Anti Identity Matrix- J and its Properties

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Abstract: In this article we have a new known matrix named Anti Identity matrix denoted by J , derive its properties and the role of Anti Identity Matrix

Keywords: Identity matrix, Characteristic Equation, Gauss Jordan Method, Normal Form, Echolon Form

I. Introduction

Definition 1.1: A square matrix J of order nXn is said to be Anti Identity matrix if J represented

as J_{nXn} = $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \iddots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

The Anti-Identity matrix of the different order for n=2, 3, 4, 5,. , .are given below.

J_{2X2} = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, J_{3X3} = $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, J_{4X4} = $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$..

II.Methodology

In this section we are going to derive the following

1. Properties of Anti Identity matrix

2.1. Properties of Anti Identity matrix

In this section we have derive properties of anti Identity matrix

Property 2.1.1: If $J_{2X2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is an Anti -Identity matrix,

$LetA = \left\{ \begin{bmatrix} a & b \\ c & b \end{bmatrix} / a = a, b = a+1, c = a+2, b = a+1, \forall a \in N \right\}$ and an equation of the form

$|A - \lambda J| = -\lambda^2 + \lambda(b+c) - 2b$

If we replace λ with matrix A is satisfied then

$$-A^2 + A(b+c) - 2bI = 2\begin{bmatrix} -c & +b \\ +c & -b \end{bmatrix}$$

Proof: Given that

$$Let A = \left\{ \begin{bmatrix} a & b \\ c & b \end{bmatrix} / a = a, b = a+1, c = a+2, b = a+1, \forall a \in N \right\}$$ and an equation of the form

$$|A - \lambda J| = -\lambda^2 + \lambda(b+c) - 2b$$

If we replace λ with matrix A is satisfied then

$$-A^2 + A(b+c) - 2bI = 2\begin{bmatrix} -c & +b \\ +c & -b \end{bmatrix}$$

---- (1)

For example (1)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

$$|A - \lambda J| = \left| \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right| = \begin{vmatrix} 1 & 2-\lambda \\ 3-\lambda & 2 \end{vmatrix} = -\lambda^2 + \lambda(2+3) - 2(2) = -\lambda^2 + 5\lambda - 4$$

If we replace λ with matrix A is satisfied then

$$-A^2 + 5A - 4I = -\begin{bmatrix} 7 & 6 \\ 9 & 10 \end{bmatrix} + 5\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -7+5-4 & -6+10-0 \\ -9+15-0 & -10+10-4 \end{bmatrix} = \begin{bmatrix} -6 & +4 \\ +6 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & +4 \\ +6 & -4 \end{bmatrix} = 2\begin{bmatrix} -3 & +2 \\ +3 & -2 \end{bmatrix}$$

For example (2)

$$A = \begin{bmatrix} 100 & 101 \\ 102 & 101 \end{bmatrix}$$

$$|A - \lambda J| = \left| \begin{bmatrix} 100 & 101 \\ 102 & 101 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right| = \begin{vmatrix} 100 & 101-\lambda \\ 102-\lambda & 101 \end{vmatrix}$$

$$= -\lambda^2 + \lambda(101+102) - 2(101) = -\lambda^2 + 203\lambda - 202$$

If we replace λ with matrix A is satisfied then

$$-A^2 + 5A - 4I = -\begin{bmatrix} 20302 & 20301 \\ 20502 & 20503 \end{bmatrix} + 203\begin{bmatrix} 100 & 101 \\ 102 & 101 \end{bmatrix} - 202\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -20302+20300-202 & -20301+20503-0 \\ -20502+20706-0 & -20503+20503-202 \end{bmatrix}$$

$$= \begin{bmatrix} -204 & +202 \\ +204 & -202 \end{bmatrix} = 2\begin{bmatrix} -102 & +101 \\ +102 & -101 \end{bmatrix}$$

Therefore

$IfA = \left\{ \begin{bmatrix} a & b \\ c & b \end{bmatrix} / \, a = a, b = a+1, c = a+2, b = a+1, \forall a \in N \right\}$ and an equation of the form

$|A - \lambda J| = -\lambda^2 + \lambda(b+c) - 2b$

If the matrix A is satisfied by A then

$$-A^2 + A(b+c) - 2bI = 2\begin{bmatrix} -c & +b \\ +c & -b \end{bmatrix}$$

Property 2.1.2: If J_{nXn} is an Anti -Identity matrix of order nXn , and let A_{nXn} be any square matrix of order n then

$J^{2m} = I_{nXn} \; \forall m \in N,$ Where I_{nXn} is an Identity matrix of order nXn

Proof: By using mathematical induction

Case (i) For n=2 and m=1,2,…k,k+1

Let J_{2X2} is an Anti -Identity matrix of order 2X2 **,**

I,e,. $J_{2X2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

For m=1

$$J^{2.1} = J^2 = J.J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For m=2

$$J^{2.2} = J^4 = J^2.J^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

For m=k ,Assume it is true true

$$J^{2.k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For m=k+1

$$J^{2.(k+1)} = J^{2k+2} = J^{2k}.J^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

For n=2 the theorem is true for every m

$\therefore \; J^{2m} = I_{nXn} \; \forall m \in N,$ Where I_{nXn} is an Identity matrix of order nXn

