Parameter-uniform hybrid numerical scheme for singularly perturbed initial value problem

Subal Ranjan Sahu

Department of Mathematics, National Institute of Technology Rourkela, Odisha-769008, India.
E-mail: subalranjan1@gmail.com

Dr. Jugal Mohapatra

Department of Mathematics, National Institute of Technology Rourkela, Odisha-769008, India.
Corresponding author: E-mail: jugal@nitrkl.ac.in

Abstract: In this article, a singularly perturbed initial value problem depending on a parameter is considered. A hybrid scheme which is a combination of the second order cubic spline scheme on the fine mesh and the midpoint upwind scheme on the coarse mesh is constructed. It is shown that the proposed scheme is second order convergent in the maximum norm. Error bounds for the numerical solution and its numerical derivative are established. A numerical example is presented which support the theoretical results.

Key words: Singular perturbation, Initial value problem, Hybrid scheme, Shishkin mesh.
Subject Classification: AMS 65L10, 65L12.

1. Introduction

In this article, we consider the following singularly perturbed initial value problem (IVP):

$$\begin{cases} Lu(x) = \varepsilon^2 u''(x) + \varepsilon p(x)u'(x) + q(x)u(x) = f(x), x \in \Omega = (0,T] \\ u(0) = \lambda, L_0 u(0) = u'(0) = \dfrac{\eta}{\varepsilon}, \end{cases} \tag{1.1}$$

where $0 < \varepsilon << 1$, is a small positive parameter, λ and η are given constant. $p(x)$, $q(x)$ and $f(x)$ are sufficiently smooth function in $(0,T]$, with $0 < \alpha \le p(x), 0 < \beta \le q(x) \le \beta^*$. Under these assumptions, the solution $u(x)$ has an exponential boundary layer near $x = 0$. This type of model problem found in quantum mechanics, fluid dynamics and other applied areas [2,4,5,10].

Amiraliyev et al. [1] proposed a fitted finite difference approximation on uniform mesh for IVP (1.1) which was first order convergent. Cen et al. [3] proposed a hybrid scheme which combines the central difference scheme on the fine part and the midpoint upwind scheme on the coarse part for IVP (1.1) on piecewise uniform Shishkin mesh. By solving the difference equation for errors and a different analogue of Gronwall's inequality [11] they got almost second order convergence for the numerical solution and the scaled numerical derivatives.

In this article, our main goal is to construct a parameter-uniform numerical scheme for solving IVP(1.1) and to approximate the solution and its derivatives. Motivated from [3], we give a fitted mesh method to solve the IVP (1.1). Based on the behavior of the exact solution, different types of layer-adapted meshes like Shishkin mesh (S mesh) and Bakhvalov-Shishkin mesh (B-S mesh) are constructed. Then we state a hybrid finite difference scheme which combines the cubic spline difference scheme on the fine mesh region with the midpoint upwind scheme on coarse mesh region [6]. Since estimation for numerical derivatives are desirable in many applied field [7,8,12], we give error bound for the numerical derivatives. Here, C denotes a generic positive constant independent of ε and the mesh parameter N. For any continuous function, the maximum norm is denoted by $\|f\|_{\overline{\Omega}} = \sup_{x\in\overline{\Omega}}|f(x)|$.

2. Properties of the solution and its derivatives

Lemma 2.1 The solution $u(x)$ and its derivatives of IVP(1.1) satisfies the following bound:

$$\|u^k(x)\| \le C\varepsilon^{-k}\left\{\phi + \max_{0\le t\le x}|f(t)| + \varepsilon^j \max_{0\le t\le x}|f(t)| + \int_0^x |f'(t)|dt\right\}, \quad (2.1)$$

for all $x \in \overline{\Omega}, 0 \le k < 6$, where $\phi = \sqrt{|\eta^2 + q(0)\lambda^2 - 2f(0)|}$, $j = \max\{0, k-2\}$.

Proof. The idea of the proof is given in [1].

We decompose the solution $u(x)$ of the IVP (1.1) into regular and singular components as: $u(x) = v(x) + w(x)$. The regular component $v(x)$ is satisfy the following sets of problem:

$$\begin{cases} Lv(x) = f(x) \\ v(0) = v_0(0) + \varepsilon v_1(0) + \varepsilon^2 v_2(0) + +\varepsilon^3 v_3(0) \\ v'(0) = v'_0(0) + \varepsilon v'_1(0) + \varepsilon^2 v'_2(0) + \varepsilon^3 v'_3(0) \end{cases} \quad (2.2)$$

