Unit - III

Recurrence Relation (Difference equation)

A Relation which involves an Independent variable x, a dependent variable y and one or more than one differences $\Delta y, \Delta^2 y, \Delta^3 y$ --- is called a Recurrence relation

$$F(x, y, \Delta y, \Delta^2 y ---) = 0 \quad — (1)$$

OR.

$$y_{n+2} - 5y_{n+1} + 6y_n = 0$$

$$y_{n+1} - 3y_n = 3^n$$

Order and degree of Recurrence Relation

The order of a Recurrence Relation is equal to the difference between Highest and lowest subscripts (arguments) divided by unit of difference interval.

$n+2-n = 2$, ie 2 order $\quad$ $n+1-n = 1$ is 1 order.

The degree of a difference eq^n is equal to the Power of Highest y_n Term. ie highest Term.

Solution of a Recurrence Relation

A Relation between the Independent variable and dependent variable is said to be a solution of a recurrence relation if this Relation satisfies the recurrence relation.

General solution:-

A general solution of a recurrence Relation of order n is a solution which Involves n arbitrary constants. A general solution is also called Complete Primitive.

Particular solution:- A Particular solution is a solution obtained from the general solution by assigning particular values to one or more arbitrary constants.

Linear Recurrence Relation

A Recurrence Relation of degree one is called Linear Recurrence Relation.

Linear Recurrence Relation with constant coefficients

form $$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + \cdots + c_k a_{r-k} = f(r) \quad \text{---} (1)$$

is a Linear Recurrence Relation of order K where $c_0\ c_1\ c_2 - c_k$ are constants.

Total solution

$$a_r = H.S + P.S$$

$$\boxed{a_r = a_r^{h} + a_r^{p}}$$

Homogeneous solution

$$a_r + b_1 a_{r-1} + b_2 a_{r-2} = 0 \quad \text{---} (1)$$

is a Linear Recurrence Relation of order 2

let $a_r = m^r$ $\quad \therefore \quad m^r + b_1 m^{r-1} + b_2 m^{r-2} = 0$

Put $r=2$ $\quad m^2 + b_1 m + b_2 = 0 \quad \text{---} (2)$

is called Auxiliary equation.

Find the Roots suppose the Roots are $m = m_1\ m_2$

Case-I If the Roots are Real and distinct.

$$a_r^{h} = c_1 (m_1)^r + c_2 (m_2)^r$$

Case-II If the Roots are Real and equal $m = m_1, m_1$

$$a_r^{h} = (c_1 + c_2 r)(m_1)^r$$

Case III If the Roots are Imaginary

$m = \alpha \pm i\beta$

$a_r^{(h)} = (c_1 \cos r\theta + c_2 \sin r\theta)(R)^r$ where $R = \sqrt{\alpha^2 + \beta^2}$

$\theta = \tan^{-1}(\beta/\alpha)$

Particular Solution

$$\boxed{a_r^{(P)} = \frac{R.h.S}{f(m)} = \frac{f(r)}{f(m)}}$$

Q. Solve the following recurrence Relations

(i) $a_r - 6a_{r-1} + 8a_{r-2} = 0$ given $a_0 = 3$, and $a_1 = 2$

ii) $2a_r - 5a_{r-1} + 2a_{r-2} = 0$ given $a_0 = 0$, $a_1 = 1$

Sol:- given $a_r - 6a_{r-1} + 8a_{r-2} = 0$ — (1)

order is 2 $\because$ R.h.S $= 0$

R.S $= 0$

Put $\boxed{a_r = m^r}$ $m^r - 6m^{r-1} + 8m^{r-2} = 0$

A.E. Put $\boxed{r = 2}$ $m^2 - 6m + 8 = 0$ — (2)

$m = 2, 4$ (Real and distinct)

$a_r^{(h)} = c_1(2)^r + c_2(4)^r$

$a_r^{(P)} = 0$ $\because$ R.h.S $= 0$

Complete Sol. $a_r = a_r^{h} + a_r^{P} = c_1(2)^r + c_2(4)^r$ — (3)

$r = 0$, $a_0 = 3$ $3 = c_1 + c_2$ — (4)

$r = 1$ $a_1 = 2$ $2 = 2c_1 + 4c_2$ — (5)

$1 = c_1 + 2c_2 \Rightarrow 1 = 3 - c_2 + 2c_2$ $\therefore c_2 = -2$ $c_1 = 5$

Then $a_r = (5)(2)^r - 2(4)^r$ Ans

Q. Solve the following recurrence Relations

i) $9a_r - 6a_{r-1} + a_{r-2} = 0$ given $a_0 = 0, a_1 = 1$

ii) $a_r + a_{r-2} = 0$ given $a_0 = 0, a_1 = 1$

iii) $a_r + 2a_{r-1} + 2a_{r-2} = 0$ given $a_0 = 0, a_1 = -1$

iv) $a_r + 6a_{r-1} + 25a_{r-2} = 0$

v) Solve $a_r - 7a_{r-2} - 6a_{r-3} = 0$ with initial condition

Q. $a_0 = 9, \quad a_1 = 10, \quad a_2 = 32,$

vi) $a_r + 3a_{r-1} + 3a_{r-2} + a_{r-3} = 0$ with $a_0 = 1, \quad a_1 = -2$ and $a_2 = -1$

Particular Solution (a_r^{P})

i) Method of Inspection (or Method of undetermined coefficients)

S.NO	Terms in $f(r)$	Trial Sol a_r
i)	b^r	$A \cdot b^r$
ii)	a Polynomial of degree K,	$A_0 + A_1 r + A_2 r^2 + \cdots + A_K r^K$.
iii)	b^r. (a Polynomial of K in r)	$b^r.(A_0 + A_1 r + A_2 r^2 + \cdots + A_K r^K)$
iv)	$\sin br$ or $\cos br$	$A \sin br + B \cos br$
(v)	$a^r \sin br$	$a^r(A \sin br + B \cos br)$
vi)	$a^r \cos br$	$a^r(A \sin br + B \cos br)$

Q. Solve $a_r - 5a_{r-1} + 6a_{r-2} = 5^r$ —— (1)

Sol:- A.E. becomes

$m^2 - 5m + 6 = 0$ —— (2)

Find the Roots $m = 2, 3$ (Real and distinct)

H.S will be.

$a_r^{(h)} = C_1(2)^r + C_2(3)^r$ —— (3)

P.S. $a_r^P = \frac{R.h.S}{\cdot}$ by Trial Method

$= A(5)^r$ Assume.

Putting in (1)

$A(5)^r - 5 \cdot A(5)^{r-1} + 6 \times A(5)^{r-2} = (5)^r$

$A\cancel{(5)^r}\left[\cancel{1} - \cancel{1} + \frac{6}{5^2}\right] = \cancel{(5)^r}$

$A\left[\frac{6}{25}\right] = 1 \quad \therefore \boxed{A = \frac{25}{6}}$

Then P.S will be $a_r^{(P)} = A(5)^r = \frac{25}{6}(5)^r$ —— (4)

Total Sol $a_r = h.s + P.s$

$$\boxed{a_r = C_1(2)^r + C_2(3)^r + \frac{25}{6}(5)^r}$$

A

Q. Solve $a_r - 2a_{r-1} = f(r)$ where $f(r) = 7r$

Determine the Particular sol.

Q. Solve the Recurrence Relation

$$a_r - 5a_{r-1} + 6a_{r-2} = 2+r \quad , \quad r \geq 2$$

with boundary Conditions $a_0 = 1$ and $a_1 = 1$

Sol:- A.E becomes $m^2 - 5m + 6 = 0$

$m = 2, 3$ (Real and diff)

$$a_r^{(h)} = C_1(2)^r + C_2(3)^r \quad \text{———} \quad (1)$$

Trial sol:-

The Particular sol (Trial sol) corresponding to the Term $2+r$ on R.h.s of is $A_0 + A_1 r$

$$a_r^{(P)} = A_0 + A r \quad \text{———} \quad (2)$$

Substituting (2) in given eqn

$$(A_0 + A_1 r) - 5(A_0 + A_1(r-1)) + 6(A_0 + A_1(r-2)) = 2+r$$

$$(A_0 - 5A_0 + 6A_0) + A_1(r - 5(r-1) + 6(r-2)) = 2+r$$

$$2A_0 + A_1(r - 5r + 5 + 6r - 12) = 2+r$$

$$2A_0 + A_1(2r - 7) = 2+r$$

$$(2A_0 - 7A_1) + 2A_1 r = 2+r$$

$2A_0 - 7A_1 = 2$ $\quad 2A_1 = 1$ $\quad \therefore A_1 = \frac{1}{2}$

$2A_0 - \frac{7}{2} = 2$ $\quad \therefore 2A_0 = 2 + \frac{7}{2} = \frac{11}{2}$ $\quad \therefore A_0 = \frac{11}{4}$

$\therefore$ P.S $\quad a_r^{(P)} = \frac{11}{4} + \frac{1}{2} r$

Total sol $\quad a_r = C_1(2)^r + C_2(3)^r + \frac{11}{4} + \frac{1}{2} r$

Ans.

Q. Solve $a_r - 2a_{r-1} + a_{r-2} = 7$

Sol:- A.E. becomes $m^2 - 2m + 1 = 0$, $m = 1, 1$ Real and equal

H.S will be. $a_r^{(h)} = (C_1 + C_2 r)(1)^r$ —— ①

By Method of undetermined coefficient its Trial solution will be $a_r = A r^2$ —— ②

The question arises that why Ar^2 has been choosen corresponding To 7 and not A or Ar, the reason is simple. Since 1 is a double root of characteristic eq^n, or we may consider like thus, since C_1 and $C_2 r$ appears in $a_r^{(h)}$

Now $A r^2 - 2A(r-1)^2 + A(r-2)^2 = 7$

$2A = 7 \quad \therefore A = 7/2$

$\therefore$ P.S will be $a_r^{(P)} = \frac{7}{2} r^2$ —— ③

Hence the Total solution of given recurrence Relation is

$$\boxed{a_r = (C_1 + C_2 r) + \frac{7}{2} r^2}$$

A

Q Solve the recurrence relation.

$$a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r, \quad r \geq 2$$

with boundary condition

Q. Solve $a_r - 4a_{r-1} + 4a_{r-2} = (r+1)^2 \quad r \geq 2$

Generating function

The Method of generating function is a Powerful Method to solve the Difference equations (Recurrence Relations)

Definition :- If y_0, y_1, y_2 —— is a sequence of a Real Numbers The function y defined for some Interval of real Numbers Containing zero, whose value at t is given by the series

$$Y(t) = \sum_{h=0}^{\infty} y_h t^h = y_0 + y_1 t + y_2 t^2 + \cdots + y_h t^h \cdots$$

is called generating function of the sequence $\{y_t\}$

To discuss the generating function method for solving a linear Difference equation; —

Consider a linear recurrence relation of first order

$$y_{h+1} = A y_h + C \quad , h = 0, 1, 2, 3 \cdots \quad (1)$$

where A and C are constant

Consider the generating function

$$Y(t) = \sum_{h=0}^{\infty} y_h t^h = y_0 + y_1 t + y_2 t^2 + \cdots \quad (2)$$

Multiplying (1) by t^h and summing from $h=0$ to $h=\infty$ we get

$$\sum_{h=0}^{\infty} y_{h+1} t^h = \sum_{h=0}^{\infty} A y_h t^h + \sum_{h=0}^{\infty} C t^h$$

$$(y_1 + y_2 t + y_3 t^2 + \cdots) = A(y_0 + y_1 t + y_2 t^2 + y_3 t^3 + \cdots \infty)$$

$$+ C\ \cancel{(y_1 + y_2 + y_3)}$$

$$C(1 + t + t^2 + t^3 + \cdots \infty)$$

$$\frac{1}{t}(y_1 t + y_2 t^2 + y_3 t^3 + \cdots) = A\,y(t) + C(1 + t + t^2 + \cdots)$$

$$\frac{1}{t}[y(t) - y_0] = A\,y(t) + C(1-t)^{-1}$$

$$y(t) - y_0 = At\,y(t) + \frac{Ct}{1-t}$$

$$y(t) - At\,y(t) = y_0 + \frac{Ct}{1-t}$$

$$y(t)(1-At) = y_0 + \frac{Ct}{1-t} \Rightarrow y(t) = \frac{y_0}{1-At} + \frac{Ct}{(1-t)(1-At)} \quad (1)$$

$$y(t) = y_0(1-At)^{-1} + \frac{Ct}{(1-t)(1-At)}$$

by Partial Fraction

$$\frac{Ct}{(1-t)(1-At)} \quad \frac{B}{1-t} + \frac{D}{(1-At)}$$

$$Ct = B(1-At) + D(1-t) \qquad \text{Put } t = 1$$

$$\boxed{B = \frac{C}{1-A}} \qquad \text{Put } t = \frac{1}{A}$$

$$\frac{C}{A} = D\left(1 - \frac{1}{A}\right) \quad \therefore \quad D = \frac{C}{A\left(1-\frac{1}{A}\right)} = \frac{C}{(A-1)} = -\frac{C}{1-A}$$

$$y(t) = y_0(1-At)^{-1} + \left(\frac{C}{1-A}\right)\left[\frac{1}{1-t} - \frac{1}{1-At}\right]$$

$$= y_0(1 + At + A^2t^2 + \cdots) + \left(\frac{C}{1-A}\right)\left[(1-t)^{-1} - (1-At)^{-1}\right]$$

$$\sum_{k=0}^{\infty} y_k t^k = y_0 \sum_{k=0}^{\infty} A^k t^k + \left(\frac{C}{1-A}\right)\left[\sum_{k=0}^{\infty} t^k - \sum_{k=0}^{\infty} A^k t^k\right]$$

$$\sum_{k=0}^{\infty} y_k t^k = y_0 \sum_{k=0}^{\infty}\left[A^k + \left(\frac{C}{1-A}\right)\left[1 - A^k\right]\right] t^k$$

Equating the coefficient of t^k both sides

$$y_k = y_0 A^k + \frac{c}{1-A}(1-A^k) \quad \text{If } A \neq 1$$

If $A = 1$

$$y(t) = \frac{y_0}{1-t} + \frac{ct}{(1-t)^2}$$

$$\sum_{k=0}^{\infty} y_n t^k = y_0 (1-t)^{-1} + ct(1-t)^{-2}$$

$$= y_0 (1+t+t^2+\cdots) + ct(1+2t+3t^2 \cdots)$$

$$= y_0 \sum_{k=0}^{\infty} t^k + c \sum_{k=0}^{\infty} k t^k$$

Equating the coefficient of t^k on both sides we have.

$$\boxed{y_k = y_0 + ck}$$

$$y_n = \begin{cases} y_0 A^k + \frac{c}{1-A}(1-A^k) & \text{If } A \neq 1 \\ y_0 + ck & \text{If } A = 1 \end{cases}$$

Q Apply Generating function Technique to solve the Initial value Problem $y_{k+1} - 2y_k = 0$ with $y_0 = 1$

Sol:- given $y_{k+1} - 2y_k = 0$ —①

Consider the generating function $Y(t)$ is given

$$Y(t) = \sum_{k=0}^{\infty} y_k t^k = y_0 + y_1 t + y_2 t^2 + \cdots$$

Multiplying by t^k of given eqn ① and summing from $k=0$ to $k=\infty$ we get

$$\sum_{k=0}^{\infty} y_{k+1} t^k - 2\sum_{k=0}^{\infty} y_k t^k = 0$$

$$(y_1 + y_2 t + y_3 t^2 + \cdots) - 2(y_0 + y_1 t + y_2 t^2 + \cdots) = 0$$

$$\frac{1}{t}[y_1 t + y_2 t^2 + y_3 t^3 + \cdots] - 2Y(t) = 0$$

$\frac{1}{t}[y(t) - y_0] - 2y(t) = 0$ given $y_0 = 1$

$\frac{1}{t}[y(t) - 1] = 2y(t)$

$y(t) = 2t y(t) + 1$ $\therefore$ $y(t) - 2t y(t) = 1$

$(1-2t) y(t) = 1$

$\therefore y(t) = \frac{1}{1-2t} = (1-2t)^{-1}$

$= 1 + 2t + (2t)^2 + (2t)^3 + \cdots$

$\sum_{k=0}^{\infty} y_k t^k = \sum_{k=0}^{\infty} (2t)^k$

equating the coefficient of t^k both sides

$\boxed{y_k = (2)^k}$ which is the Required sol.

