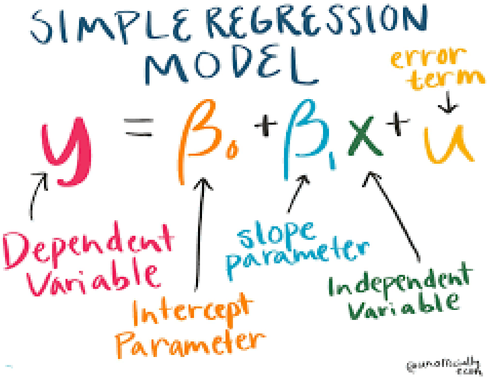
Fundamentals of Econometrics

Lecture 2: Simple Linear Regression Model



1. The Basics

2. Estimating the OLS Coefficients

3. Goodness of Fit

Section 1

The Basics

Subsection 1

Simple Linear Regression Model

Simple Linear Regression Model

$$y = \beta_0 + \beta_1 x_i + u$$

where

- y is the **dependent** variable,
- x is the **independent** variable,
- β_0 is the **intercept**,
- β_1 is the **slope**,
- and u is the error (disturbance) term.

Note: This is sometimes referred to as the **data generating process** (DGP) because we assume that the observable data follows this relationship.

Subsection 2

Simple Linear Regression Model

y	x
Dependent Variable	Independent Variable
Explained Variable	Explaining (Explanatory) Variable
Endogenous Variable	Exogenous Variable
Response Variable	Control Variable
Predicted Variable	Predictor Variable
LHS Variable	RHS Variable
Regressand	Regressor

We say we **regress** y on x.

Subsection 3

Interpreting SLR model

Interpreting SLR model

Model studies how y varies with changes in x:

$$\underset{\Downarrow}{\frac{\delta y}{\delta x} = \beta_1} \quad \text{as long as} \quad \underset{\Downarrow}{\frac{\delta u}{\delta x} = 0}$$

By how much does the dependent variable change if the independent variable is increased by one unit?	Interpretation of β_1 is only correct if all other things remain equal when the independent variable is increased by one unit.

The simple regression is rarely applicable in practice but offers a good starting point for understanding the more general multiple regression model.

Subsection 4

Examples of SLR

Example: Soybean yield and fertilizer

$$yeild_i = \beta_0 + \beta_1 fertilizer_i + u_i$$

- What else is in u?

Example: Wages and education

$$wage_i = \beta_0 + \beta_1 education_i + u$$

- wage is dollars per hour
- education is years of schooling
- What else is in u?

Subsection 5

Examples of SLR

Example: Corn yield and time

$$yeild_t = \beta_0 + \beta_1 t + u_t, \quad t = 1, \ldots, T$$

where t measures the effect of time on yield.

- What else is in u?

Note: In this case we are examining how corn yields change as a function of time (a linear trend). The implication is we have time series data

T denotes the total number of time series observations

Subsection 6

SLR Assumption #1

SLR Assumption #1

Conditional mean independence assumption:

$$\boxed{E(u_i|x) = 0}$$

- Implies that the explanatory variable must not contain information about the mean of the unobserved factors that affect the dependent.
- This assumption holds regardless of whether x is fixed or stochastic (in repeated samples).

Example: wage equation

$$wage_i = \beta_0 + \beta_1 educ_i + u$$

- u includes factors such as ability, motivation, intelligence, etc.

The conditional mean independence assumption is unlikely to hold because individuals with more education will also be more intelligent on average

Subsection 7

SLR Assumption #1

SLR Assumption #1

Population Regression Function (PRF)

- The conditional mean independence assumption implies that

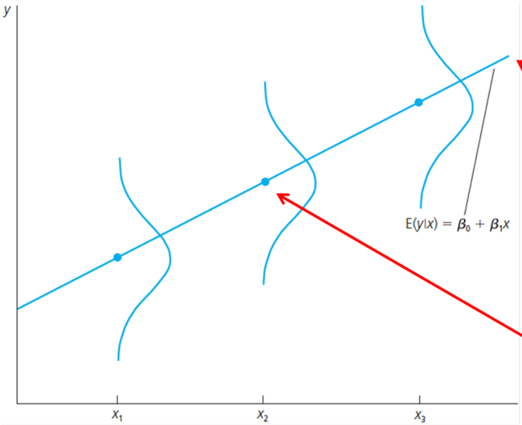
$$
\begin{aligned}
E(y|x_i) &= E(\beta_0 + \beta_1 x_i + u_i|x_i) \\
&= E(\beta_0 + \beta_1 x_i + E(u_i|x_i)) \\
&= \beta_0 + \beta_1 x_i + \cancelto{0}{E(u_i|x_i)} \\
&= \beta_0 + \beta_1 x_i
\end{aligned}
$$

- This means that the average value of the dependent variable can be expressed as a linear function of the independent variable.

Subsection 8

The SLR

The SLR



Population regression function

For individuals with $x = x_2$, the average value of y is $\beta_0 + \beta_1 x_2$

Subsection 9

SLR Assumption #2

SLR Assumption #2

Conditional mean independence assumption:

$$\boxed{E(u_i|x) = 0} \implies E(u_i) = 0$$

Subsection 10

Deriving the OLS Estimator

Deriving the OLS Estimator

One way of deriving the OLS estimator is to minimize the sum of squared residuals.

