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Solving bi-level linear fractional programming problem with interval coefficients

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Abstract In this paper, a method is developed to determine a compromise solution of bi-level linear fractional programming problem(BL-LFPP) which comprises intervals as coefficients of decision variables and constants involved both in the objective and constraint functions. Non-dominated solution at each level is separately determined by transforming fractional objectives into interval valued linear functions using variable transformation and Taylor series approximation. Goal programming method ignoring the over deviational variables for maximization goals, is applied to determine the compromise solution and range of optimal objective values. A numerical example is illustrated to demonstrate the proposed method.

Keywords Bi-level optimization · Linear fractional programming · Non-dominated solution · Interval coefficients

1 Introduction

In hierarchical organizations, bi-level programming is considered as an important problem which comprises two sequential optimization problems as upper and lower level. Upper and lower level decision makers (ULDM and LLDM) control set of decision variables independently. As decisions of both levels get

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influenced by each other, preference is given to each decision in a cooperative environment. Linear fractional programming(LFP)[11] comprises the objective function as fraction of affine (linear plus constant) functions which has wide range of applications in science, engineering, economics, finance, business, management, production, information theory and many more. Some common instances of fractional objectives are profit/cost, cost/volume, inventory/sale, debt/equity, output/employee and so on. Mishra [7] developed a method to solve BL-LFPP using analytic hierarchy process and weighting sum method. E-mam [5] proposed interactive approach to solve bi-level integer multi-objective FPP. Baky [2] used fuzzy goal programming to solve a decentralized bi-level problem. A bi-level non-linear decision making problem under fuzziness was studied by Abo-Sinna [1] whereas Baky and Abo-Sinna [3] solved a bi-level problem with multiple non-linear objectives at each level. Sakawa et al.[10] developed a fuzzy approach to solve BL-LFPP with fuzzy parameters.
This paper studies BL-LFPP with interval coefficients considering the situation that DM does not have always precise information of all the required data while formulating practical problems into suitable mathematical models. In such cases, use of a range of values in form of intervals are more appropriate instead of fixed values. The proposed method derives a compromise solution and a range of optimal objective values of the problem.
The organization of the paper is as follows: Sec.2 interprets arithmetic operations on intervals and variable transformation method is described in sec.3. Sec.4 formulates BL-LFPP with interval coefficients. The proposed solution approach is incorporated in sec.5 to determine the compromise solution. A numerical example is discussed in sec.6 and finally sec.7 includes some conclusions.

2 Preliminaries

2.1 Arithmetic operations on intervals

Consider $p, q \in I \subset \mathbb{R}$ where I denotes the set of closed intervals in the real line. If $p = [p^L, p^U]$ and $q = [q^L, q^U]$, the arithmetic operations on the intervals can be found in [8,9]. As 'I' is not totally ordered, $\preccurlyeq$ is defined as a partial ordering relation on it. $p \preccurlyeq q$ i.e., p is inferior to q or q is superior to p iff $p^L \leq q^L$ and $p^U \leq q^U$. $p \prec q$ if one of the following condition holds [12].
$\{p^L < q^L, p^U < q^U\}$ or $\{p^L < q^L, p^U \leq q^U\}$ or $\{p^L \leq q^L, p^U < q^U\}$

2.2 Variable transformation method

Charnes and Cooper [4] developed variable transformation method to determine the optimal solution of a LFPP using an extra variable and an additional constraint. Consider the following two optimization models M_1 and M_2.

(M_1): max $f(x) = \dfrac{cx+\alpha}{dx+\beta}$
subject to
$\Omega_1 = \{Ax \leq b, x \geq 0\}$

(M_2): max $g(y,z) = cy + \alpha t$
subject to
$\Omega_2 = \{dy + \beta t = 1, Ay - bt \leq 0, \quad y \geq 0, t > 0\}$

where, the transformations $t = \frac{1}{dx+\beta}$ and $y = xt$ derive (M_2) from (M_1).

Lemma 1 *[11]: For any feasible solution $(y,t) \in \Omega_2$, t is positive.*

Theorem 1 *[11]: If (y^*, t^*) is an optimal solution of (M_2) then $x^* = \frac{y^*}{t^*}$ is the optimal solution of (M_1).*

3 Problem formulation

The mathematical formulation of the bi-level LFPP (BL-LFPP) with interval coefficients can be defined in a co-operative environment as follows.

