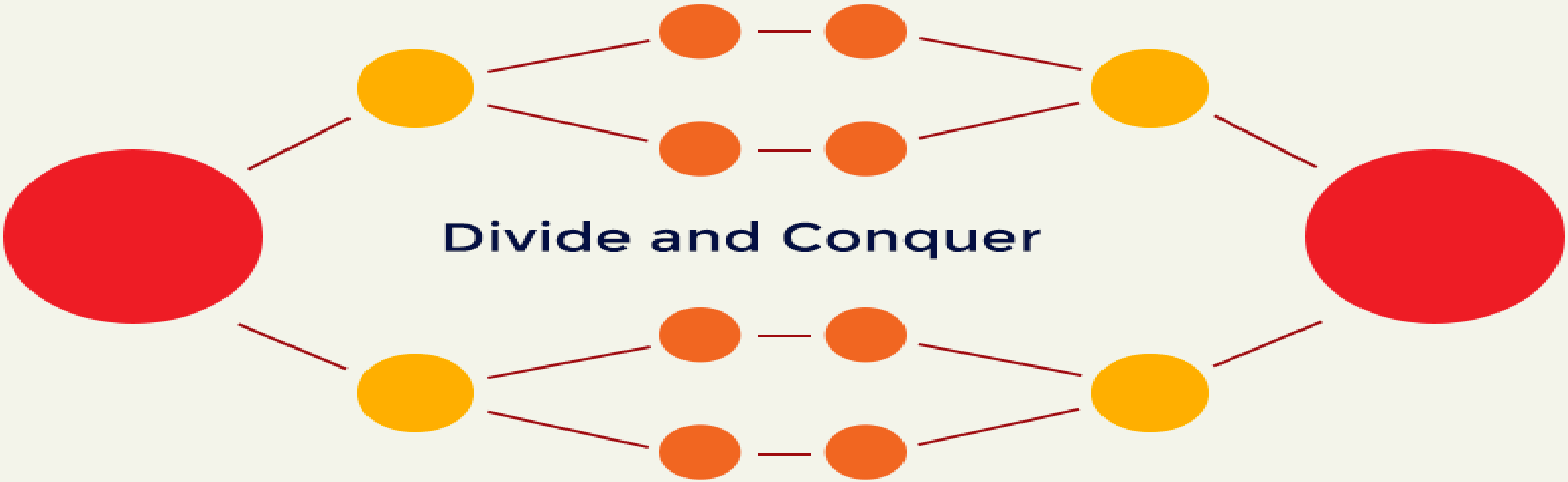
DIVIDE / CONQUER



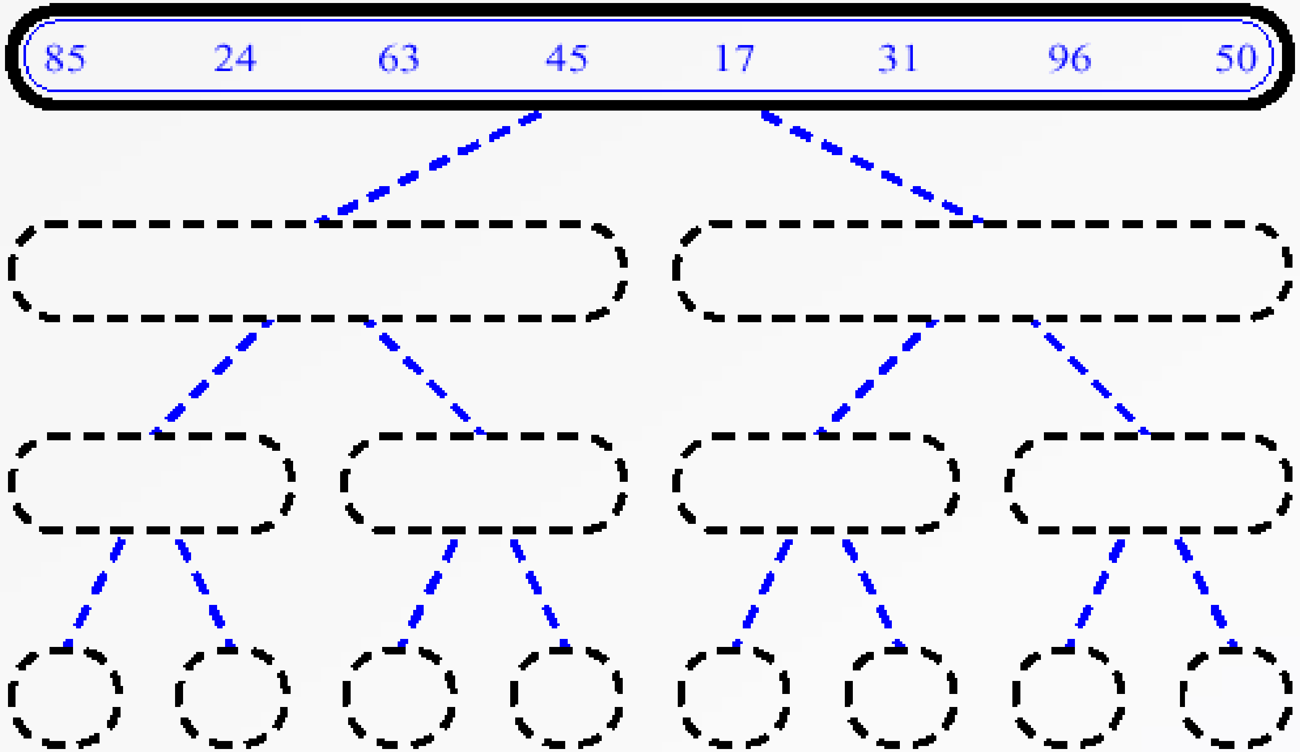
DIVIDE AND CONQUER

– **Divide** the problem into sub-problems that are similar to the original but smaller in size.

– **Conquer** the sub-problems by solving them recursively. If they are small enough, just solve them in a straightforward manner.

– **Combine** the solutions to create a solution to the original problem

MERGESORT



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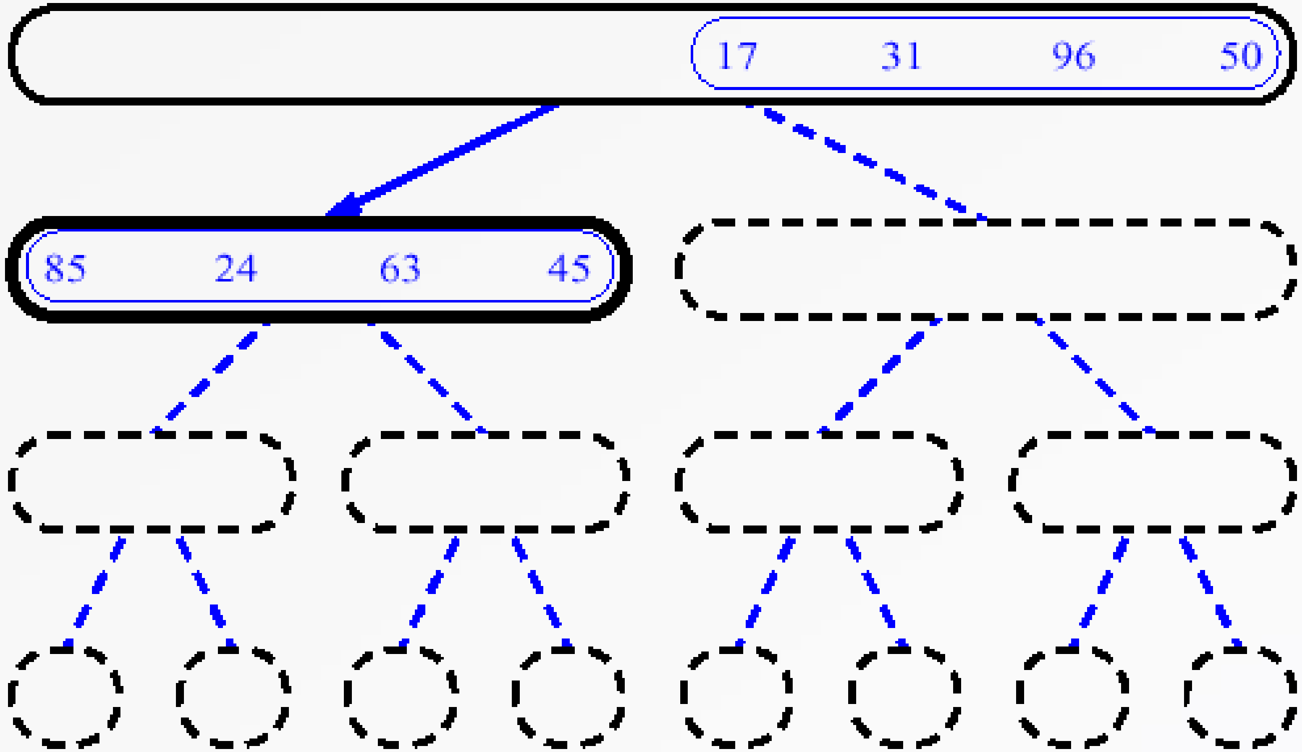

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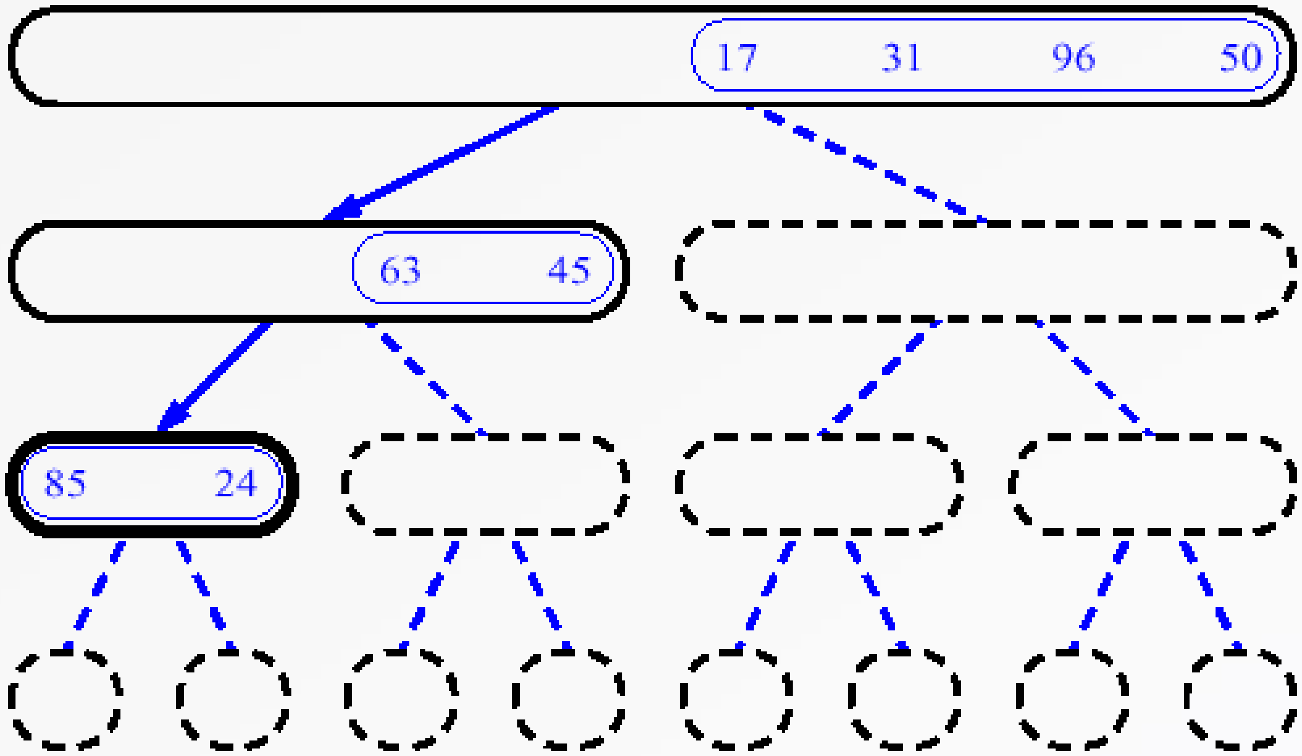

