PARAMETRIC ACCELERATED OVER RELAXATION (PAOR) METHOD

V.B. Kumar Vatti[1], *G.Chinna Rao*[2], *Srinesh S.Pai*[3]

[1]*Department of Engineering Mathematics, Andhra University, Visakhapatnam, Andhra Pradesh, India.*

[2]*Department of Engineering Mathematics, Andhra University, Visakhapatnam, Andhra Pradesh, India.*

[3]*CMT Laboratory, South Western Railways,Indian Railways, Hubli, India.*

Email: drvattivbk@yahoo.co.in

Abstract: By introducing a parameter α **,** a method named Parametric Accelerated over relaxation (PAOR) is considered and the choices of the parameters involved in PAOR method are given in terms of the eigenvalues of the Jacobi matrix. It is shown through some numerical examples that PAOR method surpasses the other methods considered in this paper.

Keywords: AOR, SOR, Gauss-Seidal, Jacobi

1. Introduction:

Many iterative methods play a major role in solving the linear system of equations

$$AX = b \tag{1.1}$$

(or)

$$(I - L - U)X = b \tag{1.2}$$

where I, L, U are unit, a strictly lower and upper triangular parts of A of order $n \times n$ and X & b are unknown and known vectors of order $n \times 1$ respectively.

In this paper, we consider PAOR method in section 2 and in section 3, the choices of the parameters associated in PAOR method are estimated. In the section 4, we consider some numerical examples to distinguish PAOR method over the other methods and concluded in section 5.

2. PAOR method:

The well known AOR method for solving (1.2) is given by

$$(I - \omega L) X^{(n+1)} = \{(1-r)I + (r-\omega)L + rU\} X^{(n)} + rb \qquad (n = 0,1,2,3,\ldots) \tag{2.1}$$

By introducing a parameter $\alpha \neq -1$ in (2.1), we have

$$\left[(1+\alpha)I-\omega L\right]X^{(n+1)}=\left\{(1+\alpha-r)I+(r-\omega)L+rU\right\}X^{(n)}+rb \qquad (n=0,1,2,3,...) \tag{2.2}$$

This method (2.2) can be called as Parametric AOR (PAOR) method and the methods such as AOR, SOR, Gauss-Seidal and Jacobi methods can be realized for the choices of α, r and ω as

$$(\alpha,r,\omega)=(0,r,\omega),(0,\omega,\omega),(0,1,1),(0,1,0) \tag{2.3}$$

respectively.

The iteration matrix of PAOR method is

$$P_{\alpha,r,\omega}=\left[(1+\alpha)I-\omega L\right]^{-1}\left\{(1+\alpha-r)I+(r-\omega)L+rU\right\} \tag{2.4}$$

Now, $P_{0,r,\omega}, P_{0,\omega,\omega}, P_{0,1,1}, P_{0,1,0}$ become the iteration matrices of AOR, SOR, Gauss-Seidal and Jacobi methods and the spectral radii of these methods are known to be

$$S(P_{0,r,w})=\begin{cases}\dfrac{\bar{\mu}^2}{\left(1+\sqrt{1-\bar{\mu}^2}\right)^2} & \text{when } \underline{\mu}=0 \text{ (or) } \sqrt{1-\bar{\mu}^2}\le 1-\underline{\mu}^2 \text{ (or) } 0<\underline{\mu}<\bar{\mu} \\ 0 & \text{when } \underline{\mu}=\bar{\mu} \\ \dfrac{\underline{\mu}\sqrt{\bar{\mu}^2-\underline{\mu}^2}}{\sqrt{1-\underline{\mu}^2}\left(1+\sqrt{1-\bar{\mu}^2}\right)} & \text{when } 0<\underline{\mu}<\bar{\mu} \ \& \ 1-\underline{\mu}^2<\sqrt{1-\bar{\mu}^2}\end{cases} \tag{2.5}$$

$$S(P_{0,\omega,\omega})=\frac{\bar{\mu}^2}{\left(1+\sqrt{1-\bar{\mu}^2}\right)^2} \tag{2.6}$$

$$S(P_{0,1,1})=\bar{\mu}^2 \tag{2.7}$$

$$S(P_{0,1,0})=\bar{\mu} \tag{2.8}$$

Theorem 2.1: If μ be the eigenvalue of Jacobi matrix $P_{0,1,0}$ and 'λ' be the eigenvalue of PAOR matrix $P_{\alpha,r,\omega}$, then μ and λ are connected by the relation

