CONSTRUCTION OF A NEW SERIES OF BALANCED INCOMPLETE BLOCK DESIGN

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ABSTRACT

Incomplete Block Designs were introduced by Yates (1936) with a restriction on the block size that all treatments are not present in the blocks k < v, which are eliminating heterogeneity to a greater extent when the number of treatments is large. Several researchers made attempts on the existence and constructions of Incomplete Block Designs This paper presents a new series of construction of Balanced Incomplete Block Design using Partially Balanced incomplete Block Designs.

Keywords: Symmetric Balanced Incomplete Block design, Efficiency, Optimality Criteria.

1. INTRODUCTION

The arrangement of ‘v’ treatments in ‘b’ blocks each sizes k_1, k_2 , …, k_b and treatment is replicated r_1, r_2, …, $r_v$blocks and n_1 pairs of treatments of first associate occurs in λ_1 blocks, n_2 pairs of treatments of second associate occurs in λ_2 blocks and so on n_m pairs of treatments of m^{th} associate occurs in λ_m blocks then the design is said to be General Incomplete Block Design. Some of the incomplete block designs commonly used are Balanced Incomplete Block Design, Partially Balanced Incomplete Block Design, etc.

Definition 1.1: The arrangement of ‘v’ treatments in ‘b’ blocks, each of size k (k <v) and each treatment appears exactly in ‘r’ blocks and each pair of treatments occurs exactly in ‘λ’ blocks is said to be a ‘Balanced Incomplete Block Design’ (BIBD). The values v, b, r, k and λ are called the parameters. The parametric relations of BIBD are: $vr = bk$, $r (k-l) = \lambda (v-1)$, $b \geq v$.

Let Y_{ij} be the observation corresponding to the i^{th} treatment occurring in the j^{th} block. The total number of treatments used for the experimentation is ‘vr’. The statistical linear model for balanced incomplete block design is $Y_{ij} = n_{ij}(\mu + \alpha_i + \beta_j + \varepsilon_{ij})$. It can be expressed in the form of general linear model $Y = X \beta + \varepsilon$ as

$$
\begin{bmatrix} Y_{11} \\ Y_{12} \\ \cdots \\ Y_{1b} \\ \hline Y_{21} \\ Y_{22} \\ \cdots \\ Y_{2b} \\ \hline \cdots \\ \hline Y_{v1} \\ Y_{v2} \\ \cdots \\ Y_{vb} \end{bmatrix}_{vb\times 1}
=
\left[\begin{array}{c|ccccc|cccc}
n_{11} & n_{11} & 0 & .. & 0 & n_{11} & 0 & \cdots & 0 \\
n_{12} & n_{12} & 0 & \cdots & 0 & 0 & n_{12} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
n_{1b} & n_{1b} & 0 & \cdots & 0 & 0 & 0 & \cdots & n_{1b} \\ \hline
n_{21} & 0 & n_{21} & \cdots & 0 & n_{21} & 0 & \cdots & 0 \\
n_{22} & 0 & n_{22} & \cdots. & 0 & 0 & n_{22} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
n_{2b} & 0 & n_{2b} & \cdots & \cdots & 0 & 0 & \cdots & n_{2b} \\ \hline
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline
n_{v1} & 0 & 0 & \cdots & n_{v1} & n_{v1} & 0 & \cdots & 0 \\
n_{v2} & 0 & 0 & \cdots & n_{v2} & 0 & n_{v2} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
n_{vb} & 0 & 0 & \cdots & n_{vb} & 0 & 0 & \cdots & n_{vb}
\end{array}\right]_{vb\times v+b+1}
\begin{bmatrix} \mu \\ \hline \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_v \\ \hline \beta_1 \\ \beta_2 \\ \cdots \\ \beta_b \end{bmatrix}_{(1+v+b)\times 1}
+
\begin{bmatrix} n_{11}\,\varepsilon_{11} \\ n_{12}\,\varepsilon_{12} \\ \cdots \\ n_{1b}\,\varepsilon_{1b} \\ \hline n_{2b}\,\varepsilon_{2b} \\ n_{22}\,\varepsilon_{22} \\ \cdots \\ n_{2b}\,\varepsilon_{2b} \\ \hline \cdots \\ \hline n_{v1}\,\varepsilon_{v1} \\ n_{v2}\,\varepsilon_{v2} \\ \cdots \\ n_{vb}\,\varepsilon_{vb} \end{bmatrix}
$$

