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Oscillatory Flow of a Circular Cylinder in a Couple-stress Fluid: Case of Resonance

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Abstract:

The flow generated due to rotary oscillations of a circular cylinder about its axis of symmetry in an incompressible Couple-stress fluid is considered. The Reynolds number for the flow is less than unity due to very slow flow and hence nonlinear convective terms in the equations of motion are neglected. A rare but distinct special case in which material constants satisfy a resonance condition is considered. The velocity component for the flow is derived. The Skin friction acting on the cylinder is evaluated and the effect of physical parameters like Reynolds number and Couple stress parameter on the Skin friction due to oscillations are shown through graphs.

***Keywords*:** Couple-stress fluid; Rotary oscillations; Resonance; Skin friction.

1. Introduction

Many authors investigated the flow of Couple-stress fluids in cylindrical geometry. Ariman et al. [1] studied Couple-stress fluids and flow of Micro-polar fluids between two concentric cylinders. Kanwal[2] studied viscous fluid flow of axisymmetric bodies generated due to rotary and longitudinal oscillations. Frater[3,4] evaluated Drag on a circular cylinder oscillating in an elastico-viscous fluid. Ravindran[5] Studied simple oscillatory flow in polar fluids. Soundalgekar et al. [6] analysed effects of Couple stresses on the oscillatory flow past an infinite plate with constant suction. Lakshmana Rao et al. [7-9] studied the oscillatory flows of circular cylinder, spheroid and elliptic cylinder in incompressible Micro-polar fluids, the main thrust of the investigation being the determination of the Drag or Couple as the case may be on the oscillating body. Lakshmana Rao et al. [10] examined Couple-stress fluid flows by analytically and computationally. Iyengar et al. [11-13] studied oscillatory flow of Micro-polar fluid generated by the rotary oscillations of approximate sphere and two concentric spheres. Calmelet-Eluhu et al. [14] studied Micro-polar fluid flow of circular cylinder generated due to longitudinal and torsional oscillations. Fetecau et al. [15] found solutions for the motion of second grade fluid due to longitudinal and torsional oscillations of circular cylinder. Anwar Kamal et al. [16], Aparna et al. [17] examined rotary oscillations of circular cylinder, permeable sphere in an incompressible Micro-polar fluid. Mehrdad Massoudi et al. [18] numerically studied the motion of second grade fluid due to longitudinal and torsional oscillations of a cylinder. Ramkissoon et al. [19], Rao et al. [20], Ramana Murthy et al. [21,22], Nagaraju et al. [23] studied oscillatory flows of circular cylinder performing longitudinal and torsional oscillations in viscous fluid, Couple-stress fluid, Micro-polar fluid. Tekasakul P *et al.* [24] studied rotary oscillations of axi-symmetric bodies in an axi-symmetric viscous flow with slip condition and obtained numerical solutions for sphere and spheroids.

Ramana Murthy *et al.* [25,26] studied case of resonance for micro-polar fluid flow generated due to rectilinear oscillations of circular cylinder and sphere. Ramana Murthy *et al.* [27] studied case of resonance for micro-polar fluid flow generated due to rotary oscillations of sphere. In this paper we propose to investigate incompressible Couple-stress fluid flow due to Circular Cylinder performing rotary oscillations.

2. Basic Equations

The basic equations of an incompressible Couple stress fluid introduced by Stokes (28) are given by:

$$div\bar{Q} = 0 \tag{1}$$

$$\rho\left(\frac{\partial\bar{Q}}{\partial\tau} + \bar{Q}.\nabla_1\bar{Q}\right) = -\nabla_1 P - \mu\nabla_1 \times \nabla_1 \times \bar{Q} - \eta\nabla_1 \times \nabla_1 \times \nabla_1 \times \nabla_1 \times \bar{Q} \tag{2}$$

where $\bar{Q}$ is fluid velocity vector, ρ is density, τ is time, μ is viscosity coefficient.
By neglecting non linear convective terms in (2), we get

$$\rho\frac{\partial\bar{Q}}{\partial\tau} = -\nabla_1 P - \mu\nabla_1 \times \nabla_1 \times \bar{Q} - \eta\nabla_1 \times \nabla_1 \times \nabla_1 \times \nabla_1 \times \bar{Q} \tag{3}$$

