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Dynamic Response of FGM Kirchhoff's Plate

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The study focuses on the strength of thin rectangular FG plate with compared to conventional materials like pure ceramic and pure metals, considering the aspects of static and dynamic analyses. A power law variation in material properties across the plate thickness is considered along with Mori-Tanaka homogenization scheme. A finite element with sixteen degrees of freedom is considered for approximation of solution. A thin plate is fabricated of particulate Aluminium silicate composites which is investigated based on classical laminate plate theory and tested to clamped plate and cantilevered plate boundary conditions. The results are generated using Matlab program and validated using ANSYS commercial FEA package.

***Keywords*:** Finite Element Method, Functionally Graded Materials, Kirchhoff's Plate, Modal Analysis.

1. Introduction

Since the launch of functionally graded materials in Japan, researchers have been studying and utilizing graded composites in various engineering applications. FGMs are of two kinds: continuous or discretely layered. FG materials performs better in terms of optimum strength with high thermal stability. Particulate FGM are far less susceptible to de-bonding and thermal fracture caused due to inter-laminar localised stresses as observed in composite laminate or short whisker FGMs. Therefore, efficiency in load transfer mechanism is retained[1]. Usually vapour deposition, powder stacking methodology, electrostatic and thermal plasma techniques are employed or fabrication of functionally graded materials[2]. The process of homogenization is understood as the determination of effective material properties of the composite based on the properties of parent materials. Usually Voigt's and Ruess method of rule of mixture is employed for the homogenization of advanced composites but when it comes to randomly oriented particulate FGM like Al-SiC, Mori-Tanaka schemes provides an effective method for prediction of in-homogenous isotropic FGM laminae[3].

2. Material Modelling

The metal ceramic composite is composed of embedded Silicon Carbide particles in an Aluminium matrix thus generating a FGM governed by power law variation in density, Young's modulus and Poisson's ratio over the plate thickness as shown in Eq. (1).

$$\boldsymbol{V_p} = \left(\frac{1}{2} + \frac{z+d}{h}\right)^{n}, \quad \left(\frac{-h}{2} \leq \mathrm{z} \leq \frac{h}{2}, \mathbf{0} \leq \boldsymbol{n} \leq \infty\right) \tag{1}$$

where, V_p is the volume fraction of reinforcement, z is lamina thickness, h is overall thickness, Power law gradient index is represented by n and d is the offset of neutral axis from geometric axis[4] of the thin rectangular plate given by Eq. (2).

$$d = \frac{\int_{-h/2}^{h/2} E(z_m)(z_m)dz_m}{\int_{-h/2}^{h/2} E(Z_m)dz_m} \tag{2}$$

where, $E(z_m)$ is the effective Young's modulus of the lamina. The Mori-Tanaka homogenization scheme in Eq. (3) -(5), given below effectively predicts the material properties of FGM.

$$\frac{\mathrm{K-Kp}}{\mathrm{Km-Kp}} = \frac{(1-\mathrm{Vp})}{1+\frac{(\mathrm{Vp})(\mathrm{Km-Kp})}{\mathrm{Kp+1.333Gp}}} \tag{3}$$

$$\frac{\mathrm{G-Gp}}{\mathrm{Gm-Gp}} = \frac{(1-\mathrm{Vp})}{1+\frac{(\mathrm{Vp})(\mathrm{Gm-Gp})}{\mathrm{Gp+B}}} \tag{4}$$

$$\mathrm{B} = \frac{\mathrm{Gp(9Kp+8Gp)}}{\mathrm{6(Kp+2Gp)}} \tag{5}$$

Using the classical laminate plate theory in which it is assumed that no deformation occurs in the neutral axis of the composite, the material matrix in plane stress condition is given by Eq. (6) -(7)

$$[Q] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{6}$$

$$\{\sigma_{xx}\ \sigma_{yy}\ \tau_{xy}\}^{\mathrm{T}} = [Q]\{\epsilon_{xx}\ \epsilon_{yy}\ \gamma_{xy}\}^{\mathrm{T}} \tag{7}$$

The differential equation of motion for the dynamic analysis of a plate without damping in the plate is given by Eq. (8). Considering variation in Poisson's ratio[5], D is the flexural rigidity as mentioned in Eq. (9). while the product of ρ and h represents value of mass per unity area.

