Discrete Mathematical Structures Tutorial

Week 4

Portions: Quantifiers

1. Fill in the blanks

Statement	When it is true?	When it is false?
$\exists x\, p(x)$	For some (at least one) a in the universe, $p(a)$ is true.	
$\forall x\, p(x)$		There is at least one replacement a from the universe for which $p(a)$ is false.
$\exists x\, \neg p(x)$	For at least one choice a in the universe, $p(a)$ is false, so its negation $\neg p(a)$ is true.	
$\forall x\, \neg p(x)$	For every replacement a from the universe, $p(a)$ is false and its negation $\neg p(a)$ is true.	

2. Translate the following into English sentences for p(x,y):x+y=17.

a) For every integer x, there exists an integer y such that x+y=17.

b) There exists an integer x, for all integers y such that x+y=17.

3.

Let $p(x)$, $q(x)$ denote the following open statements.

$$p(x):\quad x \le 3 \qquad q(x):\quad x+1 \quad \text{is odd}$$

If the universe consists of all integers, what are the truth values of the following statements?

a) $p(1)$ **b)** $q(1)$ **c)** $\neg p(3)$

d) $q(6)$ **e)** $p(7) \vee q(7)$ **f)** $p(3) \wedge q(4)$

g) $p(4)$ **h)** $\neg(p(-4) \vee q(-3))$ **i)** $\neg p(-4) \wedge \neg q(-3)$

Ans:

4.

For the universe of all integers, let $p(x)$, $q(x)$, $r(x)$, $s(x)$, and $t(x)$ be the following open statements.

$p(x)$: $x > 0$
$q(x)$: x is even
$r(x)$: x is a perfect square
$s(x)$: x is (exactly) divisible by 4
$t(x)$: x is (exactly) divisible by 5

a) Write the following statements in symbolic form.

- **i)** At least one integer is even.
- **ii)** There exists a positive integer that is even.
- **iii)** If x is even, then x is not divisible by 5.
- **iv)** No even integer is divisible by 5.
- **v)** There exists an even integer divisible by 5.
- **vi)** If x is even and x is a perfect square, then x is divisible by 4.

c) Express each of the following symbolic representations in words.

- **i)** $\forall x\,[r(x) \rightarrow p(x)]$
- **ii)** $\forall x\,[s(x) \rightarrow q(x)]$
- **iii)** $\forall x\,[s(x) \rightarrow \neg t(x)]$
- **iv)** $\exists x\,[s(x) \land \neg r(x)]$
- **v)** $\forall x\,[\neg r(x) \lor \neg q(x) \lor s(x)]$

Ans:

5. Establish the validity of the following argument.

For the universe of all people, consider the open statements

$m(x)$: x is a mathematics professor $\qquad$ $c(x)$: x has studied calculus.

Now consider the following argument.

All mathematics professors have studied calculus.
Leona is a mathematics professor.
Therefore Leona has studied calculus.

Ans:

$$\begin{array}{l} \forall x[m(x) \rightarrow c(x)] \\ \underline{m(l) \qquad\qquad} \\ \therefore c(l) \end{array}$$

Steps	**Reasons**
1) $\forall x[m(x) \rightarrow c(x)]$	Premise
2) $m(l)$	Premise
3) $m(l) \rightarrow c(l)$	Step (1) and the Rule of Universal Specification
4) $\therefore c(l)$	Steps (2) and (3) and the Rule of Detachment

6. Consider the universe of all triangles and the following open statements and prove the validity of the following argument.

$p(t)$: t has two sides of equal length

$q(t)$: t is an isosceles triangle

$r(t)$: t has two angles of equal measure.

In triangle XYZ there is no pair of angles of equal measure.
If a triangle has two sides of equal length, then it is isosceles.
If a triangle is isosceles, then it has two angles of equal measure.
Therefore triangle XYZ has no two sides of equal length.

Ans:

7. Establish the validity of the following argument.

$$\begin{array}{l} \forall x[p(x) \rightarrow q(x)] \\ \underline{\forall x[q(x) \rightarrow r(x)]} \\ \therefore \forall x[p(x) \rightarrow r(x)] \end{array}$$

Ans:

Steps	**Reasons**
1) $\forall x[p(x) \rightarrow q(x)]$	Premise
2) $p(c) \rightarrow q(c)$	Step (1) and the Rule of Universal Specification
3) $\forall x[q(x) \rightarrow r(x)]$	Premise
4) $q(c) \rightarrow r(c)$	Step (3) and the Rule of Universal Specification
5) $p(c) \rightarrow r(c)$	Steps (2) and (4) and the Law of the Syllogism
6) $\therefore \forall x[p(x) \rightarrow r(x)]$	Step (5) and the Rule of Universal Generalization

8. Establish the validity of the argument using the following open statements.

For the universe of all real numbers, consider the open statements

$$p(x):\quad 3x-7=20 \qquad q(x):\quad 3x=27 \qquad r(x):\quad x=9.$$

1) If $3x-7=20$, then $3x=27$.
2) If $3x=27$, then $x=9$.
3) Therefore, if $3x-7=20$, then $x=9$.

Ans:

9. Establish the validity of the following argument.

$$\begin{array}{l} \forall x[p(x) \vee q(x)] \\ \underline{\forall x[(\neg p(x) \wedge q(x)) \rightarrow r(x)]} \\ \therefore \forall x[\neg r(x) \rightarrow p(x)] \end{array}$$

Ans: Hint: You must assume ~r(c) as one of the assumed premise since the conclusion contains the universal quantifier.

Also ~r(c)

p(c)

$\therefore\ \forall x[\neg r(x) \rightarrow p(x)]$

10.

For each of the following (universes and) pairs of statements, use the Rule of Universal Specification, in conjunction with Modus Ponens or Modus Tollens, in order to fill in the blank line so that a valid argument results.

a) [The universe comprises all real numbers.]
All integers are rational numbers.
The real number π is not a rational number.
$\therefore$ ______________________________

b) [The universe comprises the present population of the United States.]
All librarians know the Library of Congress Classification System.
____
$\therefore$ Margaret knows the Library of Congress Classification System.

c) [The same universe as in part (b).]
____
Sondra is an administrative director.
$\therefore$ Sondra knows how to delegate authority.

11.

Determine which of the following arguments are valid and which are invalid. Provide an explanation for each answer. (Let the universe consist of all people presently residing in the United States.)

a) All mail carriers carry a can of mace.
Mrs. Bacon is a mail carrier.
Therefore Mrs. Bacon carries a can of mace.

b) All law-abiding citizens pay their taxes.
Mr. Pelosi pays his taxes.
Therefore Mr. Pelosi is a law-abiding citizen.

c) All people who are concerned about the environment recycle their plastic containers.
Margarita is not concerned about the environment.
Therefore Margarita does not recycle her plastic containers.

Ans:

12.

Establish the validity of the following argument.

$$\begin{array}{l} \forall x[p(x) \rightarrow (q(x) \wedge r(x))] \\ \underline{\forall x[p(x) \wedge s(x)]} \\ \therefore \forall x[r(x) \wedge s(x)] \end{array}$$

Ans: