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Q1 To use the inverse transform sampling, first we need to calculate the cumulative distribution function of X.

$$F(x) = \int_0^x \frac{4te^{-t^2}}{(e^{-t^2}+1)^2}\, dt$$

To calculate the integral, let us make the substitution $u = e^{-t^2} + 1$. Then,

$$du = -2te^{-t^2}\, dt \Rightarrow dt = \frac{du}{-2te^{-t^2}}$$

Rearranging gives:

$$F(x) = -2\int_0^x \frac{1}{u^2}\, du = 2\left[-\frac{1}{u}\right]_0^x = 2\left(\frac{1}{e^{-x^2}+1} - \frac{1}{1}\right)$$

Thus, we have:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{2}{e^{-x^2}+1} - 1 & \text{if } x \geq 0 \end{cases}$$

Now, let us calculate the inverse of the CDF:

$$y = \frac{e^{t^2}-1}{e^{t^2}+1}$$

This leads to:

$$e^{t^2}(y+1) = 1 + y$$

$$e^{t^2} = \frac{1+y}{y+1}$$

Taking the natural log:

$$t^2 = \ln\left(\frac{1+y}{y-1}\right)$$

$$t = \sqrt{\ln\left(\frac{1+y}{y-1}\right)}$$

Since the PDF is 0 for $x < 0$, we are interested in the positive root. For sampling from the CDF, the outputs and plots are provided in HW2Q1.ipynb.

Q2 Let $X \sim N(0,1)$ and define $Y = g(X)$, where $g(x) = \tan^{-1}(x)$ for $x \in \mathbb{R}$.

- Draw a sample $X \sim N(0,1)$ of size $n = 10^5$ and plot $g(X)$ as a histogram.
- Derive the probability density function of random variable Y. Plot this alongside the (appropriately normalized) histogram obtained in part (a).

Hint: In Python, you can use `numpy.random.normal(m,sigma,size=n)` to draw a sample containing n entries from the Gaussian distribution $N(m, \sigma^2)$.

```
#Here is the code for your reference, you can check the graphs in HW2Q2.ipynb

#first part of the question
import numpy as np
import matplotlib.pyplot as plt

np.random.seed(0)

# Sample size which was mentioned
```

```
n = 10**5
X = np.random.normal(0, 1, n)
Y = np.arctan(X)

# Plot the histogram of Y
plt.figure(figsize=(10, 6))
plt.hist(Y, bins=100, density=True, alpha=0.7, color='blue', edgecolor='black')
plt.title('Histogram of Y = arctan(X) where X ~ N(0, 1)')
plt.xlabel('Y')
plt.ylabel('Density')
plt.grid()
plt.show()




#second part of question
def fun(y):
    return (1 / np.sqrt(2 * np.pi)) * np.exp(-np.tan(y)**2 / 2) * (1 / np.cos(y)**2)

y_values = np.linspace(-np.pi/2 + 0.01, np.pi/2 - 0.01, 1000)
pdf_values = fun(y_values)

# Plotting
plt.figure(figsize=(10, 6))
plt.hist(Y, bins=100, density=True, alpha=0.5, color='blue', edgecolor='black', label='Histogram of Y')
plt.plot(y_values, pdf_values, color='red', label='PDF of Y', linewidth=2)
plt.title('Histogram and PDF of Y = arctan(X) where X ~ N(0, 1)')
plt.xlabel('Y')
plt.ylabel('Density')
plt.legend()
plt.grid()
plt.show()
```

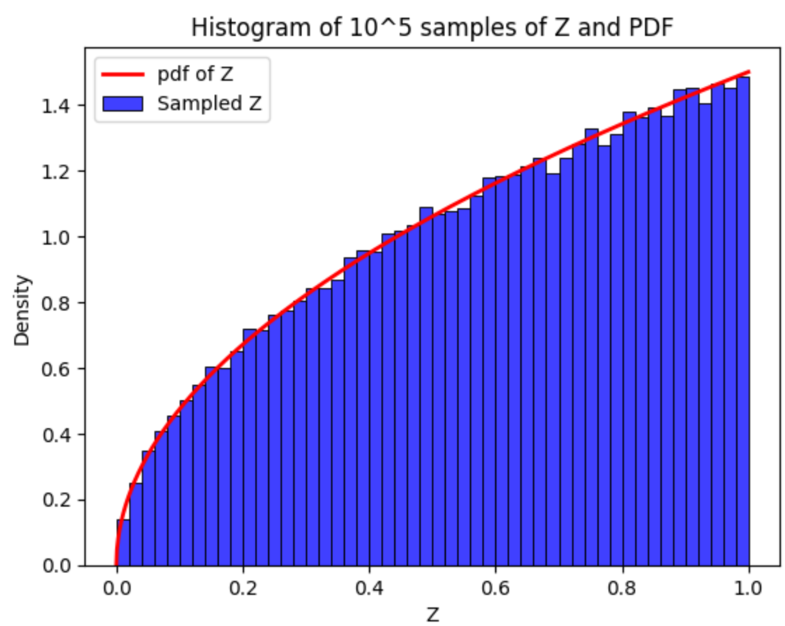
Q3 Let $X, Y \sim U(0,1)$ be independent random variables and define $Z = max(X, Y^2)$

