Parameter-uniform hybrid numerical scheme for singularly perturbed initial value problem

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This paper deals with a singularly perturbed initial value problem which depends on a parameter. A hybrid scheme which is a combination of a second order cubic spline scheme on a fine mesh and a midpoint upwind scheme on a coarse mesh is constructed. It is shown that the proposed method is second order convergent in the discrete maximum norm. Error bounds for the numerical solution and its numerical derivative are established. A numerical example is presented which support the theoretical results.

Key words: Singular perturbation, Hybrid scheme, Shishkin mesh, Initial value problem.
Subject Classification: AMS 65L12, 65L10.

1. Introduction

Consider the following singularly perturbed initial value problem (IVP):

$$\begin{cases} Ly(x) = \varepsilon^2 y''(x) + \varepsilon p(x) y'(x) + q(x) y(x) = g(x), x \in \Omega = (0,T] \\ \qquad y(0) = \lambda, L_0 y(0) = y'(0) = \dfrac{\eta}{\varepsilon}, \end{cases} \tag{1.1}$$

where $0 < \varepsilon << 1$, is a small positive parameter, λ and η are given constant. $p(x), q(x)$ and $g(x)$ are smooth function in $(0,T]$, with $0 < \alpha \le p(x), 0 < \beta \le q(x) \le \beta^*$. Taking the above assumptions into consideration, the solution $y(x)$ has an exponential boundary layer near $x = 0$. This type of model problem found in quantum mechanics, fluid dynamics and other applied areas [2,4,5,10].

Amiraliyev et al. [1] proposed a fitted finite difference approximation on uniform mesh for IVP (1.1) which was first order convergent. Cen et al. [3] proposed a hybrid scheme which combines the central difference scheme on the fine part and the midpoint upwind scheme on the coarse part for IVP (1.1) on piecewise uniform Shishkin mesh. The difference equation has been solved for errors using Gronwall's inequality [11], they got almost second order convergence for the numerical solution and the scaled numerical derivatives.

The main goal is to construct a parameter-uniform numerical scheme for solving IVP (1.1) and to approximate the solution and its derivatives. Motivated from [3], a fitted mesh method has been developed to solve the IVP (1.1). Taking different properties of the exact solution in to account, different types of layer-adapted meshes like Shishkin mesh (S mesh) and Bakhvalov-Shishkin mesh (B-S mesh) are constructed. Afterwards, we state a hybrid finite difference approximations which combines the cubic spline difference approximations on the inner mesh region with the midpoint upwind approximation on outer mesh region [6]. Since estimation for numerical derivatives are desirable in many applied field [7,8,12], we give error bound for the numerical derivatives. Here, C denotes a generic positive constant independent of ε and the mesh parameter N. For any continuous function, the maximum norm is denoted by $\|f\|_{\overline{\Omega}} = \sup_{x\in\overline{\Omega}}|f(x)|$.

2. Properties of the solution and its derivatives

Lemma 2.1 The solution $y(x)$ and its derivatives of IVP(1.1) satisfies the following bound:

$$\left\|y^l(x)\right\| \le C\varepsilon^{-l}\left\{\phi + \max_{0\le t\le x}|g(t)| + \varepsilon^j \max_{0\le t\le x}|g(t)| + \int_0^x |g'(t)|dt\right\}, \tag{2.1}$$

for all $x\in\overline{\Omega}$, $0\le l<6$, where $\phi = \sqrt{\left|\eta^2 + q(0)\lambda^2 - 2g(0)\right|}$, $j=\max\{0, l-2\}$.

Proof. The idea of the proof is given in [1].

We decompose the solution $u(x)$ of the IVP (1.1) into regular and singular components as: $y(x)=u(x)+v(x)$. The regular component $u(x)$ is satisfy the following sets of problem:

$$\begin{cases} Lu(x) = g(x) \\ u(0) = u_0(0) + \varepsilon u_1(0) + \varepsilon^2 u_2(0) + +\varepsilon^3 u_3(0) \\ u'(0) = u'_0(0) + \varepsilon u'_1(0) + \varepsilon^2 u'_2(0) + \varepsilon^3 u'_3(0) \end{cases} \tag{2.2}$$

Here u_0, u_1, u_2, u_3 respectively, the solution of the following problems:

$$\begin{cases} u_0(x) = \dfrac{g(x)}{q(x)}, \\ u_1(x) = \dfrac{1}{q(x)}\{\varepsilon u_0'' - p(x)u_0'(x)\}, \\ u_2(x) = \dfrac{1}{q(x)}\{\varepsilon u_1'' - p(x)u_1'(x)\}, \\ u_3(x) = \dfrac{1}{q(x)}\{\varepsilon u_2'' - p(x)u_2'(x)\}. \end{cases} \tag{2.3}$$

The singular components $v(x)$ is satisfy the following problem:

$$\begin{cases} Lv(x) = 0, \\ v(0) = \lambda - u(0), \\ v'(0) = \dfrac{\eta}{\varepsilon} - u'(0). \end{cases} \tag{2.4}$$

Lemma 2.2 The regular components $u(x)$ and its derivatives satisfy the following:

$$\left\|u^l(x)\right\| \le C,\ 0\le l\le 4. \tag{2.5}$$

Proof. One can find a similar kind of proof in [3].