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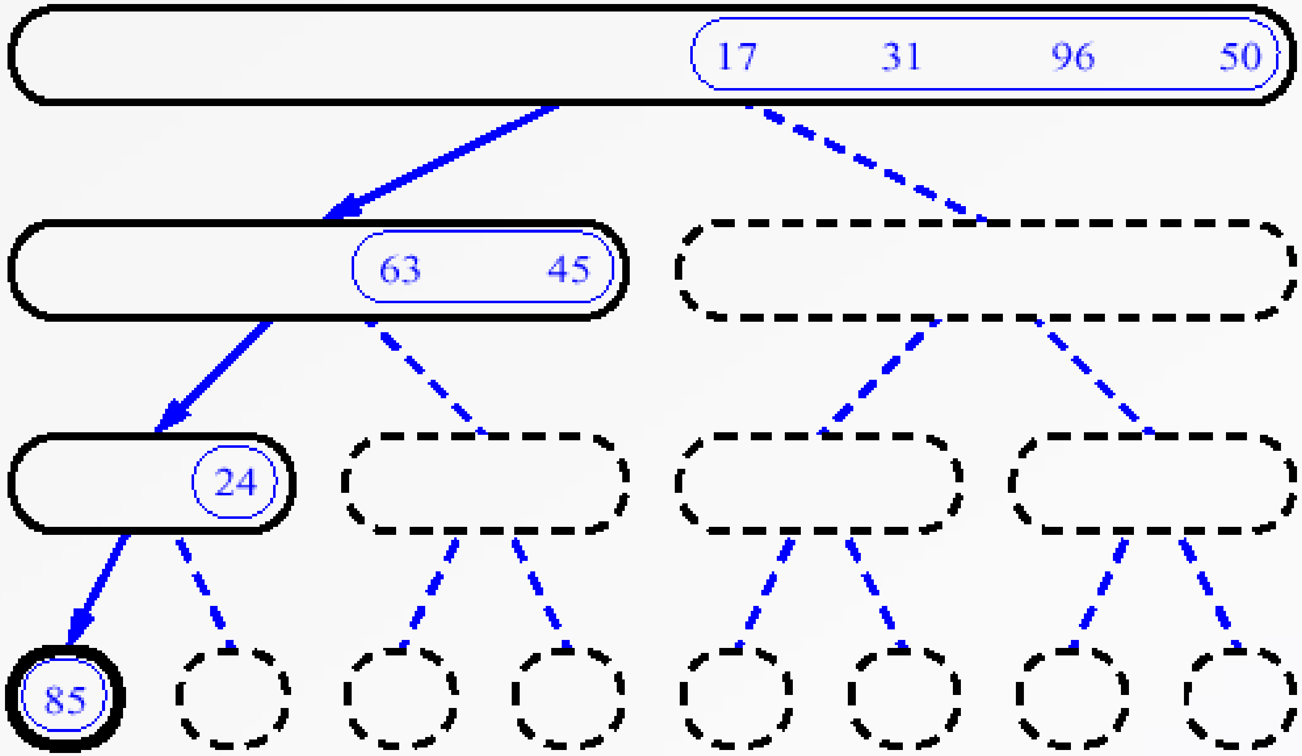

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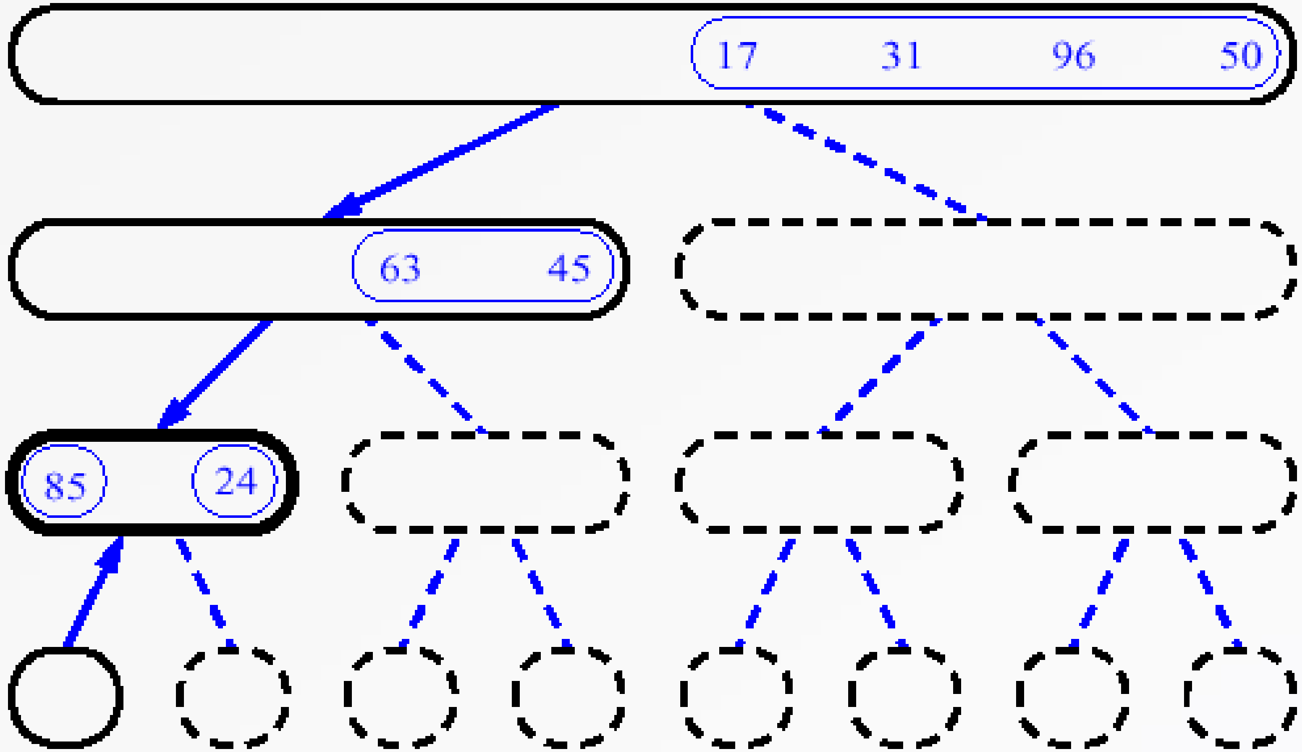

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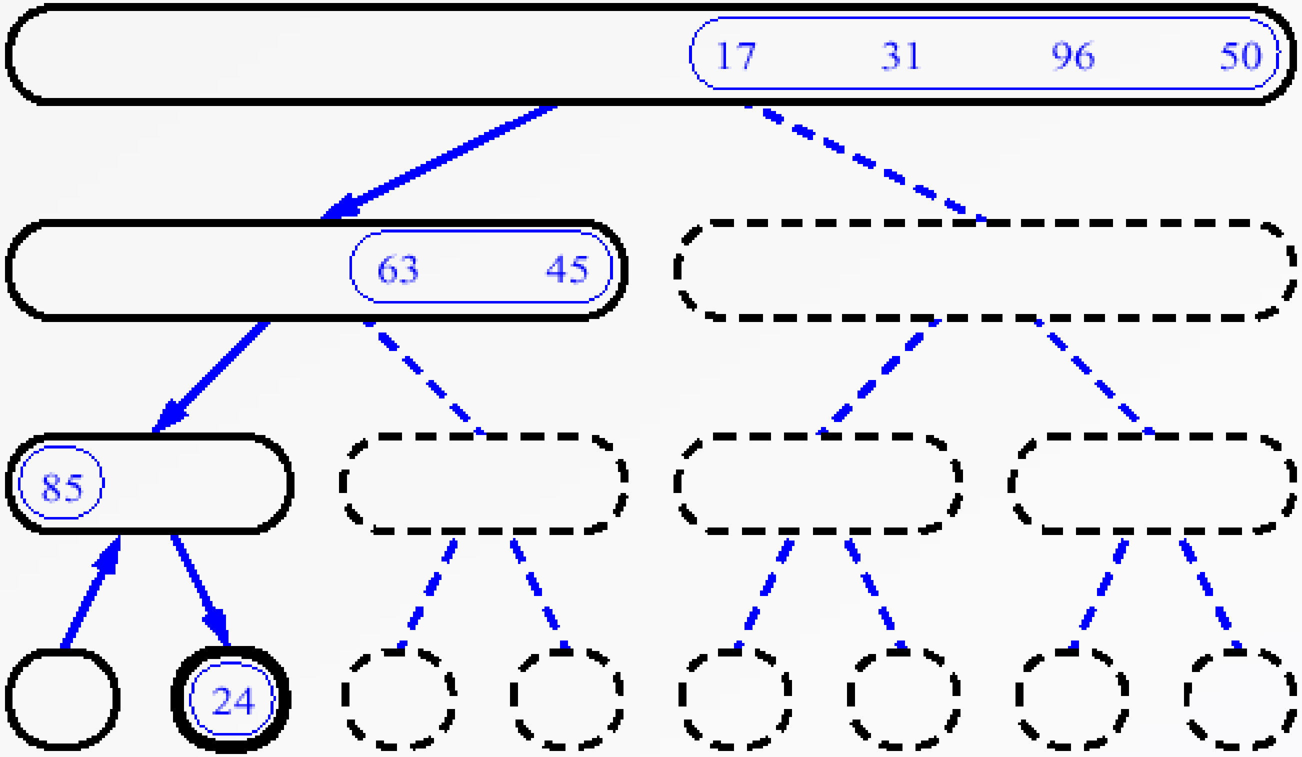

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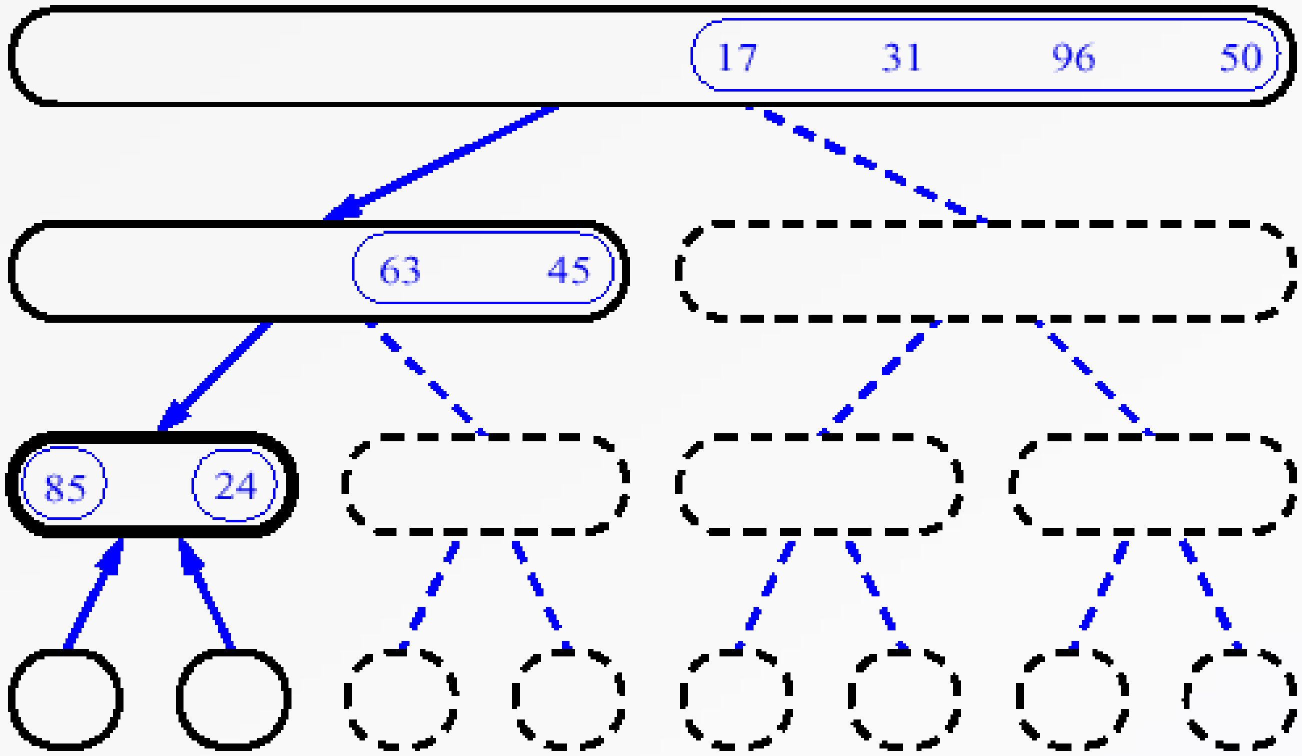