Upper level (DM_1) : $\max\limits_{X_1} \; f_1(x) = \dfrac{c_1x+\alpha_1}{d_1x+\beta_1}$

Lower level (DM_2) : $\max\limits_{X_2} \; f_2(x) = \dfrac{c_2x+\alpha_2}{d_2x+\beta_2}$

subject to

$x \in \Omega = \{A_1X_1 + A_2X_2 \leq \; b; X_1, X_2 \geq 0\}$

where, DM_i controls the set of decision variables $X_i, (i = 1,2)$ independently.
$X_i \in \mathbb{R}^{n_i}$ and $x = (X_1, X_2) = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \geq 0$, $n = n_1 + n_2$.
$A_i = (a_{kj}) \in \mathbb{R}^{m \times n_i}$, $i = 1,2$ and $b = (b_k) \in \mathbb{R}^m$, $k = 1,2,...,m$.
$c_i = (c_{ij})$, $d_i = (d_{ij}) \in \mathbb{R}^n$ and $\alpha_i, \beta_i \in \mathbb{R}$ for $i = 1,2; j = 1,2,...,n$.
Assume that $c_{ij}, d_{ij}, \alpha_i, \beta_i \in I^+$, $a_{kj}, b_j \in I$ and $d_ix + \beta_i > 0 \; \forall \; x \in \Omega$ such that, $c_{ij} = [c_{ij}^L, c_{ij}^U], d_{ij} = [d_{ij}^L, d_{ij}^U], \alpha_i = [\alpha_i^L, \alpha_i^U], \beta_i = [\beta_i^L, \beta_i^U], a_{kj} = [a_{kj}^L, a_{kj}^U], b_j = [b_j^L, b_j^U]$.

4 Proposed method of solution

The objective functions at upper and lower level of the BL-LFPP can be stated as follows:

$$f_i(x) = \frac{\sum\limits_{j=1}^{n} [c_{ij}^L, c_{ij}^U]x_j + [\alpha_i^L, \alpha_i^U]}{\sum\limits_{j=1}^{n} [d_{ij}^L, d_{ij}^U]x_j + [\beta_i^L, \beta_i^U]}, \quad i = 1,2$$

Using arithmetic operations on intervals, the objectives at upper and lower level can be transformed into interval valued functions as:

$$f_i(x) = \left[\frac{\sum\limits_{j=1}^{n} c_{ij}^L x_j + \alpha_i^L}{\sum\limits_{j=1}^{n} d_{ij}^U x_j + \beta_i^U}, \frac{\sum\limits_{j=1}^{n} c_{ij}^U x_j + \alpha_i^U}{\sum\limits_{j=1}^{n} d_{ij}^L x_j + \beta_i^L} \right] = [f_i^L(x), f_i^U(x)],\ i = 1, 2$$

The constraints of the problem BL-LFPP can be stated in the following form:
$\sum\limits_{j=1}^{n} [a_{kj}^L, a_{kj}^U] x_j \preccurlyeq [b_k^L, b_k^U],\ \ k = 1, 2, ..., m$
It can be further simplified as follows.
$\sum\limits_{j=1}^{n} a_{kj}^L x_j \leq b_k^L,\ \ \sum\limits_{j=1}^{n} a_{kj}^U x_j \leq b_k^U,\ \ k = 1, 2, ..., m$

As the BL-LFPP is considered of maximization type, the individual maximal solutions of the lower bounds $f_i^L(x)$ and upper bounds $f_i^U(x)$ of the interval valued functions $f_i(x)$ are derived at both levels. The mathematical formulation to determine the maximal solution of $f_i^L(x)$ using variable transformation method can be stated as follows.

$$\max \sum_{j=1}^{n} c_{ij}^L y_j + t\alpha_i^L \tag{1}$$

$$\text{subject to}$$
$$\sum_{j=1}^{n} d_{ij}^U y_j + t\beta_i^U = 1,\ \ \sum_{j=1}^{n} a_{kj}^L y_j \leq t b_k^L,\ \sum_{j=1}^{n} a_{kj}^U y_j \leq t b_k^U,\ \ k = 1, 2, ..., m$$
$$y_j \geq 0, t > 0$$