The regression residuals are the difference between the observed and predicted values of the dependent variable:

$$\hat{u}_i = y_i - \hat{y}_i = y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i$$

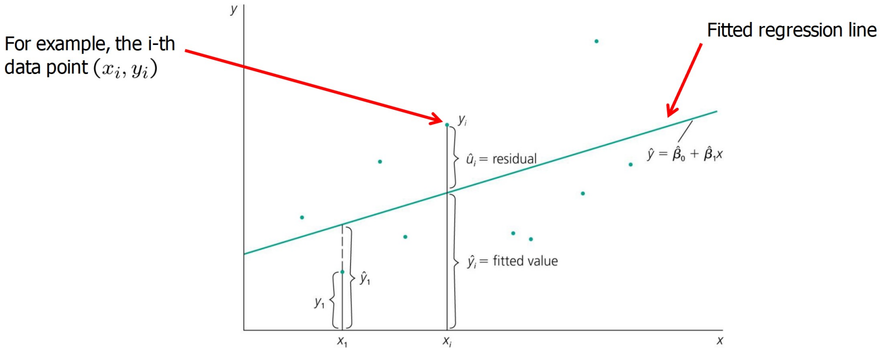
Minimizing the sum of squared residuals:

$$\underset{\beta_0,\beta_1}{min} \sum_{i=1}^{n} \hat{u}_i^2 = \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

OLS Estimators:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2};\ \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

Subsection 11



For example, the i-th data point (x_i, y_i)
Fitted regression line
y
y_i
$\hat{u}_i$ = residual
$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$
$\hat{y}_i$ = fitted value
$\hat{y}_1$
y_1
x_1
x_i
x

Subsection 12

Deriving the OLS Estimator

Deriving the OLS Estimator

$$\underset{\{\beta_0,\beta_1\}}{min}\sum_{i=1}^{n} Q = \sum_{i=1}^{n} \hat{u}_i = \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

Differentiating with respect to β_0 and β_1 and setting the derivatives equal to zero gives the OLS estimators:

$$\frac{\partial Q}{\partial \hat{\beta}_0} = \sum_{i=1}^{n} 2(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(-1) = 0 \tag{1}$$

$$\frac{\partial Q}{\partial \hat{\beta}_1} = \sum_{i=1}^{n} 2(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(-x_i) = 0 \tag{2}$$

Subsection 13

Basic properties: summation operator (Digression)

Basic properties: summation operator (Digression)

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \ldots + x_n$$

$$\sum_{i=1}^{n} \alpha = n\alpha$$

$$\sum_{i=1}^{n} \alpha x_i = \alpha x_1 + \alpha x_2 + \ldots + \alpha x_n = \alpha \sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} \alpha(x_i + y_i) = \alpha\left[\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i\right]$$

Subsection 14

Deriving the OLS Estimator

Deriving the OLS Estimator

Using these properties, we can rewrite (1) and (2) as:

$$\sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = \boxed{\sum_{i=1}^{n} \hat{u}_i = 0} \tag{3}$$

$$\sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)x_i = \boxed{\sum_{i=1}^{n} \hat{u}_i x_i = 0} \tag{4}$$

(3) **is particularly important because, as long as there is an intercept in the model, the sum of the residuals is always zero.**

(3) and (4) are the so called "Least Squares Normal Equations" or just "Normal Equations". We now have two equations and two unknowns, $\hat{\beta}_0$ and $\hat{\beta}_1$.

Subsection 15

Deriving the OLS Estimator, β_0

Deriving the OLS Estimator, β_0

$$\hat{\beta}_0 = \frac{1}{n}\sum_{i=1}^{n} y_i - \hat{\beta}_1 \frac{1}{n}\sum_{i=1}^{n} x_i \tag{5}$$

$$= \bar{y} - \hat{\beta}_1 \bar{x} \tag{6}$$

Note that (6) naturally implies that:

$$\bar{y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x}$$

Therefore, the estimated regression line always passes through the point $(\bar{x}, \bar{y})$, the sample means!

Subsection 16

Deriving the OLS Estimator

$$\boxed{\sum_{i=1}^{n}(x_i - \bar{x}) = \sum_{i=1}^{n} x_i - n\bar{x} = n\bar{x} - n\bar{x} = 0} \tag{7}$$

$$\underbrace{\sum_{i=1}^{n}(x_i - \bar{x})^2}_{var(x_i)} = \sum_{i=1}^{n}(x_i - \bar{x})x_i - \bar{x}\cancelto{0}{\sum_{i=1}^{n}(x_i - \bar{x})}$$

$$= \sum_{i=1}^{n} x_i(x_i - \bar{x}) = \sum_{i=1}^{n} x_i^2 - \bar{x}\sum_{i=1}^{n} x_i$$

$$\boxed{\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2} \tag{8}$$

Subsection 17

Deriving the OLS Estimator

Deriving the OLS Estimator

Useful Properties

$$\underbrace{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}_{\text{covariance(x,y)}} = \sum_{i=1}^{n} y_i(x_i-\bar{x}) - \bar{y}\cancelto{0}{\sum_{i=1}^{n}(x_i-\bar{x})}$$

$$= \sum_{i=1}^{n} x_i y_i - \bar{x}\sum_{i=1}^{n} y_i$$

$$\boxed{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y}) = \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i y_i \implies \sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}} \quad (9)$$

Subsection 18

Deriving the OLS Estimator, β_1

Deriving the OLS Estimator, β_1

$$\sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)x_i = 0$$

$$\sum_{i=1}^{n} x_i y_i - \hat{\beta}_0 \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \hat{\beta}_1 x_i^2 = 0$$

$$\sum_{i=1}^{n} x_i y_i = \hat{\beta}_0 \sum_{i=1}^{n} x_i + \hat{\beta}_1 \sum_{i=1}^{n} x_i^2$$