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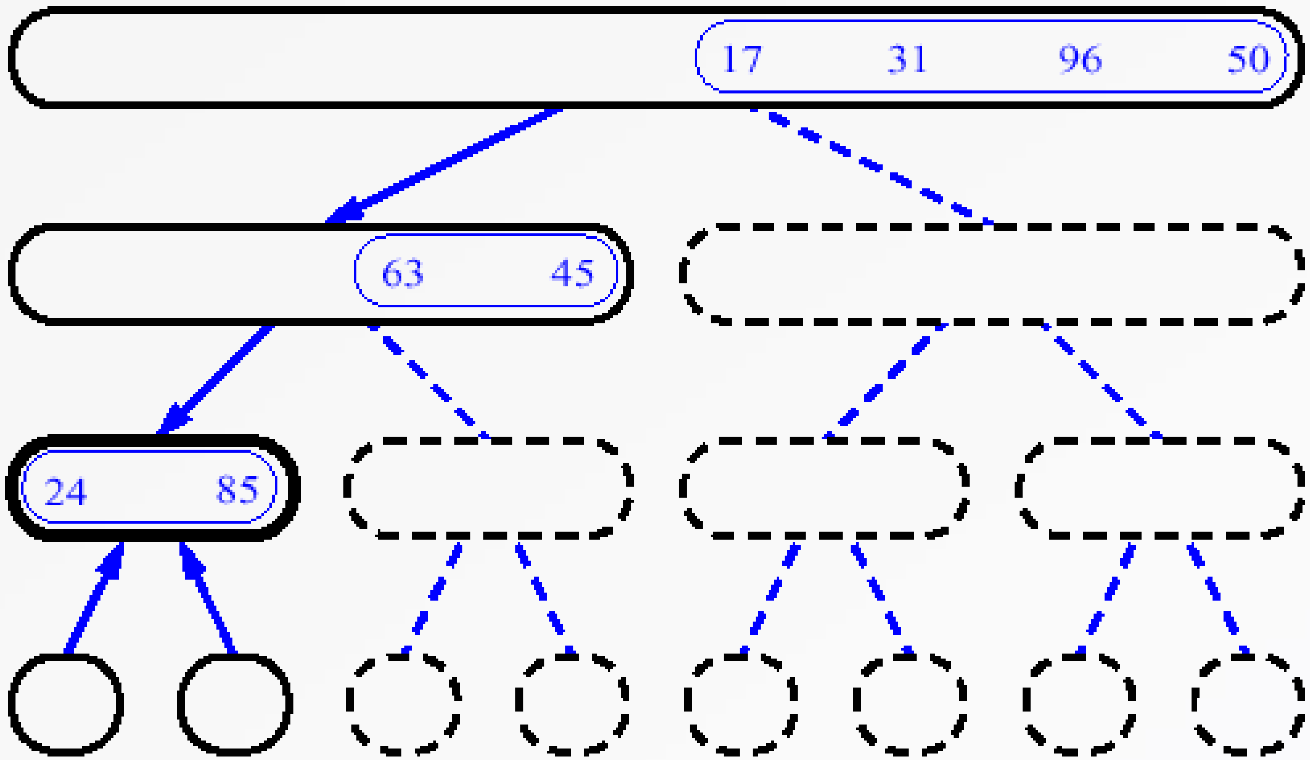

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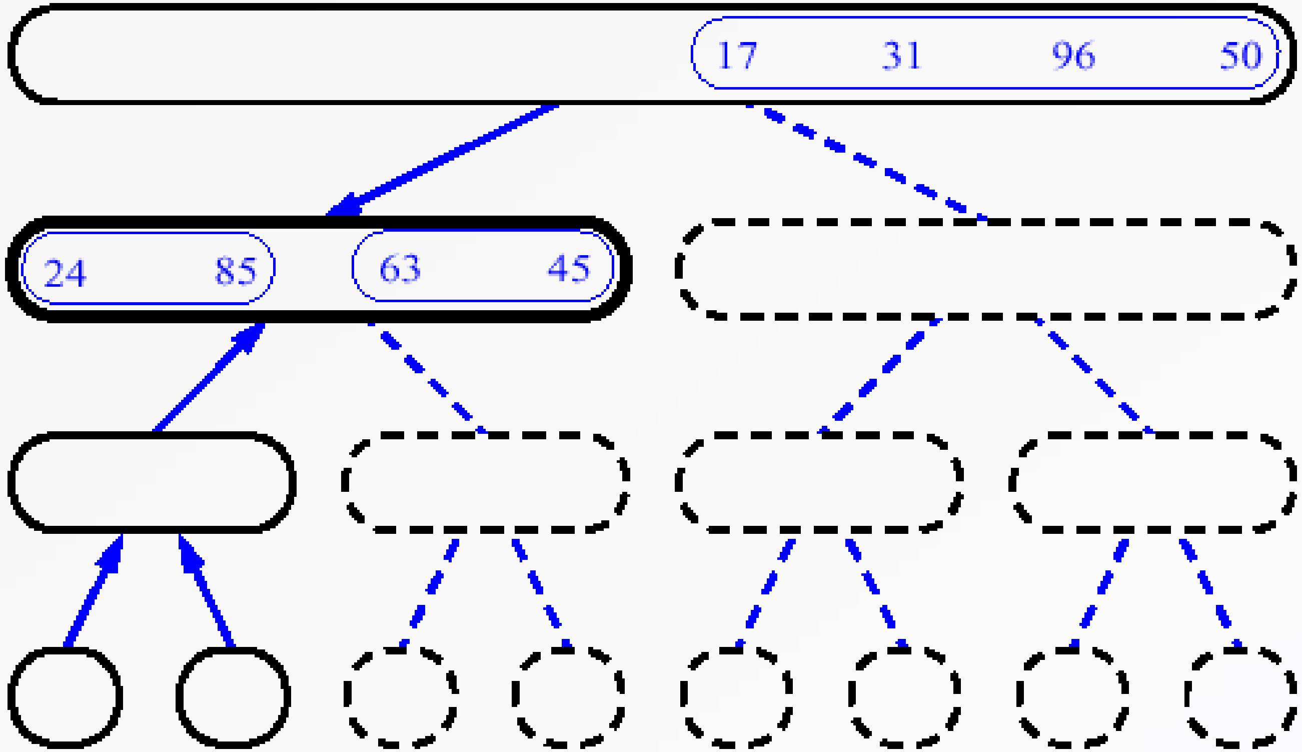

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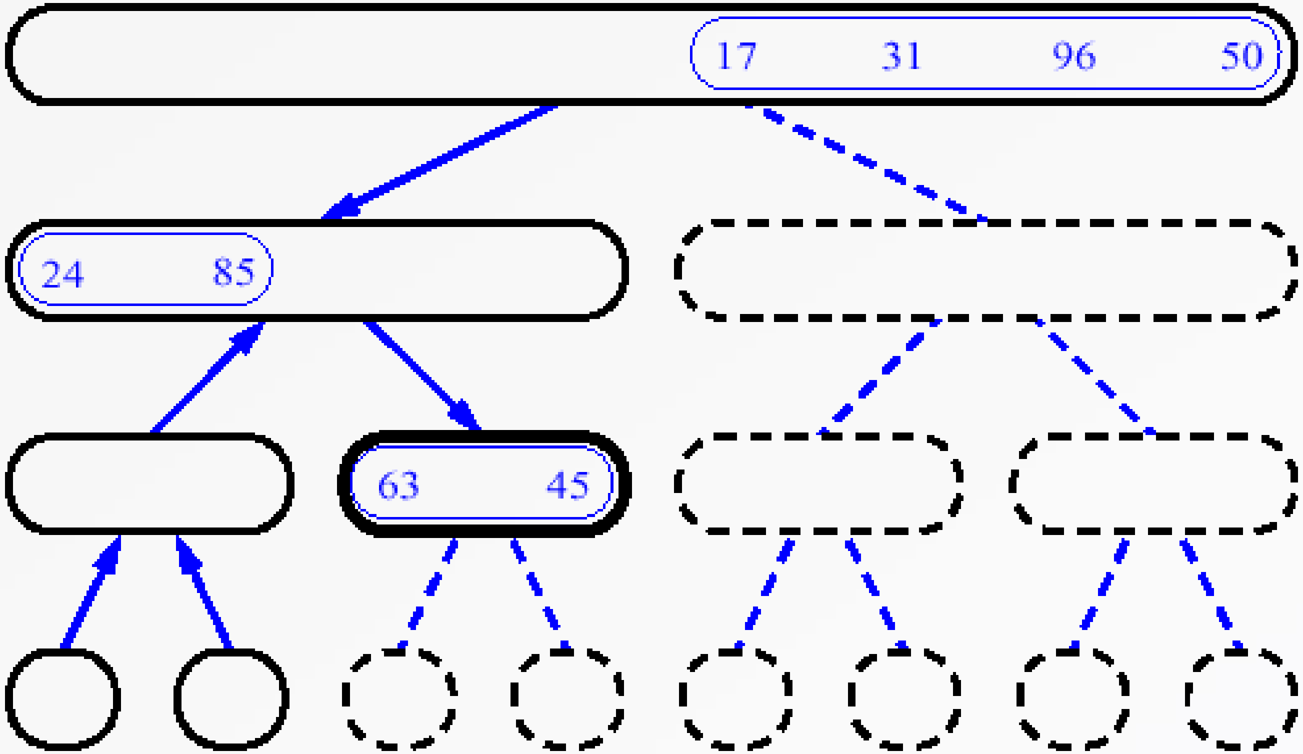

