Chaos Control Dynamics in Competitive Herbivore Species Network

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ABSTRACT

This paper studies the mathematical modelling of a competitive ecological system in which the interactions between different species are being studied in the framework of ecological systems. Both linear and non-linear interactions have been accounted in the model. Through fixed point analysis, the critical value of parameter has been evaluated after which the system enters critical phase from phase of stability and then to chaos. Bifurcation plot for variation in coefficient of indirect dependency is plotted and used to verify the different phases of evolution of the interspecies relation. The system dynamics is observed to transit from stable to chaotic state through state of critical stability. To control chaos in the competitive ecological system under master slave scheme, it is synchronized to another stable identical ecosystem. Using Lyapunov stability theorem controller are devised. The active controller is observed to completely control the chaos in the system and restore stability of the ecological system.

KEY WORDS: Competitive Species interaction, Bifurcation, Lyapunov function, Active Controller, Chaos.

INTRODUCTION

Ecosystems are fundamental units of nature in which species are connected in networks of food chain. These food chains are governed by the interactions between the species for the procurement of resources in nature and sustenance. Limited resources often lead to resource crunch which leads to competition between the species. Through competition and encounters nature strikes a balance between species growth and resource abundance. Mathematically fluctuations in case of species abundance were studied first by Volterra in 1926 [11]. Fluctuations in population evolution gradually lead to chaotic states in food chain in which species population randomly evolve with time.

Chaos was first studied in deterministic non fluidic flow by Lorenz in 1963 [6]. Chaos was first time mathematically observed in his study. Chaos has attracted the attention of several researchers since then which has led to further realization and understanding of random states in different kind of systems in various fields including ecology. The study on random impacts of complex damped systems gives an insight on significance behind observation and study of chaos [9]. The observation of chaos in different fields and their systems led to the rise of research interest of controlling it and restore stability of the system. The synchronization of chaotic Lorenz systems by using the active control mechanism discussed [1]. Active control mechanisms are one of the most widely used scheme of synchronizing the chaotic systems for controlling of chaos since then.

For physical systems like nanofluid convection models [2], energy systems [7], and complex duffing system [10] chaos and its synchronization have been studied in detail. Similarly, in ecology chaos and its control in ecosystem models have been studied. Dynamics of cooperation in competition interaction models was discussed [5]. The controlling of chaos in food chain models was demonstrated [8]. Ecological food chains can be both linear and nonlinear kind with direct and indirect species dependence. The transition to chaos in three species nonlinear model of competitive ecosystem with an intermediate competitive herbivore species is studied [3].

The problem of study comprises of modelling a competitive herbivore ecosystem where one species indirectly effects the growth of other while competing for resources in situation of lack of resource abundance. The dynamics of the system have been analysed through stability analysis from which the parametric conditions have been derived that govern the system stability and its transition to random states. The bifurcation plots and Lyapunov exponents are determined to validate the observations and derived parametric conditions. Synchronization of chaos in supply chain systems using a Lyapunov function based single controller a multistate controller has been derived [4] for the system which on activation synchronizes two such identical chaotic systems in different dynamic phases and controls chaos. The role of migration is observed to be crucial for controlling the chaos in such competitive ecosystems.

MATHEMATICAL MODELLING AND STABILITY ANALYSIS

Let us consider x_1, x_2 and x_3 represent the three population levels of three herbivore species residing in ecosystem where there is a resource shortage. The shortage of resources leads to higher mortality among these species primarily due to starvation and infightings. The species indirectly affect each other's population level in a positive way as more is the population of one herbivore species higher are its chances to be getting preyed by the predators which provides indirect refuge to the other herbivore prey species. The encounters between the two species might possibly affect the population levels of a third species in either a positive or negative manner depending on the interactions between them. The model is described by following equations:

$$\begin{aligned}\dot{x}_1 &= -a_2x_1 + b_2x_2\\ \dot{x}_2 &= -a_1x_2 + b_1x_1 - c_2x_1x_3 \qquad \ldots(1)\\ \dot{x}_3 &= -a_3x_3 + c_1x_1x_2\end{aligned}$$

where the parameters are defined as: a_1 – coefficient of decay of species 1; a_2 – coefficient of decay of species 2; a_3 – coefficient of decay of species 3; b_1 – coefficient of indirect dependency of species 2 on species 1; b_2 – coefficient of indirect dependency of species 1 on species 2; c_1 – coefficient of encounter between species 1 and species 3; c_2 – coefficient of encounter between species 1 and species 2.