Substitute $\hat{\beta}_0$ from (6)

$$\sum_{i=1}^{n} x_i y_i = (\bar{y} - \hat{\beta}_1 \bar{x}) \sum_{i=1}^{n} x_i + \hat{\beta}_1 \sum_{i=1}^{n} x_i^2$$

$$\underbrace{\boxed{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}}_{cov(x,y)} = \beta_1 \underbrace{\boxed{\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right)}}_{var(x)}$$

Subsection 19

Deriving the OLS Estimator, β_1

Deriving the OLS Estimator, β_1

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \tag{10}$$

$$\hat{\beta}_1 = \frac{cov(x, y)}{\hat{\sigma}_x^2} \tag{11}$$

Section 2

Estimating the OLS Coefficients

Subsection 1

Example: Hours studied and exam score

Example: Hours studied and exam score

Below is a sample dataset of 10 observations on the relationship between the number of hours studied and the exam score.

$$score_i = \beta_0 + \beta_1 hours_i + u_i$$

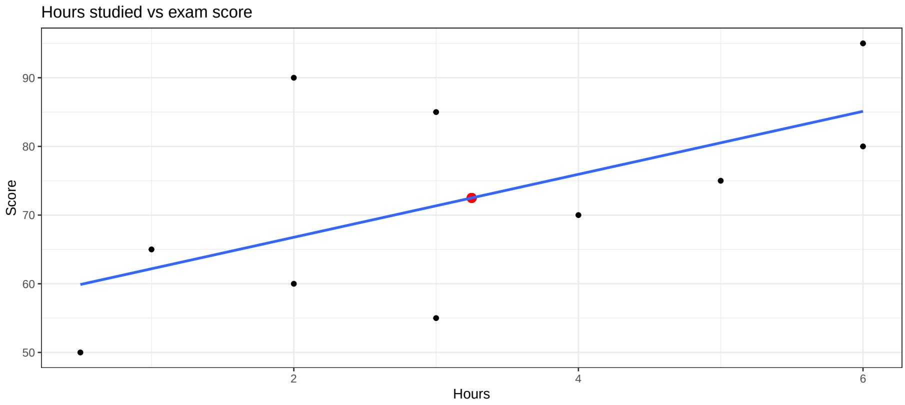
```r
hours <- c(0.5,3,2,1,4,5,6,3,2,6)
score <- c(50, 55, 60, 65, 70, 75, 80, 85, 90, 95)
df <- data.frame(hours, score)
df
```

```
##    hours score
## 1    0.5    50
## 2    3.0    55
## 3    2.0    60
## 4    1.0    65
## 5    4.0    70
## 6    5.0    75
## 7    6.0    80
## 8    3.0    85
## 9    2.0    90
## 10   6.0    95
```

Subsection 2

Let us start with visualizing the data.

```r
# Scatter plot of hours studied and exam score
ggplot(df, aes(x = hours, y = score)) +
# Add points
  geom_point() +
# Add mean of x and y to plot
  geom_point(aes(x = mean(hours), y = mean(score)), color = "red", size = 3) +
# Add regression line
  geom_smooth(method = "lm", se = FALSE) +
# Add title and axis labels
  labs(title = "Hours studied vs exam score", x = "Hours", y = "Score") +
# Change theme
  theme_bw()
```



Subsection 3

Estimating the OLS coefficients

Estimating the OLS coefficients

```r
# Calculate the means of hours and score
x_bar <- mean(df$hours)
y_bar <- mean(df$score)
# Calculate the variance of hours
var_x <- sum((df$hours - x_bar)^2)
# Calculate the covariance of hours and score
cov_xy <- sum((df$hours - x_bar) * (df$score - y_bar))
# Calculate the OLS estimator for beta_1
beta_1_hat <- cov_xy / var_x
# Calculate the OLS estimator for beta_0
beta_0_hat <- y_bar - beta_1_hat * x_bar

# Print the OLS estimators
cat("The OLS estimator for beta_0 is: ", beta_0_hat,
    "\n beta_1 is: ", beta_1_hat)
```

```
## The OLS estimator for beta_0 is:  57.59928
##  beta_1 is:  4.584838
```

Subsection 4

Estimating the OLS coefficients

Estimating the OLS coefficients

Using the `lm()` function in R, we can estimate the OLS coefficients.

```r
model1 <- lm(score ~ hours, data = df)
summary(model1)
```

```
##
## Call:
## lm(formula = score ~ hours, data = df)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -16.354  -6.561  -5.316   8.123  23.231
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   57.599      8.220   7.007 0.000112 ***
## hours          4.585      2.195   2.089 0.070158 .
## ---
```

Subsection 5

Estimating the OLS coefficients

Estimating the OLS coefficients

We hypothesize the following DGP:

$$score_i = \beta_0 + \beta_1 hours_i + u_i$$

The fitted regression line is:

$$\hat{score}_i = 57.599 + 4.585 hours_i$$

If the number of hours studied increases by one unit, the exam score is expected (predicted) to increase by 4.585 points.

Subsection 6

Computing the OLS residuals (manually)

Computing the OLS residuals (manually)

Using the OLS estimators, we can calculate the residuals.

```
# Calculate the residuals
# y - yhat = u
df$resids.man <- df$score - (beta_0_hat + beta_1_hat * df$hours)
df
```

```
##    hours score resids.man
## 1    0.5    50  -9.891697
## 2    3.0    55 -16.353791
## 3    2.0    60  -6.768953
## 4    1.0    65   2.815884
## 5    4.0    70  -5.938628
## 6    5.0    75  -5.523466
## 7    6.0    80  -5.108303
## 8    3.0    85  13.646209
## 9    2.0    90  23.231047
## 10   6.0    95   9.891697
```

What is the sum of the residuals?

Subsection 7

Computing the OLS residuals

Computing the OLS residuals

Using the `residuals()` function in R, we can calculate the residuals.

```r
# Calculate the residuals
df$resids.lm <- residuals(model1)
# Print the residuals
df
```

```
##    hours score resids.man  resids.lm
## 1    0.5    50  -9.891697  -9.891697
## 2    3.0    55 -16.353791 -16.353791
## 3    2.0    60  -6.768953  -6.768953
## 4    1.0    65   2.815884   2.815884
## 5    4.0    70  -5.938628  -5.938628
## 6    5.0    75  -5.523466  -5.523466
## 7    6.0    80  -5.108303  -5.108303
## 8    3.0    85  13.646209  13.646209
## 9    2.0    90  23.231047  23.231047
## 10   6.0    95   9.891697   9.891697
```

Subsection 8

Computing the OLS residuals

Computing the OLS residuals

Are all the residuals the same **pairwise**?

```
# Check if the residuals are the same
all.equal(df$resids.lm,df$resids.man)
```

```
## [1] TRUE
```

Subsection 9

Computing the fitted values

Computing the fitted values

We might be interested in knowing the fitted values implied by the OLS estimators.

```
# Calculate the fitted values (y-hat)
df$fit <- fitted(model1)
df$fit.man <- beta_0_hat + beta_1_hat * df$hours
df
```

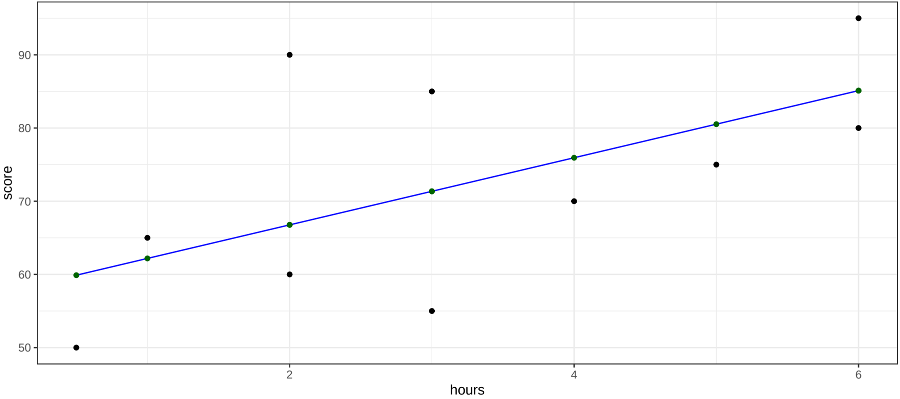
```
##   hours score resids.man  resids.lm      fit  fit.man
## 1   0.5    50  -9.891697  -9.891697 59.89170 59.89170
## 2   3.0    55 -16.353791 -16.353791 71.35379 71.35379
## 3   2.0    60  -6.768953  -6.768953 66.76895 66.76895
## 4   1.0    65   2.815884   2.815884 62.18412 62.18412
## 5   4.0    70  -5.938628  -5.938628 75.93863 75.93863
## 6   5.0    75  -5.523466  -5.523466 80.52347 80.52347
## 7   6.0    80  -5.108303  -5.108303 85.10830 85.10830
## 8   3.0    85  13.646209  13.646209 71.35379 71.35379
## 9   2.0    90  23.231047  23.231047 66.76895 66.76895
```

Subsection 10

Plotting the fitted vs actual values

Plotting the fitted vs actual values

```
# Scatter plot of hours studied and exam score
ggplot(df, aes(x = hours, y = score)) +
# Add points (use a diff point type for each obs)
  geom_point() +
# Add regression line
geom_line(aes(x = hours, y = fit), color = "blue") +
# Show points of fitted values
  geom_point(aes(x = hours, y = fit),color = "darkgreen") +
  theme_bw()
```



Section 3

Goodness of Fit

Subsection 1

Coefficient of Determination, R^2

Coefficient of Determination, R^2

How well does an explanatory variable explain the dependent variable?

More generally: How well does the regression line fit the data?

One approach is to measure the share of observed variation in y that can be explained by the variation in x.

- Total Sum of Squares (TSS) $= \sum_{i=1}^{n}(y_i - \bar{y})^2$
- Explained Sum of Squares (ESS) $= \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2$
- Residual Sum of Squares (SSR) $= \sum_{i=1}^{n}\hat{u}_i^2$

$$SST = ESS + SSR$$

Subsection 2

Coefficient of Determination, R^2

Coefficient of Determination, R^2

Hard to interpret the value of any of the sum of squares because they depend on the scale of the data. Much easier to interpret a ratio which would remove the scaling effect.

$$
\begin{aligned}
SST &= ESS + SSR \\
1 &= \frac{ESS}{SST} + \frac{SSR}{SST} \\
1 &= R^2 + \frac{SSR}{SST} \\
R^2 &= 1 - \frac{SSR}{SST}
\end{aligned}
$$

Properties of R^2:

- $0 \leq R^2 \leq 1$ (once there is an intercept in the model)
- Measures statistical correlation, not causation

Subsection 3

Coefficient of Determination, R^2

Coefficient of Determination, R^2

CEO Salary and return on equity:

$$\widehat{salary}_i = 963.191 + 18.501 roe_i$$
$$n = 209, \quad R^2 = 0.0132$$

The regression explains only 1.3% of the total variation in salaries.

Voting outcomes and campaign expenditures

$$\widehat{voteA} = 26.81 + 0.464 shareA$$
$$n = 173, \quad R^2 = 0.856$$

The percentage of total campaign expenditures accounted for by Candidate A ($shareA$) explains 85.6% of the total variation in election outcomes for candidate A.