$$\mathrm{D}\left[\frac{\partial^4 \mathrm{w}}{\partial \mathrm{x}^4} + 2\frac{\partial^4 \mathrm{w}}{\partial \mathrm{x}^2\,\partial \mathrm{y}^2} + \frac{\partial^4 \mathrm{w}}{\partial \mathrm{y}^4}\right] + \rho\mathrm{h}\frac{\partial^2 \mathrm{w}}{\partial \mathrm{t}^2} = 0 \tag{8}$$

$$\mathrm{D} = D_{ij} - {B_{ij}^2}/{A_{ij}} \quad ,(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} Q_{ij}\,(1, z, z^2)dz \tag{9}$$

The FGM plate is modelled based on the parent materials presented in Table 1.

Table. 1. Material properties of the matrix and particulate reinforcements

MATERIAL PROPERTY	**ALUMINIUM**	**SILICON-CARBIDE**
DENSITY (Kg/m3)	2700	3200
YOUNG'S MODULUS (N/m2)	72.1E9	431E9
SHEAR MODULUS(N/m2)	26.9E9	181.1E9
POISSON'S RATIO	0.34	0.19
BULK MODULUS(N/m2)	7.43E10	2.31E11
THERMAL CONDUCTIVITY (W/m-1K-1)	187	252.5

3. Finite Element Modelling

Due to the incapacity of obtaining the analytical solutions for governing equations of plate beyond the simple boundary conditions, a need for astute approximation methods such as finite element method originates. Commonly a 12 degree of freedom element is in usage for plate analysis[4]. However, this element, although computationally simple, fails to capture the deformation in the plate element accurately. Therefore, in this study a conforming element following $C^{(1)}$ continuity with 16 d.o.f is utilised for the parametric study of Al-SiC particulate FGM. There exist four displacements per node instead of three as shown in Eq. (10). The twist at plate corners is also taken into account.

$$\{W\} = \left\{w, \frac{\partial w}{\partial y}, -\frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x \partial y}\right\}_i^{\mathrm{T}}, \quad i = 1,2,3,4 \tag{10}$$

Using hermitic polynomial of cubic nature[5], the following shape functions are obtained as mentioned in set of Eq. (11).

$$\boldsymbol{N_1} = \frac{1}{a^3}(a^3 - 3ax^2 + 2x^3)\frac{1}{b^3}(b^3 - 3by^2 + 2y^3) \quad \boldsymbol{N_9} = \frac{1}{a^3}(3ax^2 - 2x^3)\frac{1}{b^3}(3bx^2 - 2y^3) \tag{11}$$

$$\boldsymbol{N_2} = \frac{1}{a^2}(a^2x - 2ax^2 + x^3)\frac{1}{b^3}(b^3 - 3by^2 + 2y^3) \quad \boldsymbol{N_{10}} = \frac{1}{a^2}(x^3 - ax^2)\frac{1}{b^3}(3bx^2 - 2y^3)$$

$$\boldsymbol{N_3} = \frac{1}{a^3}(a^3 - 3ax^2 + 2x^3)\frac{1}{b^2}(b^2x - 2by^2 + y^3) \quad \boldsymbol{N_{11}} = \frac{1}{a^3}(3ax^2 - 2x^3)\frac{1}{b^2}(y^3 - by^2)$$

$$\boldsymbol{N_4} = \frac{1}{a^2}(a^2x - 2ax^2 + x^3)\frac{1}{b^2}(b^2x - 2by^2 + y^3) \quad \boldsymbol{N_{12}} = \frac{1}{a^2}(x^3 - ax^2)\frac{1}{b^2}(y^3 - by^2)$$

$$\boldsymbol{N_5} = \frac{1}{a^3}(a^3 - 3ax^2 + 2x^3)\frac{1}{b^3}(3bx^2 - 2y^3) \quad \boldsymbol{N_{13}} = \frac{1}{a^3}(3ax^2 - 2x^3)\frac{1}{b^3}(b^3 - 3by^2 + 2y^3)$$

$$\boldsymbol{N_6} = \frac{1}{a^2}(a^2x - 2ax^2 + x^3)\frac{1}{b^3}(3bx^2 - 2y^3) \quad \boldsymbol{N_{14}} = \frac{1}{a^2}(x^3 - ax^2)\frac{1}{b^3}(b^3 - 3by^2 + 2y^3)$$