STABILITY ANALYSIS

Three fixed point for system (1) are $(0,0,0)$, $\left(\sqrt{\frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}}, \frac{a_2}{b_2}\sqrt{\frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}}, \frac{(b_1b_2-a_1a_2)}{b_2c_2}\right)$ and $\left(-\sqrt{\frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}}, -\frac{a_2}{b_2}\sqrt{\frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}}, \frac{(b_1b_2-a_1a_2)}{b_2c_2}\right)$. Jacobian of the system (1) is given as follows:

$$Jacobian = \begin{pmatrix} -a_2 & b_2 & 0\\ b_1 - c_2x_3 & -a_1 & -c_2x_1\\ c_1x_2 & c_1x_1 & -a_3\end{pmatrix}$$

<u>**CASE I**</u>: For fixed point $(0,0,0)$, the characteristic equation is $\lambda^3 + e_1\lambda^2 + e_2\lambda + e_3 = 0$, where $e_1 = (a_1 + a_2 + a_3)$, $e_2 = (a_1a_2 + a_2a_3 + a_1a_3 - b_1b_2)$, $e_3 = (-a_3)(b_1b_2 - a_1a_2)$. From Routh-Hurwitz criteria for stability $e_1 > 0, e_3 > 0$ *and* $e_1e_2 - e_3 > 0$. Thus, for stability at $(0,0,0)$ it is required that $b_1 < \left(\frac{a_1a_2}{b_2}\right) = b_0$. The system is asymptotically stable when $b_1 < b_o$, critically stable when $b_1 = b_o$ and unstable when $b_1 > b_o$.

<u>**CASE II**</u>: Fixed point $\left(\sqrt{\frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}}, \frac{a_2}{b_2}\sqrt{\frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}}, \frac{(b_1b_2-a_1a_2)}{b_2c_2}\right)$ and $\left(-\sqrt{\frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}}, -\frac{a_2}{b_2}\sqrt{\frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}}, \frac{(b_1b_2-a_1a_2)}{b_2c_2}\right)$ show invariance under the transformation $(x_1, x_2, x_3) \to (-x_1, -x_2, -x_3)$. The characteristic equation is given as $\lambda^3 + e_1\lambda^2 + e_2\lambda + e_3 = 0$, where $e_1 = (a_1 + a_2 + a_3)$, $e_2 = (a_2a_3 + \frac{b_1b_2a_3}{a_2})$, $e_3 = 2(a_3)(b_1b_2 - a_1a_2)$.

Thus, for stability it is required that $b_o < b_1 < b_c$, $b_o = \left(\frac{a_1a_2}{b_2}\right)$; $b_c = \frac{(a_1 + a_2 + a_3)(a_2a_3) + 2(a_1a_2a_3)}{a_3b_2 - (a_1 + a_3)\left(\frac{b_2a_3}{a_2}\right)}$, where, b_c is the critical value of b_1. The system is asymptotically stable when $b_o < b_1 < b_c$, critically stable when $b_1 = b_c$ and unstable at $b_c < b_1$ which leads to chaos finally.

SYNCHRONIZATION FORCONTROLLINGOFCHAOS:

For synchronization two identical system one in stable state and another in chaotic state are considered as master and slave system which are mentioned as follows:

Master System:

$$\dot{x}_1 = -a_2 x_1 + b_2 x_2$$
$$\dot{x}_2 = -a_1 x_2 + b_1 x_1 - c_2 x_1 x_3$$
$$\dot{x}_3 = -a_3 x_3 + c_1 x_1 x_2$$

Slave System :

$$\dot{y}_1 = -a_2 y_1 + b_2 y_2$$
$$\dot{y}_2 = -A_1 y_2 + B_1 y_1 - c_2 y_1 y_3$$
$$\dot{y}_3 = -a_3 y_3 + c_1 y_1 y_2$$

Error system:

$$\dot{e}_1 = \dot{y}_1 - \dot{x}_1 = -a_2 e_1 + b_2 e_2$$
$$\dot{e}_2 = \dot{y}_2 - \dot{x}_2 = -(A_1 - a_1) y_2 + (B_1 - b_1) y_1 - a_1 e_2 + b_1 e_1 - c_2 y_3 e_1 - c_2 x_1 e_3$$
$$\dot{e}_3 = \dot{y}_3 - \dot{x}_3 = -a_3 e_3 + c_1 (y_2 e_1 - x_1 e_2)$$

Lyapunov function= $\phi(e_1, e_2, e_3) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2) \Rightarrow \dot{\phi} = e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3$

For deriving the controllers for synchronization $\lim_{t\to\infty} \|e(t)\| = 0$ and $\dot{\phi} = 0$ conditions are used. The Controller 1 and Controller 2 are defined as follows:

Controller 1 :

$$u_1 = a_2(e_1) - b_2(e_2) - (e_1)$$
$$u_2 = (A_1 - a_1)(y_2) + (B_1 - b_2) y_1 + a_1 e_1 + e_2 (y_3 y_1 - x_1 x_3) - e_2$$
$$u_3 = a_3(e_3) - c_1(y_1 y_2 - x_1 x_2) - (e_3)$$

Controller 2 :

$$q_1 = -b_2(e_2)$$
$$q_2 = (A_1 - a_1)(y_2) + (B_1 - b_2) y_1 + c_2 (y_3 - x_3)$$
$$q_3 = -c_1(y_1 - x_1)$$

Using controller 1 and controller 2 the master and slave system are synchronized for controlling of chaos.

