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Numerical Solution of Coupled system of Boundary value problems by Galerkin method with different order B-splines.

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In the current paper, we present a Galerkin method with cubic, quartic B-splines for the numerical solution of highly coupled system of nonlinear boundary value problems (BVP). The method is actually applied on a linearized form of given BVP. For linearization, we used quasi linearization technique to convert the given nonlinear BVP into a sequence of linear BVPs. For each linear BVP, each unknown variable is approximate by a linear combination of cubic B-splines or quartic B-splines depending on the order of derivative for the variable that is present in the given BVP. For a variable with third order derivative present in given BVP, we used quartic B-splines where as for the variables with second order derivative, we used cubic B-splines in the approximation. Galerkin method was employed with these approximations and hence obtained the results. The convergence criterion for the solution of sequence of linear BVP is fixed at 1.0×10^{-5}. To test the efficiency of the proposed method, we employed the presented method on a problem which is available in the literature. Numerical results obtained by the proposed method are in with good agreement with the results available in the literature for the tested problem.

***Keywords*:** Coupled nonlinear boundary value problem, Galerkin method, B-spline, Numerical solution

1. Introduction

Nonlinear coupled system of boundary value problems frequently arises by the modelling of problems in the areas of engineering and applied science. Specially, in fluid dynamics modelling [1, 2, 3], the modelled problem is a coupled system of nonlinear boundary value problem in three or four unknowns. Based on the study of effects for such problems, it lead to the system of above said type into various number of variables. In the current paper, we are considering a coupled system of nonlinear BVP with four unknown variables. Solving such type of problems by analytical methods is not so easy. Hence numerical methods are popular for the solving such type of equations. Some popular methods to solve such type problems are finite difference methods, finite volume methods and finite element method. In the current paper, we propose a finite element method namely Galerkin method with B-splines to solve a general coupled system of nonlinear boundary value problems of the type

$$NL_i(t,\ F, F',\ F'', F''',\ G,\ G',\ G'',\ H,\ H',\ H'', K,\ K', K'') = 0,\quad t_0 \le t \le t_{max} \tag{1}$$

where $i = 1,2,3,4$ and NL denote a nonlinear operator. Here F, G, H, K are the unknown variables in the system along with boundary conditions

$$F(t_0) = F_0,\ F'(t_0) = F_1, F'(t_{\max}) = F_2,\ G(t_0) = G_0, G(t_{\max}) = G_1,\ H(t_0) = H_0, H(t_{\max}) = H_1$$
$$K(t_0) = K_0, K(t_{\max}) = K_1, \tag{2}$$

From the literature, we can understand that several researchers worked on various numerical methods for the solutions of this type of coupled systems[4, 5, 6,7] . Galerkin method is a method which is powerful and accurate for the numerical results of linear differential equations. In the current paper we use a linearization technique namely quasilinearization technique introduced by R.E. Bellman and R.E. Kallaba[8]. Hence a general linearized system of differential equations in four variables may be considered in the following form.

$\sum_{j=1}^{4} p_{ij}(t)\, F^{(4-j)} + \sum_{j=1}^{3} q_{ij}(t)\, G^{(3-j)} + \sum_{j=1}^{3} r_{ij}(t)\, H^{(3-j)} + \sum_{j=1}^{3} s_{ij}(t) K^{(3-j)} = T_i(t)$
$i = 1,2,3,4.$ (3)
along with boundary conditions prescribed in (2). Here the functions $p_{ij}, q_{ij}, r_{ij}, s_{ij}$ and T_i are assumed to be continuous on the interval $[t_0, t_{max}]$. This assumption assures the existence and uniqueness of solution for the problem (3) with the boundary conditions (2). In the current paper we used B-splines of different order viz, quartic B-splines and cubic B-splines to approximate the unknown variables. B-splines are very popular functions in approximation theory and these are the basis functions for the space of all Spline polynomials on a given interval. These functions approximate an unknown function more accurately when the unknown is specified at some interior points of the given interval. Also the use of B-splines in FEM, especially like Galerkin, Collocation or Petrov-Galerkin methods, is popular and comfortable for obtaining the results.

In the next section we present the Galerkin method for solving the problems of type (3) along with the definitions of B-splines (quartic, cubic). Application of the proposed method on a problem taken from literature is presented in section 3. Results and discussion is presented in section 4. The last section 5 is ended with conclusion of the findings.

