Notes on Stationary & White Noise Processes

AAEC 4484/AAEC(STAT) 5484

2024-02-07

Outline

The key idea of time series models is to exploit the past behavior of a time series to forecast its future development. This requires the future to be rather similar to the past, in other words, the past may repeat in future with certain probabilities.

This hypothesis is technically translated into an assumption of **weak stationarity**. Weak stationarity means that the variable under analysis has a stable mean and variance over time, and that the correlation of the variable with each of its own lags is also stable over time.

Any weakly stationary stochastic process can always be represented as a sum of the White noise (WN) processes, which have zero mean, uncorrelated over time and has a constant variance.

Stationarity

Now, let's first start with the definitions. The first definition is "strict or strong stationarity," which is defined as time series data have the same probability density function (PDF) over time, that is,

$$f_{y_1}(y) = f_{y_2}(y) = \cdots = f_{y_T}(y)$$

where the PDF is a function used to specify the probability of the random variable falling within a particular range of values, $Pr(a \leq X \leq b) = \int_a^b f_X(x)dx$.

If the data have the same PDF everywhere in the sample, it should have

- the same mean
- the same variance
- the same skewness (the third moment)
- the same kurtosis (the fourth moment)
- all other higher moments

However, in reality, it is really hard to find empirical data which satisfies this strict stationarity. Often times, we relax the strict stationarity for two forms of weak stationarity.

1. One is the first-order weak stationarity or the mean stationarity, which is defined as time series data have the same mean (first moment) over the sample, that is

   $$\mu_{y_1} = \mu_{y_2} = \cdots = \mu_{y_T} = \mu$$

   is constant over time.

2. Another weak stationarity is the **second-order weak stationary**, so called "covariance-stationary", defined as time series data have the same mean and variance over time, and the covariances do not depend on time, that is

   $$\begin{aligned} &\text{(i) } \mu_{y_1} = \mu_{y_2} = \cdots = \mu_{y_T} = \mu \\ &\text{(ii) } \sigma^2_{y_1} = \sigma^2_{y_2} = \cdots = \sigma^2_{y_T} = \sigma^2 \\ &\text{(iii) } \rho_{y_t, y_{t-k}} = \rho_{|k|} \end{aligned}$$

where $\rho_{y_t,y_{t-k}}$ is the autocorrelation between y_t and y_{t-k}, defined as

$$\begin{aligned}\gamma_{y_t,y_{t-k}} = \mathbf{Cov}(y_t, y_{t-k}) &= \mathbf{E}\left[(y_t - \mathbf{E}(y_t))\,(y_{t-k} - \mathbf{E}(y_{t-k}))\right] \\ &= \mathbf{E}\left[(y_t - \mu)\,(y_{t-k} - \mu)\right]\end{aligned}$$

Note γ_k is time independent of t and only depends on the distance between two observations, y_i and y_j, where $i - j = k$.

Therefore, a time series process is weakly stationary if the following three conditions are satisfied simultaneously

1. $\mathbf{E}(y_t) = \mu_y$ is constant
2. $\mathbf{Var}(y_t) = \sigma_y^2$ is constant
3. $\mathbf{Cov}(y_t, y_{t-k}) = \gamma_k$ is constant

White Noise Process

White noise is a special case of a covariance stationary variable,

i. it not only has constant mean, **but the constant mean is zero**,
ii. it also has constant variance over time.
iii. **its auto-covariance is zero as well.**

Therefore, if a random variable is a White noise, it must be covariance stationary. However, if a random variable is covariance stationary, it may not be a White noise.

Our Example

As an example, assume that

$$y_t = \varepsilon_t + c_1\varepsilon_{t-1}$$

where $\varepsilon_t \sim WN(0, \sigma^2)$ and c_1 is a non-zero constant.

We can easily show that a linear combination of WN processes might not necessarily be a WN process.

Recall that for a variable to be considered a white noise process, it must satisfy the following conditions:

1. It must have a constant mean of zero.
2. It must have a constant variance.
3. It must have zero autocorrelation at all lags.

Let us assess whether the process y_t satisfies these conditions.

1. The mean of y_t is given by

$$\mathbf{E}(y_t) = \mathbf{E}\left(\varepsilon_t + c_1\varepsilon_{t-1}\right) = \cancelto{0}{\mathbf{E}(\varepsilon_t)} + c_1\cancelto{0}{\mathbf{E}(\varepsilon_{t-1})} = 0 \checkmark\checkmark$$

Hence, y_t has a constant mean.