Here v_0, v_1, v_2, v_3 respectively, the solution of the following problems:

$$\begin{cases} v_0(x) = \dfrac{f(x)}{q(x)}, \\ v_1(x) = \dfrac{1}{q(x)}\{\varepsilon v_0'' - p(x)v_0'(x)\}, \\ v_2(x) = \dfrac{1}{q(x)}\{\varepsilon v_1'' - p(x)v_1'(x)\}, \\ v_3(x) = \dfrac{1}{q(x)}\{\varepsilon v_2'' - p(x)v_2'(x)\}. \end{cases} \quad (2.3)$$

The singular components $w(x)$ is satisfy the following problem:

$$\begin{cases} Lw(x) = 0, \\ w(0) = \lambda - v(0), \\ w'(0) = \dfrac{\eta}{\varepsilon} - v'(0). \end{cases} \quad (2.4)$$

Lemma 2.2 The regular components $v(x)$ and its derivatives satisfy the following:

$\left\| v^k(x) \right\| \le C,\ 0 \le k \le 4.$(2.5)

Proof. One can find a similar kind of proof in [3].

Lemma 2.3 The singular components $w(x)$ and its derivatives satisfy the following:

$\left\| w^k(x) \right\| \le C(\varepsilon^{3-k} + \varepsilon^{-k} e^{-mx/\varepsilon} + \varepsilon^{1-k} e^{-mx/2\varepsilon}),\ 0 \le k \le 5$, (2.6)

where $m = \begin{cases} (p(0) - \sqrt{(p^2(0) - 4q(0))})/2, & p^2(0) > 4q(0), \\ p(0)/4, & p^2(0) = 4q(0), \\ p(0)/2, & p^2(0) < 4q(0). \end{cases}$ (2.7)

Proof. The idea of the proof is given in [3].

2.1 Cubic spline approximation

Consider the cubic spline approximation on a variable mesh $\Omega^N = \{0 = x_0 < x_1 < ......x_N = 1\}$, and let $h_i = x_i - x_{i-1}$. For given value of $U(x_0), U(x_1), ...U(x_N)$ of a function $u(x)$ on Ω^N there exists an interpolating cubic spline function $S(x)$ with the following properties:
(i) $S(x)$ coincides with a polynomial of degree three on each subintervals$[x_i - x_{i-1}]$ $i = 1,.......N$.
(ii) $S(x) \in C^2(\overline{\Omega})$ (iii) $S(x_i) = U(x_i)$ for $i = 1,.......N$.
Now for $x \in [x_i - x_{i-1}]$, the cubic spline function is define as follows:

$$S(x) = \frac{(x_i - x)^3}{6h_i} M_{i-1} + \frac{(x - x_{i-1})^3}{6h_i} M_i + \left(U(x_{i-1}) - \frac{h_i^2}{6} M_{i-1} \right)\left(\frac{(x_i - x)}{6h_i} \right) + \left(U(x_i) - \frac{h_i^2}{6} M_i \right)\left(\frac{(x - x_{i-1})}{6h_i} \right).$$

where $M_i = S''(x_i)$. Now from the basic properties of the spline , it should satisfy the following : $\frac{h_i}{6} M_{i-1} + \frac{h_i + h_{i+1}}{6} M_i + \frac{h_{i+1}}{6} M_{i+1} = \frac{U(x_{i+1}) - U(x_i)}{h_{i+1}} - \frac{U(x_i) - U(x_{i-1})}{h_i}$. (2.8)

For obtaining second order approximation of the first order derivative of $u(x)$ one can refer [9]. Now for the IVP (1.1) consider the approximation:

$$\varepsilon^2 M_j + p(x_j)U'(x_j) + q(x_j)U(x_j) = f(x_j), \text{ for } j = i, i \pm 1. \quad (2.9)$$

Substituting this in (2.8), we get the following linear system of equations, for $i = 1,.......N-1$,

$L_{Cu} U_i = r^- U(x_{i-1}) + r^c U(x_i) + r^+ U(x_{i+1}) = F(x_i)$,(2.10)

where $\begin{cases} r^- = \dfrac{h_i}{6\varepsilon^2} q_{i-1} + \dfrac{1}{h_i} - \dfrac{(h_i + 2h_{i+1})}{6\varepsilon(h_i + h_{i+1})} p_{i-1} - \dfrac{h_{i+1}}{3\varepsilon h_i} p_i + \dfrac{h_{i+1}^2}{3\varepsilon h_i} p_{i+1}, \\ r^c = \dfrac{(h_i + h_{i+1})}{3\varepsilon^2} q_i - \dfrac{(h_i + h_{i+1})}{h_i h_{i+1}} + \dfrac{(h_i + h_{i+1})}{6\varepsilon h_{i+1}} p_{i-1} + \dfrac{(h_{i+1}^2 - h_i^2)}{3\varepsilon h_i h_{i+1}} p_i - \dfrac{(h_{i+1} + h_i)}{6\varepsilon h_i} p_{i+1}, \\ r^+ = \dfrac{h_{i+!}}{6\varepsilon^2} q_{i-1} + \dfrac{1}{h_{i+1}} - \dfrac{h_i^2}{6\varepsilon h_{i+1}(h_i + h_{i+1})} p_{i-1} + \dfrac{h_i}{3\varepsilon h_{i+!}} p_i + \dfrac{(h_{i+1} + h_i)}{6\varepsilon h_i (h_i + h_{i+1})} p_{i+1}, \end{cases}$

and $F(x_i)=\frac{h_i}{2}f(x_{i-1})+(h_i+h_{i+1})f(x_{i-1})+\frac{h_{i+1}}{2}f(x_{i+1})$.