2. Galerkin method with B-splines

Consider the linear system of coupled problems (3). To solve each unknown we approximate each unknown variable as a linear combinations of B-splines. Since the problem is defined on the interval $[t_0, t_{max}]$, first we consider a uniform partition on the given interval with each subinterval of length $h = \frac{t_{\max} - t_0}{n}$ where n is the number of subintervals(usually finite). The complete set of quartic B-splines that forms a basis on the given interval with the considered grid points, is $\{S_{4,-2}, S_{4,-1}, S_{4,0}, \dots, S_{4,n}, S_{4,n+1}\}$. Schoenberg[9] has proved the completeness. To access this set of quartic splines on given interval, we need to introduce 8 additional partition points outside the interval. The details are presented in the paper by Dhivya and Panthangi, MK[7]. More details can be obtained from these references[10, 11, 12]. In a similar way a complete set of cubic splines is given by $\{S_{3,-1}, S_{3,0}, \dots, S_{3,n}, S_{3,n+1}\}$. For this set of cubic splines, we take the help of six partition points outside the given interval. The definitions of quartic and cubic B-splines are given below.

$$S_{4,i}(t) = \begin{cases} \sum_{r=i-2}^{r=i+3} \frac{(t_r - t)_+^4}{\pi'(t_r)}, & t \in [t_{i-2}, t_{i+3}] \\ 0, & otherwise \end{cases} \tag{4}$$

$$S_{3,i}(t) = \begin{cases} \sum_{r=i-2}^{r=i+2} \frac{(t_r - t)_+^3}{\pi'(t_r)}, & t \in [t_{i-2}, t_{i+2}] \\ 0, & otherwise \end{cases} \tag{5}$$

Since the variable F is present at third order, it is better to approximate it by a fourth order spline curve. Thus F is approximated as

$$F(t) = \sum_{j=-2}^{n+1} f_j S_{4,j} \tag{6}$$

In a similar way we approximate the other variables with cubic B-splines as

$$G(t) = \sum_{j=-1}^{n+1} g_j S_{3,j} \tag{7}$$

$$H(t) = \sum_{j=-1}^{n+1} h_j S_{3,j} \tag{8}$$

$$K(t) = \sum_{j=-1}^{n+1} k_j S_{3,j} \tag{9}$$

In Galerkin method, we make the residual orthogonal to basis functions, and thus a weak form of integrals will be considered. To make the weak form of residual simpler, we usually consider the basis functions in the approximation of unknown variables, such that they vanish on the boundary where Dirichlet type of conditions are prescribed. For this, applying the Dirichlet type of boundary conditions to the unknown variable approximations, we get

$F(t_0) = F_0$

$\Rightarrow f_{-2} S_{4,-2}(t_0) + f_{-1} S_{4,-1}(t_0) + f_0 S_{4,0}(t_0) + f_1 S_{4,1}(t_0) = F_0$

$$\Longrightarrow f_{-2} = \frac{1}{S_{4,-2}(t_0)} \left\{F_0 - \left[f_{-1}S_{4,-1}(t_0) + f_0S_{4,0}(t_0) + f_1S_{4,1}(t_0)\right]\right\} \tag{10}$$