Q. 2 Solve $y_{k+2} - 7y_{k+1} + 10 y_k = 0$ with $y_0 = 0, y_1 = 3$ by the Method of generating function. — (1)

Sol:- Consider generating function

$y(t) = \sum_{k=0}^{\infty} y_k t^k = (y_0 + y_1 t + y_2 t^2 + \cdots)$ —— (1)

Multiplying t^k in given eq^n both sides and summing $k=0$ to $k=\infty$

Then $\sum_{k=0}^{\infty} y_{k+2} t^k - 7\sum_{k=0}^{\infty} y_{k+1} t^k + 10 \sum_{k=0}^{\infty} y_k t^k = 0$

$(y_2 + y_3 t + y_4 t^2 + \cdots) - 7(y_1 + y_2 t + y_3 t^2 + \cdots) + 10(y_0 + y_1 t + y_2 t^2 + \cdots) = 0$

$\frac{1}{t^2}(y_2 t^2 + y_3 t^3 + y_4 t^4 \cdots) - \frac{7}{t}(y_1 t + y_2 t^2 + y_3 t^3 + \cdots) + 10\, y(t) = 0$

$\frac{1}{t^2}[y(t) - y_0 - y_1(t)] - \frac{7}{t}[y(t) - y_0] + 10 y(t) = 0$

$\frac{1}{t^2}[y(t) - 3t] - \frac{7}{t}[y(t)] + 10 y(t) = 0$

$y(t) - 3t - 7t\, y(t) + 10t^2 y(t) = 0$

$(10t^2 - 7t + 1) y(t) = 3t$

$y(t) = \frac{3t}{10t^2 - 7t + 1}$

$$Y(t) = \frac{3t}{(1-2t)(1-5t)} \quad \text{by Partial fractions}$$

$$Y(t) = \frac{1}{1-5t} - \frac{1}{1-2t}$$

$$Y(t) = (1-5t)^{-1} - (1-2t)^{-1} = (1+5t+(5t)^2+\ldots) - (1+2t+(2t)^2+\ldots)$$

$$\sum_{k=0}^{\infty} y_k t^k = \sum_{k=0}^{\infty} (5t)^k - \sum_{k=0}^{\infty} (2t)^k$$

equating the coefficient of t^k both sides

$$\boxed{y_k = 5^k - 2^k}$$

Q. Solve $y_{k+2} - y_{k+1} - y_k = 0$ with $y_0 = 0$, $y_1 = 1$ by the Method of generating function.

[Ans.

$$y_k = -\frac{\sqrt{5}}{5}(-1)^k\left(\frac{\sqrt{5}-1}{2}\right)^k + \frac{\sqrt{5}}{5}\left(\frac{\sqrt{5}+1}{2}\right)^k$$

Q. Apply the generating function Method to solve the following linear Difference eqn

$$y_{k+2} - 2y_{k+1} + y_k = 2^k \quad \text{with } y_0 = 2 \text{ and } y_1 = 1$$

[Ans $y_k = 3 - 2\cdot(k+1) + 2^k$]

Binomial Theorem

It states that for any positive integer n we have

$$(x+a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \dots + {}^nC_n a^n$$

$$= \sum_{r=1}^{n} {}^nC_r x^{n-r} . a^r$$

$$(x+a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r$$

Binomial coefficient :-

${}^nC_r$ also written as $C(n,r)$ or $\binom{n}{r}$ is called Binomial coefficients where

$${}^nC_r = C(n,r) = \binom{n}{r} = \frac{\lfloor n}{\lfloor r \lfloor n-r}$$

Remarks :- when n is a positive integer, we have

i) $$(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r = \sum_{r=0}^{n} C(n,r) x^r$$

$$= C(n,0) + C(n,1)x + (n,2)x^2 + \dots + C(n,r)x^r + C(n,n)x^n$$

Here $C(n,r)$ is the coefficient of x^r in the expansion of $(1+x)^n$

(ii) $$(1-x)^n = \sum_{r=0}^{n} {}^nC_r (-1)^r x^r$$

Q. If n is a fixed positive integer, then Prove the following

i) $C(n,0)+C(n,1)+C(n,2)+ - - - + C(n,r) + - - + C(n,n)$

$$= 2^n \text{ or } \sum_{k=0}^{n} C(n,k) = 2^n$$

ii) $C(n,0) + C(n,2) + C(n,4) + - - -$

$$= C(n,1) + C(n,3) + - - = 2^{n-1}$$

iii) $C(n,1) + 2C(n,2) + 3C(n,3)$

$$+ - - - + i\,C(n,i) + - - - + n\,C(n,n) = n\,2^{n-1}$$

iv) $C(n,0) + 2C(n,1) + 2^2 C(n,2) + - - - + 2^n C(n,n) = 3^n$

v) $C^2(r,0) + C^2(r,1) + C^2(r,2) + - - - + C^2(r,i) + - - - + C^2(r,r) = C(2r,r)$

Sol:- we have

$$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r = \sum_{r=0}^{n} C(n,r)\, x^r$$

i)

$$(1+x)^n = C(n,0) + C(n,1)x + C(n,2)x^2 + - - - - - - - - \quad —①$$

Put $x=1$ ~~in~~ given eq^n ①

$$2^n = C(n,0) + C(n,1) + C(n,2) + - - - - -$$

—(ii) Proved.

ii) we have

$$(1-x)^n = \sum_{r=0}^{n} {}^nC_r (-1)^r x^r = \sum_{r=0}^{n} C(n,r)\,(-1)^r x^r$$

$$\Rightarrow C(n,0)\cancel{x} + C(n,1)x + C(n,2)x^2 - C(n,3)x^3 + C(n,4)x^4 - - - -$$

Put $x=+1$

$$+C(n,0) \cancel{+} C(n,1) + C(n,2) - C(n,3) + C(n,4) - - - = 0$$

$$C(n,0) + C(n,2) + C(n,4) + - - = C(n,1) + C(n,3) + C(n,5) + —— \quad —(iii)$$

$$2[C(n,0) + C(n,2) + C(n,4) + - - - - - -] = 2^n$$

$$C(n,0) + C(n,2) + (n,4) + - - - = \frac{2^n}{2} = 2^{n-1} \quad —(iv)$$

from ③

$$C(n,0) + C(n,2) + C(n,4) + - - = C(n,1) + C(n,3) + - - - = 2^{n-1}$$

(iii)

$$C(n,1)+2C(n,2)+3C(n,3)+\cdots = {}^nC_1+2\,{}^nC_2+3\,{}^nC_3+\cdots$$

$$n+\cancel{2}\frac{n(n-1)}{\cancel{\lfloor 2}}+3\frac{n(n-1)(n-2)}{\lfloor 3}+\cdots$$

$$n+n(n-1)+\cancel{3}\frac{n(n-1)(n-2)}{\cancel{6}\,2}+\cdots$$

$$n\left[1+n-1+\frac{(n-1)(n-2)}{2}+\cdots\right]$$

$$= n\left[{}^{n-1}C_0+{}^{n-1}C_1+{}^{n-1}C_2+\cdots\right] = n\left[(1+1)^{n-1}\right]$$

$$= n\,2^{n-1}$$

(iv) we have

$$(1+x)^n = C(n,0)+C(n,1)x+C(n,2)x^2+\cdots$$

Put $x=2$

$$(1+2)^n = C(n,0)+2C(n,1)+4C(n,2)+\cdots$$

$$3^n = C(n,0)+2C(n,1)+4C(n,2)+\cdots$$

Proved

(v) we have

$${}^rC_0+{}^rC_1x+{}^rC_2x^2+\cdots+{}^rC_{r-1}x^{r-1}+{}^rC_rx^r = (1+x)^r \quad \text{—(1)}$$

also $${}^rC_0x^r+{}^rC_1x^{r-1}+{}^rC_2x^{r-2}+\cdots+{}^rC_{r-1}x+{}^rC_r = (x+1)^r \quad \text{—(2)}$$

Multiplying (1) and (2)

$${}^rC_0^2+{}^rC_1^2+{}^rC_2^2+\cdots+{}^rC_r^2 = {}^{2r}C_r$$

Multinomial coefficient

An expression of the form $x_1 + x_2 + \cdots + x_n$, where $n \geq 3$ is called a Multinomial. We know that binomial coefficients occur when powers of a binomial (ie powers of $x_1 + x_2$) are expanded. In the similar way, when powers of a Multinomial are expanded then Multinomial coefficients occur.