Using Theorem-1, the individual maximal solution of $f_i^L(x)$ can be obtained by solving problem (1). Similarly, mathematical formulation for $f_i^U(x)$ can be modeled to determine its maximal solution. Let $x_i^{L*} = (x_{ik}^{L*}, k = 1, 2, ..., n)$ and $x_i^{U*} = (x_{ik}^{U*}, k = 1, 2, ..., n)$ be the individual maximal solutions obtained for $f_i^L(x)$ and $f_i^U(x)$ respectively. The fractional lower and upper bound functions $f_i^L(x)$ and $f_i^U(x)$ at both levels $(i = 1, 2)$ can be approximated by linear functions using their taylor series expansion about own individual maximal solutions.
These approximations can be formulated as follows.

$$f_i^L(x) \approx \tilde{f}_i^L(x) = f_i^L(x_i^{L*}) + \sum_{k=1}^{n} (x_k - x_{ik}^{L*}) \frac{\partial f_i^L(x_i^{L*})}{\partial x_k},\ i = 1, 2$$

$$f_i^U(x) \approx \tilde{f}_i^U(x) = f_i^U(x_i^{U*}) + \sum_{k=1}^{n} (x_k - x_{ik}^{U*}) \frac{\partial f_i^U(x_i^{U*})}{\partial x_k},\ i = 1, 2$$

The objective functions at upper and lower levels of the BL-LFPP can be stated as follows.

$\tilde{f}_i(x) = [\tilde{f}_i^L(x), \tilde{f}_i^U(x)],\ i = 1, 2$

Consider the following two problems where $f : \mathbb{R}^n \to I$ and $g : \mathbb{R}^n \to \mathbb{R}$.

$$(P_1): \max\ f(x) = [f^L(x), f^U(x)] \quad \text{subject to} \ \ x \in \Omega_1 \qquad (P_2): \max\ g(x) = f^L(x) + f^U(x) \quad \text{subject to} \ \ x \in \Omega_1$$

Definition 1 [12]: x^* is a non-dominated solution of the problem : $\max f(x) = [f^L(x), f^U(x)]$ subject to $x \in \Omega_1$ if there exists no $\hat{x} \in \Omega_1$ such that $f(x^*) \prec f(\hat{x})$.

Theorem 2 *[12]: If x^* is an optimal solution of (P_2) then x^* is a non-dominated solution of (P_1).*

By Theorem-2, DM$_i$ solve the following problems separately for $(i = 1, 2)$ to determine the non-dominated solutions of level-i.

$$\max\ \tilde{f}_i^L(x) + \tilde{f}_i^U(x) \tag{2}$$

$$\text{subject to}$$
$$\sum_{j=1}^{n} a_{kj}^L x_j \leq b_k^L, \ \ \sum_{j=1}^{n} a_{kj}^U x_j \leq b_k^U, \ x_j \geq 0, \ \ k = 1, 2, ..., m$$

Let x^{1*} and x^{2*} be the non-dominated solutions of upper and lower level problems respectively. As ULDM controls the set of decision variables X_1, the aspiration values X_1^* of X_1 are determined from the corresponding coordinates of the non-dominated solution x^{1*}. The aspiration values of $\tilde{f}_i^L(x)$ and $\tilde{f}_i^U(x)$ are obtained as:

At level-1: $\tilde{f}_1^L(x^{1*}) = \tilde{f}_1^{L*}$ and $\tilde{f}_1^U(x^{1*}) = \tilde{f}_1^{U*}$

At level-2: $\tilde{f}_2^L(x^{2*}) = \tilde{f}_2^{L*}$ and $\tilde{f}_2^U(x^{2*}) = \tilde{f}_2^{U*}$

Finally, the compromise solution of the BL-LFPP with interval coefficients can be obtained on solving the following problem which is formulated due to goal programming method [6] as follows.