$G(t_0) = G_0$
$\Longrightarrow g_{-1}S_{3,-1}(t_0) + g_0S_{3,0}(t_0) + g_1S_{3,1}(t_0) = G_0$
$$\Longrightarrow g_{-1} = \frac{1}{S_{3,-1}(t_0)} \left\{G_0 - \left[g_0S_{3,0}(t_0) + g_1S_{3,1}(t_0)\right]\right\} \tag{11}$$
$G(t_{max}) = G_1$
$\Longrightarrow g_{n-1}S_{3,n-1}(t_{max}) + g_nS_{3,n}(t_{max}) + g_{n+1}S_{3,n+1}(t_{max}) = G_1$
$$\Longrightarrow g_{n+1} = \frac{1}{S_{3,n+1}(t_{max})} \left\{G_1 - \left[g_{n-1}S_{3,n-1}(t_{max}) + g_nS_{3,n}(t_{max})\right]\right\} \tag{12}$$
$H(t_0) = H_0$
$\Longrightarrow h_{-1}S_{3,-1}(t_0) + h_0S_{3,0}(t_0) + h_1S_{3,1}(t_0) = H_0$
$$\Longrightarrow h_{-1} = \frac{1}{S_{3,-1}(t_0)} \left\{H_0 - \left[h_0S_{3,0}(t_0) + h_1S_{3,1}(t_0)\right]\right\} \tag{13}$$
$H(t_{max}) = H_1$
$\Longrightarrow h_{n-1}S_{3,n-1}(t_{max}) + h_nS_{3,n}(t_{max}) + h_{n+1}S_{3,n+1}(t_{max}) = H_1$
$$\Longrightarrow h_{n+1} = \frac{1}{S_{3,n+1}(t_{max})} \left\{H_1 - \left[h_{n-1}S_{3,n-1}(t_{max}) + h_nS_{3,n}(t_{max})\right]\right\} \tag{14}$$
$K(t_0) = K_0$
$\Longrightarrow k_{-1}S_{3,-1}(t_0) + k_0S_{3,0}(t_0) + k_1S_{3,1}(t_0) = K_0$
$$\Longrightarrow k_{-1} = \frac{1}{S_{3,-1}(t_0)} \left\{K_0 - \left[k_0S_{3,0}(t_0) + k_1S_{3,1}(t_0)\right]\right\} \tag{15}$$
$K(t_{max}) = K_1$
$\Longrightarrow k_{n-1}S_{3,n-1}(t_{max}) + k_nS_{3,n}(t_{max}) + k_{n+1}S_{3,n+1}(t_{max}) = K_1$
$$\Longrightarrow k_{n+1} = \frac{1}{S_{3,n+1}(t_{max})} \left\{K_1 - \left[k_{n-1}S_{3,n-1}(t_{max}) + k_nS_{3,n}(t_{max})\right]\right\} \tag{16}$$
Substitute the value of each parameter in (10)-(16) in the equations (6) to (9), we get the revised approximations for our unknown variables as
$$F(t) = F_w(t) + \sum_{j=-1}^{n+1} f_jP_{4,j}(t) \tag{17}$$
$$G(t) = G_w(t) + \sum_{j=0}^{n} g_jP_{3,j}(t) \tag{18}$$
$$H(t) = H_w(t) + \sum_{j=0}^{n} h_jP_{3,j}(t) \tag{19}$$
$$K(t) = K_w(t) + \sum_{j=0}^{n} k_jP_{3,j}(t) \tag{20}$$
where
$F_w(t) = \frac{F_0}{S_{4,-2}(t_0)} S_{4,-2}(t),$,
$G_W(t) = \frac{G_0}{S_{3,-1}(t_0)} S_{3,-1}(t) + \frac{G_1}{S_{3,n+1}(t_{max})} S_{3,n+1}(t),$
$H_W(t) = \frac{H_0}{S_{3,-1}(t_0)} S_{3,-1}(t) + \frac{H_1}{S_{3,n+1}(t_{max})} S_{3,n+1}(t),$
$K_W(t) = \frac{K_0}{S_{3,-1}(t_0)} S_{3,-1}(t) + \frac{K_1}{S_{3,n+1}(t_{max})} S_{3,n+1}(t),$
$$P_{4,j}(x) = \begin{cases} S_{4,j}(t) - \frac{S_{4,j}(t_0)}{S_{4,-2}(t_0)} S_{4,-2}(t), & j = -1,0,1 \\ S_{4,j}(t), & j = 2,3,\dots,n+1 \end{cases},$$
$$P_{3,j}(x) = \begin{cases} S_{3,j}(t) - \frac{S_{3,j}(t_0)}{S_{3,-1}(t_0)} S_{3,-1}(t), & j = 0,1 \\ S_{4,j}(t), & j = 2,3,\dots,n-2. \\ S_{3,j}(t) - \frac{S_{3,j}(t_{max})}{S_{3,n+1}(t_{max})} S_{3,n+1}(t), & j = n-1, n \end{cases}$$
Now we have $n + 3$ basis functions in F and $n + 1$ functions in each of G, H and K. Applying Galerkin method to the system of equations (3) with the new basis functions, we get system of algebraic equations in unknown parameters f_j, g_j, h_j and k_j as follows.

$$
\begin{aligned}
M_{11}\boldsymbol{f} + M_{12}\boldsymbol{g} + M_{13}\boldsymbol{h} + M_{14}\boldsymbol{k} &= N_1 \\
M_{21}\boldsymbol{f} + M_{22}\boldsymbol{g} + M_{23}\boldsymbol{h} + M_{24}\boldsymbol{k} &= N_2 \\
M_{31}\boldsymbol{f} + M_{32}\boldsymbol{g} + M_{33}\boldsymbol{h} + M_{34}\boldsymbol{k} &= N_3 \\
M_{41}\boldsymbol{f} + M_{42}\boldsymbol{g} + M_{43}\boldsymbol{h} + M_{44}\boldsymbol{k} &= N_4
\end{aligned} \tag{21}
$$

where $\boldsymbol{f} = [f_{-1}\ f_0 \ldots f_n\ f_{n+1}]^T$, $\boldsymbol{g} = [\ g_0\ g_1 \ldots g_n]^T$, $\boldsymbol{h} = [\ h_0\ h_1 \ldots h_n]^T$ and $\boldsymbol{k} = [\ k_0\ k_1 \ldots k_n]^T$.

