where as the Extended Rational Trigonometric expansion method is a semi-analytical method for the same. The solutions thus obtained can be tested in comparison with different numerical and finite element methods.

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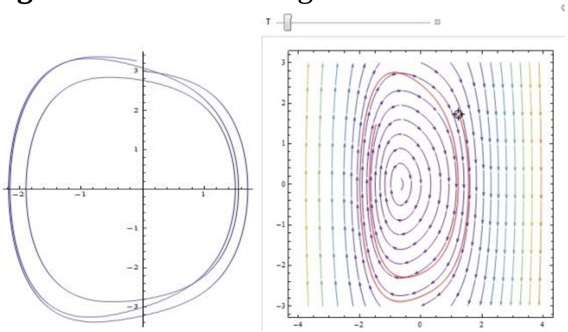
Fig.I-a and **b** Phase Diagrams for $P = R = S = -1$



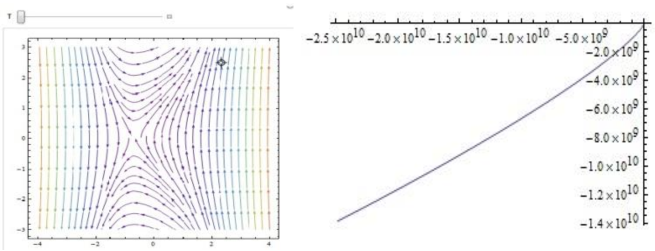


Fig.II- a and **b** Phase Diagrams for $P = R = S = 1$

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