Multinomial Theorem

It states that for any positive n, we have

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{n_1 + n_2 + \cdots + n_r = n} \binom{n}{n_1\ n_2 \cdots n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

where

$$\binom{n}{n_1\ n_2 \cdots n_r} = \frac{\lfloor n}{\lfloor n_1 \lfloor n_2 \cdots \lfloor n_r}$$

The quantity $\binom{n}{n_1, n_2 \cdots n_r}$, where $n_1 + n_2 + \cdots + n_r = n$ is called Multinomial Coefficient.

Q. Find the coefficient of $x^5 y^3 z^2$ in $(x+y+z)^{10}$

Sol:- The coefficient of $x^5, y^3 z^2$ in $(x+y+z)^{10}$ is

$$= \binom{10}{5\ 3\ 2} = \frac{\lfloor 10}{\lfloor 5 \lfloor 3 \lfloor 2} = 2520$$

A

Q. Find the coefficient of $x y^2 z^2 u^2$ in $(x+y+z+u)^7$

Sol:- Required coefficient $= \dfrac{\lfloor 7}{\lfloor 1 , \lfloor 2 , \lfloor 2 , \lfloor 2} = 630$

A

Q Find the coefficient of $x^5, y^2 z^2$ in $(x+y+z)^9$

(756)

Posets

Partially ordered sets (Poset)

A set P together with a partial order relation R in it is called partially ordered set or simply an ordered set (or poset) and it's denoted by (P, R)

The Relation R is usually denoted by the symbal $\leq$, Thus partially ordered set may be denoted by $(P, \leq)$

Note:- A Relation R on a set P is called partial order relation If

i) R is Reflexive ie aRa $\forall$ $a \in P$

ii) R is Antisymmetric : ie aRb and $bRa \Rightarrow a=b$ for same $a, b \in P$

iii) R is Transitive : ie aRb and $bRc \Rightarrow aRc$ · for $a, b, c \in P$

Q. Prove that the Relation " a divides b" If there exists an Integer c such that $ac=b$ and is denoted by $a|b$ on the set of all positive integers N is a partial order Relation.

Sol:- i) Reflexive :- $a \in N$ s.t aRa

$a|a \Rightarrow$ $a \cdot 1 = a$ $\exists$ An integer 1 .

Hence the Relation "|" is reflexive,

ii) Antisymmetric :- aRb, and $bRa \Rightarrow a=b$ for same $a, b \in N$

$\frac{a}{b} = c_1$ $\frac{b}{a} = c_2$ $\therefore$ $c_1 c_2 = 1$

$\Rightarrow c_1 = c_2 = 1$ $\Rightarrow a=b$

Hence the Relation '|' is antisymmetric

Transitive :- let $a, b, c \in N$

$\frac{a}{b}, \frac{b}{c}$ then $\frac{a}{b} = d_1$ $\frac{b}{c} = d_2$

$ad_1 = b$ $bd_2 = c$ $\Rightarrow ad_1d_2 = c$

$\Rightarrow \exists$ an integer d such that $ad = c$ where $d = d_1d_2$

$\Rightarrow a \mid c$

Hence the Relation '|' is transitive

Thus the relation '|' is a Partial order Relation on N,

**

Q. let N be the set of Positive integers, Prove that the Relation $\leq$, where $\leq$ has its usual meaning, is a Partial order Relation on N.

Q. let X be the set of all 2×2 real Matrices let

$x, y \in X$ $x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ $y = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}$

and the Relation $x \leq y$ has the Meaning

$x_1 + x_2 + x_3 + x_4 \leq y_1 + y_2 + y_3 + y_4$

Prove that $\leq$ is not a Partial order relation on the set X.

Sol:- the given Relation $\leq$ is not Anti-symmetric on the set X

because

$x \leq y, y \leq x \Rightarrow$ $x_1 + x_2 + x_3 + x_4 \leq y_1 + y_2 + y_3 + y_4$

$y_1 + y_2 + y_3 + y_4 \leq x_1 + x_2 + x_3 + x_4$

$\Rightarrow x_1 + x_2 + x_3 + x_4 = y_1 + y_2 + y_3 + y_4$

$\Rightarrow x \neq y$

For ex:- $x = \begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}$ $y = \begin{bmatrix} -2 & -3 \\ 8 & 1 \end{bmatrix} \in X$

where $x_1 + x_2 + x_3 + x_4 = 2 + 3 + 4 - 5 = 4$

$y_1 + y_2 + y_3 + y_4 = -2 - 3 + 8 + 1 = 4$; Never the less $\begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix} \neq \begin{bmatrix} -2 & -3 \\ 8 & 1 \end{bmatrix}$

$= x \neq y$

Comparability:-

let $(P,\leq)$ be a Poset. The element x and y of the Poset $(P,\leq)$ are said to be comparable if either $x\leq y$ or $y\leq x$. The elements x and y are called non-comparable if x and y are not related i.e neither $x\leq y$ nor $y\leq x$ and is written as $x \| y$.

Ex In the set of Natural Numbers N with the relation of divisibility the element 6 and 12 are comparable since 6 divides 12; but 2 and 5 are non-comparable since neither 2 divides 5 Nor 5 divides 2.

Chain:-

let $(P,\leq)$ be Partially ordered set, A subset of P is called a chain if every two elements in the subset are Related. In other words if for each Pair of elements x, y in the subset either $x\leq y$ or $y\leq x$, then the subset of P is called a chain. The chain is also known as Totally ordered set or linearly ordered set.

Ex let $\{x_1 x_2 -- x_n\}$ be a chain consisting of n elements i.e(a finite Number of elements). Because of antisymmetry and Transitivity in the

Hasse Diagram

A Partial order relation $\leq$ on a finite set P can be represented by a diagram called the Hasse Diagram

let $(P, \leq)$ be a Partially ordered set (i.e Poset)

let $a, b \in P$, then we write $a < b$ if and only if $a \leq b$ and $a \neq b$.

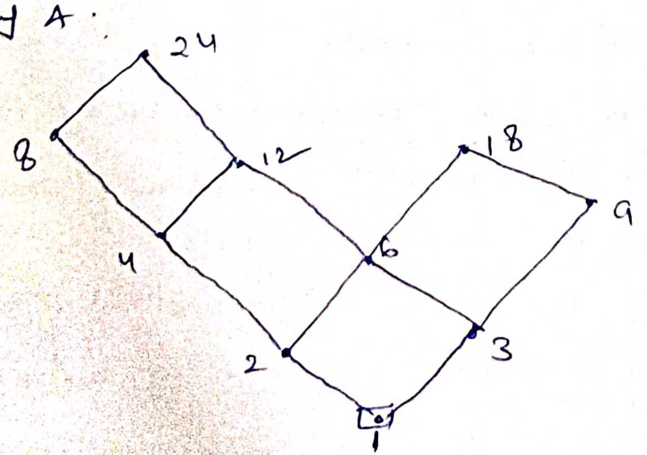
If $a < b$ then we say that b covers a iff $a \leq c \leq b \Rightarrow c = a$ or $c = b$.

In other words $b \in P$ is said to cover $a \in P$. If $a < b$, and no elements of P lies between a and b. i.e there does not exist $c \in P$, such that $a < c < b$. In this case b is also called Immediate successor of a.

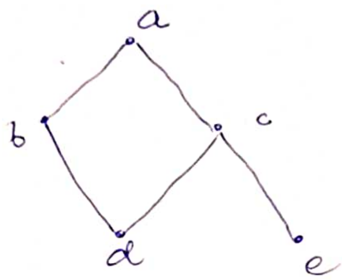
If b is an Immediate successor of a, then b is taken at a higher level than a and then a straight line is drawn to join a and b in the Hasse diagram

Ex let $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$

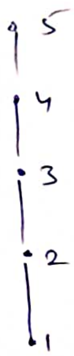
be ordered by the relation "a divides b" The Hasse diagram of A:



Q. Let $B = \{a, b, c, d, e\}$ The Hasse diagram in the set B, and a Partial order on B s.t $d \leq b$ $d \leq a$, $e \leq c$ and so on.