Here each entry of the matrices M_{ij}'s are given below.

$$
\begin{aligned}
(m_{11})_{ij} = \int_{t_0}^{t_{max}} \Big[\big(p_{11}(t) P_{4,i}(t) \big)'' P'_{4,j}(t) - \big(p_{12}(t) P_{4,i}(t) \big)' P'_{4,j}(t) \\
+ \big(p_{13}(t) P'_{4,j}(t) + p_{14}(t) P_{4,j}(t) \big) P_{4,i}(t) \Big] dt \\
+ p_{11}(t_{max})\, P_{4,i}(t_{max}) P''_{4,j}(t_{max}) \\
i = -1{:}\, n+1;\, j = -1{:}\, n+1
\end{aligned}
$$

$$
(m_{12})_{ij} = \int_{t_0}^{t_{max}} \Big[- \big(q_{11}(t) P_{4,i}(t) \big)' P'_{3,j}(t) + \big(q_{12}(t) P'_{3,j}(t) + q_{13}(t) P_{3,j}(t) \big) P_{4,i}(t) \Big] dt
$$
$$
+ q_{11}(t_{max})\, P_{4,i}(t_{max}) P'_{3,j}(t_{max})
$$
$$
i = -1{:}\, n+1;\, j = 0{:}\, n
$$

$$
(m_{13})_{ij} = \int_{t_0}^{t_{max}} \Big[- \big(r_{11}(t) P_{4,i}(t) \big)' P'_{3,j}(t) + \big(r_{12}(t) P'_{3,j}(t) + r_{13}(t) P_{3,j}(t) \big) P_{4,i}(t) \Big] dt
$$
$$
+ r_{11}(t_{max})\, P_{4,i}(t_{max}) P'_{3,j}(t_{max})
$$
$$
i = -1{:}\, n+1;\, j = 0{:}\, n
$$

$$
(m_{14})_{ij} = \int_{t_0}^{t_{max}} \Big[- \big(s_{11}(t) P_{4,i}(t) \big)' P'_{3,j}(t) + \big(s_{12}(t) P'_{3,j}(t) + s_{13}(t) P_{3,j}(t) \big) P_{4,i}(t) \Big] dt
$$
$$
+ s_{11}(t_{max})\, P_{4,i}(t_{max}) P'_{3,j}(t_{max})
$$
$$
i = -1{:}\, n+1;\, j = 0{:}\, n
$$

For second to fourth rows we can have a common expression as for each $k = 2,3,4$

$$
\begin{aligned}
(m_{k1})_{ij} = \int_{t_0}^{t_{max}} \Big[\big(p_{k1}(t) P_{3,i}(t) \big)'' P'_{4,j}(t) - \big(p_{k2}(t) P_{3,i}(t) \big)' P'_{4,j}(t) \\
+ \big(p_{k3}(t) P'_{4,j}(t) + p_{k4}(t) P_{4,j}(t) \big) P_{3,i}(t) \Big] dt \\
i = 0{:}\, n;\, j = -1{:}\, n+1
\end{aligned}
$$

$$
(m_{k2})_{ij} = \int_{t_0}^{t_{max}} \Big[- \big(q_{k1}(t) P_{3,i}(t) \big)' P'_{3,j}(t) + \big(q_{k2}(t) P'_{3,j}(t) + q_{k3}(t) P_{3,j}(t) \big) P_{3,i}(t) \Big] dt
$$
$$
i = 0{:}\, n;\ j = 0{:}\, n
$$

$$
(m_{k3})_{ij} = \int_{t_0}^{t_{max}} \Big[- \big(r_{k1}(t) P_{3,i}(t) \big)' P'_{3,j}(t) + \big(r_{k2}(t) P'_{3,j}(t) + r_{k3}(t) P_{3,j}(t) \big) P_{3,i}(t) \Big] dt
$$
$$
i = 0{:}\, n;\ j = 0{:}\, n
$$

$$
(m_{k4})_{ij} = \int_{t_0}^{t_{max}} \Big[- \big(s_{k1}(t) P_{3,i}(t) \big)' P'_{3,j}(t) + \big(s_{k2}(t) P'_{3,j}(t) + s_{k3}(t) P_{3,j}(t) \big) P_{3,i}(t) \Big] dt
$$
$$
i = 0{:}\, n;\ j = 0{:}\, n
$$