(1.1)

$Y = [\ Y_{11}\ Y_{12}\ \ldots\ Y_{1b} |\ Y_{21}\ \ Y_{22}\ \ldots\ Y_{2b}\ |\ \ldots\ |\ \ Y_{v1}\ Y_{v2}\ \ldots\ Y_{vb}\ ]'$ be the vector of responses, X be the design matrix consists the elements n_{ij}, where n_{ij} takes the values 1 or 0 based on the i^{th} treatment present or absent in the j^{th} block, $\beta = [\mu \mid \alpha_1\ \alpha_2\ ...\ \alpha_v \mid \beta_1\ \beta_2\ \ldots\ \beta_b]'$ be the vector of parameters, μ is general mean, α_i is the effect due to the i^{th} treatment, $\beta_j$is the effect due to the j^{th} block and ε be the vector random errors corresponding to Y_{ij} and follows N $(0, \sigma^2 I)$.

Let $N = ((n_{ij}))_{vxb}$ be the incidence matrix of BIBD, where n_{ij} takes one if i^{th} treatment occurs in j^{th} block and zero otherwise, i = 1, 2, ... v; j = 1, 2, ... , b. We have, $N J_{b,1} = rJ_{v,1}$; $J_{1,v} N = k J_{1,v}$ where J is a vector of unities, then we have,

$$
NN' = \begin{bmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \cdots & \cdots & \cdots & \cdots \\ \lambda & \lambda & \cdots & r \end{bmatrix} \tag{1.3}
$$

It can be expressed as $NN' = (r-\lambda)\, I_v + \lambda\, J_{vv}$.

2 CONSTRUCTION OF NEW SERIES OF BIBD'S

Detailed method for the construction of new series of Balanced Incomplete Block Designs is presented with a suitable example.

METHOD 2.1: Let N be the incidence matrix of a Symmetric Balanced Incomplete Block Design with parameters v = b, r = k, λ such that $v+\lambda = 2r$ and J be the matrix of unities. The combinatorial arrangement of the incidence matrix N is

$$
N' = \begin{bmatrix} N & J \\ J & N \end{bmatrix}
$$

The resulting is a Symmetric Balanced Incomplete Block Design with incidence matrix as N′ with parameters $v' = 2v = b'$, $r' = b+r = k'$ $\lambda' = v+\lambda$. The existences of the new series of designs are also proved.

THEOREM 2.1: A Symmetric Balanced Incomplete Block Design with parameters $v' = 2v = b'$, $r' = b+r = k'$ $\lambda' = v+\lambda$ can be constructed using the incidence matrix of a Symmetric Balanced Incomplete Block Design with parameters $v = b$, $r = k$, λ such that $v+\lambda = 2r$ with an arrangement of N and J as

$$N' = \begin{bmatrix} N & J \\ J & N \end{bmatrix}$$

Proof: Let N_{vxb} be the incidence matrix of a Symmetric Balanced Incomplete Block Design with parameters $v = b$, $r = k$, λ. Let J be the matrix of unities of order vxb. It can be observed directly from the arrangement of N and J in N′ will provides Symmetric Balanced Incomplete Block Design with parameters $v' = 2v = b'$, $r' = b+r = k'$ $\lambda' = v+\lambda$.

