Numerical Solution of Singularly Perturbed Differential-Difference Equations with Mixed Shifts using Stable Finite Difference Method

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Abstract

In this paper, an extended stable finite difference scheme is proposed for solving singularly perturbed differential-difference equations whose solutions exhibit dual layer behaviour. The reason behind opting this difference method is that, the central difference approximation is re-approximated by rewriting its error terms as a combination of first and second derivative terms and approximating them. As a result, these methods stabilize the central difference scheme while improving its accuracy. In this scheme, the original problem is transformed to an asymptotically equivalent second order singular perturbation problem. Then, second order stable central difference scheme has been applied to get a three term recurrence relation which is solved by Thomas algorithm. To illustrate the applicability of the method, numerical examples have been solved for different values of the delay parameter, advance parameter and perturbation parameter. Maximum absolute errors are computed, tabulated and compared with the results of the other methods discussed in the literature. The effect of small shifts on the solutions has been presented in graphs. Convergence of the proposed method has also been established.

**Keywords**: Singularly perturbed differential difference equation, Dual boundary layer, Central differences, Tri-diagonal system.

**NOMENCLATURE**

|  |  |  |
| --- | --- | --- |
| |  |  | | --- | --- | | perturbation parameter  delay parameter  advance parameter  *N* Number of subintervals  *h* mesh size  mesh points  , boundary functions | error terms  local truncation error | |

# INTRODUCTION

Any system involving a feedback control will almost involve time delays. These arise because a finite time is required to sense information and then react to it. If we restrict the class of delay differential equations to a class in which the highest derivative is multiplied by a small positive parameter and involving at least one delay and advanced term, then it is said to be a singularly perturbed differential difference equation. In the recent years, there has been a growing interest in the numerical treatment of such differential equations. This is due to the versatility of such type of differential equations in the mathematical modeling of various physical and biological phenomena, for example; micro scale heat transfer, hydrodynamics of liquid helium, thermo elasticity, diffusion in polymers, reaction-diffusion equations, stability, control of chaotic systems, a variety of models for physiological processes. For a detailed discussion on differential-difference equation one may refer to the books and high level monographs: Bellman and Cooke (1963), Driver (1977) and El’sgol’ts and Norkin (1973).

An extensive work had been done by Kadalbajoo and Sharma (2005), Kadalbajoo and Ramesh (2007) and Kadalbajoo and Kumar (2008) for solving singularly perturbed delay differential equations using finite differences, fitted mesh and B spline. Lange and Miura (1985, 1994) gave asymptotic approaches for linear second order differential difference equations in which the highest order derivative is multiplied by small parameter. The effect of small shifts on the oscillatory solution of the problem has been discussed by Lange and Miura (1994). Soujanya and Reddy (2016) described a numerical integration method via Simpson’s rule to solve singularly perturbed delay differential equations with layer or oscillatory behaviour. With this motivation, in this approach, with the help of Taylor series, we approximate the terms containing small shifts and the singularly perturbed differential difference equation is asymptotically replaced by singularly perturbed differential equation. In section-2, the numerical method for the solution is discussed. Section-3 is selected for the convergence analysis of the method. Numerical experiments carried out to validate the method in section-4. Finally conclusion is given in last section.

# Numerical Method

Consider a singularly perturbed differential-difference equation of the form:

 (1)

on (0, 1), under the boundary conditions

 (2)

where is a small parameter,, *b*(*x*), *c*(*x*),andare differentiable functions on (0,1) and are respectively the delay and the advance parameters. If  on , then the solution of Eq. (1) exhibits boundary layers at both ends of the interval , whereas it exhibits oscillatory behaviour for . Here, we have considered the first case where the solutions of the problem exhibit dual layers. The layer behaviour of the problem under consideration is maintained only for  and , but sufficiently small.

Using Taylor series expansion for the retarded terms in the neighborhood of the point *x,* we have

 (4)

 (5)

Using Eqs.(4) and (5) into Eq. (1), we obtain an asymptotically equivalent second order singular perturbation problem of the form

 (6)

 (7)  (8)

where

and. The transition from Eq. (1) to Eq. (6) is admitted, because of the condition that  and  are sufficiently small. Further details on the validity of this transition can be found in El’sgol’ts and Norkin (1973).