$$
\begin{aligned}
(n_1)_i = \int_{t_0}^{t_{max}} & \Big[T_1(t)P_{4,i}(t) - \left(p_{11}(t)P_{4,i}(t)\right)'' F_W'(t) + \left(p_{12}(t)P_{4,i}(t)\right)' F_W'(t) \\
& - \left(p_{13}(t)F_W'(t) + p_{14}(t)F_W(t)\right)P_{4,i}(t) + \left(q_{11}(t)P_{4,i}(t)\right)' G_W'(t) \\
& - \left(q_{12}(t)G_W'(t) + q_{13}(t)G_W(t)\right)P_{4,i}(t) + \left(r_{11}(t)P_{4,i}(t)\right)' H_W'(t) \\
& - \left(r_{12}(t)H_W'(t) + r_{13}(t)H_W(t)\right)P_{4,i}(t) + \left(s_{11}(t)P_{4,i}(t)\right)' K_W'(t) \\
& - \left(s_{12}(t)K_W'(t) + s_{13}(t)K_W(t)\right)P_{4,i}(t)\Big] dt - p_{11}(t_{max})\, P_{4,i}(t_{max})F_W''(t_{max}) \\
& - \left(p_{11}(t)P_{4,i}(t)\right)'\Big|_{t_0} F_1 + \left(p_{11}(t)P_{4,i}(t)\right)'\Big|_{t_{max}} F_2 \; - p_{12}(t_{max})P_{4,i}(t_{max})F_2 \\
& - q_{11}(t_{max})P_{4,i}(t_{max})G_W'(t_{max}) - r_{11}(t_{max})P_{4,i}(t_{max})H_W'(t_{max}) \\
& - s_{11}(t_{max})P_{4,i}(t_{max})K_W'(t_{max})
\end{aligned}
$$

$$i = -1: n+1;$$

For other column vectors, N_2, N_3, N_4, each entry is given by for $k = 2,3,4$

$$
\begin{aligned}
(n_k)_i = \int_{t_0}^{t_{max}} & \Big[T_k(t)P_{3,i}(t) - \left(p_{k1}(t)P_{3,i}(t)\right)'' F_W'(t) + \left(p_{k2}(t)P_{3,i}(t)\right)' F_W'(t) \\
& - \left(p_{k3}(t)F_W'(t) + p_{k4}(t)F_W(t)\right)P_{3,i}(t) + \left(q_{k1}(t)P_{3,i}(t)\right)' G_W'(t) \\
& - \left(q_{k2}(t)G_W'(t) + q_{k3}(t)G_W(t)\right)P_{3,i}(t) + \left(r_{k1}(t)P_{3,i}(t)\right)' H_W'(t) \\
& - \left(r_{k2}(t)H_W'(t) + r_{k3}(t)H_W(t)\right)P_{3,i}(t) + \left(s_{k1}(t)P_{3,i}(t)\right)' K_W'(t) \\
& - \left(s_{k2}(t)K_W'(t) + s_{k3}(t)K_W(t)\right)P_{3,i}(t)\Big] dt - \left(p_{k1}(t)P_{3,i}(t)\right)'\Big|_{t_0} F_1 \\
& + \left(p_{k1}(t)P_{3,i}(t)\right)'\Big|_{t_{max}} F_2
\end{aligned}
$$

$$i = 0: n;$$

Each entry in the above matrices involves of integration of terms containing the coefficients functions, their derivatives, B-splines functions (revised) and its derivatives. Since B-splines are polynomial functions in each subinterval, $[t_i, t_{i+1}]$, we approximate these integrations expressions using Gaussian quadrature formula. Based on the coefficient functions, we need to choose appropriate quadrature formula like 4 point, 5 point or 6 point formula. All these expressions are programmable using MATLAB software. Upon finding the nodal parameters f_j, g_j, h_j and k_j's we can approximate each unknown variable by the approximation formula.

Stability of the system: We can prove that each of matrix M_{ij} is positive definite and symmetric. Some detailed proofs were prescribed in the book by P.M. Prenter[12]. Hence collectively a matrix M containing all these matrices is non-singular and thus system is stable. In programming part, we propose the procedure of getting of parameters in this way.

$$\boldsymbol{u} = M^{-1}N$$

where $\boldsymbol{u} = [\boldsymbol{f}\ \boldsymbol{g}\ \boldsymbol{h}\ \boldsymbol{k}]^T$**;** $M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix}$ and $N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}$.

