Proceedings of the National Conference on Computational Modeling of Fluid Dynamics Problems (CMFDP-2019)
NIT Warangal, India – Jan 18-20, 2019
Paper No. XXX (The number assigned when the papaer is accepted)

A fourth order numerical method for singularly perturbed delay parabolic partial differential equation

L. Govindarao1[a], J. Mohapatra2[a,*]
[a]*Department of Mathematics, National Institute of Technology, Rourkela, India-769008*

*Corresponding author Email: jugal@nitrkl.ac.in (J. MOHAPATRA)

Received: Date? Accepted: Date?

This article presents a higher-order parameter uniformly convergent method for a singularly perturbed delay parabolic reaction-diffusion initial-boundary-value problem. For the discretization of the time derivative, we use the Crank-Nicolson scheme on the uniform mesh and for the spatial discretization, we use the central difference scheme on the Shishkin mesh, which provides a second order convergence rate. To enhance the order of convergence, we apply the Richardson extrapolation technique. We prove that the proposed method converges uniformly with respect to the perturbation parameter and also attains almost fourth order convergence rate. Finally, to support the theoretical results, we present some numerical experiments by using the proposed method.

Keywords: Singular perturbation, Delay parabolic problems, Crank-Nicolson scheme, Richardson extrapolation

1. Introduction

In this article, we consider the following singularly perturbed delay parabolic reaction-diffusion Initial-Boundary-Value Problem (IBVP):

$$\begin{cases} \left(\frac{\partial u}{\partial t} + L_{\varepsilon,x}\right) u(x,t) = -b(x,t)u(x,t-\tau) + f(x,t), \quad (x,\ t) \in D, \\ u(x,t) = \theta_b(x,t), \quad (x,t) \in \Gamma_b, \\ u(0,t) = \theta_l(t) \qquad on \quad \Gamma_l = \{(0,t): 0 \le t \le T\}, \\ u(0,t) = \theta_l(t) \qquad on \quad \Gamma_l = \{(0,t): 0 \le t \le T\}, \end{cases} \tag{1.1}$$

where, $L_{\varepsilon,x}u(x,t) = -\varepsilon u_{xx}(x,t) + a(x)u(x,t)$. Here $\Omega = (0,1), D = \Omega \times (0,T]$, $\Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_r$. Γ_b and Γ_r are the left and the right sides of the domain D corresponding to $x = 0$ and $x = 1$, respectively. $\Gamma_{b=}[0,1] \times [-\tau, 0]$. Also, $0 < \varepsilon \ll 1$ and $\tau > 0$ is given constant. The functions $a(x), b(x,t), f(x,t)$ on D and $\theta_b(x,t), \theta_l(t), \theta r(t)$ on Γ, are sufficiently smooth, bounded functions that satisfy, $a(x) \ge \beta \ge 0, b(x,t) > 0$ on D. The terminal time T is assumed to be $T = k\tau$ for some positive integer k. The required compatibility conditions at the corner points and the delay terms are $\theta_b(0,0) = \theta_l(0), \theta_b(1,0) = \theta r(0)$, and

$$\frac{d\theta_l(0)}{dt} - \varepsilon \frac{\partial^2 \theta_b(0,0)}{\partial x^2} + a(0)\theta_b(0,0) = -b(0,0)\theta_b(0,-\tau) + f(0,0),$$
$$\frac{d\theta_r(0)}{dt} - \varepsilon \frac{\partial^2 \theta_b(1,0)}{\partial x^2} + a(1)\theta_b(1,0) = -b(1,0)\theta_b(1,-\tau) + f(1,0),$$

under the above assumptions and compatibility conditions, problem (1.1) admits a unique solution and the solution exhibits boundary layers along x=0, x=1 (refer [1, 3]). One can refer [2, 4, 5, 10] reference therein for more details of singular perturbation.

There are few articles dealing with the theory and the numerical methods for equation (1.1). Ansari et. al [1] solved the problem (1.1) on piecewise uniform Shishkin mesh. Das and Natesan. [2] solved the delay parabolic convection diffusion problem. But most of the methods discussed above using finite difference schemes are of second order accurate. So there is a need of higher order accurate for (1.1).