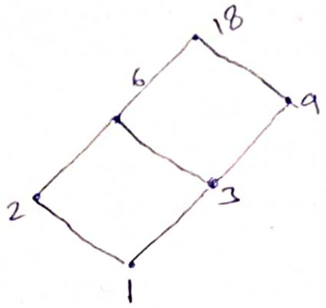
Q. The Hasse diagram of a finite linearly ordered set $P_1 = \{1, 2, 3, 4, 5\}$ with the usual relation $\leq$



Q. The set D_{18} of Positive integral divisors of 18 with the Relation "|" divides. Find its Hasse diagram.

Sol:- D_{18} is the set of divisors of 18

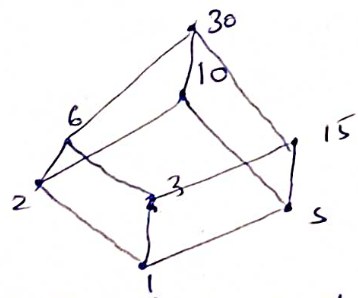
$D_{18} = \{1, 2, 3, 6, 9, 18\}$



Q. The Hasse diagram of $\{3, 5, 7\}$ with the relation "|" drawn below:-

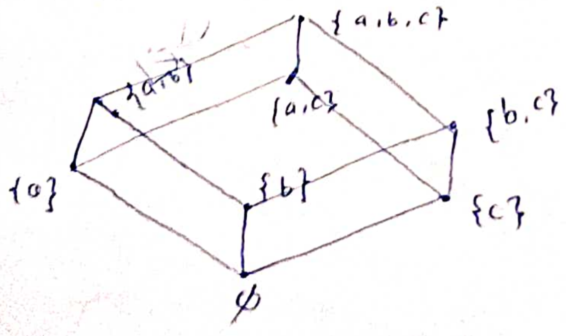
Q. The Hasse diagram of the set D_{30} of Positive integral divisors of 30 with the relation "|"

Sol:- $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$



Q. Draw the Hasse diagram of the subsets of $S = \{a, b, c\}$ with the Inclusion relation '$\subseteq$'

$P(S) = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \emptyset\}$



Maximal element:-

let $(P \leq)$ be a Partially ordered set (Poset). An element m in P is said to be Maximal element if

$x \leq m$ ~~m~~ $x \Rightarrow m = x$ (for any $x \in P$)

obviously, a Maximal element is that element which is not smaller than any element.

Ex ① let $(P, \leq)$ be a Partially ordered set, where $P = \{1, 2, 3, 4, 5\}$ and $\leq$ is the Relation of division which. Partially ordered set P. 3, 4, 5 are Maximal elements of P. because they do not divide the other

Ex ② If $P = \{1, 2, 3, 4, 5\}$ is Partially ordered by the relation $\leq$ where $\leq$ has its usual Meaning less than or equal to, then 5 is the only Maximal element in P because its is the biggest integer.

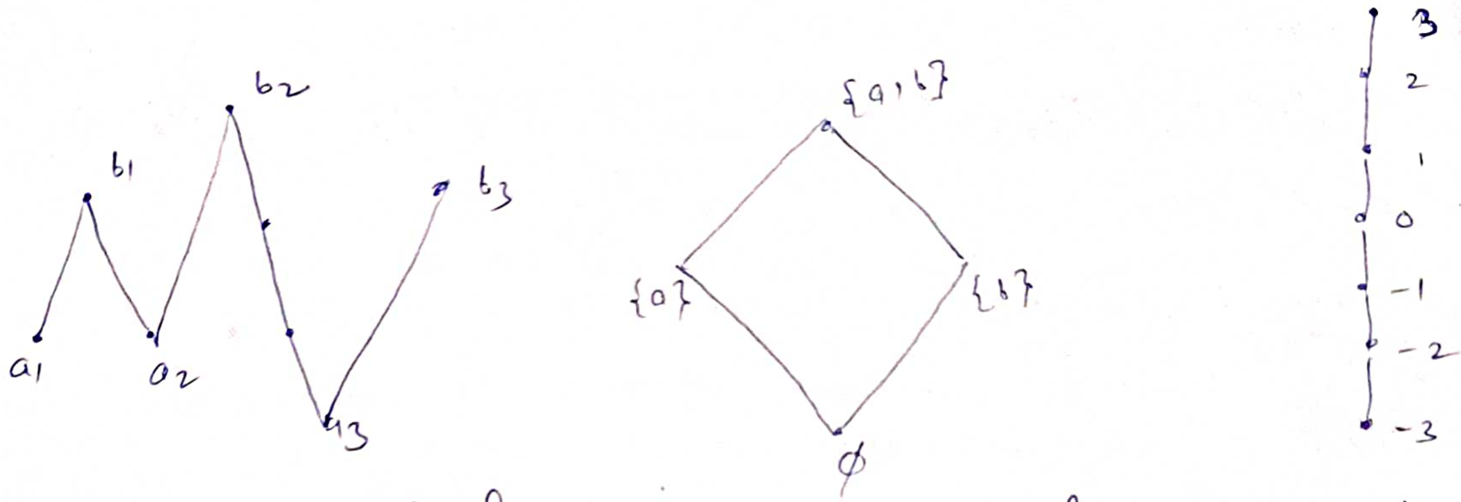
Minimal element:-

let $(P \leq)$ be a Partially ordered set. An element n in P is said to be a minimal element if for any $x \in P$

s.t x $n \leq x \Rightarrow x = n$

obviously a Minimal element is that element which is not greater than any element.

Ex on Maximal and Minimal elements



(a) a_1, a_2, a_3 — Minimal
$b_1\ b_2\ b_3$ — Maximal

(b) ϕ is Minimal
$\{a,b\}$ is Maximal

(c) $(I\ \le)$
either Maximal nor Minimal element

<u>Least upper Bound (LUB)</u>

let $(P\ \le)$ be a Partially ordered set (Poset) and $A \subseteq P$. If $u \in P$ is such an element that

$$a \le u \quad \forall \quad a \in A$$

Then u is called an upper bound of A.

If $u \in P$ is such that an element that

(i) $a \le u \ \forall \ a \in A$ (ii) $u \le v$ for all upper bounds v of A.

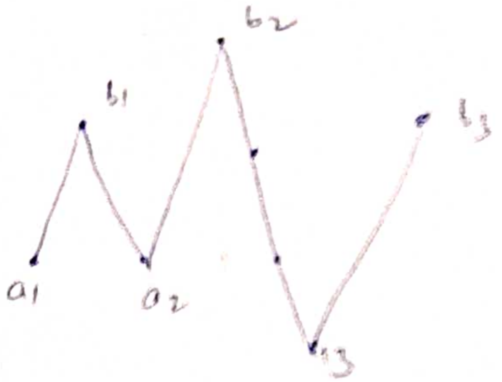
Then u is called the least upper bound or supremum of A and symbolically it is written as

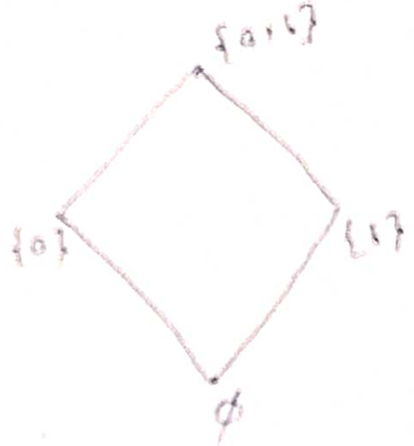
$$u = l.u.b.(A) \text{ or } u = \sup(A)$$

The supremum of a set $\{a,b\}$ of two elements is denoted by $a \vee b$ i.e

$$a \vee b = l.u.b.\{a,b\}$$

Ex are Maximal and Minimal elements





(a) a_1, a_2, a_3 — Minimal
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(b) ϕ is Minimal
$\{a,b\}$ is Maximal

(c) $(I, \leq)$
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The supremum of a set $\{a, b\}$ of two elements is denoted by $a \vee b$ i.e

$$a \vee b = l.u.b. \{a, b\}$$

Greatest lower Bound (GLB)

Let $(P, \leq)$ be a Partially ordered set (Poset) and let $A \subseteq P$

If $l \in P$ is such that

$$l \leq a \ \forall \ a \in A$$

Then l is called a lower bound of A.

