- 1 A medical test for disease D has outcomes + (positive) and (negative). We assume that
 - the probability for an individual to have the disease is 0.01
 - the probability of a positive test, given that the individual has the disease, is 0.9
 - the probability of a negative test, given that the individual does not have the disease, is 0.9

Compute the probability that an individual has the disease, given that the individual has tested positive. Comment on the quality of the test.

 $P(\text{has disease} - \text{positive test}) = P(\text{positive test} - \text{has disease}) \ P(\text{has disease}) \ / \ P(\text{positive test})$

= 0.9 * 0.01/ P(positive test)

 $P(\text{positive test}) = P(\text{positive test} - \text{has disease}) \ P(\text{has disease}) + P(\text{positive test} - \text{doesn't have disease}) \ P(\text{doesn't have disease})$

= P(positive test — has disease) P(has disease) + (1 - P(negative test — doesn't have disease)) P(doesn't have disease)

= 0.9 * 0.01 + (1 - 0.9) * .99 = 0.108

Therefore, the probability that a patient has the disease given a positive test = 0.9 * 0.01/0.108 = 0.0833 or 8.33% Therefore, the test is not very useful in diagnosing a patient.

3 Let X be a continuous real-valued random variable with the probability density function $f: \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 0 & \text{if } x < 0\\ \frac{4xe^{-x^2}}{(1+e^{-x^2})^2} & \text{if } x \ge 0 \end{cases}$$

(a) Compute the cumulative distribution function $F(x) = \mathbb{P}(X \leq x)$ of X for $x \in \mathbb{R}$

For all x < 0, F(x) = 0 trivially For all $x \ge 0$, $F(x) = \int_0^x f(u) du$ $= \int_0^x \frac{4ue^{-u^2}}{(1+e^{-u^2})^2} du$ $= \frac{-2}{e^{u^2}+1} \Big|_0^x = \frac{e^{x^2}-1}{e^{x^2}+1} = \tanh \frac{x^2}{2}$ Therefore, $F(x) = \begin{cases} 0 & x < 0 \\ \tanh(\frac{x^2}{2}) & x \ge 0 \end{cases}$

(b) Solve the quantile function of X

Given $q \in (0,1)$, $\exists x$ such that F(x) = q $\implies \tanh(\frac{x^2}{2}) = q$, $\forall q \in (0,1)$ $\implies x = \sqrt{2 \tanh^{-1}(q)}$ $\implies F_X^{-1}(q) = \sqrt{2 \tanh^{-1}(q)}$ for $q \in (0,1)$

(c) Compute the probability $\mathbb{P}(0 < X < 1)$. Which value $a \in \mathbb{R}$ satisfies $\mathbb{P}(X \le a) = 0.95$ $P(0 < X < 1) = F(1) - F(0) = \tanh(1/2) \approx 0.462$ by the definition of the quantile function, $a = F_X^{-1}(.95) = \sqrt{2 \tanh^{-1}(.95)} \approx 1.91404$

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(a) The sample space is the following set with four members:

$$\omega = \{(0,0), (1,0), (0,1), (1,1)\}$$

By simple enumeration, it follows:

$$p_X(x) = \begin{cases} 1/4 & \text{if } x = 0\\ 1/2 & \text{if } x = 1\\ 1/4 & \text{if } x = 2 \end{cases}$$

(b) Trivially, F(x) = 0 for x < 0 and F(x) = 1 for $x \ge 2$. By cumulatively summing the probabilities in the probability mass function, we get:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/4 & \text{if } 0 \le x < 1 \\ 3/4 & \text{if } 1 \le x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}$$

(c)
$$F^{-1}(q) = \begin{cases} 0 & \text{if } 0 < q \le 1/4 \\ 1 & \text{if } 1/4 < q \le 3/4 \\ 2 & \text{if } 3/4 < q < 1 \end{cases}$$