Lemma 2.3 The singular components $v(x)$ and its derivatives have the following estimates:

$$\left\|v^{l}(x)\right\| \leq C(\varepsilon^{3-l}+\varepsilon^{-l}e^{-mx/\varepsilon}+\varepsilon^{1-l}e^{-mx/2\varepsilon}),\ 0\leq l\leq 5, \tag{2.6}$$

where $$m=\begin{cases}(p(0)-\sqrt{(p^{2}(0)-4q(0))})/2, & p^{2}(0)>4q(0),\\ p(0)/4, & p^{2}(0)=4q(0),\\ p(0)/2, & p^{2}(0)<4q(0).\end{cases} \tag{2.7}$$

Proof. The idea of the proof is given in [3].

2.1 Cubic spline approximation

Consider the cubic spline approximation on a variable mesh $\Omega^{N}=\{0=x_0<x_1<.....x_N=1\}$, and let $h_i=x_i-x_{i-1}$. For given value $Y(x_0),Y(x_1),.....Y(x_N)$ of a function $y(x)$ on Ω^{N}, there exists a cubic spline function $R(x)$ with the following properties:

(i) $R(x)$ coincides with a polynomial of degree three on each subintervals $[x_i-x_{i-1}]$ $i=1,......N$.

(ii) $R(x)\in C^{2}(\overline{\Omega})$

(iii) $R(x_i)=Y(x_i)$ for $i=1,......N$.

Now for $x\in[x_i-x_{i-1}]$, the cubic spline function is define as follows:

$$R(x)=\frac{(x_i-x)^3}{6h_i}S_{i-1}+\frac{(x-x_{i-1})^3}{6h_i}S_i+\left(Y(x_{i-1})-\frac{h_i^2}{6}S_{i-1}\right)\left(\frac{(x_i-x)}{6h_i}\right)+\left(Y(x_i)-\frac{h_i^2}{6}S_i\right)\left(\frac{(x-x_{i-1})}{6h_i}\right).$$

where $S_i=R''(x_i)$. Now from the basic properties of the spline , it should satisfy the following

$$:\frac{h_i}{6}S_{i-1}+\frac{h_i+h_{i+1}}{6}S_i+\frac{h_{i+1}}{6}S_{i+1}=\frac{Y(x_{i+1})-Y(x_i)}{h_{i+1}}-\frac{Y(x_i)-Y(x_{i-1})}{h_i}. \tag{2.8}$$

For obtaining second order approximation of the first order derivative of $y(x)$ one can refer [9]. Now for the IVP (1.1) consider the approximation:

$$\varepsilon^{2}S_j+p(x_j)Y'(x_j)+q(x_j)Y(x_j)=g(x_j),\ \text{for}\ j=i,i\pm1. \tag{2.9}$$

Substituting this in (2.8), we have the following linear systems, for $i=1,......N-1$,

$$L_{Cu}Y_i=r^{-}Y(x_{i-1})+r^{c}Y(x_i)+r^{+}Y(x_{i+1})=G(x_i), \tag{2.10}$$

where $$\begin{cases} r^{-}=\dfrac{h_i}{6\varepsilon^2}q_{i-1}+\dfrac{1}{h_i}-\dfrac{(h_i+2h_{i+1})}{6\varepsilon(h_i+h_{i+1})}p_{i-1}-\dfrac{h_{i+1}}{3\varepsilon h_i}p_i+\dfrac{h_{i+1}^2}{3\varepsilon h_i}p_{i+1},\\ r^{c}=\dfrac{(h_i+h_{i+1})}{3\varepsilon^2}q_i-\dfrac{(h_i+h_{i+1})}{h_ih_{i+1}}+\dfrac{(h_i+h_{i+1})}{6\varepsilon h_{i+1}}p_{i-1}+\dfrac{(h_{i+1}^2-h_i^2)}{3\varepsilon h_ih_{i+1}}p_i-\dfrac{(h_{i+1}+h_i)}{6\varepsilon h_i}p_{i+1},\\ r^{+}=\dfrac{h_{i+!}}{6\varepsilon^2}q_{i-1}+\dfrac{1}{h_{i+1}}-\dfrac{h_i^2}{6\varepsilon h_{i+1}(h_i+h_{i+1})}p_{i-1}+\dfrac{h_i}{3\varepsilon h_{i+!}}p_i+\dfrac{(h_{i+1}+h_i)}{6\varepsilon h_i(h_i+h_{i+1})}p_{i+1},\end{cases}$$

and $G(x_i) = \frac{h_i}{2} g(x_{i-1}) + (h_i + h_{i+1}) g(x_{i-1}) + \frac{h_{i+1}}{2} g(x_{i+1}).$