Case (ii) For n=3 and m=1,2,…k,k+1

Let J_{3X3} is an Anti -Identity matrix of order 3X3**,**

I,e,. $J_{3X3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

For m=1

$$J^{2.1} = J^2 = J.J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For m=2

$$J^{2.2} = J^4 = J^2.J^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For m=k ,Assume it is true true

$$J^{2.k} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For m=k+1

$$J^{2.(k+1)} = J^{2k+2} = J^{2k}.J^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For n=3 the theorem is true for every m

$\therefore\ J^{2m} = I_{nXn}\ \forall m \in N,$ Where I_{nXn} is an Identity matrix of order nXn

Case (iii) For n=4 and m=1,2,…k,k+1

Let J_{4X4} is an Anti -Identity matrix of order 4X4**,**

Let $J_{4X4} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

For m=1

$$J^{2.1} = J^2 = J.J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For m=2

$$J^{2.2} = J^4 = J^2.J^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For m=k ,Assume it is true true

$$J^{2.k} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For m=k+1

$$J^{2.(k+1)} = J^{2k+2} = J^{2k}.J^{2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For n=4 the theorem is true for every m

$\therefore\ J^{2m} = I_{nXn}\ \forall m \in N$, Where I_{nXn} is an Identity matrix of order nXn

By Mathematical induction

For each n belongs to N the theorem is true for every m belongs to N

$\therefore\ J^{2m} = I_{nXn}\ \forall m \in N$, Where I_{nXn} is an Identity matrix of order nXn

Property 2.1.3: If J_{nXn} is an Anti -Identity matrix of order nXn, and I_{nXn} is an Identity matrix of order nXn then

i)

$|J - \lambda J| = -|I - \lambda I|$,[for n=2,3,6,7,10,11,…]

n= (2 + (-1)^n + (-1)^(n+1))*n - (1 + (-1)^n)/2, n >= 1

[Paolo P. Lava[1]]

$|J - \lambda J| = |I - \lambda I|$, [for n=4, 5, 8, 9, 12, 13,..]

n=[(2 + (-1)^n + (-1)^(n+1))*n - (1 + (-1)^n)/2]+2,n >= 1

ii) The Roots of the equations and $|J - \lambda J| = 0$ are same in any order $|I - \lambda I| = 0$ are λ=1 corresponding to the order of the matrix.

i.e, if order of matrix n=2 means λ=1,1

i.e, if order of matrix n=3 means λ=1,1,1

and respectively.

Proof:

For n=2

Let J_{2X2} is an Anti -Identity matrix of order 2X2 **,**

I,e,. $J_{2X2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

And let $I_{2X2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\left|J-\lambda J\right| = \left\|\left[\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right]-\lambda\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right\| = \left\|\left[\begin{bmatrix}0 & 1-\lambda\\1-\lambda & 0\end{bmatrix}\right]\right\| = -\lambda^2+2\lambda-1$$

$$\left|I-\lambda I\right| = \left\|\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right\| = \left\|\begin{bmatrix}1-\lambda & 0\\0 & 1-\lambda\end{bmatrix}\right\| = \lambda^2-2\lambda+1$$

$$\therefore \left|J-\lambda J\right| = -\left|I-\lambda I\right|$$

The roots of the equations

$$\lambda^2-2\lambda+1=-\lambda^2+2\lambda-1=0$$

are λ=1,1

For n=3

Let $J_{3X3} = \begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix}$

$$\left|J-\lambda J\right| = \left\|\begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix}-\lambda\begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix}\right\| = \left\|\begin{bmatrix}0 & 0 & 1-\lambda\\0 & 1-\lambda & 0\\1-\lambda & 0 & 0\end{bmatrix}\right\| = \lambda^3-3\lambda^2+3\lambda-1$$

And let $I_{3X3} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

$$\left|I-\lambda I\right| = \left\|\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}-\lambda\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right\| = \left\|\begin{bmatrix}1-\lambda & 0 & 0\\0 & 1-\lambda & 0\\0 & 0 & 1-\lambda\end{bmatrix}\right\| = -\lambda^3+3\lambda^2-3\lambda+1$$

$$=-\lambda^2+3\lambda^2-3\lambda+1=-[\lambda^2-3\lambda^2+3\lambda-1]=\left|J-\lambda J\right|$$

$$\therefore \left|J-\lambda J\right| = -\left|I-\lambda I\right|$$

The roots of the equations

$$-\lambda^3+3\lambda^2-3\lambda+1=\lambda^3-3\lambda^2+3\lambda-1=0$$

are λ=1,1,1

$$\therefore \left|J-\lambda J\right| = \left|I-\lambda I\right|$$

The roots of the equations

$$-\lambda^5+5\lambda^4-10\lambda^3+10\lambda^2-5\lambda+1=0$$

are λ=1,1,1 ,1and 1

Therefore we can prove that

$\left|J-\lambda J\right| = -\left|I-\lambda I\right|$,[for n=2,3,6,7,10,11,…]

n= (2 + (-1)^n + (-1)^(n+1))*n - (1 + (-1)^n)/2, n >= 1

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$\left| J - \lambda J \right| = \left| I - \lambda I \right|$, [for n=4, 5, 8, 9, 12, 13…]

n=[(2 + (-1)^n + (-1)^(n+1))*n - (1 + (-1)^n)/2]+2,n >= 1

III.Conclusion:

We have obtained properties of J matrix and derived its properties and found J matrix .Further theorems and its applications may be obtained

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