For Couple stress fluids, the stress components T_{ij} and Couple stress tensor M satisfy the following constitutive equations.

$$T = -PI + \lambda(\nabla_1.Q)I + \mu(\nabla_1 Q + (\nabla_1 Q)^T) + \frac{1}{2}I \times (\nabla_1.M) \tag{4}$$

$$M = mI + 2\eta\nabla_1(\nabla_1 \times Q) + 2\eta'[\nabla_1(\nabla_1 \times Q)]^T \tag{5}$$

3. Statement and Formulation of the Problem

A circular cylinder of radius a and of infinite length is performing rotary oscillations with velocity $V_0 e^{i\sigma\tau}\boldsymbol{e_\theta}$ about its axis of symmetry in an infinite vat containing incompressible Couple-stress fluid. A cylindrical coordinate system (R, θ, Z) with base vectors $(\boldsymbol{e}_R, \boldsymbol{e}_\theta, \boldsymbol{e}_Z)$ with origin on the axis of the cylinder is considered. Hence the fluid velocity will be in cross sectional plane of the cylinder containing the base vectors $(\boldsymbol{e}_R, \boldsymbol{e}_\theta)$. The velocity is assumed in the form:

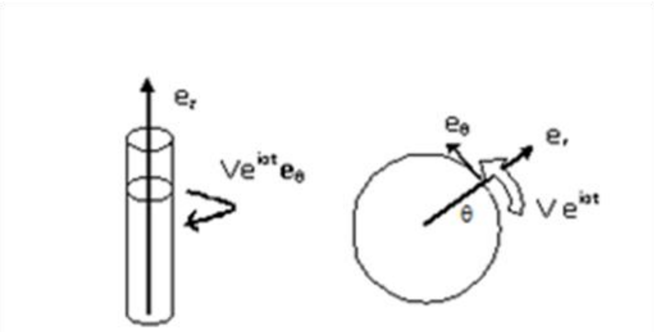


Fig 1 Geometry of the oscillating cylinder

$$\boldsymbol{Q} = V(R)\boldsymbol{e_\theta}\boldsymbol{e}^{i\sigma\tau} \tag{6}$$

The following non-dimensional scheme is introduced. Capitals and LHS terms indicate physical quantities and small letters and RHS terms indicate corresponding non-dimensional quantities.

$$R = ar,\ \ V = v.a\sigma,\ \ \boldsymbol{Q} = \boldsymbol{q}\mathrm{v}_0,\ \ P = p\rho\mathrm{v}_0^2,\ \ \tau = \frac{at}{\mathrm{v}_0} \tag{7}$$

The following are non-dimensional parameters viz, ϖ frequency parameter, S Couple stress parameter and Re Reynolds number for Couple-stress fluids.

$$\varpi = \frac{\sigma a}{\mathrm{V}_0},\ S = \frac{\mu a^2}{\eta},\ \mathrm{Re} = \frac{\rho \mathrm{V}_0 a}{\mu} \text{ which gives } \mathrm{Re}.\varpi = \frac{\rho\sigma a^2}{\mu} \tag{8}$$

By the choice of velocity field in (6) the equations of motion (3) is reduced to

$$i\sigma\rho V = -\frac{P_0}{R} + \mu D_c^2 V - \eta D_c^4 V \tag{9}$$

where $\frac{\partial P}{\partial\theta} = P_0$

Using non dimensional scheme (7) and (8) in (9) we get

$$D_c^4 v - SD_c^2 v + i\varpi Re.sv = -\frac{p_0}{r}Re.S \tag{10}$$

This equation (10) can be written as

$$(D_c^2-\lambda_1^2)(D_c^2-\lambda_2^2)v=-\frac{p_0}{r}Re.S \tag{11}$$

where $D_c^2=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}$ (12)

$\lambda_1^2+\lambda_2^2=S$ and $\lambda_1^2\lambda_2^2=i\varpi Re.S$ (13)

The solution for v if $\lambda_1\neq\lambda_2$ in (11) is given in Lakshmana Rao et al. [29]. The solution for v for the case, $\lambda_1=\lambda_2$ cannot be obtained as a limiting case of $\lambda_1\to\lambda_2$. This case is referred to as "*Resonance*". This resonance occurs if the material coefficients follow the following relation in dimensional form.

