CELL MODEL OF VISCOUS FLOW PAST A SEMIPERMEABLE CYLINDER

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Abstract. Stokes steady incompressible viscous fluid flow through a partially permeable cylinder is analytically studied taken use of cell model technique. Considered flow is divided into two regions, outer viscous fluid region and inner semipermeable region which are governed by Stokes and Darcy's law respectively. Boundary conditions for the fluid porous interface are continuity of normal component of velocity, vanishing of tangential component of velocity and continuity of pressure accompanying the boundary condition for cell surface. Exact solution and an expression for drag exerted on the cylinder is calculated using stream function. Variation of Kozeny constant against fractional void volume is studied numerically. As special case analytical and numerical results agrees with well known existing results.

Keywords: Semipermeable Cylinder, Stokes flow, Darcy's law, drag force, cell models

1 Introduction

The classical problems considering the fluid motion bearing low Reynolds number past a semipermeable membrane continues to be of much interest due to its application in the various fields like industry chemical, biomedical, meteorology, environmental engineering etc. Semipermeable membrane permits certain types of particles to pass through it that too under certain conditions i.e., basically a membrane with very low permeability.

Due to the vast application of flow past porous medium, several models have been developed. In study of flow past assemblage of porous particles there is complex interaction of numerous particles. To investigate such problem a unit cell model technique was introduced by Happel [1]. In cell model technique the problem of large number of particles is reduced to the problem considering a single particle inside a fluid envelope. Cell model proposed by Happel [2] and Kuwabara [3] are used for modelling flow along an assemblage of circular cylinders (spheres). Happel model consider the vanishing of shear stress condition on the hypothetical cell surface and Kuwabara model consider zero vorticity on the outer cell surface. Flow over a circular cylinder was first studied by Spielman and Goren [4] using Darcy-Brinkman's equation. Brown [5] discused the slow permeation of fluids past assemblage of cylinders. Viscous flow through a cylinder embedded in a porous media using Brinkman's equation was studied by Pop and Cheng [6]. Datta and Shukla [7] modelled the flow past a cylinder having slip boundary condition and obtained the drag. Stokes flow past a swarm of spherical particles in motion with arbitrary direction was evaluated by Dassios and Vafeas [8] by taking use of 3D Happel model. Kim and Yuan [9] proposed a different model in membrane filtration processes to find specific resistance of aggregated colloidal cake layers . Deo et al [10] has focused on studying the flow in circular cylinder with impermeable core for parallel and perpendicular case. The problem of micropolar fluid flow in cylindrical shell for both parallel and perpendicular case was studied by Sherief et al. [11]. Later, Srinivasacharya and Prasad [12] considered the problem of micropolar fluid past a sphere and cylinder which are embedded in a porous media. Shapovalov [14] studied viscous flow problem around a semipermeable spherical particle by considering continuity of normal component of velocity, vanishing of tangential component of velocity and continuity of pressure as boundary conditions and obtained the expression for drag which he found to be lower compared to the drag of a non porous sphere. Presently, we are extending the work done by Shapovalov to the case of perpendicular flow past a circular cylinder in cell model.

In this present study, we are considering flow past a semipermeable cylinder using Happel and Kuwabara cell model. Boundary condition used for this axisymmetric flow are continuity of normal component of velocity, vanishing of tangential component of velocity and continuity of pressure together with the condition used for cell surface. Bounded fluid region and porous region are denoted here by I, II respectively. For region I and region II, we use Stokes approximation and Darcy's law. Expression for stream function, pressure and Kozeny constant are presented. Graphs and tables are obtained for Kozeny constant with respect to fractional void volume.

2 Problem Formation

Consider the axisymmetric viscous fluid flow past a semipermeable cylinder of radius $r = b$ bounded by a cylindrical container having radius $r = a$ $(b \leq a)$ see fig 1. Outside region I and inner porous region II are denoted by i, where $i = 1, 2$ respectively. Equations for flow regions I and II governed by Stokes [1] and Darcy's law [15] are

$$\nabla \cdot \mathbf{q}^{(1)} = 0 \tag{1}$$

$$\nabla p^{(1)} + \mu \nabla \times \nabla \times \mathbf{q}^{(1)} = 0 \tag{2}$$

$$\nabla \cdot \mathbf{q}^{(2)} = 0 \tag{3}$$

$$\nabla p^{(2)} + \frac{\mu}{k} \mathbf{q}^{(2)} = 0 \tag{4}$$

where $\mathbf{q}^{(i)}$ is velocity vector, $p^{(i)}$ is pressure, μ is coefficient of viscosity for both the regions and k permeability of the porous region.