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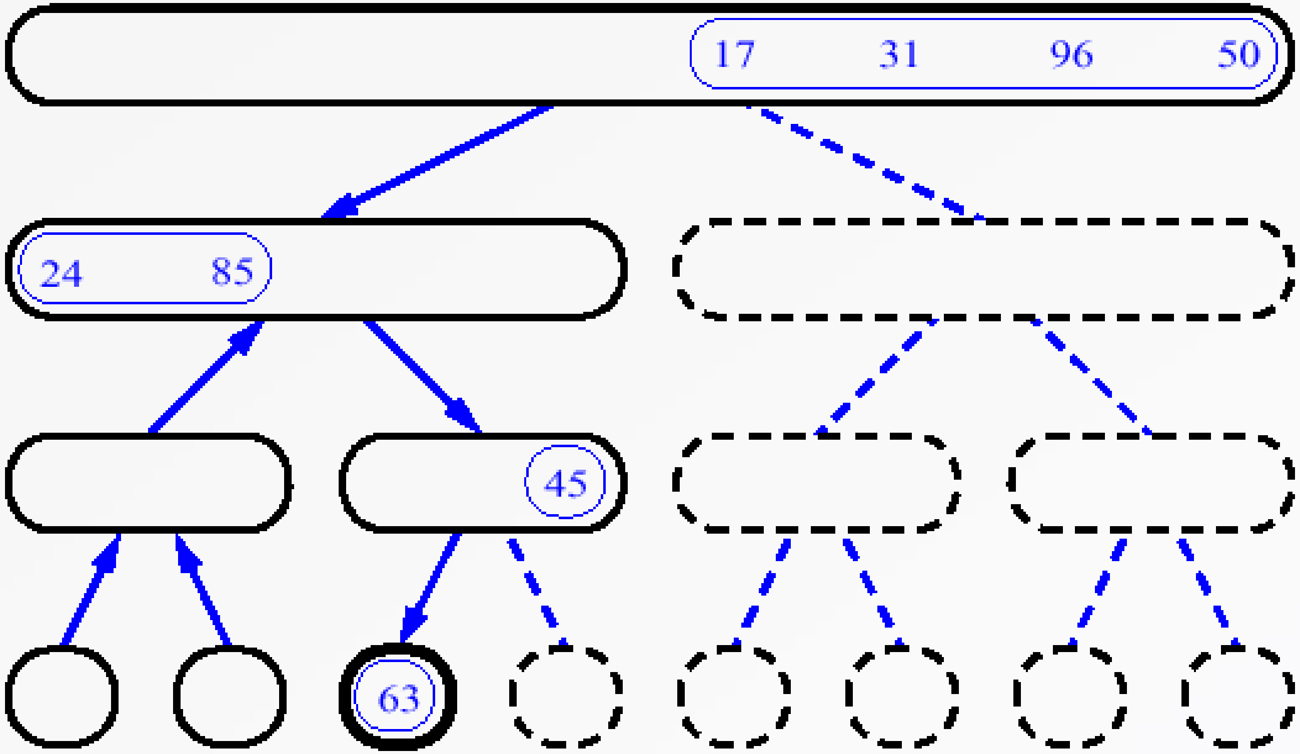

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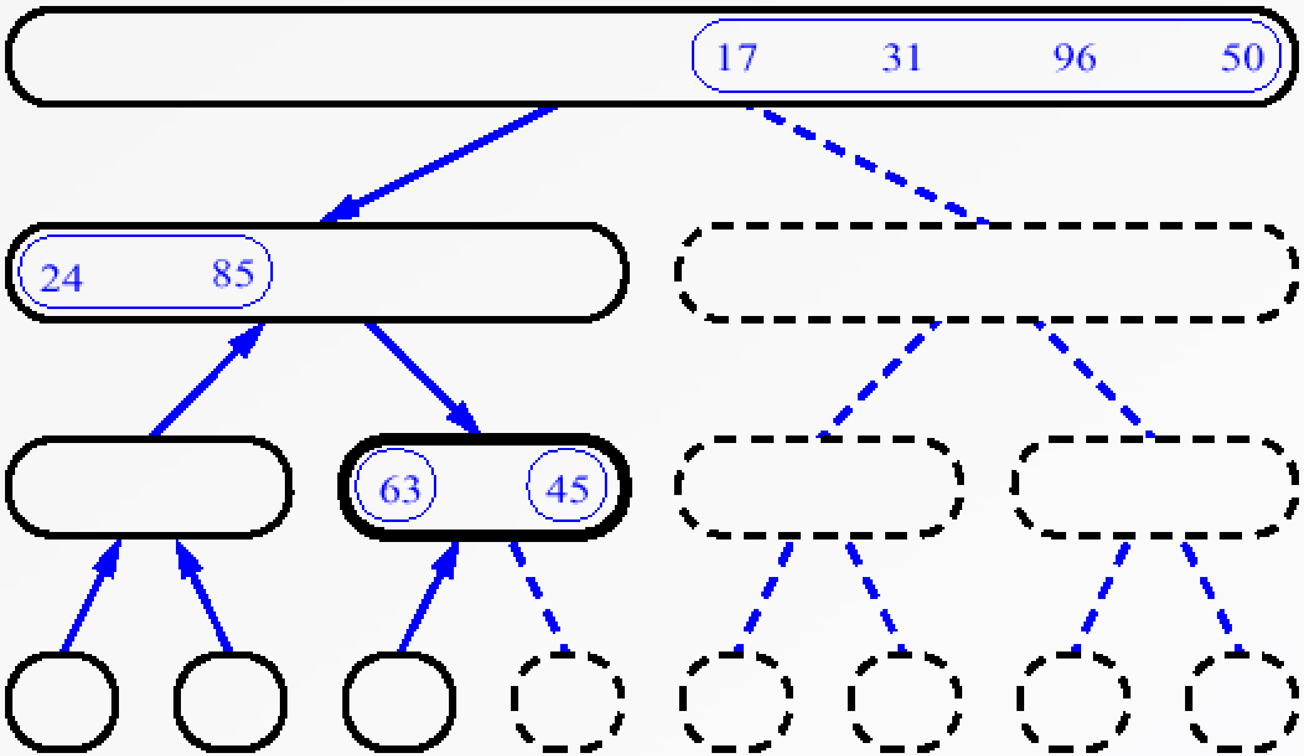

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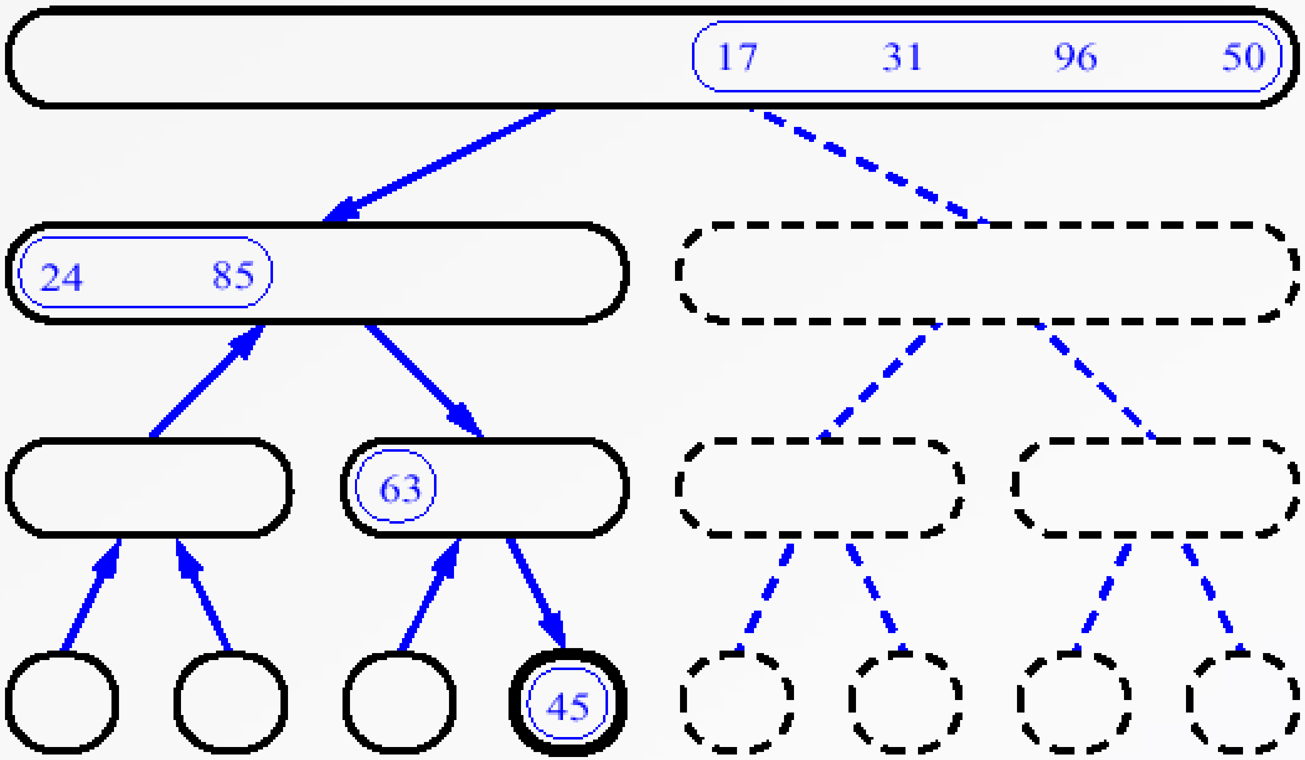

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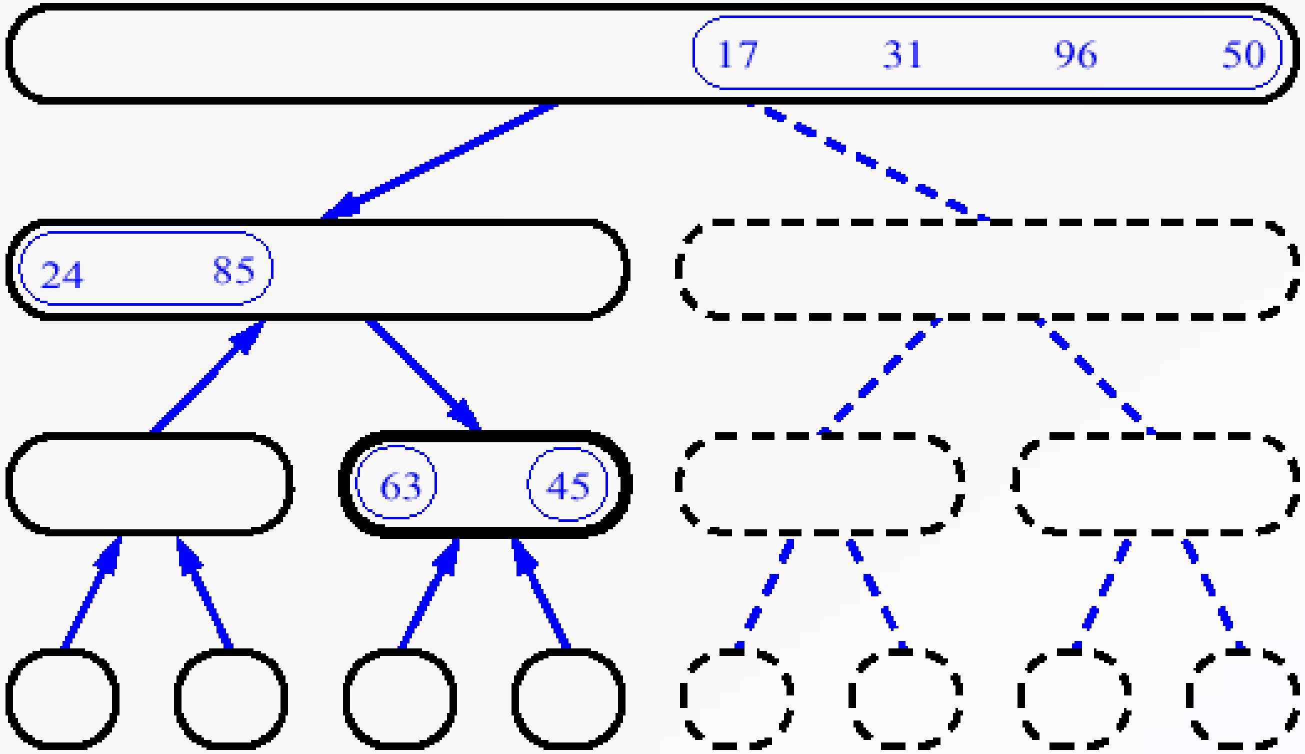