$$2\lambda^2=S=4i\varpi Re \tag{14}$$

Now the equations for v for the case of resonance is given by

$$(D_c^2-\lambda^2)^2v=-\frac{p_0}{r}Re.S \tag{15a}$$

For the case of non-resonance

$$(D_c^2-\lambda_1^2)(D_c^2-\lambda_2^2)v=-\frac{p_0}{r}Re.S \tag{15b}$$

3.1. Boundary Conditions

By no-slip condition, the non-dimensional velocity of the circular cylinder Γ is given by

$$v=1 \tag{16}$$

By hyper-stick condition,

Curl $Q_\Gamma=2\bar{e}_z$ which yields on r=1, $\frac{\partial v}{\partial r}=1$ (17)

4. Solution of the Problem

The solution of (15a), velocity function v is assumed in the form

$$v=A_1v_1+A_2v_2-\frac{1}{\lambda^4 r}p_0Re.s \tag{18a}$$

With $(D_c^2-\lambda^2)v_1=0$ and $(D_c^2-\lambda^2)^2v_2=0$ (19a)

these will yield the solutions as

$v_1=K_1(\lambda r)$ and $v_2=rK_1'(\lambda r)$ (20a)

For the case of non-resonance, the corresponding solution will be

$$v=A_1v_1+A_2v_2-\frac{1}{\lambda_1^2\lambda_2^2 r}p_0\,Re.s \tag{18b}$$

With $\left(D_c^2-\lambda_1^2\right)v_1=0$ and $\left(D_c^2-\lambda_2^2\right)v_2=0$ (19b)

these will yield the solutions as for non-resonance case as

$v_1=K_1(\lambda_1 r)$ and $v_2=K_1(\lambda_2 r)$ (20b)

The following results are useful to note.

$D_c^2v_1=\lambda^2v_1$ and $D_c^2v_2=2\lambda v_1+\lambda^2v_2$ (21a)

In case of non-resonance,

$D_c^2v_1=\lambda_1^2v_1$ and $D_c^2v_2=\lambda_2^2v_2$ (21b)

The constants A_1,A_2 are obtained by applying the boundary conditions (16) and (17) to (18a) as follows:

$$\begin{bmatrix}K_1(\lambda) & K'_1(\lambda)\\ \lambda K'_1(\lambda) & \frac{(1+\lambda^2)}{\lambda}K_1(\lambda)\end{bmatrix}\begin{bmatrix}A_1\\A_2\end{bmatrix}=\begin{bmatrix}1-\frac{ip_0}{\varpi}\\ 1+\frac{ip_0}{\varpi}\end{bmatrix} \tag{22a}$$

In the case of non-resonance, the conditions for A_1,A_2 are given by

$$\begin{bmatrix}K_1(\lambda_1) & K_1(\lambda_2)\\ \lambda_1K'_1(\lambda_1) & \lambda_2K'_1(\lambda_2)\end{bmatrix}\begin{bmatrix}A_1\\A_2\end{bmatrix}=\begin{bmatrix}1-\frac{ip_0}{\varpi}\\ 1+\frac{ip_0}{\varpi}\end{bmatrix} \tag{22b}$$

Hence we can calculate A_1 and A_2 from (22) for both the cases. Hence velocity v is known.

4.1. Skin friction acting on the Cylinder per length L

Skin friction acting on the circular cylinder is given by

$$c_f = \frac{2T_{r\theta}}{\rho v_0^2}. \tag{23}$$

Strain rate tensor is given by

$$E = \frac{1}{2}(\nabla_1 \bar{Q} + \nabla_1 \bar{Q}^T) = \begin{bmatrix} \frac{\partial U}{\partial R} & \frac{1}{2}(\frac{\partial V}{\partial R} + \frac{1}{R}\frac{\partial U}{\partial R} - \frac{V}{R}) & \frac{1}{2}(\frac{\partial W}{\partial R} + \frac{\partial U}{\partial Z}) \\ \frac{1}{2}(\frac{\partial V}{\partial R} + \frac{1}{R}\frac{\partial U}{\partial R} - \frac{V}{R}) & \frac{1}{2}(U + \frac{\partial V}{\partial \theta}) & \frac{1}{2}(\frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial Z}) \\ \frac{1}{2}(\frac{\partial W}{\partial R} + \frac{\partial U}{\partial Z}) & \frac{1}{2}(\frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial Z}) & \frac{\partial W}{\partial Z} \end{bmatrix} \tag{24}$$