$$\boldsymbol{N_7} = \frac{1}{a^3}(a^3 - 3ax^2 + 2x^3)\frac{1}{b^2}(y^3 - by^2) \quad \boldsymbol{N_{15}} = \frac{1}{a^3}(3ax^2 - 2x^3)\frac{1}{b^2}(b^2x - 2by^2 + y^3)$$

$$\boldsymbol{N_8} = \frac{1}{a^2}(a^2x - 2ax^2 + x^3)\frac{1}{b^2}(y^3 - by^2) \quad \boldsymbol{N_{16}} = \frac{1}{a^2}(x^3 - ax^2)\frac{1}{b^2}(b^2x - 2by^2 + y^3)$$

K_e and M_e being the elemental stiffness and mass matrix respectively can be calculated from Eq. (12) and Eq. (13).

$$k_{ij}^e = D^* \int_0^b \int_0^a \left[\frac{\partial^2 \mathrm{N}_i}{\partial \mathrm{x}^2}\frac{\partial^2 \mathrm{N}_j}{\partial \mathrm{x}^2} + \nu \frac{\partial^2 \mathrm{N}_i}{\partial \mathrm{x}^2}\frac{\partial^2 \mathrm{N}_j}{\partial \mathrm{y}^2} + \nu \frac{\partial^2 \mathrm{N}_i}{\partial \mathrm{y}^2}\frac{\partial^2 \mathrm{N}_j}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{N}_i}{\partial \mathrm{y}^2}\frac{\partial^2 \mathrm{N}_j}{\partial \mathrm{y}^2} + 2(1-\nu)\frac{\partial^2 \mathrm{N}_i}{\partial \mathrm{x}\, \partial \mathrm{y}}\frac{\partial^2 \mathrm{N}_j}{\partial \mathrm{x}\, \partial \mathrm{y}}\right] dxdy \tag{12}$$

$$m_{ij}^e = \rho h \int_0^b \int_0^a \left[N_i(x,y) N_j(x,y)\right] dxdy \tag{13}$$

The elemental mass and stiffness matrices are assembled to produce the global matrices ([K], [M]). The equation of motion of the plate shall yield natural frequencies $\boldsymbol{\omega}$ by solving Eq. (14).

$$([K] - \omega^2[M])\{W\} = 0 \tag{14}$$

4. Results and Discussions

As mentioned Fig. 1 FGM modelled using power law with gradient index n≥1 shows smooth transition in material properties from pure Aluminium in the first layer to pure Silicon carbide layer on the top. While in modelling based on n≤ 1, the initial volume fraction of reinforcement ranges between near about 0.2 to 0.6.