(a) **derive the probability density function of Z.**
$F_Z(z) = P(Z \leq z) = P(max(X, Y^2) \leq z) = P(X \leq z \text{ and } Y^2 \leq z)$
X and Y are independent $\implies F_Z(z) = P(X \leq z)\dot{P}(Y^2 \leq z) = P(X \leq z)\dot{P}(Y \leq \sqrt{z}) = z\sqrt{z}$ for $0 \leq z \leq 1$
$f_Z(z) = \frac{d}{dz}F_Z(z) = \frac{3\sqrt{z}}{2}$
therefore, the PDF of Z is

$$f_Z(z) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{3\sqrt{z}}{2} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x > 1 \end{cases}$$

(b) **Draw a sample of size $n = 10^5$ from the probability distribution of Z and visualize the sample as a histogram. Plot the probability density you obtained in part (a) alongside the (appropriately normalized) histogram.** See HW3Q3.ipynb to run code

```
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns

X = np.random.uniform(0, 1, 10**5)
Y = np.random.uniform(0, 1, 10**5)
Z = np.maximum(X, Y**2)

sns.histplot(Z, bins=50, kde=False, stat='density', color='blue', label="Sampled Z")
z_values = np.linspace(0, 1, 1000)
pdf_values = (3 / 2) * np.sqrt(z_values)

plt.plot(z_values, pdf_values, 'r-', label="pdf of Z", linewidth=2)

plt.title("Histogram of 10^5 samples of Z and PDF")
plt.xlabel("Z")
plt.ylabel("Density")
plt.legend()

plt.show()
```



output:

Q4 Let (X_1, X_2) be a joint random variable, and assume that

$$(\log X_1, \log X_2) \sim N\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2 & -1\\-1 & 2\end{pmatrix}\right).$$

We aim to derive the joint probability density function (PDF) of (X_1, X_2).
Recall that the bivariate Gaussian distribution $Y \sim N(\mu, C)$ for vector $\mu \in \mathbb{R}^2$ and a symmetric, positive definite

matrix $C \in \mathbb{R}^{2\times 2}$ has the probability density function given by:

$$f_Y(y) = \frac{1}{2\pi\sqrt{\det C}} \exp\left(-\frac{1}{2}(y-\mu)^T C^{-1}(y-\mu)\right), \quad y \in \mathbb{R}^2.$$

Step 1: Analysis of Provided Information

1. We are given that (X_1, X_2) is a joint random variable, where:

- Mean vector: $\mu = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- Covariance matrix: $C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

Let $(Y_1, Y_2) = (\log X_1, \log X_2)$, which follows a bivariate normal distribution:
In other words, (Y_1, Y_2) has a joint Gaussian distribution where each variable has a mean of 1, and the covariance matrix C defines how the two variables vary and correlate.

Step 2: Density Function for Y

2. The formula for the bivariate density function is given by:

$$f_Y(y) = \frac{1}{2\pi\sqrt{\det C}} \exp\left(-\frac{1}{2}(y-\mu)^T C^{-1}(y-\mu)\right),$$

where μ is the mean vector, and C is the covariance matrix. This formula has several components:

- $\frac{1}{2\pi\sqrt{\det C}}$: A normalization constant that ensures the density integrates to 1 over the entire space.
- $\exp\left(-\frac{1}{2}(y-\mu)^T C^{-1}(y-\mu)\right)$: An exponential that represents the Gaussian "shape" based on the distance between y and the mean, weighted by the inverse of the covariance matrix C.

Here, y represents a particular realization of the random variables.
3. To find the value of $\det C$, we calculate the determinant of the covariance matrix:

$$\det C = 2 \cdot 2 - (-1)(-1) = 4 - 1 = 3.$$

4. To compute the Gaussian formula, we need the inverse of the covariance matrix C^{-1}. The inverse is calculated as:

$$C^{-1} = \frac{1}{\det C}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Step 3: Substitute the Values into the Density Function

5. Now that we have $\det C$ and C^{-1}, we can plug these into the density function:

$$f_Y(y) = \frac{1}{2\pi\sqrt{3}} \exp\left(-\frac{1}{2}(y - \begin{pmatrix} 1 \\ 1 \end{pmatrix})^T \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}(y - \begin{pmatrix} 1 \\ 1 \end{pmatrix})\right).$$

6. To simplify the exponent, we expand the expression inside the exponent. The quadratic form in the exponent simplifies to:

$$(y - \begin{pmatrix} 1 \\ 1 \end{pmatrix})^T \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} (y - \begin{pmatrix} 1 \\ 1 \end{pmatrix}).$$

This term measures the "distance" from y to the mean μ, adjusted by the covariance structure.

Step 4: Transform the Density of X

7. Change of Variables: We know that $Y_1 = \log X_1$ and $Y_2 = \log X_2$, so Y can be thought of as a transformed version of X. The relationship between the densities f_X and f_Y is given by:

$$f_X(x_1, x_2) = f_Y(y_1, y_2) \cdot \left| \det(J) \right|,$$

where J is the Jacobian determinant that accounts for the change in "volume" when transforming from Y to X.
8. Compute the Jacobian Determinant: The Jacobian matrix of the transformation where $X_1 = e^{Y_1}$ and $X_2 = e^{Y_2}$ is:

$$J = \begin{pmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} \end{pmatrix} = \begin{pmatrix} e^{Y_1} & 0 \\ 0 & e^{Y_2} \end{pmatrix}.$$

The determinant is:

$$\det(J) = e^{Y_1} e^{Y_2} = X_1 X_2.$$

9. Substitute Everything into f_X: Now, we substitute f_Y and the Jacobian determinant into the density function for X:

$$f_X(x_1, x_2) = f_Y(\log x_1, \log x_2) \cdot (x_1 x_2).$$