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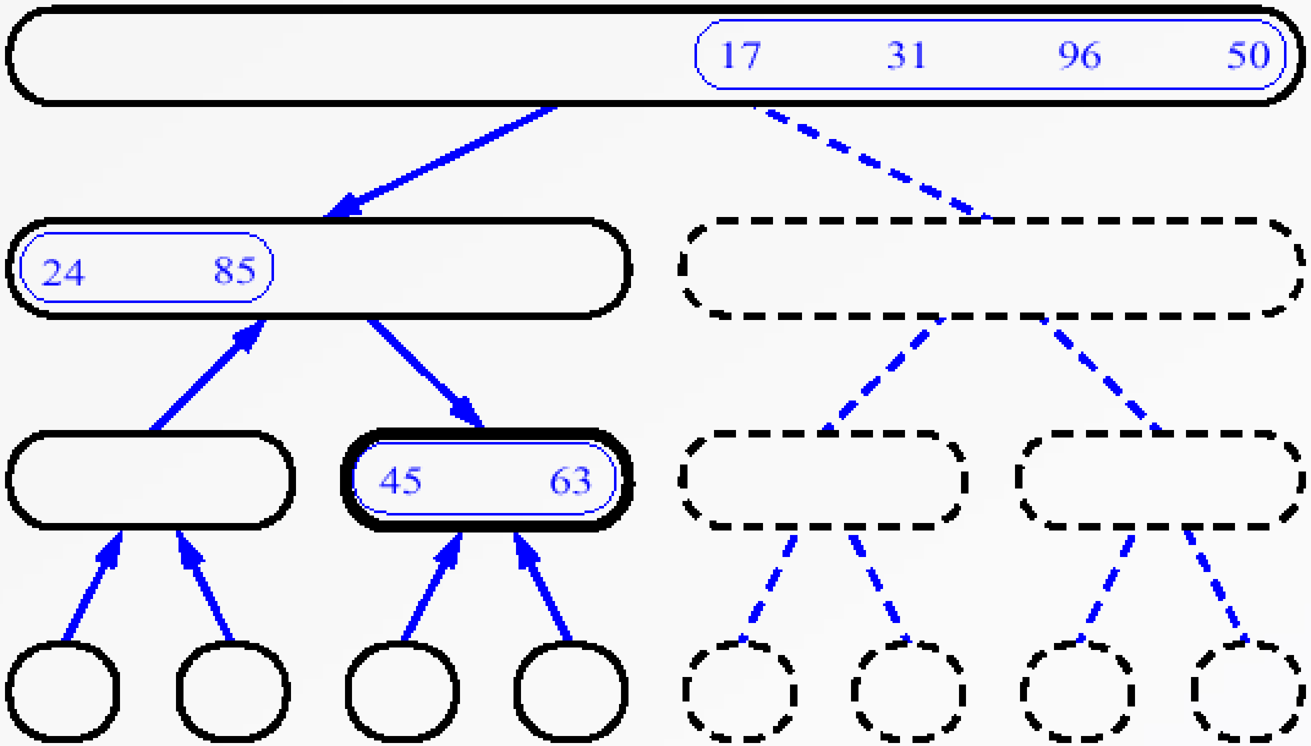

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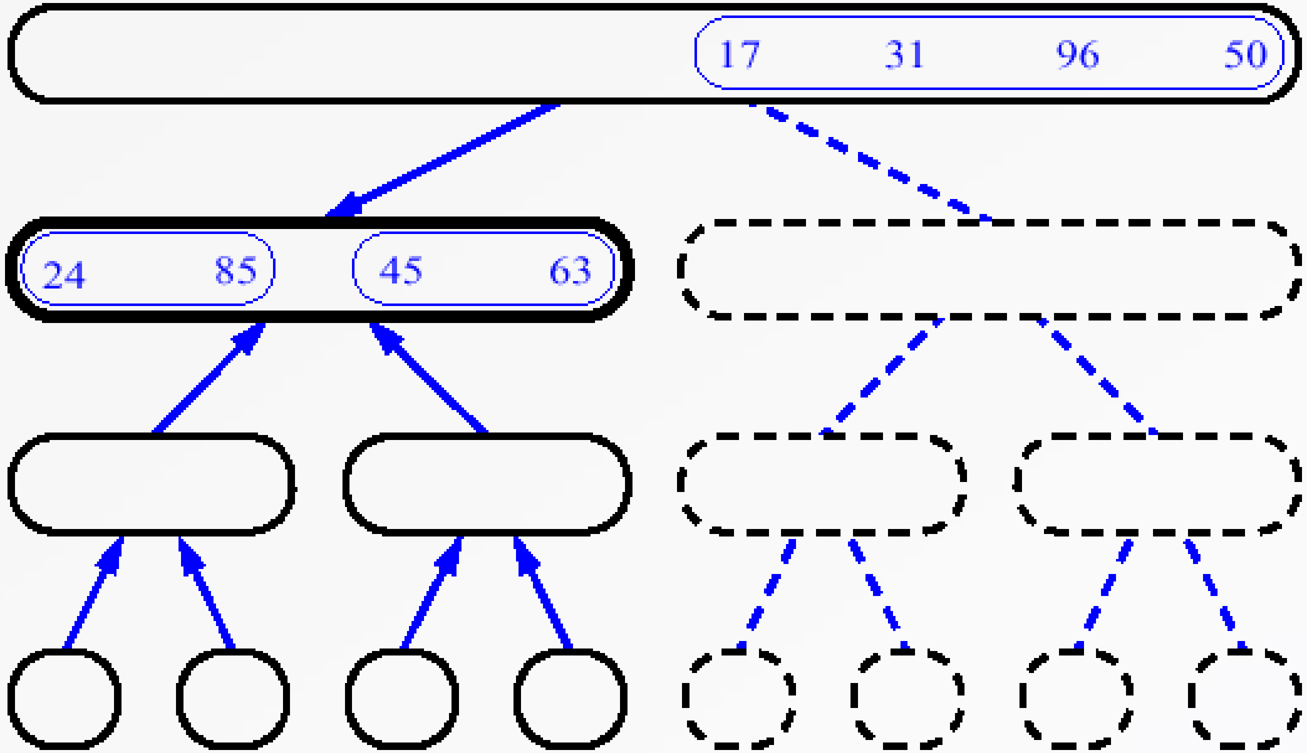

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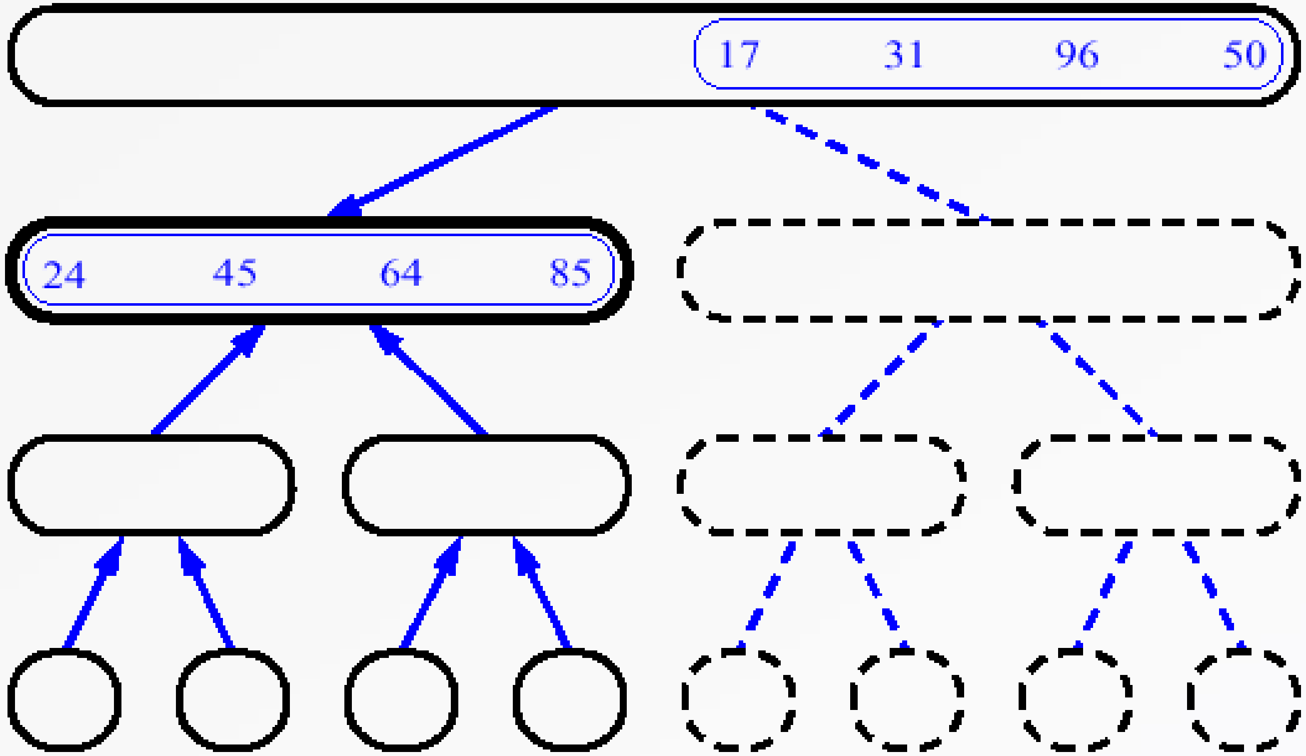

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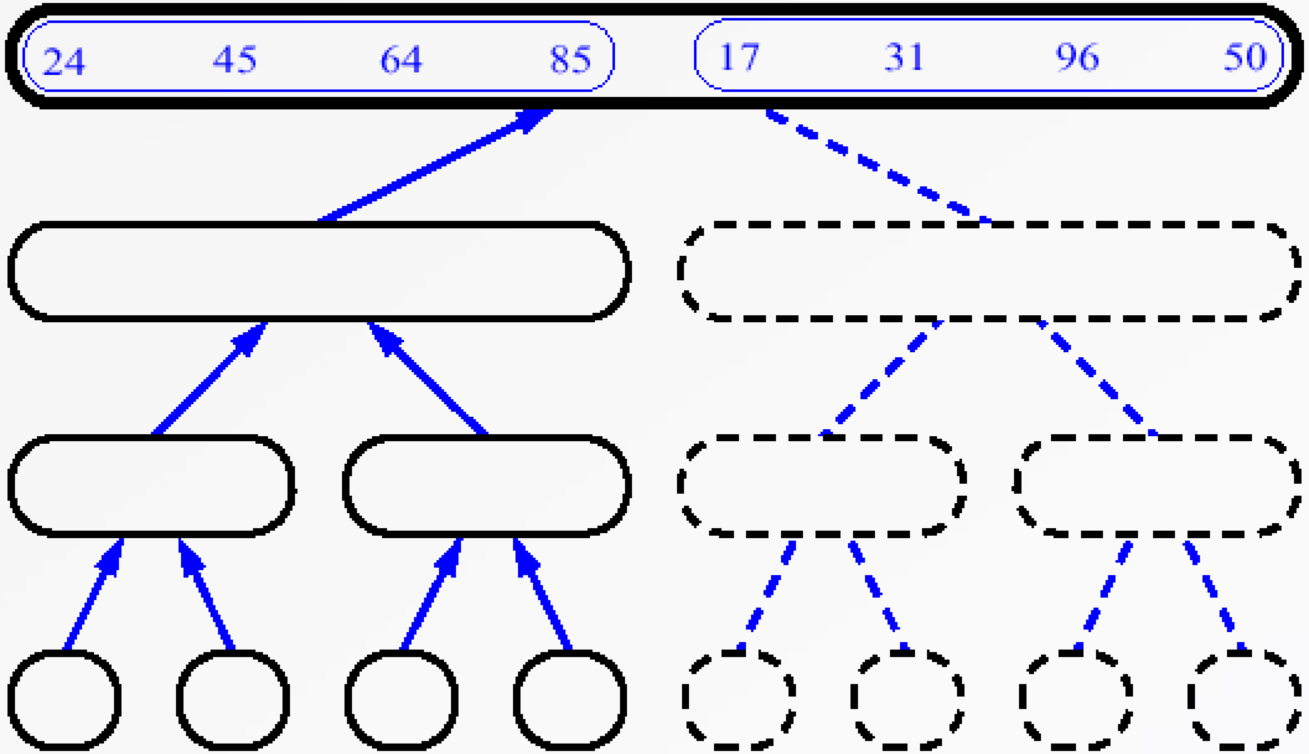