Richardson extrapolation technique is one of post processing technique used to provide a better approximate numerical solution and to increase the order of convergence. This technique is used by Mohapatra and Natesan in [5] for solving singularly perturbed delay two point BVPs while Shishkin et. al. [7] applied this idea to solve the parabolic reaction-diffusion equation. The aim of this work is to provide a fourth order convergent solution for (1.1) using the Richardson extrapolation technique. First, we use the central difference scheme for the spatial direction on Shishkin mesh and the implicit Euler method for time direction on uniform mesh. Here, we solve the problem (1.1) with N and M number of subintervals in spatial and temporal direction respectively, after that we solve (1.1) with $2N$ and $4M$ number of subintervals. Then by combining these two solutions properly, we enhance the order of convergence from second order to fourth order in spatial direction and first order to second order in time direction.

2. Time discretization

On time domain [0, T], we use uniform mesh with time step Δt, $\Omega_t^M = \{ t_n = n\Delta t, n = 0....M,\ t_M = T,\ \Delta t = T/M\}$, $\Omega_t^p = \{ t_j = j\Delta t, j = 0....p,\ t_p = \tau,\ \Delta t = \tau/p\}$, where M is number of mesh points in t-direction on the interval [0, T] and p is the number of mesh points in $[-\tau,\ 0]$. The step length Δt satisfies $p\Delta t = \tau$, where p is a positive integer, $t_n = \Delta t$, $n \geq -p$. To discretize the time variable for (1.1), we use the Crank-Nicolson method, which is given by

$$\begin{cases} \mathrm{u}^{-j} = \theta_b(x, -t_j), \quad \text{for } \mathrm{j} = 0, \dots, \mathrm{p}, \quad x \in \overline{\mathrm{D}}, \\ \left(I + \frac{\Delta t}{2} L_{\varepsilon,x}\right) u^{n+1} = \frac{\Delta t}{2}(-b^{n+1}u^{n-p+1} - b^n u^{n-p} + f^{n+1} + f^n) + \left(I - \frac{\Delta t}{2} L_{\varepsilon,x}\right) u^n, \\ u^{n+1}(0) = \theta_l(t_{n+1}),\ u^{n+1}(1) = \theta_r(t_{n+1}), \end{cases} \tag{2.1}$$

where $f^n = \mathrm{f}(\mathrm{x}, t_n)$, $c^n = \mathrm{c}(\mathrm{x}, t_n)$, $u^n = \mathrm{u}(\mathrm{x}, t_n)$ is the semidiscrete approximation to the exact solution u(x, t) of (1.1) at the time level $t_n = \Delta t$.

3. Numerical approximation

Here, we propose a numerical scheme to solve the IBVP (1.1). We discretize the IBVP (1.1) using the Crank-Nicolson scheme on a uniform mesh in time direction and the central difference scheme on a Shishkin mesh in the spatial direction. For the construction of the Shishkin mesh, one may refer [4].

3.1 Spatial discretization

Let 'σ' denotes a mesh transition parameter defined by $\sigma = min\left\{\frac{1}{4},\ \rho_0\sqrt{\varepsilon} lnN\right\}$, where $\rho_0 \geq \frac{2}{\beta}$. We divide the domain $\overline{\Omega} = [0, 1]$ into three sub-domains as $\overline{\Omega} = \overline{\Omega_l} \cup \overline{\Omega_c} \cup \overline{\Omega_r}$, where $\Omega_l = (0,\ \sigma]$, $\Omega_c = (\sigma,\ 1-\sigma]$ and $\Omega_r = (1-\sigma,\ 1]$. We assume that $N = 2^r$ with $r \geq 3$ is the total number of subintervals in the partitions of $[0, 1]$. We specify the mesh $\Omega_x^N = \{x_i \in (0, 1),\ i = 0, \dots, N\}$,

$$\text{where } x_i = \begin{cases} \frac{4i\sigma}{N}, & \text{for } i = 0,\dots,\frac{\mathrm{N}}{4}, \\ \frac{2i(1-2\sigma)}{N}, & \text{for } i = \frac{\mathrm{N}}{4}+1,\dots,\frac{\mathrm{N}}{4}, \\ \frac{4i\sigma}{N}, & \text{for } i = \frac{3\mathrm{N}}{4}+1,\dots,\mathrm{N}. \end{cases}$$