Caution: A high R-squared does not necessarily mean that the regression has a causal interpretation!

Subsection 4

```
##
## Call:
## lm(formula = score ~ hours, data = df)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -16.354  -6.561  -5.316   8.123  23.231
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   57.599      8.220   7.007 0.000112 ***
## hours          4.585      2.195   2.089 0.070158 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '
##
## Residual standard error: 12.92 on 8 degrees of freedom
## Multiple R-squared:  0.3529, Adjusted R-squared:  0.272
## F-statistic: 4.363 on 1 and 8 DF,  p-value: 0.07016
```

Subsection 5

Coefficient of Determination, R^2

Coefficient of Determination, R^2

```r
# Calculate the total sum of squares
TSS <- sum((df$score - mean(df$score))^2)
# Calculate the explained sum of squares
ESS <- sum((predict(model1) - mean(df$score))^2)
# Calculate the residual sum of squares
SSR <- sum(residuals(model1)^2)
# Calculate R-squared
R2 <- 1 - SSR / TSS
R2
```

```
## [1] 0.3528936
```

Subsection 6

Unit of Measurement

Unit of Measurement

How does changing the units of measurement affect the OLS estimators?

Example: Regress CEO salary on return on equity

Original model:

- Salary in **thousands of dollars** and roe in percent

$$\widehat{salary}_i = 963.191 + 18.501 roe_i$$

Transformed model:

- Express salary (dependent variable) in **dollars** and roe in percent

$$\widehat{salary}_i = 963,191 + 18,501 roe_i$$

Subsection 7

Unit of Measurement

Unit of Measurement

```
# library(wooldridge) # Load datasets
# Store original model
model.ceo <- lm(salary ~ roe, data = ceosal1)

# Store transformed model
model.ceo2 <- lm(salary * 1000 ~ roe, data = ceosal1)

# compare the coefficients
data.frame(Original = coef(model.ceo), Dep.Transformed = coef(model.ceo2))
```

```
##              Original Dep.Transformed
## (Intercept) 963.19134       963191.34
## roe          18.50119        18501.19
```

General Takeaways?

Subsection 8

Unit of Measurement

What happens if we were to express the independent variable in decimal form instead of percentages?

New model: Express salary (dependent variable) in **thousand of dollars** and roe in **percent/100**

```r
# Store transformed model
model.ceo3 <- lm(salary ~ I(roe/100), data = ceosal1)
coef(model.ceo3)
```

```
## (Intercept)  I(roe/100)
##    963.1913   1850.1186
```

Subsection 9

Unit of Measurement

What happens to R^2 in both cases?

```
#Extract R-squared from the original model
R2_original <- summary(model.ceo)$r.squared
#Extract R-squared from dep. variable transformed model
R2_dep <- summary(model.ceo2)$r.squared
#Extract R-squared from indep. variable transformed model
R2_ind <- summary(model.ceo3)$r.squared
# Store in a data frame
R2 <- data.frame(Original = R2_original, Dependent = R2_dep,
                 Independent = R2_ind)
R2
```

```
##     Original  Dependent Independent
## 1 0.01318862 0.01318862  0.01318862
```

Subsection 10

Functional Form

What happens if I think there are increasing returns to years of education. This might imply that:

$$wage_i = \exp \beta_0 + \beta_1 educ_i + u_i$$

If we want to estimate with OLS then the model still needs to be **linear in the parameters**. We could convert this to a linear model by taking the natural log of both sides:

$$\log{(wage_i)} = \beta_0 + \beta_1 educ_i + u_i$$

This will change the interpretation of the coefficients.

$$\beta_1 = \frac{\Delta \log{(wage)}}{\Delta educ} = \frac{1}{wage} \cdot \frac{\Delta wage}{\Delta educ} = \frac{\frac{\Delta wage}{wage}}{\Delta educ}$$