$$\left[\lambda(1+\alpha)-(1+\alpha-r)\right]^2=r\mu^2(\lambda\omega+r-\omega)$$

Proof: Let $\quad \left|P_{\alpha,r,\omega}-\lambda I\right|=0$ (or) $\left|\lambda I-P_{\alpha,r,\omega}\right|=0$

$$\left|\lambda I-\left[(1+\alpha)I-\omega L\right]^{-1}\left\{(1+\alpha-r)I+(r-\omega)L+rU\right\}\right|=0$$

$$\left|\left[\lambda(1+\alpha)-(1+\alpha-r)\right]I-\left[(\lambda\omega+r-\omega)L+rU\right]\right|=0$$

$$\Rightarrow\left|\left(\frac{\lambda(1+\alpha)-(1+\alpha-r)}{(\lambda\omega+r-\omega)^{\frac{1}{2}}r^{\frac{1}{2}}}\right)I-(L+U)\right|=0$$

If 'μ' be the eigenvalue of $(L+U)$, we have

$$\Rightarrow\frac{\lambda(1+\alpha)-(1+\alpha-r)}{(\lambda\omega+r-\omega)^{\frac{1}{2}}r^{\frac{1}{2}}}=\mu$$

$$\Rightarrow\left[\lambda(1+\alpha)-(1+\alpha-r)\right]^2=r\mu^2(\lambda\omega+r-\omega) \qquad (2.1.1)$$

Theorem 2.2: For $\omega=\frac{2(1+\alpha)}{1+\sqrt{1-\mu^2}}(\alpha\neq-1)$, the eigenvalues of the matrix $P_{\alpha,r,\omega}$ are $\lambda=\frac{r\omega\mu^2}{2(1+\alpha)^2}-\frac{r}{1+\alpha}+1$

Proof: From (2.1.1), we have

$$(1+\alpha)^2\lambda^2-\left[2(1+\alpha)^2-2r(1+\alpha)+\mu^2r\omega\right]\lambda+\left[(1+\alpha)^2-2r(1+\alpha)+\mu^2r\omega-\mu^2r^2+r^2\right]=0$$

$$\Rightarrow\lambda=\frac{\left[2(1+\alpha)^2-2r(1+\alpha)+\mu^2r\omega\right]\pm\sqrt{\Delta}}{2(1+\alpha)^2} \qquad (2.2.1)$$

where

$$\Delta=\left[2(1+\alpha)^2-2r(1+\alpha)+\mu^2r\omega\right]^2-4(1+\alpha)^2\left[(1+\alpha)^2-2r(1+\alpha)+\mu^2r\omega-\mu^2r^2+r^2\right]$$

$$=\mu^2r^2\left[\mu^2\omega^2-4(1+\alpha)\omega+4(1+\alpha)^2\right]$$

which will be zero if $\mu^2\omega^2-4(1+\alpha)\omega+4(1+\alpha)^2=0$

(Or)

if $\omega=\frac{2(1+\alpha)}{1+\sqrt{1-\mu^2}}$ for any μ.

Therefore, λ of (2.2.1) becomes

$$\lambda=\frac{r\omega\mu^2}{2(1+\alpha)^2}-\frac{r}{1+\alpha}+1$$

3. Choice of the Parameters α, r and ω:

The eigenvalues of PAOR matrix are obtained in theorem 2.2 as

$$\lambda = \left(\frac{r\omega\mu^2}{2(1+\alpha)^2}\right) - \left(\frac{r}{1+\alpha} - 1\right) \tag{3.1}$$

If the two terms of R.H.S in (3.1) are connected by the relation

$$\frac{r\omega\mu^2}{2(1+\alpha)^2} = k\left(\frac{r}{1+\alpha} - 1\right) \tag{3.2}$$

Where k is any constant, then rewriting r in terms k and k in terms of r from (3.2), we obtain

$$r = \frac{(1+\alpha)k}{k - \dfrac{\omega\mu^2}{2(1+\alpha)}} \tag{3.3}$$

and

$$k = \frac{r\omega\mu^2}{(1+\alpha)} \cdot \frac{1}{2\left[r-(1+\alpha)\right]} \tag{3.4}$$

Equating the values of $k(1+\alpha)$ obtained from (3.3) and (3.4), we get

$$\left[r-(1+\alpha)\right]\left[k - \frac{\omega\mu^2}{2(1+\alpha)}\right] = \frac{\omega\mu^2}{2} \tag{3.5}$$