We define the discretized domain $D^N = \Omega_x^N \times \Omega_t^M$ on $D, \Gamma^N = \Omega_x^N \times \Omega_t^p$ $on\, \Gamma$. Note that, whenever $\sigma = \frac{1}{4}$, the mesh is uniform and on the other hand when $\sigma = \rho_0\sqrt{\varepsilon}lnN$, the mesh is condensing near the boundaries Γ_l and Γ_r, here $x_i - x_{i-1} = 4\sigma N^{-1}$. Consider the finite difference approximation for (1.1) on domain Ω_x^N. Denote $h_j = x_j - x_{j-1}$. Given a mesh function ϕ_j, the backward and the central difference operators as:

$$\mathrm{D}_x^-\phi_j^n = \frac{\phi_j^n-\phi_{j-1}^n}{h_j}, \mathrm{D}_x^+\mathrm{D}_x^-\phi_j^n = \frac{2}{h_j+h_{j+1}}\left(\frac{\phi_{j+1}^n-\phi_j^n}{h_{j+1}} - \frac{\phi_j^n-\phi_{j-1}^n}{h_j}\right).$$

Also define the backward difference operator in time by $\mathrm{D}_t^-\phi_j^n = \frac{\phi_j^n-\phi_j^{n-1}}{\Delta t}$, where $\phi_j^n = \phi(x_i, t_n)$. We propose the following the numerical scheme to solve IBVP (1.1), the Crank-Nicolson scheme for the time derivative, and the central difference scheme for the spatial derivatives, which is defined as:

$$2\mathrm{D}_t^- U_i^{n+1} + L_\varepsilon U_i^{n+1} = -b_i^{n+1}U_i^{n-p+1} - b_i^n U_i^{n-p} + f_i^n + f_i^{n+1} - L_\varepsilon U_i^n, \tag{3.1}$$

here, $L_\varepsilon U_i^n = -\varepsilon\mathrm{D}_x^+\mathrm{D}_x^- U_i^n + \mathrm{a}_i U_i^n$, $f_i^n = \mathrm{f}(\mathrm{x}_i, t_n)$, $b_i^n = \mathrm{c}(\mathrm{x}_i, t_n)$, $\mathrm{a}_i = \mathrm{a}(\mathrm{x}_i)$, for $i = 1, 2, \dots, N-1$.

3.2 Fully discrete scheme

Using the scheme (2.1) and after rearranging the terms in (3.1), the fully discrete scheme obtained is given by,

$$\begin{cases} r_i^- U_{i-1}^{n+1} + r_i^o U_i^{n+1} + r_i^+ U_i^{n+1} = 2g_i^n, \\ U_0^{n+1} = \theta_l(t_{n+1}),\ U_0^{n+1} = \theta_r(t_{n+1}), \\ U_0^{-j} = \theta_l(\mathrm{x}_i, t_{n+1}), \text{for } j = 0, \dots, p, \text{ and } i = 1, \dots, N-1. \end{cases} \tag{3.2}$$

Here,

$$\begin{cases} r_i^- = \Delta t\left(-\frac{2\varepsilon}{\hat{h}_i \mathrm{h}_i}\right), \\ r_i^o = \Delta t\left(-\frac{2\varepsilon}{\hat{h}_i \mathrm{h}_i} + b_i^{n+1}\right) + 1, \\ r_i^+ = \Delta t\left(-\frac{2\varepsilon}{\hat{h}_i \mathrm{h}_i}\right), \end{cases}$$

for $0 < i \le N-1$, $g_i^n = \frac{\Delta t}{2}\left(-b_i^{n+1}U_i^{n-p+1} - b_i^n U_i^{n-p} + f_i^n + f_i^{n+1}\right) + \left(1 - \frac{\Delta t}{2}\right)U_i^n$. The difference equations (3.2), at each time level n+1 form a tri-diagonal system of $N-1$ equations with $N-1$ unknowns. The tri-diagonal systems have the following properties:

$$r_i^- < 0, r_i^o > 0,\ r_i^+ < 0, \text{for } i = 1, \dots, N-1.$$

These matrices have the diagonal predominance with respect to columns. To solve the tri-diagonal system, we use Thomas algorithm. For a brief details of Thomas algorithm and stability one can refer [6].

Theorem 1

Let u and U be the solutions of (1.1) and (3.1) respectively, satisfying the compatibility conditions. Then, the error of the finite difference scheme (3.1) satisfies the following estimate $\max_{i,n}|(u - U)(xi,tn)| \leq C((N^{-1}lnN)^2 + \Delta t^2)$, for $i = 1,\dots,N-1$,

where $U(x_i, t_n) = U_i^n$, for $(x_i, t_n) \in D^N$.

.