If $l \in P$ is such ~~that~~ an element that

i) $l \leq a \ \forall \ a \in A$

ii) $m \leq l$ for all lower bounds of A Then l is called the greatest lower bound or Infimum of A and symbolically it is written as

$$l = g.l.b.(A) \quad \text{or} \quad l = \text{Inf}(A)$$

Then Infimum of a set $\{a, b\}$ of two elements is denoted by $a \wedge b$ i.e

$$a \wedge b = glb\{a, b\}$$

Note:- The greatest lower bounds of a Set A may be stated as:- If a lower bound of A succeeds every lower bound of A. Then it is called the greatest lower bound of A

LATTICE

Let L be a non empty set closed under two binary operations called meet and Join denoted respectively by $\wedge$ and $\vee$. Then L is called a lattice if the following axioms hold where a, b, c are any elements in L.

[L_1] Commutative law:-

ⓐ $a \wedge b = b \wedge a$ ⓑ $a \vee b = b \vee a$

[L_2] Associative law:-

(i) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ (ii) $a \vee (b \vee c) = (a \vee b) \vee c$

[L_3] Absorption law:-

(i) $a \wedge (a \vee b) = a$ (ii) $a \vee (a \wedge b) = a$

We shall denote the lattice by $(L, \wedge, \vee)$

Q Let X, be a collection of sets closed under intersection and union then prove that $(X, \cap, \cup)$ is a lattice.

Sol:- [L_0] closure law:- by hypothesis, X is closed under intersection and union of sets in X ie If $A, B \in X$ then

$A, B \in X \Rightarrow A \cap B, A \cup B \in X \quad \forall A, B \in X$

[L_1] Commutative law:- let $A, B \in X$, then by algebra of sets

(i) $A \cap B = B \cap A \quad \forall A, B \in X$ (ii) $A \cup B = B \cup A \quad \forall A, B \in X$

Hence intersection and union of sets in X follow commutative law.

[L_2] Associative law:- let $A, B, C \in X$

(i) $(A \cap B) \cap C = A \cap (B \cap C) \quad \forall A, B, C \in X$

(ii) $(A \cup B) \cup C = (A \cup B) \cup C$

Isomorphic (Similar) ordered sets

let $(X, \leq)$ and $(Y, \leq)$ be two Partially ordered sets. A one-one onto (ie Injective) function $f: X \to Y$ is said to be a similarity Mapping from X onto Y If f Preserves the order relation. ie f satisfies the following conditions

i) If $a \leq a'$ Then $f(a) \leq f(a')$ for every pair $a, a' \in X$

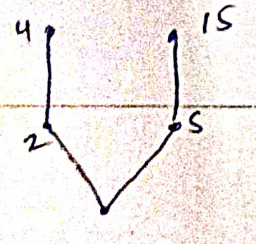
and ii) If $a \| a'$ Then $f(a) \| f(a')$ for every pair $a, a' \in X$

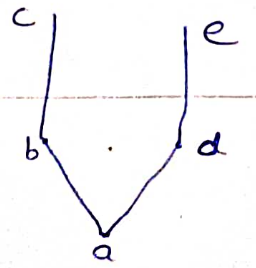
ie If a, and $a' \in X$ are non-comparable then f(a) and $f(a')$ are non-comparable.

If $f: X \to Y$ is one-one onto function and Preserves the order relation then X and Y are called Isomorphic (or similar) ordered sets. and is written as $X \cong Y$

Ex let $X = \{1, 2, 4, 5, 15\}$ be an ordered set with the relation divisibility and let $Y = \{a, b, c, d, e\}$ be Isomorphic to X. say the following function is similarly mapping from X onto Y

$f = \{(1,a)\ (2,b)\ (4,c)\ (5,d)\ (15,e)\}$





Well ordered sets

An ordered set S is called well-ordered If every subset of S has first element.

Ex. 1. The set N of Natural Numbers with the usual order $\leq$ is well ordered, since every subset of N has a first element. Moreover N itself has a first element 1.

Ex 2. The set Z of all integers with the usual order $\leq$ is not well ordered, since the set of negative integers, which has a subset of Z. has no least (first) element.

Dual statement :-

The dual of any statement in a lattice $(L \wedge \vee)$ is defined to be The statement that is obtained by interchanging $\wedge$ and $\vee$ for examp The dual of

$$a \wedge (b \vee a) = a \vee a \quad \text{is} \quad a \vee (b \wedge a) = a \wedge a$$

Ex Theorem 1 (Idempotent law)

In a Lattice $(L \wedge \vee)$ for each $a \in L$

i) $a \wedge a = a$ (ii) $a \vee a = a$

Proof :- let a be any arbitrary element of L then

(by Absorption law)

i) $a \wedge a = a \wedge (a \vee (a \wedge b)) = a$

$a \wedge a = a \quad \forall\ a \in L$

(ii) $a \vee a = a \vee (a \wedge (a \vee b))$

$= a$

$a \vee a = a \quad \forall\ a \in L$

PDNF and PCNF without Truth Table

Principal Disjunctive Normal form

P, Q → propositionals.

Minterms :- $P \wedge Q$ $P \wedge \sim Q$, $\sim P \wedge Q$ $\sim P \wedge \sim Q$

Maxterms :- $P \vee Q$ $P \vee \sim Q$ $\sim P \vee Q$ $\sim P \vee \sim Q$

PDNF :- Sum of minterms DNF :- Sum of elementary Product

$(\sim P \vee \sim Q \vee P)$

PCNF :- Product of Max terms CNF :- Product of elementary sum

1). $P \leftrightarrow Q \equiv (P \wedge Q) \vee (\sim P \wedge \sim Q)$] DNF

$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$

$P \to Q \equiv (\sim P \vee Q)$ } CNF

2) Introduce missing factors 3) Avoid duplication.

Q Obtain the PDNF of the following

(i) $P \vee \sim Q$ (ii) $P \to Q$ (iii) $Q \vee (P \vee \sim Q)$

Sol :- $(P \vee \sim Q) = P \wedge (Q \vee \sim Q) \vee (\sim Q \wedge (\sim P \vee P))$

$= (P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim Q \wedge \sim P) \vee (\sim Q \wedge P)$

$= (P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$ $P \vee P \equiv P$

b) $p \to q = \sim p \vee q$

$= [\sim p \wedge (q \vee \sim q)] \vee [q \wedge (p \vee \sim p)]$

$= (\sim p \wedge q) \vee (\sim p \vee \sim q) \vee (q \wedge p) \vee (q \wedge \sim p)$

$= (\sim p \wedge q) \vee (\sim p \vee \sim q) \vee (p \wedge q)$ A

(d) $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

Sol:- $[(p \wedge q) \wedge (r \vee \sim r)] \vee [(\sim p \wedge r) \wedge (\sim q \vee q)] \vee [(q \wedge r) \wedge (\sim p \vee p)]$

$[(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r)] \vee [(\sim p \wedge r \wedge \sim q) \vee (\sim p \wedge r \wedge q)] \vee [(q \wedge r \wedge \sim p) \vee (q \wedge r \wedge p)]$

$[(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge r)$

A

Q. Obtain the PCNF of the following

(a) $p \wedge \sim q$ (b) $p \leftrightarrow q$ (c) $p \wedge q$ (D) $(\sim p \to r) \wedge (q \leftrightarrow p)$

(a) $p \wedge \sim q \equiv [p \vee (q \wedge \sim q)] \wedge [\sim q \vee (p \wedge \sim p)]$

$= (p \vee q) \wedge (p \vee \sim q) \wedge (\sim q \vee p) \wedge (\sim q \vee \sim p)]$

$= (p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee \sim q)$ A

(b) $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$ A

(c) $p \wedge q \equiv [p \vee (q \wedge \sim q)] \wedge [q \vee (\sim p \wedge p)]$

$(p \vee q) \wedge (p \vee \sim q) \wedge [(q \vee \sim p) \wedge (q \vee p)$

$(p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q)$ A

Partial order in a Lattice :-

Let L be a Lattice, then we define a Partial order $\leq$ on L. as follows

$$a \leq b \text{ if } a \wedge b = a$$

Analogously we could define

$$a \leq b \text{ If } a \vee b = b$$

Theorem ②

Let L be a Lattice. Then

i) $a \wedge b = a$ iff $a \vee b = b$

ii) The Relation $a \leq b$ (defined by $a \wedge b = a$ or $a \vee b = b$) is a Partial order on L)

Proof:- Let $(L, \wedge, \vee)$ be a Lattice and $a, b \in L$ are arbitrary elements.