For this present problem,we get strain rate tensor as

$$E = \begin{bmatrix} 0 & \frac{1}{2}(\frac{\partial V}{\partial R} - \frac{V}{R}) & 0 \\ \frac{1}{2}(\frac{\partial V}{\partial R} - \frac{V}{R}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{25}$$

$$M = \begin{bmatrix} m & 0 & 2\eta\frac{\partial C}{\partial R} \\ 0 & m & 0 \\ 2\eta'\frac{\partial C}{\partial R} & 0 & m \end{bmatrix} where\ C = \frac{\partial V}{\partial R} + \frac{V}{R} \tag{26}$$

$$\nabla_1 . M = \frac{2\eta}{R}\frac{\partial}{\partial R}\left(R\frac{\partial C}{\partial R}\right)\bar{e}_Z \tag{27}$$

$$\text{And } \mathrm{I} \times (\nabla_1 . M) = \begin{bmatrix} 0 & -\frac{2\eta}{R}\frac{\partial}{\partial R}\left(R\frac{\partial C}{\partial R}\right) & 0 \\ \frac{2\eta}{R}\frac{\partial}{\partial R}\left(R\frac{\partial C}{\partial R}\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{28}$$

By substituting (26), (27) and (28) in (4) and simplifying we get

$$T_{R\theta} = \left\{\mu\left(\frac{\partial V}{\partial R} - \frac{V}{R}\right) - \eta\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\left(\frac{\partial V}{\partial R} + \frac{V}{R}\right)\right)\right\}e^{i\varpi t} \tag{29}$$

Using non dimensional scheme (7) and (8) in (29) we get

$$T_{r\theta} = \frac{V_0\eta}{a^3}\left[S\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) - \frac{1}{r}\frac{\partial}{\partial r}\ (rD_c^2 v)\right]. \tag{30}$$

The Skin friction acting on the circular cylinder (after deleting the factor $e^{i\varpi t}$) is obtained as:

$$c_f = \frac{1}{Re.S}\left[S\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) - \frac{1}{r}D_c^2 v - \frac{\partial}{\partial r}D_c^2 v\right]$$

On r=1, $c_f = \frac{-1}{Re.S}[D_c^2 v + \frac{\partial}{\partial r}D_c^2 v]$ (31)

Using the boundary conditions and $\lambda K_1'(\lambda) + K_1(\lambda) = -\lambda K_0(\lambda)$ we have ;

In the resonance case, the Skin friction is given by

$$c_f = \frac{2\lambda^2}{Re.S}[1 - A_2 K_0(\lambda)] \tag{32a}$$

In the non-resonance case, the Skin friction is given by

$$c_f = \frac{1}{Re.S}[\lambda_1^3 K_0(\lambda_1))A_1 + \lambda_2^3 K_0(\lambda_2))A_2] \tag{32b}$$

5. Results and Discussions

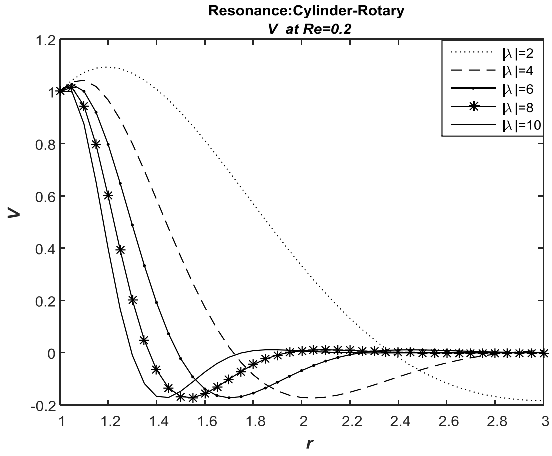
The roots of $x^2 - Sx + i\varpi ReS = 0$ are taken as the values of λ^2.