4. Richardson extrapolation technique

To increase the accuracy of the numerical solutions of the scheme, we use Richardson extrapolation technique. To apply the technique, we solve the discrete problem (3.1) on the fine mesh $D^{2N} = \Omega_x^{2N} \times \Omega_t^{4M}$. with $2N$ mesh intervals in the spatial direction and $4M$ mesh intervals in the time direction, where $\overline{\Omega}_x^{2N}$ is a piecewise uniform Shishkin mesh having the same transitions points as $\overline{\Omega}_x^N$ and obtained by bisecting each mesh interval of $\overline{\Omega}_x^N$. Clearly, $D^N = \{(x_i, t_n)\} \subset D^{2N} = \{(\bar{x}_i, \bar{t}_n)\}$. Therefore, the corresponding mesh the mesh is $\overline{\Omega}_x^N = \{\bar{x}_i \in (0,1),\ i = 0,\dots,2N\}$, where

$$\bar{x}_i = \begin{cases} \frac{2i\sigma}{N}, & for\ \ i = 0, \dots, N/2, \\ \frac{i(1-2\sigma)}{N}, & for\ \ i = \frac{N}{2} + 1, \dots, N/2, \\ \frac{2i\sigma}{N}, & for\ \ i = \frac{3N}{2} + 1, \dots, 2N. \end{cases}$$

Let $\overline{U}(\bar{x}_i, \bar{t}_n)$ solutions of the discrete problems (3.1) on the mesh D^{2N} using the same transition point. Therefore, we use the following extrapolation formula

$$U_{ext}(x_i, t_n) = \frac{1}{3}(4\overline{U} - u)(x_i, t_n), \quad \text{for } (x_i, t_n) \in D^N. \tag{4.1}$$

Theorem 2

Let u be the solution of the continuous problem (1.1) and U_{ext}be the solution obtained by the Richardson extrapolation technique (4.1) by solving the discrete problem (3.1) on two meshes D^N and D^{2N}. Then we have the following error bound associated with U_{ext} :

$\max_{i,n}|(u - U_{ext})(x_i, t_n)| \leq C((N^{-1}lnN)^4 + \Delta t^4)$, for $i = 1,\dots,N-1$.

5. Numerical results and discussions

Example 5.1 Consider the following singularly perturbed delay parabolic IBVP:

$$\begin{cases} u_t - \varepsilon u_{xx} + \frac{(1+x)^2}{2}u = t^3 - u(x, t-1), (x,t) \in (0,1) \times (0,2], \\ u(x,t) = 0,\ (x,t) \in [0,1] \times [-1,0], \\ u(0,t) = 0, u(1,t) = 0, t \in [0,2]. \end{cases} \tag{5.1}$$

The exact solution of (5.1) is unknown. To obtain the pointwise errors and to verify the ε – uniform convergence of the proposed scheme, we use the double mesh principle. Let $\widetilde{U}(x_i, t_n)$ be the numerical solution obtained on the fine mesh $\widetilde{D}^{2N} = \widetilde{\Omega}_x^{2N} \times \widetilde{\Omega}_t^{2M}$with $2N$ mesh intervals in the spatial direction and $2M$ mesh intervals in the t-direction, where $\Omega_x^{2N}$is piecewise-uniform Shishkin mesh as like Ω_x^N with the same transition parameter. Now for each ε, we calculate the maximum point wise error by

$$E_\varepsilon^{N,\Delta t} = \max_{(x_i,t_n)\in D^N} \left|(U - \tilde{U}^{2N,\Delta t})(x_i,t_n)\right|,$$

and the corresponding order of convergence by $P_\varepsilon^{N,\Delta t} = log_2\left(\frac{E_\varepsilon^{N,\Delta t}}{E_\varepsilon^{2N,\Delta t/2}}\right)$.

The numerical solutions of (5.1) is plotted in Figure 1 for various values of ε.These figures confirm the existence of boundary layers near $x = 0$ and $x = 1$.The calculated maximum pointwise errors $E_\varepsilon^{N,\Delta t}$ and the rate of convergence $P_\varepsilon^{N,\Delta t}$ for Example (5.1) by using central difference scheme on space and the Crank-Nicolson scheme on time scale is presented in Table 1. Clearly, it shows that the rate of convergence is almost fourth order after extrapolation.

Table. 1. $E_\varepsilon^{N,\Delta t}$and $P_\varepsilon^{N,\Delta t}$ generated on S-mesh by using the Crank-Nicolson scheme.