i) only If condition: suppose $a \wedge b = a$ then

$$\begin{aligned} a \vee b &= (a \wedge b) \vee b & \because \underline{a \wedge b = a} \\ &= b \vee (a \wedge b) & \text{commutative law} \\ &= b \vee (b \wedge a) & \text{by Absorption law} \\ a \vee b &= b \end{aligned}$$

Therefore $a \wedge b = a$ ——— ①

If Condition :- Suppose $a \wedge b = b$ then

$a \vee b = b$ then

$a \wedge b = a \wedge (a \vee b)$ $\because a \vee b = b$

$= a$ by Absorption law

Therefore $a \vee b = b \Rightarrow a \wedge b = a$ —— (2)

from (1) and (2) we obtain

$a \wedge b = a \Leftrightarrow a \vee b = b$

$a \wedge b = a$ If and only If $a \vee b = b$

(ii) Suppose the relation $\leq$ is defined as follows

$a \leq b$ If $a \wedge b = a$

Now we see that

i) Reflexivity :- for each $a \in L$ we have

$a \wedge a = a$ (Idempotent law)

$a \leq a \quad \forall\, a \in L$

Hence the relation $\leq$ is reflexive in L

ii) Anti-Symmetry :- If $a, b \in L$ and $a \leq b$, $b \leq a$ then

$a \leq b,\ b \leq a \Rightarrow a \wedge b = a,\ b \wedge a = b$

$\Rightarrow a = b$ $\because [a \wedge b = b \wedge a]$

Hence the relation $\leq$ is anti-symmetric in L.

iii) Transitivity :- If $a, b, c \in L$ are such that $a \leq b$, $b \leq c$, then

$a \leq b,\ b \leq c \Rightarrow a \wedge b = a,\ b \wedge c = b$ (Replacing b with $b \wedge c$)

$\Rightarrow a \wedge (b \wedge c) = a$ (Associative law : using (2a))

$\Rightarrow (a \wedge b) \wedge c = a$ $\because [a \wedge b = a]$

$\Rightarrow a \wedge c = a$

$\Rightarrow a \leq c$

Hence the relation $\leq$ is Transitive in L

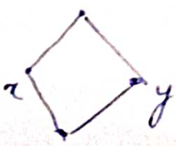
Lattice

A Poset $(A \leq)$ is called Lattice if every pair of element of x and y has both a least upper bound (LUB) and greatest lower bound (glb).

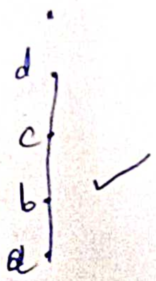
$glb\{a \cdot b\} = a \wedge b$ {meet of a & b}

$lub\{a \cdot b\} = a \vee b$ {Join of a & b}

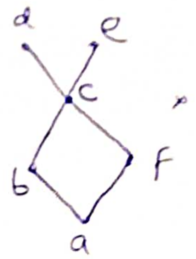
Every chain is a lattice.



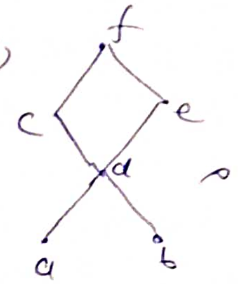
i)



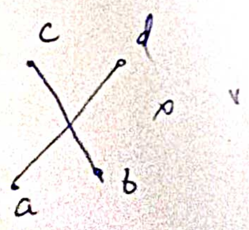
(ii)



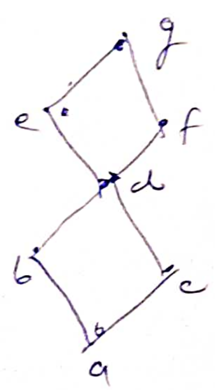
(iii)



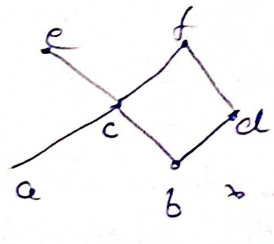
(iv)



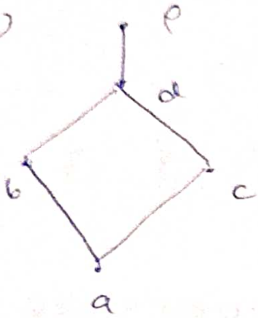
(v)



(vi)



Q. 1)



lub Table

∨	a	b	c	d	e
a	a	b	c	d	e
b	b	b	d	d	e
c	c	d	c	d	e
d	d	d	d	d	e
e	e	e	e	e	e

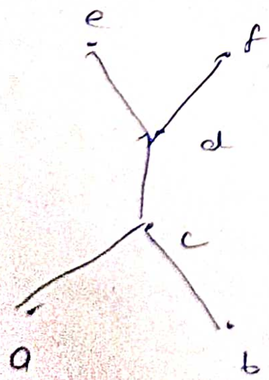
glb Table

glb:-

∧	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	b	b
c	a	a	c	c	c
d	a	b	c	d	d
e	a	b	c	d	e

This Post is a lattice.

(2)



lub Table

∨	a	b	c	d	e	f
a	a	c	c	d	e	f
b	c	b	c	d	e	f
c	c	c	c	d	e	f
d	d	d	d	d	e	f
e	e	e	e	e	e	-
f	f	f	f	f	-	f

glb:-

∧	a	b	c	d	e	f
a	a	-	a	a	a	a
b	-	b	b	b	b	b
c	a	b	c	c	c	c
d	a	b	c	d	d	d
e	a	b	c	d	e	~~d~~
f	a	b	c	d	d	f

Since all the entry of lub and glb are not Completed, This Poset is not a lattice

I) Idempotent :-

i) $a \vee a = a$ (ii) $a \wedge a = a$

II) Commutative :-

i) $a \vee b = b \vee a$ (ii) $a \wedge b = b \wedge a$

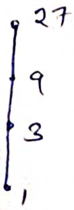
III) Associative law

i) $a \vee (b \vee c) = (a \vee b) \vee c$ (ii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

IV) Absorption law :-

i) $a \vee (a \wedge b) = a$ ii) $a \wedge (a \vee b) = a$

check given Hasse diagram is a Lattice or not.



i) Idempotent :-

$a = 3$, i) $a \vee a = a$

$3 \vee 3 = lub\{3,3\} = 3$

$3 \wedge 3 = glb\{3,3\} = 3$

ii) Commutative :-

let $a = 1$ $b = 3$

$a \vee b = lub\{1,3\} = 3$

$b \vee a = lub\{3,1\} = 3$

$a \wedge b = glb\{1,3\} = 1$

$b \wedge a = glb\{1,3\} = 1$

iii) Associative :- let $a = 1, b = 3, c = 9$

$(a \vee (b \vee c) = a \vee (b \vee c) = 1 \vee (3 \vee 9) = 1 \vee 9 = 9$

$(a \vee b) \vee c = (1 \vee 3) \vee 9 = 3 \vee 9 = 9$

(iv) Absorption law :- let $a = 1, b = 3$

$a \vee (a \wedge b) = 1 \vee (1 \wedge 3) = 1 \vee 1 = 1$ Bounded

Hence given Hasse diagram is a lattice,

Types of Lattice :-

Sub Lattice

let $(L, \leq)$ be a Lattice, A subset M of L is said to be a Sub-Lattice of L If M is closed with respect to Meet $(\wedge)$ and Join $(\vee)$ which means that for each pair of elements $x, y \in M$, $x \wedge y$ and $x \vee y$ are contained in M.

In other words, a non empty subset M of a lattice L is said to be a sub-lattice of L If M itself is a lattice with respect to the binary operations meet $(\wedge)$ and Join $(\vee)$ of L

Ex let N be the set of positive integers and let there be defined a relation '$|$' on N, where the meaning of $a|b$ is 'a divides b' ie there exists an integer n such that $(b = na$, let D_m be the set of all divisors of m, Proved that

(a) $(N '|')$ is a lattice

(b) D_m is a sub-lattice of the Lattice N.

(

(b) let $D_m = \{x \in N : x | m\}$ that is D_m be the set of all divisors of the positive integer m

The binary relation meet $(\wedge)$ and Join $(\vee)$ on D_m are defined as follows

$$x \wedge y = H.C.F\{x, y\} \qquad x \vee y = L.C.M\{x, y\}$$

Then it is clear that for each pair of elements x, y in D

$$x, y \in D_m \Rightarrow x \wedge y,\ x \vee y \in D_m$$

That is D_m is closed with respect to the binary operation $\wedge$ and $\vee$

Hence D_m is a sub-lattice of the Lattice N.