Hence for non-resonance $\lambda = \sqrt{\frac{S \pm \sqrt{S^2 - 4S.Re.i\omega}}{2}}$ and for resonance $\lambda = \sqrt{\frac{S}{2}}$ (33)

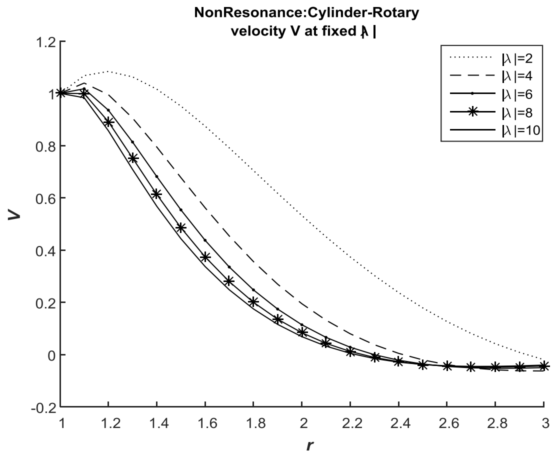
Here ϖ *and* Re are choosen independently, with Re≪1 and ω≫1 such that ω.Re is not negligibly small (say>1) then λ is obtained from (33). Then A_1 and A_2 and hence V and Drag are obtained. To get physical quantities, the corresponding real part of the quantities are taken.

5.1. Velocity

When |λ| is fixed, for resonance Re and ϖ cannot vary independently. From Fig 2, for resonance we observe that as |λ| increases, velocity drastically decreases near to the cylinder and takes negative values. But for non-resonance, as $|\lambda|$ increases, velocity decreases slowly and takes positive values only.



(a)



(b)

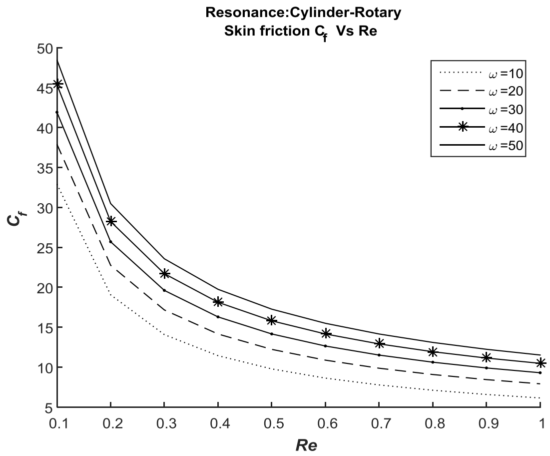
Fig. 2. Velocity at fixed values of $|\lambda|$ for the case of (a) resonance and (b) non-resonance

5.2. Skin friction

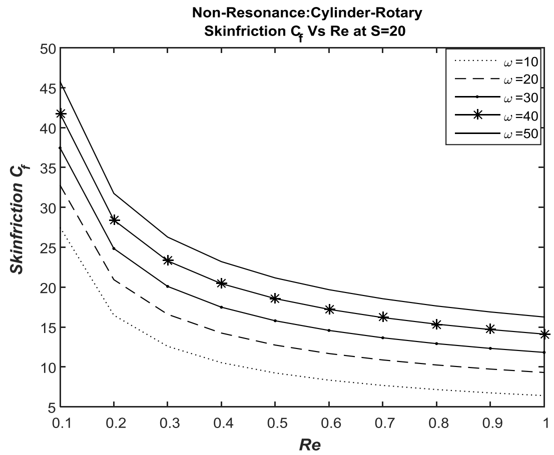
From Fig. 3, the Skin friction is almost same for resonance and non-resonance.

Form Fig. 4, we notice that Skin friction at lower values of Re, is more for resonance but at values near to Re=1, Skin friction for resonance is less than the corresponding values of non-resonance case.

From Fig. 5, we notice that as ϖ increases, skin-friction also increases. But when Re increases, skin-friction decreases, since $Re\varpi$ comes as a unit in the Skin friction. In this case for resonance and non-resonance, Skin friction is almost same. This is because ϖ does not appear explicitly in the formula for skin-friction.