Discretizing the domain [0, 1] into *N* +1 mesh points with constant mesh length *h.* Let  be the mesh points. Then, . For convenience, denote ,  and . Assuming that  has continuous fourth derivatives on , the following central difference formulae for  and  can be obtained at :

 (9)

 (10)

where, , and  for 

Using Eqs. (9) and (10) into Eq. (6), we get the difference equation in a form that includes the  error term for . That is,

 (11)

where .

Further, differentiating Eq. (6), we have:

 (12)

Using Eq. (12) into Eq. (11), which have the stabilizing effect, using central difference formulae for and finally, we get the stable central difference scheme as follows:

 (13)

where



is the local truncation error. Eq. (13) is a three term recurrence relation of form:

;  (14)

where

 ,

 , 

This tri-diagonal system has been solved using Thomas Algorithm.

# Convergence Analysis

Writing the tri-diagonal system Eq. (14) in matrix-vector form, we obtain

 (15)

where, is a tri-diagonal matrix of order *N*-1 , with



 is a column vector with ,  with local truncation error  (16)

where .We also have

 (17) anddenote the actual solution and the local truncation error respectively.

Using Eqs. (15) and (16), we get

 (18)

Thus, the error equation is

 (19)

where .

Let  be the sum of elements of the *ith* row of *A*, then we have



Since and , for sufficiently small  the matrix *A* is irreducible and monotone. Then it follows that exists and its elements are non negative.

Hence, from Eq. (19), we get

 (20)

Also

 (21)

Let  be theelement of .

Since ,we have

 (22)

from the theory of matrices. Therefore, it follows that

 (23)

where . We define

 and

.

Therefore, using Eq.(16), Eq. (20), and Eq.(23), we get

 (24)

Eq. (24) implies that

 (25)

Where



and is some number between  and .

Hence, using Eq. (25) we get



i.e., the proposed method gives second order convergence on uniform mesh.

## Numerical Experiments

To demonstrate the applicability of the method, we have applied it on two cases of the problems of the form Eq. (1)-(2) with constant coefficients.

The exact solution of the problem Eq. (1)-(2) is 

where**,** 

**Example 1.** Consider the model problem given by equations (1)-(3) with 

The numerical results are given in Tables 1-2 for  =0.1 and different values of the delay and advance parameters.

**Example 2.**  Consider the model problem given by equations (1)-(3) with



The numerical results are given in Tables 3 for  =0.1 and different values of the delay and advance parameters.

## Conclusion

Singularly perturbed differential-difference equations whose solutions exhibit dual layer are solved using a stable second order method. The proposed method has been implemented on model examples for different values of the perturbation, delay and advance parameters. Maximum absolute errors are computed, tabulated and compared with Kadalbajoo and Sharma (2005) to show the efficiency of the method. The effect of small shifts on the solutions has been plotted in graphs. From the results, it is observed that the present method approximates the exact solution very well. From the graphs, it has been observed that by increasing the value of either δ or η, the thickness of the left boundary layer decreases and that of the right boundary layer increases. Thus, it can be concluded that a negative shift is more dominant than a positive shift. Theoretical convergence of the method has also been established.

**Table 1. Maximum absolute errors in the solution of Example 1 for ,**  \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ *N*\   

\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

Present method

0.00 2.208e-003 2.212e-005 2.212e-007 2.213e-009

0.03 1.540e-003 1.543e-005 1.544e-007 1.543e-009

0.06 1.044e-003 1.050e-005 1.050e-007 1.050e-009

0.09 6.956e-004 7.006e-006 7.007e-008 0.0000014

Results in Kadalbajoo and Sharma (2005)

0.00 0.004070 0.0004280 0.00004302 0.00000430

0.03 0.002370 0.0002586 0.00002608 0.00000261

0.06 0.000453 0.0000667 0.00000689 0.00000069

0.09 0.001684 0.0001482 0.00001463 0.00000146

\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

**Table 2. Maximum absolute errors in the solution of Example 1 for ,** 

*\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ N*\   

*\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_*

Present method

0.00 1.736e-003 1.745e-005 1.745e-007 6.044e-010

0.03 1.194e-003 1.197e-005 1.197e-007 7.003e-010

0.06 7.970e-004 8.034e-006 8.034e-008 1.050e-009

0.09 5.246e-004 5.307e-006 5.308e-008 1.543e-009

Results in Kadalbajoo and Sharma (2005)

0.00 0.0077728 0.0008525 0.0000859 0.0000086

0.03 0.0030685 0.0003926 0.0000401 0.0000040

0.06 0.0031451 0.0002386 0.0000230 0.0000023

0.09 0.0107514 0.0010400 0.0001036 0.0000103

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