3. The difference scheme

In this section, to approximate IVP (1.1), we introduce a hybrid scheme on a non-uniform mesh of N intervals Ω^N. Let σ denotes a mesh transition parameter defined by $\sigma = \min\left\{\frac{T}{2}, \frac{4}{m}\ln N\right\}$, which divide Ω^N into two subdomains with $N/2$ equal subintervals. Define a mesh generating function ζ with $\zeta(0)=0$ and $\zeta(1/2)=\ln N$ .Then the mesh points are given by

$$x_i = \begin{cases} \frac{2\varepsilon}{\beta}\zeta(t_i), & i = 0,1,\ldots\ldots\ldots N/2, \\ 1-\left(1-\frac{2\varepsilon}{\beta}\ln N\right)\left(\frac{2(N-i)}{N}\right) & i = N/2+1,\ldots\ldots\ldots N. \end{cases} \tag{3.1}$$

where $t_i = i/N$. Define a new function χ closely related to ζ with $\zeta = -\ln\chi$, and satisfies $\chi(0)=1$ and $\chi(1/2) = N^{-1}$. Then the characterizing function χ is given by

$$\chi(t) = \begin{cases} e^{-2(\ln N)t} & \text{(S mesh)}, \\ 1-2(1-N^{-1})t & \text{(B-S mesh)}. \end{cases}$$

3.1 Hybrid difference scheme

In this scheme, we use the cubic spline approximation define in (2.10) in the fine mesh region and the midpoint upwind approximations on the coarse mesh region of Ω^N.

Let $L_{Mp}Y_i = \frac{2\varepsilon^2}{h_i + h_{i+1}}\left(\frac{Y_{i+1}-Y_i}{h_{i+1}} - \frac{Y_i - Y_{i-1}}{h_i}\right) + p_{i-1/2}\left(\frac{Y_i - Y_{i-1}}{h_i}\right) + q_{i-1/2}\overline{Y}_i = g_{i-1/2}$, (3.2)

where $Y_i = Y(x_i)$, $\overline{Y}_i = (Y_{i-1}+Y_i)/2$ and $p_{i-1/2} = p((x_{i-1}+x_i)/2)$; similar for $q_{i-1/2}$ and $g_{i-1/2}$.
Thus the hybrid finite approximation for IVP (1.1) takes of the form:

$$L_H Y_i = \begin{cases} L_{Cu}Y_i, & 1 \le i < .N/2, \\ L_{Mp}Y_i & N/2 \le i < N-1. \end{cases} \tag{3.3}$$

and $L_0 Y_0 = \frac{-Y_2 + 4Y_1 - Y_0}{-(x_2 - x_1) + 4(x_1 - x_0)} = \frac{\eta}{\varepsilon}, \qquad Y(0) = \lambda.$ (3.4)

Proposition 3.1 Let y and Y_i be the solutions of the IVP (1.1) and the discrete problem (3.3-3.4) respectively. Then, the parameter uniform estimate is given by:
$\|y_i - Y_i\| \le CN^{-2}\ln^2 N$, (S mesh)
$\|y_i - Y_i\| \le CN^{-2}$, (B-S mesh)

Proposition 3.2 Let y and Y_i be the solutions of the IVP(1.1) and the discrete problem (3.3-3.4) respectively. Then, the following parameter error estimate for the scaled numerical derivatives:

$$\varepsilon\left|\frac{Y_{i+1}-Y_i}{h_{i+1}}-y'_{i+1/2}\right|\le CN^{-2}\ln^2 N \text{, (S mesh)}$$

$$\varepsilon\left|\frac{Y_{i+1}-Y_i}{h_{i+1}}-y'_{i+1/2}\right|\le CN^{-2} \text{, (B-S mesh)}$$

