1 A medical test for disease D has outcomes + (positive) and - (negative). We assume that

- **the probability for an individual to have the disease is 0.01**
- **the probability of a positive test, given that the individual has the disease, is 0.9**
- **the probability of a negative test, given that the individual does not have the disease, is 0.9**

Compute the probability that an individual has the disease, given that the individual has tested positive. Comment on the quality of the test.

P(has disease | positive test) = P(positive test | has disease) P(has disease) / P(positive test)
= $0.9 * 0.01$/ P(positive test)
P(positive test) = P(positive test | has disease) P(has disease) + P(positive test | doesn't have disease) P(doesn't have disease)
= P(positive test | has disease) P(has disease) + (1 - P(negative test | doesn't have disease)) P(doesn't have disease)
$= 0.9 * 0.01 + (1 - 0.9) * .99 = 0.108$
Therefore, the probability that a patient has the disease given a positive test $= 0.9 * 0.01/0.108 = 0.0833$ or 8.33%
Therefore, the test is not very useful in diagnosing a patient.

3 Let X be a continuous real-valued random variable with the probability density function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 0 & \textbf{if } x < 0 \\ \frac{4xe^{-x^2}}{(1+e^{-x^2})^2} & \textbf{if } x \geq 0 \end{cases}$$

(a) **Compute the cumulative distribution function $F(x) = \mathbb{P}(X \leq x)$ of X for $x \in \mathbb{R}$**
For all $x < 0$, $F(x) = 0$ trivially
For all $x \geq 0, F(x) = \int_0^x f(u)du$
$= \int_0^x \frac{4ue^{-u^2}}{(1+e^{-u^2})^2} du$
$= \frac{-2}{e^{u^2}+1}|_0^x = \frac{e^{x^2}-1}{e^{x^2}+1} = \tanh \frac{x^2}{2}$
Therefore, $F(x) = \begin{cases} 0 & x < 0 \\ \tanh(\frac{x^2}{2}) & x \geq 0 \end{cases}$

(b) **Solve the quantile function of X**
Given $q \in (0, 1), \exists x$ such that $F(x) = q$
$\implies \tanh(\frac{x^2}{2}) = q, \forall q \in (0, 1)$
$\implies x = \sqrt{2 \tanh^{-1}(q)}$
$\implies F_X^{-1}(q) = \sqrt{2 \tanh^{-1}(q)}$ for $q \in (0, 1)$

(c) **Compute the probability $\mathbb{P}(0 < X < 1)$. Which value $a \in \mathbb{R}$ satisfies $\mathbb{P}(X \leq a) = 0.95$**
$P(0 < X < 1) = F(1) - F(0) = \tanh(1/2) \approx 0.462$
by the definition of the quantile function, $a = F_X^{-1}(.95) = \sqrt{2 \tanh^{-1}(.95)} \approx 1.91404$

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(a) The sample space is the following set with four members:

$$\omega = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$$

By simple enumeration, it follows:

$$p_X(x) = \begin{cases} 1/4 & \text{if } x = 0 \\ 1/2 & \text{if } x = 1 \\ 1/4 & \text{if } x = 2 \end{cases}$$

(b) Trivially, $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x \geq 2$. By cumulatively summing the probabilities in the probability mass function, we get:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/4 & \text{if } 0 \leq x < 1 \\ 3/4 & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

(c) $$F^{-1}(q) = \begin{cases} 0 & \text{if } 0 < q \leq 1/4 \\ 1 & \text{if } 1/4 < q \leq 3/4 \\ 2 & \text{if } 3/4 < q < 1 \end{cases}$$