$$\min\ \sum_{i=1}^{2} d_{iL}^- + \sum_{i=1}^{2} d_{iL}^+ + \sum_{i=1}^{2} d_{iU}^- + \sum_{i=1}^{2} d_{iU}^+ + (e^- + e^+) \tag{3}$$

$$\text{subject to}$$
$$\tilde{f}_i^L(x) + d_{iL}^- - d_{iL}^+ = \tilde{f}_i^{L*}, \ \tilde{f}_i^U(x) + d_{iU}^- - d_{iU}^+ = \tilde{f}_i^{U*}, \ i = 1, 2$$
$$X_1 + e^- - e^+ = X_1^*$$
$$\sum_{j=1}^{n} a_{kj}^L x_j \leq b_k^L, \ \ \sum_{j=1}^{n} a_{kj}^U x_j \leq b_k^U, \ \ k = 1, 2, ..., m$$
$$x_j \geq 0, \ d_{iL}^-, d_{iL}^+, \ d_{iU}^-, d_{iU}^+, \ e^-, e^+ \geq 0$$
$$d_{iL}^-.d_{iL}^+ = 0, \ d_{iU}^-.d_{iU}^+ = 0, \ e^-.e^+ = 0$$

Since over deviation from the aspiration level shows the state of complete achievement, the over deviational variables d_{iL}^+ and d_{iU}^+ can be ignored for the objective functions of maximization types. Therefore, the above formulation (3) can be simplified as follows:

$$\min\ \sum_{i=1}^{2} d_{iL}^- + \sum_{i=1}^{2} d_{iU}^- + (e^- + e^+) \tag{4}$$

$$\begin{gathered}\text{subject to}\\ \tilde{f}_i^L(x)+d_{iL}^- \geq \tilde{f}_i^{L*},\ \tilde{f}_i^U(x)+d_{iU}^- \geq \tilde{f}_i^{U*},\ i=1,2\\ X_1+e^- - e^+ = X_1^*\\ \sum_{j=1}^n a_{kj}^L x_j \leq b_k^L,\ \sum_{j=1}^n a_{kj}^U x_j \leq b_k^U,\ \ k=1,2,...,m\\ x_j \geq 0,\ d_{iL}^-, d_{iU}^-,\ e^-, e^+ \geq 0,\ e^-.e^+ = 0\end{gathered}$$

5 Numerical example

To illustrate the solution procedure, consider the following bi-level LFPP in a co-operative environment.

Upper level: $\max\limits_{x_1}\ f_1(x)=\dfrac{[2,3]x_1+[5,7]x_2+[1,2]x_3+[1,2]}{[3,5]x_1+[2,6]x_2+[2,3]x_3+[2,4]}$

Lower level: $\max\limits_{x_2,x_3}\ f_2(x)=\dfrac{[2,5]x_1+[4,7]x_2+[3,5]x_3+[3,4]}{[1,2]x_1+[3,5]x_2+[5,7]x_3+[4,5]}$

subject to

$[-1,1]x_1+[1,1]x_2+[-1,1]x_3 \preccurlyeq [-1,5]$

$[-2,3]x_1+[-1,-1]x_2+[1,2]x_3 \preccurlyeq [-1,7],\ \ x_1,x_2,x_3 \geq 0$

Using arithmetic operations on intervals, the objective at both levels are formulated as follows.

Upper level: $\max\limits_{x_1}\ f_1(x)=\left[\dfrac{2x_1+5x_2+x_3+1}{5x_1+6x_2+3x_3+4},\dfrac{3x_1+7x_2+2x_3+2}{3x_1+2x_2+2x_3+2}\right]$

Lower level: $\max\limits_{x_2,x_3}\ f_2(x)=\left[\dfrac{2x_1+4x_2+3x_3+3}{2x_1+5x_2+7x_3+5},\dfrac{5x_1+7x_2+5x_3+4}{x_1+3x_2+5x_3+4}\right]$

subject to

$$\Omega=\begin{cases}x_1+x_2+x_3\leq 5,\ x_1-x_2+x_3\geq 1,\\ 3x_1-x_2+2x_3\leq 7,\ 2x_1+x_2-x_3\geq 1,\ x_1,x_2,x_3\geq 0\end{cases}$$

According to the proposed method, the objectives are approximated by linear functions as $f_i(x)=[\tilde{f}_i^L(x),\tilde{f}_i^U(x)]$ at level $i=1,2$.
At upper level ($i=1$):
$\tilde{f}_1^L(x)=-0.0298x_1+0.0630x_2-0.0255x_3+0.5104$
$\tilde{f}_1^U(x)=-0.1870x_1+0.2701x_2-0.1246x_3+1.6647$
At lower level ($i=2$):
$\tilde{f}_2^L(x)=0.0181x_1-0.0023x_2-0.127x_3+0.7598$
$\tilde{f}_2^U(x)=0.1893x_1-0.0473x_2-0.5917x_3+2.0652$