Subsection 11

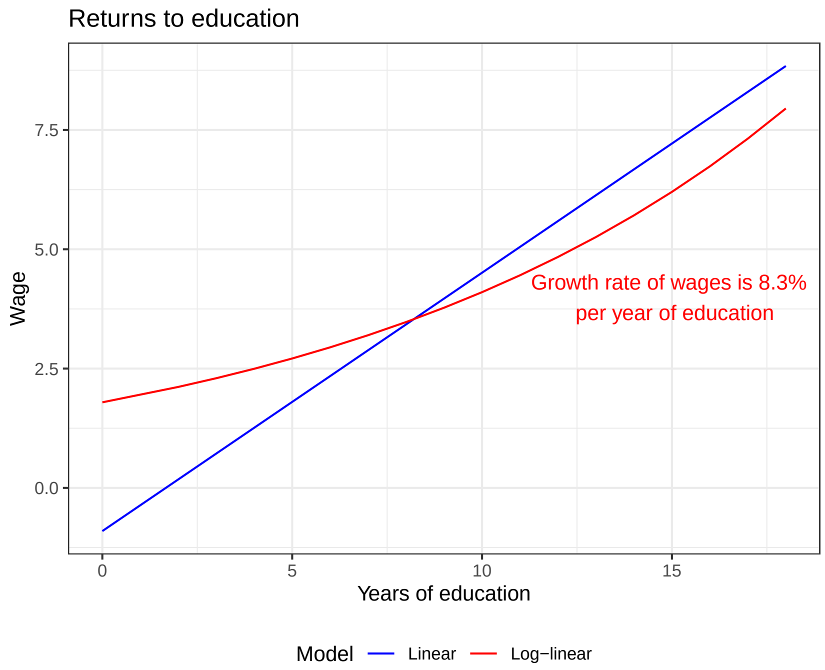
Functional Form

Example: returns of education to wage

```
wage.lin <- lm(wage ~ educ, data = wage1)
wage.log <- lm(log(wage) ~ educ, data = wage1)

# Plot the model fits
ggplot(wage1) +
  geom_line(aes(x = educ, y = fitted(wage.lin), color = "blue")) +
  # Add the log-linear model, must exponentiate the fitted values
  geom_line(aes(x = educ, y = exp(fitted(wage.log)), color = "red")) +
  labs(title = "Returns to education", x = "Educ", y = "Wage") +
  # Add a legend
  scale_color_manual(name = "Model", values = c("blue", "red"),
                     labels = c("Linear", "Log-linear")) +
  # Add an arrow and text to quadratic line
  annotate("text",x = 15, y = 4,
           label = "Growth rate of wages is 8.3% \n per year of educ.",
           color = "red") +
  theme_bw() +
  theme(legend.position = "bottom")
```

Subsection 12


Returns to education
Wage
7.5
5.0
2.5
0.0
0
5
10
15
Years of education
Growth rate of wages is 8.3%
per year of education
Model
Linear
Log−linear

Subsection 13

Functional Form

Example: CEO salary and firm sales

What about a model where logs appear on both sides?

$$log(salary) = \beta_0 + \beta_1 log(sales) + u$$

- Again, this changes the interpretation of the coefficients.

$$\beta_1 = \frac{\Delta \log(salary)}{\Delta \log(sales)} = \frac{\frac{\Delta salary}{salary}}{\frac{\Delta sales}{sales}}$$

- A 1% increase in sales is associated with a β_1% increase in salary.
- This is a measure of the **elasticity** of salary with respect to sales, whereas the semi-log form assumes a **semi-elasticity**.

Subsection 14

$$\widehat{\log(salary)} = 4.822 + 0.257 \log(sales)$$

```
##
## Call:
## lm(formula = log(salary) ~ log(sales), data = ceosal1)
##
## Coefficients:
## (Intercept)   log(sales)
##      4.8220       0.2567
```

As sales increase by 1%, CEO salary is expected to increase by 0.257%.

Subsection 15

Functional Form

Functional Form

The functional forms can be summarized as follows:

- level-level: $wage_i = \beta_0 + \beta_1 educ_i + u_i$
- log-level (semi-log): $\log(wage_i) = \beta_0 + \beta_1 educ_i + u_i$
- level-log (semi-log): $wage_i = \beta_0 + \beta_1 \log(educ_i) + u_i$
- log-log: $\log(wage_i) = \beta_0 + \beta_1 \log(educ_i) + u_i$

You should be able to interpret the coefficients in each of these models.

Subsection 16

Properties of OLS Estimators

Properties of OLS Estimators

The estimated regression coefficients are random variables because they are calculated from a random sample.

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}, \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

Our data is random and depends on particular sample that has been drawn.

This raises the question of what the estimators will estimate on average and how large will their variability be in repeated samples.

$$E(\hat{\beta}_1) \stackrel{?}{=} \beta_1, \; E(\hat{\beta}_0) \stackrel{?}{=} \beta_0 \quad var(\hat{\beta}_1) \stackrel{?}{=} \sigma^2_{\hat{\beta}_1}, \quad var(\hat{\beta}_0) = ?, \; var(\hat{\beta}_1) = ?$$

Subsection 17

Digression

Digression

Show that β_1 is a random variable since it depends on the sample drawn.

$$
\begin{aligned}
\hat{\beta}_1 &= \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} \\
&= \frac{\sum_{i=1}^n (x_i - \bar{x}) y_i}{\sum_{i=1}^n (x_i - \bar{x})^2} \\
&= \frac{\sum_{i=1}^n (x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} (\beta_0 + \beta_1 x_i + u_i) \\
&= \beta_0 \frac{\cancelto{0}{\sum_{i=1}^n (x_i - \bar{x})}}{\sum_{i=1}^n (x_i - \bar{x})^2} + \beta_1 \frac{\sum_{i=1}^n (x_i - \bar{x}) x_i}{\sum_{i=1}^n (x_i - \bar{x})^2} + \frac{\sum_{i=1}^n (x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} u_i \\
&= \beta_1 + \frac{\sum_{i=1}^n (x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} u_i
\end{aligned}
$$

Similar logic holds for $\hat{\beta}_0$

Subsection 18

OLS Assumptions

OLS Assumptions

1. **SLR.1 Linearity in parameters:** The model is linear in the parameters reflecting the assumption that the true relationship between the dependent and independent variables (in the population) is indeed linear.
2. **SLR.2 Random sampling:** The data (each x, y pair) are a random sample from the population of interest. Each data point therefore follows the population equation...
3. **SLR.3 Sample variation in the independent variable:** The independent variable has some variation in the sample: $\implies \sum_{i=1}^{n}(x_i - \bar{x})^2 > 0$.
4. **SLR.4 Zero conditional mean:** The value of the explanatory variable must not contain information about the mean of the unobserved factors that might affect y: $\implies E(u_i|x_i) = 0$.

Subsection 19

OLS Assumptions

Unbiasedness of OLS Estimators:

SLR.1 - SLR.4 $\implies E(\hat{\beta}_0) = \beta_0,\ E(\hat{\beta}_1) = \beta_1$

Interpretation: If we were to draw many samples from the population and estimate the regression line in each sample, the average of the estimated slopes would be the true population slope.

- The estimated coefficients may be smaller or larger, depending on the sample that is the result of a random draw. Some could be well-off from their true values too.
- However, on average, they will be equal to the values that characterize the true relationship between y and x in the population.