Now, fixing ω in (3.5) as

$$\omega = \omega^* = \frac{2(1+\alpha)}{1+\sqrt{1-\mu^2}} \tag{3.6}$$

and multiplying and dividing the R.H.S term of (3.5) by

$$\omega + \frac{\bar{\mu}^2 - \underline{\mu}^2}{2}$$

where $\underline{\mu}$ and $\bar{\mu}$ are the smallest and largest eigenvalues of $P_{0,1,0}$, we obtain

$$\left[r-(1+\alpha)\right]\left[k - \frac{\omega^*\mu^2}{2(1+\alpha)}\right] = \left[\omega^* + \frac{\bar{\mu}^2 - \underline{\mu}^2}{2}\right]\left[\frac{\dfrac{\omega^*\mu^2}{2}}{\omega^* + \dfrac{\bar{\mu}^2 - \underline{\mu}^2}{2}}\right] \tag{3.7}$$

Equating equation (3.7) of the first term in LHS with first term in RHS and similarly the second term in LHS with second term in RHS, we get

$$r = 1 + \alpha + \omega^* + \frac{\bar{\mu}^2 - \underline{\mu}^2}{2} \tag{3.8}$$

$$k = 1 - \sqrt{1 - \bar{\mu}^2} + \frac{\frac{\omega^* \mu^2}{2}}{\omega^* + \frac{\bar{\mu}^2 - \underline{\mu}^2}{2}} \tag{3.9}$$

It is observed and verified that if $k > 1$, then r should be taken as r in (3.8) and if $k < 1$, then r should be taken as $\frac{r}{2}$.

We summarize above results by giving the choices of ω and r for various values of α :

Type 1: when $\underline{\mu} = \bar{\mu}$ and $k = 1$

$$\omega = \frac{2}{1 + \sqrt{1 - \bar{\mu}^2}} \quad \& \quad r = \frac{(1 + \alpha)}{\sqrt{1 - \bar{\mu}^2}} \tag{3.10}$$

Type 2: when $\underline{\mu} \neq \bar{\mu}$ and $k > 1$

$$\omega = \frac{2}{1 + \sqrt{1 - \bar{\mu}^2}} \quad \& \quad r = 1 + \alpha + \omega^* + \frac{\bar{\mu}^2 - \underline{\mu}^2}{2} \tag{3.11}$$

Type 3: when $\underline{\mu} \neq \bar{\mu}$ and $k < 1$

$$\omega = \frac{2}{1 + \sqrt{1 - \bar{\mu}^2}} \quad \& \quad r = \left(1 + \alpha + \omega^* + \frac{\bar{\mu}^2 - \underline{\mu}^2}{2}\right) / 2 \tag{3.12}$$

4. Numerical examples:

Example 4.1: For the matrix $\begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}$ considered by A. Hadjimos [2]

$$\underline{\mu} = \frac{2\sqrt{2}}{3} = \bar{\mu}.$$

It is estimated that

$$S\left(P_{\alpha,3,1.5}\right)=0=S\left(P_{0,3,1.5}\right)<S\left(P_{0,1.5,1.5}\right)=0.5<S\left(P_{0,1,1}\right)=0.88889<S\left(P_{0,1,0}\right)=0.94281$$

Example 4.2: For the matrix $A=\begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{71}{10} & \frac{113}{10} \\ \frac{16}{5} & \frac{1}{5} & 1 & 0 \\ 2 & \frac{1}{5} & 0 & 1 \end{bmatrix}$ considered by G.Avdelas and A. Hadjimos [1], $\underline{\mu}=\frac{\sqrt{23}}{5}, \overline{\mu}=\frac{\sqrt{24}}{5}$. It is calculated from the iteration matrices that

$$S\left(P_{\alpha,2.68666667,\frac{5}{3}}\right)=0.56893<S\left(P_{0,\frac{5}{3},\frac{5}{3}}\right)=\frac{2}{3}<S\left(P_{0,1,1}\right)=0.96<S\left(P_{0,1,0}\right)=0.9798$$

It is interesting to note that

$S\left(P_{0,-\frac{5}{4},\frac{5}{3}}\right)$ happens to be 1.30703262 but not $\frac{\sqrt{46}}{12}$ as mentioned by G.Avdelas and A. Hadjimos[1].

5. Conclusion:

It is observed in many examples including the above two that spectral radius of PAOR method lesser than the other methods for any $\alpha \neq -1$ and hence this method plays a major role in the rate of convergence over the other methods considered in this paper.

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