The non-dominated solution at upper level problem is obtained as $x^{1*}=(0.6667,2,2.3333)$ where $\tilde{f}_1^{L*}=0.5570$ and $\tilde{f}_1^{U*}=1.7895$. The aspiration value for x_1 controlled by ULDM is ascertained as $x_1^*=0.6667$. The non-dominated solution at lower level problem is obtained as $x^{2*}=(3,2,0)$ where

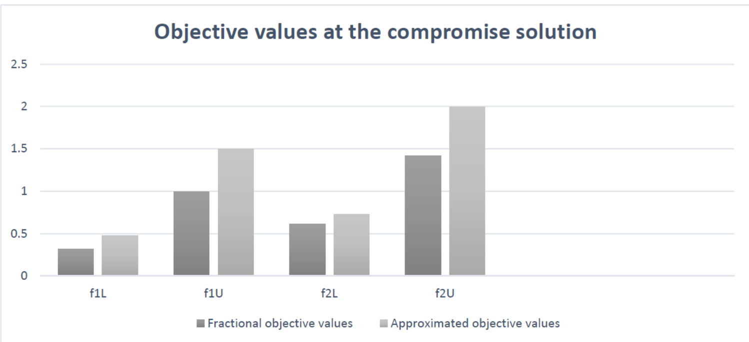
$\tilde{f}_2^{L*} = 0.8095$ and $\tilde{f}_2^{U*} = 2.5385$.Using goal programming method, the problem is formulated as follows.

$$
\begin{gathered}
\min \; \sum_{i=1}^{2} d_{iL}^{-} + \sum_{i=1}^{2} d_{iU}^{-} + (e^{-} + e^{+}) \\
\text{subject to} \\
(-0.0298x_1 + 0.0630x_2 - 0.0255x_3 + 0.5104) + d_{1L}^{-} \geq 0.5570 \\
(0.0181x_1 - 0.0023x_2 - 0.127x_3 + 0.7598) + d_{2L}^{-} \geq 0.8095 \\
(-0.1870x_1 + 0.2701x_2 - 0.1246x_3 + 1.6647) + d_{1U}^{-} \geq 1.7895 \\
(0.1893x_1 - 0.0473x_2 - 0.5917x_3 + 2.0652) + d_{2U}^{-} \geq 2.5385 \\
x_1 + e^{-} - e^{+} = 0.6667 \\
x_1 + x_2 + x_3 \leq 5, \;\; x_1 - x_2 + x_3 \geq 1, \;\; 3x_1 - x_2 + 2x_3 \leq 7, \;\; 2x_1 + x_2 - x_3 \geq 1 \\
x_1, x_2, x_3 \geq 0 \\
x_1, x_2, x_3 \geq 0, \; d_{1L}^{-}, d_{2L}^{-}, d_{1U}^{-}, d_{2U}^{-}, \; e^{-}, e^{+} \geq 0, \; e^{-}.e^{+} = 0
\end{gathered}
$$

On solving the above problem, the compromise solution is obtained as $x^* = (0.6667, 0, 0.3333)$ where the values of the lower and upper bound functions of the fractional and approximated linear objectives are shown in the following table and bar graph.

Table 1 Objective values at the compromise solution

$Level : i$	$f_i^L(x^*)$	$\tilde{f}_i^L(x^*)$	$f_i^U(x^*)$	$\tilde{f}_i^U(x^*)$
$Level : i = 1$	0.3200	0.4820	1	1.4985
$Level : i = 2$	0.6154	0.7295	1.4211	1.9942



The values of the fractional objectives $[f_i^L(x^*), f_i^U(x^*)], i = 1, 2$ are considered as the range of optimal objective values.

6 Conclusion

In hierarchical organizations, if DMs have no precise information about the resources then intervals are more appropriate to be considered instead of fixed values. In this view, this paper develops a method to solve a BL-LFPP with interval coefficients and optimal range of objective values are determined for the objectives of both levels as the solution. This method is also useful for BL-LFPP with fuzzy coefficients (linear, triangular, trapezoidal etc.) as fuzzy numbers can be transformed into interval forms using fuzzy α-cuts. Numerical example is discussed to illustrate the solution procedure.

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