3. The difference scheme

In this section, to approximate IVP (1.1), we introduce a hybrid scheme on a non-uniform mesh of N intervals Ω^N. Let σ denotes a mesh transition parameter defined by $\sigma=\min\left\{\frac{T}{2},\frac{4}{m}\ln N\right\}$, which divide Ω^N into two subdomains with $N/2$ equal subintervals. Define a mesh generating function ϕ with $\phi(0)=0$ and $\phi(1/2)=\ln N$. Then the mesh points are given by

$$x_i=\begin{cases}\frac{2\varepsilon}{\beta}\phi(t_i), & i=0,1,...............N/2,\\ 1-\left(1-\frac{2\varepsilon}{\beta}\ln N\right)\left(\frac{2(N-i)}{N}\right) & i=N/2+1,...........N.\end{cases}\tag{3.1}$$

where $t_i=i/N$. let us define a new function ψ closely related to ϕ with $\phi=-\ln\psi$, and satisfies $\psi(0)=1$ and $\psi(1/2)=N^{-1}$. Then the characterizing function ψ is given by

$$\psi(t)=\begin{cases}e^{-2(\ln N)t} & \text{(S mesh)},\\ 1-2(1-N^{-1})t & \text{(B-S mesh)}.\end{cases}$$

3.1 Hybrid difference scheme

In this scheme, we use the cubic spline approximation define in (2.10) in the fine mesh regionand the midpoint upwind scheme on the coarse mesh region of Ω^N.

Let $L_{Mp}U_i=\frac{2\varepsilon^2}{h_i+h_{i+1}}\left(\frac{U_{i+1}-U_i}{h_{i+1}}-\frac{U_i-U_{i-1}}{h_i}\right)+p_{i-1/2}\left(\frac{U_i-U_{i-1}}{h_i}\right)+q_{i-1/2}\overline{U}_i=f_{i-1/2}$, (3.2)

where $U_i=U(x_i)$, $\overline{U}_i=(U_{i-1}+U_i)/2$ and $p_{i-1/2}=p((x_{i-1}+x_i)/2)$; similar for $q_{i-1/2}$ and $f_{i-1/2}$. Thus the hybrid finite scheme for IVP (1.1) takes of the form:

$$L_HU_i=\begin{cases}L_{Cu}U_i, & 1\le i<.N/2,\\ L_{Mp}U_i & N/2\le i<N-1.\end{cases}\tag{3.3}$$

and $L_0U_0=\frac{-U_2+4U_1-U_0}{-(x_2-x_1)+4(x_1-x_0)}=\frac{\eta}{\varepsilon}$, $\quad U(0)=\lambda$. (3.4)

Proposition 3.1 Let u and U_i be respectively the solutions of the IVP (1.1) and the discrete problem (3.3-3.4). Then, we have the following parameter uniform estimate:

$\|u_i-U_i\|\le CN^{-2}\ln^2 N$,(S mesh)

$\|u_i-U_i\|\le CN^{-2}$, (B-S mesh)

Proposition 3.2 Let u and U_i be respectively the solutions of the IVP(1.1)and the discrete problem (3.3-3.4). Then, we have the following parameter error estimate for the scaled numerical derivatives:

$$\varepsilon\left|\frac{U_{i+1}-U_i}{h_{i+1}}-u'_{i+1/2}\right| \le CN^{-2}\ln^2 N \text{ ,(S mesh)}$$

$$\varepsilon\left|\frac{U_{i+1}-U_i}{h_{i+1}}-u'_{i+1/2}\right| \le CN^{-2} \text{ ,(B-S mesh)}$$

4. Numerical results and discussions

Example 4.1 Consider the singularly perturbed IVP:

$$\varepsilon^2 u''(x)+\varepsilon(3+x\sin(x))u'(x)+(1+e^x)u(x)=f(x), 0<x\le 1,$$
$$u(0)=2, u'(0)=1-\frac{1}{2\varepsilon},$$

where $f(x)$ is chosen in such a way that the exact solution is $u(x)=1+x-x^2+e^{-x/2\varepsilon}$. The error of the difference scheme is measured in the discrete maximum norm. For any value of N the maximum pointwise errors E^N and the scaled errorsof numerical derivatives D^N is defined by:

$$E^N=\max_{1\le i\le N}\left|u(x_i)-U_i\right|, D^N=\max_{1\le i\le N}\left|\frac{U_{i+1}-U_i}{h_{i+1}}-u'_{i+1/2}\right|$$

Where U_i is obtained by the proposed methods. Then the corresponding rate of convergence are given by: $r_E=\dfrac{\ln E^N-\ln E^{2N}}{\ln(2\ln N/\ln(2N))}$, $r_D=\dfrac{\ln D^N-\ln D^{2N}}{\ln(2\ln N/\ln(2N))}$.