Subsection 20

Unbiasedness of OLS Estimators (Adopted from URFIE, Heiss)

Unbiasedness of OLS Estimators (Adopted from URFIE, Heiss)

```
set.seed(1234567) # Set the random seed

# set sample size and number of simulations
n <- 1000; r <- 10000
# set true parameters: betas and sd of u
b0 <- 1; b1 <- 0.5; su <- 2
# initialize b0hat and b1hat to store results later:
b0hat <- numeric(r); b1hat <- numeric(r)

x <- rnorm(n,4,1)# Draw a sample of x, fixed over replications:

for(j in 1:r) {# repeat r times:
# Draw a sample of y:
  u <- rnorm(n,0,su) # u changes with each draw
  y <- b0 + b1*x + u
# estimate parameters by OLS and store them in the vectors
  bhat <- coefficients( lm(y~x) )
  b0hat[j] <- bhat["(Intercept)"]; b1hat[j] <- bhat["x"]
}
```

Subsection 21

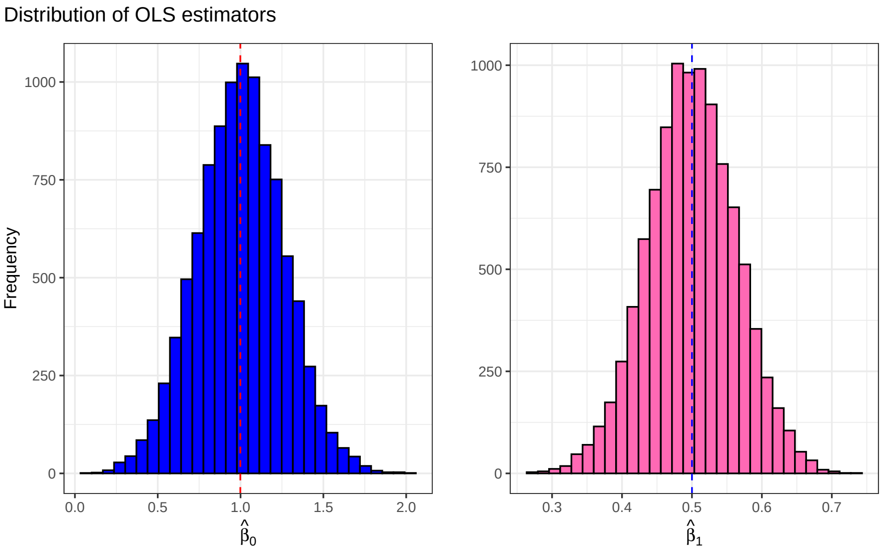
```r
#library(patchwork)

p1 <- ggplot() +
  geom_histogram(aes(x = b0hat), fill = "blue",
                 color = "black", bins = 30) +
  labs(title = NULL, x = expression(hat(beta)[0]), y = "Frequency") +
  geom_vline(xintercept = mean(b0hat), color = "red",
             linetype = "dashed") +
  theme_bw()

p2 <- ggplot() +
  geom_histogram(aes(x = b1hat), fill = "hotpink",
                 color = "black", bins = 30) +
  labs(title = NULL, x = expression(hat(beta)[1]), y = "") +
  geom_vline(xintercept = mean(b1hat), color = "blue",
             linetype = "dashed") +
  theme_bw()

(p1|p2) + plot_annotation(title = "Distribution of OLS estimators")
```

Subsection 22


Distribution of OLS estimators
Frequency
1000
750
500
250
0
0.0
0.5
1.0
1.5
2.0
$\hat{\beta}_0$
1000
750
500
250
0
0.3
0.4
0.5
0.6
0.7
$\hat{\beta}_1$

Subsection 23

Proving Unbiasedness of OLS Estimators

Proving Unbiasedness of OLS Estimators

Recall from earlier that:

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_{i=1}^{n}(x_i - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} u_i$$

taking conditional expectations:

$$E(\hat{\beta}_1 | x_i) = \beta_1 + \frac{\sum_{i=1}^{n}(x_i - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \cancelto{0}{E(u_i | x_i)}$$

$$E(\hat{\beta}_1 | x_i) = \beta_1$$

Therefore, $\hat{\beta}_1$ is an unbiased estimator of the population slope coefficient, β_1.

Subsection 24

In the case of $\hat{\beta}_0$:

$$\begin{aligned}
\hat{\beta}_0 &= \frac{1}{n}\sum_{i=1}^{n} y_i - \hat{\beta}_1 \frac{1}{n}\sum_{i=1}^{n} x_i \\
&= \frac{1}{n}\sum_{i=1}^{n}\Big(\beta_0 + \beta_1 x_i + u_i\Big) - \hat{\beta}_1 \frac{1}{n}\sum_{i=1}^{n} x_i \\
&= \frac{1}{n}\cdot n\beta_0 + \frac{1}{n}\Big(\beta_1 - \hat{\beta}_1\Big)\sum_{i=1}^{n} x_i + \frac{1}{n}\sum_{i=1}^{n} u_i
\end{aligned}$$

taking conditional expectations:

$$\begin{aligned}
E(\hat{\beta}_0|x_i) &= \beta_0 + \boxed{\Big(\beta_1 - E(\hat{\beta}_1|x_i)\Big)}\frac{1}{n}\sum_{i=1}^{n} x_i + \frac{1}{n}\sum_{i=1}^{n} E\Big(u_i|x_i\Big) \\
E(\hat{\beta}_0|x_i) &= \beta_0
\end{aligned}$$

Hence, $\hat{\beta}_0$ is an unbiased estimator of the population intercept term.

