3 Let X be a continuous real-valued random variable with the probability density function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 0 & \textbf{if } x < 0 \\ \frac{4xe^{-x^2}}{(1+e^{-x^2})^2} & \textbf{if } x \geq 0 \end{cases}$$

(a) **Compute the cumulative distribution function $F(x) = \mathbb{P}(X \leq x)$ of X for $x \in \mathbb{R}$**

For all $x < 0$, $F(x) = 0$ trivially

For all $x \geq 0, F(x) = \int_0^x f(u)du$

$= \int_0^x \frac{4ue^{-u^2}}{(1+e^{-u^2})^2}du$

$= \frac{-2}{e^{u^2}+1}|_0^x = \frac{e^{x^2}-1}{e^{x^2}+1} = \tanh \frac{x^2}{2}$

Therefore, $F(x) = \begin{cases} 0 & x < 0 \\ \tanh(\frac{x^2}{2}) & x \geq 0 \end{cases}$

(b) **Solve the quantile function of X**

Given $q \in (0,1), \exists x$ such that $F(x) = q$

$\implies \tanh(\frac{x^2}{2}) = q, \forall q \in (0,1)$

$\implies x = \sqrt{2\tanh^{-1}(q)}$

$\implies F_X^{-1}(q) = \sqrt{2\tanh^{-1}(q)}$ for $q \in (0,1)$

(c) **Compute the probability $\mathbb{P}(0 < X < 1)$. Which value $a \in \mathbb{R}$ satisfies $\mathbb{P}(X \leq a) = 0.95$**

$P(0 < X < 1) = F(1) - F(0) = \tanh(1/2) \approx 0.462$

by the definition of the quantile function, $a = F_X^{-1}(.95) = \sqrt{2\tanh^{-1}(.95)} \approx 1.91404$

4

(a) The sample space is the following set with four members:

$$\omega = \{(0,0),(1,0),(0,1),(1,1)\}$$

By simple enumeration, it follows:

$$p_X(x) = \begin{cases} 1/4 & \text{if } x = 0 \\ 1/2 & \text{if } x = 1 \\ 1/4 & \text{if } x = 2 \end{cases}$$

(b) Trivially, $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x \geq 2$. By cumulatively summing the probabilities in the probability mass function, we get:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/4 & \text{if } 0 \leq x < 1 \\ 3/4 & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

(c) $F^{-1}(q) = \begin{cases} 0 & \text{if } 0 < q \leq 1/4 \\ 1 & \text{if } 1/4 < q \leq 3/4 \\ 2 & \text{if } 3/4 < q < 1 \end{cases}$