3. Numerical Example

To test the efficiency of the proposed method we consider a modelled problem in[1]. The considered problem is in four variables and it is nonlinear. The problem is

$$f''' + B_1 h' + ff'' + G_r\theta + G_m\varphi - Kf' - Mf' = 0$$
$$\lambda h'' - 2\frac{\lambda}{G_1}(2h + f'') + f'h + fh' = 0;$$
$$\theta'' + Prf\theta' + PrDu\varphi'' + PrEc(f'')^2 = 0\ ;$$
$$\varphi'' + Scf\varphi' + ScSr\theta'' - \tau(\theta'\varphi' + \theta''\varphi) = 0 \tag{22}$$

The boundary conditions are given by

$$f' = 1,\ f = V_0,\ h = -sf'', \theta = 1, \varphi = 1, \qquad at \quad \eta = 0$$
$$f' = 0,\ h = 0, \theta = 0,\ \varphi = 0 \qquad at \quad \eta \to \infty \tag{23}$$

In the paper[1], it was given that η as tending to max value can be considered as $\eta_{max} = 6$ and the same had considered.

The nonlinear system of equations (22) is converted to a sequence of linear system of differential equations using quasilinearization technique[8]. The converted is mentioned below.

$$f_{n+1}''' + f_n f_{n+1}'' - (K + M)f_{n+1}' + f_n'' f_{n+1} + B_1 h_{n+1}' + G_r\theta_{n+1} + G_m\varphi_{n+1} = f_n f_n''$$
$$-2\frac{\lambda}{G_1} f_{n+1}'' + h_n f_{n+1}' + h_n' f_{n+1} - \lambda h_{n+1}'' + f_n h_{n+1}' + \left(f_n' - 4\frac{\lambda}{G_1}\right) h_{n+1} = f_n h_n' + h_n f_n'$$
$$2PrEcf_n'' f_{n+1}'' + Pr\theta_n f_{n+1} + \theta_{n+1}'' + Prf_n\theta_{n+1}' + PrDu\varphi_{n+1}'' = PrEc(f_n'')^2 + Prf_n\theta_n'$$
$$Sc\varphi_n f_{n+1} + (ScSr - \tau\varphi_n)\theta_{n+1}'' - \tau\varphi_n'\theta_{n+1}' + \varphi_{n+1}'' + Scf_n\varphi_{n+1}' - \tau\theta_n''\varphi_{n+1} =$$
$$Sc\varphi_n f_n - \tau\varphi_n'\theta_n' + \theta_n''\varphi_n \tag{24}$$

Here f_n, h_n, θ_n and $\varphi_n$are the n^{th} approximation to the solutions f, h, θ and φ respectively. We employed the proposed method on the equations (24) along with boundary conditions (23). Here we took a partition over the given subinterval [0,6] with $N = 10$. Using additional grid points outside the given interval we programmed each B-spline function as per the requirement.

3. Results and discussion

Numerical solutions of unknown parameters were obtained by the proposed method with very good accuracy and less number of computations. Here the convergence criterion for the solutions of the sequence of problems (24) was set at 1.0×10^{-5}. We know that this convergence happens at the rate of second order[8]. The graphical solutions obtained by the proposed method are presented below along with the graphs that are actually in [1].

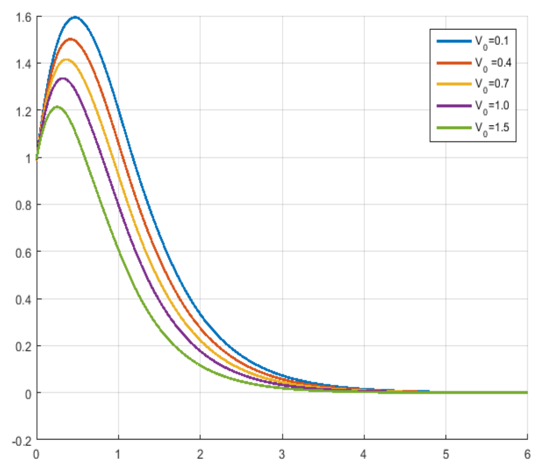

Fig1: Effect of V_0 on velocity profile by proposed method

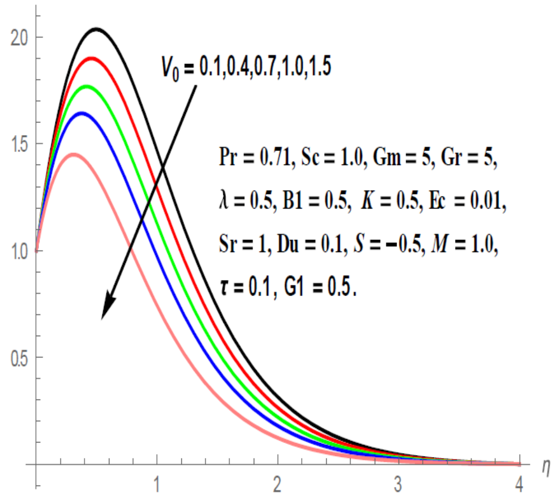

Fig2: Effect of V_0 on velocity profile in the paper[1].