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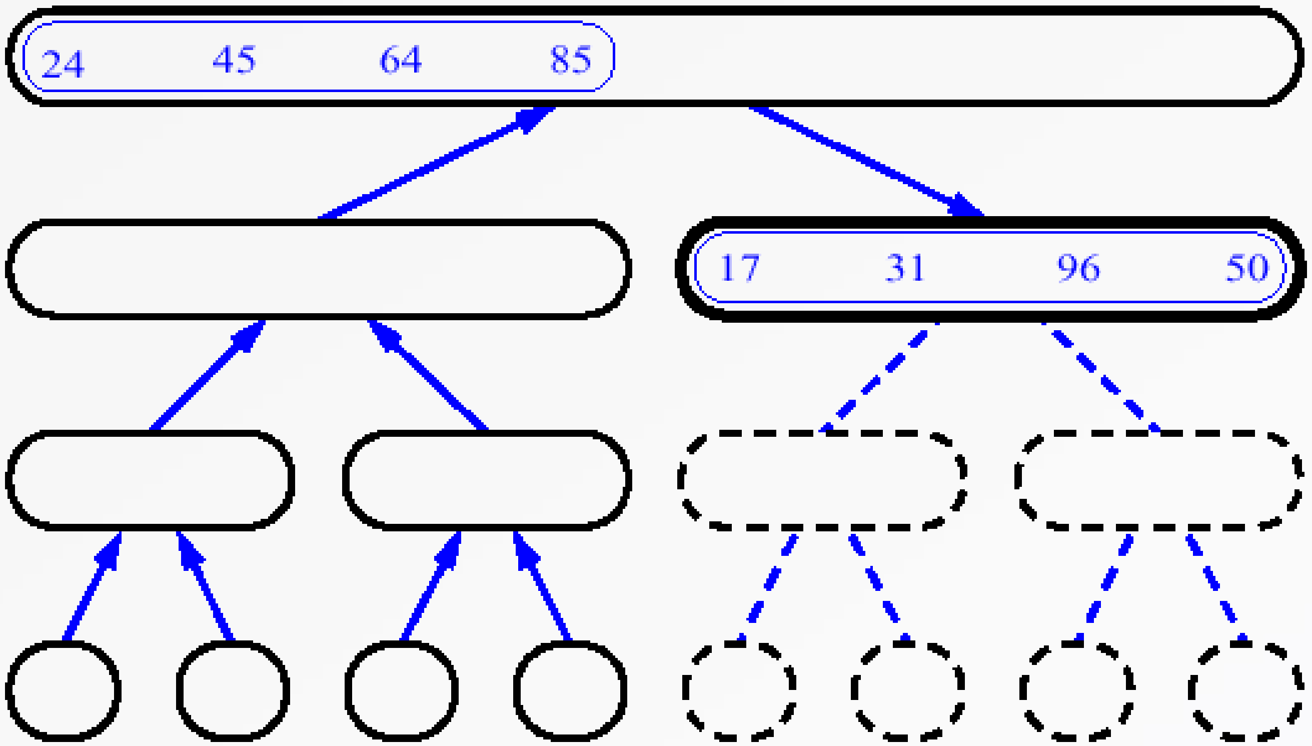

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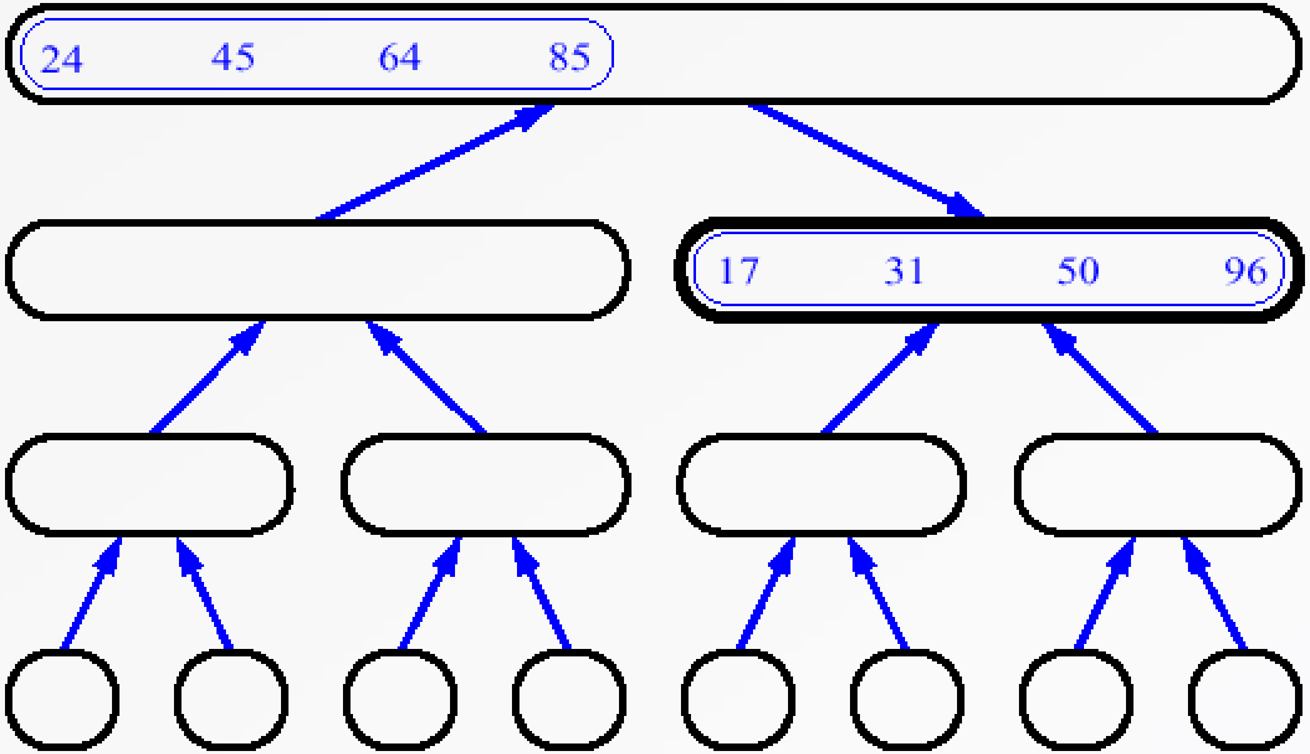

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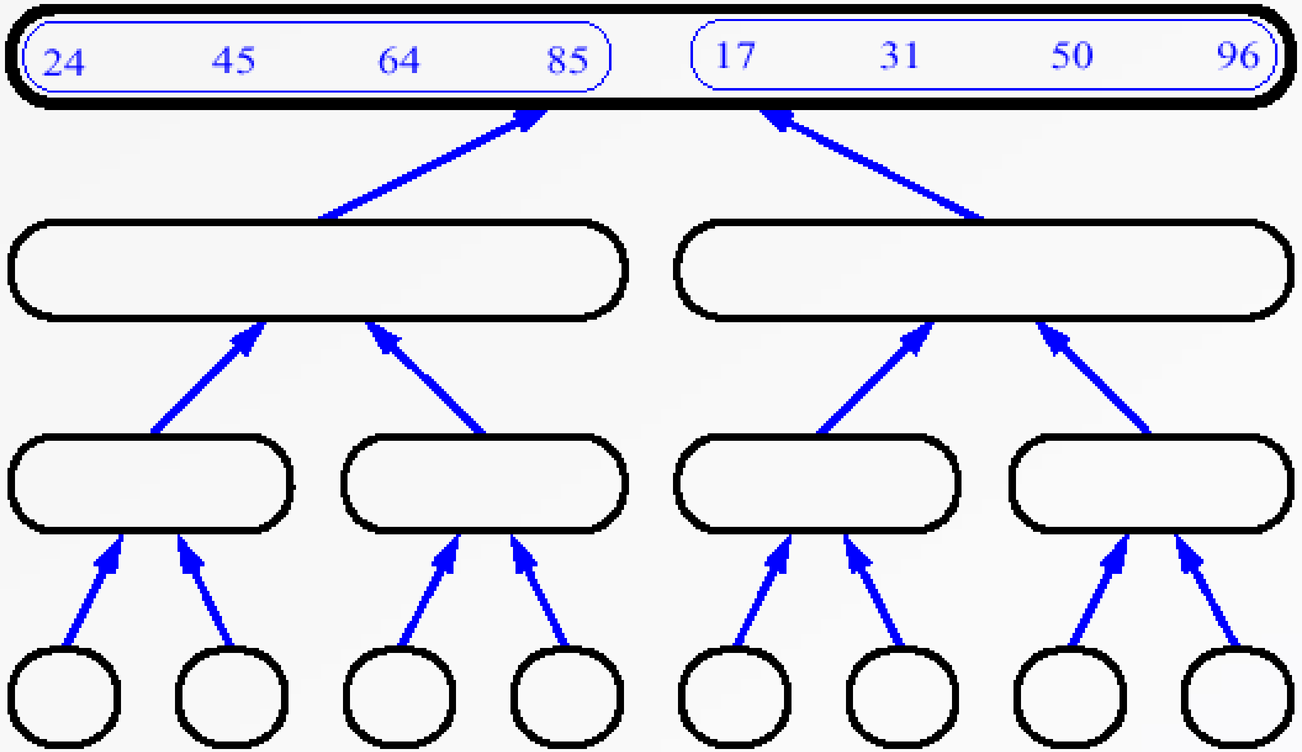

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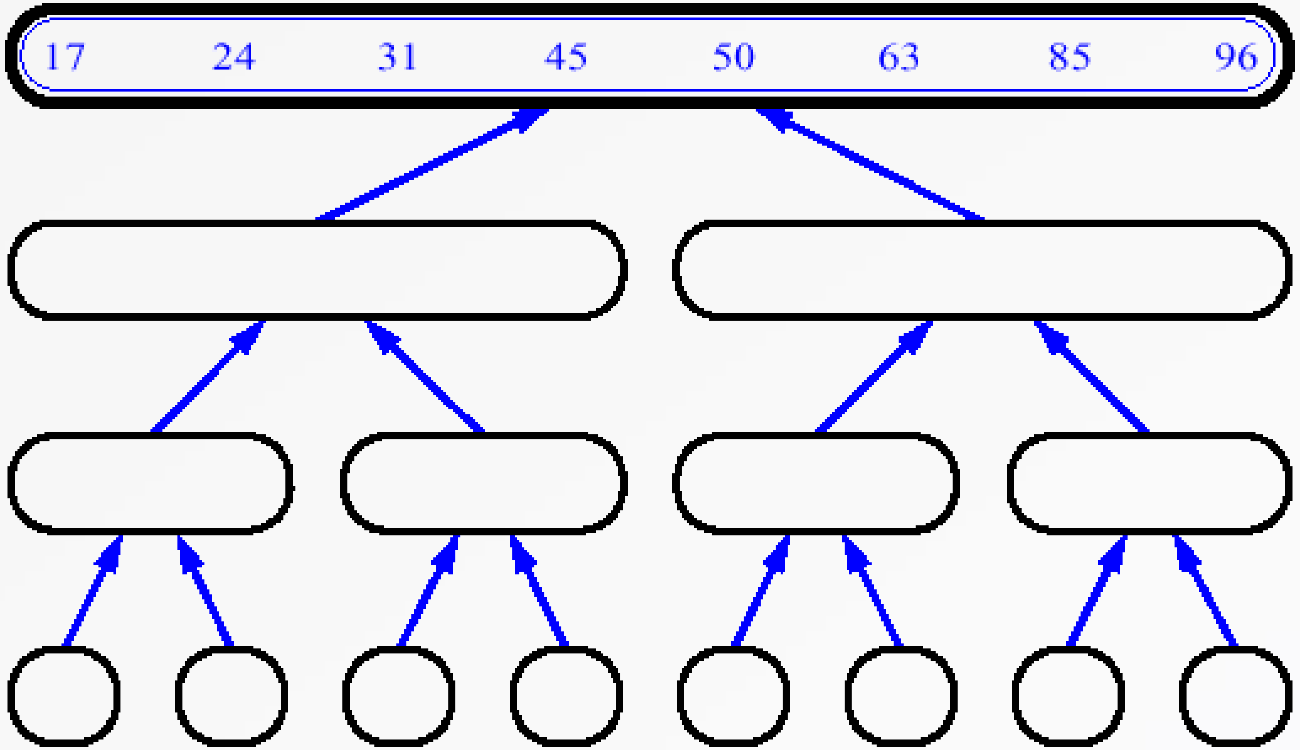