RESULTS AND DISCUSSION

Numerical simulation of the above system is carried out for different values of the fixed parameters, $a_1 = 1, a_2 = 5,\ a_3 = 1, b_2 = 6, c_1 = 2, c_2 = 1$ and varying parameter b_1. For the fixed parameter values, one gets $b_o = \left(\frac{a_1 a_2}{b_2}\right) = 0.83$ while $b_c = \frac{(a_1 + a_2 + a_3)(a_2 a_3) + 2(a_1 a_2 a_3)}{a_3 b_2 - (a_1 + a_3)\left(\frac{b_2 a_3}{a_2}\right)} = 11.5$. For different values of b_1 the following conditions are observed:

- when $b_1 = 6,$ the condition $b_1 > b_o$ and $b_1 < b_c$ is satisfied and system is observed to be in stable state at point $(x_1, x_2, x_3) = (1.8, 1.1, 5.2)$;
- when $b_1 = 11,$ the condition $b_1 > b_o$ and $b_1 \approx b_c$ is so critically stable state is observed at $(x_1, x_2, x_3) = (\pm 2.5, \pm 2.0, 10.2)$;
- when $b_1 = 16$, $b_1 > b_o$ and $b_1 > b_c$ the system is in chaotic state at point $(x_1, x_2, x_3) = (\pm 3.0, \pm 2.5, 15.2)$.

In figure 1, the transition of dynamic state from stable to chaotic phase can be observed through the bifurcation diagrams for variation in b_1.

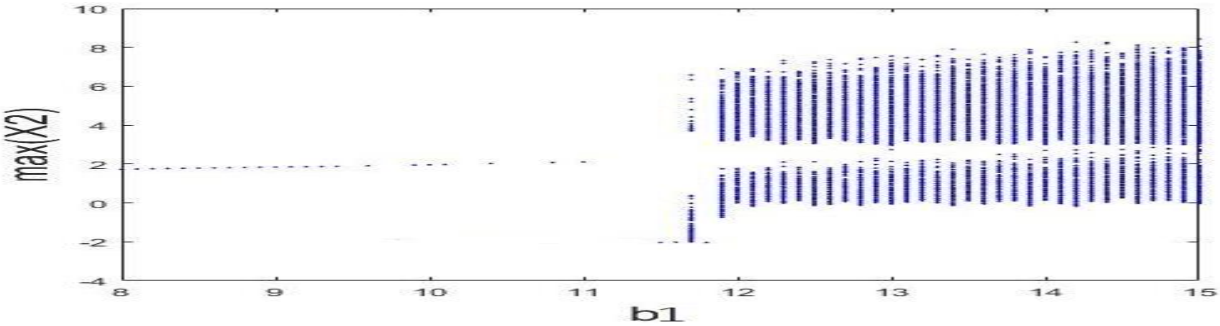

Figure1: Bifurcation diagram for variation in b_1 parameter

From Figure 2 and Figure 3 it is evident that the Controller 1 and Controller 2 derived from Condition 1 and Condition 2 of synchronization between master and slave system are perfectly synchronizing and controlling chaos.