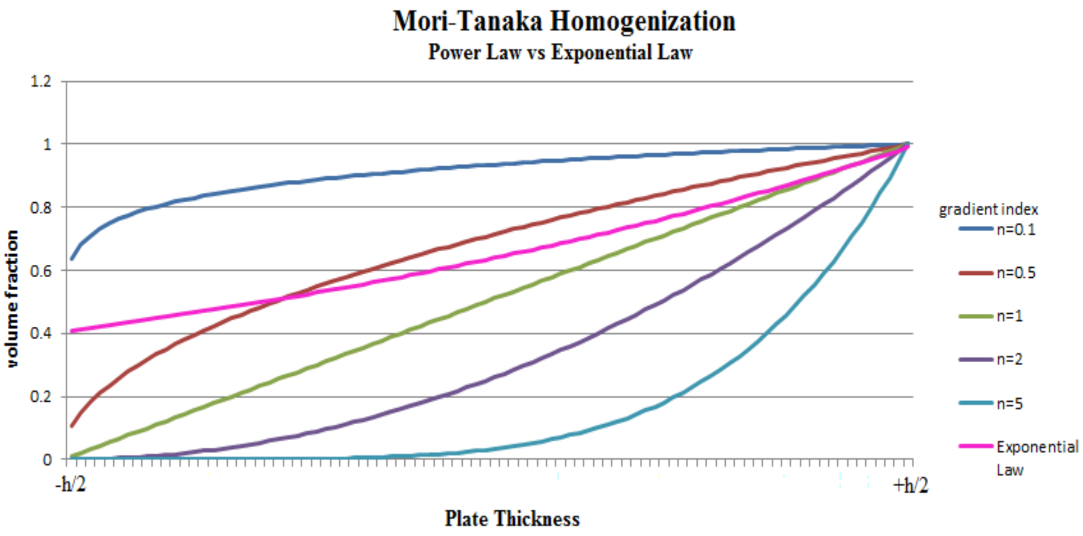

Fig. 1. Variation in volume fraction over the plate thickness using Mori-Tanaka Homogenization. *(colour print)*

The chosen method for FG plate is Mori-Tanaka scheme with power-law variation (with gradient index=1). The plate dimensions are thus: Thickness: 6.3e-3m and Area: (0.1x0.1) m^2. This FG plate is tested against pure aluminium and pure SiC plate under an external pressure on 2Gpa are plotted in Table 2.

Table. 2. Deformations in Plates with different materials and strength determinations in (metres)

Boundary conditions	Pure Aluminium	Ceramic SiC	FGM
CCCC	0.16113	2.90E-02	6.35E-02
CFFF	15.487	2.7341	6.08

The natural frequency of the FGM plate is obtained using 16 d.o.f element and 12 d.o.f element for a mesh size of 20x20 divisions is shown in Table 3 and mode shapes are portrayed in Fig 2.

Table. 3. Natural Frequency of FGM plate using different Finite element

Boundary Conditions	**Mode number**	**AnSys validation**	**Conforming element (16 d.o.f)**	**Non-conforming element (12 d.o.f)**
CCCC	1st	8406.3	8327.3	9517
	2nd	16650	16514	19393
CFFF	1st	839.43	833.12	921.18
	2nd	2013.2	2033.9	2265

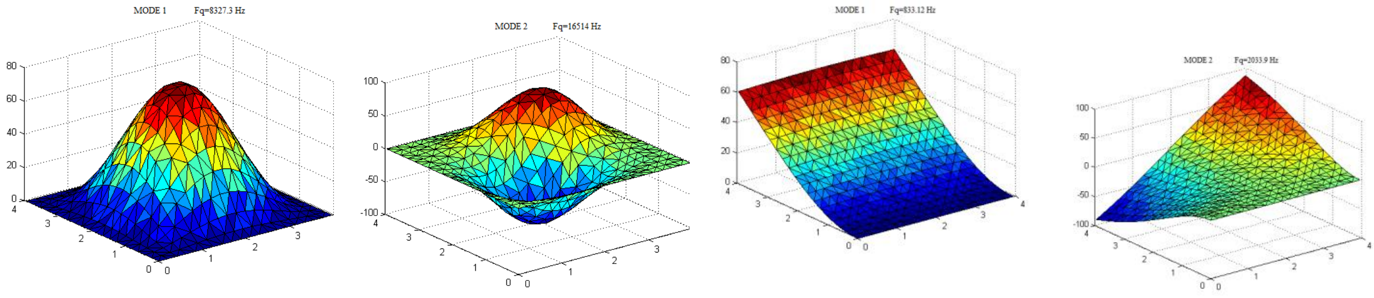

Fig. 2 First and second mode shapes of Clamped plate and cantilever plate respectively. *(colour print)*

2. Conclusion

The stiffness of FG materials is larger than aluminium. It imbibes the optimum properties of ceramic plate as it can be applied in high temperature environment. FG materials should be modelled based on power law variation with gradient index greater than 1 as they show smooth transition in material properties in an FGM as it is expected to vary from pure aluminium to pure ceramic. Mori-Tanaka homogenization scheme renders an effective in-homogenous isotropic material which can be studied as isotropic conventional materials. The application of transformed bending matrix with consideration to variation in Poisson’s ratio is the accurate way to model FGM. This fact was long ignored in previous literatures. The conforming finite element is better able to capture the actual plate deformations which were under presented by non-conforming finite element. These elements are too stiff and therefore overestimate the natural frequency of the structures which add to material cost. Therefore, it is mathematically viable to utilise conforming element for computational modelling of engineering structure in favour of result accuracy and quick convergence than non-conforming finite element with 12 degrees of freedom.

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