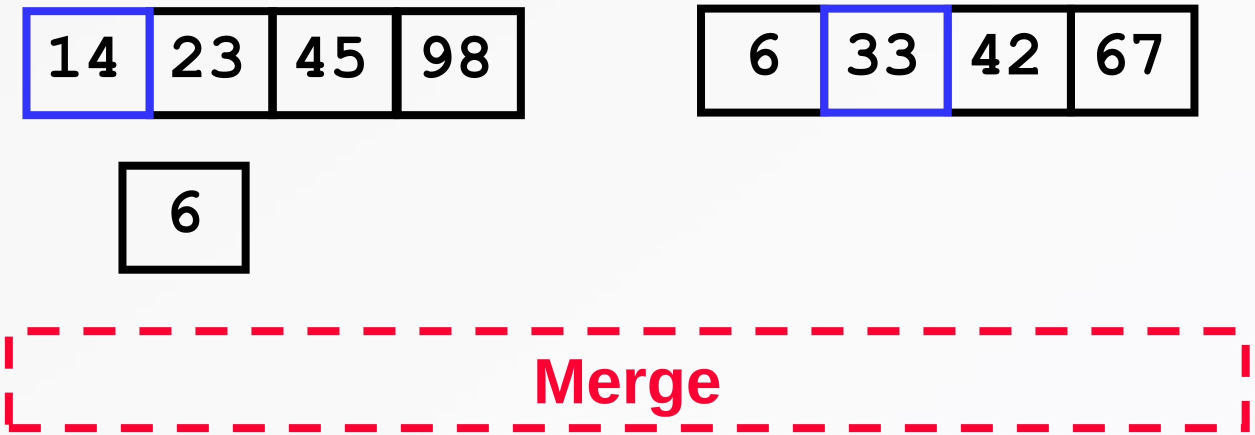
14	23	45	98

6	33	42	67

14	23	45	98

6	33	42	67

Merge

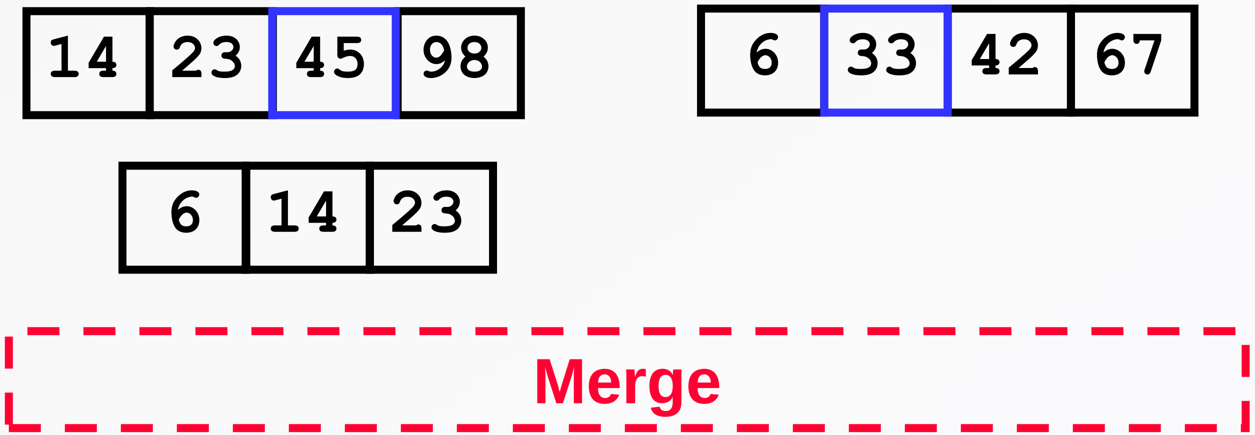

14 23 45 98
6 33 42 67
6
Merge

14	23	45	98

6	33	42	67

6	14

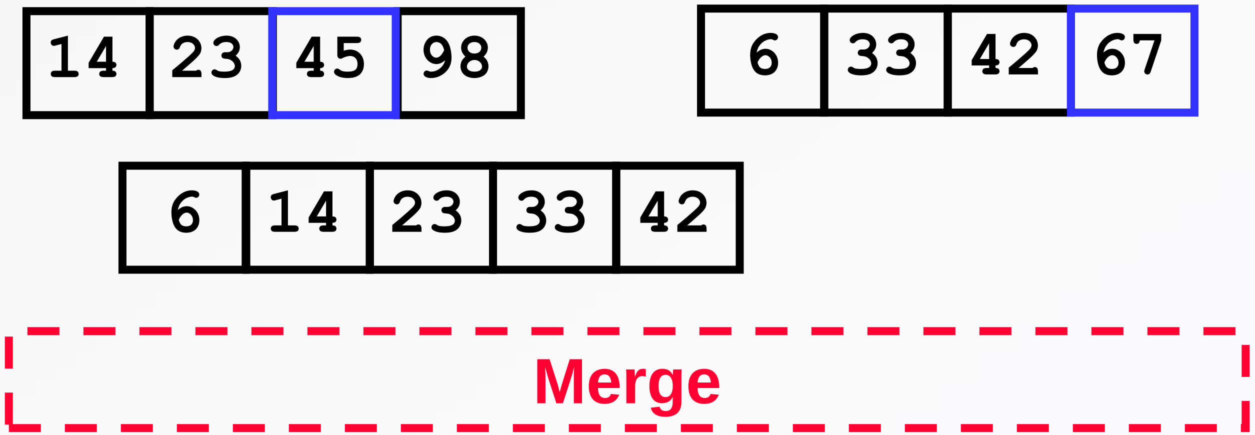
Merge


14 23 45 98
6 33 42 67
6 14 23
Merge

14	23	45	98		6	33	42	67

6	14	23	33

Merge


14 23 45 98
6 33 42 67
6 14 23 33 42
Merge

14	23	45	98

6	33	42	67

6	14	23	33	42	45

Merge

14	23	45	98		6	33	42	67

6	14	23	33	42	45	67

Merge

14	23	45	98

6	33	42	67

6	14	23	33	42	45	67	98

Merge

ALGORITHM

```
MergeSort(A[0....n-1])
  if n>1
      Copy A[0.... (n/2) -1] to B[ 0.... (n/2) -1]
      Copy A[(n/2).....n-1]] to C[ 0.... (n/2) -1]
 MergeSort(B[0....n-1])
 MergeSort(C[0....n-1])
 Merge(B,C,A)
```

```
Merge(B [0....p-1], C[0....q-1], A[0....p+q-1])
 i←0, j←0, k←0
while i<p & j<q do
 if B[i] <= C[j]
   A[K] ← B[i] ; i++;
 else
   A[K] ← C[j] ; j++;
K ←K+1 ;

//Copy left over elements
if i==p
 copy C[j......q-1] to A[k..... p+q-1]
else
 copy B[j......p-1] to A[k..... p+q+1]
```

8 3 2 9 7 1 5 4

MergeSort(A[0....n-1]) → T(n)

if n>1

Copy A[0.... (n/2) -1] to B[0.... (n/2) -1]

Copy A[0.... (n/2) -1] to C[0.... (n/2) -1]

MergeSort(B[0....n-1]) → T(n/2)

MergeSort(C[0....n-1]) → T(n/2)

Merge(B,C,A) → T(n)

$$T(n) \le \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{T\left(\lceil n/2 \rceil\right)}_{\text{solve left half}} + \underbrace{T\left(\lfloor n/2 \rfloor\right)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

T(n)= 2T(n/2)

+cn

APPROACHES TO SOLVE RECURSION

Approach 1:

1. Intuitive solution to recurrence is to “unroll” the recursion, accounting for the running time of first few levels.

2. Identify a pattern that can be continued as the recursion expands.

3. Sum the running times over all levels of the recursion and thereby arrives at a total running time.

Step1: Analyze the first few levels.

- 1st level of recursion → Single problem of size n → O(n)
- 2nd level of recursion → 2 problems each of size n/2 → O(n/2)
- 3rd level of recursion → 4 problems each of size n/4 → O(n/4)

Step 2: Identifying the pattern.

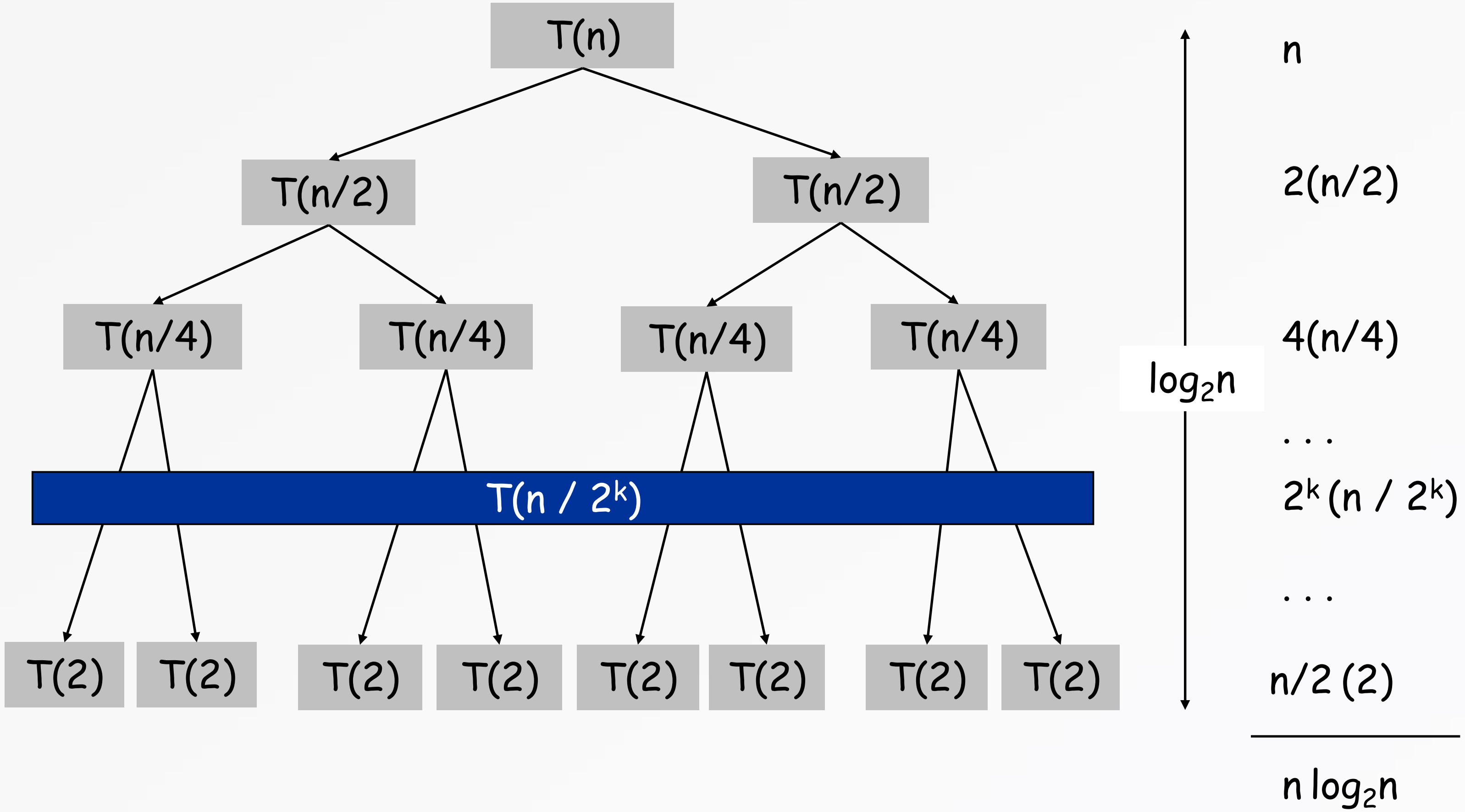
- At level *j* of the recursion, the number of subproblems are now a total of $\mathbf{2^j}$
- Each problem has shrunk in size by factor of 2 “j” time → $n/\mathbf{2^j}$

Step 3: Summing overall levels of recursion.