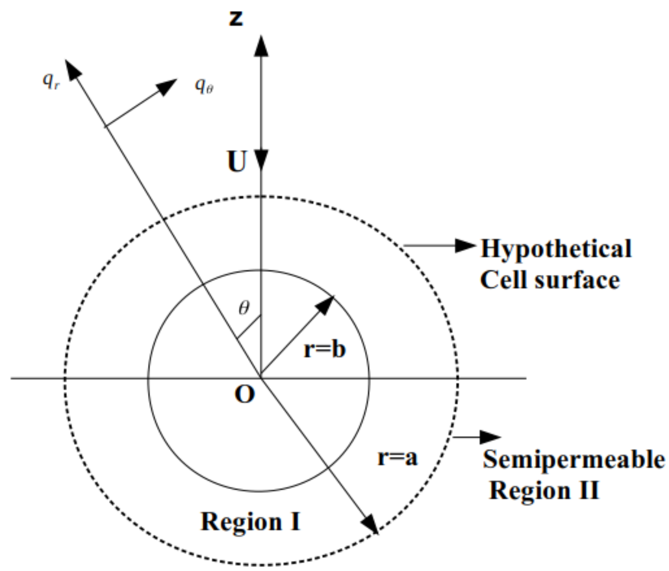

Fig. 1. Co-ordinate of semipermeable cylinder in cell model

Non-dimensional variable are considered to convert the governing equations into dimensionless form as

$$r = b\,\tilde{r}, \mathbf{q}^{(i)} = U\,\tilde{\mathbf{q}}^{(i)},\ \nabla = \frac{\tilde{\nabla}}{b},\ p^{(i)} = \frac{\mu U}{b}\tilde{p}^{(i)}, \tag{5}$$

and substituting them in Eqs (1) to (4) and then dropping the tildes we get

$$\nabla \cdot \mathbf{q}^{(1)} = 0 \tag{6}$$

$$\nabla p^{(1)} + \nabla \times \nabla \times \mathbf{q}^{(1)} = 0 \tag{7}$$

$$\nabla \cdot \mathbf{q}^{(2)} = 0 \tag{8}$$

$$\nabla p^{(2)} + \beta^2 \mathbf{q}^{(2)} = 0 \tag{9}$$

Where, $\beta^2 = \frac{b^2}{k}$.

Let (r, θ, z) be the co-ordinate in cylindrical system with z axis towards the axis of cell surface. As the fluid flow is two dimensional and quasi steady, so all the quantities involved in fluid flow does'nt depend on z. Velocity vectors are expressed as

$$\mathbf{q}^{(i)} = q_r^{(i)}(r, \theta)\,\mathbf{e_r} + q_\theta^{(i)}(r, \theta)\,\mathbf{e}_{\boldsymbol{\theta}},\ \ i = 1, 2 \tag{10}$$

Let, the surface of the cylinder be $r = b$. We define stream functions satisfying continuity equations $\psi^{(i)}$; $i = 1, 2$ for the flow in bounded and porous region of the semipermeable cylinder respectively.

Components of velocity in terms of stream function are

$$q_r^{(i)} = \frac{1}{r}\frac{\partial \psi^{(i)}}{\partial \theta},\ q_\theta^{(i)} = -\frac{\partial \psi^{(i)}}{\partial r};\ \ i = 1, 2 \tag{11}$$

Using Eqs (11) and then eliminating pressure terms from Eqs. (7) and (9) we obtain

$$\nabla^4 \psi^{(1)} = 0, \tag{12}$$

$$\nabla^2 \psi^{(2)} = 0. \tag{13}$$

Where $\nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$ is the Laplacian operator.