In Table 1, we present the maximum pointwise error and the corresponding rate for $\varepsilon=1e-4$ and $\varepsilon=1e-8$. Similarly, Table 2 displays the scaled error and the corresponding rate r_D. The results clearly indicate that the proposed scheme is uniformly convergent of order two. To have a proper visualization of the rate of a convergence, the log log plots of the E^N and D^N are shown in Figure 1.

In this article, a singularly perturbed second order IVP is considered. We presented the hybrid scheme on Shishkin type meshes (S mesh and B-S mesh). The proposed hybrid scheme is a combination of cubic spline approximation on the inner region and the midpoint upwind approximation on the outer region. The proposed method generates a second order ε-uniform convergence rate of the numerical solution and the scaled numerical derivatives. The efficiency of the proposed method can be easily seen from the numerical results and the approximation coincides with thetheoretical results.

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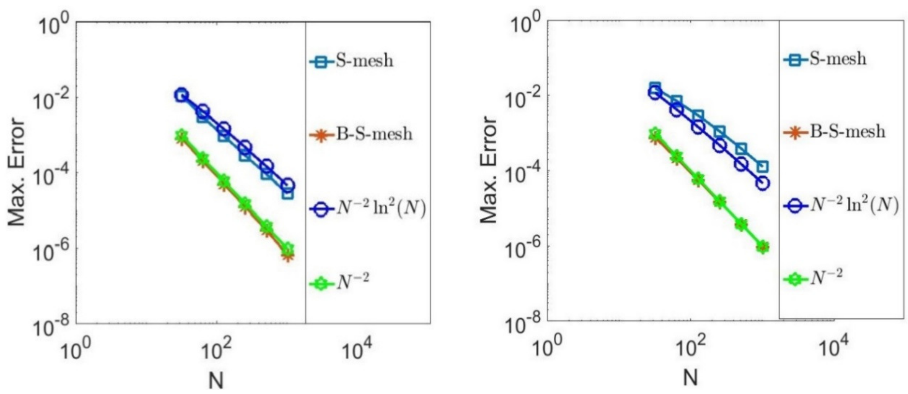

(a) Maximum error (b) Maximum scaled error

Fig1: Loglog plot for Example 4.1

Table1: E^N and r_E generated by the hybrid scheme for Example 4.1.

N	S mesh		B-S mesh	
	$\varepsilon = 1e-4$	$\varepsilon = 1e-8$	$\varepsilon = 1e-4$	$\varepsilon = 1e-8$
32	1.0797e-2 2.4748	1.0797e-2 2.4748	7.9889e-4 2.6732	8.0394e-4 2.6629
64	3.0497e-3 2.2184	3.0497e-3 2.2184	2.0391e-4 2.5678	2.0617e-4 2.5474
128	9.2248e-4 2.0653	9.2249e-4 2.0665	5.1092e-5 2.5055	5.2215e-5 2.4657
256	2.9041e-4 2.0031	2.9022e-4 2.0019	1.2572e-5 2.4877	1.3138e-5 2.4038
512	9.1725e-5 1.9885	9.1726e-5 2.0359	3.0046e-6 2.5416	3.2953e-6 2.3558

Table2: D^N and r_D generated by the hybrid scheme for Example 4.1.

N	S mesh		B-S mesh	
	$\varepsilon = 1e-4$	$\varepsilon = 1e-8$	$\varepsilon = 1e-4$	$\varepsilon = 1e-8$
32	1.5617e-2 1.5422	1.5617e-2 1.5422	7.6177e-4 2.5062	7.6177e-4 2.5062
64	7.1034e-3 1.6511	7.1034e-3 1.6511	2.1174e-4 2.4513	2.1175e-4 2.4503
128	2.9172e-3 1.7480	2.9173e-3 1.7479	5.6525e-5 2.4115	5.6525e-5 2.4114
256	1.0968e-3 1.8299	1.0969e-3 1.8300	1.4661e-5 2.3754	1.4661e-5 2.3754
512	3.8273e-4 1.8927	3.8273e-4 1.8920	3.7377e-6 2.3413	3.7378e-6 2.3409