- The number of times the input must be halved to reduce the size of n to 1 is **log n**
- There are totally “n” levels of recursion → **O(n logn)**

ANALYSIS

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$



SUBSTITUTING A SOLUTION INTO THE MERGESORT RECURSION

$T(n) = O(n \log n)$

$T(n) = \text{c. } n \log n$

$T(n/2) = \text{c. } n/2 \log n/2$

$T(n) = 2T\ (n/2) + O(n)$

$T(n) = 2T\ (n/2) + cn$

$T(n) = 2.\ \mathbf{c.\ n/2 \log n/2} + cn$

$T(n) = cn.\ [\log n - 1] + cn$

$T(n) = cn.\log n - cn + cn$

$T(n) = cn.\log n$

$\boldsymbol{T(n) = O(n\log n)}$

Analysis

- Basic operation - key comparison.
- Best Case, Worst Case, Average Case exists?
  - Execution does not depend on the order of the data
  - Best case and average case runtime are the same as worst case runtime.
- Worst case:
  - During key comparison, neither of the two arrays becomes empty before the other one contains just one element

Analysis – Worst Case

- Assuming for simplicity that total number of elements **n** is a power of 2, the recurrence relation for the number of key comparisons $C(n)$ is

$$C(n) = 2C(n/2) + C_{merge}(n) \quad \text{for } n > 1, \quad C(1) = 0.$$

- $C_{merge}(n)$ - the number of key comparisons performed during the merging stage.
- At each step, exactly one comparison is made, total comparisons are (n-1)

$$C_{worst}(n) = 2C_{worst}(n/2) + n - 1 \quad \text{for } n > 1, \quad C_{worst}(1) = 0.$$

Analysis

$$C_{worst}(n) = 2C_{worst}(n/2) + n - 1 \quad \text{for } n > 1, \quad C_{worst}(1) = 0.$$

- Here a = 2, b = 2, f (n) = n-1 = Θ (n) => d = 1.
- Therefore $2 = 2^1$, case 2 holds in the master theorem
- $\boldsymbol{C_{worst}}$ (n) = Θ (n^d log n) = Θ (n^1 log n) = Θ (n log n)
- Therefore **C_{worst}(n) = Θ (n log n)**

Master theorem

It states that, in recurrence equation T(n) = aT(n/b) + f(n),

If f(n)∈ Θ (n^d) where d ≥ 0 then

$$T(n) \in \begin{cases} \Theta(n^d) & \text{if } a < b^d, \\ \Theta(n^d \log_b n) & \text{if } a = b^d, \\ \Theta(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$

Quick Sort

Extra!

- It is a Divide and Conquer method
- Sorting happens in Divide stage itself.
- C.A.R. Hoare (also known as Tony Hore), prominent British computer scientist invented quicksort.

Quick Sort

- Quicksort divides (or partitions) array according to the value of some pivot element A[s]
- Divide-and-Conquer:
  - If n=1 terminate (every one-element list is already sorted)
  - If n>1, partition elements into two; based on pivot element

$$\underbrace{A[0]\dots A[s-1]}_{\text{all are } \le A[s]} \quad A[s] \quad \underbrace{A[s+1]\dots A[n-1]}_{\text{all are } \ge A[s]}$$

Quick Sort

$$\underbrace{A[0] \ldots A[s-1]}_{\text{all are } \leq A[s]} \quad A[s] \quad \underbrace{A[s+1] \ldots A[n-1]}_{\text{all are } \geq A[s]}$$

ALGORITHM *Quicksort*$(A[l..r])$

//Sorts a subarray by quicksort
//Input: Subarray of array $A[0..n-1]$, defined by its left and right
// indices l and r
//Output: Subarray $A[l..r]$ sorted in nondecreasing order

if $l < r$
$s \leftarrow Partition(A[l..r])$ //s is a split position
$Quicksort(A[l..s-1])$
$Quicksort(A[s+1..r])$

How do we partition?

- There are several different strategies for selecting a pivot and partitioning.
- We use the sophisticated method suggested by C.A.R. Hoare, the inventor of quicksort.
- Select the subarray’s first element: **p = A[l].**
- Now scan the subarray from both ends, comparing the subarray’s elements to the pivot.

How do we partition?

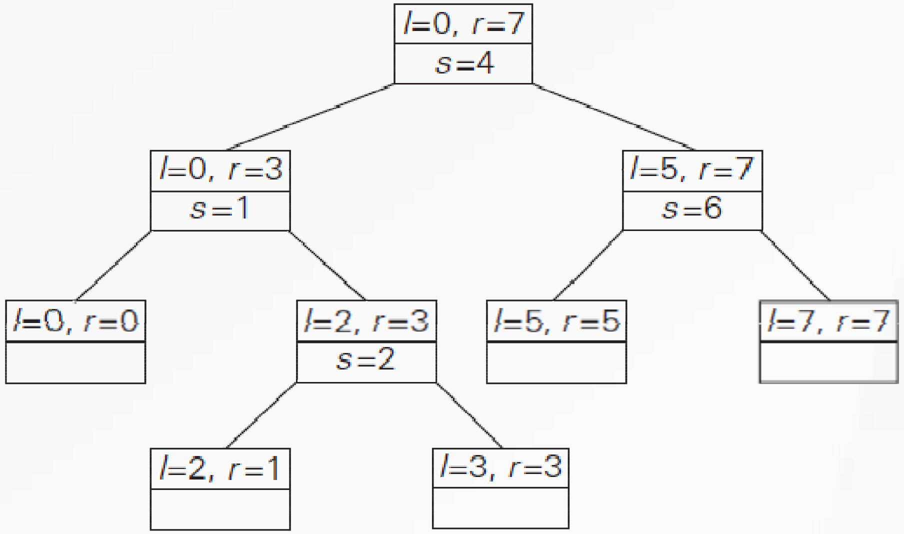
ALGORITHM $HoarePartition(A[l..r])$

//Partitions a subarray by Hoare's algorithm, using the first element
// as a pivot
//Input: Subarray of array $A[0..n-1]$, defined by its left and right
// indices l and r $(l < r)$
//Output: Partition of $A[l..r]$, with the split position returned as
// this function's value

$p \leftarrow A[l]$
$i \leftarrow l;\ j \leftarrow r+1$

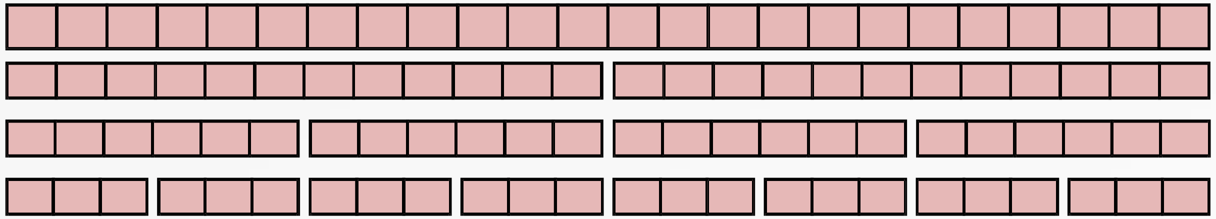
repeat
  repeat $i \leftarrow i+1$ **until** $A[i] \geq p$
  repeat $j \leftarrow j-1$ **until** $A[j] \leq p$
  swap($A[i]$, $A[j]$)
until $i \geq j$
swap($A[i]$, $A[j]$) //undo last swap when $i \geq j$
swap($A[l]$, $A[j]$)
return j

0	1	2	3	4	5	6	7
5	i 3	1	9	8	2	4	j 7
5	3	1	i 9	8	2	j 4	7
5	3	1	i 4	8	2	j 9	7
5	3	1	4	i 8	j 2	9	7
5	3	1	4	i 2	j 8	9	7
5	3	1	4	j 2	i 8	9	7
2	3	1	4	5	8	9	7
2	i 3	1	j 4				
2	i 3	j 1	4				
2	i 1	j 3	4				
2	j 1	i 3	4				
1	2	3	4				
1							
		3	i j 4				
		j 3	i 4				
			4				
					8	i 9	j 7
					8	i 7	j 9
					8	j 7	i 9
					7	8	9
					7		
							9



Analysis

- Basic Operation : Key Comparison
- Best case exists
  - all the splits happen in the middle of subarrays,
  - So the depth of the recursion in $\log_2 n$



$$C_{best}(n) = 2C_{best}(n/2) + n \quad \text{for } n > 1, \quad C_{best}(1) = 0.$$

– As per Master Theorem, $C_{best}(n) \in \Theta(n \log_2 n)$;

Analysis

- Worst Case
  - Splits will be skewed to the extreme
  - This happens if the input is already sorted
- In the worst case, partitioning always divides the size n array into these three parts:
  - A length one part, containing the pivot itself
  - A length zero part, and
  - A length n-1 part, containing everything else
- Recurring on the length n-1 part requires (in the worst case) recurring to depthn-1

Analysis

- Worst Case

Analysis

Worst Case

- if A[0..n − 1] is a strictly increasing array and we use A[0] as the pivot,
  - the left-to-right scan will stop on A[1] while the right-to- left scan will go all the way to reach A[0], indicating the split at position 0
  - n + 1 comparisons required

Total comparisons

$$C_{worst}(n) = (n+1) + n + \cdots + 3 = \frac{(n+1)(n+2)}{2} - 3 \in \Theta(n^2).$$

Analysis

Average Case

- Let $C_{avg}(n)$ be the average number of key comparisons made by quicksort on a randomly ordered array of size n.
- A partition can happen in any position s $(0 \le s \le n-1)$
- n+1 comparisons are required for partition.
- After the partition, the left and right subarrays will have s and $n - 1 - s$ elements, respectively.

Analysis

Average Case

- Assuming that the partition split can happen in each position s with the same probability 1/n, we get

$$C_{avg}(n) = \frac{1}{n}\sum_{s=0}^{n-1}[(n+1) + C_{avg}(s) + C_{avg}(n-1-s)] \quad \text{for } n > 1,$$

$$C_{avg}(0) = 0, \quad C_{avg}(1) = 0.$$

$$C_{avg}(n) \approx 2n \ln n \approx 1.39n \log_2 n.$$