The tangential and normal stresses components are given by

$$\tau_{r\theta}^{(1)} = \mu \left[\frac{1}{r^2}\frac{\partial^2 \psi^{(1)}}{\partial \theta^2} + \frac{1}{r}\frac{\partial \psi^{(1)}}{\partial r} - \frac{\partial^2 \psi^{(1)}}{\partial r^2} \right], \tag{14}$$

$$\tau_{rr}^{(1)} = -p^{(1)} + \frac{2\mu}{r}\left[\frac{\partial^2 \psi^{(1)}}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial \psi^{(1)}}{\partial \theta} \right], \tag{15}$$

3 Required Boundary Conditions

Boundary conditions for the present formulated problem of semipermeable particle i.e., particle satisfying Darcy's law for the governing equations are continuity of normal component of velocity, vanishing of tangential component of velocity, continuity of pressure [13, 14] together with boundary condition for cell models.

At the interface between fluid-porous region of the cylinder $r = b$, the boundary conditions are

$$q_r^{(1)} = q_r^{(2)}, \tag{16}$$

$$q_\theta^{(1)} = 0, \tag{17}$$

$$p^{(1)} = p^{(2)}, \tag{18}$$

and at cell surface of cylindrical envelope $r = a$, the boundary conditions are

$$q_r^{(1)} + \cos\theta = 0, \tag{19}$$

For Happel model

$$t_{r\theta}^{(1)} = 0, \tag{20}$$

For Kuwabara model

$$curl\,\mathbf{q}^{(1)} = 0. \tag{21}$$

The boundary condition on the inner surface $r = 1$ using stream function $\psi^{(i)}$, where $i = 1, 2$ are

$$\frac{\partial \psi^{(1)}}{\partial \theta} = \frac{\partial \psi^{(2)}}{\partial \theta}, \tag{22}$$

$$\frac{\partial \psi^{(1)}}{\partial r} = 0, \tag{23}$$

$$p^{(1)} = p^{(2)}, \tag{24}$$

and on cell surface $r = \lambda^{-1}$ ($\lambda = b/a$)

$$\frac{\partial \psi^{(1)}}{\partial \theta} + r\cos\theta = 0, \tag{25}$$

For Happel model

$$\left(\nabla^2 - 2\frac{\partial^2}{\partial r^2}\right)\psi^{(1)} = 0, \tag{26}$$

For Kuwabara model

$$\nabla^2 \psi^{(1)} = 0. \tag{27}$$

4 Solution part

The expression of solution for the region I and II obtained by are

$$\psi^{(1)} = \left[A\,r + B\,r^3 + \frac{C}{r} + D\,r\ln r\right]\sin\theta \tag{28}$$

$$\psi^{(2)} = E\,r\sin\theta \tag{29}$$

The expression for pressures in both the regions are given as

$$p^{(1)} = \left(8B\,r - \frac{2D}{r}\right)\cos\theta \tag{30}$$

$$p^{(2)} = -\beta^2 E\,r\cos\theta \tag{31}$$

Where A, B, C, D and E are arbitrary constants to be determined.

5 Drag force acting on the body

Drag force acting on the cylinder due to the viscous flow can be evaluated by using the formula

$$F_D = \int_0^{2\pi} \left[\tau_{rr}\cos\theta - \tau_{r\theta}\sin\theta\right]_{r=1} r\,d\theta$$

By substituting Eqs (28) in the above integral we get

$$F_D = 4\pi\mu U D \tag{32}$$

Drag acting on the semipermeable cylinder in presence of cylindrical cell is

– Happel Model

$$F_D = -8\pi\mu U\left[-\frac{\left(\left(\beta^2-4\right)\ \lambda^4+\beta^2\right)}{\left(2\left(\beta^2-4\right)\ \ln\left(\frac{1}{\lambda}\right)+\beta^2+14\right)\ \lambda^4+\beta^2\ \left(2\ln\left(\frac{1}{\lambda}\right)-1\right)+2}\right] \tag{33}$$

– Kuwabara Model

$$F_D = -8\pi\mu U\left[\frac{2\beta^2}{4\left(\beta^2+2\right)\ \lambda^2+\beta^2\ \left(4\ln\left(\frac{1}{\lambda}\right)-3\right)-\left(\beta^2-4\right)\ \lambda^4+4}\right] \tag{34}$$