The method is illustrated in the example 2.1

EXAMPLE 2.1: Let N_{4x4} be the incidence matrix of Balanced Incomplete Block Design with parameters $v = 4 = b$, $r = 3 = k$, $\lambda = 2$.

$$N = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix};$$

The arrange the incidence matrices N and J in N′ as

$$N' = \begin{bmatrix} N & J \\ J & N \end{bmatrix} = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{array}\right]$$

The resulting design N′ is the incidence matrix of a SBIBD with parameters $v' = 8 = b'$, $r' = 7 = k'$, $\lambda' = 6$.

3. EFFICIECY AND OPTIMALITY CRITERIA

The efficiency of any treatment contrast is defined as the ratio of the variance of the estimate of a treatment contrast to the variance of the estimate of the same treatment contrast through an orthogonal design with the same vector of replications and same variance per plot. This ratio is called efficiency factor.

3.1 Efficiency Criteria: If N is the incidence matrix of a Balanced Incomplete Block Design with parameters v, b, r, k, λ. The least square estimate of the parameter is $\hat{\beta} = (X'X)^{-1}X'Y$ after imposing restrictions under H_0,

$$\begin{bmatrix}\hat{\mu}\\ \hat{\alpha}\\ \hat{\gamma}\end{bmatrix} = \begin{bmatrix}1 & n^{-1}RJ_{1,v} & n^{-1}KJ_{1,b}\\ 0 & R-NK^{-1}N' & 0\\ 0 & 0 & B-N'R^{-1}N\end{bmatrix}^{-1}\begin{bmatrix}G/n\\ Q\\ P\end{bmatrix} \quad (3.1)$$

$Q = T–N K^{-1}B$ is the vector of adjusted treatment totals and the estimate of $\hat{\alpha}$ is $Q = C\hat{\alpha}$ where $C = R – NK^{-1}N'$. The C matrix is a singular, positive definite, symmetric matrix, of rank v-1, satisfies the property that $C J_{vx1} = J_{1xv} C = 0$.

$$|C| = \left[\frac{[r(k-1)+\lambda]}{k}\right]^{v-1} \cdot \left[\frac{[r(k-1)-\lambda(v-1)]}{k}\right]$$

Due to its rank is v-1, from the characteristic equation $| C - \theta I |= 0$, It has 'v-1' repeated roots

$$\theta = \left[\frac{[r(k-1)+\lambda]}{k}\right] = \frac{\lambda v}{k} \quad (3.4.4)$$

The efficiency of BIBD over RBD is the ratio of variance contrast between the two designs with the same number of replications is

$$E = \frac{V(\hat{\alpha}_i - \hat{\alpha}_i)_{RBD}}{V(\hat{\alpha}_i - \hat{\alpha}_i)_{BIBD}} = \frac{2\sigma^2/r}{2k\sigma^2/v\lambda} = \frac{\lambda v}{rk} \quad (3.4.8)$$

$\therefore$ The efficiency factor of BIBD is $E = \frac{\lambda v}{rk}$.

Efficiency of BIBD over RBD $E = \lambda v / rk = 1 - [(1/k) - \lambda/kr] < 1$ i.e, BIBD is less efficient than R.B.D.

Note: Yates (1940) noted that, the cases may arise in which the use of ordinary randomized blocks will be more efficient than the use of Balanced Incomplete Block Designs, where as lattice designs can never be less efficient than ordinary randomized blocks. This advantage is shared by the resolvable incomplete Block Designs.

A series of symmetric Balanced Incomplete Block Designs derived using the method 2.1 and their efficiencies are presented in Table 3.1.