4. Numerical results and discussions

Example 4.1 Consider the singularly perturbed IVP:

$$\varepsilon^2 y''(x)+\varepsilon(3+x\sin(x))y'(x)+(1+e^x)y(x)=g(x),\quad 0<x\le 1,$$
$$y(0)=2, y'(0)=1-\frac{1}{2\varepsilon},$$

where $g(x)$ is chosen in such a way that the exact solution is $y(x)=1+x-x^2+e^{-x/2\varepsilon}$. The error of the difference approximation is measured in the discrete maximum norm. For any value of N the maximum pointwise errors E^N and the scaled errors of numerical derivatives D^N is defined by:

$$E^N=\max_{1\le i\le N}\left|y(x_i)-Y_i\right|,\qquad D^N=\max_{1\le i\le N}\left|\frac{Y_{i+1}-Y_i}{h_{i+1}}-y'_{i+1/2}\right|$$

where U_i is obtained by the proposed methods. Then the corresponding rate of convergence are given by: $r_E=\dfrac{\ln E^N-\ln E^{2N}}{\ln(2\ln N/\ln(2N))}$, $r_D=\dfrac{\ln D^N-\ln D^{2N}}{\ln(2\ln N/\ln(2N))}$.

Table 1 represents the maximum pointwise error and the corresponding rate for $\varepsilon=1e-4$ and $\varepsilon=1e-8$. Similarly, Table 2 displays the scaled error and the corresponding rate r_D. The results clearly indicate that the proposed scheme is uniformly convergent of order two. To have a proper visualization of the rate of a convergence, the log log plots of the E^N and D^N are shown in Figure 1.

In this paper, a singularly perturbed second order IVP is considered. We presented the hybrid scheme on Shishkin type meshes (S mesh and B-S mesh). The proposed hybrid scheme is a combination of cubic spline approximation on the inner region and the midpoint upwind approximation on the outer region. The proposed method generates a second order ε-uniform convergence rate of the numerical solution and the scaled numerical derivatives. The efficacy of the proposed scheme can be easily seen from the numerical results and the approximation coincides with the theoretical results.

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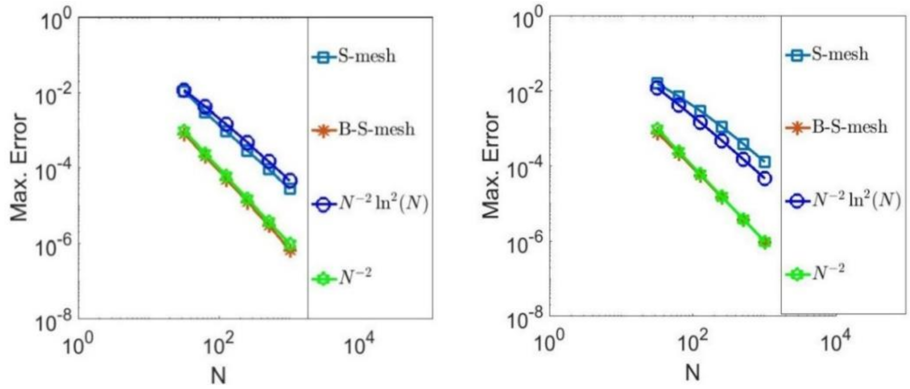

(a) Maximum error (b) Maximum scaled error

Fig1: Loglog plot for Example 4.1

Table1: E^N and r_E generated by the hybrid scheme for Example 4.1.

N	S mesh		B-S mesh	
	$\varepsilon = 1e-4$	$\varepsilon = 1e-8$	$\varepsilon = 1e-4$	$\varepsilon = 1e-8$
32	1.0797e-2 2.4748	1.0797e-2 2.4748	7.9889e-4 2.6732	8.0394e-4 2.6629
64	3.0497e-3 2.2184	3.0497e-3 2.2184	2.0391e-4 2.5678	2.0617e-4 2.5474
128	9.2248e-4 2.0653	9.2249e-4 2.0665	5.1092e-5 2.5055	5.2215e-5 2.4657
256	2.9041e-4 2.0031	2.9022e-4 2.0019	1.2572e-5 2.4877	1.3138e-5 2.4038
512	9.1725e-5 1.9885	9.1726e-5 2.0359	3.0046e-6 2.5416	3.2953e-6 2.3558

Table2: D^N and r_D generated by the hybrid scheme for Example 4.1.

N	S mesh		B-S mesh	
	$\varepsilon = 1e-4$	$\varepsilon = 1e-8$	$\varepsilon = 1e-4$	$\varepsilon = 1e-8$
32	1.5617e-2 1.5422	1.5617e-2 1.5422	7.6177e-4 2.5062	7.6177e-4 2.5062
64	7.1034e-3 1.6511	7.1034e-3 1.6511	2.1174e-4 2.4513	2.1175e-4 2.4503
128	2.9172e-3 1.7480	2.9173e-3 1.7479	5.6525e-5 2.4115	5.6525e-5 2.4114
256	1.0968e-3 1.8299	1.0969e-3 1.8300	1.4661e-5 2.3754	1.4661e-5 2.3754
512	3.8273e-4 1.8927	3.8273e-4 1.8920	3.7377e-6 2.3413	3.7378e-6 2.3409