5.1 Special results

If $\beta \to \infty$ i.e., permeability $k = 0$, it behaves as solid cylinder and the drag force is

– Happel model

$$F_D = -8\pi\mu U b\left[\frac{(\lambda^4+1)}{\lambda^4+2(\lambda^4+1)\ln\left(\frac{1}{\lambda}\right)-1}\right] \tag{35}$$

– Kuwabara model

$$F_D = -8\pi\mu U b\left[\frac{2}{4\lambda^2+4\ln\left(\frac{1}{\lambda}\right)-\lambda^4-3}\right] \tag{36}$$

which agrees with the results obtained in Happel and Brenner [1].

Expression for Kozeny constant [18, **?**] given by Kozeny carman [17] for flow past a cylinder with perpendicular flow is

$$k_z = -\frac{\varepsilon^3 F_D}{4(1-\varepsilon)} \tag{37}$$

where

$$\varepsilon = (1-\frac{\pi b^2}{\pi a^2}) = (1-\gamma) = (1-\lambda^2)$$

The variation of Kozeny constant k_z with respect to fractional void volume ε is shown by Fig. 2 and table 1 for different values of permeability parameter k_1 ($k_1 = \frac{1}{\beta^2}$). It is interesting to note that for semipermeable cylindrical particle the value of Kozeny constant for fixed permeability value increases for increasing void volume. Moreover, for increasing permeability the decrease in the value of Kozeny constant is observed. It is noticed that when fractional void volume ε is small, Kozeny constant is weaker and Kozeny constant increases at high rate as ε approaches to 1.

From given table the values of Kozeny constant for solid cylinder and semipermeable cylinder for the case of perpendicular flow are obtained. The results for the case of solid cylinder are identical to the values obtained by Deo et al.[10]

and Saad [19]. New result for the case of semipermeable cylinder is presented. By giving attention to the obtained results it is seen that the magnitude of Kozeny constant for semipermeable cylinder are extremely less compared to solid cylinder in cell model. However these values are relatively higher for the case of Kuwabara model compared to Happel model.

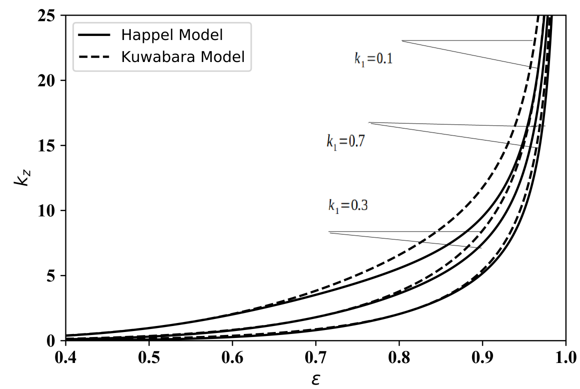

Fig. 2. Variation of Kozeny constant against fractional void volume with varying permeability for $\lambda = (1-\varepsilon)^{1/2}$

Table 1. Magnitude of Kozeny constant against fractional void volume for solid cylinder($\beta \to \infty$) and semipermeable cylinder($\beta = (0.4)^{-1}$)

	k_z			
ε	Solid		Semipermeable	
	Happel model	Kuwabara model	Happel model	Kuwabara model
0.1	5.72530	6.16400	0.00137	0.00192
0.2	5.50758	6.36223	0.01475	0.01923
0.3	5.35972	6.60756	0.06837	0.08237
0.4	5.30187	6.92065	0.22660	0.25097
0.5	5.36785	7.33706	0.61545	0.63439
0.6	5.62053	7.92424	1.41455	1.41456
0.7	6.19507	8.82976	2.76247	2.85915
0.8	7.45963	10.461	4.79917	5.38812

6 Conclusion

Present work has been done for finding the exact solution of the problem related to stokes flow through a semipermeable cylinder. Stokes and Darcy's equations governing the flow field for the present problem are solved analytically and an expression for drag exerted on the semipermeable cylindrical particle is calculated. Numerical study of variation of Kozeny constant against void fraction is presented and it matches with earlier well known results in reduction case. From the above investigation it is seen that the drag exerted on semipermeable cylinder is lower than the drag on non-porous cylinder.

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