Subsection 25

Variance of the OLS Estimators

Variance of the OLS Estimators

- As we noted earlier, depending on the sample, the estimates will be nearer or farther away from the true population values.
- How far can we expect our estimates to be away from the true population values on average (= sampling variability)?
- Sampling variability is measured by the estimator's variances

$$var(\hat{\beta}_0), \quad var(\hat{\beta}_1)$$

OLS Assumption:

5. **SLR.5 Homoskedasticity:** The error term has the same variance conditional on the independent variable:

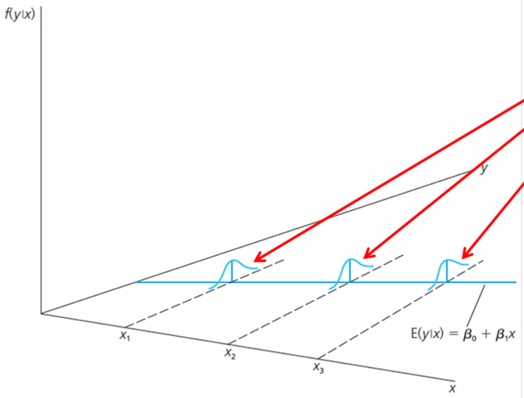
$$\sigma^2 = var(u_i|x_i).$$

That is, the value of the xs must contain no information about the variability of the unobserved factors (variables in u).

Subsection 26

Graphical illustration of homoskedasticity

Graphical illustration of homoskedasticity

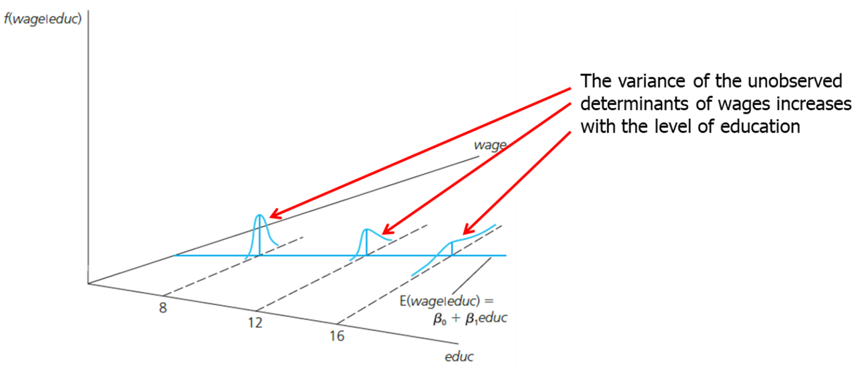


The variability of the unobserved influences does not depend on the value of the explanatory variable

Subsection 27

Graphical illustration of heteroskedasticity

Graphical illustration of heteroskedasticity



Subsection 28

Variance of the OLS Estimators

Variance of the OLS Estimators

SLR.1 – SLR.5

$$\implies var(\hat{\beta}_0) = \frac{\sigma^2}{n} \cdot \frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}, \quad var(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

Key Takeaway: The sampling variability of the estimated regression coefficients will be the higher the larger the variability of the unobserved factors, and lower, the higher the variation in the explanatory variable.

Subsection 29

Variance of the OLS Estimators

Proof of $var(\hat{\beta}_1)$:

$$\hat{\beta}_1 = \beta_1 + \frac{\sum\limits_{i=1}^{n}(x_i - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} u_i$$

running the variance operator through:

$$var(\hat{\beta}_1) = var\left(\beta_1 + \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{\sum_{i=1}^{n}\left((x_i - \bar{x})^2\right)^2} var(u_i)\right)$$

$$var(\hat{\beta}_1) = 0 + \frac{\sigma^2}{\sum\limits_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\sigma^2}{SST_x}$$

Subsection 30

Proof of $var(\hat{\beta}_0)$:

I leave this to you!

Subsection 31

Estimating the Error Variance

Estimating the Error Variance

Like the values of the parameters, the value of σ^2 is unknown and must be estimated from the data.

$$var(u_i|x_i) = \sigma^2 = E(\hat{u}_i - \bar{\hat{u}})^2 = E(\hat{u}_i)^2$$

$$\tilde{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{u}_i - \bar{\hat{u}}\right)^2 = \frac{1}{n}\sum_{i=1}^{n}\hat{u}_i^2 \tag{12}$$

$$\boxed{\hat{\sigma}^2 = \frac{1}{n-2}\sum_{i=1}^{n}\hat{u}_i^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2} \tag{13}$$

- (12) is the sample variance of the residuals, however it **is biased**.
- (13) is the **unbiased** estimator of the population variance of the error term and is obtained by subtracting the number of estimated regression parameters from the sample size (number of obs).

Subsection 32

Variance of the OLS Estimators

Variance of the OLS Estimators

Substituting (13) into the respective formulas for the variances of the OLS estimators, we get:

$$
\begin{aligned}
\widehat{var(\hat{\beta}_0)} &= \frac{\hat{\sigma}^2}{n} \cdot \frac{\sum\limits_{i=1}^{n} x_i^2}{SST_x} \\
\widehat{var(\hat{\beta}_1)} &= \frac{\hat{\sigma}^2}{SST_x} \\
se(\hat{\beta}_0) = \sqrt{var(\hat{\beta}_0)} &= \sqrt{\hat{\sigma}^2/n \cdot \sum_{i=1}^{n} x_i^2/SST_x} \\
se(\hat{\beta}_1) = \sqrt{var(\hat{\beta}_1)} &= \sqrt{\hat{\sigma}^2/SST_x}
\end{aligned}
$$

The estimated standard deviations of the regression coefficients are called “standard errors”. They measure how precisely the regression coefficients are estimated.