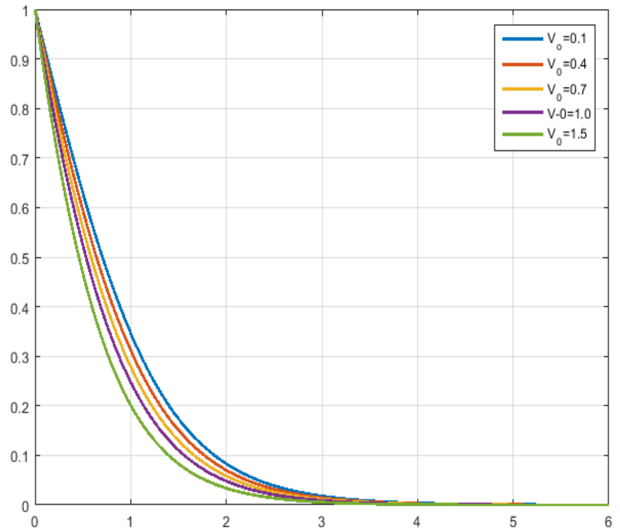

Fig3: Effect of V_0 on temperature profile by proposed method

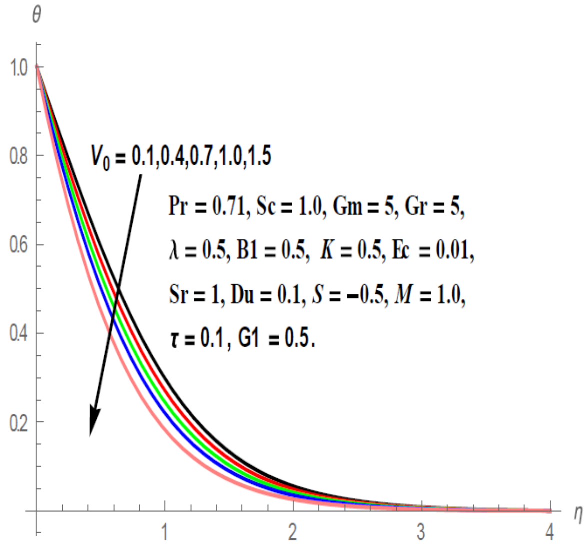

Fig4: Effect of V_0 on temperature profile in the paper[1]

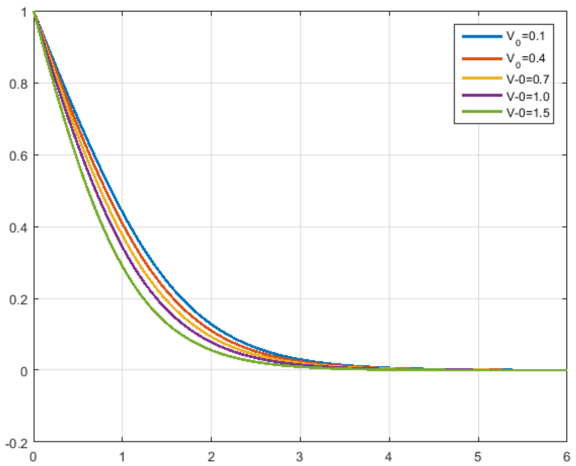

Fig5: Effect of V_0 on concentration profile by the proposed method.

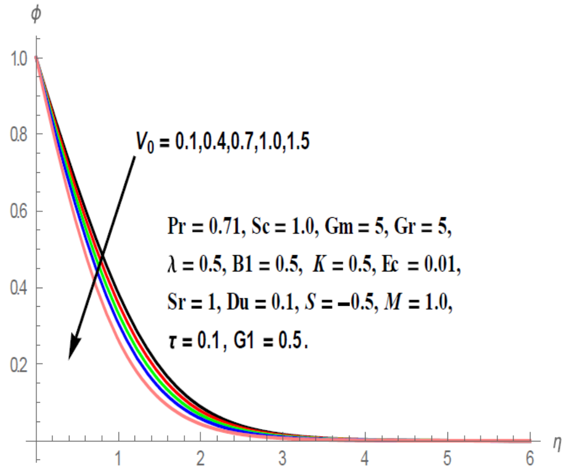

Fig6: Effect of V_0 on concentration profile in the paper[1]

Figure 1, figure 3 and figure 5 are the graphs by the proposed method that shows the effect the parameter V_0 on velocity, temperature and concentration profiles. The results obtained by the proposed method are matching with required profiles that are in Figure 2, figure 4 and figure 6 respectively. Also we are presenting the effect of another parameter M on the above said profiles in the figures Figure 7, figure 9 and figure 11. These graphs are compared with the graphs figure 8 , figure 10 and figure 12 which are taken from [1]. Results obtained are in very good agreement with the required one.