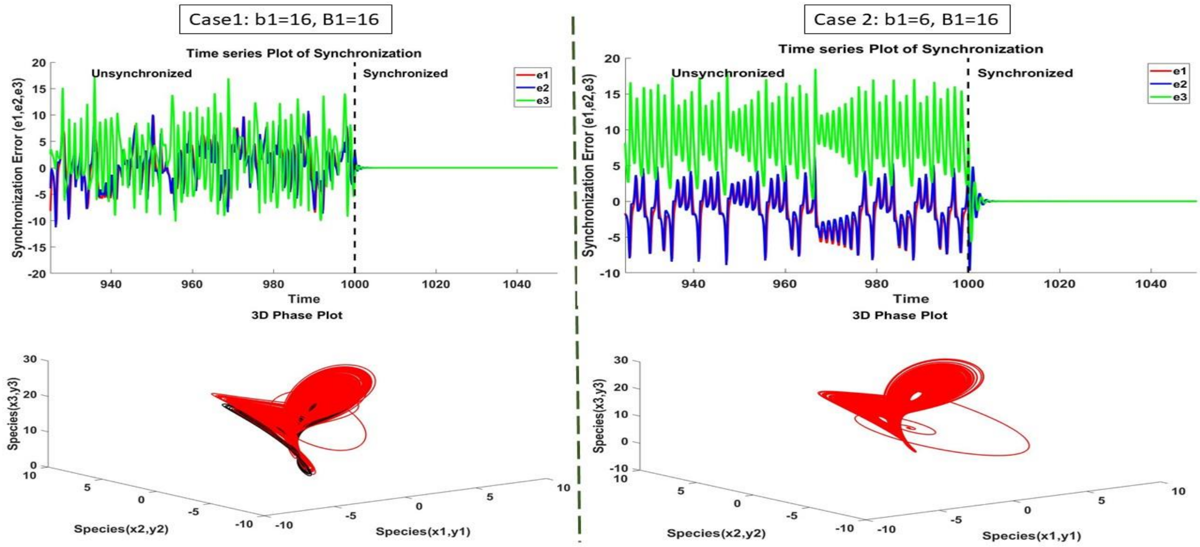

Figure 2: Synchronization in Case 1 and Synchronization with chaos control of chaos in Case 2 on activation of controller at t=1000

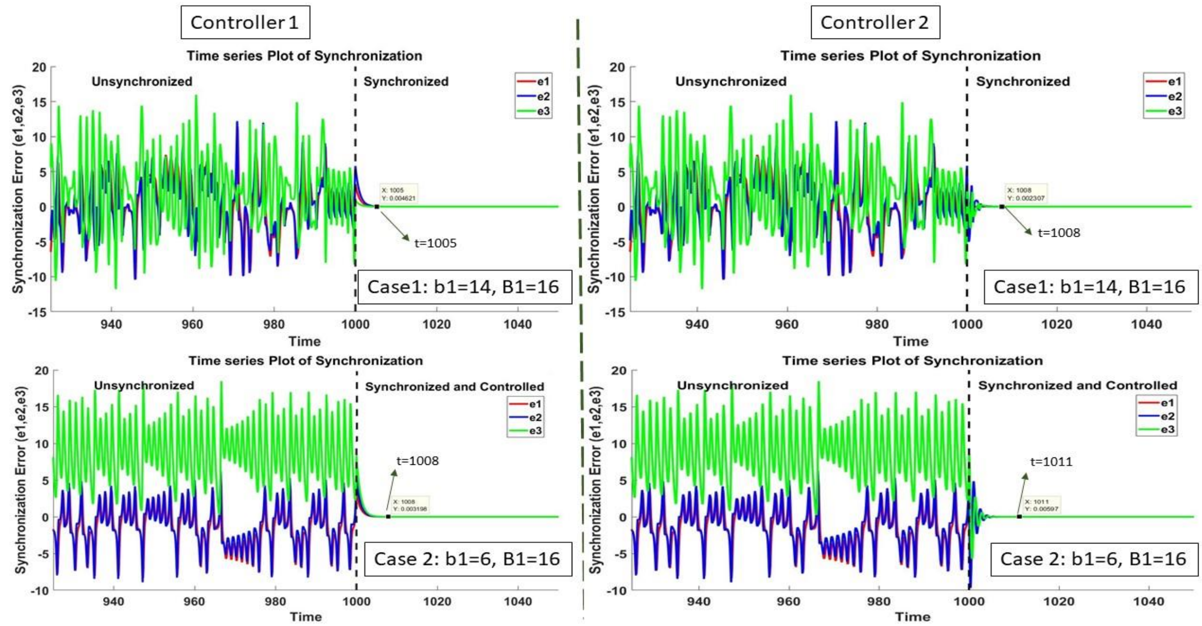

Figure 3: Synchronization in Case 1 and Synchronization with chaos control of chaos in Case 2 on activation of Controller 1 and Controller 2 at t=1000

It is further observed that Controller 1 is faster than Controller 2 by three-unit time due to presence of more migration terms, details of which are mentioned in Table 1. This highlights the importance of migration in stabilizing the population levels ensuring stability of the system.

Table1: Time taken by Controllers to synchronize and control chaos after activation at t=1000

Controller	Time for Case1: Synchronization	Time for Case2: Synchronization+ Control
Controller 1	1005	1008
Controller 2	1008	1011

CONCLUSION

In this paper, the herbivore competitive ecosystem in a resource crunch environment is studied. From stability analysis parametric condition governing the transition of dynamic state of system is derived. All the three phases: stable, intermittence and chaotic states are observed. Using Lyapunov function, the controllers are designed which synchronize the chaotic system successfully for control of chaos. Controller 1 is observed to be faster than Controller 2 in controlling chaos due to presence of more population migration terms. It can be concluded that migration plays a crucial role in stabilizing the population levels and restoring the stability of the system which susceptible to fluctuations to ecological disturbances.

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