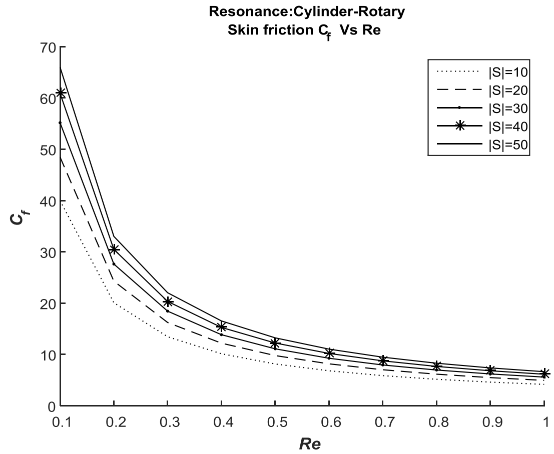


(a)

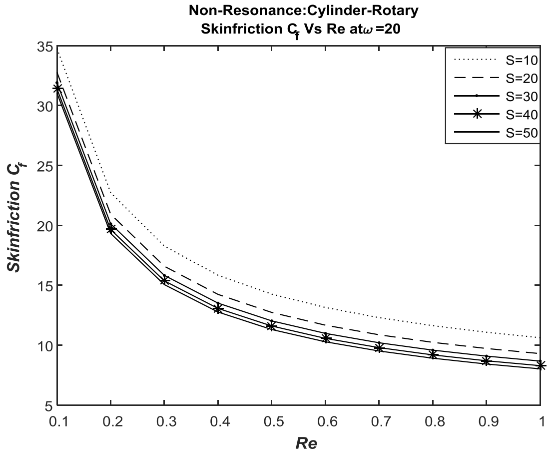


(b)

Fig. 3. Skin friction Vs Re for the case of (a) resonance and (b) non-resonance

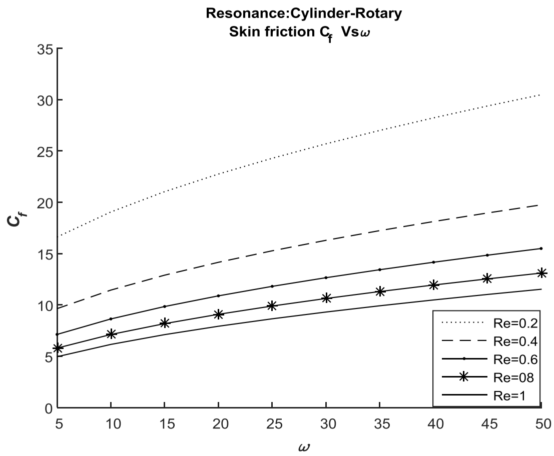


(a)

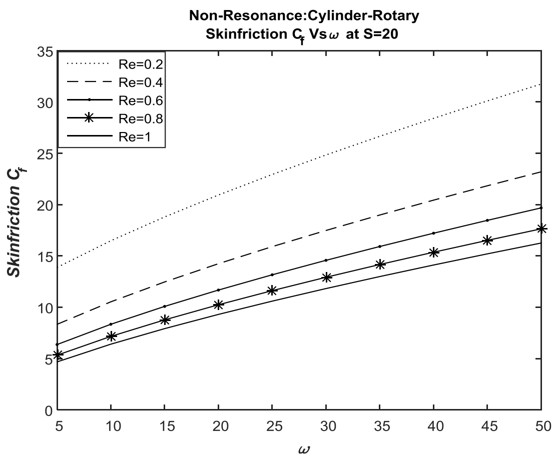


(b)

Fig. 4. Skin friction Vs Reynolds number Re at different values of Couple stress parameter S, for the case of (a) resonance and (b) non-resonance.



(a)



(b)

Fig. 5. Skin friction Vs frequency parameter ϖ different values of Reynolds number Re for the case of (a) resonance and (b) non-resonance.

6. Conclusions

We observe that for resonance,

i) when $|\lambda|$ is fixed, the skin-friction decreases drastically in comparison with non-resonance.

ii) Velocity changes from positive values to negative values near to the cylinder. for non-resonance, velocity takes positive values only.

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