Ex let $L = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be ordered by the Relation '|' where $x|y$ Means 'x divides y' show that D_{24} the set of all divisors of the integer 24 of L is a sub Lattice of the Lattice (L, |),

Sol:- The given Lattice is

$L = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$

which is Partially ordered by the Relation '|'

The set of all divisors of $24 \in L$ is the set D_{24} given by $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$

Now we defined two binary operation Meet ($\wedge$) and Join ($\vee$) in D as follows.

$x \wedge y = H.C.F\{x, y\}$ $\qquad x \vee y = L.C.M\{x, y\}$

$\wedge$	1	2	3	4	6	8	12	24
1	1	1	1	1	1	1	1	1
2	1	2	1	2	2	2	2	2
3	1	1	3	1	3	1	3	3
4	1	2	1	4	2	4	4	4
6	1	2	3	2	6	2	6	6
8	1	4	1	4	2	8	4	8
12	1	2	3	4	6	4	12	12
24	1	2	3	4	6	8	12	24

$\vee$	1	2	3	4	6	8	12	24
1	1	2	3	4	6	8	12	24
2	2	2	6	4	6	8	12	24
3	3	6	3	12	6	24	12	24
4	4	4	12	4	12	8	12	24
6	6	6	6	12	6	24	12	24
8	8	8	24	8	24	8	24	24
12	12	12	12	12	12	24	12	24
24	24	24	24	24	24	24	24	24

Since all 64-64 entries of Composition Tables of $\wedge$ and $\vee$ are elements of D_{24} and so for each Pair of elements $x, y \in D_{24}$ we have $x, y \in D_{24} \Rightarrow x \wedge y, x \vee y \in D_{24}$ Hence D_{24} is Closed under $\wedge$ and $\vee$

Hence $(D_{24}, |)$ is a sublattice of the lattice $(L, |)$

Complete order

A linear order $\le$ on a set x is called complete if every non-empty subset of x which is bounded above have a supremum in x (which need not belong to that subset)

Ex every well order is complete.

Complete Lattice

let $(L, \le)$ be a lattice. Then L is said to be complete if every subset A (finite or infinite) of L, $\wedge A$ and $\vee A$ exist in L. Thus in every complete lattices $(L, \le)$ there exist a greatest element g and a least element f.

for further discussion we shall use $a \wedge b$ is place of $\wedge\{a,b\}$ and $a \vee b$ in place of $\vee\{a,b\}$ for $a, b \in L$

Distributive lattice :-

i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Permutation

Suppose an event E can occur in m different ways and associated with each way of occurring of E another event F can occur in n different ways. Then the Total Number of occurrence of the two events in the given order is $m \times n$.

Addition Principle If an event E can occur in m ways and another event F can occur in n ways and suppose that both cannot occur together, then E or F can occur in $m+n$ ways.

Permutation :-

A Permutation is an arrangement of objects in a definite order.

Permutation of n different objects. The Number of Permutations of n objects Taken all at a Time denoted by the Symbol ^nP_n is given by

$$\boxed{^nP_n = \underline{|n}}$$

$$\underline{|n} = n(n-1)(n-2) - - - - 3 \cdot 2 \cdot 1$$

The number of Permutation of m objects Taken r at a time in $0 \le r \le m$ denoted by ^mP_r is given by m.

$$^mP_r = \frac{\lfloor m}{\lfloor m-r} \quad \text{we assume that } \lfloor 0 = 1$$

when repetition of objects is allowed the Number of Permutations of n Things Taken all at a time when repetion of objects is allowed is n^n.

Q. (1) Find the Number of different Permutations of the set $\{1,2\}$

Sol:- the set $\{1,2\}$ has two different elements 1 and 2

Thus there are $2!$ ie 2 Permutations

their Permutations are 12 and 21.

Ex Find the Number different Permutations of the set $\{1,2,3\}$

Sol:- Total Number of Permutations $= {}^3P_3 = \frac{\lfloor 3}{\lfloor 0} = 6$.

Clearly the Permutations are

123, 132, 312 321, 213, 231,

Permutation with Repetition

Total Number of Permutations when out of m objects in a set. p, objects are exactly alike of one kind (ie p objects are repeated q, objects are exactly alike of second kind, r objects are alike of third kind, and the remaining $m-(p+q+r)$ are dis. objects are dissimilar, is $= \frac{\lfloor m}{\lfloor p, \lfloor q, \lfloor r}$

Q How many numbers can be formed taking only 3 out of the 5 digits 3, 4, 5, 6, 7 ?

Sol:- Clearly each arrangement of 3 digits will give us a new number.

$\therefore$ Required Number $= {}^5P_3 = \frac{\lfloor 5}{\lfloor 2} = 5 \times 4 \times 3 = 60$ Ans

Q In how Many ways can the letters of the word 'ASPERITY' be arranged. Also find the number of ways so that the vowels may never be separated.

Sol:- The word 'ASPERITY' Contains 8 distinct letter

i) Total number of Arrangements $= {}^8P_8 = \lfloor 8 = 40320$

ii) If the vowels are never separated then let us Put vowels within brackets (AEI), then we have only 6 Things NAMELY S, P, R T Y, (AEI) to arrange.

The Number of arrangement of the letters in which vowels stand together in the order (AEI)

$= {}^6P_6 = \lfloor 6 = 720$

Q. In how many ways can the letters of the word 'ALLAHABAD' be arranged?

Sol:- The word 'ALLAHABAD' has 4 A's, 2 L's and three different letters namely H, B and D

Hence the Total Number of Arrangements of the nine letters

$$= \frac{\lfloor 9}{\lfloor 4 . \lfloor 2} = \frac{9 \times 8 \times 7 \times 6 \times 5}{2} = 7560$$

Ans

Circular Permutations:-

The Permutations discussed above may be called linear Permutations since the objects are being arranged in a line. If instead, we arrange objects in a circle or in a simple closed curve, then the Permutations so obtained are called circular Permutations.

To obtain number of circular Permutation when n objects are placed in a circle, we fix one object and arrange remaining (n-1) objects, Thus the Total Number of circular Permutation

$= \lfloor n-1$

Ex In how many ways ~~in which~~ can 9 Persons be seated at a round Table?

Sol:- Total Number of ways in which one of the nine Persons occupy a fixed Position $= 8! = 40320$

Ans

Q. If ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5$ find n.

Combination

An unordered arrangement or unordered selection of objects from a finite set of objects is called a combination.

~~Form~~ From the definition it follows that the order of objects in a combination does not matter while in Permutation order of objects matters.

The number of Combinations (or selections) of r $(r \leq n)$ objects taken from a set of n objects is denoted by ^nC_r or by $C(n,r)$ and is given by

$$^nC_r = C(n,r) = \frac{\lfloor n}{\lfloor r \lfloor n-r}$$ [Repetition of objects is not allowed]

Theorem :- To Prove that

$$^nP_r = \lfloor r \cdot {}^nC_r$$

$$L.H.S = {}^nP_r = \frac{\lfloor n}{\lfloor n-r} = \frac{\lfloor r \cdot \lfloor n}{\lfloor r \lfloor n-r} = \lfloor r \, {}^nC_r$$

$$= R.H.S$$

Theorem (2)

To Prove that

$$^{n}C_{r} = {}^{n}C_{n-r}$$

Proof:-

$$^{n}C_{r} = \frac{\lfloor n}{\lfloor r \, \lfloor n-r} \quad \text{--- (1)}$$

$$^{n}C_{n-r} = \frac{\lfloor n}{\lfloor n-r \, \lfloor \cancel{n}-\cancel{n}+r} = \frac{\lfloor n}{\lfloor r \, \lfloor n-r} = {}^{n}C_{r} \quad \text{--- (2)}$$

from (1) and (2)

$$\boxed{^{n}C_{n-r} = {}^{n}C_{r}}$$

Theorem (3)

To Prove that

$$^{n}C_{r} = {}^{n-1}C_{r} + {}^{n-1}C_{r-1}$$

Theorem (4)

let n and r be Positive integers with $r \leq n$ to Prove that

$$^{n+1}C_{r} = {}^{n}C_{r} + {}^{n}C_{r-1}$$ (Replace $n \to n+1$ Theorem (3))