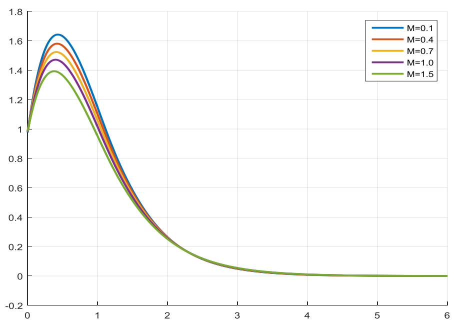

Fig7: Effect of M on velocity profile by proposed method

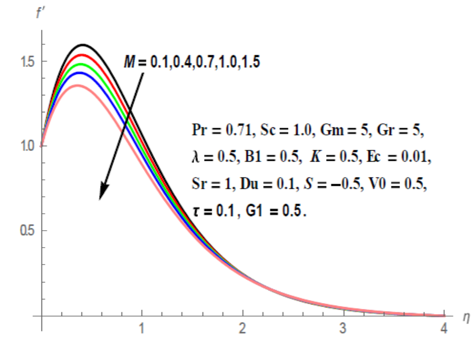

Fig8: Effect of M on velocity profile in the paper[1].

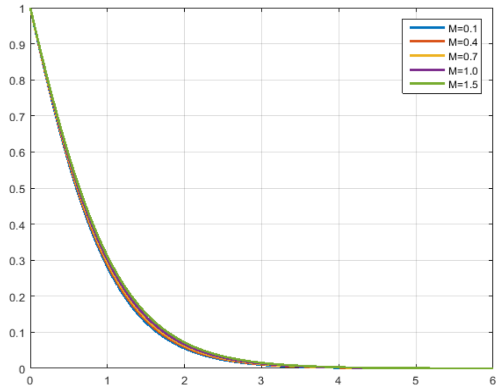

Fig9: Effect of M on temperature profile by proposed method

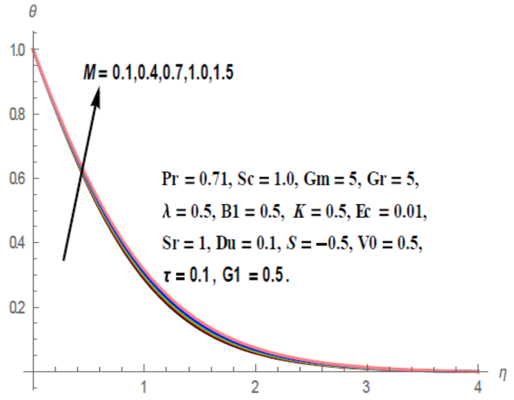

Fig10: Effect of M on temperature profile in the paper[1]

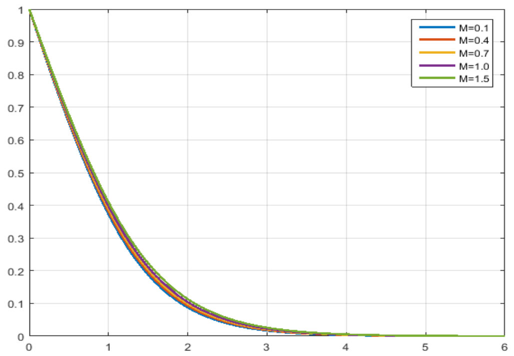

Fig11: Effect of M on concentration profile in the paper[1]

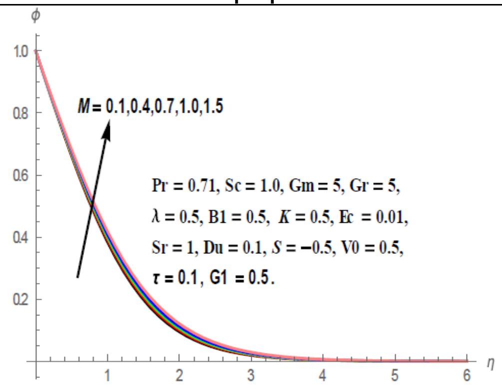

Fig12: Effect of M on concentration profile in the paper[1]

4. Conclusion

In the current paper we propose a Galerkin method with different orders of B-splines to solve coupled system of nonlinear boundary value problems. The proposed method is stable and gives accurate answers for the unknown variables in the considered problem with less number of computations and within shorter time than any other methods that are in use. We illustrated the method proposed by solving a problem which is available in literature. The results obtained are presented in graphical representation and they are in good agreement with the required answers.

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