2. The variance of y_t is given by

$$\begin{aligned}\mathbf{Var}(y_t) &= \mathbf{E}\left[(y_t - \mathbf{E}(y_t))^2\right] \\ &= \mathbf{E}\left(\varepsilon_t + c_1\varepsilon_{t-1}\right)^2 \\ &= \mathbf{E}(\varepsilon_t^2) + c_1^2\mathbf{E}(\varepsilon_{t-1}^2) + 2c_1\cancelto{0}{\mathbf{E}(\varepsilon_t\varepsilon_{t-1})};\end{aligned}$$

Since $\varepsilon_t \sim WN(0,\sigma^2)$, such that $\mathbf{E}(\varepsilon_t^2) = \mathbf{E}(\varepsilon_{t-1}^2) = \sigma^2$ and $\mathbf{E}(\varepsilon_t\varepsilon_{t-1}) = 0$, and hence

$$\begin{aligned}\mathbf{Var}(y_t) &= \sigma^2 + c_1^2\sigma^2 \\ &= \left(1 + c_1^2\right)\sigma^2 \checkmark\checkmark\end{aligned}$$

Therefore, y_t has a constant variance.

3. The autocovariances of y_t are as follows:

$$\begin{aligned}
\gamma_0 &= \mathbf{Cov}(y_t, y_t) = \mathbf{Var}(y_t) \\
\gamma_1 &= \mathbf{Cov}(y_t, y_{t-1}) = \mathbf{E}\left[(y_t - \mathbf{E}(y_t))(y_{t-1} - \mathbf{E}(y_{t-1}))\right] \\
&= \mathbf{E}(y_t y_{t-1}) \\
&= \mathbf{E}(\varepsilon_t + c_1\varepsilon_{t-1})\left(\varepsilon_{t-1} + c_1\varepsilon_{t-2}\right) \\
&= \cancelto{0}{\mathbf{E}\left(\varepsilon_t\varepsilon_{t-1}\right)} + c_1\mathbf{E}\left(\varepsilon_{t-1}^2\right) + c_1\mathbf{E}\left(\varepsilon_t\varepsilon_{t-2}\right) + c_2^2\cancelto{0}{\mathbf{E}\left(\varepsilon_{t-1}\varepsilon_{t-2}\right)} \\
&= c_1\sigma^2 \times \\
\gamma_2 &= \mathbf{Cov}(y_t, y_{t-2}) = \mathbf{E}(y_t - \mathbf{E}(y_t))(y_{t-2} - \mathbf{E}(y_{t-2})) \\
&= \mathbf{E}(y_t y_{t-2}) \\
&= \mathbf{E}(\varepsilon_t + c_1\varepsilon_{t-1})\left(\varepsilon_{t-2} + c_1\varepsilon_{t-3}\right) \\
&= \cancelto{0}{\mathbf{E}\left(\varepsilon_t\varepsilon_{t-2}\right)} + c_1\cancelto{0}{\mathbf{E}\left(\varepsilon_{t-1}\varepsilon_{t-2}\right)} + c_1\cancelto{0}{\mathbf{E}\left(\varepsilon_t\varepsilon_{t-3}\right)} + c_2^2\cancelto{0}{\mathbf{E}\left(\varepsilon_{t-1}\varepsilon_{t-3}\right)} \\
&= 0 \\
&\vdots \\
\gamma_k &= 0, \forall k > 1
\end{aligned}$$

We can quickly compute the autocorrelation coefficients as follows:

$$\begin{aligned}
\rho_0 &= \frac{\gamma_0}{\gamma_0} = 1 \\
\rho_1 &= \frac{\gamma_1}{\gamma_0} = \frac{c_1\sigma^2}{\left(1 + c_1^2\right)\sigma^2} = \frac{c_1}{1 + c_1^2} \times \\
\rho_2 &= \frac{\gamma_2}{\gamma_0} = \frac{0}{\left(1 + c_1^2\right)\sigma^2} = 0 \\
&\vdots \\
\rho_k &= 0, \forall k > 1
\end{aligned}$$

y_t has a covariance that is independent of time. However, the autocorrelation of y_t is not zero at all lags. Therefore, y_t is not a WN process.