ε	Extrapolation	Number of intervals N				
		32/10	64/40	128/160	256/640	
1e-2	Before	1.0812e-2 2.0702	2.5746e-3 2.0313	6.2981e-4 2.0212	1.5515e-4 2.0121	3.8465e-5
	After	1.0917e-6 3.8939	7.3444e-8 3.9746	4.6717e-9 3.9937	2.9325e-10 3.9984	1.8348e-11
1e-4	Before	9.0217e-3 1.4189	3.3741e-3 1.5281	1.1699e-3 1.5559	3.9791e-4 1.6554	1.2631e-4
	After	3.3446e-4 2.7042	5.1321e-5 3.0377	6.2498e-6 3.4893	5.5654e-7 3.9206	3.6751e-8
1e-8	Before	9.0217e-3 1.4189	3.3741e-3 1.5281	1.1699e-3 1.5559	3.9791e-4 1.6554	1.2631e-4
	After	3.3446e-4 2.7042	5.1321e-5 3.0377	6.2498e-6 3.2276	6.6720e-7 3.3140	6.7088e-8

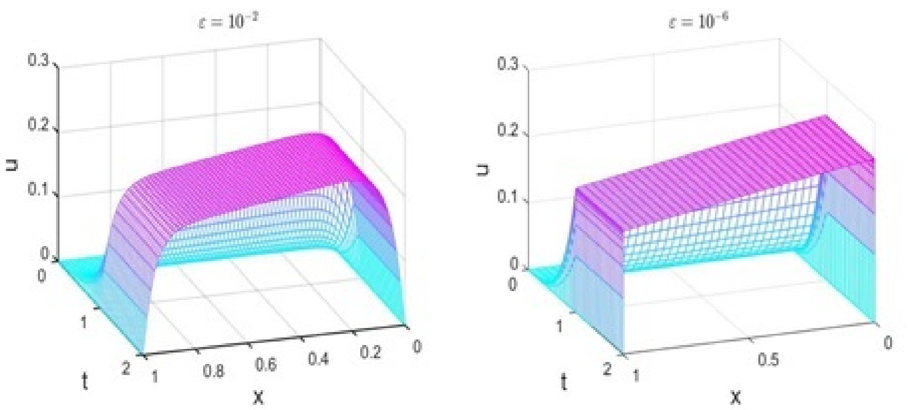

Fig. 1: Surface plots of the numerical solution for Example 5.1.

Acknowledgement

The authors express their sincere thanks to DST, Govt. of India for supporting this work under the research grant EMR/2016/005805.

References

1. A. R. Ansari, S. A. Bakr and G. I. Shishkin, A parameter-robust finite difference method for singularly perturbed delay parabolic partial differential equations, J. Comput. Appl. Math. 205, 552–566, (2007)
2. K. Bansal, P. Rai and K. K. Sharma, Numerical treatment for the class of time dependent singularly perturbed parabolic problems with general shift arguments, Differ Equ Dyn Syst, 25(2), 327-346, (2015)
3. A. Das and S. Natesan, Second-order uniformly convergent numerical method for singularly perturbed delay parabolic partial differential equations, Int. J. Comput. Math. 95 (3), 490–510, (2018)
4. P. A. Farrell, A. F. Hegarty, J. J. H. Miller, E. O'Riordan and G. I. Shishkin, Robust Computational Techniques for Boundary Layers, Chapman & Hall/CRC Press, Boca Raton, FL (2000)
5. L. Govindarao and J. Mohapatra, A second order numerical method for singularly perturbed delay parabolic partial differential equation, Engineering Computations, DOI (10.1108/EC-08-2018-0337)
6. M. Kumar and S. C. Sekhara Rao, High order parameter-robust numerical method for time, dependent singularly perturbed reaction-diffusion problems, Computing 90 (1-2), 15–38, (2010)
7. J. J. H. Miller, E. ORiordan, and I.G. Shishkin, Fitted Numerical Methods for Singular Perturbation Problems, World Scientific, Singapore (1996)
8. J. Mohapatra and S. Natesan, Uniformly convergent second-order numerical method for singularly perturbed delay differential equations, Neural Parallel Sci. Comput. 30, 353–370, (2008)
9. N. Raji Reddy and J. Mohapatra, An efficient numerical method for singularly perturbed two point boundary value problems exhibiting boundary layers, Natl. Acad. Sci. Lett. 4(38), 355–359, (2015)
10. A. A. Salama and D. G. Al-Amerya, A higher order uniformly convergent method for singularly perturbed delay parabolic partial differential equations , Int. J. Comput. Math., 94(12), 2520-2546, (2017)
11. G. I. Shishkin and L. P. Shishkina, A Richardson scheme of the decomposition method for solving singularly perturbed parabolic reaction-diffusion equation, Comput. Math. Math. Phys. 50(12), 2003–2022, (2010)