Table 3.1: New series of Derived SBIBD

	Symmetric BIBD			Derived Symmetric BIBD			Efficiency of SBIBD's	
	v = b	r = k	λ	v′ = b′	r′ = k′	λ′	E = λv/rk	E′=λ′v′/r′k′
1	4	3	2	8	7	6	0.6666	0.9795
2	5	4	3	10	9	8	0.9375	0.9876
3	6	5	4	12	11	10	0.960	0.9917
4	7	6	5	14	13	12	0.9722	0.9940
5	8	7	6	16	15	14	0.9795	0.9955
6	10	9	8	20	19	18	0.9876	0.9972

3.2 Optimality Criteria: The C matrix of Balanced Incomplete Block Design is singular, positive definite, symmetric, of rank v-1. Its characteristic roots are '0' and 'λv/k' with multiplicity 1 and v-1. Its various Optimality criteria are derived and presented below.

1. A - optimality Criteria = Trace $[(X'X)^{-1}]_D = \lambda v(v-1)/k$
2. D – optimality Criteria = Sup $|(X'X)^{-1}|_D = \{[r(k-1) + \lambda]/k - \theta\}^{v-1}.\{[r(k-1) - \lambda(v-1)]/k-\theta\}$
3. E – optimality Criteria = Maximum Eigen value = $\lambda v / k$.

Table 3.2 Efficiency and Optimality Criteria of BIBD's

No	v	b	r	k	λ	E=λv/rk	E-opt	D-opt	A-opt	G-opt	I-opt
1	4	6	3	2	1	0.666	1.6660	0.020800	6	0.416	1.66
2	4	4	3	3	2	0.888	3.1100	0.111100	9	0.77	3.11
3	9	12	4	3	1	0.750	2.7500	0.000012	12	0.3055	2.75
4	9	36	8	2	1	0.560	1.2050	0.000001	16	0.1338	1.205
5	9	12	3	4	1	0.750	4.0909	0.000355	11	0.4545	4.0909
6	9	12	8	6	5	0.930	2.6875	0.00000031	48	0.2986	2.6875
7	6	15	5	2	1	0.600	1.3500	0.000097	10	0.225	1.35
8	5	10	4	2	1	0.625	1.4580	0.001540	8	0.2916	1.458
9	6	15	10	4	6	0.900	1.2750	0.000024	40	0.2125	1.275
10	13	13	3	4	1	1.083	6.0660	0.000016	15	0.4666	6.066
11	13	13	4	4	1	0.812	4.06250	8503056	16	0.3125	4.0625
12	11	11	5	5	2	0.880	3.3730	0.000000	25	0.3066	3.373
13	5	10	4	2	1	0.625	1.4583	0.00154	8	0.2916	1.4583
14	5	5	4	4	3	0.937	4.0625	0.0625	16	0.8125	4.0625
15	5	10	6	3	3	0.833	1.3888	0.00068	18	0.2777	1.3888
16	6	15	6	2	1	0.500	1.350	0.00009	10	0.225	1.350
17	6	10	5	3	2	0.800	1.733	0.00027	15	0.2888	1.733
18	6	6	5	5	4	0.960	5.040	0.0400	25	0.840	5.040
19	6	6	5	5	6	1.440	4.971	0.0285	35	0.828	4.970
20	7	7	3	3	1	0.777	3.111	0.0017	9	0.4444	3.111
21	7	7	4	4	2	0.875	3.111	0.00097	16	0.4444	3.111
22	7	21	6	2	1	0.583	1.283	0.00001	12	0.1832	1.283
23	8	28	7	2	1	0.571	1.238	0.00007	14	0.154	1.238
24	8	14	7	4	3	0.857	1.785	0.0000021	28	0.2231	1.785

25	8	8	7	7	6	0.979	7.020	0.0200	49	0.8775	7.020
26	10	18	9	5	4	0.888	7.020	0.0000001	45	0.702	7.020
27	10	15	9	6	5	0.925	2.268	0.00000007	54	0.2268	2.2608
28	11	11	6	6	3	0.916	3.361	0.00000047	36	0.3055	3.361
29	11	55	10	2	1	0.550	1.161	0.000000000	20	0.1055	1.161
30	12	44	11	3	2	0.727	1.252	0.000000000	39	0.1043	1.252

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